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A combinatorial proof of the Burdzy–Pitman conjecture*

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Abstract

First, we prove the following sharp upper bound for the number of high degree differences in bipartite graphs. Let (U, V, E) be a bipartite graph with $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$. For $n \geq k > \frac{n}{2}$ we show that

$$\sum_{1 \leq i,j \leq n} \mathbb{1}\Big\{ |\deg(u_i) - \deg(v_j)| \geq k \Big\} \leq 2k(n-k).$$

Second, as a corollary, we confirm the Burdzy–Pitman conjecture about the maximal spread of coherent and independent vectors: for $\delta \in (\frac{1}{2}, 1]$ we prove that

$$\mathbb{P}(|X - Y| \geq \delta) \leq 2\delta(1-\delta)$$

for all random vectors (X, Y) satisfying $X = \mathbb{P}(A|\mathcal{G})$ and $Y = \mathbb{P}(A|\mathcal{H})$ for some event A and independent σ-fields $\mathcal{G}$ and $\mathcal{H}$.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that a random vector (X, Y) defined on this probability space is coherent if there exist sub σ-fields $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ and an event $A \in \mathcal{F}$, such that

$$X = \mathbb{E}(\mathbb{1}_A|\mathcal{G}), \qquad Y = \mathbb{E}(\mathbb{1}_A|\mathcal{H}).$$

We will also say that the joint distribution of such (X, Y) is coherent on $[0, 1]^2$. Hereinafter, we write $(X, Y) \in \mathcal{C}$ or $\mu \in \mathcal{C}$ to indicate that the vector (X, Y) or a distribution μ is coherent. By abuse of notation, $\mathcal{C}$ will be used to denote a family of vectors and a family of distributions

As suggested in [6], a coherent vector can be interpreted as objective opinions of two autonomous experts about the odds of some random event A. In this context, we interpret $\mathcal{G}$ and $\mathcal{H}$ as different information sources that are available to the experts. Motivated by this application, it is natural to ask about the maximal possible spread of coherent opinions. Accordingly, Burdzy and Pal [1] proved that for any $\delta \in (\frac{1}{2}, 1]$ and

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$(X,Y) \in \mathcal{C}$ the probability $\mathbb{P}(|X-Y| \geq \delta)$ that the difference between coherent random variables exceeds a given threshold δ is bounded above by $\frac{2(1-\delta)}{2-\delta}$. They go on to show that this bound is sharp and it is attained by a random vector (X,Y) with X and Y being dependent random variables. We will write

$$\mathcal{C}_{\mathcal{I}} = \{(X,Y) \in \mathcal{C} : \; X \perp Y\},$$

to denote the family of those coherent vectors whose components are independent. In this paper we prove the following claim stated as a conjecture by Burdzy and Pitman in [2].

Theorem 1.1. *If $\delta \in (\frac{1}{2}, 1]$ and $(X,Y) \in \mathcal{C}_{\mathcal{I}}$ then*

$$\mathbb{P}(|X-Y| \geq \delta) \; \leq \; 2\delta(1-\delta). \tag{1.1}$$

Moreover, the bound $2\delta(1-\delta)$ is optimal.

In other words, Theorem 1.1 provides a sharp upper bound on the maximal spread of coherent opinions in the special case of two experts with access to independent sources of information. Let us point out that restricting δ to $(\frac{1}{2}, 1]$ does not diminish generality of the result. Consider $X' = \mathbb{1}_A$ and $Y' = \mathbb{P}(A)$ for an arbitrary event A with $\mathbb{P}(A) = \frac{1}{2}$. It is easy to see that $(X', Y') \in \mathcal{C}$. In this case, $\mathbb{P}(|X'-Y'| \geq \frac{1}{2}) = 1$. Hence, for all $\delta \in [0, \frac{1}{2}]$ the problem is trivial.

Let us briefly describe our approach and the organization of the paper. Although there are known alternative characterizations of coherent distributions [6, 7, 9], let us quote [2]:

> For reasons we do not understand well, these general characterizations seem to be of little help in establishing the evaluations of $\epsilon(\delta)$ [i.e. $\mathbb{P}(|X-Y| \geq \delta)$] discussed above, or in settling a number of related problems about coherent distributions [...].

It is our belief that this is indeed so because of the underlying combinatorial nature of these problems. Discretization and combinatorial techniques appeared already in [1, 5]. Moreover, it is a remarkable fact that the properties of two-dimensional coherent vectors are closely related to the properties of degree sequences of bipartite graphs. An intriguing example of this phenomenon can be found in [12]. Therefore, in order to take advantage of the combinatorial nature of the claim made in Theorem 1.1, we start by discussing its graph-theoretic version. More precisely, we prove the following theorem.

Theorem 1.2. *Let $G = (U, V, E)$ be a bipartite graph with an equal bipartition, i.e.*

$$U = \{u_1, u_2, \ldots, u_n\}, \quad V = \{v_1, v_2, \ldots, v_n\},$$

for some $n \in \mathbb{Z}_+$. For $n \geq k > \frac{n}{2}$ we have

$$\sum_{1 \leq i,j \leq n} \mathbb{1}\Big\{|\deg(u_i) - \deg(v_j)| \geq k\Big\} \; \leq \; 2k(n-k). \tag{1.2}$$

Note that the trivial upper bound n^2 is the best possible upper bound in the case $k \leq \frac{n}{2}$. The proof of Theorem 1.2, given in Section 2, is based on an idea similar to the spread bounding theorem of Erdős, Chen, Rousseau and Schelp – see [8, 3]. Later in the same section we provide an elementary example showing that the bound (1.2) is sharp. In Section 3 we show how to transform the Theorem 1.1 to Theorem 1.2. To this end, we make use of an appropriate sampling construction, similar in spirit to [11]. The key idea is to approximate a fixed coherent distribution with a randomly generated sequence of graphs. We then apply Theorem 1.2 to each of the graphs in the sequence and obtain (1.1) by passing to the limit.

2 Number of high degree differences in bipartite graphs

Let $G = (U, V, E)$ be a bipartite graph with an equal bipartition, that is a triplet

$$U = \{u_1, u_2, \dots, u_n\}, \quad V = \{v_1, v_2, \dots, v_n\},$$

and

$$E \subset U \times V,$$

for some fixed $n \in \mathbb{Z}_+$. Let us fix a natural number k satisfying $n \geq k > \frac{n}{2}$. Hereinafter, we denote the degree sequences of G as $(\alpha_i)_{i=1}^n$ and $(\beta_j)_{j=1}^n$, i.e., $\alpha_i = \deg(u_i)$ and $\beta_j = \deg(v_j)$ for all $1 \leq i, j \leq n$. Without loss of generality we also assume that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n,$$

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n.$$

We start with an observation similar to the spread bounding theorem of Erdős et al. – see [8].

Lemma 2.1. *There exist* $s, t \in \{1, 2, \dots, n-k+1\}$ *such that* $\alpha_s \leq \beta_{s+k-1} + k - 1$ *and* $\beta_t \leq \alpha_{t+k-1} + k - 1$.

Proof. We will prove only the existence of s, as the case of t is analogous. Assume for the sake of contradiction that such a number s does not exists. Therefore, the total number of edges incident to $u_1, u_2, \dots, u_{n-k+1}$ is at least $\beta_k + \beta_{k+1} + \cdots + \beta_n + k(n-k+1)$. Observe that at least $k(n-k+1)$ of these edges go to vertices $v_1, v_2, \dots, v_{k-1}$. Let us denote

$$\tilde{E} := E \cap \Big(\{u_1, u_2, \dots, u_{n-k+1}\} \times \{v_1, v_2, \dots, v_{k-1}\}\Big).$$

We have just shown that $|\tilde{E}| \geq k(n-k+1)$. On the other hand, we clearly have

$$|\tilde{E}| \leq (k-1)(n-k+1),$$

which is a contradiction. □

Proof of Theorem 1.2. For $1 \leq i, j \leq n$, let us call (i, j) an $\mathcal{A}$-pair if $\alpha_i \geq \beta_j + k$. Analogously, let us call (i, j) a $\mathcal{B}$-pair if $\beta_j \geq \alpha_i + k$. Since $k > \frac{n}{2}$, we have $\alpha_i > \frac{n}{2}$ for all $\mathcal{A}$-pairs (i, j) and $\alpha_i < \frac{n}{2}$ for all $\mathcal{B}$-pairs (i, j). As a consequence, there exists an $i_0 \in \{1, 2, \dots, n+1\}$ such that:

1. $i \leq i_0 - 1$ for any $\mathcal{A}$-pair (i, j),
2. $i \geq i_0$ for any $\mathcal{B}$-pair (i, j).

Analogously, there exists $j_0 \in \{1, 2, \dots, n+1\}$ such that:

3. $j \leq j_0 - 1$ for any $\mathcal{B}$-pair (i, j),
4. $j \geq j_0$ for any $\mathcal{A}$-pair (i, j).

Observe that by Lemma 2.1,

5. for any $\mathcal{A}$-pair (i, j) either $i < s$ or $j > s + k - 1$,
6. for any $\mathcal{B}$-pair pair (i, j) either $j < t$ or $i > t + k - 1$.

We will now show that conditions 1–6 imply that the total number of $\mathcal{A}$-pairs and $\mathcal{B}$-pairs is at most $2k(n-k)$. Let us fix $i_0, j_0 \in \{1, 2, \dots, n+1\}$. First, we will show that it is sufficient to consider only s and t such that $s, t \in \{1, n-k+1\}$ because these values of s and t are optimal in the sense that they maximize the total number of pairs (i, j) fulfilling all conditions 1–6.

Note that the variable s appears only in the 5-th condition and thus the value of s is not relevant for bounding the number of $\mathcal{B}$-pairs. Moreover, observe that if $i_0 \le n-k+1$, then for $s = n-k+1$ condition 5 is automatically fulfilled and thus $s = n-k+1$ is an optimal value. Similarly, if $j_0 \ge k+1$, then for $s = 1$ condition 5 is also automatically fulfilled and $s = 1$ is an optimal value. Finally, let us assume that $i_0 \ge n-k+2$ and $j_0 \le k$. In this case, the restrictions imposed by condition 5 remove exactly $(i_0 - s)(s + k - j_0)$ additional pairs. Therefore, as the last expression is a concave function of $s \in [1, n-k+1]$, it is minimized in one of the endpoints. Hence we may assume that $s = 1$ or $s = n-k+1$, as desired. Analogously, we show that $t = 1$ or $t = n-k+1$ is optimal. There are four possible cases now:

a. $s = 1,\ t = n-k+1$. We have $j \ge k+1$ for all $\mathcal{A}$-pairs and $j \le n-k$ for all $\mathcal{B}$-pairs (i, j). Thus any i participates in at most $n-k$ of $\mathcal{A}$-pairs and in at most $n-k$ of $\mathcal{B}$-pairs. Therefore, since a fixed vertex can not participate in both types of pairs, every i participates overall in at most $n-k$ pairs. As a consequence, the total number of pairs does not exceed $n(n-k) < 2k(n-k)$.

b. $s = n-k+1,\ t = 1$. This case is symmetric to the previous one.

c. $s = 1,\ t = 1$. We have $j \ge k+1$ for all $\mathcal{A}$-pairs and $i \ge k+1$ for all $\mathcal{B}$-pairs (i, j). Let us denote $a := \max(k+1, j_0)$ and $b := \max(k+1, i_0)$. Then the total number of $\mathcal{A}$-pairs is bounded by $(n-a+1)(b-1)$, while the total number of $\mathcal{B}$-pairs is at most $(n-b+1)(a-1)$. Notice, that for $a, b \in [k+1, n+1]$ the sum

$$S := (n-a+1)(b-1) + (n-b+1)(a-1),$$

is bilinear and it is maximized at one of four corners. For $a = b = k+1$, we get $S = 2k(n-k)$. For, say $a = n+1$, we get $S = n(n-b+1) \leqslant n(n-k) < 2k(n-k)$.

d. $s = n-k+1,\ t = n-k+1$. This case is analogous to c.

Hence we have shown that Theorem 1.2 holds in all cases. This ends the proof. $\square$

We end this section with an example showing that the upper bound $2k(n-k)$ in (1.2) cannot be improved. Note that a straightforward modification of this example shows that $2\delta(1-\delta)$ in (1.1) is also sharp.

Example 2.2. Consider $n, k \in \mathbb{Z}_+$, with $n \ge k > \frac{n}{2}$. Let $G_{n,k} = (U, V, E)$, where $U = V = \{1, 2, \dots, n\}$ and

$$E = \{(u, v) \in U \times V :\ \max(u, v) \le k\}.$$

We clearly have

$$\sum_{1 \le i,j \le n} \mathbb{1}\Big\{ |\deg(u_i) - \deg(v_j)| \ge k \Big\} = 2k(n-k).$$

Moreover, one can check that inequality (1.2) becomes an equality exactly for those graphs G that are isomorphic to $G_{n,k}$ or to its complement $\overline{G}_{n,k}$. This follows easily from the proof of Theorem 1.2 and we leave the details to interested reader.

3 Proof of the Burdzy–Pitman conjecture

By $\mathcal{C}_{\mathcal{I}}(m)$ we denote the set of $(X,Y)\in\mathcal{C}_{\mathcal{I}}$ such that both X and Y take at most m different values.

Proposition 3.1. *Let (X,Y) be coherent and independent, and let m be a positive integer. Then there exists $(X_m,Y_m)\in\mathcal{C}_{\mathcal{I}}(m)$, such that $|X-X_m|\le\frac{1}{m}$ and $|Y-Y_m|\le\frac{1}{m}$, almost surely.*

The proof of the above Proposition can be found in [4, 1]. In what follows, fix any $\delta\in(\frac{1}{2},1]$.

Proposition 3.2. *To prove Theorem 1.1 it is enough to verify it for all $(X,Y)\in\mathcal{C}_{\mathcal{I}}(m)$, $m\ge 1$.*

Proof. Fix $(X,Y)\in\mathcal{C}_{\mathcal{I}}$ and choose (X_m,Y_m) as in Proposition 3.1. By the triangle inequality we get

$$\mathbb{P}(|X-Y|\ge\delta)\ \le\ \mathbb{P}(|X_m-Y_m|\ge\delta-2/m).$$

Thus, assuming that Theorem 1.1 is true for all $(X,Y)\in\ \cup_{m=1}^{\infty}\mathcal{C}_{\mathcal{I}}(m)$, for m large enough so that $\delta-2/m>1/2$, we obtain

$$\mathbb{P}(|X-Y|\ge\delta)\ \le\ 2(\delta-2/m)(1-\delta+2/m).$$

Letting $m\to\infty$ completes the proof. □

We are now able to prove our main result.

Proof of Theorem 1.1. Fix $(X,Y)\in\bigcup_{m=1}^{\infty}\mathcal{C}_{\mathcal{I}}(m)$. There exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$, independent sub σ-fields $\mathcal{G},\mathcal{H}\subset\mathcal{F}$ and an event $A\in\mathcal{F}$, such that $X=\mathbb{E}(\mathbb{1}_A|\mathcal{G})$ and $Y=\mathbb{E}(\mathbb{1}_A|\mathcal{H})$. Furthermore, for some $N,M\in\mathbb{Z}_+$, we may suppose that X takes values $x_1,x_2,\dots,x_N$ on sets $G_1,G_2,\dots,G_N$ and Y takes values $y_1,y_2,\dots,y_M$ on sets $H_1,H_2,\dots,H_M$. We can also assume without loss of generality that

$$\mathcal{G}=\sigma\Big(G_1,G_2,\dots,G_N\Big),$$

$$\mathcal{H}=\sigma\Big(H_1,H_2,\dots,H_M\Big).$$

For $1\le i\le N$ and $1\le j\le M$, let $p_i=\mathbb{P}(G_i),q_j=\mathbb{P}(H_j)$ and

$$\rho_{i,j}=\frac{\mathbb{P}(G_i\cap H_j\cap A)}{\mathbb{P}(G_i\cap H_j)}.$$

Then by independence we have $\mathbb{P}(G_i\cap H_j)=p_iq_j$ and

$$x_i=\sum_{j=1}^{M}q_j\rho_{i,j},\qquad 1\le i\le N, \tag{3.1}$$

$$y_j=\sum_{i=1}^{N}p_i\rho_{i,j},\qquad 1\le j\le M. \tag{3.2}$$

First, we show how to construct a sequence of bipartite graphs $G_n=(U_n,V_n,E_n)$ with $|U_n|=|V_n|=n$, such that:

(C1) there are $p_in+O(n^{3/4})$ vertices in U_n of degree $x_in+O(n^{3/4})$, $i=1,2,\dots,N$,

(C2) there are $q_jn+O(n^{3/4})$ vertices in V_n of degree $y_jn+O(n^{3/4})$, $j=1,2,\dots,M$, where by $O(n^{3/4})$ we denote any quantity bounded in magnitude by $Cn^{3/4}$ for some constant $C < \infty$ independent of n, N, M, i and j.

Fix $n \geq 1$ and choose n independent points $u_1, u_2, \ldots, u_n$ in the initial space Ω (distributed according to $\mathbb{P}$) and for $1 \leq i \leq n$ denote $\alpha_i = s$ if $u_i \in G_s$. In other words, $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an i.i.d. sample from the set $\{1, 2, \ldots, N\}$ with weights $p_1, p_2, \ldots, p_N$, respectively. We can think about this sample as a randomly generated sequence of labels. Let $A_s = \sum_{i=1}^n \mathbb{1}_{\{\alpha_i = s\}}$ be the number of labels equal to s, $1 \leq s \leq N$. Observe that A_s is the sum of n independent Bernoulli random variables. Hence, by Hoeffding's inequality [10], we have

$$\mathbb{P}(|A_s - np_s| \geq nr) \;\leq\; 2 \cdot e^{-2nr^2},$$

for all positive r. Consequently, setting $r = n^{-1/4}$ we get

$$\mathbb{P}(|A_s - np_s| \geq n^{3/4}) \;\leq\; 2 \cdot e^{-2\sqrt{n}}.$$

Thus, for large n, with high probability we have $|A_s - np_s| < n^{3/4}$ simultaneously for all $1 \leq s \leq N$.

Analogously, we choose points $v_1, v_2, \ldots, v_n$ and generate an i.i.d. sample $(\beta_1, \beta_2, \ldots, \beta_n)$ from the set $\{1, 2, \ldots, M\}$ with weights $q_1, q_2, \ldots, q_M$. If $B_t = \sum_{j=1}^n \mathbb{1}_{\{\beta_j = t\}}$ for $1 \leq t \leq M$ then, for large n, with high probability $|B_t - nq_t| < n^{3/4}$ for all t simultaneously.

Given the points $(u_i)_{i=1}^n$ and $(v_j)_{j=1}^n$ and the corresponding labels $(\alpha_i)_{i=1}^n$ and $(\beta_j)_{j=1}^n$, we will generate independently edges of a random bipartite graph (U_n, V_n, E_n), where $U_n = \{u_1, u_2, \ldots, u_n\}$ and $V_n = \{v_1, v_2, \ldots, v_n\}$. The subscripts on $\mathbb{P}_{\alpha,\beta}$ and $\mathbb{E}_{\alpha,\beta}$ will denote conditioning on $(\alpha_i)_{i=1}^n$ and $(\beta_j)_{j=1}^n$.

1. Generate independent indicator random variables $Z_{i,j}$ for $1 \leq i, j \leq n$ satisfying

$$\mathbb{P}_{\alpha,\beta}(Z_{i,j} = 1) = 1 - \mathbb{P}_{\alpha,\beta}(Z_{i,j} = 0) = \rho_{\alpha_i, \beta_j},$$

2. for $1 \leq i, j \leq n$, set $(u_i, v_j) \in E_n$ iff $Z_{i,j} = 1$.

Hence, $Z_{i,j} = \mathbb{1}_{\{(u_i, v_j) \in E_n\}}$. For $1 \leq i \leq n$,

$$\mathbb{E}_{\alpha,\beta} \deg(u_i) \;=\; \mathbb{E}_{\alpha,\beta}\left(\sum_{j=1}^n Z_{i,j}\right) \;=\; \sum_{t=1}^M B_t \rho_{\alpha_i, t} \;=\; \sum_{t=1}^M \Big(nq_t + O(n^{3/4})\Big)\rho_{\alpha_i, t},$$

and hence, by (3.1),

$$\mathbb{E}_{\alpha,\beta} \deg(u_i) \;=\; nx_{\alpha_i} + O(n^{3/4}). \tag{3.3}$$

Similarly, for $1 \leq j \leq n$, by (3.2) we get

$$\mathbb{E}_{\alpha,\beta} \deg(v_j) \;=\; ny_{\beta_j} + O(n^{3/4}). \tag{3.4}$$

We apply Hoeffdings's inequality again to obtain

$$\mathbb{P}_{\alpha,\beta}\Big(|\deg(u_i) - \mathbb{E}_{\alpha,\beta} \deg(u_i)| \geq n^{3/4}\Big) \;\leq\; 2 \cdot e^{-2\sqrt{n}}, \tag{3.5}$$

and

$$\mathbb{P}_{\alpha,\beta}\Big(|\deg(v_j) - \mathbb{E}_{\alpha,\beta} \deg(v_j)| \geq n^{3/4}\Big) \;\leq\; 2 \cdot e^{-2\sqrt{n}}, \tag{3.6}$$

for all $i, j \in \{1, 2, \ldots, n\}$. Note that the concentration rates (3.5) and (3.6) are exponential in $\sqrt{n}$. Thus, since n is large, with high probability all these inequalities hold simultaneously. Then, by (3.3) and (3.4), we have $\deg(u_i) = nx_{\alpha_i} + O(n^{3/4})$ and $\deg(v_j) = ny_{\beta_j} + O(n^{3/4})$ for all $i, j \in \{1, 2, \ldots, n\}$ with high probability. This, together with bounds on $(A_s)_{s=1}^N$ and $(B_t)_{t=1}^M$, proves that (deterministic) G_n's satisfying conditions (C1)-(C2) exist for large n.

In what follows, we add additional subscripts and write $u_i^{(n)}$ and $v_j^{(n)}$ for generic elements of U_n and V_n, respectively. We can now write

$$\mathbb{P}(|X-Y|\geq\delta) \;=\; \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} \mathbb{1}_{\{|x_i-y_j|\geq\delta\}}\cdot p_i q_j$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\sum_{\substack{1\leq i\leq N\\ 1\leq j\leq M}} \mathbb{1}_{\{|nx_i-ny_j|\geq n\delta\}}\cdot\Big(p_i n+O(n^{3/4})\Big)\Big(q_j n+O(n^{3/4})\Big). \tag{3.7}$$

By the triangle inequality

$$|nx_{\alpha_i}-ny_{\beta_j}| \;\leq\; |\deg(u_i^{(n)})-\deg(v_j^{(n)})|+2\cdot O(n^{3/4}),$$

for all $i,j\in\{1,2,\dots,n\}$. This and (C1)-(C2) imply that we can bound the right hand side of (3.7) by

$$\leq\; \limsup_{n\to\infty}\frac{1}{n^2}\sum_{1\leq i,j\leq n}\mathbb{1}\Big\{|\deg(u_i^{(n)})-\deg(v_j^{(n)})|\geq n\delta-2O(n^{3/4})\Big\}.$$

Finally, applying Theorem 1.2 to bipartite graphs G_n, we obtain

$$\leq\; \limsup_{n\to\infty}\frac{1}{n^2}\cdot 2\Big(n\delta-2O(n^{3/4})\Big)\Big(n-n\delta+2O(n^{3/4})\Big) \;=\; 2\delta(1-\delta),$$

which ends the proof. □

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