Electronic Journal of Probability

Electron. J. Probab. **27** (2022), article no. 160, 1–27.
ISSN: 1083-6489 https://doi.org/10.1214/22-EJP887

Isomorphism theorems, extended Markov processes and random interlacements

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Abstract

Several questions concerning the Gaussian free field on $\mathbb{Z}^d$ ($d \geq 3$) are solved thanks to a Dynkin-type isomorphism theorem established by Sznitman [29]. This isomorphism theorem relates the Gaussian free field to random interlacements and has the same spirit as the generalized second Ray-Knight theorem [11]. We show here that this isomorphism theorem is actually the generalized second Ray-Knight theorem written for a Markov process which is an extension of the continuous time simple random walk on $\mathbb{Z}^d$. As a result, the occupation times of random interlacements are the local time processes of this extended Markov process. More generally, for any given transient Markov process $(X_t)_{t\geq 0}$ with an unbounded state space and finite symmetric 0-potential densities, we construct an extended Markov process $(Y_t)_{t\geq 0}$ with a recurrent point. The generalized second Ray-Knight theorem applied to $(Y_t)_{t\geq 0}$ leads to an identity connecting the Gaussian free field associated to $(X_t)_{t\geq 0}$ to the local time process of $(Y_t)_{t\geq 0}$. Besides symmetry is not required from a transient Markov process to admit an extended Markov process with a recurrent point. Given a transient Markov process, we explore the connections between its associated Kuznetsov processes, its quasi-processes, its extended Markov process and its random interlacements.

Keywords: Markov process; excessive measure; local time; Gaussian free fields; isomorphism theorem; random interlacements; Kuznetsov process; quasi-process.
MSC2020 subject classifications: 60A10; 60G05; 60G07; 60G15; 60G53; 60G57; 60J25; 60J35; 60J40; 60J45; 60J55.
Submitted to EJP on December 3, 2021, final version accepted on November 30, 2022.

1 Introduction

Several questions concerning the Gaussian free field on $\mathbb{Z}^d$ ($d \geq 3$) are solved thanks to a Dynkin-type isomorphism theorem established by Sznitman [29] (see for example [6], [31], [1], [27] or [7]). This isomorphism theorem relates the Gaussian free field on any transient weighted graph G to continuous-time random interlacements as follows.

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Denote by $\mathcal{V}$ the vertex set of G. For simplicity assume that each edge of G has weight 1, then for any fixed $u > 0$:

$$(\frac{1}{2}(\eta_x + \sqrt{2u})^2,\ x \in \mathcal{V}) \overset{\text{(law)}}{=} (\frac{1}{2}\eta_x^2 \ +\ L_{x,u}\ ,\ x \in \mathcal{V}) \tag{1.1}$$

where $(\eta_x)_{x\in\mathcal{V}}$ is a Gaussian free field on G, namely a centered Gaussian process with covariance the Green function of the continuous time simple symmetric random walk on G; and the field $(L_{x,u})_{x\in\mathcal{V}}$, independent of η, is the field of the occupation times of the random interlacements on G, at level u. The random interlacements on G at level u is a Poisson point process with intensity $u\,\mu$ where μ is a measure on the space of doubly-infinite nearest neighbors trajectories on G modulo time-shift (for a full description see [29]).
Sznitman's identity (1.1) is of the same type as the generalized second Ray-Knight theorem [11]. Indeed, for any recurrent symmetric Markov process X with state space E, with local time process $(L_t^x, x \in E, t \geq 0)$, fix a point o in E, define: $T_o = \inf\{t \geq 0 : X_t = o\}$ and $\tau_u = \inf\{t \geq 0 : L_t^o > u\}$, for $u > 0$. Then according to [11] one has, given $(X_0 = o)$:

$$(\frac{1}{2}(\eta_x + \sqrt{2u})^2,\ x \in E) \overset{\text{(law)}}{=} (\frac{1}{2}\eta_x^2 \ +\ L_{\tau_u}^x\ ,\ x \in E) \tag{1.2}$$

where $(\eta_x, x \in E)$ is a centered Gaussian process independent of X with covariance g_{T_o} the Green function of X killed at T_o.
In fact, the similarity of (1.1) to (1.2) can be understood through its proof. Indeed (1.1) is established in [29] thanks to an approximation of $(L_{x,u}, x \in \mathcal{V})$ by a sequence of local time processes, each of the above type $L_{\tau_u}^{\cdot}$, and thus each satisfying (1.2) for some Gaussian process.

Our primary aim was to better understand the connection between (1.1) and (1.2), and actually we are going to show that (1.1) is a special case of (1.2). To do so, we answer the following more general question:
Consider a transient symmetric Markov process $(X_t, t \geq 0)$ with state space E, and local time process $(L_t^x(X), x \in E, t \geq 0)$. Can one exhibit a recurrent symmetric Markov process $(Y_t, t \geq 0)$ with state space $E \cup \{\delta\}$, where δ is a point outside E, and local time process $(L_t^x(Y), x \in E \cup \{\delta\}, t \geq 0)$ such that for every a in E:

$$((L_\infty^x(X), x \in E)|X_0 = a) \overset{\text{(law)}}{=} ((L_{T_\delta}^x(Y), x \in E)|Y_0 = a) \tag{1.3}$$

where $T_\delta = \inf\{t \geq 0 : Y_t = \delta\}$?

A positive answer to (1.3), allows to say that the Green function of X is equal to the Green function of Y killed at T_δ. Hence, using (1.2) for Y given $(Y_0 = \delta)$, one obtains that the Gaussian free field associated to X, $(\eta_x)_{x\in E}$, satisfies:

$$(\frac{1}{2}(\eta_x + \sqrt{2u})^2,\ x \in E) \overset{\text{(law)}}{=} (\frac{1}{2}\eta_x^2 \ +\ L_{\tau_u}^x(Y)\ ,\ x \in E), \tag{1.4}$$

with $\tau_u = \inf\{t \geq 0 : L_t^\delta(Y) > u\}$.
The connections between Y and X are not limited to (1.3), which is just a consequence of the construction of Y.

Coming back to the case when X is the continuous time simple random walk on a transient weighted graph, one hence obtains a representation of the occupation times of the corresponding random interlacements on this graph (see section 4). This interpretation might help to visualize differently questions involving this occupation times.

Section 2 is devoted to the construction of the process $(Y_t)_{t\geq 0}$. For that we use the results of Taksar [32], [33]. Given a semigroup $(P_t)_{t\geq 0}$ that corresponds to a Markov process on E such that its life time is finite and has no atom, and a finite excessive measure ν for $(P_t)_{t\geq 0}$, Taksar constructs a semigroup $(\bar{P}_t)_{t\geq 0}$ that is larger than $(P_t)_{t\geq 0}$ in the sense that for any nonnegative f on E: $\bar{P}_t f(x) \geq P_t f(x),\ \forall x \in E, \forall t > 0$, and for which ν is invariant. He shows that there exists then a stationary Markov process indexed by $I\!R$ with state space $E \cup \{\delta\}$, one-dimensional distribution ν and semigroup $(\bar{P}_t)_{t\geq 0}$, such that the Kuznetsov process associated to $\{\nu, (P_t)_{t\geq 0}\}$ is its subprocess in E. Heuristically, this implies that a Markov process indexed by $I\!R_+$ with semigroup $(\bar{P}_t)_{t\geq 0}$ killed at the first hitting time of δ should be equal in law to a Markov process indexed by $I\!R_+$ with semigroup $(P_t)_{t\geq 0}$. In order to establish this identity in law, we directly construct a Markov process $(Y_t)_{t\geq 0}$ with semigroup $(\bar{P}_t)_{t\geq 0}$.
To do so, we start from a given Markov process $(X_t)_{t\geq 0}$ with semigroup $(P_t)_{t\geq 0}$ and finite excessive reference measure ν, satisfying Taksar's conditions of finite life time without atom. Under this sole assumption, we construct in section 2.1 a process $(Y_t)_{t\geq 0}$ on $E \cup \{\delta\}$ such that X has the same law as Y killed at T_δ and show that the law of Y_t given $(Y_0 = x)$ is $\bar{P}_t(x, \cdot)$. The set of times $\{t : Y_t = \delta\}$ is the range of a subordinator with 0 drift, with which Taksar defined the semigroup $(\bar{P}_t)_{t\geq 0}$. The construction of Y does not require symmetry from X nor the existence of local times. In general $(Y_t)_{t\geq 0}$ is not a right process. Nevertheless we shall show in section 2.2, that it satisfies the simple Markov property at all fixed times $t > 0$, and the strong Markov property at some stopping times. This will allow us to establish the isomorphism theorem (1.2) for $(Y_t)_{t\geq 0}$ when X is symmetric and admits a local time process (in section 3). Finally, in section 2.3 we show how to get around the assumption that ν is finite and relax the assumption of finite life time for $(X_t)_{t\geq 0}$ when X is transient and has a local time process.
When X is symmetric, transient and has a local time process, one of the consequences of the existence of this extended Markov process Y is the relation (1.4). We precisely state it in section 3 and present some illustrations in section 4. In particular, we show how to recover a percolation result for the Gaussian free field on $\mathbb{Z}^d$ due to Bricmont et al [3] with its extension to any infinite connected graph.

Having established (1.3) and (1.4), we go beyond these identities in section 5. In section 5, we consider the Kuznetsov process associated to $\{\nu, (P_t)_{t\geq 0}\}$ for ν excessive for $(P_t)_{t\geq 0}$ and explain its connections with its quasi-process as constructed by Fitzsimmons [14], the excursion measure from δ of Y and Sznitman's random interlacements [29] in a general setting. To do so we first have to set a definition for the random interlacements associated to $\{\nu, (P_t)_{t\geq 0}\}$. This definition requires paths continuity from $(X_t)_{t\geq 0}$ the Markov process with semigoup $(P_t)_{t\geq 0}$. We show that if ν is finite and X has a finite life time with no atom, the random interlacements associated to $\{\nu, (P_t)_{t\geq 0}\}$ correspond to the excursion process from δ of Y modulo time shift. If this condition is not satisfied, one still has a correspondence under the additional assumption that X is transient and has a local time process. Namely the occupation time field of the random interlacements at level u associated to $\{\nu, (P_t)_{t\geq 0}\}$ coincides with the local time process of Y at the first time its local time at δ exeeds u.

Finally we mention that to establish (1.2) for Y, we did not require symmetry from Y, the symmetry of X was actually sufficient for that. Nevertheless, we show in section 6, using among others the connection between the Kuznetsov process and the quasi-process, that when $(P_t)_{t\geq 0}$ has a symmetric resolvent so does $(\bar{P}_t)_{t\geq 0}$.

2 Extension of Borel right processes

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, I\!P_x; x \in E)$ be a Borel right process taking values in a Borel space $(E, \mathcal{E})$. This means that $(X_t)_{t\geq 0}$ is stochastic process on a probability space $(\Omega, \mathcal{F}, I\!P)$ with right continuous paths; θ_t denotes the usual shift operator ($X_s \circ \theta_t = X_{t+s}$; $\forall t, s \geq 0$) and for every x in E, $(X_t)_{t\geq 0}$ satisfies the strong Markov property with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ (which is augmented and right continuous) under the probability $I\!P_x$ such that $I\!P_x(X_0 = x) = 1$.
Denote by $(P_t)_{t\geq 0}$ the semigroup of X. We do not assume that $P_t 1 = 1$. One defines the life time ζ of X by: $I\!P_x(\zeta > t) = P_t 1(x), \forall t > 0, \forall x \in E$.
Let ν be an excessive measure for $(P_t)_{t\geq 0}$ (i.e. ν is a σ-finite measure on $\mathcal{E}$ and for every nonnegative $\mathcal{E}$ measurable function f: $\nu P_t f \leq \nu f$).
For instance, if $E = \mathbb{Z}^d, d \geq 3$, and $(X_t)_{t\geq 0}$ is the time continuous simple symmetric random walk on $\mathbb{Z}^d$, then X has an infinite life time, its semigroup is defined by $P_t(x, y) = e^{t(P-I)}(x, y)$ for $x, y \in \mathbb{Z}^d$, where P is the transition matrix of the discrete simple symmetric random walk on $\mathbb{Z}^d$ and I the identity on $\mathbb{Z}^d$. The semigroup $(P_t)_{t\geq 0}$ is symmetric and admits the counting measure on $\mathbb{Z}^d$ as invariant (hence excessive) measure.

We are first looking for a Markov process $(Y_t)_{t\geq 0}$ on $E \cup \{\delta\}$ ($\delta \notin E$) that would extend X in the sense that for every x in E: $((X_t, t < \zeta) | X_0 = x) \overset{(\text{law})}{=} ((Y_t, t < T_\delta) | Y_0 = x)$ and such that starting from δ, Y visits E a.s.
Denote by $(\bar{P}_t)_{t\geq 0}$ the semigroup of such Y. Then one immediately checks that $(\bar{P}_t)_{t\geq 0}$ must be larger than $(P_t)_{t\geq 0}$ (i.e. for every nonnegative $\mathcal{E}$ measurable function f: $\bar{P}_t f(x) \geq P_t f(x), \forall x \in E$).
Under additional assumptions on $(P_t)_{t\geq 0}$ and ν, Taksar [32, 33] has constructed a semigroup $(\bar{P}_t)_{t\geq 0}$ larger than $(P_t)_{t\geq 0}$, so that $\bar{P}_t 1(x) = 1$ for all $x \in E$ and ν is invariant for $(\bar{P}_t)_{t\geq 0}$. For our needs, and also for the sake of completeness, in section 2.1, we shall give explicitly the construction of $(\bar{P}_t)_{t\geq 0}$ and of the process Y that extends X with law at time t equal to $\bar{P}_t(x, \cdot)$. We shall start by quoting Theorems 1 and 2 of [33], combined into one theorem, adapted to our setting.

Theorem 2.1. (Taksar)*Let $(P_t)_{t\geq 0}$ be a Borel right semigroup on a Borel space $(E, \mathcal{E})$ such that for all x in E, $t \to P_t 1(x)$ is continuous in t on $I\!R_+$. Let ν be a finite $(P_t)_{t\geq 0}$ excessive measure on $(E, \mathcal{E})$. Then there exists a semigroup $(\bar{P}_t)_{t\geq 0}$ which is larger than $(P_t)_{t\geq 0}$ and satisfies $\bar{P}_t(x, E) = 1$ and for which ν is invariant. If in addition ν is a minimal excessive measure then $(\bar{P}_t)_{t\geq 0}$ is unique up to a ν-null set.*

An excessive measure m for $(P_t)_{t\geq 0}$ is minimal if for every couple of excessive measures for $(P_t)_{t\geq 0}$, (m_1, m_2), such that: $m = m_1 + m_2$, then necesseraly m_1 and m_2 are proportional to m.

Remark 2.2. For our purpose one will actually exploit Theorem 2.1 only in the case when ν is purely excessive (when ν is invariant, the semigroup $(\bar{P}_t)_{t\geq 0}$ provided by Theorem 2.1 is $(P_t)_{t\geq 0}$ itself). An excessive measure ν for $(P_t)_{t\geq 0}$ is purely excessive if for all x in E: $\int_E \nu(dx) P_t 1(x) \to 0$ as $t \to \infty$. This is equivalent to $\int_E \nu(dx) I\!P_x[\zeta = \infty] = 0$. Hence if ζ is finite $I\!P_x$ a.s. $\forall x \in E$, and ν is finite excessive then ν is purely excessive. Note that $t \to P_t 1(x)$ is continuous on $I\!R_+$ for all x in E iff ζ has no atom $I\!P_x$ a.s. To obtain this property, sufficient conditions compatible with non infinite life time, will be given in Proposition 2.10.

Section 2.1 below establishes the following proposition which is a rough description of the extended process Y.

Proposition 2.3. *Let X be a Borel right process with semigroup $(P_t)_{t\geq 0}$ such that for all $x \in E$, $t \to P_t 1(x)$ is continuous in t on $\mathbb{R}_+$ and there exists a finite purely excessive measure ν with respect to $(P_t)_{t\geq 0}$. There exist then a Poisson point process e on $\mathbb{R}_+$ with values in the space of right continuous functions from $(0,\infty)$ into $(E,\mathcal{E})$ and a subordinator $(\tau_t)_{t\geq 0}$ on $\mathbb{R}_+$, both independent of X, such that the process $(Y_t)_{t\geq 0}$ defined by*

$$Y_t = \begin{cases} X_t & t < \zeta \\ \delta & t \in \zeta + M \\ e_s(t - \tau_{s-} - \zeta) & \zeta + \tau_{s-} < t < \zeta + \tau_s \end{cases} \tag{2.1}$$

where $M = \{s \in \mathbb{R}_+ : \tau_t = s \text{ for some } t \in \mathbb{R}_+\}$ and $\zeta + M = \{t : t = \zeta + s \text{ for some } s \in M\}$,
is a Markov process with semigroup $(\bar{P}_t)_{t\geq 0}$ admitting δ for recurrent point and satisfying for every a in E:

$$((X_t, t < \zeta)|X_0 = a) \overset{(\text{law})}{=} ((Y_t, t < T_\delta)|Y_0 = a).$$

Each term of the above description is precisely defined in section 2.1. In particular the Poisson point process (PPP in the sequel) e is defined by (2.4) and (2.3), the subordinator $(\tau_t)_{t\geq 0}$ by (2.5) and the semigroup $(\bar{P}_t)_{t\geq 0}$ by (2.10), (2.11) and (2.7).
Note that Proposition 2.3 does not require the existence of a local time process for X nor symmetry.
Note also that the continuous time simple symmetric random walk on $\mathbb{Z}^d$ does not fulfill the assumptions of Proposition 2.3. Nevertheless one can obtain (1.3) for this Markov process. Indeed Corollary 2.4 below relaxes the assumption of finite purely excessive measure under the condition of transience and existence of a local time process.
More precisely, consider a Borel right process X with a reference measure ν (i.e. for every $\alpha \geq 0$, U^α, the α-potential of X, is such that $U^\alpha(x,.)$ is absolutely continuous with respect to ν). Let $(u^\alpha(x,y), (x,y) \in E \times E)$ be its α-potential densities with respect to ν. We will simply write $u(x,y)$ for $u^0(x,y)$. When $u(x,y) < \infty$ for all $x, y \in E$, then the two following properties are satisfied:

- X is transient (i.e. for every y in E: $\sup\{t \geq 0 : X_t = y\} < \infty, \mathbb{P}_x\ a.s. \forall x \in E$).
- X admits a local time process with respect to ν.

In this case we shall denote the local time process by $(L_t^x(X), x \in E, t \geq 0)$ and normalize it so that

$$u^\alpha(x,y) = E_x[\int_0^\infty e^{-\alpha t} dL_t^y(X)] \quad \text{for all } x, y \in E. \tag{2.2}$$

The resolvent equation

$$u^\alpha(x,y) = u^\beta(x,y) + (\beta - \alpha)U^\alpha u^\beta(x,y)$$

where U^α is the α-potential operator of X, guarantees that the normalization of $(L_t^x(X), x \in E, t \geq 0)$ for one α is good for all $\alpha \geq 0$.
Corollary 2.4 uses the notion of standard process. A Markov process X is standard if for every stopping time T, every x in E, for every increasing sequence of stopping times $(T_n)_{n\geq 0}$ converging $\mathbb{P}_x$ a.s. to T, then X_{T_n} tends to X_T on $\{T < \zeta\}$ $\mathbb{P}_x$ a.s.
For q nonnegative function on E, we denote by $\nu \cdot q$ the measure on $\mathcal{E}$ defined by $\nu \cdot q(A) = \int_A q(x)\nu(dx)$, $A \in \mathcal{E}$.

Corollary 2.4. *Let X be a Borel right process on E with finite 0-potential densities with respect to an excessive reference measure ν. Assume that either **(a)** X is a standard process or **(b)** E is discrete. Then there exists a positive function q on E such that $\nu \cdot q$*

is finite, and there exists a Markov process $(Y_t)_{t\geq 0}$ *with state space* $E\cup\{\delta\}$ $(\delta\notin E)$ *admitting a local time process on* E, $(L^x_t(Y), x\in E, t\geq 0)$*, with respect to* $\nu\cdot q$ *and* δ *as a recurrent point, satisfying for every* a *element of* E*:*

$$((L^x_\infty(X), x\in E)|X_0=a) \overset{\text{(law)}}{=} ((L^x_{T_\delta}(Y), x\in E)|Y_0=a).$$

Corollary 2.4 is established in section 2.3. Still, for a full positive answer to (1.3) one needs the following corollary which is established in section 6.

Corollary 2.5. *For* X *satisfying the assumptions of* Corollary 2.4*, assume moreover that the* 0*-potential densities of* X *are symmetric, then* Y *is a symmetric recurrent Markov process.*

2.1 Proof of Proposition 2.3

Section 2.1 is entirely devoted to the proof of Proposition 2.3.

To define the PPP e, one needs first to define its intensity. To do so we note that since ν is a purely excessive measure there exists a unique entrance law $(m_t)_{t>0}$ with respect to $(P_t)_{t\geq 0}$, such that $\nu(f)=\int_0^\infty m_t(f)dt$ and $m_sP_{t-s}=m_t$, $0<s<t$ (see [9]). Let P_* be the measure on $(\Omega,\mathcal{F},\mathcal{F}_t)$ with one dimensional distributions at times t equal to $(m_t)_{t>0}$ and transition function $(P_t)_{t\geq 0}$, the transition function of X. Under P_*, the coordinate process $(Z_t)_{t\geq 0}$ is a Borel right process on $(0,\infty)$ with values in E and finite dimensional distributions given by

$$P_*(Z_{t_1}\in A_1,..,Z_{t_n}\in A_n)=\int_{A_1} m_{t_1}(dx_1)\int_{A_2} P_{t_2-t_1}(x_1,dx_2)..\int_{A_n} P_{t_n-t_{n-1}}(x_{n-1},dx_n) \tag{2.3}$$

for $0<t_1\leq ..\leq t_n$ and $A_1,..,A_n\in\mathcal{E}$.
The existence of P_* is a consequence of the compatibility conditions satisfied by the finite-dimensional distributions given by (2.3).
Under P_*, $(Z_t)_{t\geq 0}$ has right continuous sample paths in $(0,\infty)$ and satisfies the strong Markov property at all $(\mathcal{F}_t)_{t\geq 0}$ stopping times T such that $0<T<\zeta$. Moreover it has a finite life time since one has:

$$P_*[\,\zeta\,]=P_*[\int_0^{+\infty} 1_{\zeta>t}dt]=\int_0^{+\infty} m_t(E)dt=\nu(E)<\infty.$$

Let $e=(e_t,\mathcal{G},\mathcal{G}_t,Q)$ be a PPP on $\mathbb{R}_+$, taking values in the space of right continuous functions from $(0,\infty)$ to $(E,\mathcal{E})$ and life time ζ_e, with rate measure (intensity)

$$dt\times P_*. \tag{2.4}$$

We assume that e is independent of X. Define

$$\tau_t=\sum_{s\leq t}\zeta_{e_s}. \tag{2.5}$$

One computes: $Q(e^{-\alpha\tau_t})=\exp\{-t\,P_*(1-e^{-\alpha\zeta})\}$. Since $P_*(\zeta=+\infty)=0$, it follows that $(\tau_t,\mathcal{G}_t,Q)$ is a subordinator with $Q(\tau_t<\infty)=1$ for all $t\in\mathbb{R}_+$. Let Q_y be the law of e obtained by shifting the time 0 to the point y, thus moving its starting point to the point y. In particular, note that under Q_y: $\tau_0=y$, and for a Borel set $A\subset\mathbb{R}_+$, $\quad Q_y(\tau_t\in A)=Q(\tau_t\in A-y)$.
We set

$$M=\{s\in\mathbb{R}_+:\tau_t=s \text{ for some } t\in\mathbb{R}_+\} \tag{2.6}$$

and

$$\ell_t = \inf\{s : \tau_s > t\}. \tag{2.7}$$

If the first point of increase of $(\tau_t)_{t\geq 0}$ is strictly positive, then $(\tau_t)_{t\geq 0}$ is a compound Poisson process and e has a finite number of points in any finite interval. If the first point of increase of $(\tau_t)_{t\geq 0}$ is equal to 0 then every point of $\mathbb{R}_+$ is a point of right increase of $(\tau_t)_{t\geq 0}$ that is $t \to \tau_t$ is strictly increasing and e has a countable number of points in any finite interval.

In the case when $(\tau_t)_{t\geq 0}$ is a compound Poisson process, $(\ell_t)_{t\geq 0}$ is a sum of i.i.d. exponentially distributed random variables. Otherwise, $(\ell_t)_{t\geq 0}$ is a continuous process. In both cases $t \to \ell_t$ increases on M and for every t, ℓ_t is a $(\mathcal{G}_t)_{t\geq 0}$ stopping time.

Using [23], we know that M is a regenerative set and $(\ell_t)_{t\geq 0}$ a local time at M. We specify below the choice of filtration and shift operators with respect to which this properties hold. Set

$$\bar{\mathcal{F}}_t = \mathcal{G}_{\ell_t} \tag{2.8}$$

and for $t > 0$, define the shift operator $\hat{\theta}_t$ by $\hat{\theta}_t e = ((\hat{\theta}_t e)_s, s \geq 0)$ with
$(\hat{\theta}_t e)_s = e_{\ell_t+s} = (e_{\ell_t+s}(u), u \in (0, \tau_{\ell_t+s} - \tau_{(\ell_t+s)-}))$ for $s > 0$, and
$(\hat{\theta}_t e)_0 = (e_{\ell_t}(u), u \in (0, \tau_{\ell_t} - t))$.

One obtains:

- $\tau_s(\hat{\theta}_t) = \tau_{\ell_t+s} - t$
- $M \circ \hat{\theta}_t = (M - t) \cap \mathbb{R}_+$
- $\ell_{t+s} = \ell_t + \ell_s \circ \hat{\theta}_t$

which shows that $\{M, \bar{\mathcal{F}}_t, \hat{\theta}_t, Q\}$ is a regenerative set as defined by Maisonneuve [23] and that $(\ell_t)_{t\geq 0}$ is the local time at M.

Besides, since under Q, e is a PPP, one has for every $(\mathcal{G}_t)_{t\geq 0}$ stopping time T:

$$Q(f(e_{T+.})|\mathcal{G}_T) = Q(f(e)) \text{ a.s. on } \{T < \infty\}$$

where $e_{T+.}$ are the points of the PPP during the time interval (T, ∞). More generally, for every $(\bar{\mathcal{F}}_t)_{t\geq 0}$ stopping time T that falls in M, one has a.s. on $\{T < \infty\}$:

$$Q[f((\hat{\theta}_T e)_s, s > 0)|\bar{\mathcal{F}}_T] = Q[f(e_s)_{s>0})] \tag{2.9}$$

(this follows from the fact that for T $(\bar{\mathcal{F}}_t)_{t\geq 0}$-stopping time, ℓ_T is a $(\mathcal{G}_t)_{t\geq 0}$-stopping time).

We are now ready to define the extended process $(Y_t)_{t\geq 0}$ according to (2.1):

$$Y_t = \begin{cases} X_t & t < \zeta \\ \delta & t \in \zeta + M \\ e_s(t - \tau_{s-} - \zeta) & \zeta + \tau_{s-} < t < \zeta + \tau_s \end{cases}$$

where $\zeta + M = \{t : t = \zeta + s \text{ for some } s \in M\}$.

The process $(Y_t)_{t\geq 0}$ is constructed by first running X until its death time, then run the PPP e that is independent of X, taking values in the space of right continuous functions with values in E, and stitch its pieces together. The regenerative set M, defined by (2.6), is the range of a subordinator with no drift and whose jumps are the life times of the pieces of the PPP e. We have defined the process Y to be equal to δ on this set M.

We now define $\bar{P}_t$ for every t, as follows. For $f \in \mathcal{E}$, f is extended to $E \cup \{\delta\}$ by $f(\delta) = 0$.

$$\bar{P}_t f(x) = P_t f(x) + \int_0^t \mathbb{P}_x(\zeta \in dy) Q_y(\int_0^t P_*(f(Z_{t-s}))d\ell_s) \tag{2.10}$$

and

$$\bar{P}_t f(\delta) = Q(\int_0^t P_*(f(Z_{t-s}))d\ell_s). \tag{2.11}$$

We shall show that $(\bar{P}_t)_{t\geq 0}$ is the transition semigroup that defines the law of the extended process $(Y_t)_{t\geq 0}$.

Step 1 The first step consists in checking that $(\bar{P}_t)_{t\geq 0}$ is a transition semigroup. For that we take advantage of the fact that (2.10) coincides with the expression (2.2.2) in [33] set by Taksar in the case when ν is purely excessive. Consequently we can use Theorem 2.2.1 of Taksar [33] (which relies on the fact that ν is finite purely excessive and $P_t 1(x)$ is continuous in t), to claim that

1. For any $\Gamma \in \mathcal{E} \quad \nu(\Gamma) = \int_E \nu(dx)\bar{P}_t(x,\Gamma)$
2. $\bar{P}_t(x,E) = 1$
3. $\int_E \bar{P}_s(x,dy)\bar{P}_t(y,\Gamma) = \bar{P}_{s+t}(x,\Gamma)$

and hence that for all $t,x \quad \bar{P}_t(x,\delta) = 1 - \bar{P}_t(x,E) = 0$.
In particular, one obtains that $(\bar{P}_t)_{t\geq 0}$ is a transition semigroup and that ν is invariant for it.

Step 2 One computes the law of Y_t given $(Y_0 = x)$. The result is Lemma 2.6 below.

Lemma 2.6. *The law of Y_t conditioned on $\{Y_0 = x\}$ is given by* (2.10) *and* (2.11)*.*

Proof. Let f be a measurable function on E, extended to $E \cup \{\delta\}$ by $f(\delta) = 0$. For x in E:

$$\begin{aligned} \mathbb{E}(f(Y_t)|Y_0 = x) &= \mathbb{E}(f(Y_t)1_{\{t<\zeta\}}|Y_0 = x) + \mathbb{E}(f(Y_t)1_{\{t\geq\zeta\}}|Y_0 = x) \\ = P_t f(x) \quad &+ \quad \int_{y=0}^t P_x(\zeta \in dy) Q\left(\sum_{y+\tau_{s-}\leq t<y+\tau_s} 1_{t-\tau_{s-}-y<\zeta_{e_s}} f(e_s(t-\tau_{s-}-y))\right) \end{aligned}$$

Recall now that $dt \times P_*$ is the $(\mathcal{G}_t)_{t\geq 0}$ compensator of e, and it is, by the definition of the compensator, the dual predictable projection with respect to $(\mathcal{G}_t)_{t\geq 0}$ of the sum of its points (one usually refers to this property as the "compensation formula" [4]). Therefore:

$$\begin{aligned} &Q \sum_{y+\tau_{s-}<t<y+\tau_s} 1_{t-\tau_{s-}-y<\zeta_{e_s}} f(e_s(t-\tau_{s-}-y)) \\ &= \quad Q_y \sum_{\tau_{s-}<t<\tau_s,} 1_{t-\tau_{s-}<\zeta_{e_s}} f(e_s(t-\tau_{s-})) \\ &= \quad Q_y \sum_{\tau_{s-}<\tau_s} 1_{[0,\ell_t]}(s) 1_{t-\tau_{s-}<\zeta_{e_s}} f(e_s(t-\tau_{s-})) \\ &= \quad \int Q_y(d\omega) \int_0^\infty ds\ 1_{[0,\ell_t(\omega)]}(s) \int P_*(d\omega') f(\omega'(t-\tau_{s-}(\omega))) 1_{t-\tau_{s-}(\omega)<\zeta(\omega')}. \end{aligned}$$

We have used here the fact that since ℓ_t is a $(\mathcal{G}_t)_{t\geq 0}$ stopping time $(1_{[0,\ell_t]}(s))_{s>0}$ is a $(\mathcal{G}_t)_{t\geq 0}$ predictable process and so is $(\tau_{s-})_{s>0}$ since it is $(\mathcal{G}_t)_{t\geq 0}$ adapted and left continuous. Finally we recall that $\tau_{s-} < \tau_s$ only for a countable number of times and therefore the last expression is equal to

$$\int Q_y(d\omega) \int_0^\infty ds\ 1_{[0,\ell_t(\omega)]}(s) \int P_*(d\omega') f(\omega'(t-\tau_s(\omega))) 1_{t-\tau_s(\omega)<\zeta(\omega')},$$

and hence to

$$Q_y \int_0^t P_*(f(Z(t-s))d\ell_s.$$

Summing it all up we have shown that

$$\mathbb{E}(f(Y_t)|Y_0 = x) = P_t f(x) + \int_{y=0}^{t} \mathbb{P}_x(\zeta \in dy) Q_y \int_0^t P_*(f(Z(t-s)) d\ell_s,$$

as appears in (2.10).

Given $Y_0 = \delta$, Y enters E immediately. The above computation leads similarly to (2.11). □

Step 3 Finally we have to check that Y satisfies the simple Markov property at each $t \geq 0$. This follows from its construction: first note that $(Y_t, 0 \leq t < \zeta)$ is Markovian and that $(Y_t, 0 \leq t < \zeta)$ is independent of $(Y_{t+\zeta}, t \geq 0)$. One has: $Y_{t+\zeta} = e_{\ell_t}(t - \tau_{(\ell_t)-})$, hence $Y_{t+\zeta}$ is $\bar{\mathcal{F}}_t$ measurable. Since: $Y_{t+s+\zeta} = e_{\ell_{t+s}}(t + s - \tau_{(\ell_{t+s})-})$

- on $\{\ell_{t+s} > \ell_t\}$: using (2.9), $Y_{t+s+\zeta} = \hat{\theta}_t(e_{\ell_s}(s - \tau_{(\ell_s)-}))$ is independent of $\bar{\mathcal{F}}_t$ and hence of $(Y_{u+\zeta}, u \leq t)$,

- on $\{\ell_{t+s} = \ell_t\}$: $Y_{t+s+\zeta} = e_{\ell_t}(t + s - \tau_{(\ell_t)-})$ and one uses the Markov property of e_{ℓ_t} on $(0, \tau_{\ell_t} - \tau_{(\ell_t)-})$ at time $t - \tau_{(\ell_t)-}$.

Hence $(Y_{t+\zeta}, t \geq 0)$ is Markovian and so is Y.

To establish that δ is a recurrent point for Y, just note that $\tau_t < \infty$ a.s. for every t. □

2.2 Strong Markov properties of the extended process Y

We saw in section 2.1 that Y satisfies the simple Markov property at each $t \geq 0$. Its state space is $(E \cup \{\delta\}, \mathcal{E} \vee \delta)$. Since $(P_t)_{t \geq 0}$ is assumed to be a Borel right semigroup, Y has a version with right continuous paths outside $\zeta + M$, and satisfies the strong Markov property at all stopping times with graphs in the complement of the set of times when Y is equal to δ. On the set of times when Y is equal to δ, it may not even be right continuous. To remedy this, one could possibly use a Ray Knight compactification of E, but then one may possibly need to enlarge E by more than one point. Since we have constructed our process by first running X until its death and then running the PPP e, one can define $\tilde{\mathcal{G}}_t = \mathcal{F}_\infty \vee \mathcal{G}_t$, and the measure $\tilde{P}_x = \mathbb{P}_x \times Q$ and $\tilde{Q} = \tilde{P}_x(\cdot\,|\mathcal{F}_\infty)$. Define now $\tilde{\tau}_s = \zeta + \tau_s,\ s \geq 0$, and similarly $\tilde{\ell}_t = \inf\{s : \tilde{\tau}_s > t\}$. Note that $\tilde{\ell}_t = 0$ for all $t \leq \tilde{\tau}_0$. Define also for $t \geq \zeta$: $\tilde{e}_t = e_{t-\zeta}$. Set: $\tilde{\mathcal{F}}_t = \tilde{\mathcal{G}}_{\tilde{\ell}_t}$. By the fact that the range of $(\tilde{\tau}_t)_{t \geq 0}$ is the range of a regenerative set moved by ζ, it follows that if a $(\tilde{\mathcal{G}}_{\tilde{\ell}_t})_{t \geq 0}$ stopping time T has a graph in the range of $(\tilde{\tau}_t)_{t \geq 0}$

$$\tilde{Q}(f(\tilde{e}_{T+\cdot})|\tilde{\mathcal{F}}_T) = \tilde{Q}(f(e_{T-\zeta+\cdot})|\tilde{\mathcal{F}}_T) = Q(f(e))$$

by (2.9). In particular, for any $t \geq 0$

$$\tilde{Q}(f(\tilde{e}_{\tilde{\tau}_t+\cdot})|\tilde{\mathcal{F}}_t) = Q(f(e))$$

because $\tilde{\tau}_t$ is a $(\tilde{\mathcal{F}}_t)$ stopping time that falls in $\zeta + M$. The same is true when γ, is an exponentially distributed random variable, independent of $\tilde{\mathcal{G}}_\infty$, and

$$\tilde{\tau}_\gamma = \inf\{t : \tilde{\ell}_t > \gamma\},$$

because $\tilde{\tau}_\gamma$ is a $(\tilde{\mathcal{F}}_t)_{t \geq 0} \vee \sigma_\gamma$ stopping time that falls in the range of $\tilde{\tau}$. Integrating with respect to the distribution of γ one can show that the law of $e_{\tilde{\tau}_\gamma+\cdot}$ given $\tilde{\mathcal{F}}_{\tilde{\tau}_\gamma}$ is again the law of e under Q. Thus Y satisfies the strong Markov property at $\tilde{\tau}_s$ for $s \geq 0$ and also at $\tilde{\tau}_\gamma$ where γ is an exponentially distributed random variable independent of $\tilde{\mathcal{G}}_\infty$.

Remark 2.7. Assume that X admits a local time process with respect to ν. Then Y also admits a local time process on E with respect to ν that we denote by $(L_t^x(Y), x \in E, t \geq 0)$. Note that for any $s > 0$, any $y \in E$: $E_\delta(L_{\tilde{\tau}_s}^y(Y)) = sP_*(L_\zeta^y)$.

It follows from Lemma 2.8 below that:

$$E_x(L^y_{\bar{\tau}_\gamma}(Y)) = u(x,y) + \mathbb{E}(\gamma)P_*(L^y_\zeta) = u(x,y) + \mathbb{E}(\gamma). \tag{2.12}$$

Lemma 2.8. *For every* $x \in E$, $\quad P_*(L^x_\zeta) = 1$.

Proof. By monotone convergence, one has

$$P_*(L^x_\zeta) = \lim_{t\downarrow 0} P_*(\ 1_{\zeta>t}\ L^x_\zeta o\theta_t).$$

Note that: $P_*(1_{\zeta>t}\ L^x_\zeta o\theta_t) = \int_E m_t(dy)u(y,x)$. Remember that for all $0<t<s$, we have: $m_tP_{s-t} = m_s$. Hence we have:

$$m_tU = \int_t^\infty m_s ds. \tag{2.13}$$

By taking the Radon Nikodym derivatives on both sides of (2.13), one obtains:

$$\int_E m_t(dy)u(y,x) = \frac{d(\int_t^\infty m_s ds)}{d\nu}(x)$$

Consequently:

$$\begin{aligned} P_*(L^x_\zeta) &= \lim_{t\downarrow 0} P_*(1_{\{\zeta>t\}}L^x(\theta_t)) \\ &= \lim_{t\downarrow 0}\int_E m_t(dy)u(y,x) = \lim_{t\downarrow 0}\frac{d\left(\int_t^\infty m_s(\cdot)ds\right)}{d\nu}(x) \\ &= \frac{d\nu}{d\nu}(x) = 1. \end{aligned}$$

□

Remark 2.9. Taksar [32, 33] relates the transition semigroup $\bar{P}_t(x,\cdot)$ to a covering process of a Kuznetsov process $Z = (\mathcal{W},\mathcal{G},\mathcal{G}_t,Z_t,\sigma_t,\mathbf{Q}_\nu)_{t\in\mathbb{R}}$. Here under $\mathbf{Q}_\nu$, $(Z_t)_{t\in\mathbb{R}}$ is a stationary Markov process on $(E,\mathcal{E})$ with random times of birth and death, with one dimensional distribution at time t equal to ν and transition semigroup $(P_t)_{t\geq 0}$ (for more details see section 5). He constructs a Markov process $(Z_t,\bar{P})_{t\in\mathbb{R}}$ on $E\cup\{\delta\}$ such that $\bar{P}_t(x,E) = 1, \forall x\in E$, and which "covers" the process $(Z_t,\mathbf{Q}_\nu)_{t\in\mathbb{R}}$. Heuristically this means that for any $s_1<s_2<..<s_n$ in $\mathbb{R}$ and $A_1,...,A_n\in\mathcal{E}$

$$\mathbf{Q}_\nu(Z_{s_1}\in A_1, Z_{s_2}\in A_2,..,Z_{s_n}\in A_n) = \bar{P}(Z_{s_1}\in A_1,..,Z_{s_n}\in A_n,\ [s_1,s_n]\cap\bar{M}=\emptyset), \tag{2.14}$$

where $\bar{M} = \{t\in\mathbb{R} : Z_t = \delta\}$. But this identity would require the strong Markov property from $(Z_t,\bar{P})_{t\in\mathbb{R}}$ at some special random times.
Unlike Taksar, we have constructed a covering process $(Y_t)_{t\geq 0}$ for the Markov process $(X_t)_{t\geq 0}$, using the PPP $(e_t)_{t\geq 0}$. In Lemma 2.6 we have shown that Taksar's semigroup $(\bar{P}_t)_{t\geq 0}$ determines the law of Y. Thanks to sections 2.1 and 2.2, we can claim that (2.14) is correct.

2.3 From transient with local times to a process satisfying Taksar's conditions

Let $X = (\Omega,\mathcal{F},\mathcal{F}_t,X_t,\theta_t,\mathbb{P}_x)$ be a Borel right process with an excessive measure ν. So far, using Remark 2.2, which provides sufficient conditions to use Proposition 2.3, we have constructed an extended Markov process Y for X under the following conditions:

- $t\to\mathbb{P}_x(\zeta>t)$ is continuous in t for all $x\in E$.
- The life time of X is $\mathbb{P}_x$ a.s. finite for each $x\in E$.
- The excessive measure ν is finite.

The following proposition provides sufficient conditions for the realization of the first item.

Proposition 2.10. ***(i)*** *Let X be a standard process with state space E admitting a local time process. Then for every x in E, $I\!P_x$ a.s. ζ has no atom.*

(ii) *Let X be a Borel right process with a discrete state space E, admitting a finite excessive measure ν. Then for every x in E, $I\!P_x$ a.s. ζ has no atom.*

Proof. (i) First note the following property for every $t, s > 0$

$$I\!P_x[\zeta > t+s] = \int_E I\!P_x[X_s \in dy] I\!P_y[\zeta > t]. \tag{2.15}$$

Denote by f_s the function defined by $f_s(x) = I\!P_x[\zeta > s]$. If $t > u$, then $f_t \leq f_u$. For every $t > 0$, (2.15) is equivalent to $P_t f_s(x) = f_{t+s}(x)$, hence: $P_t f_s \leq f_s$. Moreover thanks to the right continuity of $t \to I\!P_x[\zeta > t]$ on $I\!R_+$ one has: $\lim_{t \downarrow 0} P_t f_s = f_s$. Consequently the function f_s is excessive with respect to $(P_t)_{t \geq 0}$.
Let $(s_n)_{n>0}$ be an increasing sequence converging to $t > 0$. The sequence $(f_{s_n})_{n>0}$ is a decreasing sequence of excessive functions converging to the function $x \to I\!P_x[\zeta \geq t]$. According to Theorem 3.6 in Chap.3, p.81 in [2], the set $\{x \in E : I\!P_x[\zeta > t] < I\!P_x[\zeta \geq t]\}$ is a subset of points of E which are not regular for X. By assumption all the points of E are regular for X, hence for every x in E and every $t > 0$: $I\!P_x[\zeta = t] = 0$.
(ii) From (2.15) one obtains by monotone convergence for every x in E the right continuity of $t \to I\!P_x[\zeta > t]$ on $I\!R_+$. One also obtains by letting t increase and dominated convergence:

$$I\!P_x[\zeta \geq t+s] = \int_E I\!P_x[X_s \in dy] I\!P_y[\zeta \geq t],$$

which leads with (2.15) to

$$I\!P_x[\zeta = t+s] = \int_E I\!P_x[X_s \in dy] I\!P_y[\zeta = t]. \tag{2.16}$$

By definition, the excessive measure ν satisfies: $\nu P_s f \leq \nu f$, for every $s > 0$ and every bounded nonnegative function f. For $f(x) = I\!P_x[\zeta = t]$, one obtains thanks to (2.16): $I\!P_\nu[\zeta = t+s] \leq I\!P_\nu[\zeta = t]$. Consequently if for some $u > 0$ $I\!P_\nu[\zeta = u] > 0$ then for every $t \in (0, u]$: $I\!P_\nu[\zeta = t] > 0$. Since ν is finite, the function $t \to I\!P_\nu[\zeta > t]$ is right continuous with left limits, hence the set of discontinuities is at most countable. Consequently: $I\!P_\nu[\zeta = t] = 0, \forall t$. Since E is discrete, one obtains: $\forall x \in E, I\!P_x[\zeta = t] = 0$. □

Proof of Corollary 2.4. Suppose we are given a process X which has finite 0-potential densities $(u(x,y), (x,y) \in E \times E)$ with respect to ν excessive reference measure.
In case ν is not finite note that there always exists a positive function h on E such that $\int_E h(x)\nu(dx) < \infty$.
Set: $q(x) = \frac{h(x)}{u(x,x)} \wedge h(x)$. One obtains then: $Uq(x) < \infty$ for all $x \in E$ and $\nu \cdot q < \infty$. Set

$$A_t = \int_0^t q(X_s)ds, \; S_t = \inf\{s : A_s > t\} \text{ and } \tilde{X}_t = X_{S_t} \quad t \geq 0. \tag{2.17}$$

Denote by $\zeta_{\tilde{X}}$ and ζ_X the respective life times of $\tilde{X}$ and X.
$I\!E_x(\zeta_{\tilde{X}}) = I\!E_x(A_\infty) = Uq(x) < \infty$, so that $I\!P_x$ a.s., $\tilde{X}$ has a finite life time. Further, the measure $\nu \cdot q$ is an excessive reference measure for $\tilde{X}$ (see for example [20]) and the 0-potential measure of $\tilde{X}$ is equal to $u(x,y)q(y)\nu(dy)$ and therefore its density with respect to $\nu \cdot q$, is equal to $u(x,y)$, the same 0-potential density as that of X. Further for any x in E the total accumulated local time of $\tilde{X}$ at x in $[0, \zeta_{\tilde{X}})$ is equal to the total accumulated local time of X at x in $[0, \zeta_X)$.

Making use of Proposition 2.10, one can obtain the continuity of "$t \to I\!P_x[\tilde{\zeta} > t]$" with one of the additional assumptions below:

(a) X is a standard process admitting a local time process.

(b) The state space of X is discrete.

Indeed, in each case $\tilde{X}$ satisfies the same assumption. Under (b) this is obvious. Under the assumption (a), note that $\tilde{X}$ is a standard process too and that it admits local times with respect to the reference measure $\nu \cdot q$. Since $\nu \cdot q$ is finite, by Proposition 2.10 "$t \to P_x[\zeta_{\tilde{X}} > t]$" is hence continuous in both cases.

The process $\tilde{X}$ satisfies all the assumptions of Proposition 2.3. One hence obtains the existence of a Markov process on $E \cup \{\delta\}$ admitting δ as recurrent point such that for every a element of E:

$$((\tilde{X}_t, t < \zeta_{\tilde{X}} | \tilde{X}_0 = a) \overset{(\text{law})}{=} ((Y_t, t < T_\delta) | Y_0 = a),$$

and hence in particular

$$((L^x_\infty(X), x \in E) | X_0 = a) \overset{(\text{law})}{=} ((L^x_{T_\delta}(Y), x \in E) | Y_0 = a),$$

which establishes Corollary 2.4. □

Remark 2.11. As Taksar explains it in [33], if the life time has atoms then the construction of the semigroup $(\bar{P}_t)_{t\geq 0}$ might require to enlarge E by more than one point.

3 Isomorphism theorem for the extension of transient symmetric Markov processes

It is now time to use the construction of section 2. Consider a transient Borel right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, I\!P_x)$ with an excessive measure ν. Assume that X has finite symmetric 0-potential densities $(u(x,y), (x,y) \in E \times E)$ with respect to ν. Let $(L^x_t, x \in E, t \geq 0)$ be its local time process normalized as in (2.2). Thanks to section 2.3, by a time change and a change of reference measure, one obtains $\tilde{X}$, a transient Borel right process with finite life time and same total accumulated local time process and 0-potential densities as X (but with respect to a new reference measure). We assume moreover either that X is standard or E is discrete to guarantee that the life time of $\tilde{X}$ has no atom (see section 2.3).

Using section 2, we obtain Y, a Markov process that extends $\tilde{X}$, with transition semigroup $(\bar{P}_t)$, with a local time process $(\bar{L}^x_t, x \in E \cup \{\delta\}, t \geq 0)$, which admits δ as recurrent point and satisfies:

$$(L^x_\infty, x \in E) \overset{(\text{law})}{=} (\bar{L}^x_{\tilde{\tau}_0}, x \in E) \tag{3.1}$$

where $\tilde{\tau}_r = \inf\{s \geq 0 : \bar{L}^\delta_s > r\}$, $r \geq 0$.

We shall call Y the extended X.

The process $(Y_t)_{t\geq 0}$ satisfies all the required properties to make use of an isomorphism theorem for non necessarily symmetric Markov processes established in [12] (Corollary 3.5). Indeed the identity presented by Corollary 3.5 in [12] requires the existence of a recurrent point (one chooses the point δ), the strong Markov property at time $\tilde{\tau}_\gamma$, for γ exponential time independent of Y (see section 2.2) and for all x, y in E: $I\!E_x[\bar{L}^y_{\tilde{\tau}_\gamma}] = u(x,y) + I\!E[\gamma]$ (see (2.12)).

Denote by $g_{\tilde{\tau}_0}$ the Green function of Y killed at the first hitting time of δ, and by $g_{\tilde{\tau}_\gamma}$, the Green function of Y killed at $\tilde{\tau}_\gamma$. Note that $(g_{\tilde{\tau}_0}(x,y), (x,y) \in E \times E) = (u(x,y), (x,y) \in E \times E)$ and that $g_{\tilde{\tau}_0}(x,\delta) = g_{\tilde{\tau}_0}(\delta,x) = 0$.

Let $(\phi(x), x \in E \cup \{\delta\})$ and $(\psi(x), x \in E \cup \{\delta\})$ be two permanental process with index 2 and respective kernels $g_{\tilde{\tau}_0}$ and $g_{\tilde{\tau}_\gamma}$. We choose ϕ independent of Y. According to Corollary 3.5 in [12], we have given $(Y_0 = \delta)$:

$$(\frac{1}{2}\psi(x), x \in E \cup \{\delta\} \,|\, \psi(\delta) = 2r) \overset{\text{(law)}}{=} (\frac{1}{2}\phi(x) + \bar{L}^x_{\tilde{\tau}_r}, x \in E \cup \{\delta\}). \tag{3.2}$$

Since $g_{\tilde{\tau}_0}$ is symmetric, $(\phi(x), x \in E) = (\eta_x^2, x \in E)$ where $(\eta_x, x \in E)$ is a centered Gaussian process with covariance u. Setting: $\eta_\delta = 0$, one obtains:

$$(\phi(x), x \in E \cup \{\delta\}) \overset{\text{(law)}}{=} (\eta_x^2, x \in E \cup \{\delta\}). \tag{3.3}$$

Similarly, since: $(g_{\tilde{\tau}_\gamma}(x,y), (x,y) \in (E \cup \{\delta\})^2) = (g_{\tilde{\tau}_0}(x,y) + \mathbb{E}[\gamma], (x,y) \in (E \cup \{\delta\})^2)$ (see (2.12)), one obtains:

$$(\psi(x), x \in E \cup \{\delta\}) \overset{\text{(law)}}{=} ((\eta_x + N)^2, x \in E \cup \{\delta\}), \tag{3.4}$$

where N is a centered Gaussian variable with variance $\mathbb{E}[\gamma]$ independent of $(\eta_x)_{x \in E}$. One then easily shows that:

$$((\eta_x + N)^2, x \in E \cup \{\delta\}) \,|N^2 = 2r) \overset{\text{(law)}}{=} ((\eta_x + \sqrt{2r})^2, x \in E \cup \{\delta\}). \tag{3.5}$$

Together (3.2), (3.3), (3.4) and (3.5), lead to the following result.

Theorem 3.1. *For $(\eta_x, x \in E)$ centered Gaussian process with covariance $(u(x,y), (x,y) \in E \times E)$ the 0-potential densities of X, independent of Y, we have given $(Y_0 = \delta)$:*

$$(\frac{1}{2}\eta_x^2 \ + \ \bar{L}^x_{\tilde{\tau}_r}, x \in E) \overset{\text{(law)}}{=} (\frac{1}{2}(\eta_x + \sqrt{2r})^2, x \in E).$$

The following proposition is a consequence of Theorem 3.1.

Proposition 3.2. *Let $(X_t)_{t \geq 0}$ be a transient symmetric Borel right process with metric state space E. If the local time process of X is continuous then for every $r \geq 0$, the process $(\bar{L}^x_{\tilde{\tau}_r}(Y), x \in E)$ is continuous.*

Proof. If the local time process of X is continuous, then one has immediately the continuity of $(\bar{L}^x_{\tilde{\tau}_0}(Y), x \in E, t \geq 0)$. Besides, thanks to [25], $(\eta_x, x \in E)$, the centered Gaussian process with covariance $(u(x,y))$, the 0-potential density of X, is continuous. Choosing η independent of Y, we have for every $r > 0$, under $\bar{P}_\delta$:

$$\frac{1}{2}\eta^2 + \bar{L}_{\tilde{\tau}_r}(Y) \overset{\text{(law)}}{=} \frac{1}{2}(\eta + \sqrt{2r})^2$$

which gives immediately the continuity of the process $(\bar{L}^x_{\tilde{\tau}_r}(Y), x \in E)$ for every $r > 0$. □

4 A tractable tool for Gaussian free fields on graphs

Let G be an infinite, locally finite graph with vertex set $\mathcal{V}$ and edge set $\mathfrak{E}$. Let $(X_t)_{t \geq 0}$ be a continuous time transient Markov chain on $\mathcal{V}$ with transition probability $(p(x,y), (x,y) \in \mathcal{V}^2)$ such that $p(x,y) > 0$ if $[x,y] \in \mathfrak{E}$ and $x \neq y$. Assume that X is reversible, that is there exists a measure λ with support equal to $\mathcal{V}$, such that $\lambda p = p^t \lambda$. Denote by $(u(x,y), (x,y) \in \mathcal{V}^2)$ the 0-potential density of X with respect to the measure λ. Then $(u(x,y), (x,y) \in \mathcal{V}^2)$ is symmetric positive definite. Note that $u(x,y) = \frac{u_0(x,y)}{\lambda(y)}$, where $(u_0(x,y), (x,y) \in \mathcal{V}^2)$ is the 0-potential densities of X with respect to the counting measure on $\mathcal{V}$, the measure that gives the same weight 1 to every point of $\mathcal{V}$. Denote by

$(\eta_x, x \in \mathcal{V})$ a centered Gaussian process with covariance $(u(x,y),(x,y) \in \mathcal{V}^2)$ independent of X. We say that $(\eta_x, x \in \mathcal{V})$ is a Gaussian free field on G associated with X.
Denote by Y the extension process of X. Applying Theorem 3.1 to X, one obtains given $(Y_0 = \delta)$:

$$(\frac{1}{2}(\eta_x + \sqrt{2r})^2,\ x \in \mathcal{V}) \overset{\text{(law)}}{=} (\frac{1}{2}\eta_x^2\ +\ \bar{L}^x_{\bar{\tau}_r}(Y)\ ,\ x \in \mathcal{V}). \tag{4.1}$$

In [29], under the additional assumption that: $\sum_{y \in \mathfrak{E}} p(x,y) = 1$ ((0.1) in [29]), Sznitman establishes the following identity. For every $r \geq 0$:

$$(\frac{1}{2}(\eta_x + \sqrt{2r})^2,\ x \in \mathcal{V}) \overset{\text{(law)}}{=} (\frac{1}{2}\eta_x^2\ +\ L_{x,r}\ ,\ x \in \mathcal{V}),$$

where $(L_{x,r},\ x \in \mathcal{V})$ is the field of occupation times of random interlacements at level r (see [29] for a precise description of this field).
In view of (4.1) one obtains:

$$(L_{x,r}, x \in \mathcal{V}) \overset{\text{(law)}}{=} ((\bar{L}^x_{\bar{\tau}_r}(Y), x \in \mathcal{V}) | Y_0 = \delta). \tag{4.2}$$

We can extend $(\eta_x, x \in \mathcal{V})$ to the edges of G in order to obtain an extension of (4.1) to $\bigcup_{e \in \mathfrak{E}} e$. We abuse the notation by saying that $x \in G$ for $x \in \bigcup_{e \in \mathfrak{E}} e$. For $e \in \mathfrak{E}$ and $x \in \mathcal{V}$, we write $e \sim x$ to mean that the edge e is adjacent to the vertex x. For x, y in $\mathcal{V}$, we set $C(x,y) = \lambda(x)p(x,y)$, $C(x,y)$ is symmetric in x and y. We keep the assumption $\sum_{y \in \mathcal{V}} p(x,y) = 1$ and shall use a Brownian motion W on a weighted metric graph as defined by Folz [16]. Brownian motions on metric graphs generalize the definition of Walsh Brownian motion. They can be obtained from a real valued Brownian motion $(B_t)_{t \geq 0}$. One needs moreover two positive functions on $\mathfrak{E}$, a length function ℓ and a weight function w. The length function allows to define a metric d on G by identifying an edge e with $[0, \ell(e)]$ and d restricted to e with the euclidian distance on $[0, \ell(e)]$. For x, y in G, $d(x,y)$ is then the length of the shortest path in G joining x to y.
To give a quick description of W, one considers the excursions set of B around 0. One starts B at some vertex a. Each excursion is performed on an edge e adjacent to a with probability $\frac{w(e)}{\sum_{f \sim a} w(f)}$ (W moves along the edge e of length $\ell(e)$ as $|B|$ on $[0, \ell(e)]$). At the first time W hits a new vertex, one repeats the same procedure from the new vertex and independently of the past. Starting from an inside point of an edge e, W moves as $|B|$ on $[0, \ell(e)]$ until one vertex is hit. The obtained process W is continuous and admits a local time process $(L^z_t(W), z \in G, t \geq 0)$ with respect to the measure μ on G such that $\mu_{|_e} = m(e) Leb_{|_{[0,\ell(e)]}}$ (with $m(e) > 0, \forall e \in \mathfrak{E}$). Define $T = \inf\{t \geq 0 : W_t \in \mathcal{V} \setminus \{W_0\}\}$. One chooses the three functions ℓ, w and m in order to have:

(i) $\mathbb{P}_x[W_T = y] = p(x,y)$.
(ii) $\mathbb{E}_x[L^x_T(W)] = \frac{1}{\lambda(x)}$.
(iii) $(L^z_t(W), z \in G, t \geq 0)$ is continuous with respect to $d \times Leb(\mathbb{R}_+)$.

Theorem 4.1 in [16] requires for the realization of (i) that there exists $\alpha > 0$ such that for every $[x,y] \in \mathfrak{E}$: $\alpha C(x,y) = \frac{w([x,y])}{\ell([x,y])}$. Condition (iii) requires that for every vertex a and every adjacent edge e: $\frac{m(e)}{\sum_{f \sim a} m(f)} = \frac{w(e)}{\sum_{f \sim a} w(f)} = \frac{1}{\deg(x)}$ (see e.g. [19] Theorem 2.1 (2.14)). One hence obtains that there exists $\gamma, \beta > 0$: $\forall e \in \mathfrak{E}$, $w(e) = \beta$ and $m(e) = \gamma$.
Finally for (ii), under $\mathbb{P}_x$ for $t \in [0,T]$: $L^x_t(W) = \lim_{\varepsilon \to 0} \frac{1}{\mu(B(0,\varepsilon))} \int_0^t 1_{\{W_s \in B(0,\varepsilon)\}} ds$, where $B(0,\varepsilon) = \{z \in G : d(x,z) \leq \varepsilon\}$. This leads to

$$L^x_t(W) = \frac{1}{\sum_{e \sim x} m(e)} L^0_t(|B|).$$

Since $(|B_t| - L_t^0(B), t \geq 0)$ is a local martingale given $(B_0 = 0)$, then $(|B_t| - L_t^0(B)), 0 \leq t \leq T)$ is a martingale, hence $\mathbb{E}_0[|B_T|] = \mathbb{E}[L_T^0(B)]$, which gives thanks to condition (i):

$$\begin{aligned} \mathbb{E}_0[L_T^0(B)] &= \sum_{y:[x,y]\in\mathfrak{E}} \mathbb{P}_x[W_T = y]\ell([x,y]) = \sum_{y:[x,y]\in\mathfrak{E}} p(x,y)\ell([x,y]) \\ &= \frac{1}{\lambda(x)} \sum_{y:[x,y]\in\mathfrak{E}} C(x,y)\ell([x,y]). \end{aligned}$$

Hence (ii) is equivalent for every vertex x to:

$$\frac{1}{\lambda(x)} = \frac{1}{\sum_{e\sim x} m(e)} \frac{2}{\lambda(x)} \sum_{[x,y]\in\mathfrak{E}} C(x,y)\ell([x,y]),$$

which leads to: $\alpha\gamma = 2\beta$.
Consequently (i)(ii)(iii) is equivalent to the following condition.

$$\exists\, \beta,\gamma > 0 : \; w(e) = \beta, \; m(e) = \gamma, \; \ell(e) = \frac{\gamma}{2\, C(e)}, \; \forall e \in \mathfrak{E}. \tag{4.3}$$

On can take: $m(e) = 1$, $w(e) = 1$ and $\ell(e) = \frac{1}{2C(e)}$, $\forall e \in \mathfrak{E}$. But note that other choices are possible (e.g. $\forall e \in \mathfrak{E}$, $m(e) = 2$, $w(e) = 1$ and $\ell(e) = \frac{1}{C(e)}$).
Whatever the choice of β,γ satisfying (4.3) the obtained process W has a continuous local time process and is such that its restriction to $\mathcal{V}$ has the law of X. Since X is transient, W is transient.
One obtains an extension of $(\eta_x, x \in \mathcal{V})$ to G, by defining $(\eta_x, x \in G)$ as a centered Gaussian process with covariance $(\mathbb{E}_x[L_\infty^y(W)], (x,y) \in G \times G)$.
Denote by Y_G the extended process of W, Theorem 3.1 gives conditionally on $(Y_G(0) = \delta)$:

$$(\frac{1}{2}(\eta_x + \sqrt{2r})^2, \; x \in G) \overset{(\text{law})}{=} (\frac{1}{2}\eta_x^2 \; + \; L^x_{\tilde{\tau}_r}(Y_G), \;\; x \in G). \tag{4.4}$$

When the graph G is $\mathbb{Z}^d$ ($d \geq 3$), to avoid confusion between the graph and its vertex set, we denote it by $\tilde{\mathbb{Z}}^d$.
In the case when G is $\tilde{\mathbb{Z}}^d$ and $(X_t)_{t\geq 0}$ is a simple symmetric random walk on $\mathbb{Z}^d$, Lupu has established the following identity (Proposition 6.3 in [22])

$$(\frac{1}{2}(\eta_x + \sqrt{2r})^2, \; x \in \tilde{\mathbb{Z}}^d) \overset{(\text{law})}{=} (\frac{1}{2}\eta_x^2 \; + \; L^x(\tilde{I}^r), \;\; x \in \tilde{\mathbb{Z}}^d),$$

where $(L^x(\tilde{I}^r), x \in G)$ is the occupation time process of the extension to $\tilde{\mathbb{Z}}^d$ of the random interlacement at level r associated to the continuous time simple symmetric random walk on $\mathbb{Z}^d$.
Together with (4.4), this identity shows that:

$$(L^x(\tilde{I}^r), \; x \in \tilde{\mathbb{Z}}^d) \overset{(\text{law})}{=} ((L^x_{\tilde{\tau}_r}(Y_{\tilde{\mathbb{Z}}^d}), \;\; x \in \tilde{\mathbb{Z}}^d) \mid Y_{\tilde{\mathbb{Z}}^d}(0) = \delta). \tag{4.5}$$

This identification, which implies (4.2) for the simple symmetric random walk on $\mathbb{Z}^d$, can also be directly obtained by using Proposition 5.3 established in section 5.3.

Example 4.1. The case of the graph $\mathbb{Z}^d$ has been intensively studied. We remind that the vertex set is denoted by $\mathbb{Z}^d$ and the whole graph by $\tilde{\mathbb{Z}}^d$. The set $\tilde{\mathbb{Z}}^d$ is obtained by connecting by an edge any couple of points x, y of $\mathbb{Z}^d$ such that the euclidian distance between x and y equals 1.
Assume that $d \geq 3$. Let X be the continuous time simple symmetric random walk on $\mathbb{Z}^d$, X is hence transient. In this particular case: $C(x,y) = p(x,y) = \frac{1}{2d}$. Consider the

weighted metric graph $(\tilde{\mathbb{Z}}^d, \ell, w)$ such that for every edge e of the graph $\tilde{\mathbb{Z}}^d$: $\ell(e) = 1$ and $w(e) = 1$. Let W be a Brownian motion on this weighted metric graph. Denote by $(L^x_t(W), x \in \tilde{\mathbb{Z}}^d, t \geq 0)$ the local time process of W w.r.t. the measure μ such that $m(e) = \frac{1}{d}$ for every edge of $\tilde{\mathbb{Z}}^d$. This way the three conditions (i), (ii), (iii) are satisfied.

Set $g(x,y) = I\!E_x[L^y_\infty(W)]$. There exists a centered Gaussian process $(\eta_x, x \in \tilde{\mathbb{Z}}^d)$ with covariance $(g(x,y), (x,y) \in \tilde{\mathbb{Z}}^d \times \tilde{\mathbb{Z}}^d)$. By restricting to $\mathbb{Z}^d$ this Gaussian process one obtains the so-called Gaussian free field on $\mathbb{Z}^d$, $(\eta_x, x \in \mathbb{Z}^d)$.
There exists a critical parameter $h_*(d)$ in $[-\infty, +\infty]$ such that if $h < h_*(d)$ a.s. the set $\{x \in \mathbb{Z}^d : \eta(x) \geq h\}$ contains an infinite connected component and if $h > h_*(d)$ a.s. it does not. In [3], Bricmont et al show that $h_*(d) \geq 0$. Several authors have given alternative proofs of this result ([27], [22], [13]).
We show now how to recover the result of [3] by using $Y_{\tilde{\mathbb{Z}}^d}$ the extended Markov process of W. Thanks to Theorem 3.1, we have given $(Y_{\tilde{\mathbb{Z}}^d}(0) = \delta)$:

$$(\frac{1}{2}\eta_x^2 + L^x_{\tilde{\tau}_r}(Y_{\tilde{\mathbb{Z}}^d}), x \in \tilde{\mathbb{Z}}^d) = (\frac{1}{2}(\eta_x + \sqrt{2r})^2, x \in \tilde{\mathbb{Z}}^d). \tag{4.6}$$

Since the local time process of W is continuous, making use of [24], we know that $(\eta_x, x \in \tilde{\mathbb{Z}}^d)$ is continuous.
To lighten the notation we shall use the following: we write Y for $Y_{\tilde{\mathbb{Z}}^d}$ and if A is a subset of $\tilde{\mathbb{Z}}^d$, we write $A \infty cc$, for "A contains an infinite connected component". We have for any $t > 0$:

$$\begin{aligned}
I\!P_\delta \quad [\quad & \{x \in \tilde{\mathbb{Z}}^d : L^x_{\tilde{\tau}_r}(Y)) > 0\} \infty cc] = I\!P_\delta[\{Y(s), 0 \leq s \leq \tilde{\tau}_r\} \infty cc] \\
\geq \quad & I\!P_\delta[\tilde{\tau}_r > t;\ \{Y(s), 0 \leq s \leq \tilde{\tau}_r\} \infty cc] \\
\geq \quad & I\!P_\delta[\tilde{\tau}_r > t;\ I\!P_{Y_t}[\{Y(s), 0 \leq s \leq T_\delta\} \infty cc]\ ] \\
= \quad & \int_E I\!P_\delta[\tilde{\tau}_r > t;\ Y_t \in dx]\ \ I\!P_x[\{Y(s), 0 \leq s \leq T_\delta\} \infty cc]\ ] \\
= \quad & I\!P_\delta[\tilde{\tau}_r > t],
\end{aligned}$$

since $I\!P_x[\{Y(s), 0 \leq s \leq T_\delta\} \infty cc] = 1$, for every $x \in E$.
Hence, one obtains: $I\!P_\delta[\{x \in \tilde{\mathbb{Z}}^d : L^x_{\tilde{\tau}_r}(Y)) > 0\} \infty cc] \geq I\!P_\delta[\tilde{\tau}_r > t]$, for every $t > 0$. By letting t tend to 0, this leads to: $I\!P_\delta[\{x \in \tilde{\mathbb{Z}}^d : L^x_{\tilde{\tau}_r}(Y)) > 0\} \infty cc] = 1$.
Consequently, using (4.6), a.s. $\{x \in \tilde{\mathbb{Z}}^d : (\eta_x + \sqrt{2r})^2 > 0\}$ contains an unbounded connected component. Since η is continuous, this implies that for any $h > 0$ a.s. either $\{x \in \tilde{\mathbb{Z}}^d : \eta_x > -h\}$ contains an unbounded connected component or $\{x \in \tilde{\mathbb{Z}}^d : \eta_x < -h\}$ contains an unbounded connected component. By symmetry:

$$I\!P[\{x \in \tilde{\mathbb{Z}}^d : \eta_x < -h\} \infty cc] = I\!P[\{x \in \tilde{\mathbb{Z}}^d : \eta_x > h\} \infty cc]$$

and hence: $I\!P[\{x \in \tilde{\mathbb{Z}}^d : \eta_x < -h\} \infty cc] \leq I\!P[\{x \in \tilde{\mathbb{Z}}^d : \eta_x > -h\} \infty cc]$. One obtains for every $h > 0$: $I\!P[\{x \in \tilde{\mathbb{Z}}^d : \eta_x > -h\} \infty cc] \geq 1/2$. Consequently: $h_* \geq 0$.

On can substitute to the graph $\mathbb{Z}^d$ any transient, locally finite, infinite connected graph G (as introduced at the beginning of this section) and make use similarly of the Brownian motion W on the corresponding weighted metric graph satisfying (4.3) with $w(e) = 1, \forall e \in \mathfrak{E}$. One obtains similarly that

$$\sup\{h \in I\!R : I\!P[\{x \in \mathcal{V} : \eta_x \geq h\} \infty cc] > 0\} \geq 0.$$

This remark has been obtained differently in [27], [13] and in a more general setting in [8].

5 Connection of the extended processes with Kuznetsov processes, quasi-processes and Sznitman's random interlacements

Consider a Borel right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t X_t, \theta_t, P_x; x \in E)$ on a Borel state space $(E, \mathcal{E})$ with semigroup $(P_t)_{t\geq 0}$. Let ν be a measure on $(E, \mathcal{E})$ that is excessive for $(P_t)_{t\geq 0}$. Let $\mathcal{W}$ denote the space of paths ω from $I\!R$ to $E \cup \Delta$ which are right continuous and E valued on some open interval $(b(\omega), d(\omega))$ and $\omega(t) = \Delta$ outside the interval. $b(\omega)$ is called the birth time of the path and $d(\omega)$ its death time. As presented by Fitzsimmons in [14] one can associate with $\{\nu, (P_t)_{t\geq 0}\}$, two Markov processes on $\mathcal{W}$: a Kuznetsov process and a quasi-process. In the sections 5.1 and 5.2 below, we remind their definition and give some of their connections with the extended process of X when it exists. Concerning random interlacements, one has first to define them. Indeed Sznitman has defined random interlacements for Brownian motion in $I\!R^d$ [30] and for continuous (and discrete) time reversible nearest neighbors random walks on locally finite, connected graphs [29], but not for a general X. In section 5.3, we extend his definition to a continuous Borel process X under the assumption that X admits a weak dual with respect to ν. We then establish connections of the random interlacements of X with its quasi-process and with its extended Markov process when it exists.
We remind that X is in weak duality with respect to ν, with $\hat{X}$ another Borel right process with state space E and semigoup $(\hat{P}_t)_{t\geq 0}$, if for every $t \geq 0$, every $\mathcal{E}$ measurable nonnegative functions f and g:

$$\int_E P_t f(x) g(x) \nu(dx) = \int_E f(x) \hat{P}_t g(x) \nu(dx). \tag{5.1}$$

By choosing $f = 1_E$ or $g = 1_E$, note that a σ-finite measure ν which satisfies (5.1) is necessarily excessive for $(P_t)_{t\geq 0}$ and $(\hat{P}_t)_{t\geq 0}$.
Notation with $\hat{}$ will refer to $\hat{X}$ (e.g. $\hat{\zeta}$ denotes the life time of $\hat{X}$).
We denote by $(Z_t)_{t\in I\!R}$ the coordinate process on $\mathcal{W}$: $Z_t(\omega) = \omega(t)$. We define the σ-fields $\mathcal{G} = \sigma\{Z_t : t \in I\!R\}$, $\mathcal{G}_t = \sigma\{Z_s : s \leq t\}$, and the shift operators σ_t on $\mathcal{W}$: $\sigma_t \omega(s) = \omega(t+s),\ s, t \in I\!R$. The constant trajectory Δ is denoted by Δ.

5.1 Kuznetsov process

The Kuznetsov measure $\mathbf{Q}_\nu$ on $\mathcal{W} \setminus \{\Delta\}$ is defined by:

$$\begin{aligned} \mathbf{Q}_\nu(Z_{t_1} \in A_1, Z_{t_2} \in A_2, \cdots Z_{t_n} \in A_n) &= \\ \int_{A_1} \nu(dx_1) \int_{A_2} P_{t_2 - t_1}(x_1, dx_2) \quad .. & \quad \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \end{aligned} \tag{5.2}$$

for $-\infty < t_1 < t_2 < .. < t_n < +\infty$ and $A_1, .., A_n \in \mathcal{E}$.
Under $\mathbf{Q}_\nu$ the coordinate process $(Z_t)_{t\in I\!R}$ is hence a stationary Markov process with one dimensional distribution at time t equal to ν and transition semigroup $(P_t)_{t\geq 0}$. Since ν is excessive, the measure $\mathbf{Q}_\nu$ is unique (see Kuznetsov [21]).
When ν is purely excessive (i.e. $\int_E \nu(dx) P_t 1(x) \to_{t\to\infty} 0$), there exists an entrance law $(m_t)_{t>0}$ such that

$$\nu(f) = \int_0^\infty m_t(f) dt.$$

Define P_* by

$$P_*(Z_{t_1} \in A_1, \cdots Z_{t_n} \in A_n) = \int_{A_1} m_{t_1}(dx_1) \cdots \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \tag{5.3}$$

for $0 < t_1 \leq .. \leq t_n$ and extend its definition to negative t_i's by setting $m_s = 0$ for $s < 0$. One checks using (5.2),(5.3) that for such ν, $\mathbf{Q}_\nu$ is given by

$$\mathbf{Q}_\nu(Z_{t_1} \in A_1, Z_{t_2} \in A_2, \cdots Z_{t_n} \in A_n) = \int_{-\infty}^{t_1} P_*(Z_{t_1-t} \in A_1, Z_{t_2-t} \in A_2, \quad .., \quad Z_{t_n-t} \in A_n)dt. \tag{5.4}$$

That is, the birth time is chosen according to the Lebesgue measure on $I\!R$, and the law of Z after the birth is given by P_*. As described in Remark 2.9, Taksar [32, 33] has constructed the covering process so that $(Z_t, \bar{P})_{t\in I\!R}$ "covers" the process $(Z_t, \mathbf{Q}_\nu)_{\mathbf{t}\in\mathbf{R}}$.

When the process X is in weak duality with a Borel right process $\hat{X}$ with respect to ν, the measure $\mathbf{Q}_\nu$ satisfies the following property thanks to a construction of Mitro [26]. Denote by $(\hat{P}_t)_{t\geq 0}$ the semigroup of $\hat{X}$. For every $u \in I\!R$, $0 < t_1 < .. < t_n$ and $0 < s_1 < .. < s_k$, the finite-dimensional laws of $\mathbf{Q}_\nu$ satisfy

$$\begin{aligned}\mathbf{Q}_\nu \quad & [Z_u \in A_0, Z_{t_1+u} \in A_1, .., Z_{t_n+u} \in A_n, \, Z_{(u-s_1)_-} \in B_1, ..., Z_{(u-s_k)_-} \in B_k] \\ = \quad & \int_{A_0} \nu(dx) \int_{A_1} P_{t_1}(x, dx_1) .. \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \\ & \int_{B_1} \hat{P}_{s_1}(x, dy_1) .. \int_{B_k} \hat{P}_{s_k - s_{k-1}}(y_{k-1}, dy_k)\end{aligned} \tag{5.5}$$

for every $A_0, A_1, .., A_n, B_1, .., B_k$ in $\mathcal{E}$.
Define $\hat{Z}$ by $\hat{Z}_t = Z_{(-t)-}, \; t \in I\!R$. Then using (5.5) one obtains:

$$\mathbf{Q}_\nu[F(\hat{Z}_t, t \in I\!R)] = \hat{\mathbf{Q}}_\nu[F(Z_t, t \in I\!R)] \tag{5.6}$$

where $\hat{\mathbf{Q}}_\nu$ is the Kuznetsov measure associated to $\{\nu, (\hat{P}_t)_{t\geq 0}\}$.

5.2 Quasi-process

The quasi-processes were introduced by Hunt [18] in a discrete setting and then set and named by Weil [34] in continuous time. Let $\mathcal{A}$ denote the σ-algebra of (σ_t) shift invariant events in $\mathcal{G}$. The quasi-process associated with $\{\nu, (P_t)_{t\geq 0}\}$, is the measure $\mathbf{P}_\nu$ on $(\mathcal{W}, \mathcal{A})$ that is determined by the conditions:

$$\mathbf{P}_\nu \left(\int_{I\!R} f(Z_t) dt \right) = \nu(f), \tag{5.7}$$

for any nonnegative measurable function f on E, and

$$\begin{aligned}&\text{for any intrinsic stopping time S, } \{Z_{S+t}, \; t > 0\} \text{ under} \\ &\mathbf{P}_\nu(\, \cdot \, ; \; S \in I\!R) \; \text{ is Markovian with semigroup } (P_t)_{t\geq 0},\end{aligned} \tag{5.8}$$

where by intrinsic stopping time, one means a $(\mathcal{G}_t)$-stopping time that satisfies

$$b \leq S < d \text{ on } \{S < +\infty\}, \text{ and } S = t + S \circ \sigma_t \text{ for all } \; t \in I\!R.$$

A $\mathcal{G}$-measurable random time $S : \mathcal{W} \to [-\infty, \infty]$ is called a stationary time if it satisfies $S = t + S \circ \sigma_t$ for all $t \in I\!R$.
A measure m on $\mathcal{E}$ is dissipative for $(P_t)_{t\geq 0}$ if for any nonnegative function f such that $\int_E f(x)m(dx) < \infty$, one has: $\int_0^\infty P_t f dt < \infty \;\; m \; a.e.$
It is shown in [14] that when the excessive measure ν is dissipative, $\mathbf{Q}_\nu$ almost everywhere, there exists a stationary time S^* such that $\mathbf{Q}_\nu(S^* \notin I\!R) = 0$. Whatever the choice of such a stationary time S^*, it is shown in [14] that the quasi-process $\mathbf{P}_\nu$ on $(\mathcal{W}, \mathcal{A})$ satisfies:

$$\mathbf{P}_\nu(A) = \mathbf{Q}_\nu(A; 0 < S^* < 1) \qquad A \in \mathcal{A} \tag{5.9}$$

where S^* is a stationary time such that $\mathbf{Q}_\nu(S^* \notin I\!R) = 0$.
One can invert (5.9) (see (2.12) in [14]) and obtain for any $\mathcal{G}$ measurable H

$$\mathbf{Q}_\nu(H) = \mathbf{P}_\nu(\int_{I\!R} dt\ H \circ \sigma_t). \tag{5.10}$$

Since ν is σ-finite, $\mathbf{Q}_\nu$ is σ-finite. Using Theorem 1 in Dynkin [10], one hence knows that (5.10) determines $\mathbf{P}_\nu$ on $\mathcal{A}$.
Suppose now that ν is purely excessive. Then ν has to be dissipative (see [15]) and moreover we know that P_* exists. Besides (5.4) can be rewritten as

$$\begin{aligned}\mathbf{Q}_\nu(Z_{t_1} \in A_1, Z_{t_2} \in A_2, \cdots Z_{t_n} \in A_n) &= \\ \int_{-\infty}^{+\infty} P_*(Z_{t_1+t} \in A_1, Z_{t_2+t} \in A_2, \quad .., \quad Z_{t_n+t} \in A_n)dt,\end{aligned}$$

(using that $m_s = 0$, for $s < 0$) which leads to

$$\mathbf{Q}_\nu(H) = P_*(\int_{I\!R} dt\ H \circ \sigma_t).$$

Consequently, thanks to Theorem 1 in [10] one obtains

$$P_*(A) = \mathbf{P}_\nu(A),\ \forall A \in \mathcal{A}. \tag{5.11}$$

Now note that for any $t_1 < t_2 < .. < t_n$ and any $t \in I\!R$, one has:
$(F(Z_{t_1+b}, .., Z_{t_n+b});\ t_n < d - b)(\sigma_t) = (F(Z_{t_1+b}, .., Z_{t_n+b});\ t_n < d - b)$.
Hence (5.11) leads to:

$$\mathbf{P}_\nu(F(Z_{t_1+b}, .., Z_{t_n+b}); t_n < d - b) = P_*(F(Z_{t_1}, .., Z_{t_n}); t_n < \zeta) \tag{5.12}$$

since the birth time under P_* is 0.

5.3 Random interlacements

In [28], Rosen sets a definition of the random interlacements at level u of a transient Borel right process X with potential densities with respect to an excessive measure ν such that ν is dissipative. He defines it as the PPP with intensity $u\mathbf{P}_\nu$. In section 1 of [28] it is implicit that in the case when X is a Brownian motion in $I\!R^d$ ($d \geq 3$ and ν is the Lebesgue measure), this definition should coincide with Sznitman's one [30]. A proof of this equality is given by Dereich and Döring in [5].
Here we start directly from Sznitman's definition of random interlacements for Brownian motion and set it for non necessarily symmetric, continuous transient Borel processes.
The framework is the triplet $\{\nu, ((P_t)_{t\geq 0}, (\hat{P}_t)_{t\geq 0})\}$ where $(P_t)_{t\geq 0}$ and $(\hat{P}_{t\geq 0})$ are the respective semigroups of X and $\hat{X}$ two continuous transient Borel processes in weak duality with respect to the σ-measure ν. The measure ν is hence excessive for $(P_t)_{t\geq 0}$ and $(\hat{P}_t)_{t\geq 0})$ (see the remark following (5.1)).
We make use of the notion of capacitary measure associated to the transient continuous Markov process $\hat{X}$, defined as follows. For B compact subset of E, let $\hat{L}_B$ be the last time $\hat{X}$ hits B. The capacitary measure $\hat{e}_B$ of B associated to $\hat{X}$ (also called the equilibrium measure of B) is the Revuz measure of the homegenous random measure $1_{\{0<\hat{L}_B<\zeta\}}\delta_{\hat{L}_B}(dt)$ with respect to ν (see [17]). This means that $\hat{e}_B$ satisfies for every nonnegative f with compact support:

$$\hat{e}_B(f) = \sup_{t>0} \frac{1}{t} \int_E \nu(dx) \hat{I\!E}_x[f(\hat{X}_{\hat{L}_B}) 1_{\hat{L}_B \leq t},\ 0 < \hat{L}_B < \zeta].$$

Definition 5.1. *For $u > 0$ the random interlacements at level u associated to $\{\nu, ((P_t)_{t\geq 0}, (\hat{P}_t)_{t\geq 0})\}$ is a PPP with intensity measure $u\mu_\nu$ where μ_ν is the measure on $(\mathcal{W}, \mathcal{A})$ such that $\mu_\nu(\omega \equiv \Delta) = 0$, characterized by the following properties:*

- *for any compact subset B of E, define $H_B = \inf\{t \in (b(\omega), d(\omega)) : \omega(t) \in B\}$ with $\inf \emptyset = +\infty$, then*
$$\mu_\nu[\omega_{H_B} \in dx, H_B < \infty] = \hat{e}_B(dx) \tag{5.13}$$
where $\hat{e}_B$ is the capacitary measure of B associated to $\hat{X}$;
- *for every couple of $\mathcal{A}$ measurable functionals (F_1, F_2)*

$$\begin{aligned}\mu_\nu \quad & [F_1(\omega(H_B + t), t \geq 0);\; F_2(\omega((H_B - t),)t \geq 0);\; H_B < \infty] \\ = \quad & \int_E \hat{e}_B(dx) I\!P_x[F_1(X_t, t \geq 0)] \hat{I\!P}_x[F_2(\hat{X}_t, t \geq 0) | \hat{X}(0, \infty) \cap B = \emptyset].\end{aligned} \tag{5.14}$$

For the above definition to make sense, one has to check existence and unicity of the measure μ_ν. One easily checks the unicity of such measure μ_ν. Existence is given by the following theorem which gives also the connections of μ_ν with the quasi-process and the extended process associated to $\{\nu, (P_t)_{t\geq 0}\}$.

Theorem 5.1. *Let X and $\hat{X}$ be continuous Markov processes in weak duality with respect to a σ-finite measure ν. Assume that $\hat{X}$ is transient, then we have*

$$\mathbf{P}_\nu = \mu_\nu \;. \tag{5.15}$$

If moreover ν is purely excessive then we have

$$\mathbf{P}_\nu = \mu_\nu = P_{*|_{\mathcal{A}}} \;. \tag{5.16}$$

From (5.13) and (5.14), one does not see how to obtain ν from μ_ν. In view of Theorem 5.1 it becomes obvious since for every nonnegative measurable function f on E

$$\nu(f) = \mu_\nu(\int_{I\!R} f(\omega(t))dt).$$

Proof. It is sufficient to show that $\mathbf{P}_\nu$ satisfies (5.13) and (5.14). First note that for any compact subset B of E, H_B is an intrinsic stopping time. Indeed it is stationary and on $\{H_B < +\infty\}$: $b \leq H_B < d$. Hence under $\mathbf{P}_\nu(.; H_B < \infty)$ $(Z_{H_B+t}, t > 0)$ is Markovian with semigroup $(P_t)_{t\geq 0}$, which implies

$$\mathbf{P}_\nu[F_1(Z_{H_B+t}, t > 0).; H_B < \infty] = \mu_\nu[F_1(Z_{H_B+t}, t > 0).; H_B < \infty].$$

Since $\hat{X}$ is transient, for every compact subset B one has according to (13.12) in [17]

$$\mathbf{Q}_\nu[F(H_B, Z_{H_B}); H_B < d] = \int_B \int_{-\infty}^{\infty} F(t, x)\; dt\; \hat{e}_B(dx)$$

or equivalently ((13.13) in [17])

$$\mathbf{Q}_\nu[H_B \in dt, Z_{H_B} \in dx] = dt\; \hat{e}_B(dx) \tag{5.17}$$

where $\hat{e}_B$ is the capacitary measure of B with respect to $\hat{X}$.
In particular, one has

$$\mathbf{Q}_\nu[0 < H_B \leq 1; f(Z_{H_B})] = \int_B f(x)\hat{e}_B(dx). \tag{5.18}$$

Note that when X or $\hat{X}$ is assumed to be transient the excessive measure ν is dissipative (see Remark 4.9 c) in [15]). Since ν is dissipative, one can make use of (5.9). Hence there exists a stationary time S^* such that $\mathbf{Q}_\nu(S^* \notin I\!R) = 0$ and in particular:

$$\begin{aligned} \mathbf{P}_\nu[H_B < \infty; f(Z_{H_B})] &= \mathbf{Q}_\nu[H_B < \infty; f(Z_{H_B}); 0 < S^* \leq 1] \\ &= \mathbf{Q}_\nu[0 < H_B \leq 1; f(Z_{H_B})] \end{aligned}$$

thanks to the switching property of Proposition 2.4 in [14], which together with (5.18), establishes (5.13) for $\mathbf{P}_\nu$.
Note that the capacity of B relative to $\hat{X}$ satisfies: $\hat{C}(B) = \mathbf{P}_\nu[H_B < \infty]$.
We check now (5.14).

$$\begin{aligned} \mathbf{P}_\nu \quad & [F_1(Z(H_B + t), t \geq 0);\ F_2(Z(H_B - t), t > 0);\ H_B < \infty] \\ = \quad & \mathbf{Q}_\nu[F_1(Z(H_B + t), t \geq 0);\ F_2(Z(H_B - t), t > 0);\ H_B < \infty;\ 0 < S^* \leq 1] \\ = \quad & \mathbf{Q}_\nu[F_1(Z(H_B + t), t \geq 0);\ F_2(Z(H_B - t), t > 0);\ 0 < H_B \leq 1]. \end{aligned}$$

The random time H_B is a stopping time for $\mathbf{Q}_\nu$ (i.e. for every $t \in I\!R$, $\{H_B \leq t\} \in \mathcal{G}_t$). From [26] (clearly stated by (10.12) in [17]), under $\mathbf{Q}_\nu$, one has Markov property at time H_B, under the following form:

$$\begin{aligned} \mathbf{Q}_\nu \quad & [F_1(Z(H_B + t), t \geq 0);\ F_2(Z(H_B - t), t \geq 0);\ 0 < H_B \leq 1] \\ = \quad & \mathbf{Q}_\nu[0 < H_B \leq 1, P_{Z(H_B)}[F_1(X_s, s \geq 0)] F_2(Z((H_B - t)), t \geq 0)]. \end{aligned}$$

Using the joint law given by (5.17) one obtains:

$$\begin{aligned} \mathbf{Q}_\nu \quad & [F_1(Z(H_B + t), t \geq 0);\ F_2(Z(H_B - t), t \geq 0);\ 0 < H_B \leq 1] \\ = \quad & \int_0^1 dt \int_B \hat{e}_B(dx)\ P_x[F_1(X_s, s \geq 0)]\ \mathbf{Q}_\nu[F_2(Z(t - s), s \geq 0) | Z_t = x, H_B = t]. \end{aligned}$$

Set: $\lambda_B = \sup\{s \in (b, d) : Z_s \in B\}$. Using (5.6) and (5.5) one obtains for every x in B

$$\begin{aligned} \mathbf{Q}_\nu[F_2(Z(t - s), s \geq 0) \quad | \quad & Z_t = x, H_B = t] = \hat{\mathbf{Q}}_\nu[F_2(Z(s + t), s \geq 0) | Z_t = x,\ \lambda_B = t] \\ = \quad & \hat{\mathbf{Q}}_\nu[F_2(Z(s + t), s \geq 0) | Z_t = x,\ Z(t, \infty) \cap B = \emptyset] \\ = \quad & \hat{P}_x[F_2(\hat{X}(s), s > 0) |\ \hat{X}(0, \infty) \cap B = \emptyset], \end{aligned}$$

which is independent of t. Consequently, we have:

$$\begin{aligned} \mathbf{Q}_\nu \quad & [F_1(Z(H_B + t), t \geq 0);\ F_2(Z((H_B - t)), t \geq 0);\ 0 < H_B \leq 1] \\ = \quad & \int_B \hat{e}_B(dx) P_x[F_1(X_s, s \geq 0)] \hat{P}_x[F_2(\hat{X}(s), s > 0) |\ \hat{X}(0, \infty) \cap B = \emptyset] \end{aligned}$$

which gives (5.14) for $\mathbf{P}_\nu$.
If moreover ν is purely excessive then P_* exists and (5.16) is a consequence of (5.11). □

Remark 5.2. **(i)** In case ν is finite, X has a finite life time and $t \to P_x(\zeta_X > t)$ is continuous for all x in E, the extended Markov process $(Y, \bar{P})$ associated to $\{\nu, (P_t)_{t \geq 0}\}$ is well defined (thanks to Remark 2.2 and Proposition 2.3). The state space of Y is $E \cup \{\delta\}$ and the excursion process from δ (or from M) of Y is a PPP with intensity $dt \times P_*$. According to (5.16) the random interlacements at level u associated to $\{\nu, ((P_t)_{t \geq 0}, (\hat{P}_t))_{t \geq 0}\}$ is a PPP with intensity $uP_{*|_{\mathcal{A}}}$ and as such it can be interpreted as the excursion process of Y from δ modulo time-shift. The parameter u turns here to be a time parameter t, the index of $(\tilde{\tau}_t)_{t \geq 0}$, the inverse of the local time at δ. In particular, if X admits local times with respect to ν, then the field of occupation times of the random interlacements at level u associated to $\{\nu, ((P_t)_{t\geq 0}, (\hat{P}_t)_{t\geq 0})\}$ is equal in law to the local time process of Y at the first time the local time at δ exceeds u.

(ii) In case ν is infinite or X has an infinite life time with positive probability, we assume moreover that X has finite 0-potential densities with respect to ν. Keeping the notation of section 2.3, $\tilde{X}$ is a transient continuous Markov process with finite life time and same total accumulated local time process and 0-potential densities as X (but with respect to a new reference measure $\nu \cdot q$) and Y is a Markov process that extends $\tilde{X}$ (the extended X). Denote by $(\tilde{P}_t)_{t\geq 0}$ the semigroup of $\tilde{X}$. Using Theorem 3.1 in [20], we know that $\tilde{X}$ admits a weak dual with respect to $\nu \cdot q$ which is also continuous. Denote its semigroup by $(\hat{\tilde{P}}_t)_{t\geq 0}$. Then in view of (i), the local time process of Y at the first time its local time at δ exceeds u is equal in law to the field of occupation times of the random interlacements at level u associated to $\{\nu \cdot q, ((\tilde{P}_t)_{t\geq 0}, (\hat{\tilde{P}}_t)_{t\geq 0})\}$.
But note that the random interlacements at level u associated with $\{\nu, ((P_t)_{t\geq 0}, (\hat{P}_t)_{t\geq 0})\}$ are also well defined. These two random interlacements have the same total occupation times. Indeed denote by $\mathbf{P}_\nu$ (resp. $\tilde{\mathbf{P}}_{\nu\cdot q}$) the quasi-process associated to $\{\nu, (P_t)_{t\geq 0}\}$ (resp. $\{\nu \cdot q, (\tilde{P}_t)_{t\geq 0}\}$). One obtains using the argument of the proof of Lemma 2.1 in [28]

$$\mathbf{P}_\nu[\prod_{i=1}^{n}(L_d^{y_i} - L_b^{y_i})] = \sum_{\sigma\in\mathcal{S}_n} u(y_{\sigma(1)}, y_{\sigma(2)})...u(y_{\sigma(n-1)}, y_{\sigma(n)}),$$

where $\mathcal{S}_n$ is the set of all the permutations of $\{1, ..., n\}$.
In the same manner, one has also

$$\tilde{\mathbf{P}}_{\nu\cdot q}[\prod_{i=1}^{n}(L_d^{y_i} - L_b^{y_i})] = \sum_{\sigma\in\mathcal{S}_n} u(y_{\sigma(1)}, y_{\sigma(2)})...u(y_{\sigma(n-1)}, y_{\sigma(n)}).$$

Consequently one obtains:

$$\mathbf{P}_\nu[\prod_{i=1}^{n}(L_d^{y_i} - L_b^{y_i})] = \tilde{\mathbf{P}}_{\nu\cdot q}[\prod_{i=1}^{n}(L_d^{y_i} - L_b^{y_i})]. \tag{5.19}$$

Then remark that the field of the occupation times of the random interlacements at level u associated to $\{\nu, ((P_t)_{t\geq 0}, (\hat{P}_t)_{t\geq 0})\}$ is a PPP with intensity the law of the total local time process under $u\mathbf{P}_\nu$. This last remark together with (5.19) implies that the local time process of Y at the first time its local time at δ exceeds u is equal in law to the field of occupation times of the random interlacements at level u associated to $\{\nu, ((P_t)_{t\geq 0}, (\hat{P}_t)_{t\geq 0})\}$.

From Remark 5.2 merges an identity connecting the extended Markov process to the random interlacements whatever the total weight of ν and the finiteness of the life time of X.

Proposition 5.3. *Let X be a transient continuous Markov process with semigroup $(P_t)_{t\geq 0}$ in weak duality with respect to a σ-finite measure ν with a Markov process with semigroup $(\hat{P}_t)_{t\geq 0}$. Assume that X has finite 0-potential densities with respect to ν. Then the field of the occupation time of the random interlacement at level u associated to $\{\nu, ((P_t), (\hat{P}_t))\}$ equals in law the local time process of the extended X at the first time its local time at δ exceeds u.*

Note that Brownian motions on metric graphs (see section 4) satisfy the assumption of Proposition 5.3. In the case when E is the graph $\mathbb{Z}^d$, one hence directly obtains (4.5) by using Proposition 5.3.

Remark 5.4. To set a proper definition of random interlacements for a general transient Borel right process X in weak duality with $\hat{X}$ with respect to a σ-finite measure ν, one would have to define μ_ν by the two following properties:

$$\mu_\nu[\omega_{H_B} \in dx, \omega_{H_B-} \in dy, H_B < \infty] = \epsilon_B(dxdy) \tag{5.20}$$

where ϵ_B is a measure on $E \times E$ with first marginal $\hat{e}_B(dx)$ where $\hat{e}_B$ is the capacitary measure of B associated to $\hat{X}$

and for every couple of $\mathcal{A}$ measurable functionals (F_1, F_2)

$$\begin{aligned} \mu_\nu \quad & [F_1(\omega(H_B+t), t \geq 0);\ F_2(\omega((H_B-t)_-,)t \geq 0);\ H_B < \infty] \\ = \quad & \int_{E\times E} \epsilon_B(dxdy) I\!P_x[F_1(X_t, t \geq 0)] \hat{I\!P}_y[F_2(\hat{X}_t, t \geq 0) | \hat{X}(0,\infty) \cap B = \emptyset]. \end{aligned} \tag{5.21}$$

In order to extend Theorem 5.1 to this framework, one must have:

$$\epsilon_B(dxdy) = \mathbf{P}_\nu[Z_{H_B} \in dx, Z_{H_B-} \in dy, H_B < \infty].$$

Using the same argument as in the proof of Theorem 5.1, one then would obtain

$$\epsilon_B(dxdy) = \mathbf{Q}_\nu[Z_{H_B} \in dx, Z_{H_B-} \in dy, 0 < H_B < 1].$$

This last fact would be sufficient to show that $\mathbf{P}_\nu$ satisfies (5.21) similarly as in the proof of Theorem 5.1.
The missing point in order to set a satisfying definition for μ_ν with (5.20) and (5.21), is an expression of ϵ_B in terms of X or $\hat{X}$ instead of $\mathbf{Q}_\nu$. We are currently working on this definition together with its implications.

6 Preserving symmetry

In section 2, we consider a transient Markov process admitting finite symmetric 0-potential densities with respect to an excessive measure ν and show the existence of an extended Markov process Y satisfying (1.3) and Theorem 3.1. Note that we did not require symmetry from Y to establish Theorem 3.1. Nevertheless in this section we would like to show that the extended semigroup remains symmetric.
We first assume that X satisfies the assumptions of Proposition 2.3.
Let $U^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt$ be the α-potential of X, and denote by $V^\alpha f(x) = \int_0^\infty e^{-\alpha t} \bar{P}_t f(x) dt$, the α-potential of Y.

Lemma 6.1. *Let X be a transient Borel right process with semigroup $(P_t)_{t\geq 0}$ such that for all $x \in E$, $t \to P_t 1(x)$ is continuous in t on $I\!R_+$ and there exists a finite purely excessive reference measure ν with respect to $(P_t)_{t\geq 0}$. For every x in E*

$$V^\alpha f(x) = U^\alpha f(x) + E_x(e^{-\alpha\zeta}) \frac{m_\alpha(f)}{\Psi(\alpha)}$$

and

$$V^\alpha f(\delta) = \frac{m_\alpha(f)}{\Psi(\alpha)}$$

where $m_\alpha(f) = \int_0^\infty e^{-\alpha t} m_t(f) dt$, with (m_t) the entrance law associated to ν and $\Psi(\alpha)$ the Levy exponent of the subordinator (τ_s) which equals $\alpha m_\alpha(1)$.

Proof of Lemma 6.1. To compute the α-potentials of $(P_t)_{t\geq 0}$ and $(\bar{P}_t)$ we take Laplace transforms with respect to α from both sides of (2.10). Let $f \geq 0$ be an $\mathcal{E}$ measurable function. For any x in E, one has:

$$\begin{aligned}
V^\alpha f(x) &= \int_0^\infty e^{-\alpha t} P_t f(x)dt + \int_0^\infty e^{-\alpha t}\left(\int_{y=0}^t P_x(\zeta\in dy) Q_y \int_0^t P_*(f(Z_{t-s}))d\ell_s\right)dt \\
&= \int_0^\infty e^{-\alpha t} P_t f(x)dt + \int_0^\infty e^{-\alpha t}\left(\int_{y=0}^t P_x(\zeta\in dy) Q_y \int_0^t m_{t-s}(F)d\ell_s\right)dt \\
&= \int_0^\infty e^{-\alpha t} P_t f(x)dt + \int_0^\infty e^{-\alpha t}\left(\int_{y=0}^t P_x(\zeta\in dy) Q \int_y^t m_{t-s}(F)d\ell_{s-y}\right)dt \\
&= \int_0^\infty e^{-\alpha t} P_t f(x)dt + \int_0^\infty e^{-\alpha t}\left(\int_{y=0}^t P_x(\zeta\in dy) Q \int_0^{t-y} m_{t-u-y}(F)d\ell_u\right)dt \\
&= \int_0^\infty e^{-\alpha t} P_t f(x)dt \\
&\quad + \int_0^\infty e^{-\alpha y} P_x(\zeta\in dy)\left(\int_0^\infty dv\, e^{-\alpha v} Q\int_0^v m_{v-u}(F)d\ell_u\right) \\
&= U^\alpha f(x) + E_x(e^{-\alpha\zeta}) m_\alpha(f) Q(\int_0^\infty e^{-\alpha u} d\ell_u) \\
&= U^\alpha f(x) + E_x(e^{-\alpha\zeta}) m_\alpha(f) Q(\int_0^\infty e^{-\alpha\tau_s} ds) = U^\alpha f(x) + E_x(e^{-\alpha\zeta})\frac{m_\alpha(f)}{\Psi(\alpha)}.
\end{aligned}$$

One shows similarly using (2.11) that $V^\alpha f(\delta) = \frac{m_\alpha(f)}{\Psi(\alpha)}$. To show that $\Psi(\alpha) = \alpha m_\alpha(1)$, just note that: $P_*(1-e^{-\alpha\zeta}) = \alpha\int_0^\infty e^{-\alpha t} P_*(\zeta > t)dt$. □

We are going to show that if U^α has symmetric densities with respect to ν so does V^α. Note that whatever the measure λ on $E\cup\{\delta\}$, if V^α has densities $(v^\alpha(x,y),(x,y)\in(E\cup\{\delta\})^2)$ with respect to λ, the value of $v^\alpha(x,\delta)$ can be freely chosen since: $V^\alpha f(x) = \int_{E\cup\{\delta\}} v^\alpha(x,y)f(y)\lambda(dy) = \int_E v^\alpha(x,y)f(y)\lambda(dy)$.

Proposition 6.2. *Let X be a transient Borel right process satisfying the assumptions of* Proposition 2.3. *Assume moreover that the densities $(u^\alpha(x,y),(x,y)\in E\times E)$ of U^α with respect to ν are symmetric. Then V^α has symmetric densities with respect to ν given for x,y in E by:*

$$v^\alpha(x,y) = u^\alpha(x,y) + \frac{1}{\Psi(\alpha)} E_x(e^{-\alpha\zeta})E_y(e^{-\alpha\zeta})$$

and

$$v^\alpha(\delta,y) = \frac{1}{\Psi(\alpha)} E_y(e^{-\alpha\zeta}).$$

Proof. We first show that

$$m_\alpha(f) = \int_E E_y(e^{-\alpha\zeta})f(y)\nu(dy). \tag{6.1}$$

The fact that the Radon Nikodym derivative of m_α with respect to ν at the point x is equal to $E_x(e^{-\zeta})$ follows from the theory of Kuznetsov processes and quasi-processes. The definition of the Kuznetsov measure $\mathbf{Q}_\nu$ and the quasi-process $\mathbf{P}_\nu$ have been reminded in section 5. Using successively (5.12) and (5.9) one has:

$$\begin{aligned}
m_\alpha(f) = \int_0^\infty e^{-\alpha t} m_t(f)dt = P_*\int_0^\infty e^{-\alpha s} f(Z_s)ds = \mathbf{P}_\nu(\int_b^d e^{-\alpha(s-b)} f(Z_s)ds) \\
= \mathbf{Q}_\nu(0\le b\le 1; \int_b^d e^{-\alpha(s-b)} f(Z_s)ds). \qquad (6.2)
\end{aligned}$$

By time reversal and symmetry, one has:

$$\mathbf{Q}_\nu(0 \le b \le 1; \int_b^d e^{-\alpha(s-b)} f(Z_s)ds) = \mathbf{Q}_\nu(-1 \le d \le 0;\ \int_b^d e^{-\alpha d(\sigma_s)} f(Z_s)ds).$$

We use (2.1) and (2.3) in [14] to note that for every $a, b \in I\!R$ such that $b - a = 1$:

$$\mathbf{Q}_\nu(-1 \le d \le 0;\ \int_b^d e^{-\alpha d(\sigma_s)} f(Z_s)ds) = \mathbf{Q}_\nu(a \le d \le b;\ \int_b^d e^{-\alpha d(\sigma_s)} f(Z_s)ds),$$

and choose $a = 0$ and $b = 1$, to obtain:

$$\mathbf{Q}_\nu(0 \le b \le 1; \int_b^d e^{-\alpha(s-b)} f(Z_s)ds) = \mathbf{Q}_\nu(0 \le d \le 1;\ \int_b^d e^{-\alpha d(\sigma_s)} f(Z_s)ds),$$

and then thanks to (5.9) one can substitute $(0 \le b \le 1)$ to $(0 \le d \le 1)$:

$$\mathbf{Q}_\nu(0 \le b \le 1; \int_b^d e^{-\alpha(s-b)} f(Z_s)ds) = \mathbf{Q}_\nu(0 \le b \le 1;\ \int_b^d e^{-\alpha d(\sigma_s)} f(Z_s)ds).$$

By the Markov property of $(Z_s)_{s\in I\!R}$ under $\mathbf{Q}_\nu$ one finally obtains:

$$\begin{aligned}
\mathbf{Q}_\nu(0 \le b \le 1; \int_b^d e^{-\alpha(s-b)} f(Z_s)ds) &= \mathbf{Q}_\nu(0 \le b \le 1;\ \int_b^d E_{Z_s}(e^{-\alpha\zeta}) f(Z_s)ds) \\
&= \mathbf{P}_\nu \int_b^d E_{Z_s}(e^{-\alpha\zeta}) f(Z_s)ds \\
\text{(thanks to (5.11))} \quad &= P_* \int_0^\infty E_{Z_s}(e^{-\alpha\zeta}) f(Z_s)ds \\
&= \int_0^\infty \int_E E_x(e^{-\alpha\zeta}) f(x) m_s(dx)ds \\
&= \int_E E_x(e^{-\alpha\zeta}) f(x)\nu(dx),
\end{aligned}$$

which together with (6.2) gives (6.1). Proposition 6.2 is an immediat consequence of Lemma 6.1 and (6.1). □

Proof of Corollary 2.5. Assume that X satisfies the assumptions of Corollary 2.5. Making use of section 2.3, we know that $\tilde{X}$ has the same 0-potential densities as X but with respect to $\nu{\cdot}q$. Consequently $\tilde{X}$ satisfies the assumptions of Proposition 6.2. The extended process Y of X which is actually the one associated to $\tilde{X}$ is hence also symmetric with respect to $\nu \cdot q$. Besides Y admits δ as recurrent point. With the notation of the proof of Corollary 2.4, we know that for all x in E: $P_x(\zeta_{\tilde{X}} < \infty) = 1$. Consequently for Y: $\forall x \in E$, $P_x(T_\delta < \infty) = 1$. Making use e.g. of the arguments of Lemma 3.6.13 in [25], this is sufficient to claim that Y is recurrent. □

References

[1] Abächerli, A. and Sznitman, A.-S.: Level-set percolation for the Gaussian free field on a transient tree. *Ann. Inst. Henri Poincaré Probab. Stat. 54, no. 1, 173–201* (2018). MR3765885

[2] Blumenthal R.M. and Getoor R.K.: Markov processes and potential theory. *Pure and Applied Mathematics, vol 29, Academic Press, New York - London* (1968). MR0264757

[3] Bricmont J., Lebowitz J.L. and Maes C.: Percolation in strongly correlated systems: the massless Gaussian field. *J. Statist. Phys. 48, (5-6), 1249–1268* (1987). MR0914444

[4] Dellacherie C.: Capacités et processus stochastiques. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 67. Springer-Verlag, Berlin-New York* (1972). MR0448504

[5] Dereich S. and Döring L.: Random interlacements via Kuznetsov measures. arXiv:1501.00649 (2015).

[6] Drewitz A., Prévost A. and Rodriguez P.-F.: The sign clusters of the massless Gaussian free field percolates on $Z^d, d \geq 3$ (and more) *Comm. Math. Phys. 362, no. 2, 513–546* (2018). MR3843421

[7] Drewitz A., Prévost A. and Rodriguez P.-F.: Critical exponents for a percolation model on transient graphs. arXiv:2101.05801 (2021).

[8] Drewitz A., Prévost A. and Rodriguez P.-F.: Cluster capacity fucntionals and isomorphism theorems for Gaussian free fields. arXiv:2101.05800. *To appear in PTRF*. MR4421175

[9] Dynkin E.B.: Minimal excessive measures and functions.*Trans. Amer. Math. Soc. 258, 217–244* (1980). MR0554330

[10] Dynkin E.B.: An application of flows to time shift and time reversal in stochastic processes. *Trans. Amer. Math. Soc. 287, 613–619* (1985). MR0768728

[11] Eisenbaum N., Kaspi H., Marcus M., Rosen J. and Shi Z.: A Ray-Knight Theorem for symmetric Markov processes. *Annals of Probability, 28, 1781–1796* (2000). MR1813843

[12] Eisenbaum N. and Kaspi H.: On permanental processes. *Stoch. Proc. and Appl. 119, 5,1401–1415* (2009). MR2513113

[13] Eisenbaum N.: Some percolations involving Gaussian free fields. *Electron. Commun. Probab. 26, 11* (2021). MR4236681

[14] Fitzsimmons P.J.: On a connection between Kuznetsov processes and quasi-processes. *Seminar on Stochastic Processes 1987, 123–133. Birkhauser Boston, Basel* (1988). MR1046413

[15] Fitzsimmons P.J. and Maisonneuve B.: Excessive measures and Markov processes with random birth and death. *Probab. Th. Rel. Fields 72, 319–336* (1986). MR0843498

[16] Folz M.: Volume growth and stochastic completeness of graphs. *Trans. Amer. Math. Soc., 366 (4) 2089–2119* (2014). MR3152724

[17] Getoor R.K. and Sharpe M. J.: Naturality, standardness and weak duality for Markov processes. *Z. Wahrsch. Verw. Gebiete 67, 1–62* (1984). MR0756804

[18] Hunt G.A.: Markov chain and Martin boundaries. *Ill. J. Math. 4, 313–340* (1960). MR0123364

[19] Ichiba T., Karatzas I., Prokaj, V. and Yan, M.: Stochastic integral equations for Walsh semimartingales. *Ann. Inst. Henri Poincaré Probab. Stat. 54 no. 2, 726–756* (2018). MR3795064

[20] Kaspi H. and Mitro J.B.: Homogeneous random measures for Markov processes in weak duality: a study via entrance boundary. *Stochastic processes and their Applications 29, 2, 291–308* (1988). MR0958506

[21] Kuznetsov S.E.: Construction of Markov processes with random birth and death.*Th. Prob. Appl. 571–574* (1974). MR0343376

[22] Lupu T.: From loop clusters and random interlacements to the free field. *Annals of Probab. 44, 3, 2117–2146* (2016). MR3502602

[23] Maisonneuve B.: Subordinators regenerated. *Seminar on Stochastic processes 1986 - Birkhauser, Boston* (1987). MR0902432

[24] Marcus M.J and Rosen J: Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Probab. 20, no. 4, 1603–1684* (1992). MR1188037

[25] Marcus M.J. and Rosen J.: Markov processes, Gaussian processes, and local times. *Cambridge University Press* (2006). MR2250510

[26] Mitro J.B.: Dual Markov processes: construction of a useful auxiliary process. *Z. Wahrsch. Verw. Gebiete 47, no. 2, 139–156* (1979). MR0523166

[27] Rodriguez P.-F. and Sznitman A.-S.: Phase transition and level-set percolation for the Gaussian free field. *Comm. Math. Phys. 320 (2) 571–601* (2013). MR3053773

[28] Rosen J.: Intersection local times for interlacements. *Stochastic Processes and their Applications, 124, 1849–1880* (2014). MR3170227

[29] Sznitman A.-S.: An isomorphism theorem for random interlacements. *Electron. Commun. Probab. 17, 9* (2012). MR2892408

[30] Sznitman A.-S.: On scaling limits and Brownian interlacements. *Bull. Braz. Math. Soc. (N.S.) 44 (2013), no. 4, 555–592* (2013). MR3167123

[31] Sznitman, A.-S.: Coupling and an application to level-set percolation of the Gaussian free field. *Electron. J. Probab. 21 (2016), Paper No. 35, 26 pp* (2016). MR3492939

[32] Taksar M.I.: Subprocesses of stationary Markov processes. *Zeit. fur Wahr. verw. Geb. 55, 275–296* (1981). MR0608022

[33] Taksar M.I.: Enhancing of semigroups. *Zeit. fur Wahr. verw. Geb. 63, 445–462* (1983). MR0705616

[34] Weil M.: Quasi-processes. *Séminaire de Probabilités IV, 216–239, LNM 124, ed Springer-Verlag* (1970). MR0266305