

Simple sufficient condition for inadmissibility of Moran’s single-split test*

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Abstract: Suppose that a statistician observes two independent variates X_1 and X_2 having densities $f_i(\cdot; \theta) \equiv f_i(\cdot - \theta)$, $i = 1, 2$, $\theta \in \mathbb{R}$. His purpose is to conduct a test for

$$H : \theta = 0 \text{ vs. } K : \theta \in \mathbb{R} \setminus \{0\}$$

with a pre-defined significance level $\alpha \in (0, 1)$. Moran (1973) suggested a test which is based on a single split of the data, *i.e.*, to use X_2 in order to conduct a one-sided test in the direction of X_1 . Specifically, if b_1 and b_2 are the $(1 - \alpha)$ 'th and α 'th quantiles associated with the distribution of X_2 under H , then Moran's test has a rejection zone

$$(a, \infty) \times (b_1, \infty) \cup (-\infty, a) \times (-\infty, b_2)$$

where $a \in \mathbb{R}$ is a design parameter. Motivated by this issue, the current work includes an analysis of a new notion, *regular admissibility* of tests. It turns out that the theory regarding this kind of admissibility leads to a simple sufficient condition on $f_1(\cdot)$ and $f_2(\cdot)$ under which Moran's test is inadmissible.

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1. Introduction

Assume that a statistician observes iid one-dimensional Gaussians with mean $\theta \in \mathbb{R}$ and unit variance. The purpose of the statistician is to test $H : \theta = 0$ versus $K : \theta \in \mathbb{R} \setminus \{0\}$ with a pre-defined significance level $\alpha \in (0, 1)$. In [17], Moran provided a discussion regarding a test which is based on data-splitting. Namely, divide the data into two sub-samples. If the mean of the first sub-sample is positive, then use the mean of the second sub-sample to conduct an α -level upper-tailed Z-test. Otherwise, use the mean of the second sub-sample to conduct an α -level lower-tailed Z-test. Intuitively, it seems reasonable to expect that once θ is far away from the origin, then Moran's test leads to some gain relative to an α -level two-sided Z-test. Interestingly, this intuition fails because Moran's suggestion is an unbiased test and it is well-known that the two-sided Z-test is a uniformly most powerful (UMP) unbiased test.

Of-course, this test might be applied also when the data are not distributed according to a Gaussian distribution. The current research query is whether Moran's test is inadmissible under general assumptions regarding the data distribution? Specifically, assume that the mean of the i 'th ($i = 1, 2$) sub-sample has a distribution with a density function $f_i(\cdot; \theta) \equiv f_i(\cdot - \theta)$, $\theta \in \mathbb{R}$. In this setup, let b_1 and b_2 be the $(1 - \alpha)$ 'th and α 'th quantiles of the distribution whose density is $f_2(\cdot; 0)$. Then, Moran's test for $H : \theta = 0$ versus $K : \theta \in \mathbb{R} \setminus \{0\}$ has a rejection zone

$$(a, \infty) \times (b_1, \infty) \cup (-\infty, a) \times (-\infty, b_2) \quad (1.1)$$

where $a \in \mathbb{R}$ is a design parameter. In this work, it is shown that the following condition is sufficient for inadmissibility of Moran's test.

Condition 1.

- (a) For every $i = 1, 2$, $f_i(\cdot)$ is continuous, bounded and positive on \mathbb{R} .
 (b) For every $\theta \in (0, \infty)$:

$$\lim_{x \rightarrow \infty} \frac{f_2(x - \theta)}{f_2(x)} = \infty \quad , \quad \lim_{x \rightarrow -\infty} \frac{f_2(x - \theta)}{f_2(x)} < \infty. \quad (1.2)$$

(c) For every $\theta \in (-\infty, 0)$:

$$\lim_{x \rightarrow \infty} \frac{f_2(x - \theta)}{f_2(x)} < \infty \quad , \quad \lim_{x \rightarrow -\infty} \frac{f_2(x - \theta)}{f_2(x)} = \infty . \quad (1.3)$$

In particular, when $f_2(\cdot)$ is even, (b) and (c) may be unified into the condition

$$\lim_{x \rightarrow \infty} \frac{f_2(x - \theta)}{f_2(x)} = \infty \quad , \quad \forall \theta \in (0, \infty) . \quad (1.4)$$

Then, it is easy to apply this condition for specific examples of data distributions (including the Gaussian case) in order to prove that Moran's test is inadmissible for them.

An additional contribution of this work is by showing that Moran's test for the multi-dimensional Gaussian setup is inadmissible. More precisely, assume that the observations are d -dimensional Gaussians ($2 \leq d < \infty$) with mean $\theta \in \mathbb{R}^d$ and an identity covariance matrix. Then, the purpose of the statistician is to test

$$H : \theta = \mathbf{0} \quad \text{vs.} \quad K : \theta \in \mathbb{R}^d \setminus \{\mathbf{0}\} \quad (1.5)$$

where $\mathbf{0}$ is the zero-vector in \mathbb{R}^d . Let \bar{X}_1 be the mean of the first sub-sample. Then, Moran [17] suggested to use the second sub-sample in order to test a simple hypothesis in the direction of \bar{X}_1 . That is, consider a most powerful (MP) test for the simple hypothesis testing

$$H : \theta = \mathbf{0} \quad , \quad K(\bar{X}_1) : \theta = \frac{\bar{X}_1}{\|\bar{X}_1\|} \quad (1.6)$$

with a pre-defined significant level $\alpha \in (0, 1)$ where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^d . Standard likelihood-ratio calculations yield that the rejection zone of this test equals to

$$\left\{ x_1, x_2, \dots, x_n \in \mathbb{R}^d \text{ s.t. } \bar{x}_1 \neq \mathbf{0} ; \frac{\bar{x}_1 \cdot \bar{x}_2}{\|\bar{x}_1\|} > D \right\} \quad (1.7)$$

where \cdot denotes dot product, $\bar{x}_1 \equiv \frac{1}{m} \sum_{i=1}^m x_i$, $\bar{x}_2 \equiv \frac{1}{k} \sum_{i=m+1}^n x_i$ and $D > 0$ is a constant which is determined uniquely by the vector (α, m, n) .

Motivated by these applications, the theoretical contribution of this work is due to an analysis of a new notion, *regular admissibility* of tests. Section 2 includes a detailed description of a general model in which the theory regarding regular admissibility is established. Roughly speaking, the main result of Section 3 is that any test which satisfies regular admissibility is an MP test in a simple hypothesis testing problem of the original null H versus a Bayes-mixture of simple alternatives under K . In Section 4, it is shown how the fundamental Lemma of Neyman and Pearson naturally leads to a sufficient condition for inadmissibility. This condition is applied in order to show that Moran's test is inadmissible. Specifically, Section 5 includes an application of the results from the previous sections to the proof of Condition 1. Section 6 is about another application to the multi-dimensional Gaussian case. Section 7 is a short discussion

regarding the way in which the current results should be interpreted including several directions for further research. Finally, Section 8 includes all the proofs for the claims in this paper.

1.1. Related literature

In [17], Moran wrote that data-splitting is an approach in which much of natural science proceeds, *i.e.*, one examines a large body of evidence, notices certain empirical features, and then proceeds to test if these are real. Despite this fact, as mentioned by Romano *et al.* [18], it turns out that the statistical literature does not contain much work in this direction. Besides [17], there is another work of Cox [6, 7] which is about an application of data-splitting to an hypothesis testing regarding the mean of a Gaussian population with a sparse alternative.

Naturally, tests which are based on data-splitting are exposed to criticism because they are not invariant to permutations of the data. In order to relieve this effect, much of the modern research in this area is focused on tests which are based on many splits (unlike [6, 7] and [17] which are based on a single split). Namely, from each sub-sample it is possible to extract a p-value. Then, the question has become how to combine these p-values? For several works in this direction see, *e.g.*, [8, 9, 18] and the references therein.

Recently, Vovk *et al.* [23, 24] suggested to use e-values instead of p-values. By doing so, they claimed that the problem of using a test which is not invariant to permutations of the data becomes much less serious. For more information about e-values, see [21].

Another branch of literature which is related to the current work is about complete class theorems for hypothesis testing problems with simple null versus a composite alternative. Several works in this direction are *e.g.*, [4, 5, 16]. In addition, some relevant surveys regarding admissibility are [13, 20]. Importantly, note that there is a literature about admissibility and complete class theorems in the broader context of decision theory. An introduction to this topic is given in Section 2 of [10]. For a more advanced treatment, see, *e.g.*, [1, 14] and the references therein.

2. Preliminaries

Consider a measurable space (Ω, \mathcal{H}) and a finite-dimensional Banach space $(\Theta, \|\cdot\|)$. In particular, let $\mathbf{0}$ be the zero vector which is associated with Θ . Then, it is known that both Θ and $\Theta_K \equiv \Theta \setminus \{\mathbf{0}\}$ are locally compact and σ -compact (For exact definitions, see, *e.g.*, Section 2.3 and Section 2.16 in [19]). In addition, assume that X is a (Ω, \mathcal{H}) -measurable function which receives values in some metric space \mathcal{X} . Then, for every $\theta \in \Theta$, let P_θ be a probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ where \mathcal{B} is the Borel σ -field of the corresponding topological space. Then, assume that a statistician who does not know the value of θ observes X which is a realization of P_θ (here there is a slight an abuse of notation). His

intention is to test

$$H : \theta = \mathbf{0} \text{ vs. } K : \theta \in \Theta_K \quad (2.1)$$

with a pre-defined significance level $\alpha \in (0, 1)$.

Denote

$$\Phi \equiv \{\phi : \mathcal{X} \rightarrow [0, 1] ; \phi \text{ is a Borel function}\} \quad (2.2)$$

which is the set of all tests and for every $\phi \in \Phi$ define the power function

$$\beta_\phi(\theta) \equiv \int_{\mathcal{X}} \phi(x) dP_\theta(x) \quad , \quad \forall \theta \in \Theta \quad (2.3)$$

and its complement $\gamma_\phi \equiv 1 - \beta_\phi$. In particular, note that for every $\theta \in \Theta_K$, $\gamma_\phi(\theta)$ equals to the risk which is associated with the test ϕ under θ . In addition, notice that if $t \in (0, 1)$ and $\phi_1, \phi_2 \in \Phi$, then

$$\phi_t \equiv t\phi_1 + (1 - t)\phi_2 \in \Phi \quad (2.4)$$

and

$$\beta_{\phi_t}(\theta) = t\beta_{\phi_1}(\theta) + (1 - t)\beta_{\phi_2}(\theta) \quad , \quad \forall \theta \in \Theta. \quad (2.5)$$

Now, for every $\Gamma \in \{\Theta, \Theta_K\}$ let $\mathcal{C}(\Gamma)$ be the set of all functions $f : \Gamma \rightarrow \mathbb{R}$ which are continuous on Γ . Then, denote the space of continuous functions which vanish in infinity by $\mathcal{C}_0(\Gamma)$. Namely, $f \in \mathcal{C}_0(\Gamma)$ if and only if (iff) $f \in \mathcal{C}(\Gamma)$ and for every $\epsilon > 0$, there exists a compact set $K_f(\epsilon) \subseteq \Gamma$ such that

$$\{\theta \in \Gamma ; |f(\theta)| \geq \epsilon\} \subseteq K_f(\epsilon). \quad (2.6)$$

Similarly, $\mathcal{C}_c(\Gamma)$ is a notation for the space of all compactly supported functions. Namely, $f \in \mathcal{C}_c(\Gamma)$ iff $f \in \mathcal{C}(\Gamma)$ and there is a compact set $K_f \subseteq \Gamma$ such that

$$\{\theta \in \Gamma ; f(\theta) \neq 0\} \subseteq K_f. \quad (2.7)$$

The following simple lemma will be helpful later on. In order to make its statement, for every function $f : \Theta \rightarrow \mathbb{R}$, let $f|_{\Theta_K}$ be the restriction of f to Θ_K .

Lemma 1. *There exists a function $f \in \mathcal{C}_0(\Theta)$ such that:*

1. $f|_{\Theta_K} \in \mathcal{C}_0(\Theta_K)$.
2. $f(\mathbf{0}) < f(\theta)$ for every $\theta \in \Theta_K$.

3. Regular admissibility and α -level Bayes tests

Consider the following definitions:

Definition 1. Let

$$\Phi(\alpha) \equiv \{\phi \in \Phi ; \beta_\phi(\mathbf{0}) = \alpha\}. \quad (3.1)$$

Then, $\phi \in \Phi(\alpha)$ is a *regular* test (of level α) iff $\gamma_\phi \in \mathcal{C}_0(\Theta)$. In addition, let $\Phi_0(\alpha)$ be the set of all regular tests (of level α).

Definition 2. Let $\phi \in S \subseteq \Phi(\alpha)$. Then, ϕ is an admissible test in S iff for every $\phi' \in S$, the condition

$$\beta_\phi(\theta) \leq \beta_{\phi'}(\theta) \quad , \quad \forall \theta \in \Theta_K \quad (3.2)$$

implies that

$$\beta_\phi(\theta) = \beta_{\phi'}(\theta) \quad , \quad \forall \theta \in \Theta_K . \quad (3.3)$$

Otherwise, ϕ is inadmissible in S . Correspondingly, denote the set of all admissible tests in S by $\mathcal{A}_S(\alpha)$.

In particular, $\mathcal{A}(\alpha) \equiv \mathcal{A}_{\Phi(\alpha)}(\alpha)$ is the set of all admissible tests. In addition, denote the set $\mathcal{A}_0(\alpha) \equiv \mathcal{A}_{\Phi_0(\alpha)}(\alpha)$ and each test $\phi \in \mathcal{A}_0(\alpha)$ is said to have the property of regular admissibility.

Definition 3. Let Π be the set of all probability measures on $(\Theta_K, \mathcal{B}(\Theta_K))$ and consider some $\pi \in \Pi$. In addition, let $S \subseteq \Phi(\alpha)$ be such that for every $\phi' \in S$, $\theta \mapsto \gamma_{\phi'}(\theta)$ is a Borel function on Θ_K . Then, $\phi \in S$ is an α -level π -Bayes test in S iff

$$\int_{\Theta_K} \beta_\phi(\theta) d\pi(\theta) \geq \int_{\Theta_K} \beta_{\phi'}(\theta) d\pi(\theta) \quad , \quad \forall \phi' \in S . \quad (3.4)$$

Remark 1. To the best of the author's knowledge, the concept of a regular test as described in Definition 1 is new. Generally speaking, this form of regularity reflects the expectation that the power function of a sensible test should tend to one as the alternative diverges from the null.

Remark 2. Regular admissibility is a concept which is derived from the standard concept of admissibility (see, *e.g.*, Equation (6.18) in [15]) by ignoring all tests which are not regular in the sense of Definition 1. For other concepts of admissibility which are related to hypothesis testing, see, *e.g.*, Section 6.7 in [15] and [5] with the references therein.

Remark 3. Albeit the notion which is introduced in Definition 3 resembles the standard definition of a Bayes rule (see, *e.g.*, Equation (1.15) in [10]), there is a difference. To see this, take π which is a probability measure on $(\Theta, \mathcal{B}(\Theta))$ such that $\pi(\Theta_K) = 1$. In addition, denote the restriction of π to Θ_K by $\pi|_{\Theta_K}$. It is straightforward that a statistician with prior belief which is represented by π will reject the null hypothesis for sure. On the other hand, if ϕ is an α -level $\pi|_{\Theta_K}$ -Bayes test, then $\beta_\phi(\mathbf{0}) = \alpha < 1$ which makes it different.

Theorem 1. *If $\phi \in \mathcal{A}_0(\alpha)$, then there exists a regular probability measure $\pi \in \Pi$ for which ϕ is an α -level π -Bayes test in $\Phi_0(\alpha)$.*

Remark 4. In fact, the measure π which is defined in the statement of Theorem 1 stems from an application of the Riesz-Markov-Kakutani representation theorem. Therefore, it is regular in the sense that it satisfies the regularity properties which appear in the statement of this theorem (see Section 2.14 in [19]). Importantly, the regularity of π is not important for the analysis to follow in the sequel.

It is well-known (see, *e.g.*, Problem 3.20 in [15]) that any α -level π -Bayes test is admissible. Consequently, Theorem 1 implies the following corollary.

Corollary 1. $\mathcal{A}_0(\alpha) \subseteq \mathcal{A}(\alpha)$.

4. Inadmissibility of regular tests

Assume that $(B, \theta) \mapsto P_\theta(B)$ is a transition kernel on $\Theta_K \times \mathcal{B}(\mathcal{X})$. Specifically, this means that for every $B \in \mathcal{B}(\mathcal{X})$, $\theta \mapsto P_\theta(B)$ is a Borel function on Θ_K and for every $\theta \in \Theta$, P_θ is a probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. In addition, assume that there exists a σ -finite measure λ on \mathcal{X} such that

$$P_\theta \ll \lambda, \quad \forall \theta \in \Theta \quad (4.1)$$

where \ll is a symbol for absolute continuity. Correspondingly, for every $\theta \in \Theta$ let f_θ be the nonnegative version of the Radon-Nikodym derivative of P_θ with respect to λ . Then, for every probability measure $\pi \in \Pi$, define a probability measure \mathcal{P}_π on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ such that

$$\mathcal{P}_\pi(B) = \int_{\Theta} P_\theta(B) d\pi(\theta), \quad \forall B \in \mathcal{B}(\mathcal{X}). \quad (4.2)$$

In particular, \mathcal{P}_π may be considered as a Bayes-mixture of simple alternatives under K (this terminology is taken from Birnbaum [3]) and (4.1) implies that $\mathcal{P}_\pi \ll \lambda$.

Definition 3 actually asserts that an α -level π -Bayes test in $\Phi(\alpha)$ is an MP test of the simple hypothesis testing problem:

$$H : X \sim P_0 \quad \text{vs.} \quad K_\pi : X \sim \mathcal{P}_\pi. \quad (4.3)$$

Therefore, the fundamental lemma of Neyman and Pearson (see Theorem 3.2.1(iii) in [15]) might be carried out under the assumption that there is no trivial test for (4.3). Specifically, if $\phi \in \Phi(\alpha)$ is an α -level π -Bayes test in $\Phi(\alpha)$, then there exists a constant $C \equiv C_\pi \in \mathbb{R}$ such that

$$\phi(x) = \begin{cases} 1 & C f_0(x) < \int_{\Theta_K} f_\theta(x) d\pi(\theta) \\ 0 & C f_0(x) > \int_{\Theta_K} f_\theta(x) d\pi(\theta) \end{cases}, \quad \lambda\text{-a.s.} \quad (4.4)$$

Furthermore, define ϕ_π^* to be the α -level π -Bayes test which is determined as the solution of the optimization in Section 6 of [11]. In particular, once $f_\theta(x) > 0$, for every $\theta \in \Theta$ and $x \in \mathcal{X}$, then C which appears in (4.4) is positive. Moreover, in such a case, there exist $C_\pi^* \in (0, \infty)$ and $\tau_\pi^* \in [0, 1]$ such that

$$\phi_\pi^*(x) = \mathbf{1}_{\{L_\pi(x) > C_\pi^*\}} + \tau_\pi^* \mathbf{1}_{\{L_\pi(x) = C_\pi^*\}}, \quad \forall x \in \mathbb{R}^d \quad (4.5)$$

where

$$L_\pi(x) \equiv \int_{\mathbb{R}^d \setminus \{0\}} \frac{f_\theta(x)}{f_0(x)} d\pi(\theta), \quad \forall x \in \mathbb{R}^d. \quad (4.6)$$

The following corollary includes a sufficient condition for inadmissibility of a regular test. It stems immediately from the above-mentioned discussion with an application of Theorem 1.

Corollary 2. Let $\phi \in \Phi_0(\alpha)$ and assume that there are sets $\Pi_1, \Pi_2 \subseteq \Pi$ such that:

1. $\Pi = \Pi_1 \cup \Pi_2$.
2. $\Pi_1 \subseteq \{\pi \in \Pi ; \phi_\pi^* \in \Phi_0(\alpha)\}$.
3. For every $\pi \in \Pi_2$, there exists $\phi'_\pi \in \Phi_0(\alpha)$ such that

$$\int_{\Theta_K} \beta_\phi(\theta) d\pi(\theta) < \int_{\Theta_K} \beta_{\phi'_\pi}(\theta) d\pi(\theta). \tag{4.7}$$

If for every $\pi \in \Pi_1$, there is no trivial test for (4.3) and there is no $C \in \mathbb{R}$ for which (4.4) is satisfied, then $\phi \in \Phi_0(\alpha) \setminus \mathcal{A}_0(\alpha)$.

Remark 5. Notice that $\Phi_0(\alpha) \subseteq \Phi(\alpha)$. Therefore, a statement that $\phi \in \Phi_0(\alpha)$ is inadmissible in $\Phi_0(\alpha)$ is more informative than a statement which asserts that it is inadmissible in $\Phi(\alpha)$.

Remark 6. It is possible to have π for which $\phi_\pi^* \notin \Phi_0(\alpha)$. For example, consider the special case where

1. $\Theta = \mathcal{X} = \mathbb{R}$.
2. For every $\theta \in \mathbb{R}$, assume that $X \sim \mathcal{N}(\theta, 1)$ under P_θ .
3. π is a Dirac measure on $\{\theta_0\}$ for some $\theta_0 \in \mathbb{R} \setminus \{0\}$.

Then, ϕ_π^* is a one-sided Z-test in the direction of θ_0 . Clearly, this test does not belong to $\Phi_0(\alpha)$ because $\theta \mapsto \gamma_{\phi_\pi^*}(\theta)$ does not vanish in infinity.

4.1. Specific framework

It turns out that for the applications to be discussed in the upcoming sections, Corollary 2 may not be applied directly. However, these applications can be phrased as special cases of a more specific framework in which it is possible to derive a more practical result. Specifically, consider the special case where $\Theta = \mathbb{R}^p$, $\mathcal{X} = \mathbb{R}^d$, $1 \leq d, p < \infty$ and λ is the Lebesgue measure on \mathbb{R}^d . In addition, let P be a probability measure on (Ω, \mathcal{H}) and consider $U : \Omega \rightarrow \mathbb{R}^d$ which is a (Ω, \mathcal{H}) -measurable function. Assume that the distribution function of U (with respect to P) is absolutely continuous and denote the corresponding density by $f_U(\cdot)$. In particular, assume that $f_U(u) > 0$ for every $u \in \mathbb{R}^d$.

For every $\theta \in \mathbb{R}^p$, let $T_\theta : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a bijective differentiable function such that for every $u \in \mathbb{R}^d$:

- (T1) $\theta \mapsto T_\theta(u)$ is continuous.
- (T2) $\|T_\theta(u)\| \rightarrow \infty$ as $\|\theta\| \rightarrow \infty$ where $\|\cdot\|$ is the Euclidean norm in the proper space (\mathbb{R}^d or \mathbb{R}^p).

Then, for every $\theta \in \mathbb{R}^p$ define $X \equiv X(\theta) \equiv T_\theta(U)$. Correspondingly, for every $\theta \in \mathbb{R}^p$, P_θ is the push-forward probability measure which is induced by $X(\theta)$. Hence, the Jacobian theorem implies that for every $\theta \in \mathbb{R}^p$, P_θ has a density with respect to Lebesgue measure on \mathbb{R}^d which is given by

$$f_\theta(x) \equiv f_U [T_\theta^{-1}(x)] |\det J_{T_\theta^{-1}}(x)|, \quad \forall x \in \mathbb{R}^d \quad (4.8)$$

where T_θ^{-1} is the inverse of T_θ and $J_{T_\theta^{-1}}$ is the Jacobian matrix associated with T_θ^{-1} . Importantly, by the above-mentioned assumptions, deduce that for every $\theta \in \mathbb{R}^p$, $f_\theta(\cdot)$ is positive on \mathbb{R}^d . This implies that for every $\theta \in \mathbb{R}^p$ the distribution of the observations under P_θ is supported on \mathbb{R}^d . Consequently, there is no trivial test for (4.3).

For the statement and proof of the following theorem, let $\mathbf{1}_A$ be a notation of an indicator function which is supported on a set A . In addition, for every $a, b \in \mathbb{R}$, let $a \wedge b \equiv \min\{a, b\}$ and denote the expectation operator with respect to P by E .

Theorem 2. *Let V be an open set in \mathbb{R}^d such that $\phi = \mathbf{1}_V$ satisfies the following conditions:*

- (I) $\beta_\phi(\mathbf{0}) = \alpha$.
- (II) $\beta_\phi(\theta) \rightarrow 1$ as $\|\theta\| \rightarrow \infty$.

Then, $\phi \in \Phi_0(\alpha)$.

Furthermore, assume that $\Pi_1 \subseteq \Pi$ is such that for every $\pi \in \Pi_1$ the following conditions are satisfied:

- (i) $L_\pi(\cdot)$ is continuous on \mathbb{R}^d .
- (ii) There is no $C \in \mathbb{R}$ for which (4.4) is satisfied.

Under these assumptions, if for every $\pi \in \Pi \setminus \Pi_1$, there exists $\phi' \in \Phi_0(\alpha)$ such that

$$\int_{\Theta_K} \beta_\phi(\theta) d\pi(\theta) < \int_{\Theta_K} \beta_{\phi'}(\theta) d\pi(\theta), \quad (4.9)$$

then $\phi \in \Phi_0(\alpha) \setminus \mathcal{A}_0(\alpha)$.

In the rest of this section, consider the setup in which the forthcoming examples are defined. Namely, consider the case in which $f \equiv f_U(\cdot)$ is a bounded continuous density function and $T_\theta(u) = u + \theta$ for every $u, \theta \in \mathbb{R}^d$. Under these assumptions the following proposition holds.

Proposition 1. *Let π be a probability measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and consider an α level test $\phi(x) \equiv \mathbf{1}_V(x)$ such that*

$$V \equiv \left\{ x \in \mathbb{R}^d; \int_{\mathbb{R}^p \setminus \{\mathbf{0}\}} \frac{f(x - \theta)}{f(x)} d\pi(\theta) > C \right\} \quad (4.10)$$

for some constant $C \in (0, \infty)$ which is determined in order to preserve the significance level. If $\mathbb{R}^d \setminus V$ is bounded, then $\phi \in \mathcal{A}_0(\alpha)$.

Remark 7. Proposition 1 is about a sufficient condition for existence of a regular admissible test. Some important special cases are associated with $f(\cdot)$ which is radial. In such cases it is natural to take π which is the normalized Lebesgue measure on a unit ball centred at the origin. In particular, standard

calculations yield that this choice of π implies that $\mathcal{A}_0(\alpha) \neq \emptyset$ when $f(\cdot)$ is a Gaussian density. Furthermore, if $d = 1$ it is also sensible to consider π such that $\pi(\{a\}) = \pi(\{-a\}) = \frac{1}{2}$ for some $a \in (0, \infty)$.

5. One-dimensional general case

Let $\theta \in \mathbb{R}$ and consider two independent random variables U_1 and U_2 which are not necessarily identically distributed. In particular, assume that for every $i = 1, 2$, the distribution of U_i is absolutely continuous with a continuous density function $f_{U_i} : \mathbb{R} \rightarrow (0, M)$ for some $0 < M < \infty$. In addition, define

$$X(\theta) \equiv X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} U_1 + \theta \\ U_2 + \theta \end{bmatrix} \quad (5.1)$$

and let P_θ be the push-forward probability measure which is induced by $X(\theta)$.

Then, a statistician who does not know the value of θ (but knows the distribution of U) observes X and wants to test

$$H : \theta = 0 \quad , \quad K : \theta \neq 0 \quad (5.2)$$

with a pre-defined significance level $\alpha \in (0, 1)$. For this purpose, he considers a test

$$\phi(x) \equiv \mathbf{1}_{(a, \infty) \times (b_1, \infty) \cup (-\infty, a) \times (-\infty, b_2)}(x_1, x_2) \quad , \quad \forall x = (x_1, x_2) \in \mathbb{R}^2 \quad (5.3)$$

for some $a \in \mathbb{R}$ where b_1 and b_2 are the $(1 - \alpha)$ 'th and α 'th quantiles of the distribution of U_2 , *i.e.*,

$$\begin{aligned} b_1 &\equiv \inf \{t \in \mathbb{R}; P(U_2 \leq t) \geq 1 - \alpha\} \quad , \quad (5.4) \\ b_2 &\equiv \inf \{t \in \mathbb{R}; P(U_2 \leq t) \geq \alpha\} \quad . \end{aligned}$$

Note that ϕ is a test which is based on data-splitting in the sense that X_1 is used in order to determine the direction of a one-sided test to be performed through the statistic X_2 . In particular, regardless of the value of X_1 , observe that under H , the test to be performed through X_2 leads to a rejection with probability α which means that $\phi \in \Phi(\alpha)$.

Theorem 3.

1. $\phi \in \Phi_0(\alpha)$.
2. Assume that

$$\lim_{x \rightarrow \infty} \frac{f_{U_2}(x - \theta)}{f_{U_2}(x)} = \infty \quad , \quad \lim_{x \rightarrow -\infty} \frac{f_{U_2}(x - \theta)}{f_{U_2}(x)} < \infty \quad , \quad \forall \theta \in (0, \infty) \quad (5.5)$$

and

$$\lim_{x \rightarrow \infty} \frac{f_{U_2}(x - \theta)}{f_{U_2}(x)} < \infty \quad , \quad \lim_{x \rightarrow -\infty} \frac{f_{U_2}(x - \theta)}{f_{U_2}(x)} = \infty \quad , \quad \forall \theta \in (-\infty, 0) \quad . \quad (5.6)$$

Then, $\phi \in \Phi_0(\alpha) \setminus \mathcal{A}_0(\alpha)$.

Remark 8. When U_2 has a distribution which is symmetric with respect to the origin, *i.e.*, $f_{U_2}(x) = f_{U_2}(-x)$ almost everywhere on \mathbb{R} , the conditions (5.5) and (5.6) may be unified into the condition

$$\lim_{x \rightarrow \infty} \frac{f_{U_2}(x - \theta)}{f_{U_2}(x)} = \infty, \quad \forall \theta \in (0, \infty). \quad (5.7)$$

Especially, in the Gaussian case, *i.e.*, when

$$f_{U_2}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \forall x \in \mathbb{R}, \quad (5.8)$$

simple algebra implies that

$$\frac{f_{U_2}(x - \theta)}{f_{U_2}(x)} \propto e^{\theta x}, \quad \forall \theta \in \mathbb{R} \setminus \{0\}. \quad (5.9)$$

Hence, Theorem 3 yields that Moran's test is inadmissible in the Gaussian case.

Remark 9. It makes sense to think about a model in which ξ_1, \dots, ξ_n is an iid sequence of random variables. Then, a statistician who does not know the value of $\theta \in \mathbb{R}$ observes $\rho_i = \xi_i + \theta$, $i = 1, 2, \dots, n$. His intention is to test (5.2) with a procedure which is based on a single split of the data. What is the relation between this setup and the model which was described so far in this section? In this setup X_1 and X_2 may be viewed as statistics which are calculated respectively from the first and second sub-samples. Assume that the first sub-sample is (ρ_1, \dots, ρ_m) and the second sub-sample is $(\rho_{m+1}, \dots, \rho_n)$ for some $1 \leq m < n$. Then, the requirement is that $U_1 \equiv X_1 - \theta$ and $U_2 \equiv X_2 - \theta$ satisfy the following conditions:

1. U_1 is determined uniquely by (ξ_1, \dots, ξ_m) and U_2 is determined uniquely by $(\xi_{m+1}, \dots, \xi_n)$.
2. The distribution of (U_1, U_2) is free of θ .

Some examples of statistics which satisfy these conditions are *e.g.*, sample means, sample quantiles and sample extreme values. Notably, in Section 2 of [17], Moran regards the case in which X_1 and X_2 are the sample means of the two sub-samples.

6. Multi-dimensional Gaussian case

Let $d \geq 1$ and assume that U_1, U_2, \dots, U_n is an iid sequence of standard d -dimensional Gaussians. In addition, for every $1 \leq i \leq n$, define $X_i \equiv X_i(\theta) \equiv U_i + \theta$. A statistician observes X_1, X_2, \dots, X_n and his purpose is to test

$$H : \theta = \mathbf{0}, \quad K : \theta \in \mathbb{R}^d \setminus \{\mathbf{0}\} \quad (6.1)$$

where $\mathbf{0}$ is the zero-vector in \mathbb{R}^d . In this model, Moran's test ϕ is as follows: The first step is to compute the mean of the first sub-sample, that is $\bar{X}_1 \equiv \frac{1}{m} \sum_{i=1}^m X_i$

for some $1 \leq m < n$. Then, given the computation result, use the second subsample X_{m+1}, \dots, X_n in order to test a simple hypothesis in the direction of \bar{X}_1 . That is, consider

$$H : \theta = \mathbf{0} \quad , \quad K(\bar{X}_1) : \theta = \frac{\bar{X}_1}{\|\bar{X}_1\|} \quad (6.2)$$

with a pre-defined significant level $\alpha \in (0, 1)$ where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^d . The general idea for this test was presented by Moran in [17] and recently, DiCiccio analysed the power of this test (see, Section 2.1.2 in [8]).

In practice, straightforward calculation of the likelihood ratio (note that x_1 is considered as a constant) implies that the rejection zone of ϕ equals to

$$R_\phi \equiv \left\{ x_1, x_2, \dots, x_n \in \mathbb{R}^d \text{ s.t. } \bar{x}_1 \neq \mathbf{0} ; \frac{\bar{x}_1 \cdot \bar{x}_2}{\|\bar{x}_1\|} > D \right\} \quad (6.3)$$

where \cdot denotes dot product, $\bar{x}_1 \equiv \frac{1}{m} \sum_{i=1}^m x_i$, $\bar{x}_2 \equiv \frac{1}{n-m} \sum_{i=m+1}^n x_i$ and $D > 0$ is a constant which is determined uniquely by the vector (α, n, m) .

Theorem 4. $\phi \in \Phi_0(\alpha) \setminus \mathcal{A}_0(\alpha)$.

6.1. Beyond Gaussian assumption

Assume that $n = 2$ and U has a density function $f(\cdot)$ which is not necessarily Gaussian. Then, it is natural to ask whether the test ϕ which has been defined earlier in this section is inadmissible? Honestly, the initial effort was to prove a multi-dimensional version of Theorem 3. Since the multi-dimensional setup is related with a continuum of directions, a generalization of the proof of Theorem 3 is not straightforward.

Then, it is constructive to see whether the current proof of Theorem 4 might be extended? A careful reading of this proof yields that it remains valid as long as the set

$$\{(x_1, x_2) \in \mathbb{R}^{2d} ; L_\pi(x_1, x_2) \leq C\} \quad (6.4)$$

is convex for every $\pi \in \Pi$ and $C \in (0, \infty)$ where

$$L_\pi(x_1, x_2) = \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \frac{f(x - \theta)}{f(x)} d\pi(\theta) \quad , \quad \forall (x_1, x_2) \in \mathbb{R}^{2d} .$$

Clearly, this condition is satisfied whenever the likelihood ratio $f(x - \theta)/f(x)$ is convex in x for every $\theta \in \mathbb{R}^d$. While initially this gave some hope for finding a non-Gaussian density for which the current proof works, a recent result [12] had implied that the situation is different.

7. Discussion

From the perspective of classical decision theory, an inadmissible test should not be used because there is another test which is better. Accordingly, the

inadmissibility results which appear in this work are not encouraging applied statisticians to apply Moran's test for real data. The purpose of this short section is to discuss the implications of the current results more deeply along with some suggestions for further research.

Primarily, knowing that Moran's test is inadmissible should be considered as an initial step toward the pursuit for a better test. For example, in the one-dimensional Gaussian case, a combination of Theorem 3 and the theory regarding unbiased tests implies that a two-sided Z-test is better than Moran's test. However, it is not clear how to find a better test in the general case, *e.g.*, when Condition 1 is satisfied. This practical question remains open.

Furthermore, it is possible that in certain setups, Moran's test performs quite well (at least for alternatives which are distant from the null). In such cases, there is no strong incentive for practitioners to look for a better test. In Section 2.1.2 of [8], there is an effort to compare the power of Moran's test with the chi-square test in the multi-dimensional Gaussian setup. It might be good to keep on in this direction by assessing the performance of Moran's test when the data does not have a Gaussian distribution.

It is also reasonable to consider a statistician who is willing to pay in terms of power for testing the 'correct' hypothesis. It is interesting to see how to phrase a formal model which is consistent with the preferences of such a statistician. Then, the challenge will be to figure out whether Moran's test is admissible in this new framework. Possibly, an inspiration for such a model might come from some model-selection frameworks in which the statistician is willing to give up some part of the data in favour of conducting a statistical inference on a better model. Another branch of literature which might be related regards constrained statistical inference and an analysis of type III error (for more information, see *e.g.*, [22]).

This work also motivates further investigation of the notion of regular admissibility. In particular, it is interesting to look whether exists an example in which there is no regular admissible test, *i.e.*, $\mathcal{A}_0(\alpha) = \emptyset$. Furthermore, it is instructive to see whether analogous result to the one which appears in Proposition 1 may be derived under more general assumptions.

Finally, it is intriguing to think whether exists a multi-dimensional density $f(\cdot)$ under which the conclusion of Theorem 4 remains valid. The discussion in Section 6.1 yields that a derivation of such a result should be a consequence of a new approach for the present problem.

8. Proofs

8.1. Proof of Lemma 1

Consider a function

$$f(\theta) \equiv \begin{cases} \|\theta\| & \|\theta\| \leq 1 \\ e^{-(\|\theta\|-1)} & \|\theta\| > 1 \end{cases}, \quad \forall \theta \in \Theta. \quad (8.1)$$

It is given that Θ is a finite-dimensional and hence it is enough to check that for every $\epsilon > 0$, the set $\{\theta \in \Theta; f(\theta) \geq \epsilon\}$ is closed, bounded and contained in Θ_K . This as well as the other requirements of the lemma are easy to verify. ■

8.2. Proof of Theorem 1

Denote

$$\Delta(\phi) \equiv \{f \in \mathcal{C}(\Theta_K) ; f(\theta) \leq \gamma_\phi(\theta) , \forall \theta \in \Theta_K\} \quad (8.2)$$

and observe that

$$\mathcal{S} \equiv \{\gamma_{\phi'}|_{\Theta_K} ; \phi' \in \Phi_0(\alpha)\} \subseteq \mathcal{C}(\Theta_K). \quad (8.3)$$

It is given that $\phi \in \mathcal{A}_0(\alpha)$ and hence

$$\left(\Delta(\phi) \setminus \{\gamma_\phi|_{\Theta_K}\}\right) \cap \mathcal{S} = \emptyset. \quad (8.4)$$

In fact, it can be verified that both \mathcal{S} and $\Delta(\phi) \setminus \{\gamma_\phi|_{\Theta_K}\}$ are convex sets. Especially, since $\gamma_\phi(\theta) \geq 0$ for every $\theta \in \Theta_K$, then

$$f(\theta) = -1 , \quad \forall \theta \in \Theta_K \quad (8.5)$$

is an interior point of $\Delta(\phi) \setminus \{\gamma_\phi|_{\Theta_K}\}$ with respect to the norm

$$\|f\|_{\infty|\Theta_K} \equiv \sup_{\theta \in \Theta_K} |f(\theta)| , \quad \forall f \in \mathcal{C}(\Theta_K). \quad (8.6)$$

Therefore, the basic separation theorem (see Theorem 3.5.13 and Corollary 3.5.14(a) in [2]) implies that there exists a non-zero continuous linear functional $\mathcal{F}\{\cdot\}$ on $\mathcal{C}(\Theta_K)$ such that

$$\mathcal{F}\{f\} \leq \mathcal{F}\{\gamma\} , \quad \forall f \in \Delta(\phi) \setminus \{\gamma_\phi|_{\Theta_K}\} , \quad \gamma \in \mathcal{S}. \quad (8.7)$$

Now, consider some $f \in \mathcal{C}(\Theta_K)$ which is nonpositive and not identically zero. Then, $f + \gamma_\phi|_{\Theta_K} \in \Delta(\phi) \setminus \{\gamma_\phi|_{\Theta_K}\}$ and hence

$$\mathcal{F}\{f\} + \mathcal{F}\{\gamma_\phi|_{\Theta_K}\} = \mathcal{F}\{f + \gamma_\phi|_{\Theta_K}\} \leq \mathcal{F}\{\gamma_\phi|_{\Theta_K}\}. \quad (8.8)$$

Moreover, $\mathcal{F}\{\cdot\}$ is continuous and hence it is also bounded, *i.e.*, there exists a constant $M \in (0, \infty)$ such that

$$|\mathcal{F}\{f\}| \leq M\|f\|_{\infty|\Theta_K} , \quad \forall f \in \mathcal{C}(\Theta_K). \quad (8.9)$$

Therefore, since $\|\gamma_\phi|_{\Theta_K}\|_{\infty|\Theta_K} \leq 1$, deduce that $|\mathcal{F}\{\gamma_\phi|_{\Theta_K}\}| < \infty$ and hence $\mathcal{F}\{\cdot\}$ is positive, *i.e.*,

$$\mathcal{F}\{f\} \geq 0 , \quad \forall f \in \mathcal{C}(\Theta_K) \text{ s.t. } f(\theta) \geq 0 , \quad \forall \theta \in \Theta_K. \quad (8.10)$$

$(\Theta_K, \|\cdot\|)$ is a special case of locally compact σ -compact Hausdorff space. Thus, Riesz-Markov-Kakutani representation theorem (see Section 2.14 in [19]) states that there exists a regular Borel measure μ on Θ_K such that

$$\mathcal{F}\{f\} = \int_{\Theta_K} f(\theta) d\mu(\theta) \quad , \quad \forall f \in \mathcal{C}_c(\Theta_K). \quad (8.11)$$

Since $\mathcal{F}\{\cdot\}$ is non-zero functional, then μ is a non-zero measure. In addition, since Θ_K is σ -compact, then there exists a sequence of non-empty compact sets $K_1 \subseteq K_2 \subseteq \dots \subseteq \Theta_K$ such that $\cup_{n=1}^{\infty} K_n = \Theta_K$. Therefore, since Θ_K is an open set, then Urysohn's lemma (see Section 2.12 in [19]) implies that for every $n \geq 1$ there exists a function $g_n \in \mathcal{C}_c(\Theta_K)$ such that

1. $0 \leq g_n(\theta) \leq 1$ for every $\theta \in \Theta_K$.
2. $g_n(\theta) = 1$ for every $\theta \in K_n$.

Thus, observe that for every $n \geq 1$

$$\begin{aligned} \mu(K_n) &= \int_{K_n} d\mu(\theta) \\ &\leq \int_{\Theta} g_n(\theta) d\mu(\theta) = \mathcal{F}\{g_n\} \leq M \|g_n\|_{\infty|\Theta_K} = M < \infty. \end{aligned} \quad (8.12)$$

This means that by taking $n \rightarrow \infty$, continuity of measure (from below) implies that $\mu(\Theta_K)$ is finite.

Let f be some arbitrary function in $\mathcal{C}_0(\Theta_K)$. It is well-known that $\mathcal{C}_c(\Theta_K)$ is dense in $\mathcal{C}_0(\Theta_K)$ with respect to the norm $\|\cdot\|_{\infty|\Theta_K}$. This means that there exists a sequence $(h_n)_{n \geq 1} \subseteq \mathcal{C}_c(\Theta_K)$ such that $\|f - h_n\|_{\infty|\Theta_K} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, since $\mathcal{F}\{\cdot\}$ is continuous, then $\mathcal{F}\{h_n\} \rightarrow \mathcal{F}\{f\}$ as $n \rightarrow \infty$. In addition, $\|f\|_{\infty|\Theta_K} < \infty$ and hence

$$\sup_{n \geq 1} \|h_n\|_{\infty|\Theta_K} = \sup_{n \geq 1} \sup_{\theta \in \Theta_K} |h_n(\theta)| < \infty. \quad (8.13)$$

At the same time, the above-mentioned uniform convergence implies pointwise convergence $h_n \rightarrow f$ as $n \rightarrow \infty$ on Θ_K . Thus, recalling that μ is a finite measure, then bounded convergence theorem yields that

$$\mathcal{F}\{h_n\} = \int_{\Theta_K} h_n(\theta) d\mu(\theta) \xrightarrow{n \rightarrow \infty} \int_{\Theta_K} f(\theta) d\mu(\theta). \quad (8.14)$$

Hence, since f is an arbitrary element in $\mathcal{C}_0(\Theta_K)$, then the uniqueness of the limit implies that

$$\mathcal{F}\{f\} = \int_{\Theta_K} f(\theta) d\mu(\theta) \quad , \quad \forall f \in \mathcal{C}_0(\Theta_K). \quad (8.15)$$

Now, define a new non-zero linear functional

$$\mathcal{G}\{f\} \equiv \mathcal{F}\left\{f|_{\Theta_K}\right\} \quad (8.16)$$

on a normed vector-space $(\mathcal{C}(\Theta), \|\cdot\|_{\infty|\Theta})$ where

$$\|f\|_{\infty|\Theta} \equiv \sup_{\theta \in \Theta} |f(\theta)|, \quad \forall f \in \mathcal{C}(\Theta). \tag{8.17}$$

Observe that for every $f \in \mathcal{C}(\Theta)$,

$$|\mathcal{G}\{f\}| = |\mathcal{F}\{f|_{\Theta_K}\}| \leq M \sup_{\theta \in \Theta_K} |f(\theta)| \leq M \|f\|_{\infty|\Theta} \tag{8.18}$$

which means that $\mathcal{G}\{\cdot\}$ is bounded and hence also continuous. Thus, as it was shown for $\mathcal{F}\{\cdot\}$, it is possible to show that $\mathcal{G}\{\cdot\}$ is positive. Consequently, since $(\Theta, \|\cdot\|)$ is a special case of a locally compact σ -compact Hausdorff space, Riesz-Markov-Kakutani representation theorem can be applied once again. This time in order to show that there exists a regular Borel measure ν on Θ such that

$$\mathcal{G}\{f\} = \int_{\Theta} f(\theta) d\nu(\theta), \quad \forall f \in \mathcal{C}_c(\Theta). \tag{8.19}$$

In particular, ν is non-zero and finite by analogous arguments to those which were used in order to justify these properties of μ . Furthermore, analogous arguments to those which lead to (8.15), imply that

$$\mathcal{G}\{f\} = \int_{\Theta} f(\theta) d\nu(\theta), \quad \forall f \in \mathcal{C}_0(\Theta). \tag{8.20}$$

Now, assume by contradiction that ν is a Dirac measure on $\{\mathbf{0}\}$. Notice that Lemma 1 implies that there exists a bounded function $f \in \mathcal{C}_0(\Theta)$ such that:

1. $f|_{\Theta_K} \in \mathcal{C}_0(\Theta_K)$.
2. $f(\mathbf{0}) < f(\theta)$ for every $\theta \in \Theta_K$.

Then, a contradiction follows from

$$f(\mathbf{0}) = \mathcal{G}\{f\} = \mathcal{F}\{f|_{\Theta_K}\} = \int_{\Theta_K} f|_{\Theta_K}(\theta) d\mu(\theta) > f(\mathbf{0}).$$

Fix some $\phi' \in \Phi_0(\alpha)$ and for every $n \geq 1$, let

$$f_n(\theta) \equiv \gamma_{\phi}(\theta) - \frac{1}{n}, \quad \forall \theta \in \Theta. \tag{8.21}$$

Clearly, for every $n \geq 1$, $f_n|_{\Theta_K} \in \Delta(\phi) \setminus \{\gamma_{\phi}\} \subseteq \mathcal{C}(\Theta_K)$ and observe that

$$\|\gamma_{\phi}|_{\Theta_K} - f_n|_{\Theta_K}\|_{\infty|\Theta_K} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{8.22}$$

Hence, since $\mathcal{F}\{\cdot\}$ is a continuous functional on $\mathcal{C}(\Theta_K)$, then $\mathcal{F}\{f_n|_{\Theta_K}\} \rightarrow \mathcal{F}\{\gamma_{\phi}|_{\Theta_K}\}$ as $n \rightarrow \infty$. Due to (8.7), deduce that $\mathcal{F}\{f_n|_{\Theta_K}\} \leq \mathcal{F}\{\gamma_{\phi'}|_{\Theta_K}\}$ for every $n \geq 1$. Thus, taking the limit $n \rightarrow \infty$ yields that $\mathcal{F}\{\gamma_{\phi}|_{\Theta_K}\} \leq \mathcal{F}\{\gamma_{\phi'}|_{\Theta_K}\}$ and hence

$$\int_{\Theta} \gamma_{\phi}(\theta) d\nu(\theta) = \mathcal{G}\{\gamma_{\phi}\} = \mathcal{F}\{\gamma_{\phi}|_{\Theta_K}\} \tag{8.23}$$

$$\leq \mathcal{F}\{\gamma_{\phi'}|_{\Theta_K}\} = \mathcal{G}\{\gamma_{\phi'}\} = \int_{\Theta} \gamma_{\phi'}(\theta) d\nu(\theta).$$

Consequently, the generality of ϕ' implies that

$$\int_{\Theta} \beta_{\phi}(\theta) d\nu(\theta) \geq \int_{\Theta} \beta_{\phi'}(\theta) d\nu(\theta) \quad , \quad \forall \phi' \in \Phi_0(\alpha). \quad (8.24)$$

Finally, since ν is a non-zero finite regular Borel measure on Θ which is not concentrated on $\{\mathbf{0}\}$ and $\beta_{\phi}(\mathbf{0}) = \beta_{\phi'}(\mathbf{0}) = \alpha$, then the result follows by setting π to be the restriction of ν to Θ_K with a proper normalization. ■

Remark 10. Regarding the proof of Theorem 1, one may be wondering why not to take π to be μ ? To answer this question, for simplicity assume that $\Theta = \mathbb{R}$ which means that $\Theta_K = \mathbb{R} \setminus \{0\}$. In addition, consider a test $\phi \in \Phi_0(\alpha)$ such that $\theta \mapsto \gamma_{\phi}(\theta)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. In such a case, for every $0 < \epsilon < 1 - \alpha$, the set $\{\theta \in \Theta_K; \gamma_{\phi}(\theta) \geq \epsilon\}$ is not closed and hence not compact. This means that $\gamma_{\phi} \notin \mathcal{C}_0(\Theta_K)$ and consequently the proof of Theorem 1 would be incorrect if one took π which equals to μ .

8.3. Proof of Theorem 2

Observe that for every $\theta \in \mathbb{R}^p$,

$$\beta_{\phi}(\theta) = E \mathbf{1}_V [T_{\theta}(U)] . \quad (8.25)$$

Thus, since V is open, due to **(T1)**, bounded convergence theorem implies that $\theta \mapsto \beta_{\phi}(\theta)$ is continuous. This with the assumptions **(I)** and **(II)** imply that $\phi \in \Phi_0(\alpha)$.

Consider some arbitrary $\pi \in \Pi_1$ and for every $\zeta, \eta > 0$ define a test

$$\phi_{\zeta, \eta}(x) \equiv \left[\mathbf{1}_{\{\|x\| < \zeta\}} \frac{\phi_{\pi}^*(x)\zeta}{1 + \zeta} + \mathbf{1}_{\{\|x\| > \eta\}} \phi(x) + \frac{1}{\eta} \right] \wedge 1 \quad , \quad \forall x \in \mathbb{R}^d. \quad (8.26)$$

Now, observe that for every $\theta \in \mathbb{R}^p$,

$$\beta_{\phi_{\zeta, \eta}}(\theta) = E \left\{ \mathbf{1}_{\{\|X(\theta)\| < \zeta\}} \frac{\phi_{\pi}^*[X(\theta)]\zeta}{1 + \zeta} + \mathbf{1}_{\{\|X(\theta)\| > \eta\}} \phi[X(\theta)] + \frac{1}{\eta} \right\} \wedge 1 \quad (8.27)$$

and hence bounded convergence theorem implies that for every $\theta \in \mathbb{R}^p$, $(\zeta, \eta) \mapsto \beta_{\phi_{\zeta, \eta}}(\theta)$ is a continuous mapping on \mathbb{R}_{++}^2 . Notice that for every $\eta > 0$, bounded convergence theorem yields that

$$\beta_{\phi_{\zeta, \eta}}(\mathbf{0}) \xrightarrow{\zeta \uparrow \infty} E \left[\phi_{\pi}^*(U) + \mathbf{1}_{\{\|U\| > \eta\}} \phi(U) + \frac{1}{\eta} \right] \wedge 1 > E \phi_{\pi}^*(U) = \alpha. \quad (8.28)$$

In particular, to see why the inequality in (8.28) holds, recall that $\beta_{\phi_{\pi}^*}(\mathbf{0}) = \alpha < 1$ which implies that

$$P[\phi_{\pi}^*(U) = 1] < 1. \quad (8.29)$$

In addition, for every $\zeta > 0$ bounded convergence yields that

$$\beta_{\phi_{\zeta, \eta}}(\mathbf{0}) \xrightarrow{\eta \uparrow \infty} E \mathbf{1}_{\{\|U\| < \zeta\}} \frac{\phi_{\pi}^*(U)\zeta}{1 + \zeta} < \beta_{\phi_{\pi}^*}(\mathbf{0}) = \alpha \tag{8.30}$$

and hence the intermediate value theorem implies that there is a sequence $(\zeta_n, \eta_n)_{n \geq 1} \subset \mathbb{R}_{++}^2$ such that:

1. $(\zeta_n, \eta_n) \rightarrow (\infty, \infty)$ as $n \rightarrow \infty$.
2. $\beta_{\phi_{\zeta_n, \eta_n}}(\mathbf{0}) = \alpha, \forall n \geq 1$.

For every $n \geq 1$, denote $\phi_n \equiv \phi_{\zeta_n, \eta_n}$ and observe that there is a pointwise convergence $\phi_n \rightarrow \phi_{\pi}^*$ as $n \rightarrow \infty$. Therefore, bounded convergence theorem implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \beta_{\phi_n}(\theta) d\pi(\theta) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \phi_n(x) d\mathcal{P}_{\pi}(x) \\ &= \int_{\mathbb{R}^d} \phi_{\pi}^*(x) d\mathcal{P}_{\pi}(x) \\ &= \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \beta_{\phi_{\pi}^*}(\theta) d\pi(\theta) > \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \beta_{\phi}(\theta) d\pi(\theta) \end{aligned} \tag{8.31}$$

where the inequality is justified by **(ii)** with the help of the fundamental lemma of Neyman and Pearson (recall that there is no trivial test under the current model assumptions). This means that there is $N \geq 1$ such that $\beta_{\phi_N}(\mathbf{0}) = \alpha$ and for which

$$\int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \beta_{\phi_N}(\theta) d\pi(\theta) > \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \beta_{\phi}(\theta) d\pi(\theta). \tag{8.32}$$

Hence, since π is an arbitrary element in Π_1 , then Theorem 1 implies that it is left to show that $\gamma_{\phi_N} \in \mathcal{C}_0(\mathbb{R}^d)$. To this end, notice that for every $\theta \in \mathbb{R}^p$

$$E \mathbf{1}_{\{\|X(\theta)\| > \eta_N\}} \phi[X(\theta)] \leq \beta_{\phi_N}(\theta) \leq 1 \tag{8.33}$$

and it is to be shown that the left hand-side tends to one as $\|\theta\| \rightarrow \infty$. Initially, observe that **(T2)** implies the following pointwise convergence (on Ω):

$$|\mathbf{1}_{\{\|X(\theta)\| > \eta_N\}} \phi[X(\theta)] - \phi[X(\theta)]| \rightarrow 0 \text{ as } \|\theta\| \rightarrow \infty. \tag{8.34}$$

Consequently,

$$|E \mathbf{1}_{\{\|X(\theta_N)\| > \eta\}} \phi[X(\theta)] - \beta_{\phi}(\theta)| \leq E |\mathbf{1}_{\{\|X(\theta)\| > \eta_N\}} \phi[X(\theta)] - \phi[X(\theta)]| \tag{8.35}$$

and the right hand-side tends to zero as $\|\theta\| \rightarrow \infty$ by bounded convergence theorem. Hence, due to **(II)**, deduce that

$$E \mathbf{1}_{\{\|X(\theta)\| > \eta\}} \phi[X(\theta)] \rightarrow 1 \text{ as } \|\theta\| \rightarrow \infty. \tag{8.36}$$

Now, **(i)** implies that

$$R \equiv \{x \in \mathbb{R}^d; L_{\pi}(x) > C_{\pi}^*\} , Q \equiv \{x \in \mathbb{R}^d; L_{\pi}(x) < C_{\pi}^*\} \tag{8.37}$$

are open sets. In addition, observe that for every $x \in \mathbb{R}^d$, $\phi_N(x)$ equals to

$$\left\{ \zeta_N \frac{\mathbf{1}_{R \cap B}(x) + \tau_\pi^* [\mathbf{1}_B(x) - \mathbf{1}_{R \cap B}(x) - \mathbf{1}_{Q \cap B}(x)]}{\zeta_N + 1} + \mathbf{1}_{V \cap D}(x) + \frac{1}{\eta} \right\} \wedge 1 \quad (8.38)$$

where $B \equiv \{x \in \mathbb{R}^d; \|x\| < \zeta_N\}$ and $D \equiv \{x \in \mathbb{R}^d; \|x\| > \eta_N\}$. Since V is an open set, then each of the indicators which appear in (8.38) is supported on an open set. This means that the same arguments which were introduced in order to prove continuity of $\theta \mapsto \beta_\phi(\theta)$ imply that $\theta \mapsto \beta_{\phi_N}$ is continuous. ■

8.4. Proof of Proposition 1

Due to Problem 3.20 in [15], deduce that $\phi \in \mathcal{A}(\alpha)$ and hence it is left to show that $\phi \in \Phi_0(\alpha)$. To this end, recall that due to the assumptions on $f(\cdot)$, $\beta_\phi(\cdot)$ is continuous. In addition, observe that $\mathbb{R}^d \setminus V$ is bounded and hence $\mathbf{1}_V(U + \theta) \rightarrow 1$, P -a.s. as $\|\theta\| \rightarrow \infty$ and the result follows. ■

8.5. Proof of Theorem 3

It is easy to verify that ϕ satisfies the assumptions of the first part of Theorem 2 from which the assertion follows.

In order to prove the second assertion, consider a probability measure $\pi \in \Pi$ which is concentrated on $(0, \infty)$ and define a test

$$\phi_\zeta^+(x_1, x_2) \equiv \mathbf{1}_{(\zeta, \infty) \times (b_1, \infty) \cup (-\infty, \zeta) \times (-\infty, b_2)}(x_1, x_2) \quad , \quad \forall (x_1, x_2) \in \mathbb{R}^2 \quad (8.39)$$

which is parametrized by $\zeta \in (-\infty, a)$. Notice that for every $\zeta \in (-\infty, a)$, $\phi_\zeta^+ \in \Phi_0(\alpha)$ just like $\phi \in \Phi_0(\alpha)$.

Since $f_{U_2}(\cdot)$ is supported on \mathbb{R} , then the definitions of b_1 and b_2 imply that

$$P_\theta(X_2 < b_2) < \alpha < P_\theta(X_2 > b_1) \quad , \quad \forall \theta \in (0, \infty). \quad (8.40)$$

Therefore, since for every $\theta \in (0, \infty)$, X_1 and X_2 are independent and $P_\theta(X_1 = a) = 0$, then

$$\begin{aligned} P_\theta(X_2 > b_1) &= P_\theta(X_1 > a) P_\theta(X_2 > b_1) + P_\theta(X_1 < a) P_\theta(X_2 > b_1) \\ &> P_\theta(X_1 > a) P_\theta(X_2 > b_1) + P_\theta(X_1 < a) P_\theta(X_2 < b_2) \\ &= \beta_\phi(\theta) \quad , \quad \forall \theta \in (0, \infty). \end{aligned} \quad (8.41)$$

Note that for every $\theta \in (0, \infty)$, bounded convergence implies that

$$\begin{aligned} \beta_{\phi_\zeta^+}(\theta) &= E \mathbf{1}_{(\zeta, \infty) \times (b_1, \infty) \cup (-\infty, \zeta) \times (-\infty, b_2)} [X_1(\theta), X_2(\theta)] \\ &\rightarrow P_\theta(X_2 > b_1) \quad \text{as } \zeta \rightarrow -\infty. \end{aligned} \quad (8.42)$$

Therefore, an additional application of bounded convergence theorem yields that

$$\lim_{\zeta \rightarrow -\infty} \int_{(0, \infty)} \beta_{\phi_\zeta^+}(\theta) d\pi(\theta) = \int_{(0, \infty)} P_\theta(X_2 > b_1) d\pi(\theta)$$

$$> \int_{(0, \infty)} \beta_\phi(\theta) d\pi(\theta) \quad (8.43)$$

where the inequality is justified by (8.41). This means that for any $\pi \in \Pi$ which is concentrated on $(0, \infty)$, ϕ is not an α -level π -Bayes test in $\Phi_0(\alpha)$. Similarly, an analogous result can be made for any $\pi \in \Pi$ which is concentrated on $(-\infty, 0)$.

Thus, it is left to consider $\pi \in \Pi$ which is a probability measure on $\mathbb{R} \setminus \{0\}$ such that

$$\pi((-\infty, 0)) \wedge \pi((0, \infty)) > 0. \quad (8.44)$$

Primarily, for every $\theta \in \mathbb{R}$, the joint likelihood of the data is given by

$$l_\theta(x_1, x_2) \equiv f_{U_1}(x_1 - \theta)f_{U_2}(x_2 - \theta) \quad , \quad \forall (x_1, x_2) \in \mathbb{R}^2. \quad (8.45)$$

In particular, it is given that $f_{U_1}(\cdot)$ and $f_{U_2}(\cdot)$ are positive continuous functions on \mathbb{R} . Therefore, deduce that for every $\theta \in \mathbb{R} \setminus \{0\}$, the likelihood ratio

$$(x_1, x_2) \mapsto \frac{l_\theta(x_1, x_2)}{l_0(x_1, x_2)} \quad (8.46)$$

is continuous on \mathbb{R}^2 . Consequently, since for every $(x_1, x_2) \in \mathbb{R}^2$, $\theta \mapsto l_\theta(x_1, x_2)$ is bounded, then bounded convergence theorem leads to the conclusion that

$$L_\pi(x_1, x_2) = \int_{\mathbb{R} \setminus \{0\}} \frac{l_\theta(x_1, x_2)}{l_0(x_1, x_2)} d\pi(\theta) \quad , \quad \forall (x_1, x_2) \in \mathbb{R}^2 \quad (8.47)$$

is continuous.

Now, observe that for every $(x_1, x_2) \in \mathbb{R}^2$

$$L_\pi(x_1, x_2) = \int_{(-\infty, 0)} \frac{l_\theta(x_1, x_2)}{l_0(x_1, x_2)} d\pi(\theta) + \int_{(0, \infty)} \frac{l_\theta(x_1, x_2)}{l_0(x_1, x_2)} d\pi(\theta). \quad (8.48)$$

In addition, fix some $x_1 \neq 0$ and notice that (5.5) and (5.6) with bounded convergence theorem imply that

$$\lim_{x_2 \rightarrow \pm\infty} L_\pi(x_1, x_2) = \infty. \quad (8.49)$$

This means that for every $C \in (0, \infty)$, there exist $-\infty < w_1 < w_2 < \infty$ such that

$$(-\infty, w_1) \cup (w_2, \infty) \subseteq \{x_2 \in \mathbb{R} ; L_\pi(x_1, x_2) > C\}. \quad (8.50)$$

On the other hand,

$$\{x_2 \in \mathbb{R} ; \phi(x_1, x_2) = 1\} = \begin{cases} (b_1, \infty) & x_1 > 0 \\ (-\infty, b_2) & x_1 < 0 \end{cases} \quad (8.51)$$

and hence the result follows by an application of Theorem 2. ■

8.6. Proof of Theorem 4

For simplicity and w.l.o.g. consider the case where $n = 2$ and $m = 1$. In addition, as mentioned in Remark 8, Theorem 3 implies the result for the special case where $d = 1$ and hence consider the case where $2 \leq d < \infty$.

By construction, $\phi \in \Phi(\alpha)$. In addition, notice that for every $\theta \neq \mathbf{0}$

$$\begin{aligned} \frac{X_1(\theta) \cdot X_2(\theta)}{\|X_1(\theta)\|} &= \frac{(U_1 + \theta) \cdot (U_2 + \theta)}{\|U_1 + \theta\|} \\ &= \frac{U_1 \cdot U_2 + \theta \cdot (U_1 + U_2) + \|\theta\|^2}{\|U_1 + \theta\|} \\ &\geq \frac{U_1 \cdot U_2 - \|\theta\| \|U_1 + U_2\| + \|\theta\|^2}{\|U_1 + \theta\|} \end{aligned} \quad (8.52)$$

and observe that

$$0 < \frac{\|U_1 + \theta\|}{\|\theta\|} \leq \frac{\|U_1\|}{\|\theta\|} + 1. \quad (8.53)$$

Therefore, deduce that

$$\frac{X_1(\theta) \cdot X_2(\theta)}{\|X_1(\theta)\|} \xrightarrow{w.p.1} \infty \text{ as } \|\theta\| \rightarrow \infty \quad (8.54)$$

and hence $\beta_\phi(\theta) \rightarrow 1$ as $\|\theta\| \rightarrow \infty$. Also, observe that $(x_1, x_2) \mapsto \frac{x_1 \cdot x_2}{\|x_1\|}$ is a continuous function on $\{x_1, x_2 \in \mathbb{R}^d; x_1 \neq \mathbf{0}\}$ which means that R_ϕ is an open set. Therefore, by the first part of Theorem 2, deduce that $\phi \in \Phi_0(\alpha)$.

Now, consider some $\pi \in \Pi$ and observe that the same arguments which appear in the proof of Theorem 3 may be used here in order to show that $(x_1, x_2) \mapsto L_\pi(x_1, x_2)$ is continuous. In addition, standard algebra implies that for every $x_1, x_2 \in \mathbb{R}^d$

$$L_\pi(x_1, x_2) = \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \exp \left\{ -\frac{1}{2} \sum_{i=1,2} [\|\theta\|^2 - 2\theta \cdot x_i] \right\} d\pi(\theta).$$

Since expectation is an operator which preserves convexity, deduce that $(x_1, x_2) \mapsto L_\pi(x_1, x_2)$ is convex on \mathbb{R}^{2d} . Consequently, for every $C \in (0, \infty)$ the set

$$\{(x_1, x_2) \in \mathbb{R}^{2d}; L_\pi(x_1, x_2) \leq C\} \quad (8.55)$$

is convex.

Now, assume by contradiction that ϕ is an α -level π -Bayes test. Therefore, up to a null set, the acceptance zone of ϕ is convex. More precisely, this implies that the Lebesgue measure of the set of points $(u, v) \in \mathbb{R}^{2d}$ for which there exist $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^{2d}$ such that

$$1. (u, v) = \frac{(u_1, v_1) + (u_2, v_2)}{2},$$

2. $\frac{u \cdot v}{\|u\|} > D$,
3. $\frac{u_i \cdot v_i}{\|u_i\|} < D$, $\forall i = 1, 2$,

is zero. In order to obtain a contradiction, denote

$$u_1 \equiv \frac{\mathbf{e}_1}{\sqrt{2}} + \frac{\mathbf{e}_2}{\sqrt{2}} \quad , \quad u_2 \equiv \frac{\mathbf{e}_1}{\sqrt{2}} - \frac{\mathbf{e}_2}{\sqrt{2}} \quad (8.56)$$

and $v_1 \equiv v_2 \equiv \delta D \mathbf{e}_1$ for some $1 < \delta < \sqrt{2}$ where \mathbf{e}_i is the i 'th ($i = 1, 2$) element in the standard basis of \mathbb{R}^d . In particular observe that for every $i = 1, 2$,

$$\frac{u_i \cdot v_i}{\|u_i\|} = \frac{\delta D}{\sqrt{2}} < D. \quad (8.57)$$

In addition, define

$$u \equiv \frac{u_1 + u_2}{2} = \frac{\mathbf{e}_1}{\sqrt{2}} \quad , \quad v \equiv \frac{v_1 + v_2}{2} = \delta D \mathbf{e}_1 \quad (8.58)$$

and notice that

$$\frac{u \cdot v}{\|u\|} = \frac{\delta D / \sqrt{2}}{1 / \sqrt{2}} = \delta D > D. \quad (8.59)$$

Thus, a continuity argument yields that there exists $\delta > 0$ such that

$$\frac{\left(\frac{\tilde{u}_1 + \tilde{u}_2}{2}\right) \cdot \left(\frac{\tilde{v}_1 + \tilde{v}_2}{2}\right)}{\left\|\frac{\tilde{u}_1 + \tilde{u}_2}{2}\right\|} > D \quad , \quad \frac{\tilde{u}_1 \cdot \tilde{v}_1}{\|\tilde{u}_1\|} < D \quad , \quad \frac{\tilde{u}_2 \cdot \tilde{v}_2}{\|\tilde{u}_2\|} < D \quad (8.60)$$

for every $(\tilde{u}_i, \tilde{v}_i) \in B_\delta((u_i, v_i))$, $i = 1, 2$ where $B_\delta(y)$ refers to an Euclidean ball with radius $\delta > 0$ around $y \in \mathbb{R}^{2d}$. Thus, the result follows by Theorem 2. ■

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References

- [1] ASGHARIAN, M., and NOORBALOOCHI, S. (1998). Note on a fundamental relationship between admissible and Bayesian decision rules. *Statistics: A Journal of Theoretical and Applied Statistics*, **31**(1), 21–34. [MR1625613](#)
- [2] ASH, R. B. (2014). *Real Analysis and Probability: Probability and Mathematical Statistics: a Series of Monographs and Textbooks*. Academic press.
- [3] BIRNBAUM, A. (1954). Combining independent tests of significance. *Journal of the American Statistical Association*, **49**(267), 559-574. [MR0065101](#)

- [4] BROWN, L. D., and MARDEN, J. I. (1989). Complete class results for hypothesis testing problems with simple null hypotheses. *The Annals of Statistics*, **17**(1), 209–235. [MR0981446](#)
- [5] BROWN, L. D., and MARDEN, J. I. (1992). Local admissibility and local unbiasedness in hypothesis testing problems. *The Annals of Statistics*, 832–852. [MR1165595](#)
- [6] COX, D. R. (1975). A note on data-splitting for the evaluation of significance levels. *Biometrika*, **62**(2), 441–444. [MR0378189](#)
- [7] COX, D. R., SPJØTVOLL, E., JOHANSEN, S., VAN ZWET, W. R., BITHELL, J. F., BARNDORFF-NIELSEN, O., and KEULS, M. (1977). The role of significance tests [with discussion and reply]. *Scandinavian Journal of Statistics*, 49–70.
- [8] DICICCIO, C. J. (2018). *Hypothesis Testing Using Multiple Data Splitting*. Stanford University. [MR4144697](#)
- [9] DICICCIO, C. J., DICICCIO, T. J., and ROMANO, J. P. (2020). Exact tests via multiple data splitting. *Statistics and Probability Letters*, **166**. [MR4122109](#)
- [10] FERGUSON, T. S. (2014). *Mathematical statistics: A decision theoretic approach* (Vol. 1). Academic press. [MR0215390](#)
- [11] JACOBOVIC, R., and KELLA, O. (2020). Minimizing a stochastic convex function subject to stochastic constraints and some applications. *Stochastic Processes and their Applications*, **130**(11), 7004–7018. [MR4158810](#)
- [12] JACOBOVIC, R., and KELLA, O. (2022). A characterization of normality via convex likelihood ratios. *Statistics and Probability Letters*, **186**. [MR4400084](#)
- [13] JOHNSTONE, I. M. (2019). Larry Brown’s work on admissibility. *Statistical Science*, **34**(4), 657–668. [MR4048596](#)
- [14] KOZEK, A. (1982). Towards a calculus for admissibility. *The Annals of Statistics*, 825–837. [MR0663435](#)
- [15] LEHMANN, E. L., and ROMANO, J. P. (2006). *Testing statistical hypotheses*. Springer Science & Business Media. [MR2135927](#)
- [16] MARDEN, J. I. (1982). Minimal complete classes of tests of hypotheses with multivariate one-sided alternatives. *The Annals of Statistics*, 962–970. [MR0663447](#)
- [17] MORAN, P. A. (1973). Dividing a sample into two parts a statistical dilemma. *Sankhyā: The Indian Journal of Statistics, Series A*, **35**, 329–333. [MR0518783](#)
- [18] ROMANO, J. P., and DICICCIO, C. (2019). *Multiple data splitting for testing*. Department of Statistics, Stanford University.
- [19] RUDIN. W. (1987). *Real and complex analysis*. McGraw-Hill. [MR0924157](#)
- [20] RUKHIN, A. L. (1995). Admissibility: Survey of a concept in progress. *International Statistical Review/Revue Internationale de Statistique*, 95–115.
- [21] SHAFER, G. (2020). Testing by betting: A strategy for statistical and scientific communication. *To be read before the Royal Statistical Society on September, 9*. [MR3925711](#)
- [22] SILVAPULLE, M. J., and SEN, P. K. (2005). *Constrained statistical in-*

- ference: Inequality, order and shape restrictions.* John Wiley & Sons. [MR2099529](#)
- [23] VOVK, V., and WANG, R. (2019). Combining e-values and p-values. *arXiv preprint arXiv:1912.06116*. [MR4298879](#)
- [24] VOVK, V. (2020). A note on data splitting with e-values: online appendix to my comment on Glenn Shafer's "Testing by betting". *arXiv preprint arXiv:2008.11474*. [MR4255905](#)