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Fluctuations of two-dimensional stochastic heat equation and KPZ equation in subcritical regime for general initial conditions*

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Abstract

The Kardar-Parisi-Zhang equation (KPZ equation) is solved via Cole-Hopf transformation $h = \log u$, where u is the solution of the multiplicative stochastic heat equation(SHE). In [CD20, CSZ20, G20], they consider the solution of two dimensional KPZ equation via the solution u_ε of SHE with the flat initial condition and with noise which is mollified in space on scale in ε and its strength is weakened as $\beta_\varepsilon = \hat{\beta}\sqrt{\frac{2\pi}{-\log \varepsilon}}$, and they prove that when $\hat{\beta} \in (0,1)$, $\frac{1}{\beta_\varepsilon}(\log u_\varepsilon - \mathbb{E}[\log u_\varepsilon])$ converges in distribution as a random field to a solution of Edwards-Wilkinson equation.

In this paper, we consider a stochastic heat equation u_ε with a general initial condition u_0 and its transformation $F(u_\varepsilon)$ for F in a class of functions $\mathfrak{F}$, which contains $F(x) = x^p$ $(0 < p \leq 1)$ and $F(x) = \log x$. Then, we prove that $\frac{1}{\beta_\varepsilon}(F(u_\varepsilon(t,\cdot)) - \mathbb{E}[F(u_\varepsilon(t,\cdot))])$ converges in distribution as a random field to a centered Gaussian field jointly in finitely many $F \in \mathfrak{F}$, t, and u_0. In particular, we show the fluctuations of solutions of stochastic heat equations and KPZ equations jointly converge to solutions of SPDEs which depend on u_0.

Our main tools are Itô's formula, the martingale central limit theorem, and the homogenization argument as in [CNN22]. To this end, we also prove a local limit theorem for the partition function of intermediate disorder $2d$ directed polymers.

Keywords: KPZ equation, stochastic heat equation; Edwards-Wilkinson equation; local limit theorem for polymers; stochastic calculus.
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1 Introduction and main result

The KPZ equation is an SPDE formally given by

$$\frac{\partial}{\partial t}h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \beta\xi(t,x), \tag{1.1}$$

where ξ is space-time white noise on $[0,\infty)\times\mathbb{R}^d$. This SPDE is ill-posed since ∇h is no longer a function and the non-linear term $|\nabla h|^2$ cannot make sense.

For $d=1$, Bertini and Giacomin formulated the solution of (1.1) via Cole-Hopf transformation $h=\log u$ [BG97], where u is the solution of stochastic heat equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta u(t,x)\xi(t,x). \tag{1.2}$$

When we consider a space-regularized multiplicative stochastic heat equation for $d=2$, we will scale the disorder strength as

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{1}{2}\Delta u_\varepsilon + \beta_\varepsilon u_\varepsilon \xi_\varepsilon, \quad u_\varepsilon(0,x) = u_0(x), \tag{1.3}$$

where $\beta_\varepsilon = \hat\beta\sqrt{\frac{2\pi}{-\log\varepsilon}}$ with $\hat\beta \geq 0$, and ξ_ε is a mollification in space of ξ such that $\xi_\varepsilon \Rightarrow \xi$ as $\varepsilon\to 0$, i.e.,

$$\xi_\varepsilon(t,x) = (\xi(t,\cdot)\star\phi_\varepsilon)(x) = \int \phi_\varepsilon(x-y)\xi(t,y)\mathrm{d}y,$$

with $\phi_\varepsilon(x) = \varepsilon^{-2}\phi(\varepsilon^{-1}x)$ and ϕ being a smooth, non-negative, compactly supported function on $\mathbb{R}^2$, such that $\int\phi(x)\mathrm{d}x = 1$ and $\phi(-x)=\phi(x)$. Let $h_\varepsilon = \log u_\varepsilon$. Then, we find by Itô's formula that h_ε satisfies the SPDE

$$\frac{\partial h_\varepsilon}{\partial t} = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta_\varepsilon\xi_\varepsilon\ , \quad h_\varepsilon(0,x) = h_0(x), \tag{1.4}$$

where C_ε is the constant depending on ε given by:

$$C_\varepsilon = \frac{\beta_\varepsilon^2 V(0)}{2\varepsilon^2}, \tag{1.5}$$

where $V(x) = \int_{\mathbb{R}^2}\phi(x-y)\phi(y)\mathrm{d}y$. Caravenna, Sun, and Zygouras proved that if the initial condition is flat, i.e., $h_\varepsilon(0,x) = h_0(x)\equiv 0$, and $\hat\beta\in(0,1)$, then $\beta_\varepsilon^{-1}\left(h_\varepsilon - \mathbb{E}[h_\varepsilon]\right)$ converges in distribution as a random field to the solution of Edwards-Wilkinson equation [CSZ20]. We remark that Chatterjee and Dunlap addressed the tightness of $\beta_\varepsilon^{-1}\left(h_\varepsilon - \mathbb{E}[h_\varepsilon]\right)$ [CD20] and Gu obtained Edwards-Wilkinson limit in $\hat\beta\in(0,\beta_0)$ for some $\beta_0\leq 1$ [G20].

In this paper, we will look at the fluctuations of u_ε for general initial conditions and its composition with functions in a certain class. Let $\mathfrak{C}$ be a set of continuous functions satisfying

$$0 < \inf_{x\in\mathbb{R}^2} u_0(x) \leq \sup_{x\in\mathbb{R}^2} u_0(x) < \infty, \text{ or equivalently } \|\log u_0\|_\infty < \infty. \tag{1.6}$$

Let $\mathfrak{F}$ be a set of functions $F\in C^3((0,\infty))$ such that there exists a constant $C=C_F>0$ such that for any $x\in(0,\infty)$,

$$|F'(x)| \leq C(x^{-1}+1), \quad |F''(x)| \leq C(x^{-2}+1), \quad |F'''(x)| \leq C(x^{-3}+1). \tag{1.7}$$

We note that $\mathfrak{F}$ contains x^p $(0<p\leq 1)$, $\log x$, $\sin x$, $\cos x$, e^{-x}. In this paper, we focus on the fluctuation of $u_\varepsilon^{(F)} := F(u_\varepsilon)$.

We remark that u_ε is a process indexed by u_0 and $\hat\beta$, so we should write $u_\varepsilon = u_\varepsilon^{(\hat\beta,u_0)}$ and $u_\varepsilon^{(F)} = u_\varepsilon^{(F,\hat\beta,u_0)}$. However, we omit $\hat\beta$ and u_0 for simplicity of notation when it is clear from the context.

We denote by $\mathcal{C}_c^\infty$ the set of infinitely differentiable, compactly supported functions on $\mathbb{R}^2$.

Moreover, we introduce a family of centered Gaussian fields

$$\left\{\mathscr{U}_t(f,F,\hat\beta,u_0) : t\geq 0, f\in C_c^\infty, F\in\mathfrak{F}, \beta\in(0,1), u_0\in\mathfrak{C}\right\} \tag{1.8}$$

with covariance

$$\begin{aligned}&\mathbb{E}\left[\mathscr{U}_t(f,F,\hat\beta,u_0)\mathscr{U}_{t'}(f',F',\hat\beta',u_0')\right]\\ &=\frac{1}{1-\hat\beta\hat\beta'}\int_0^{t\wedge t'}\mathrm{d}\sigma\int \mathrm{d}x\mathrm{d}y f(x)f'(y)I(x)I'(y)\int \mathrm{d}z\rho_\sigma(x,z)\rho_\sigma(y,z)\bar u(t-\sigma,z)\bar u(t'-\sigma,z),\end{aligned} \tag{1.9}$$

where $\rho_t(x) = (2\pi t)^{-1}e^{-\frac{|x|^2}{2t}}$ is the heat kernel, $\rho_t(x,y)=\rho_t(x-y)$, and $\bar u(t,x) = \int \rho_t(x,y)u_0(y)\mathrm{d}y$. Also, $I(x) = I^{(t,F,\hat\beta,u_0)}(x)$ is given in (1.11) below and$I'(x) = I^{(t',F',\hat\beta',u_0')}(x)$.

Theorem 1.1. *Suppose $u_0^{(1)},\cdots,u_0^{(n)}\in\mathfrak{C}$, $\hat\beta^{(1)},\cdots,\hat\beta^{(n)}\in(0,1)$ and $F_1,\cdots,F_n\in\mathfrak{F}$. For $t_1,\cdots,t_n\geq 0$ and $f_1,\cdots,f_n\in\mathcal{C}_c^\infty$, the following convergence holds jointly as $\varepsilon\to 0$*

$$\begin{aligned}&\frac{1}{\beta_\varepsilon^{(i)}}\int_{\mathbb{R}^2} f_i(x)\left(u_\varepsilon^{(F_i,u_0^{(i)},\hat\beta^{(i)})}(t_i,x) - \mathbb{E}\left[u_\varepsilon^{(F_i,u_0^{(i)},\hat\beta^{(i)})}(t_i,x)\right]\right)\mathrm{d}x\\ &\qquad\xrightarrow{(d)}\mathscr{U}_{t_i}(f_i,F_i,\hat\beta^{(i)},u_0^{(i)})\ ,\end{aligned}$$

where $\{\mathscr{U}_{t_i}(f_i,F_i,\hat\beta^{(i)},u_0^{(i)}) : i=1,\dots,n\}$ are centered Gaussian random fields defined in (1.8)*.*

We have discussed so far the fluctuation of the random fields. On the other hand, the one-point distribution $h_\varepsilon(t,x)$ is also studied well. The following convergence is proved in [CSZ17b, Theorem 2.15]: for any $t>0$ and $x\in\mathbb{R}^2$,

$$h_\varepsilon(t,x)\Rightarrow\begin{cases}X_{\hat\beta} := \sigma(\hat\beta)Z - \frac{1}{2}\sigma^2(\hat\beta) & \text{if } 0\leq\hat\beta<1,\\ 0 & \text{if } \hat\beta\geq 1,\end{cases} \tag{1.10}$$

where Z is a standard Gaussian random variable and $\sigma(\hat\beta) = \sqrt{\log\frac{1}{1-\hat\beta^2}}$.

Then, the function $I(x)$ appears in Theorem 1.1 is given by

$$I(x) = I^{(t,F,\hat\beta,u_0)}(x) = \mathbb{E}\left[F'\left(e^{X_{\hat\beta}}\bar u(t,x)\right)e^{X_{\hat\beta}}\right] = \mathbb{E}\left[F'\left(e^{X_{\hat\beta}+\sigma^2(\hat\beta)}\bar u(t,x)\right)\right], \tag{1.11}$$

where $X_{\hat\beta}$ is as in (1.10).

Remark 1.2. The centered Gaussian fields defined in (1.8) can be constructed explicitly as follows. Let $\left\{\left(\mathscr{V}^{(\hat\beta,u_0)}(t,x)\right)_{(t,x)\in[0,\infty)\times\mathbb{R}^2} : \hat\beta\in(0,1),\, u_0\in\mathfrak{C}\right\}$ be solutions of the following Edwards-Wilkinson type equations: $\mathscr{V}^{(\hat\beta,u_0)}(0,x)\equiv 0$ and

$$\partial_t\mathscr{V}^{(\hat\beta,u_0)}(t,x) = \frac{1}{2}\Delta\mathscr{V}^{(\hat\beta,u_0)}(t,x) + \bar u(t,x)\xi_{\hat\beta}(t,x), \tag{1.12}$$

where $\xi_{\hat\beta}(t,x)=\sum_{n=0}^\infty(\hat\beta)^n\,\xi^{(n)}(t,x)$ with an independent sequence of space-time white noises $\{\xi^{(n)}(t,x)\}_{n\geq 0}$ for $\hat\beta\in(0,1)$. We remark that $\xi_{\hat\beta}$ is space-time white noise with $\mathbb{E}\left[\xi_{\hat\beta}(t,x)\xi_{\hat\gamma}(t',x')\right]=\frac{1}{1-\hat\beta\hat\gamma}\delta_{t,t'}\delta_{x,x'}$. Then, by Duhamel's principle, $\mathscr{V}^{(\hat\beta_i,u_0^{(i)})}$ is given by

$$\mathscr{V}^{(\hat\beta^{(i)},u_0^{(i)})}(t,x)=\int_0^t\int_{\mathbb{R}^2}\rho_{t-s}(x,y)\bar u^{(i)}(s,y)\xi_{\hat\beta_i}(\mathrm{d}s,\mathrm{d}y),\tag{1.13}$$

and the centered Gaussian field given by

$$\mathscr{V}_{t_i}(f_i,F_i,\hat\beta_i,u_0^{(i)})=\int_{\mathbb{R}^2}\mathrm{d}xI^{(i)}(x)f_i(x)\mathscr{V}^{(\hat\beta_i,u_0^{(i)})}(t,x)\tag{1.14}$$

has the covariance structure (1.9).

Example 1.3. For some typical choice of $F\in\mathfrak{F}$, $u_\varepsilon^{(F)}=F(u_\varepsilon)$ and $\mathcal{U}^{(F)}$ are the solutions of SPDEs:

- If $F(x)=x^p$ $(0<p\leq 1)$, then $u_\varepsilon^{(F)}(0,x)=u_0(x)^p$ and $u_\varepsilon^{(F)}$ satisfies

$$\partial_t u_\varepsilon^{(F)}=\frac{1}{2}\Delta u_\varepsilon^{(F)}-\frac{(p-1)}{2p}\frac{|\nabla u_\varepsilon^{(F)}|^2}{u_\varepsilon^{(F)}}+\frac{\beta_\varepsilon^2V(0)p(p-1)u_\varepsilon^{(F)}}{2\varepsilon^2}+\beta_\varepsilon u_\varepsilon^{(F)}\xi_\varepsilon.$$

  Also, we consider the solution $\mathscr{U}^{(F,u_0,\hat\beta)}(t,x)$ of $\mathscr{U}(0,x)\equiv 0$ and

$$\partial_t\mathscr{U}=\frac{1}{2}\Delta\mathscr{U}+\frac{p(p-1)}{2}|\nabla\log\bar u|^2\mathscr{U}+(1-p)\nabla\log\bar u\cdot\nabla\mathscr{U}+\frac{p\bar u(t,x)}{(1-\hat\beta^2)^{\frac{p^2-p+1}{2}}}\xi(t,x).$$

- If $F(x)=\log x$ (the KPZ equation), then $u_\varepsilon^{(F)}(0,x)=\log u_0(x)$ and

$$\partial_t u_\varepsilon^{(F)}=\frac{1}{2}\Delta u_\varepsilon^{(F)}+\frac{1}{2}|\nabla u_\varepsilon^{(F)}|^2-\frac{\beta_\varepsilon^2V(0)}{2\varepsilon^2}+\beta_\varepsilon\xi_\varepsilon.$$

  Also, we consider the solution $\mathscr{U}^{(F,u_0,\hat\beta)}(t,x)$ of $\mathscr{U}(0,x)\equiv 0$ and

$$\partial_t\mathscr{U}=\frac{1}{2}\Delta\mathscr{U}+\nabla\log\bar u\cdot\nabla\mathscr{U}+\frac{\bar u(t,x)}{(1-\hat\beta^2)^{\frac{1}{2}}}\xi(t,x).$$

Then, it is easy to see from (1.12) and (1.14) that $\mathscr{U}_t(f,F,u_0,\hat\beta)\overset{(d)}{=}\int f(x)\mathscr{U}^{(F,u_0,\hat\beta)}(t,x)\mathrm{d}x$, where $\mathscr{U}_t$ is a Gaussian process introduced in (1.8).

From the above remark, we have the following.

Corollary 1.4. *Suppose $\hat\beta<1$. As $p\to 0$, $p^{-1}\mathscr{U}_t(f,x^p,\hat\beta,u_0)$ converges to $\mathscr{U}_t(f,\log x,\hat\beta,u_0)$ in distribution.*

Remark 1.5. In [DGRZ20], they study the fluctuations of the transformation $F(u_\varepsilon)$ for higher dimensional case $d\geq 3$ with F, its derivative and second derivative growing at most $x^{-p}+x^p$. They proved that there exists a constant β_p such that the Gaussian fluctuation holds for $\hat\beta\in(0,\beta_p)$. Our assumption on F is slightly different from theirs.

To analyze u_ε, we use the Feynman-Kac representation given in [BC95, Section 2] where they considered the case $d=1$ but it is easily modified for $d\geq 2$:

$$u_\varepsilon(t,x)=\mathrm{E}_x\left[\exp\left(\beta_\varepsilon\int_0^t\int_{\mathbb{R}^2}\phi_\varepsilon(B_s-y)\xi(t-s,y)\mathrm{d}s\mathrm{d}y-\frac{\beta_\varepsilon^2tV(0)}{2\varepsilon^2}\right)u_0\left(B_t\right)\right],$$

where we denote by P_x and E_x the law and the expectation with respect to two dimensional Brownian motion $B = \{B_t\}_{t\geq 0}$ starting from x.

Due to the time-reversal invariance and the scaling invariance of space-time white noise and the scale invariance of Brownian motion, $\varepsilon B_{\varepsilon^{-2}s} \stackrel{d}{=} B_s$, $\{u_\varepsilon(t,x) : x \in \mathbb{R}^2\}$ has the same distribution as

$$\begin{aligned}
&\mathrm{E}_x\left[\exp\left(\beta_\varepsilon \int_0^t \int_{\mathbb{R}^2} \phi_\varepsilon(B_s - y)\xi(s,y)\mathrm{d}s\mathrm{d}y - \frac{\beta_\varepsilon^2 t V(0)}{2\varepsilon^2}\right) u_0\left(B_t\right)\right] \\
&= \mathrm{E}_{\frac{x}{\varepsilon}}\left[\exp\left(\beta_\varepsilon \int_0^{\frac{t}{\varepsilon^2}} \int_{\mathbb{R}^2} \phi\left(B_s - \frac{y}{\varepsilon}\right)\xi(s,y)\mathrm{d}s\mathrm{d}y - \frac{\beta_\varepsilon^2 t V(0)}{2\varepsilon^2}\right) u_0\left(\varepsilon\, B_{\frac{t}{\varepsilon^2}}\right)\right]. \qquad (1.15)
\end{aligned}$$

In particular, for the flat initial condition, u_ε has the same distribution as *the partition function* $\mathcal{Z}_{\frac{t}{\varepsilon^2}}\left(\frac{x}{\varepsilon}\right)$ of continuum directed polymers, where $\mathcal{Z}_t(x)$ is given by

$$\mathcal{Z}_t^{\beta_\varepsilon}(x) = \mathrm{E}_x\left[\Phi_t^{\beta_\varepsilon}\right],$$

and for $t \geq 0$, $\beta \in (0,\infty)$

$$\Phi_t^\beta = \Phi_t^\beta(B,\xi) := \exp\left(\beta \int_0^t \int_{\mathbb{R}^2} \phi(y - B_s)\,\xi(\mathrm{d}s,\mathrm{d}y) - \frac{\beta^2 t V(0)}{2}\right).$$

Thus, we can reduce the problem on the law of u_ε to the partition function of continuum directed polymers. Such connections between SHE (and KPZ equation) and directed polymers have been already pointed out in [KPZ86] and used in a number of studies on SHE and KPZ equation [BC95, BG97, MSZ16, GRZ18, MU17, DGRZ20, CSZ17a, CSZ17b, CSZ19a, CSZ19b, CSZ20, CSZ21, CCM20, CCM19, CNN22, CN21, LZ20, CC22].

Remark 1.6. Edwards-Wilkinson type fluctuations for the KPZ equation for $d = 2$ have been obtained in [CSZ20] and [G20] with the flat initial condition. In [CSZ20], the problem was reduced to the SHE via approximating $\log u_\varepsilon$ by "$u_\varepsilon - 1$" and the authors obtained the Gaussian fluctuation by combining the Wiener chaos expansion and the fourth moment theorem. In [G20], Gaussian fluctuation was obtained by Malliavin calculus and the second order Poincaré inequality.

In Theorem 1.1, we obtain the Gaussian fluctuations for the general initial conditions and multi-dimensional parameters. Our proof uses a martingale CLT (see Theorem 3.3) via Itô's lemma and homogenization.

Remark 1.7. The Gaussian fluctuations for partition functions of discrete directed polymers [LZ20] and solutions of the SHE [MSZ16, GRZ18, DGRZ18b, CNN22] and the KPZ equation [MU17, DGRZ20, CNN22] in $d \geq 3$ have been proved, where the disorder strength is given by $\hat{\beta}\varepsilon^{\frac{d-2}{2}}$ for $d \geq 3$. The fluctuations of and around the stationary solutions $\mathfrak{u}(\varepsilon)$ of KPZ equation in $d \geq 3$ was proved in [CCM19]. The space-time stationary solution was constructed in [DS80]. In [CCM20], they also proved the point-wise and space-time fluctuation results for $F(\frac{u_\varepsilon}{\mathfrak{u}(\varepsilon)} - 1)$ with general F.

The Gaussian fluctuations for a nonlinear stochastic heat equation with Gaussian multiplicative noise that is white in time and smooth in space [GL20] and the counterpart for $d = 2$ is stated in [DG20] and [T22].

Remark 1.8. For the critical case ($\hat{\beta} = 1$), it is proved that one-point distribution $u_\varepsilon(t,x)$ converges to 0 in [CSZ17b] but is tight and has non-trivial subsequential limit in the sense of random field in [GQT19, CSZ19b]. Recently, it was proved that u_ε converges to the unique random field in [CSZ21].

Note: Throughout the paper and if clear from the context, the constant C that appears in successive upper-bounds may take different values.

Organization of the article The main idea of Gaussian fluctuation is the same as in [CNN22]. Section 2 is devoted to proving key properties of partition functions of directed polymers, L^2-boundedness, boundedness of negative moments, and local limit theorem. Section 3 is dedicated to the proof of Theorem 1.1. In subsection 3.1, we give a rough proof strategy and explain a heuristic idea of Gaussian fluctuation. The rigorous proof starts from subsection 3.2.

2 Some estimates for partition functions

In this section, we discuss some properties of partition functions of continuum directed polymers in random environment.

Hereafter, we set

$$\begin{aligned} &T = T_\varepsilon := \varepsilon^{-2}, \\ &\beta = \beta_\varepsilon := \hat{\beta}\sqrt{\frac{2\pi}{\log \varepsilon^{-1}}} = \hat{\beta}\sqrt{\frac{4\pi}{\log T}}, \text{ and} \\ &\gamma = \gamma_\varepsilon = \hat{\gamma}\sqrt{\frac{2\pi}{\log \varepsilon^{-1}}} = \hat{\gamma}\sqrt{\frac{4\pi}{\log T}}. \end{aligned} \tag{2.1}$$

Throughout the paper, we write the subscript ε in T_ε and β_ε in each statement to emphasize its dependence but we often omit the subscript ε in the proofs for simplicity.

2.1 L^p-bound of partition functions

First, we remark that for $x, y \in \mathbb{R}^2$,

$$\begin{aligned} &\mathbb{E}\left[\mathrm{E}_x[\Phi^\beta_{tT}]\mathrm{E}_y[\Phi^\gamma_{tT}]\right] \\ &= 1 + \sum_{n=1}^\infty \beta^n \gamma^n \int_{0<s_1<\cdots<s_n<tT} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^n \left(V(\sqrt{2}x_i)\rho_{s_i - s_{i-1}}(x_{i-1}, x_i)\right) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x}, \end{aligned} \tag{2.2}$$

where $\mathbb{E}$ and $\mathbb{P}$ denote the expectation and the probability with respect to the white noise ξ and we set $x_0 = \frac{x-y}{\sqrt{2}}$, $s_0 = 0$ and $\mathrm{d}\mathbf{s} = \mathrm{d}s_1 \cdots \mathrm{d}s_n$, $\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$. This representation is obtained from the general property of the white noise:

$$\mathbb{E}\left[\exp\left(\int_0^t \int_{\mathbb{R}^2} f(t,x)\xi(\mathrm{d}s, \mathrm{d}x)\right)\right] = \exp\left(\frac{1}{2}\int_0^t \int_{\mathbb{R}^2} f(t,x)^2 \mathrm{d}s\mathrm{d}x\right). \tag{2.3}$$

Indeed, we have

$$\begin{aligned} &\mathbb{E}\left[\mathrm{E}_x[\Phi^\beta_{tT}]\mathrm{E}_y[\Phi^\gamma_{tT}]\right] \\ &= \mathrm{E}_x \otimes \mathrm{E}_y \left[\exp\left(\int_0^t (\beta\phi(B_s - y) + \gamma\phi(\tilde{B}_s - y))^2 \mathrm{d}s\mathrm{d}y - \frac{(\beta^2+\gamma^2)V(0)t}{2}\right)\right] \\ &= \mathrm{E}_x \otimes \mathrm{E}_y \left[\exp\left(\beta\gamma \int_0^t V(B_s - \tilde{B}_s)\mathrm{d}s\right)\right] \\ &= \mathrm{E}_{\frac{x-y}{\sqrt{2}}}\left[\exp\left(\beta\gamma\int_0^t V(\sqrt{2}B_s)\mathrm{d}s\right)\right], \end{aligned} \tag{2.4}$$

and the Taylor expansion $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ gives (2.2)

For simplicity, we write $\mathrm{E} = \mathrm{E}_0$ for the expectation of a Brownian motion starting at 0.

Lemma 2.1. *Suppose $\hat\beta,\hat\gamma \in (0,1)$ and fix $t>0$. Then,*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\mathrm{E}[\Phi^{\beta_\varepsilon}_{tT_\varepsilon}]\mathrm{E}[\Phi^{\gamma_\varepsilon}_{tT_\varepsilon}]\right] = \frac{1}{1-\hat\beta\hat\gamma}, \tag{2.5}$$

$$\sup_{\varepsilon\le 1}\sup_{s\le tT_\varepsilon}\mathbb{E}\left[\mathrm{E}^{s,0}_{0,0}\left[\Phi^{\beta_\varepsilon}_s\right]^2\right]<\infty, \tag{2.6}$$

where $\mathrm{P}^{t,y}_{0,x}$ and $\mathrm{E}^{t,y}_{0,x}$, denote the probability measure and the expectation of the Brownian bridge from $(0,x)$ to (t,y) in $\mathbb{R}^2$.

Remark 2.2. [CSZ17b, Theorem 2.15] proved (2.5) when $\hat\beta=\hat\gamma$, by reducing the problem to directed polymers in random environments, which can be easily modified for $\hat\beta\neq\hat\gamma$, but we give a direct proof in this paper.

Notation 2.3. *We call*

$$\mathcal{Z}^{t,y}_{0,x}=\mathcal{Z}^{(\beta)}_{0,x;t,y}:=\mathrm{E}^{t,y}_{0,x}\left[\Phi^{\beta}_t\right]$$

the point-to-point partition function of continuum directed polymers.

Proof of (2.5). We have from (2.2)

$$\begin{aligned}&\mathbb{E}\left[\mathrm{E}\left[\Phi^{\beta}_{tT}\right]\mathrm{E}\left[\Phi^{\gamma}_{tT}\right]\right]\\ &=1+\sum_{n=1}^{\infty}\beta^n\gamma^n\int_{0<s_1<\cdots<s_n<tT}\int_{(\mathbb{R}^2)^n}\prod_{i=1}^{n}\left(V(\sqrt{2}x_i)\rho_{s_i-s_{i-1}}(x_{i-1},x_i)\right)\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x},\end{aligned}\tag{2.7}$$

with $x_0=0$. We first consider the upper bound:

$$\varlimsup_{\varepsilon\to 0}\mathbb{E}\left[\mathrm{E}[\Phi^{\beta}_{tT}]\mathrm{E}[\Phi^{\gamma}_{tT}]\right]\le\frac{1}{1-\hat\beta\hat\gamma}.$$

Let us consider the function

$$r_s=\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}V(\sqrt{2}y)\rho_s(x,y)\mathrm{d}y\ge 0. \tag{2.8}$$

Since $\int_{\mathbb{R}}\rho_s(x,y)\mathrm{d}y=1$, $\sup_{s>0}|r_s|\le\|V\|_\infty$. Moreover, using $\int V(x)\mathrm{d}x=1$, we obtain

$$r_s=\frac{1}{4\pi s}\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}V(y)e^{-\frac{|x-y|^2}{4s}}\mathrm{d}y\le\frac{1}{4\pi s}\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}}V(y)\mathrm{d}y=\frac{1}{4\pi s}. \tag{2.9}$$

Hence, (2.7) is bounded from above by

$$\begin{aligned}&1+\sum_{n=1}^{\infty}\beta^n\gamma^n\left(\int_0^{tT}r_s\mathrm{d}s\right)^n\\ &\le 1+\sum_{n=1}^{\infty}\beta^n\gamma^n\left(\frac{\log(tT)}{4\pi}+\|V\|_\infty\right)^n=1+\sum_{n=1}^{\infty}\hat\beta^n\hat\gamma^n\left(1+\frac{\log t+4\pi\|V\|_\infty}{\log T}\right)^n\\ &\to\sum_{n=0}^{\infty}(\hat\beta\hat\gamma)^n=\frac{1}{1-\hat\beta\hat\gamma},\end{aligned}$$

as $\varepsilon\to 0$, where the convergence is absolute since $\hat\beta\hat\gamma<1$.

Next, we consider the lower bound. Let

$$\mathbf{T}_n=\{(s_1,\cdots,s_n)\in(0,tT)^n: s_1<\cdots<s_n,\ s_i-s_{i-1}>1,\ \forall i\in\{1,\cdots,n\}\}.$$

Then, since each term is non-negative, it is enough to show for fixed $L \in \mathbb{N}$,

$$\varliminf_{\varepsilon \to 0} \left(1 + \sum_{n=1}^{L} \beta^n \gamma^n \int_{\mathbf{T}_n} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^{n} \left(V(\sqrt{2}x_i) \rho_{s_i - s_{i-1}}(x_{i-1}, x_i) \right) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x} \right) \geq \sum_{n=0}^{L} (\hat{\beta}\hat{\gamma})^n. \quad (2.10)$$

For fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_{\mathbf{T}_n} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^{n} \left(V(\sqrt{2}x_i) \rho_{s_i - s_{i-1}}(x_{i-1}, x_i) \right) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{T}_n} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^{n} \left(\frac{1}{2\pi(s_i - s_{i-1})} V(\sqrt{2}x_i) - \bar{r}_{s_i - s_{i-1}}(x_{i-1}, x_i) \right) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{T}_n} \prod_{i=1}^{n} \frac{1}{4\pi(s_i - s_{i-1})} \mathrm{d}\mathbf{s} + A_\varepsilon^n, \qquad (2.11)
\end{aligned}$$

where

$$\bar{r}_s(x,y) = \frac{1}{2\pi s} V(\sqrt{2}y) - V(\sqrt{2}y)\rho_s(x,y) = \frac{1}{2\pi s} \left(1 - \exp\left(-\frac{|y-x|^2}{2s} \right) \right) V(\sqrt{2}y) \geq 0,$$

$$A_\varepsilon^n = \sum_{k=1}^{n} (-1)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \int_{\mathbf{T}_n} \int_{(\mathbb{R}^2)^n} \prod_{i \neq j_1, \cdots, j_k} \frac{V(\sqrt{2}x_i)}{4\pi(s_i - s_{i-1})} \prod_{j = j_1, \cdots, j_k} \bar{r}_{s_j - s_{j-1}}(x_{j-1}, x_j) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x}.$$

Let $D_V = \overline{\{x \in \mathbb{R}^2 : \ V(\sqrt{2}x) \neq 0\}}$ be the support of V with scale $1/\sqrt{2}$, which is compact. Note that

$$\begin{aligned}
\sup_{x \in D_V} \int_{\mathbb{R}^2} \bar{r}_s(x,y) \mathrm{d}y &= (2\pi s)^{-1} \sup_{x \in D_V} \int_{\mathbb{R}^2} \left(1 - \exp\left(-\frac{|y-x|^2}{2s} \right) \right) V(\sqrt{2}y) \mathrm{d}y \\
&\leq (2\pi s)^{-1} |D_V| \sup_{x,y \in D_V} \left(1 - \exp\left(-\frac{|y-x|^2}{2s} \right) \right) \\
&\leq C(s^{-2} \wedge 1),
\end{aligned}$$

with some $C = C(V) \geq 1 \vee \|V\|_\infty$, where $|D_V|$ is the volume of D_V. Therefore, we have

$$\begin{aligned}
|A_\varepsilon^n| &\leq C^{n+1} \sum_{k=1}^{n} \sum_{j_1 < j_2 < \cdots < j_k} \int_{\mathbf{T}_n} \int_{(\mathbb{R}^2)^n} \prod_{i \neq j_1, \cdots, j_k} \frac{\mathrm{d}s_i}{s_i - s_{i-1}} \\
&\leq C^{n+1} \sum_{k=1}^{n} n^k \left(\log(tT) \right)^{n-k} \\
&\leq (Cn)^{n+1} (\log(tT))^{n-1},
\end{aligned}$$

and hence we have for any fixed $L > 0$

$$\sum_{n=1}^{L} \beta^n \gamma^n A_\varepsilon^n = \sum_{n=1}^{L} \hat{\beta}^n \hat{\gamma}^n \left(\frac{4\pi}{\log T} \right)^n A_\varepsilon^n \to 0, \qquad (2.12)$$

as $\varepsilon \to 0$. Also,

$$\int_{\mathbf{T}_n} \prod_{i=1}^{n} \frac{1}{4\pi(s_i - s_{i-1})} \mathrm{d}\mathbf{s} \geq \left(\int_1^{\frac{tT}{n}} \frac{1}{4\pi s} \mathrm{d}s \right)^n = \left(\frac{\log(tT/n)}{4\pi} \right)^n,$$

and hence we have

$$\varliminf_{\varepsilon \to 0} \sum_{n=1}^{L} \beta^n \gamma^n \int_{\mathbf{T}_n} \prod_{i=1}^{n} \frac{1}{4\pi(s_i - s_{i-1})} \mathrm{d}\mathbf{s} \geq \sum_{n=1}^{L} \hat{\beta}^n \hat{\gamma}^n. \qquad (2.13)$$

Then, (2.11), (2.12) and (2.13) yield (2.10). $\square$

Proof of (2.6). We obtain by the same manner as in (2.2) that

$$\begin{aligned}\mathbb{E}\left[\mathrm{E}_{0,0}^{tT,0}\left[\Phi_{tT}^{\beta}\right]^2\right] &= \mathrm{E}_{0,0}^{tT,0}\left[\exp\left(\beta^2\int_0^{tT} V(\sqrt{2}B_u)\mathrm{d}u\right)\right]\\ &= 1+\sum_{n=1}^{\infty}\beta^{2n}\int_{0<s_1<\cdots<s_n<tT}\int_{(\mathbb{R}^2)^n}\left(\prod_{i=1}^n V(\sqrt{2}x_i)\rho_{s_i-s_{i-1}}(x_{i-1},x_i)\right)\frac{\rho_{tT-s_n}(x_n)}{\rho_{tT}(0)}\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x},\end{aligned} \tag{2.14}$$

where we use the orthogonal transformation invariance of Brownian bridges in the first equation. We have for $s,t>0$, by the Markov property of Brownian motion,

$$\int_{\mathbb{R}}\rho_s(x,y)\rho_t(y)V(\sqrt{2}y)\mathrm{d}y \le \|V\|_\infty\int_{\mathbb{R}}\rho_s(x,y)\rho_t(y)\mathrm{d}y \le \|V\|_\infty\rho_{s+t}(x),$$

and by $\int_{\mathbb{R}}V(\sqrt{2}y)\mathrm{d}y = 1/2$,

$$\begin{aligned}\int_{\mathbb{R}}\rho_s(x,y)\rho_t(y)V(\sqrt{2}y)\mathrm{d}y &= \frac{1}{4\pi st}\int_{\mathbb{R}}V(\sqrt{2}y)\exp\left(-\frac{|x-y|^2}{2s}-\frac{|y|^2}{2t}\right)\\ &= \rho_{s+t}(x)\,\frac{s+t}{2\pi st}\int_{\mathbb{R}}V(\sqrt{2}y)\exp\left(-\frac{(s+t)|y+*|^2}{2st}\right)\mathrm{d}y\\ &\le \rho_{s+t}(x)\,\frac{s+t}{2\pi st}\int_{\mathbb{R}}V(\sqrt{2}y)\mathrm{d}y = \frac{s+t}{4\pi st}\rho_{s+t}(x).\end{aligned} \tag{2.15}$$

Putting things together with $C=4\pi\|V\|_\infty$, we have

$$\int_{\mathbb{R}}\rho_s(x,y)\rho_t(y)V(\sqrt{2}y)\mathrm{d}y \le \frac{1}{4\pi}\left(C\wedge\frac{s+t}{st}\right)\rho_{s+t}(x).$$

Using this successively, we can bound each term of (2.14) as

$$\begin{aligned}&\beta^{2n}\int_{0<s_1<\cdots<s_n<tT}\int_{(\mathbb{R}^2)^n}\left(\prod_{i=1}^n V(\sqrt{2}x_i)\rho_{s_i-s_{i-1}}(x_{i-1},x_i)\right)\frac{\rho_{tT-s_n}(x_n)}{\rho_{tT}(0)}\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x}\\ &\le\left(\frac{\beta^2}{4\pi}\right)^n\int_{0<s_1<\cdots<s_n<tT}\prod_{i=1}^n\left(C\wedge\frac{tT-s_{i-1}}{(s_i-s_{i-1})(tT-s_i)}\right)\mathrm{d}\mathbf{s}\\ &=\left(\frac{\beta^2}{4\pi}\right)^n\int_{0<s_1<\cdots<s_n<tT}\prod_{i=1}^n\left(C\wedge\left(\frac{1}{s_i-s_{i-1}}+\frac{1}{tT-s_i}\right)\right)\mathrm{d}\mathbf{s},\end{aligned} \tag{2.16}$$

where we set $s_0=0$ and $s_{n+1}=tT$ and we have used $\frac{1}{tT-s_i}+\frac{1}{s_i-s_{i-1}}=\frac{tT-s_{i-1}}{(s_i-s_{i-1})(tT-s_i)}$ in the last line. We write $\log_+(x)=\log(x\vee 1)$ and $C_1=2C$. We use the following integral estimate: for $s<tT$ and $k\ge 0$,

$$\begin{aligned}&\int_s^{tT}(C_1+\log_+(tT-u))^k\left(C\wedge\left(\frac{1}{u-s}+\frac{1}{tT-u}\right)\right)\mathrm{d}u\\ &\le 2C(C_1+\log_+(tT-s))^k+\int_{s+1}^{tT-1}(C_1+\log_+(tT-u))^k\left(\frac{1}{u-s}+\frac{1}{tT-u}\right)\mathrm{d}u\\ &\le C_1(C_1+\log_+(tT-s))^k\\ &+(C_1+\log_+(tT-s))^k\int_{s+1}^{tT-1}\frac{1}{u-s}\mathrm{d}u-(k+1)^{-1}\left[(C_1+\log_+(tT-u))^{k+1}\right]_{s+1}^{tT-1}\\ &\le C_1(C_1+\log_+(tT-s))^k\\ &+(C_1+\log_+(tT-s))^k\log_+(tT-s)+(k+1)^{-1}(C_1+\log_+(tT-s))^{k+1}\end{aligned}$$

$$= \frac{k+2}{k+1}(C_1 + \log_+(tT-s))^{k+1}.$$

Using this, (2.16) can be successively bounded from above as

$$\int_{0<s_1<\cdots<s_n<tT} \prod_{i=1}^{n} \left(C \wedge \left(\frac{1}{s_i - s_{i-1}} + \frac{1}{tT - s_i} \right) \right) \mathrm{d}\mathbf{s} \le (n+1)(C_1 + \log tT)^n.$$

Together with (2.14) and (2.16), using $\beta = \hat{\beta}\sqrt{\frac{4\pi}{\log T}}$ with $\hat{\beta} < 1$, we have

$$\varlimsup_{\varepsilon\to 0} \mathbb{E}\left[\mathrm{E}_{0,0}^{tT,0}\left[\Phi_{tT}^{\beta} \right]^2 \right] \le \varlimsup_{\varepsilon\to 0} \sum_{n=0}^{\infty} \left(\frac{\beta^2}{4\pi} \right)^n (n+1)(C_1 + \log tT)^n = \sum_{n=0}^{\infty} (n+1)\hat{\beta}^{2n} < \infty. \tag{2.17}$$

□

Remark 2.4. The right most summation in (2.17) is actually the limit of the second moment of point-to-point partition functions $\mathcal{Z}_{0,x}^{tT_\varepsilon,y}$. It is a consequence from the local limit theorem (Theorem 2.10) and (2.5).

The following lemma is a consequence of Lemma 2.1 combining hypercontractivity of chaos expansion:

Lemma 2.5. *[CSZ20, (5.11)] Fix $\hat{\beta} \in (0,1)$. There exists $p_{\hat{\beta}} > 2$ such that for any $2 \le p < p_{\hat{\beta}}$ and for $t \ge 0$*

$$\varlimsup_{\varepsilon\to 0} \mathbb{E}\left[\mathrm{E}_x \left[\Phi_{tT_\varepsilon}^{\beta_\varepsilon} \right]^p \right] < \infty.$$

Remark 2.6. Recently, it has been proved that the limit exists for any $p \ge 2$ and any $\hat{\beta} \in (0,1)$ in [CZ21, LZ21] for the discrete setting. If the same holds for the continuous setting, one may make the condition of $\mathfrak{F}$, (1.9), mild so that F has a polynomial growth at infinity.

Lemma 2.7. *Suppose $\hat{\beta} \in (0,1)$ and fix $t > 0$. Then,*

$$\sup_{\varepsilon\le 1} \sup_{x\in\mathbb{R}^2} \sup_{s\le tT_\varepsilon} \mathrm{E}_{0,0}^{s,x}\left[\exp\left(\beta_\varepsilon^2 \int_0^s V(\sqrt{2}B_u)\mathrm{d}u \right) \right] < \infty. \tag{2.18}$$

Proof. By (2.6), (2.14), for $s \le tT$,

$$\begin{aligned}
\mathrm{E}_{0,0}^{s,z}\left[e^{\beta^2 \int_0^s V(\sqrt{2}B_u)\mathrm{d}u} \right] &= \mathbb{E}[\mathrm{E}_{0,0}^{s,z}[\Phi_s^{\beta}]\mathrm{E}_{0,0}^{s,0}[\Phi_s^{\beta}]] \\
&\le \mathbb{E}[\mathrm{E}_{0,0}^{s,0}[\Phi_s^{\beta}]^2] = \mathrm{E}_{0,0}^{s,0}\left[e^{\beta^2 \int_0^s V(\sqrt{2}B_u)\mathrm{d}u} \right] \\
&\le \sup_{\varepsilon\le 1} \left[\exp\left(\beta_\varepsilon^2 \int_0^{tT_\varepsilon} V(\sqrt{2}B_u)\mathrm{d}u \right) \right] < \infty,
\end{aligned}$$

where in the first inequality, we have used the remark below and the Cauchy-Schwarz inequality. □

Remark 2.8. By the shear invariance of environment, we have

$$\mathrm{E}_{0,x}^{t,y}\left[\Phi_t^{\beta} \right] \overset{(d)}{=} \mathrm{E}_{0,0}^{t,0}\left[\Phi_t^{\beta} \right],$$

for any $t > 0$ and $x, y \in \mathbb{R}^2$.

We end this subsection by presenting the boundedness of negative moments of partition functions:

Lemma 2.9. *[CSZ20, (5.12), (5.13), (5.14)] Let $\hat{\beta} \in (0,1)$ and fix $t > 0$. For any $p \geq 0$ and $x \in \mathbb{R}^2$,*

$$\sup_{s\in[0,t]} \mathbb{E}\left[\left(\mathcal{Z}^{\beta_\varepsilon}_{sT_\varepsilon}(x)\right)^{-p}\right] < \infty.$$

2.2 Local limit theorem

In this subsection, we give an estimate of the local limit theorem for partition functions. To describe the statement, we introduce the time-reversed partition function of time horizon ℓ, $\overleftarrow{\mathcal{Z}}^{\beta}_{T,\ell}(z)$:

$$\overleftarrow{\mathcal{Z}}^{\beta}_{t,\ell}(z) = \mathrm{E}_z\left[\exp\left\{\beta\int_{t-\ell}^{t}\int_{\mathbb{R}^2}\phi(B_{t-s}-y)\xi(\mathrm{d}s,\mathrm{d}y) - \frac{\beta^2 V(0)\ell}{2}\right\}\right].$$

Theorem 2.10 (Local limit theorem for polymers)**.** *Fix $t > 0$.Then, for all $\hat{\beta} < 1$ there exists a positive constant $C = C_{\hat{\beta}}$ such that for $4 \leq L \leq tT_\varepsilon$, $1 \leq \ell \leq \frac{L}{4}$ and for all $x \in \mathbb{R}^d$,*

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{E}^{L,x}_{0,0}[\Phi^{\beta_\varepsilon}_L] - \mathcal{Z}^{\beta_\varepsilon}_\ell(0)\overleftarrow{\mathcal{Z}}^{\beta_\varepsilon}_{L,\ell}(x)\right)^2 \\
&\leq \begin{cases} C\frac{\ell}{L} + C\beta_\varepsilon^2\left(\log\frac{L}{\ell} + \frac{|x|\log\ell}{L(T)} + \frac{|x|^2\ell}{L(T)^2}\right) & |x| \leq \sqrt{L\log L} \\ C & |x| \geq \sqrt{L\log L}\end{cases}.
\end{aligned}$$

Remark 2.11. The theorem states that the point-to-point partition function from $(0,x)$ to $(L(T),y)$ is approximated by the product of the partition function from $(0,x)$ with length ℓ and the time-reversed partition function from $(L(T),y)$ with length ℓ in L^2-sense. In the proof of Theorem 1.1, L appears as a length of partition function and ℓ will be chosen appropriately such that $\ell \ll L$. For $d \geq 3$, the reader may refer to [CNN22, S95, V06]. Also, a similar result was independently obtained for directed polymers in random environment in $d = 2$ by Gabriel [G21].

Notation 2.12. *Fix $R_V > 0$ such that $\operatorname{supp} V \subset B(0,R_V)$.*

The proof is composed of three lemmas (Lemma 2.13, 2.14, 2.15).

Lemma 2.13. *Fix $t > 0$. There exists a positive constant $C = C_{\hat{\beta}}$ such that for $4 \leq L \leq tT_\varepsilon$ and $1 \leq \ell \leq \frac{L}{4}$,*

$$\sup_{x\in\mathbb{R}^2}\mathbb{E}\left[\left(\mathrm{E}^{L,x}_{0,0}[\Phi^{\beta}_L] - \mathrm{E}^{L,x}_{0,0}[\Phi^{\beta}_\ell\Phi^{\beta}_{L-\ell,L}]\right)^2\right] \leq C\beta_\varepsilon^2\log\frac{L}{\ell},$$

where

$$\Phi^{\beta}_{s,t} = \exp\left(\beta\int_s^t\int_{\mathbb{R}^2}\phi(y-B_u)\xi(\mathrm{d}u,\mathrm{d}y) - \frac{\beta^2V(0)(t-s)}{2}\right).$$

Proof. Since $B^{(1)}_s - B^{(2)}_s \overset{(d)}{=} \sqrt{2}B_s$ for two independent Brownian motions, by $1-e^{-x} \leq x$ for $x \geq 0$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathrm{E}^{L(T),x}_{0,0}[\Phi^{\beta}_L - \Phi^{\beta}_\ell\Phi^{\beta}_{L-\ell,L}]\right)^2\right] \\
&= \mathrm{E}^{L,0}_{0,0}\left[\mathrm{e}^{\beta^2\int_0^L V(\sqrt{2}B_s)ds} - \mathrm{e}^{\beta^2\int_0^\ell V(\sqrt{2}B_s)ds}\mathrm{e}^{\beta^2\int_{L-\ell}^L V(\sqrt{2}B_s)ds}\right]
\end{aligned}$$

$$\leq \mathrm{E}_{0,0}^{L,0}\left[e^{\beta^2\int_0^L V(\sqrt{2}B_s)ds}\,\beta^2\int_\ell^{L-\ell}V(\sqrt{2}B_s)\mathrm{d}s\right].$$

The last expectation equals

$$\beta^2\int_{[\ell,L-\ell]\times\mathbb{R}^2}V(\sqrt{2}z)\frac{\rho_s(z)\rho_{L-s}(z)}{\rho_L(0)}\mathrm{E}_{0,0}^{s,z}\left[e^{\beta^2\int_0^s V(\sqrt{2}B_u)\mathrm{d}u}\right]\mathrm{E}_{0,z}^{L-s,0}\left[e^{\beta^2\int_0^{tT-s}V(\sqrt{2}B_u)\mathrm{d}u}\right]\mathrm{d}z\mathrm{d}s$$
$$\leq \beta^2\left(\sup_{s\leq L}\sup_{|z|\leq R_V}\mathrm{E}_{0,0}^{s,z}\left[e^{\beta^2\int_0^s V(\sqrt{2}B_u)\mathrm{d}u}\right]\right)^2\int_\ell^{L-\ell}\int_{\mathbb{R}^2}V(\sqrt{2}z)\frac{\rho_s(z)\rho_{L-s}(z)}{\rho_L(0)}\mathrm{d}s\mathrm{d}z,$$

where the supremum in the last line is finite by Lemma 2.7. Finally, by (2.15),

$$\begin{aligned}\int_\ell^{L-\ell}\int_{\mathbb{R}^2}\frac{\rho_s(z)\rho_{L-s}(z)}{\rho_L(0)}V\left(\sqrt{2}z\right)\mathrm{d}s\mathrm{d}z &\leq \frac{1}{4\pi}\int_\ell^{L-\ell}\frac{L}{s(L-s)}\mathrm{d}s\\ &\leq \frac{1}{2\pi}\log\frac{L-\ell}{\ell}\leq \log\frac{L}{\ell},\end{aligned}$$

and the statement of the lemma follows. □

By the shear invariance of Brownian bridge, Brownian motion, and noise, we have

$$\begin{aligned}&\mathrm{E}_{0,0}^{L,x}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}^\beta\right]-\mathcal{Z}_\ell^\beta(0)\overleftarrow{\mathcal{Z}}_{L,\ell}^\beta(x)\\ &\overset{(d)}{=}\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}^\beta\right]\\ &-\mathrm{E}_0\left[\Phi_\ell^\beta\left(B_\cdot-\frac{x}{L}\cdot\right)\right]\mathrm{E}_0\left[\exp\left(\beta\int_{L-\ell}^L\phi\left(B_{L-s}+\frac{(L-s)x}{L}-y\right)\xi(\mathrm{d}s,\mathrm{d}y)-\frac{\beta^2V(0)\ell}{2}\right)\right],\end{aligned}$$

where $B_\cdot-\frac{x}{L}\cdot$ is a Brownian motion with drift $-\frac{x}{L}$.

Define

$$\begin{aligned}A_{L,\ell}&:=\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}^\beta\right]-\mathrm{E}_0\left[\Phi_\ell^\beta\right]\overleftarrow{\mathcal{Z}}_{L,\ell}^\beta(0),\\ B_{L,\ell,x}&:=\mathrm{E}_0\left[\Phi_\ell^\beta\right]-\mathrm{E}_0\left[\Phi_\ell^\beta\left(B_\cdot-\frac{x}{L}\cdot\right)\right].\end{aligned}$$

Lemma 2.14. *There exists a positive constant $C=C_{\hat\beta}$ such that for $4\leq L\leq tT_\varepsilon$ and $1\leq \ell\leq\frac{L}{4}$,*

$$\mathbb{E}\left[(A_{L,\ell})^2\right]\leq C\frac{\ell}{L}.$$

Proof. Since

$$\begin{aligned}\mathbb{E}\left[A_{L,\ell}^2\right]&=\left(\mathbb{E}\left[\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}^\beta\right]^2\right]-\mathbb{E}\left[\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}\right]\mathrm{E}_0\left[\Phi_\ell^\beta\right]\overleftarrow{\mathcal{Z}}_{L,\ell}(0)\right]\right)\\ &\quad+\left(\mathbb{E}\left[\mathrm{E}_0\left[\Phi_\ell^\beta\right]\overleftarrow{\mathcal{Z}}_{L,\ell}(0)^2\right]-\mathbb{E}\left[\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}\right]\mathrm{E}_0\left[\Phi_\ell^\beta\right]\overleftarrow{\mathcal{Z}}_{L,\ell}(0)\right]\right)\\ &=: A_1+A_2,\end{aligned}$$

it is enough to prove the absolute value of each term is bounded by $C\frac{\ell}{L}$.

(2.3) yields that

$$\begin{aligned}&\mathbb{E}\left[\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}^\beta\right]^2\right]\\ &=\mathrm{E}_{0,0}^{L,0}\otimes\mathrm{E}_{0,0}^{L,0}\left[\exp\left(\beta^2\int_0^\ell V(B_s-\widetilde{B}_s)ds+\beta^2\int_{L-\ell}^L V(B_s-\widetilde{B}_s)ds\right)\right]\end{aligned}$$

$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, \rho_\ell(x)\rho_\ell(y)\frac{\rho_{L-2\ell}(y-x)}{\rho_L(0)} \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}z\mathrm{d}w \rho_\ell(z)\rho_\ell(w)\frac{\rho_{L-2\ell}(z-w)}{\rho_L(0)}$$
$$\times \mathrm{E}_{0,0}^{\ell,x}\otimes \mathrm{E}_{0,0}^{\ell,z}\left[\exp\left(\beta^2\int_0^\ell V(B_s-\widetilde{B}_s)ds\right)\right]\mathrm{E}_{0,0}^{\ell,y}\otimes \mathrm{E}_{0,0}^{\ell,w}\left[\exp\left(\beta^2\int_0^\ell V(B_s-\widetilde{B}_s)ds\right)\right],$$

and

$$\mathbb{E}\left[\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta \Phi_{L-\ell,L}\right]\mathrm{E}_0\left[\Phi_\ell^\beta\right]\overleftarrow{\mathcal{Z}}_{L,\ell}(0)\right]$$
$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, \rho_\ell(x)\rho_\ell(y)\frac{\rho_{L-2\ell}(y-x)}{\rho_L(0)} \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}z\mathrm{d}w \rho_\ell(z)\rho_\ell(w)$$
$$\times \mathrm{E}_{0,0}^{\ell,x}\otimes \mathrm{E}_{0,0}^{\ell,z}\left[\exp\left(\beta^2\int_0^\ell V(B_s-\widetilde{B}_s)ds\right)\right]\mathrm{E}_{0,0}^{\ell,y}\otimes \mathrm{E}_{0,0}^{\ell,w}\left[\exp\left(\beta^2\int_0^\ell V(B_s-\widetilde{B}_s)ds\right)\right],$$

where $B=\{B_s : 0\le s\le \ell\}$ and $\widetilde{B}=\{\widetilde{B}_s : 0\le s\le\ell\}$ are independent Brownian bridges with the law $\mathrm{P}_{0,0}^{\ell,u}$ $(u=x,y,z,w)$. Then, it is easy to see that

$$A_1$$
$$= \mathbb{E}\left[\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta\Phi_{L-\ell,L}^\beta\right]^2\right] - \mathbb{E}\left[\mathrm{E}_{0,0}^{L,0}\left[\Phi_\ell^\beta \Phi_{L-\ell,L}\right]\mathrm{E}_0\left[\Phi_\ell^\beta\right]\overleftarrow{\mathcal{Z}}_{L,\ell}(0)\right]$$
$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, \rho_\ell(x)\rho_\ell(y)\frac{\rho_{L-2\ell}(y-x)}{\rho_L(0)} \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}z\mathrm{d}w \rho_\ell(z)\rho_\ell(w)\left(\frac{\rho_{L-2\ell}(z-w)}{\rho_L(0)}-1\right)$$
$$\times \mathrm{E}_{0,0}^{\ell,x}\otimes \mathrm{E}_{0,0}^{\ell,z}\left[\exp\left(\beta^2\int_0^\ell V(B_s-\widetilde{B}_s)ds\right)\right]\mathrm{E}_{0,0}^{\ell,y}\otimes \mathrm{E}_{0,0}^{\ell,w}\left[\exp\left(\beta^2\int_0^\ell V(B_s-\widetilde{B}_s)ds\right)\right].$$

Combining with (2.18) gives

$$\begin{aligned}|A_1| &\le C\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathrm{d}z\mathrm{d}w\rho_\ell(z)\rho_\ell(w)\left|\frac{\rho_{L-2\ell}(z-w)}{\rho_L(0)}-1\right| \\ &= C\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathrm{d}z\mathrm{d}w\rho_\ell(z)\rho_\ell(w)\left|\frac{L}{L-2\ell}\left(\exp\left(-\frac{|z-w|^2}{L-2\ell}\right)-1\right)+\frac{2\ell}{L-2\ell}\right| \\ &\le C\left(\frac{2L\ell}{(L-2\ell)^2}+\frac{2\ell}{L-2\ell}\right)\le C\frac{\ell}{L},\end{aligned}$$

where we have used $1-e^{-x}\le x$ for $x\ge 0$ in the third line. Also, the same argument holds for $|A_2|$. □

Lemma 2.15. *Fix $t>0$. For all $\hat{\beta}<1$ there exists a positive constant $C=C_{\hat{\beta}}$ such that for $4\le L\le tT_\varepsilon$, $1\le\ell\le\frac{L}{4}$ and all $x\in\mathbb{R}^d$,*

$$\mathbb{E}\left[B_{L,\ell,x}^2\right]\le\begin{cases}C\beta_\varepsilon^2\left(\frac{|x|\log\ell}{L}+\frac{|x|^2\ell}{L^2}\right) & |x|\le\sqrt{L\log L},\\ C & |x|\ge\sqrt{L\log L}.\end{cases}$$

Proof. For $|x|\ge\sqrt{L\log L}$, it is trivial from (2.5). We assume $|x|<\sqrt{L\log L}$. Combining (2.3) and transformation of Brownian motions yield that

$$\begin{aligned}\mathbb{E}\left[B_{L,\ell,x}^2\right] &= 2E\left[\exp\left(\beta^2\int_0^\ell V(\sqrt{2}B_s)ds\right)-\exp\left(\beta^2\int_0^\ell V\left(\sqrt{2}B_s-\frac{xs}{L}\right)ds\right)\right] \\ &= 2\sum_{n=1}^\infty\beta^{2n}\int_{0<t_1<\cdots<t_n<\ell}\int_{(\mathbb{R}^2)^n}\mathbf{dsdx}\prod_{i=1}^n V(\sqrt{2}x_i)\end{aligned}$$

$$\times \left(\prod_{i=1}^{n} \rho_{s_i - s_{i-1}}(x_{i-1}, x_i) - \prod_{i=1}^{n} \rho_{s_i - s_{i-1}} \left(x_i - x_{i-1} - \frac{x(s_i - s_{i-1})}{L} \right) \right),$$

where we set $x_0 = 0$. When we use the relation

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \sum_{j=1}^{n} \left(\prod_{i=1}^{j-1} b_i \right) (a_j - b_j) \left(\prod_{k=j+1}^{n} a_k \right),$$

and recall the notation r_s from (2.8), we have

$$\begin{aligned}
&\mathbb{E}\left[B_{L,\ell,x}^2\right] \\
&\leq 2 \sup_{z \in B(0,R_V)} \beta^2 \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \left| \rho_s(y-z) - \rho_s(y - z - \frac{xs}{L}) \right| \mathrm{d}y\mathrm{d}s \\
&\qquad \times \sum_{n=1}^{\infty} \sum_{k=1}^{n} \beta^{2n-2} \left(\sup_{z \in \mathbb{R}} \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \rho_s(y-z) \mathrm{d}y\mathrm{d}s \right)^{n-1} \\
&\leq 2 \sum_{n=1}^{\infty} n \beta^{2(n-1)} \left(\int_0^\ell r_s \mathrm{d}s \right)^{n-1} \sup_{z \in B(0,R_V)} \beta^2 \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \left| \rho_s(y-z) - \rho_s\left(y - z - \frac{xs}{L}\right) \right| \mathrm{d}y\mathrm{d}s \\
&\leq C\beta^2 \sup_{z \in B(0,R_V)} \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \left| \rho_s(y-z) - \rho_s\left(y - z - \frac{xs}{L}\right) \right| \mathrm{d}y\mathrm{d}s, \qquad (2.19)
\end{aligned}$$

where we have used the estimate (2.9) in the last line. Also, we have for $y, z \in B(0, R_V)$ and for $s > 0$

$$\begin{aligned}
&\left| \rho_s(y-z) - \rho_s\left(y - z - \frac{xs}{L}\right) \right| \\
&= \rho_s(y-z) \left| 1 - \exp\left(\frac{\langle y-z, x \rangle}{L} - \frac{|x|^2 s}{2L^2} \right) \right| \\
&\leq \rho_s(y-z) \left(\frac{R_V|x|}{L} \exp\left(\frac{2R_V|x|}{L} - \frac{|x|^2 s}{2L^2} \right) + \frac{R_V|x|}{L} + \frac{|x|^2 s}{2L^2} \right) \\
&\leq C\rho_s(y-z) \left(\frac{|x|}{L} + \frac{|x|^2 s}{L^2} \right),
\end{aligned}$$

where we denote by $\langle x, y \rangle$ the inner product of x and $y \in \mathbb{R}^2$ and we use $e^x - 1 \leq xe^x$ if $x \geq 0$ and $1 - e^x \leq -x$ if $x < 0$ in the last line. For $|x| \leq \sqrt{L \log L}$,

$$\mathbb{E}\left[B_{L,\ell,x}^2\right] \leq C\beta^2 \left(\frac{|x| \log \ell}{L} + \frac{|x|^2 \ell}{L(T)^2} \right). \qquad \square$$

Putting things together, we conclude the proof of Theorem 2.10.

We also use the following lemma later.

Lemma 2.16. *For fixed $t > 0$ and $\hat{\beta} \in (0,1)$, there exists a positive constant C such that for $x \in \mathbb{R}^2$ and for $1 \leq \ell \leq tT_\varepsilon$*

$$\mathbb{E}\left[\left(\mathrm{E}_x\left[\Phi_\ell^{\beta_\varepsilon}\right] - \mathrm{E}_0\left[\Phi_\ell^{\beta_\varepsilon}\right] \right)^2 \right] \leq \begin{cases} C\beta_\varepsilon^2 (1 + |x|^2) & |x| \leq \sqrt{\log \ell}, \\ C & |x| > \sqrt{\log \ell}. \end{cases} \qquad (2.20)$$

Proof. For $|x| \geq \sqrt{\log \ell}$, it is trivial from (2.5). We suppose $|x| < \sqrt{\log \ell}$. Using the same argument as in (2.19), with the convention $x_0 = 0$,

$$\mathbb{E}\left[\left(\mathrm{E}_x\left[\Phi_\ell^{\beta_\varepsilon}\right] - \mathrm{E}\left[\Phi_\ell^{\beta_\varepsilon}\right] \right)^2 \right]$$

$$
\begin{aligned}
&= E\left[\exp\left(\beta^2\int_0^\ell V(\sqrt{2}B_s)ds\right) - \exp\left(\beta^2\int_0^\ell V\left(x+\sqrt{2}B_s\right)ds\right)\right] \\
&= \sum_{n=1}^\infty \beta^{2n}\int_{0<t_1<\cdots<t_n<\ell}\int_{\mathbb{R}^{2n}} \mathbf{dsdx}\prod_{i=1}^n V(\sqrt{2}x_i) \\
&\qquad \times\left(\prod_{i=1}^n \rho_{s_i-s_{i-1}}(x_i-x_{i-1}) - \rho_{s_i-s_{i-1}}(x_1-x)\prod_{i=2}^n \rho_{s_i-s_{i-1}}(x_i-x_{i-1})\right) \\
&\leq C\beta^2\left(1+\sup_{z\in B(0,R_V)}\int_1^\ell\int_{\mathbb{R}^2} V(\sqrt{2}y)\left|\rho_s(y-z)-\rho_s(y-z+x)\right|\mathrm{d}y\mathrm{d}s\right).
\end{aligned}
$$

Also, we have that for $y,z\in B(0,R_V)$ and $s\geq 1$,

$$
\begin{aligned}
|\rho_s(y-z)-\rho_s(y-z+x)| &= \rho_s(y-z)\left|1-\exp\left(-\frac{2\langle y-z,x\rangle+|x|^2}{2s}\right)\right| \\
&\leq \frac{C}{s}\rho_s(y-z)\left(\exp\left(\frac{R_V^2}{s}\right)+|x|+|x|^2\right) \\
&\leq \frac{C}{s}\rho_s(y-z)\left(1+|x|^2\right).
\end{aligned}
$$

Since

$$
\int_1^\ell\int_{\mathbb{R}^2} V(\sqrt{2}y)\frac{1}{s}\rho_s(y-z)\mathrm{d}y\mathrm{d}s \leq \int_1^\ell\int_{\mathbb{R}^2} s^{-2}V(\sqrt{2}y)\mathrm{d}y\mathrm{d}s \leq 1,
$$

we have

$$
\mathbb{E}\left[\left(\mathrm{E}_x\left[\Phi_\ell^{\beta_\varepsilon}\right]-\mathrm{E}_0\left[\Phi_\ell^{\beta_\varepsilon}\right]\right)^2\right] \leq C\beta^2\left(1+|x|^2\right). \qquad \square
$$

3 Proofs of Theorem 1.1

For fixed $t>0$ and for $u_0\in\mathfrak{C}$, let us define the martingale

$$
s\to\mathcal{W}_s(x)=\mathcal{W}_s^{(t,T,\hat{\beta},u_0)}(x)=\mathrm{E}_x\left[\Phi_s^\beta(B)\,u_0\left(\frac{B_{tT}}{\sqrt{T}}\right)\right]
$$

with respect to the filtration $\{\mathcal{F}_s : 0\leq s\leq tT\}$ associated to the white noise ξ. Then, it follows from the Feynman-Kac formula (see (1.15)) that for each $(t,x)\in[0,\infty)\times\mathbb{R}^2$

$$
u_\varepsilon^{(T,\hat{\beta},u_0)}(t,x)\overset{(d)}{=}\mathcal{W}_{tT}^{(t,T,\hat{\beta},u_0)}(\sqrt{T}x). \tag{3.1}
$$

We omit some superscripts t, T, $\hat{\beta}$, and u_0 to make notations simple when they are easily understood from the contexts.

Since both $\|u_0^{-1}\|_\infty$ and $\|u_0\|_\infty$ are finite,

$$
\|u_0^{-1}\|_\infty^{-1}\mathcal{Z}_s(x)\leq\mathcal{W}_s(x)\leq\|u_0\|_\infty\mathcal{Z}_s(x).
$$

Hereafter, we use this without any comment.

Itô's formula yields that for each $x\in\mathbb{R}^2$,

$$
\mathcal{W}_s^{(t,T,\hat{\beta},u_0)}(x)=\bar{u}(t,x)+\int_0^s \mathrm{d}\mathcal{W}_u^{(t,T,\hat{\beta},u_0)}(x), \tag{3.2}
$$

$$
\mathcal{W}_s^{(t,T,\hat{\gamma},v_0)}(x)=\bar{v}(t,x)+\int_0^s \mathrm{d}\mathcal{W}_u^{(t,T,\hat{\gamma},v_0)}(x), \tag{3.3}
$$

with

$$\langle \mathcal{W}^{(\hat\beta,u_0)}(x),\mathcal{W}^{(\hat\gamma,v_0)}(y)\rangle_s = \int_0^s \beta\gamma \mathrm{E}_x\otimes\mathrm{E}_y\left[V(B_u-\widetilde{B}_u)\Phi_u^\beta(B)\Phi_u^\gamma(\widetilde{B})u_0\left(\frac{B_{tT}}{\sqrt{T}}\right)v_0\left(\frac{\widetilde{B}_{tT}}{\sqrt{T}}\right)\right]\mathrm{d}u, \tag{3.4}$$

for each $x,y\in\mathbb{R}^2$, where $\mathrm{E}_x\otimes\mathrm{E}_y$ denotes the expectation of two independent Brownian motions B and $\widetilde{B}$ starting from x and y.

Then, using Itô's formula, we see that for $F\in\mathfrak{F}$, $F(\mathcal{W}_s(x))$ has the following semi-martingale representation

$$\begin{aligned} F(\mathcal{W}_s^{(\hat\beta,u_0)}(x)) = F(\bar{u}(t,x)) &+ \int_0^s F'(\mathcal{W}_u^{(\hat\beta,u_0)}(x))\mathrm{d}\mathcal{W}_u^{(\hat\beta,u_0)}(x) \\ &+ \frac{1}{2}\int_0^s F''(\mathcal{W}_u^{(\hat\beta,u_0)}(x))\mathrm{d}\langle \mathcal{W}^{(\hat\beta,u_0)}(x)\rangle_u, \end{aligned} \tag{3.5}$$

and we denote by

$$G_s^{(t,T,F,\hat\beta,u_0)}(x) = G_s(x) = \int_0^s F'(\mathcal{W}_u^{(\hat\beta,u_0)}(x))\mathrm{d}\mathcal{W}_u^{(\hat\beta,u_0)}(x), \tag{3.6}$$

$$H_s^{(t,T,F,\hat\beta,u_0)}(x) = H_s(x) = \int_0^s F''(\mathcal{W}_u^{(\hat\beta,u_0)}(x))\mathrm{d}\langle \mathcal{W}^{(\hat\beta,u_0)}(x)\rangle_u. \tag{3.7}$$

First, we will prove the fluctuations of the martingale parts converge to centered Gaussian random variables.

Proposition 3.1. *Suppose* $u_0^{(1)},\cdots,u_0^{(n)}\in\mathfrak{C}$, $\hat\beta^{(1)},\cdots,\hat\beta^{(n)}\in(0,1)$ *and* $F_1,\cdots,F_n\in\mathfrak{F}$.

For any test functions $f_1,\cdots,f_n\in C_c^\infty(\mathbb{R}^2)$ *and* $t\geq 0$, *as* $\varepsilon\to 0$

$$\left\{\frac{1}{\beta_\varepsilon^{(i)}}\int_{\mathbb{R}^2} f_i(x)G_{T_\varepsilon t}^{\left(t,T_\varepsilon,F_i,\hat\beta^{(i)},u_0^{(i)}\right)}(\sqrt{T_\varepsilon}x)\mathrm{d}x\right\}_{i=1,\cdots,n} \xrightarrow{(d)} \left\{\mathscr{U}(t,f_i,F_i,\hat\beta^{(i)},u_0^{(i)})\right\}_{i=1,\cdots,n}, \tag{3.8}$$

where $\left\{\mathscr{U}(t,f_i,F_i,\hat\beta^{(i)},u_0^{(i)})\right\}_{i=1,\cdots,n}$ *are centered Gaussian random variables with covariance given by* (1.9) *at* $t_i=t_j=t$.

Then, we will prove that the Itô correction terms are negligible in the limit:

Proposition 3.2. *For any* $t>0$, $\hat\beta\in(0,1)$, $u_0\in\mathfrak{C}$ *and* $F\in\mathfrak{F}$, *as* $\varepsilon\to 0$,

$$\frac{1}{\beta_\varepsilon}\int_{\mathbb{R}^2} f(x)\left(H_{T_\varepsilon t}^{(F)}(\sqrt{T_\varepsilon}x)-\mathbb{E}\left[H_{T_\varepsilon t}^{(F)}(\sqrt{T_\varepsilon}x)\right]\right)\xrightarrow{L^1} 0. \tag{3.9}$$

Proposition 3.1 and Proposition 3.2 combined with (3.1) and (3.5) imply Theorem 1.1 for 1-dimensional in time. Thus, Gaussian limit comes from the martingale part of $\int f(x)F(\mathcal{W}_s(\sqrt{T_\varepsilon}x))\mathrm{d}x$.

3.1 Proof of Proposition 3.1 and heuristics

First, we introduce the key theorem to prove the convergence of martingale to Gaussian process in this paper:

Theorem 3.3. *[JS87, Theorem 3.11 in Chap. 8], [EK86, Theorem 1.4 in Chap. 7]*

For each $n\geq 1$, *let* $\mathcal{F}^n=\{\mathcal{F}_t^n: t\geq 0\}$ *be a filtration and let* $X^{(n)}=(X_t^{(n,d)},\ldots,X_t^{(n,d)})$ *be an* $\mathbb{R}^d$*-valued continuous* $\mathcal{F}^n$*-martingale with* $X_0^{(n)}=0$. *Suppose that there exists a* $d\times d$ *positive definite matrix-valued continuous function* $c=\{c_{ij}(t)\}_{i,j=1}^d$ *such that for each* $t\geq 0$, $\langle X^{(n,i)},X^{(n,j)}\rangle_t\to c_{ij}(t)$ *in probability. Then,* $X^{(n)}\xrightarrow{(d)} X$, *where* $X=(X_t^{(1)},\cdots,X_t^{(d)})$ *is an* $\mathbb{R}^d$*-valued Gaussian process with* $\langle X^{(i)},X^{(j)}\rangle_t=c_{ij}(t)$.

Remark 3.4. Theorem 3.3 is simplified from the original one for our convenience.

Thus, it suffices to focus our analysis on the cross-variation of martingales.

In the following, we give a heuristic idea of the proof of Proposition 3.1 and we will discuss only the quadratic variation because it is easy to modify the following argument to the cross-variation. Also, we fix $\hat{\beta} \in (0, 1)$ and omit the superscript $\hat{\beta}$ in notations when it is clear from the context.

We fix $\delta \in (0, \frac{1}{100})$ arbitrary. By the local limit theorem (Theorem 2.10), we may expect that for large s,

$$\begin{aligned} \mathcal{W}_s(x) &= \mathrm{E}_x \left[\Phi_s \, u_0 \left(\frac{B_{tT}}{\sqrt{T}} \right) \right] \\ &= \int_{\mathbb{R}^2} \rho_s(z - x) \mathrm{E}^{s,z}_{0,x} [\Phi_s] \mathrm{E}_z \left[u_0 \left(\frac{B_{tT-s}}{\sqrt{T}} \right) \right] \mathrm{d}z \\ &\approx \int_{\mathbb{R}^2} \rho_s(z - x) \mathcal{Z}_{s\ell(T)}(x) \overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z) \mathrm{E}_z \left[u_0 \left(\frac{B_{tT-s}}{\sqrt{T}} \right) \right] \mathrm{d}z, \end{aligned} \tag{3.10}$$

where we set

$$\ell(T) = \exp\left(- (\log T)^{\frac{1}{2} - \delta} \right). \tag{3.11}$$

Here, we choose $\ell(T)$ such that the last integrals in (3.37) in the proof of Lemma 3.16 tends to 0.

Moreover, one may expect that the last term of (3.10) is approximated in some sense by

$$\mathcal{Z}_{s\ell(T)}(x) \int_{\mathbb{R}^2} \rho_s(z - x) \mathbb{E}\left[\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z) \right] \mathrm{E}_z \left[u_0 \left(\frac{B_{tT-s}}{\sqrt{T}} \right) \right] \mathrm{d}z = \mathcal{Z}_{s\ell(T)}(x) \bar{u}(t, T^{-\frac{1}{2}} x), \tag{3.12}$$

since $\mathbb{E}\left[\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z) \right] = 1$ and Lemma 2.16 may imply that $(\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(x))_{x \in \mathbb{R}^2}$ are asymptotically independent and homogenization occurs.

Therefore, one may observe for $F \in \mathfrak{F}$ that for large s

$$\begin{aligned} & F'(\mathcal{W}_s(x)) \mathrm{d}\mathcal{W}_s(x) \\ &= \beta F'(\mathcal{W}_s(x)) \int_{\mathbb{R}^2} \xi(\mathrm{d}s, \mathrm{d}b) \mathrm{E}_x \left[\phi(B_s - b) \Phi_s \, u_0 \left(\frac{B_{tT}}{\sqrt{T}} \right) \right] \\ &= \beta F'(\mathcal{W}_s(x)) \int_{\mathbb{R}^2} \xi(\mathrm{d}s, \mathrm{d}b) \int_{\mathbb{R}^2} \rho_s(z - x) \phi(z - b) \mathrm{E}^{s,z}_{0,x} [\Phi_s] \, \mathrm{E}_z \left[u_0 \left(\frac{B_{tT-s}}{\sqrt{T}} \right) \right] \mathrm{d}z \\ &\approx \beta F'(\mathcal{Z}_{s\ell(T)}(x) \bar{u}(t, T^{-\frac{1}{2}} x)) \mathcal{Z}_{s\ell(T)}(x) \\ &\qquad \times \int_{\mathbb{R}^2} \xi(\mathrm{d}s, \mathrm{d}b) \int_{\mathbb{R}^2} \rho_s(z - x) \phi(z - b) \overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z) \, \mathrm{E}_z \left[u_0 \left(\frac{B_{tT-s}}{\sqrt{T}} \right) \right] \mathrm{d}z, \end{aligned} \tag{3.13}$$

where we have used (3.12) and the local limit theorem in the third line. We denote by

$$I^{(T)}_s(x) = I^{(t,T,F,\hat{\beta},u_0)}_s(x) = F'\left(\mathcal{Z}^{\beta}_{s\ell(T)}(\sqrt{T}x) \bar{u}(t, x) \right) \mathcal{Z}^{\beta}_{s\ell(T)}(\sqrt{T}x).$$

Thus, we may approximate the cross variation of $F'(\mathcal{W}_s(x)) \mathrm{d}\mathcal{W}(x)$ as

$$\begin{aligned} & F'(\mathcal{W}_s(x)) F'(\mathcal{W}_s(y)) \mathrm{d}\langle \mathcal{W}(x), \mathcal{W}(y) \rangle_s \\ &= \mathrm{d}s \beta^2 F'(\mathcal{W}_s(x)) F'(\mathcal{W}_s(y)) \int_{(\mathbb{R}^2)^2} \mathrm{d}z_1 \mathrm{d}z_2 \rho_s(x, z_1) \rho_s(y, z_2) V(z_1 - z_2) \mathrm{E}^{s,z_1}_{0,x} [\Phi_s] \, \mathrm{E}^{s,z_2}_{0,y} [\Phi_s] \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \times \mathrm{E}_{z_1}\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right]\mathrm{E}_{z_2}\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right] \\
&\approx \mathrm{d}s\beta^2 F'\left(\mathcal{Z}_{s\ell(T)}(x)\bar{u}(t,T^{-\frac{1}{2}}x)\right)F'\left(\mathcal{Z}_{s\ell(T)}(y)\bar{u}(t,T^{-\frac{1}{2}}y)\right)\mathcal{Z}_{s\ell(T)}(x)\mathcal{Z}_{s\ell(T)}(y) \\
&\times \int_{(\mathbb{R}^2)^2}\mathrm{d}z_1\mathrm{d}z_2\rho_s(x,z_1)\rho_s(y,z_2)V(z_1-z_2)\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_1)\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_2) \\
&\qquad\qquad\qquad\qquad \times \mathrm{E}_{z_1}\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right]\mathrm{E}_{z_2}\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right],
\end{aligned}
$$

where we have used (3.13) in the approximation. Changing the variables as $s = T\sigma$, $x = \sqrt{T}x'$, and $y = \sqrt{T}y'$, this is equal to

$$
\begin{aligned}
&\mathrm{d}\sigma\beta^2 I_s^{(T)}(x')I_s^{(T)}(y')\int_{(\mathbb{R}^2)^2}\mathrm{d}z\mathrm{d}w\rho_\sigma(x',z)\rho_\sigma(y',z-T^{-\frac{1}{2}}w)V(w) \\
&\qquad \times \overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(\sqrt{T}z)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(\sqrt{T}z-w)\bar{u}(t-\sigma,z)\bar{u}(t-\sigma,z-T^{-\frac{1}{2}}w).
\end{aligned}
$$

Also, neglecting $T^{-1/2}w$, we may approximate it by

$$
\mathrm{d}\sigma\beta^2 I_{T\sigma}^{(T)}(x')I_{T\sigma}^{(T)}(y')\int_{(\mathbb{R}^2)^2}\mathrm{d}z\mathrm{d}w\rho_\sigma(x',z)\rho_\sigma(y',z)V(w)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(\sqrt{T}z)^2\bar{u}(t-\sigma,z)^2.
$$

Roughly speaking, $\overleftarrow{\mathcal{Z}}(\sqrt{T}z)$ and $\overleftarrow{\mathcal{Z}}(\sqrt{T}z')$ are independent if the distance between z and z' is sufficiently large. So integrating with respect to z, the law of large numbers can be applied and $\overleftarrow{\mathcal{Z}}^2$ would be replaced by $\mathbb{E}\left[\overleftarrow{\mathcal{Z}}^2\right]$ so that the cross variation $F'(\mathcal{W}_s(x))\mathrm{d}\mathcal{W}(x)$ would be approximated by

$$
\mathrm{d}\sigma\beta^2 I_{T\sigma}^{(T)}(x')I_{T\sigma}^{(T)}(y')\int_{\mathbb{R}^2}\mathrm{d}z\rho_\sigma(x',z)\rho_\sigma(y',z)\mathbb{E}\left[\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(\sqrt{T}z)^2\right]\bar{u}(t-\sigma,z)^2.
$$

Thus, the quadratic variation of $\frac{1}{\beta_\varepsilon}\int_{\mathbb{R}^2} f(x)G_{sT}(x)\mathrm{d}x$ would be approximated as

$$
\begin{aligned}
&\int_0^{sT}\int_{(\mathbb{R}^2)^2} f(x)f(y)F'(\mathcal{W}_u(\sqrt{T}x))F'(\mathcal{W}_u(\sqrt{T}y))\mathrm{d}\langle\mathcal{W}(\sqrt{T}x),\mathcal{W}(\sqrt{T}y)\rangle_u, \\
&\approx \int_{sT}\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}yf(x)f(y)I_s^{(T)}(x')I_s^{(T)}(y') \\
&\qquad\qquad \times \int_{\mathbb{R}^2}\mathrm{d}z\rho_\sigma(x',z)\rho_\sigma(y',z)\mathbb{E}\left[\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(z)^2\right]\bar{u}(t-\sigma,z)^2.
\end{aligned}
$$

Moreover, $F'(\mathcal{Z}\bar{u})\mathcal{Z}$ terms in $I_s^{(T)}(\cdot)$ would be replaced by its expectations in the same manner.

Due to (1.10) and (2.5), we have

$$
\begin{aligned}
&\int_0^{sT}\int_{(\mathbb{R}^2)^2} f(x)f(y)F'(\mathcal{W}_u(\sqrt{T}x))F'(\mathcal{W}_u(\sqrt{T}y))\mathrm{d}\langle\mathcal{W}(\sqrt{T}x),\mathcal{W}(\sqrt{T}y)\rangle_u \\
&\approx \frac{1}{1-\hat{\beta}^2}\mathbb{E}\left[F'(e^{X_{\hat{\beta}}-\frac{1}{2}\sigma^2(\hat{\beta})}\bar{u}(t,x))e^{X_{\hat{\beta}}-\frac{1}{2}\sigma^2(\hat{\beta})}\right]\mathbb{E}\left[F'(e^{X_{\hat{\beta}}-\frac{1}{2}\sigma^2(\hat{\beta})}\bar{u}(t,y))e^{X_{\hat{\beta}}-\frac{1}{2}\sigma^2(\hat{\beta})}\right] \\
&\times \int_0^s\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}yf(x)f(y)\int_{\mathbb{R}^2}\mathrm{d}z\rho_\sigma(x,z)\rho_\sigma(y,z)\bar{u}(t-\sigma,z)^2, \qquad (3.14)
\end{aligned}
$$

and Theorem 3.3 implies that the limit process is a Gaussian process with covariance function (3.14).

To make this rough idea rigorous, we introduce a martingale increment $\mathrm{d}\mathcal{M}_s^{(t,T,F,\hat{\beta},u_0)}(x)$ for fixed $t>0$, $x\in\mathbb{R}^2$, $\hat{\beta}\in(0,1)$, $F\in\mathfrak{F}$, and $u_0\in\mathfrak{C}$ as

$$\begin{aligned}
&\mathrm{d}\mathcal{M}_s(x)=\mathrm{d}\mathcal{M}_s^{(t,T,F,\beta,u_0)}\\
&=\beta F'(\mathcal{Z}^\beta_{s\ell(T)}(x)\bar{u}(t,T^{-\frac{1}{2}}x))\mathcal{Z}^\beta_{s\ell(T)}(x)\\
&\qquad\times\int_{\mathbb{R}^2}\xi(\mathrm{d}s,\mathrm{d}b)\int_{\mathbb{R}^2}\rho_s(z-x)\phi(z-b)\overleftarrow{\mathcal{Z}}^\beta_{s,s\ell(T)}(z)\,\mathrm{E}_z\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right]\mathrm{d}z,
\end{aligned}\tag{3.15}$$

and set

$$\mathcal{M}_s(x)=\mathcal{M}_s^{(t,T,F,\beta,u_0)}(x):=\begin{cases}\displaystyle\int_{tT_\varepsilon\ell(T_\varepsilon)}^s\mathrm{d}\mathcal{M}_u(x) & (s\geq tT_\varepsilon\ell(T_\varepsilon)),\\ 0 & (0\leq s\leq tT_\varepsilon\ell(T_\varepsilon)).\end{cases}$$

Heuristically, $d\mathcal{M}_s(x)$ is an approximation of $dG_s(x)$ following (3.13).

The following proposition computes the covariances of $\mathcal{M}_s$:

Proposition 3.5. *Suppose $u_0^{(1)},\cdots,u_0^{(n)}\in\mathfrak{C}$, $\hat{\beta}^{(1)},\cdots,\hat{\beta}^{(n)}\in(0,1)$ and $F_1,\cdots,F_n\in\mathfrak{F}$. For any test function $f_1,\cdots,f_n\in C_c^\infty(\mathbb{R}^2)$, as $\varepsilon\to0$*

$$\frac{1}{\beta_\varepsilon^{(i)}}\int_{\mathbb{R}^2}f_i(x)\,\mathcal{M}^{(i)}_{T_\varepsilon t}(\sqrt{T_\varepsilon}x)\mathrm{d}x\xrightarrow{(d)}\mathscr{U}(t,f_i,F_i,\hat{\beta}^{(i)},u_0^{(i)}).\tag{3.16}$$

The following proposition states that $\mathrm{d}G_s(x)$ can be replaced by $\mathrm{d}\mathcal{M}_s(x)$, which concludes the proof of Proposition 3.1:

Proposition 3.6. *For any test function f and $s>0$,*

$$\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\left|\int_{\mathbb{R}^2}f(x)\left(G^{(t,T,F,\beta,u_0)}_{sT_\varepsilon}(\sqrt{T_\varepsilon}x)-\mathcal{M}^{(t,T,F,\beta,u_0)}_{sT_\varepsilon}(\sqrt{T_\varepsilon}x)\right)\mathrm{d}x\right|\right]\to0,$$

as $\varepsilon\to0$.

The proof of Proposition 3.5 is given in the following subsection and the proof of Proposition 3.6 is given in Subsection 3.3.

3.2 Proof of Proposition 3.5

We will focus only on the quadratic variation of $G_{sT}(x\sqrt{T})$ to make our arguments simple. Readers can easily replace the quadratic variation by the cross variation.

To prove Proposition 3.5, we will show the following two lemmas:

Lemma 3.7. *Let $0<\tau_0\leq\tau\leq t$. Then, as $\varepsilon\to0$,*

$$\begin{aligned}
&\frac{1}{\beta_\varepsilon^2}\int_{T_\varepsilon\tau_0}^{T_\varepsilon\tau}\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}yf(x)f(y)\mathrm{d}\left\langle\mathcal{M}(x_{T_\varepsilon}),\mathcal{M}(y_{T_\varepsilon})\right\rangle_s\\
&\xrightarrow{L^1}\frac{1}{1-\hat{\beta}^2}\int_{\tau_0}^{\tau}\mathrm{d}s\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}yf(x)f(y)I(x)I(y)\int_{\mathbb{R}^2}\mathrm{d}z\rho_\sigma(x-z)\rho_\sigma(y-z)\bar{u}(t-\sigma,z)^2,
\end{aligned}\tag{3.17}$$

where we set $x_{T_\varepsilon}=x\sqrt{T_\varepsilon}$ for $x\in\mathbb{R}^2$ and $I(x)=I^{(t,F,\hat{\beta},u_0)}(x)$ is defined by (1.11).

Lemma 3.7 with Theorem 3.3 implies that the centered martingale $\left(\int_{\mathbb{R}^2}\frac{1}{\beta}f(x)(\mathcal{M}_{T\tau}(x_T)-\mathcal{M}_{T\tau_0}(x_T))\,\mathrm{d}x\right)_{\tau_0\leq\tau\leq t}$ converges in distribution to a Gaussian process with covariance given by the RHS of (3.17).

Lemma 3.8.

$$\lim_{\tau_0\to0}\varlimsup_{\varepsilon\to0}\mathbb{E}\left[\frac{1}{\beta_\varepsilon^2}\left(\int f(x)\mathcal{M}_{T_\varepsilon\tau_0}(x_{T_\varepsilon})\mathrm{d}x\right)^2\right]=0.\tag{3.18}$$

Letting $\tau_0\to0$ and $\tau=t$, the RHS of (3.17) is exactly the covariance function of the Gaussian process $\mathscr{U}_t(f,F,\hat{\beta},u_0)$. Combining it with Lemma 3.8 implies Proposition 3.5.

3.2.1 Proof of Lemma 3.7

The proof of Lemma 3.7 is divided into several steps.

Recall that $x_T = \sqrt{T}\,x$. We can write by the Markov property and (3.4) that

$$\begin{aligned}
&\frac{1}{\beta^2}\int_{T\tau_0}^{T\tau}\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}y f(x)f(y)\mathrm{d}\left\langle \mathcal{M}(x_T),\mathcal{M}(y_T)\right\rangle_s \mathrm{d}x\mathrm{d}y\\
&=\int_{T\tau_0}^{T\tau}\mathrm{d}s\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}y f(x)f(y)I_s^{(T)}(x)I_s^{(T)}(y)\\
&\qquad\times\int_{(\mathbb{R}^2)^2}\mathrm{d}z_1\mathrm{d}z_2\rho_s(z_1-x_T)\rho_s(z_2-y_T)V(z_1-z_2)\\
&\qquad\qquad\times\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_1)\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_2)\mathrm{E}_{z_1}\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right]\mathrm{E}_{z_2}\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right]\\
&=T\int_{\tau_0}^{\tau}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}y f(x)f(y)I_{T\sigma}^{(T)}(x)I_{T\sigma}^{(T)}(y)\\
&\qquad\times\int_{(\mathbb{R}^2)^2}\mathrm{d}z\mathrm{d}w\rho_\sigma(z-x)\rho_\sigma(w-y)V(z_T-w_T)\\
&\qquad\qquad\times\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(z_T)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\bar{u}(t-\sigma,z)\bar{u}(t-\sigma,w).
\end{aligned}$$

We define for $x,y\in\mathbb{R}^2$ and $\tau_0\le\sigma\le\tau$

$$\begin{aligned}
\Psi_\sigma^T(x,y)=T\int_{(\mathbb{R}^2)^2}&\mathrm{d}z\mathrm{d}w\rho_\sigma(z-x)\rho_\sigma(w-y)V(z_T-w_T)\\
&\times\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(z_T)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\bar{u}(t-\sigma,z)\bar{u}(t-\sigma,w).
\end{aligned}$$

Lemma 3.9. *For each $x,y\in\mathbb{R}^2$, $\sigma\in[\tau_0,\tau]$,*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\left|\Psi_\sigma^{T_\varepsilon}(x,y)-\Psi_\sigma(x,y)\right|\right]=0,$$

where

$$\Psi_\sigma(x,y)=\frac{1}{1-\hat\beta^2}\int_{\mathbb{R}^2}\mathrm{d}w\rho_\sigma(w-x)\rho_\sigma(w-y)\bar{u}(t-\sigma,z)^2.$$

Combining this with Lemma 2.9 and (1.7), we can see by the dominated convergence theorem that

$$\begin{aligned}
&\mathbb{E}\left[\int_{\tau_0}^{\tau}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\left|f(x)f(y)I_{\sigma T}^{(T)}(x)I_{\sigma T}^{(T)}(y)\right|\left|\Psi_\sigma^T(x,y)-\Psi_\sigma(x,y)\right|\right]\\
&=\int_{\tau_0}^{\tau}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}|f(x)f(y)|\,\mathbb{E}\left[\left|I_{\sigma T}^{(T)}(x)I_{\sigma T}^{(T)}(y)\right|\right]\mathbb{E}\left[\left|\Psi_\sigma^T(x,y)-\Psi_\sigma(x,y)\right|\right]\to 0.
\end{aligned}$$

Lemma 3.10. *For any test function $f\in C_c^\infty(\mathbb{R}^2)$,*

$$\begin{aligned}
&\int_{\tau_0}^{\tau}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}y f(x)f(y)I_{\sigma T_\varepsilon}^{(T_\varepsilon)}(x)I_{\sigma T_\varepsilon}^{(T_\varepsilon)}(y)\Psi_\sigma(x,y)\\
&\approx_{L^1}\int_{\tau_0}^{\tau}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}y f(x)f(y)I(x)I(y)\Psi_\sigma(x,y),
\end{aligned}$$

as $\varepsilon\to 0$, where the $\approx_{L^p}$ sign means that the difference between the left and the right sides goes to 0 in L^p-sense.

Proof of Lemma 3.9. (Step 1) We first prove that for each $x, y \in \mathbb{R}^2$ and $\sigma \in [\tau_0, \tau]$,

$$\begin{aligned}\Psi_\sigma^T(x,y) &\approx_{L^1} \int_{\mathbb{R}^2} \mathrm{d}w\mathrm{d}v\, \rho_\sigma(w-x)\rho_\sigma(w-y)V(v)\overleftarrow{\mathcal{Z}}_{T\sigma,\ell(T\sigma)}(w_T)^2\bar{u}(t-\sigma,w)^2 \\ &= \int_{\mathbb{R}^2} \mathrm{d}w \rho_\sigma(w-x)\rho_\sigma(w-y)\overleftarrow{\mathcal{Z}}_{T\sigma,\ell(T\sigma)}(w_T)^2\bar{u}(t-\sigma,w)^2.\end{aligned} \tag{3.19}$$

Letting $z = w + \frac{v}{\sqrt{T}}$,

$$\begin{aligned}\Psi_\sigma^T(x,y) = \int_{(\mathbb{R}^2)^2} &\mathrm{d}w\mathrm{d}v\, \rho_\sigma(w-y)V(v)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\bar{u}(t-\sigma,w) \\ &\times \rho_\sigma(w+\frac{v}{\sqrt{T}}-x)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T+v)\bar{u}\left(t-\sigma, w+\frac{v}{\sqrt{T}}\right).\end{aligned}$$

We note that

$$\begin{aligned}&\Bigg|\rho_\sigma\left(w+\frac{v}{\sqrt{T}}-x\right)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T+v)\bar{u}\left(t-\sigma,w+\frac{v}{\sqrt{T}}\right) \\ &\qquad\qquad - \rho_\sigma(w-x)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\bar{u}(t-\sigma,w)\Bigg| \\ &\leq \left|\rho_\sigma\left(w+\frac{v}{\sqrt{T}}-x\right)-\rho_\sigma(w-x)\right|\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T+v)\bar{u}\left(t-\sigma,w+\frac{v}{\sqrt{T}}\right) \\ &\quad + \rho_\sigma(w-x)\left|\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T+v)-\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\right|\bar{u}(t-\sigma,w) \\ &\quad + \rho_\sigma(w-x)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\left|\bar{u}\left(t-\sigma,w+\frac{v}{\sqrt{T}}\right)-\bar{u}(t-\sigma,w)\right|,\end{aligned}$$

all of which converge to 0 as $T \to \infty$ uniformly for $v \in \mathrm{supp}(V)$ in L^1 by Lemma 2.16 and continuity of $\rho_\sigma(\cdot)$ and $\bar{u}(t-\sigma,\cdot)$. Since $\bar{u}$ and ρ_σ is bounded for $\sigma \in [\tau_0, \tau]$, using (2.5) and $\int V(v)\mathrm{d}v = 1$, we have (3.19) by the dominated convergence theorem.

Since we have $\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(w) \overset{(d)}{=} \mathcal{Z}_{s\ell(T)}(w)$ for each $w \in \mathbb{R}^2$ and $s > 0$, it is enough from (2.5) to show that for each $\sigma \in [\tau_0, \tau]$ and $x, y \in \mathbb{R}^2$,

$$\begin{aligned}&\int_{\mathbb{R}^2} \mathrm{d}w \rho_\sigma(w-x)\rho_\sigma(w-y)\mathcal{Z}_{T\sigma\ell(T)}(w_T)^2\bar{u}(t-\sigma,w)^2 \\ &\approx_{L^1} \int_{\mathbb{R}^2} \mathrm{d}w \rho_\sigma(w-x)\rho_\sigma(w-y)\mathbb{E}\left[\mathcal{Z}_{T\sigma\ell(T)}(w_T)^2\right]\bar{u}(t-\sigma,w)^2.\end{aligned} \tag{3.20}$$

(Step 2) We prove that for each $\sigma \in [\tau_0, \tau]$ and $w \in \mathbb{R}^2$,

$$\mathbb{E}\left[\left(\mathcal{Z}_{T\sigma\ell(T)}(w_T)\right)^2 - \left(\left(\widetilde{\mathcal{Z}}_{T\sigma\ell(T),\ell'(\sigma,T)}(w_T)\right)^2 \wedge (\ell(T))^{-\frac{1}{2}}\right)\right] \to 0, \tag{3.21}$$

where we define for $t \geq 0$, $r > 0$, $z \in \mathbb{R}^2$,

$$\widetilde{\mathcal{Z}}_{t,r}(z) = \widetilde{\mathcal{Z}}^\beta_{t,r}(z) = \mathrm{E}_z\left[\Phi_t^\beta(B) : \mathtt{F}_{t,r}(B,z)\right], \tag{3.22}$$

and $\mathtt{F}_{t,r}(B,z)$ is the event that Brownian motion B does not escape from the open ball $B(z,r) = \{x \in \mathbb{R}^2 : |x-y| < r\}$ up to time t;

$$\mathtt{F}_{t,r}(B,z) = \{B_s \in B(z,r) \text{ for any } s \in [0,t]\},$$

and we set

$$\ell'(\sigma,T) = \sqrt{T\sigma}\ell(T)^{\frac{1}{4}}. \tag{3.23}$$

Also, we denote by $\mathcal{V}^{\beta}_{T,\sigma}(w) = \left(\widetilde{\mathcal{Z}}^{\beta}_{(T\sigma\ell(T),\ell'(\sigma,T))}(w_T)\right)^2 \wedge (\ell(T))^{-\frac{1}{2}}$ for simplicity. The cut-off in the definition of $\widetilde{\mathcal{Z}}$ is introduced so that $\widetilde{\mathcal{Z}}_{t,r}(z)$ and $\widetilde{\mathcal{Z}}_{t,r}(z')$ are independent if $|z-z'| > 2r + 2R_\phi$. We find

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathcal{Z}_{T\sigma\ell(T)}(w_T)\right)^2 - \mathcal{V}^{\beta}_{T,\sigma}(w)\right] \\
&\leq \mathbb{E}\left[\left(\mathcal{Z}_{T\sigma\ell(T)}(w_T)\right)^2 - \left(\widetilde{\mathcal{Z}}_{T\sigma\ell(T),\ell'(\sigma,T)}(w_T)\right)^2\right] \\
&+ \mathbb{E}\left[\left(\widetilde{\mathcal{Z}}_{T\sigma\ell(T),\ell'(\sigma,T)}(w_T)\right)^2 : \left(\widetilde{\mathcal{Z}}_{T\sigma\ell(T),\ell'(\sigma,T)}(w_T)\right)^2 \geq (\ell(T))^{-\frac{1}{2}}\right],
\end{aligned}$$

and the last term tends to 0 as $T \to \infty$ and thus $\ell(T) \to 0$ by Lemma 2.5. Furthermore, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathcal{Z}_{T\sigma\ell(T)}(w_T)\right)^2 - \left(\widetilde{\mathcal{Z}}_{T\sigma\ell(T),\ell'(\sigma,T)}(w_T)\right)^2\right] \\
&\leq 4\mathbb{E}\left[\left(\mathcal{Z}_{T\sigma\ell(T)}(w_T)\right)^2\right] \mathbb{E}\left[\mathrm{E}_{w_T}\left[\Phi_{T\sigma\ell(T)}(B) : \mathtt{F}_{T\sigma\ell(T),\ell'(\sigma,T)}(B,w_T)^c\right]^2\right],
\end{aligned}$$

and this is bounded from above due to a similiar computation to (2.4) and Lemma 2.1 by

$$C\mathrm{E}_{w_T} \otimes \mathrm{E}_{w_T}\left[\exp\left(\beta^2 \int_0^{T\sigma\ell(T)} V(B_s - B'_s)\mathrm{d}s\right) : \mathtt{F}_{T\sigma\ell(T),\ell'(\sigma,T)}(B,w_T)^c\right]. \tag{3.24}$$

Also, it is easy to see from Hölder's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that (3.24) is bounded from above by

$$\begin{aligned}
&C\mathrm{E}_{w_T} \otimes \mathrm{E}_{w_T}\left[\exp\left(\beta^2 p \int_0^{T\sigma\ell(T)} V(B_s - B'_s)\mathrm{d}s\right)\right]^{1/p} \mathrm{P}_{w_T}\left(\mathtt{F}_{T\sigma\ell(T),\ell'(\sigma,T)}(B,w_T)^c\right)^{1/q} \\
&\leq C\mathrm{P}_{w_T}\left(\mathtt{F}_{T\sigma\ell(T),\ell'(\sigma,T)}(B,w_T)^c\right)^{1/q} \to 0,
\end{aligned}$$

where in the last line we have used the fact that there exists a constant $p>1$ such that

$$\varlimsup_{T\to\infty} \mathrm{E}_x \otimes \mathrm{E}_x\left[\exp\left(\beta^2 p \int_0^{T\sigma\ell(T)} V(B_s - B'_s)\mathrm{d}s\right)\right] < \infty. \tag{3.25}$$

Thus, (3.21) follows.

Then, (3.21) implies that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{d}w \rho_\sigma(w-x)\rho_\sigma(w-y)\left(\mathcal{Z}_{T\sigma\ell(T)}(w_T)\right)^2 \bar{u}(t-\sigma,w)^2 \\
&\approx_{L^1} \int_{\mathbb{R}^2} \mathrm{d}w \rho_\sigma(w-x)\rho_\sigma(w-y)\mathcal{V}^{\beta}_{T,\sigma}(w)\bar{u}(t-\sigma,w)^2,
\end{aligned} \tag{3.26}$$

and

$$\int_{\mathbb{R}^2} \mathrm{d}w \rho_\sigma(w-x)\rho_\sigma(w-y)\mathbb{E}\left[\mathcal{V}^{\beta}_{T,\sigma}(w)\right]\bar{u}(t-\sigma,w)^2 \approx_{L^1} \text{RHS of (3.20)}. \tag{3.27}$$

(Step 3) We end the proof by showing that the right hand side in (3.26) is approximated by the left hand side in (3.27) in the L^1-sense.

First, we remark that if $|w_T - w'_T| > 2(\ell'(\sigma,T) + R_\phi)$, then

$$\mathrm{Cov}\left(\mathcal{V}^{\beta}_{T,\sigma}(w_T), \mathcal{V}^{\beta}_{T,\sigma}(w'_T)\right) = 0,$$

where R_ϕ is a constant with $\mathrm{supp}\phi \subset B(0,R_\phi)$.

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^2} \mathrm{d}w \rho_\sigma(w-x)\rho_\sigma(w-y)\left(\mathcal{V}^\beta_{T,\sigma}(w_T)-\mathbb{E}\left[\mathcal{V}^\beta_{T,\sigma}(w_T)\right]\right)\bar{u}(t-\sigma,w)^2\right)^2\right]\\
&=\int_{(\mathbb{R}^2)^2}\mathrm{d}w\mathrm{d}w'\rho_\sigma(w-x)\rho_\sigma(w-y)\rho_\sigma(w'-x)\rho_\sigma(w'-y)\\
&\qquad\times \mathrm{Cov}\left(\mathcal{V}^\beta_{T,\sigma}(w_T),\mathcal{V}^\beta_{T,\sigma}(w'_T)\right)\bar{u}(t-\sigma,w)^2\bar{u}(t-\sigma,w')^2\\
&\leq \int_{|w_T-w'_T|\leq 2(\ell'(\sigma,T)+R_V)}\mathrm{d}w\mathrm{d}w'\rho_\sigma(w-x)\rho_\sigma(w-y)\rho_\sigma(w'-x)\rho_\sigma(w'-y)\\
&\qquad\times \mathbb{E}\left[\mathcal{V}^\beta_{T,\sigma}(0)^2\right]\bar{u}(t-\sigma,w)^2\bar{u}(t-\sigma,w')^2\\
&\leq C\ell(T)^{\frac12}\mathbb{E}\left[\mathcal{V}^\beta_{T,\sigma}(0)^2\right].
\end{aligned}
$$

Thus, it is enough to show that

$$\lim_{T\to\infty}\ell(T)^{\frac12}\mathbb{E}\left[\mathcal{V}^\beta_{T,\sigma}(0)^2\right]=0,$$

which follows from Lemma 2.5 and the following:

Lemma 3.11. *[CN21, Lemma 3.3] Let $(X_k)_{k\in\mathbb{N}}$ be a non-negative, uniformly integrable family of random variables. Then, for any sequence $a_k\to\infty$, $a_k^{-1}\mathbb{E}[(X_k\wedge a_k)^2]\to 0$ as $k\to\infty$.* □

Proof of Lemma 3.10. The proof is essentially the same as in Lemma 3.9. Indeed, we can approximate $I^{(T)}_{\sigma T}(x)$ by $\mathbb{E}\left[F'(\mathcal{Z}_{T\sigma\ell(T)})\mathcal{Z}_{T\sigma\ell(T)}\right]$ due to the same arguments as in (Step 2) and (Step 3) in the proof of Lemma 3.9 and (1.7). In particular, we remark that its expectation converges to $I(x)$ due to (1.10) and (1.7). We omit the detail. □

3.2.2 Proof of Lemma 3.8

Proof of Lemma 3.8. By (1.7) and (2.5), for $s\leq tT_\varepsilon$,

$$\mathbb{E}\left[(I^{(T)}_s(x))^2\right]\leq C\mathbb{E}\left[\left(1+\mathcal{Z}_{s\ell(T)}(x_T)\right)^2\right]\leq C_t,$$

with some constant $C=C_t>0$ independent of x,s,ε. Hence, we have

$$
\begin{aligned}
&\mathbb{E}\left[\frac{1}{\beta^2}\left(\int f(x)\mathcal{M}_{T\tau_0}(x_T)\mathrm{d}x\right)^2\right]\\
&=\int_{t\ell(T)}^{\tau_0}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}y f(x)f(y)\mathbb{E}\left[I^{(T)}_s(x)I^{(T)}_s(y)\right]\\
&\qquad\times\int_{\mathbb{R}^2}\mathrm{d}w\int_{\mathbb{R}^2}\mathrm{d}v\rho_\sigma\left(w+\frac{v}{\sqrt{T}}-x\right)\rho_\sigma(w-y)V(v)\\
&\qquad\quad\times\mathbb{E}\left[\overleftarrow{\mathcal{Z}}^\beta_{T\sigma,T\sigma\ell(T)}(w_T+v)\overleftarrow{\mathcal{Z}}^\beta_{T\sigma,T\sigma\ell(T)}(w_T)\right]\bar{u}\left(t-\sigma,w+\frac{v}{\sqrt{T}}\right)\bar{u}(t-\sigma,w)\\
&\leq C\int_{t\ell(T)}^{\tau_0}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}x\mathrm{d}y|f(x)f(y)|\int_{\mathbb{R}^2}\mathrm{d}w\int_{\mathbb{R}^2}\mathrm{d}v\rho_\sigma\left(w+\frac{v}{\sqrt{T}}-x\right)\rho_\sigma(w-y)V(v)\\
&\leq C\left(\tau_0-t\ell(T)\right),
\end{aligned}
$$

where we have used $\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(x) \stackrel{(d)}{=} \mathcal{Z}_{T\sigma,T\sigma\ell(T)}(x)$ with (2.5) and $\|u_0\|_\infty < \infty$ in the second line, and

$$\begin{aligned}&\int_{(\mathbb{R}^2)^2} \mathrm{d}x\mathrm{d}y|f(x)f(y)| \int_{\mathbb{R}^2} \mathrm{d}w \int_{\mathbb{R}^2} \mathrm{d}v\rho_\sigma\left(w+\frac{v}{\sqrt{T}}-x\right)\rho_\sigma(w-y)V(v)\\ &= \int_{(\mathbb{R}^2)^2} \mathrm{d}x\mathrm{d}vV(v)|f(y)| \int_{\mathbb{R}^2} \mathrm{d}x|f(x)|\rho_{2\sigma}\left(y-x+\frac{v}{\sqrt{T}}\right) \leq \|f\|_1\|f\|_\infty\end{aligned}$$

in the last line. □

3.3 Proof of Proposition 3.6

For $s \geq tT_\varepsilon\ell(T_\varepsilon)$, writing

$$G_s(x) - \mathcal{M}_s(x) = G_{tT_\varepsilon\ell(T_\varepsilon)}(x) + (G_s(x) - G_{tT_\varepsilon\ell(T_\varepsilon)}(x) - \mathcal{M}_s(x)), \tag{3.28}$$

the last term is a martingale which is equal to 0 at time $s = tT_\varepsilon\ell(T_\varepsilon)$.

The time $tT\ell(T_\varepsilon)$ is much smaller than tT so that the first term is negligible.

Lemma 3.12.

$$\lim_{\varepsilon\to 0}\frac{1}{\beta_\varepsilon^2}\mathbb{E}\left[\left(\int f(x)G_{tT_\varepsilon\ell(T_\varepsilon)}(x_{T_\varepsilon})\mathrm{d}x\right)^2\right] = 0.$$

Then, we will prove that the second term in (3.28) converges to 0 in the following sense:

Lemma 3.13.

$$\lim_{\varepsilon\to 0}\frac{1}{\beta_\varepsilon^2}\mathbb{E}\left[\left(\int f(x)\left(\left(G_{tT_\varepsilon}(x_{T_\varepsilon}) - G_{tT_\varepsilon\ell(T_\varepsilon)}(x_{T_\varepsilon})\right) - \mathcal{M}_{tT_\varepsilon}(x_{T_\varepsilon})\right)\mathrm{d}x\right)^2\right] = 0.$$

Lemma 3.12 and Lemma 3.13 conclude Proposition 3.6.

3.3.1 Proof of Lemma 3.12

To prove Lemma 3.12, we will introduce a new martingale: let

$$n(T) = \sqrt{\ell(T)} = \exp\left(-\frac{1}{2}(\log T)^{\frac{1}{2}-\delta}\right)$$

and

$$\widetilde{\mathcal{W}}_s(x) = \mathrm{E}_x\left[\Phi_s(B)u_0\left(\frac{B_{tT}}{\sqrt{T}}\right) : \mathtt{F}_{tT\ell(T),\sqrt{tTn(T)}}(B,x)\right], \tag{3.29}$$

where recall the notation $\mathtt{F}$ from (3.22). $\widetilde{W}_s(x)$ is the partition function in which the underlying Brownian motion is constrained to a tube up to time $tT\ell(T)$. This enables us to use the independence of $\widetilde{W}(x)$ and $\widetilde{W}(y)$ if the distance between x and y is sufficiently large.

Lemma 3.12 is concluded by the following two lemmas.

Lemma 3.14. *For any $f \in C_c^\infty(\mathbb{R}^2)$, $t > 0$, $x \in \mathbb{R}^2$, and $\hat{\beta} \in (0,1)$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\left|G_{tT_\varepsilon\ell(T_\varepsilon)}(x) - \int_0^{tT_\varepsilon\ell(T_\varepsilon)} F'(\mathcal{W}_u(x))\mathrm{d}\widetilde{\mathcal{W}}_u(x)\right|\right] = 0.$$

Lemma 3.15. *For any $f \in C_c^\infty(\mathbb{R}^2)$, $t > 0$, and $\hat{\beta} \in (0,1)$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\left|\int_{\mathbb{R}^2}\mathrm{d}xf(x)\int_0^{tT_\varepsilon\ell(T_\varepsilon)} F'(\mathcal{W}_u(x_{T_\varepsilon}))\mathrm{d}\widetilde{\mathcal{W}}_u(x_{T_\varepsilon})\right|\right] = 0.$$

Proof of Lemma 3.14. We have

$$\begin{aligned}
&\mathbb{E}\left[\left|G_{tT\ell(T)}(x)-\int_0^{tT\ell(T)}F'(\mathcal{W}_u(x))\mathrm{d}\widetilde{\mathcal{W}}_u(x)\right|^2\right]\\
&=\mathbb{E}\left[\left(\int_0^{tT\ell(T)}F'\left(\mathcal{W}_u(x)\right)\mathrm{d}\mathcal{W}_u(x)-\int_0^{tT\ell(T)}F'\left(\mathcal{W}_u(x)\right)\mathrm{d}\widetilde{\mathcal{W}}_u(x)\right)^2\right]\\
&=\mathbb{E}\left[\int_0^{tT\ell(T)}F'\left(\mathcal{W}_u(x)\right)^2\mathrm{d}\left\langle\mathcal{W}(x)-\widetilde{\mathcal{W}}(x)\right\rangle_u\right].
\end{aligned}$$

Then, the last expectation is written by

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{tT\ell(T)}\mathrm{d}uF'\left(\mathcal{W}_u(x)\right)^2\right.\\
&\qquad\left.\times\mathrm{E}_x\otimes\mathrm{E}_x\left[V(B_u-\widetilde{B}_u)\Phi_u(B_u)\Phi_s(\widetilde{B}_u)u_0\left(\frac{B_{tT}}{\sqrt{T}}\right)u_0\left(\frac{\widetilde{B}_{tT}}{\sqrt{T}}\right):A_T(B,\widetilde{B},x)\right]\right],
\end{aligned}\tag{3.30}$$

where we set

$$A_T(B,\widetilde{B},x):=\mathtt{F}_{tT\ell(T),\sqrt{tTn(T)}}(B,x)^c\cap\mathtt{F}_{tT\ell(T),\sqrt{tTn(T)}}(\widetilde{B},x)^c.$$

By using $|V(B_s-\widetilde{B}_s)|\le\|V\|_\infty$ and (1.7), (3.30) is bounded from above by

$$\begin{aligned}
&\|V\|_\infty\mathbb{E}\left[\int_0^{tT\ell(T)}\mathrm{d}uF'\left(\mathcal{W}_u(x)\right)^2\left(\mathcal{W}_u(x)-\widetilde{\mathcal{W}}_u(x)\right)^2\right]\\
&\le C\mathbb{E}\left[\int_0^{tT\ell(T)}\mathrm{d}u\left(\frac{1}{\mathcal{W}_u(x)}+1\right)^2\left(\mathcal{W}_u(x)-\widetilde{\mathcal{W}}_u(x)\right)^2\right]\\
&\le C\int_0^{tT\ell(T)}\mathrm{d}u\mathbb{E}\left[\frac{\mathcal{W}_u(x)-\widetilde{\mathcal{W}}_u(x)}{\mathcal{W}_u(x)}+\mathcal{W}_u(x)-\widetilde{\mathcal{W}}_u(x)+\left(\mathcal{W}_u(x)-\widetilde{\mathcal{W}}_u(x)\right)^2\right].
\end{aligned}\tag{3.31}$$

It is easy to see that

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathcal{W}_u(x)-\widetilde{\mathcal{W}}_u(x)\right)^2\right]\\
&=\mathrm{E}_x\otimes\mathrm{E}_x\left[\exp\left(\int_0^u\beta^2V(B_s-\widetilde{B}_s)\mathrm{d}s\right)u_0\left(\frac{B_{tT}}{\sqrt{T}}\right)u_0\left(\frac{\widetilde{B}_{tT}}{\sqrt{T}}\right):A_T(B,\widetilde{B},x)\right]\\
&\le C\mathrm{E}_x\otimes\mathrm{E}_x\left[\exp\left(\int_0^u\beta^2V(B_s-\widetilde{B}_s)\mathrm{d}s\right):A_T(B,\widetilde{B},x)\right].
\end{aligned}\tag{3.32}$$

Then, Hölder's inequality yields that

$$\begin{aligned}
&\mathrm{E}_x\otimes\mathrm{E}_x\left[\exp\left(\int_0^u\beta^2V(B_s-\widetilde{B}_s)\mathrm{d}s\right):A_T(B,\widetilde{B},x)\right]\\
&\le\mathrm{E}_x\otimes\mathrm{E}_x\left[\exp\left(\int_0^u p\beta^2V(B_s-\widetilde{B}_s)\mathrm{d}s\right)\right]^{\frac{1}{p}}\mathrm{P}_x\left(\mathtt{F}_{tT\ell(T),\sqrt{tTn(T)}}(B,x)^c\right)^{\frac{2}{q}},
\end{aligned}\tag{3.33}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ are chosen such that

$$\varlimsup_{T\to\infty}\mathrm{E}_x\otimes\mathrm{E}_x\left[\exp\left(p\beta^2\int_0^{Tt}V(B_u-\widetilde{B}_u)\mathrm{d}u\right)\right]<\infty.$$

Hence, (3.32) tends to 0 as $T \to \infty$ since we know

$$\mathrm{P}_x\left(F_{tT\ell(T),\sqrt{tTn(T)}}(B,x)^c\right) \le C\exp\left(-\frac{n(T)}{4\ell(T)}\right), \tag{3.34}$$

which decays faster than any polynomial of T. By using the Cauchy-Schwarz inequality with Lemma 2.9, we can see that (3.31) converges to 0 as $T \to \infty$. ☐

Proof of Lemma 3.15. It is enough to show that

$$\begin{aligned}
&\frac{1}{\beta^2}\mathbb{E}\left[\left(\int_{\mathbb{R}^2}\mathrm{d}xf(x)\int_0^{tT\ell(T)}F'(\mathcal{W}_s(x_T))\mathrm{d}\widetilde{\mathcal{W}}_s(x_T)\right)^2\right]\\
&=\frac{1}{\beta^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathrm{d}x\mathrm{d}yf(x)f(y)\mathbb{E}\left[\int_0^{tT\ell(T)}F'(\mathcal{W}_s(x_T))F'(\mathcal{W}_s(y_T))\mathrm{d}\langle\widetilde{\mathcal{W}}(x_T),\widetilde{\mathcal{W}}(y_T)\rangle_s\right]\to 0,
\end{aligned}$$

as $T \to \infty$. Since $\widetilde{\mathcal{W}}_s(x)$ and $\widetilde{\mathcal{W}}_s(y)$ are independent and hence $\mathrm{d}\langle\widetilde{\mathcal{W}}(x),\widetilde{\mathcal{W}}(y)\rangle_u = 0$ if $|x-y| > 2(\sqrt{tTn(T)}+R_\phi)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^2}\mathrm{d}xf(x)\int_0^{tT\ell(T)}F'(\mathcal{W}_s(x_T))\mathrm{d}\widetilde{\mathcal{W}}_s(x_T)\right)^2\right]\\
&\le Cn(T)\mathbb{E}\left[\int_0^{tT\ell(T)}F'(\mathcal{W}_s(x))^2\mathrm{d}\langle\widetilde{\mathcal{W}}(x)\rangle_s\right]\\
&\le Cn(T)\mathbb{E}\left[\int_0^{tT\ell(T)}\left(\frac{1}{\mathcal{W}_s(x)^2}+1\right)\mathrm{d}\langle\widetilde{\mathcal{W}}(x)\rangle_s\right].
\end{aligned} \tag{3.35}$$

By construction of $\widetilde{W}_s(x)$, we have

$$Cn(T)\mathbb{E}\left[\int_0^{tT\ell(T)}\left(\frac{1}{\mathcal{W}_s(x)^2}+1\right)\mathrm{d}\langle\widetilde{\mathcal{W}}(x)\rangle_s\right]\le Cn(T)\mathbb{E}\left[\int_0^{tT\ell(T)}\left(\frac{1}{\mathcal{W}_s(x)^2}+1\right)\mathrm{d}\langle\mathcal{W}(x)\rangle_s\right],$$

and furthermore, by applying Itô's lemma to $\log\mathcal{W}(x)$ and $\mathcal{W}(x)^2$, it is bounded by

$$Cn(T)\mathbb{E}\left[\log\bar{u}(t,x)-\log\mathcal{W}_{tT\ell(T)}+\mathcal{W}_{tT\ell(T)}(x)^2\right]\le Cn(T).$$

Thus, Lemma 3.15 is concluded. ☐

3.3.2 Proof of Lemma 3.13

Define

$$\mathrm{d}\mathcal{L}_s(x) = \beta_\varepsilon\mathcal{Z}^\beta_{s\ell(T)}(x)\int_{\mathbb{R}^2}\mathrm{d}z\xi(\mathrm{d}s,\mathrm{d}b)\int_{\mathbb{R}^2}\rho_s(z-x)\phi(z-b)\overleftarrow{\mathcal{Z}}^\beta_{s,s\ell(T)}(z)\,\mathrm{E}_z\left[u_0\left(\frac{B_{tT-s}}{\sqrt{T}}\right)\right].$$

Then, we can see that

$$\mathrm{d}\mathcal{M}_s(x_T) = F'\left(\mathcal{Z}^\beta_{s\ell(T)}\bar{u}(t,x)\right)\mathrm{d}\mathcal{L}_s(x_T), \tag{3.36}$$

for $s \ge tT\ell(T)$ and $x \in \mathbb{R}^2$.

Recalling the definition of G from (3.6), we know that

$$\begin{aligned}
&G_{tT_\varepsilon}(x) - G_{tT_\varepsilon\ell(T_\varepsilon)}(x) - \mathcal{M}_s(x)\\
&=\int_{tT_\varepsilon\ell(T_\varepsilon)}F'(\mathcal{W}_s(x))d\mathcal{W}_s(x) - \int_{tT_\varepsilon\ell(T_\varepsilon)}F'(\mathcal{Z}^\beta_{s\ell(T_\varepsilon)}\bar{u}(t,x))d\mathcal{L}_s(x).
\end{aligned}$$

Lemma 3.13 follows from the next two lemmas.

Lemma 3.16. *For $t > 0$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\left|\int \mathrm{d}x f(x)\left(\int_{tT_\varepsilon\ell(T_\varepsilon)}^{tT_\varepsilon}F'(\mathcal{W}_s(x_{T_\varepsilon}))\mathrm{d}\mathcal{W}_s(x_{T_\varepsilon})-\int_{tT_\varepsilon\ell(T_\varepsilon)}^{tT_\varepsilon}F'(\mathcal{W}_s(x_{T_\varepsilon}))\mathrm{d}\mathcal{L}_s(x_{T_\varepsilon})\right)\right|\right]=0.$$

Lemma 3.17. *For $t > 0$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\left|\int \mathrm{d}x f(x)\left(\int_{tT_\varepsilon\ell(T_\varepsilon)}^{tT_\varepsilon}F'(\mathcal{W}_s(x_{T_\varepsilon}))\mathrm{d}\mathcal{L}_s(x_{T_\varepsilon})-\mathcal{M}_{tT_\varepsilon}(x_{T_\varepsilon})\right)\right|\right]=0.$$

Proof of Lemma 3.16. By the Burkholder-Davis-Gundy inequality and the trivial bound for martingale $|M_T| \leq \max_{0\leq t\leq T}|M_t|$, we have

$$\begin{aligned}&\frac{1}{\beta}\mathbb{E}\left[\left|\int \mathrm{d}x f(x)\left(\int_{tT\ell(T)}^{tT}F'(\mathcal{W}_s(x_T))\mathrm{d}\mathcal{W}_s(x_T)-\int_{tT\ell(T)}^{tT}F'(\mathcal{W}_s(x_T))\mathrm{d}\mathcal{L}_s(x_T)\right)\right|\right]\\&\leq \frac{C}{\beta}\int \mathrm{d}x|f(x)|\mathbb{E}\left[\left(\int_{tT\ell(T)}^{tT}F'(\mathcal{W}_s(x_T))^2\mathrm{d}\left\langle \mathcal{W}(x_T)-\mathcal{L}(x_T)\right\rangle_s\right)^{\frac12}\right].\end{aligned}$$

By using the Cauchy-Schwarz inequality, the right hand side is bounded from above by

$$\begin{aligned}&\frac{C}{\beta}\int \mathrm{d}x|f(x)|\mathbb{E}\left[\sup_{0\leq s\leq tT}\mathcal{Z}_s(0)^{-2}+1\right]^{\frac12}\mathbb{E}\left[\int_{tT\ell(T)}^{tT}\mathrm{d}\left\langle \mathcal{W}(x_T)-\mathcal{L}(x_T)\right\rangle_s\right]^{\frac12}\\&\leq \frac{C}{\beta}\int \mathrm{d}x|f(x)|\mathbb{E}\left[\int_{tT\ell(T)}^{tT}\mathrm{d}\left\langle \mathcal{W}(x_T)-\mathcal{L}(x_T)\right\rangle_s\right]^{\frac12},\end{aligned}$$

where we have used Doob's inequality and Lemma 2.9 in the inequality.

By boundedness of $\bar{u}$, we can see from the definitions of $\mathcal{W}$ and $\mathcal{L}$ that for $x\in\mathbb{R}^2$,

$$\begin{aligned}&\frac{1}{\beta^2}\mathbb{E}\left[\int_{tT\ell(T)}^{tT}\mathrm{d}\left\langle \mathcal{W}(x_T)-\mathcal{L}(x_T)\right\rangle_s\right]\\&=\int_{tT\ell(T)}^{tT}\mathrm{d}s\int_{(\mathbb{R}^2)^2}\mathrm{d}z_1\mathrm{d}z_2\rho_s(z_1)\rho_s(z_2)V(z_1-z_2)\bar{u}(tT-s,z_1)\bar{u}(tT-s,z_2)\\&\qquad\times\mathbb{E}\left[\left(\mathrm{E}_{0,0}^{s,z_1}[\Phi_s]-\mathcal{Z}_{s\ell(T)}(0)\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_1)\right)\left(\mathrm{E}_{0,0}^{s,z_2}[\Phi_s]-\mathcal{Z}_{s\ell(T)}(0)\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_2)\right)\right]\\&\leq C\int_{tT\ell(T)}^{tT}\mathrm{d}s\int_{(\mathbb{R}^2)^2}\mathrm{d}z_1\mathrm{d}z_2\rho_s(z_1)\rho_s(z_2)V(z_1-z_2)\\&\qquad\times\mathbb{E}\left[\left|\mathrm{E}_{0,0}^{s,z_1}[\Phi_s]-\mathcal{Z}_{s\ell(T)}(0)\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_1)\right|\left|\mathrm{E}_{0,0}^{s,z_2}[\Phi_s]-\mathcal{Z}_{s\ell(T)}(0)\overleftarrow{\mathcal{Z}}_{s,s\ell(T)}(z_2)\right|\right].\end{aligned}$$

By the Cauchy-Schwarz inequality and $|xy|\leq\frac{x^2+y^2}{2}$, the right hand side is bounded by

$$\begin{aligned}&\frac{C}{2}\int_{t\ell(T)}^{t}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}w\mathrm{d}v\rho_\sigma(w)\rho_\sigma(w+\frac{v}{\sqrt{T}})V(v)\\&\qquad\times\left(\mathbb{E}\left[\left(\mathrm{E}_{0,0}^{T\sigma,w_T}[\Phi_{T\sigma}]-\mathcal{Z}_{s\ell(T)}(0)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\right)^2\right]\right.\\&\qquad\quad\left.+\mathbb{E}\left[\left(\mathrm{E}_{0,0}^{T\sigma,w_T+v}[\Phi_{T\sigma}]-\mathcal{Z}_{T\sigma\ell(T)}(0)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T+v)\right)^2\right]\right).\end{aligned}$$

By changing variables $w+\frac{v}{\sqrt{T}}=w'$ for the second term and the symmetry of V, it is equal to

$$C\int_{t\ell(T)}^{t}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}w\mathrm{d}v\rho_\sigma(w)\rho_\sigma(w+\frac{v}{\sqrt{T}})V(v)\mathbb{E}\Big[\Big(\mathrm{E}_{0,0}^{T\sigma,w_T}\left[\Phi_{T\sigma}\right]-\mathcal{Z}_{T\sigma\ell(T)}(0)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(w_T)\Big)^2\Big],$$

and furthermore Lemma 2.10 allows us to bound it from above by

$$\begin{aligned}
&C\int_{t\ell(T)}^{t}\mathrm{d}\sigma\int_{|w_T|\le\sqrt{\sigma T\log(\sigma T)}}\mathrm{d}w\frac{\rho_\sigma(w)}{\sigma}\\
&\qquad\times\left(\ell(T)-\frac{\log\ell(T)}{\log T}+\frac{\sqrt{\log(\sigma T)}\log(\sigma T\ell(T))}{\sqrt{\sigma T}\log T}+\frac{\ell(T)\log(\sigma T)}{\log T}\right)\\
&\qquad+C\int_{t\ell(T)}^{t}\mathrm{d}\sigma\int_{|w_T|\ge\sqrt{\sigma T\log(\sigma T)}}\mathrm{d}w\frac{\rho_\sigma(w)}{\sigma}\\
&\le C\int_{t\ell(T)}^{t}\mathrm{d}\sigma\frac{1}{\sigma}\left(\ell(T)-\frac{\log\ell(T)}{\log T}+\frac{\sqrt{\log(tT)}\log(tT\ell(T))}{\sqrt{\sigma T}\log T}+\frac{\ell(T)\log(tT)}{\log T}\right)\\
&\qquad+C\int_{t\ell(T)}^{t}\frac{1}{\sqrt{\sigma^3T}}\mathrm{d}\sigma.
\end{aligned}\tag{3.37}$$

Since both terms in the last line tend to 0 as $T\to\infty$ from the definition of $\ell(T)$, Lemma 3.16 is concluded. ☐

Proof of Lemma 3.17. We have from (3.36),

$$\begin{aligned}
&\frac{1}{\beta}\mathbb{E}\left[\left|\int f(x)\left(\int_{tT\ell(T)}^{tT}F'(\mathcal{W}_s(x_T))\mathrm{d}\mathcal{L}_s(x_T)-\mathcal{M}_{tT}(x_T)\right)\mathrm{d}x\right|\right]\\
&\le\frac{1}{\beta}\int|f(x)|\mathbb{E}\left[\left|\int_{tT\ell(T)}^{tT}F'(\mathcal{W}_s(x_T))\mathrm{d}\mathcal{L}_s(x_T)-\mathcal{M}_{tT}(x_T)\right|\right]\mathrm{d}x\\
&\le\frac{1}{\beta}\int\mathrm{d}x|f(x)|\mathbb{E}\left[\int_{tT\ell(T)}^{tT}\left(F'(\mathcal{W}_s(x_T))-F'(\mathcal{Z}_{s\ell(T)}(x_T)\bar{u}(t,x))\right)^2\mathrm{d}\langle\mathcal{L}(x_T)\rangle_s\right]^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\int_{tT\ell(T)}^{tT}\left(F'(\mathcal{W}_s(x_T))-F'(\mathcal{Z}_{s\ell(T)}(x_T)\bar{u}(t,x))\right)^2\mathrm{d}\langle\mathcal{L}(x_T)\rangle_s\right]\\
&=\beta^2\int_{t\ell(T)}^{t}\mathrm{d}\sigma\int_{(\mathbb{R}^2)^2}\mathrm{d}z\mathrm{d}v\rho_\sigma(z)\rho_\sigma(z+\frac{v}{\sqrt{T}})V(v)\mathrm{E}_{z_T}\left[u_0\left(\frac{B_{tT-\sigma T}}{\sqrt{T}}\right)\right]\mathrm{E}_{z_T+v}\left[u_0\left(\frac{B_{tT-\sigma T}}{\sqrt{T}}\right)\right]\\
&\quad\times\mathbb{E}\Big[\left(F'(\mathcal{W}_{T\sigma}(x_T))-F'(\mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x))\right)^2\\
&\qquad\qquad\times\mathcal{Z}_{T\sigma\ell(T)}(x_T)^2\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(z_T)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(z_T+v)\Big].
\end{aligned}$$

By (1.7), Lemma 2.5 and Lemma 2.9, we can choose $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left|F'(\mathcal{W}_{T\sigma}(x_T))-F'(\mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x))\right|^{2q}\right]^{\frac{1}{q}}\\
&\quad\times\mathbb{E}\left[\mathcal{Z}_{T\sigma\ell(T)}(x_T)^{2p}\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(z_T)^p\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(z_T+v)^p\right]^{\frac{1}{p}}\\
&\le C,
\end{aligned}$$

for some constant $C>0$ uniformly in $t\ell(T)\le\sigma\le t$, $x\in\mathbb{R}^2$ and $0<\varepsilon<\frac{1}{2}$.

Since we have $\left|F'(\mathcal{W}_{T\sigma}(x_T)) - F'(\mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x))\right| = \left|\int_{\mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x)}^{\mathcal{W}_{T\sigma}(x_T)} F''(w)\mathrm{d}w\right|$, (1.7) and Hölder's inequality yield

$$\begin{aligned}
&\mathbb{E}\left[\left|F'(\mathcal{W}_{T\sigma}(x_T)) - F'(\mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x))\right|^{2q}\right] \\
&\leq C\mathbb{E}\left[\left|\frac{1}{\mathcal{W}_{T\sigma}(x_T)} + \frac{1}{\mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x)} + 1\right|^{2q-1} \left|F'(\mathcal{W}_{T\sigma}(x_T)) - F'(\mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x))\right|\right] \\
&\leq C\mathbb{E}\left[\sup_{t\ell(T)\leq\sigma\leq t}\left\{\left|\frac{1}{\mathcal{Z}_{T\sigma}(x_T)} + \frac{1}{\mathcal{Z}_{T\sigma\ell(T)}(x_T)} + 1\right|^{2q-1}\left|\frac{1}{\mathcal{Z}_{T\sigma}(x_T)^2} + \frac{1}{\mathcal{Z}_{T\sigma\ell(T)}(x_T)^2} + 1\right|\right\}\right. \\
&\qquad\qquad \left.\times\left|\mathcal{W}_{T\sigma}(x_T) - \mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x)\right|\right].
\end{aligned}$$

Moreover, Doob's inequality, the Cauchy-Schwarz inequality, and Lemma 2.9 allow us to bound the right most term from above by

$$C\mathbb{E}\left[\left(\mathcal{W}_{T\sigma}(x_T) - \mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x)\right)^2\right]^{1/2}.$$

We have

$$\begin{aligned}
\mathcal{W}_{T\sigma}(x_T) &\approx_{L^2} \mathcal{Z}_{T\sigma\ell(T)}(x_T)\int p_\sigma(x,y)\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(y)\bar{u}(t-\sigma,y)\mathrm{d}y \\
&\approx_{L^2} \mathcal{Z}_{T\sigma\ell(T)}(x_T)\int p_\sigma(x,y)\mathbb{E}\left[\overleftarrow{\mathcal{Z}}_{T\sigma,T\sigma\ell(T)}(y)\right]\bar{u}(t-\sigma,y)\mathrm{d}y = \mathcal{Z}_{T\sigma\ell(T)}(x_T)\bar{u}(t,x),
\end{aligned}$$

where we have used Theorem 2.10 in the first approximation and a homogenization argument as in Lemma 3.9 in the second approximation. Thus, Lemma 3.17 follows by the dominated convergence theorem. □

3.4 Proof of Proposition 3.2

Our goal is to prove that

$$\frac{1}{\beta_\varepsilon}\int_{\mathbb{R}^2}\mathrm{d}x f(x)\left(\int_0^{tT_\varepsilon} F''(\mathcal{W}_s(x_{T_\varepsilon}))\mathrm{d}\langle\mathcal{W}(x_{T_\varepsilon})\rangle_s - \mathbb{E}\left[\int_0^{tT_\varepsilon} F''(\mathcal{W}_s(x_{T_\varepsilon}))\mathrm{d}\langle\mathcal{W}(x_{T_\varepsilon})\rangle_s\right]\right) \xrightarrow{L^1} 0. \tag{3.38}$$

For simplicity of notation, we set $t=1$ hereafter.

The proof is composed of four steps. In the first step, we will investigate that the influence at large time is negligible in the following sense:

Lemma 3.18 (Step 1)**.**

$$\lim_{\varepsilon\to0}\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\int_{\mathbb{R}^2}\mathrm{d}x f(x)\int_{T_\varepsilon\ell(T_\varepsilon)}^{T_\varepsilon} F''(\mathcal{W}_s(x_{T_\varepsilon}))\mathrm{d}\langle\mathcal{W}(x_{T_\varepsilon})\rangle_s\right] = 0. \tag{3.39}$$

In the second step, we will see that the contributions of "large" $\mathcal{W}(x)$ and "small" $\widetilde{\mathcal{W}}(x)$ are negligible. To do this, we introduce a stopping time:

$$\tau_{T_\varepsilon} = \tau_{T_\varepsilon}(x) := \inf\left\{s\geq 0 : W_s(x) + \widetilde{\mathcal{W}}_s(x)^{-1} > \frac{1}{\ell(T_\varepsilon)^{\frac{1}{20}}}\right\}\wedge T_\varepsilon\ell(T_\varepsilon).$$

Lemma 3.19 (Step 2)**.** *For any* $x\in\mathbb{R}^2$*,*

$$\lim_{\varepsilon\to0}\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\int_{\tau_{T_\varepsilon}(x)}^{T_\varepsilon\ell(T_\varepsilon)} |F''(\mathcal{W}_s(x))|\mathrm{d}\langle\mathcal{W}(x)\rangle_s\right] = 0.$$

In Lemma 3.22 below, we will see that the probability $\tau_{T_\varepsilon} < T_\varepsilon \ell(T_\varepsilon)$ is much smaller than β_ε^{-1}.

In the third step, we will prove that the contributions from $\mathcal{W}(x)$ and $\widetilde{\mathcal{W}}(x)$ are asymptotically identified.

Lemma 3.20 (Step 3)**.** *For any $x \in \mathbb{R}^2$,*

$$\lim_{\varepsilon\to 0} \frac{1}{\beta_\varepsilon} \mathbb{E}\left[\left| \int_0^{\tau_{T_\varepsilon}(x)} F''(\mathcal{W}_s(x)) \mathrm{d}\langle \mathcal{W}(x)\rangle_s - \int_0^{\tau_{T_\varepsilon}(x)} F''(\widetilde{\mathcal{W}}_s(x)) \mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s \right|\right] = 0.$$

In the last, we will prove the remainder is also negligible.

Lemma 3.21 (Step 4)**.**

$$\lim_{\varepsilon\to 0} \frac{1}{\beta_\varepsilon^2} \mathbb{E}\left[\left(\int_{\mathbb{R}^2} \mathrm{d}x f(x) \left(\int_0^{\tau_{T_\varepsilon}} F''(\widetilde{\mathcal{W}}_u(x_{T_\varepsilon})) \mathrm{d}\langle \widetilde{\mathcal{W}}(x_{T_\varepsilon})\rangle_u - \mathbb{E}\left[\int_0^{\tau_{T_\varepsilon}} F''(\widetilde{\mathcal{W}}_u(x_{T_\varepsilon})) \mathrm{d}\langle \widetilde{\mathcal{W}}(x_{T_\varepsilon})\rangle_u\right]\right)\right)^2\right]$$
$$= 0.$$

Putting these lemmas together, Proposition 3.2 is concluded.

Proof of Lemma 3.18. With δ from (3.11), for $C > (\|u_0\|_\infty + \|u_0^{-1}\|_\infty)^4$,

$$\begin{aligned}
&\mathbb{E}\left[\int_{T\ell(T)}^T |F''(\mathcal{W}_s(x_{T_\varepsilon}))| \, \mathrm{d}\langle \mathcal{W}(x_T)\rangle_s\right] \\
&\leq C\mathbb{E}\left[\sup_{T\ell(T)\leq s\leq T} \left(1 + \mathcal{Z}_s(x)^{-2}\right) \int_{T\ell(T)}^T \mathrm{d}\langle \mathcal{Z}(x_T)\rangle_s\right] \\
&\leq 2C\mathbb{E}\left[\sup_{T\ell(T)\leq s\leq T} \mathcal{Z}_s(x)^{-2} \mathbf{1}\left\{\sup_{T\ell(T)\leq s\leq T} \mathcal{Z}_s(x)^{-1} > (\log T)^{\delta/4}\right\} \int_{T\ell(T)}^T \mathrm{d}\langle \mathcal{Z}(x_T)\rangle_s\right] \\
&\qquad + 2C(\log T)^{\delta/2} \mathbb{E}\left[\mathbf{1}\left\{\sup_{T\ell(T)\leq s\leq T} \mathcal{Z}_s(x)^{-1} \leq (\log T)^{\delta/4}\right\} \int_{T\ell(T)}^T \mathrm{d}\langle \mathcal{Z}(x_T)\rangle_s\right].
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, Doob's inequality, and Hölder's inequality, the first expectation is bounded from above by

$$\begin{aligned}
&C\mathbb{P}\left(\sup_{T\ell(T)\leq s\leq T} \mathcal{Z}_s(x)^{-1} > (\log T)^{\delta/4}\right)^{\frac{1}{2p}} \mathbb{E}\left[\frac{1}{\mathcal{Z}_T(x)^{4p}}\right]^{\frac{1}{2p}} \mathbb{E}\left[\left(\int_{T\ell(T)}^T \mathrm{d}\langle \mathcal{Z}(x_T)\rangle_s\right)^q\right]^{\frac{1}{q}} \\
&\leq C\frac{1}{(\log T)^2} \mathbb{E}\left[\mathcal{Z}_T(x)^{-\frac{16p}{\delta}}\right]^{\frac{1}{2p}} \mathbb{E}\left[\mathcal{Z}_T(x)^{-4p}\right]^{\frac{1}{2p}} \mathbb{E}\left[\mathcal{Z}_T(x)^{2q}\right]^{\frac{1}{q}} \\
&\leq C(\log T)^{-2},
\end{aligned}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\sup_{\varepsilon<1} \mathbb{E}\left[\mathcal{Z}_{T_\varepsilon}(x)^{2q}\right] < \infty$ from Lemma 2.5. On the other hand, the second expectation can be bounded from above by

$$\begin{aligned}
&(\log T)^{\delta/2} \mathbb{E}\left[\int_{T\ell(T)}^T \mathrm{d}\langle \mathcal{Z}(x_T)\rangle_s\right] \\
&= (\log T)^{\delta/2} \int_{T\ell(T)}^T \beta^2 \mathrm{E}_0\left[V(\sqrt{2}B_s) \exp\left(\int_0^s \beta^2 V(\sqrt{2}B_u)\mathrm{d}u\right)\right] \mathrm{d}s \\
&= (\log T)^{\delta/2} \int_{T\ell(T)}^T \beta^2 \int_{\mathbb{R}^2} \mathrm{d}x V(\sqrt{2}x)\rho_s(x) \mathrm{E}_{0,0}^{s,x}\left[\exp\left(\int_0^s \beta^2 V(\sqrt{2}B_u)\mathrm{d}u\right)\right] \mathrm{d}s \\
&\leq \beta^2 (\log T)^{\delta/2} (\log(T) - \log(T\ell(T)))
\end{aligned}$$

$$\le -\beta^2(\log T)^{\delta/2}\log \ell(T),$$

where we have used Lemma 2.7 in the third line and $\beta^2(\log T)^{\delta/2}\log\ell(T)\to 0$ as $\varepsilon\to 0$ as desired. □

Let us define the event

$$A_{T_\varepsilon}(x)=\{\tau_{T_\varepsilon}(x)=T_\varepsilon\ell(T_\varepsilon)\}=\left\{\mathcal{W}_s(x)+\widetilde{\mathcal{W}}_s(x)^{-1}\le \ell(T_\varepsilon)^{-\frac{1}{20}} \text{ for all } s\le T_\varepsilon\ell(T_\varepsilon)\right\}.$$

Before the proof of Lemma 3.19, we give an estimate of the probability of $(A_T(x))^c$.

Lemma 3.22. *There exists a constant $C>0$ such that for $\varepsilon\in(0,1)$ and $x\in\mathbb{R}^2$,*

$$\mathbb{P}\left(A_{T_\varepsilon}(x)^c\right)\le C\ell(T_\varepsilon).$$

Proof. We have

$$\begin{aligned}&\mathbb{P}\left(A_T(x)^c\right)\\&\le \mathbb{P}(\mathcal{W}_s(x)>(2\ell(T))^{-\frac{1}{20}},\,\exists s\in[0,T\ell(T)])+\mathbb{P}(\widetilde{\mathcal{W}}_s(x)<2\ell(T)^{\frac{1}{20}},\,\exists s\in[0,T\ell(T)]).\end{aligned}$$

The first term is bounded from above by $2\|u_0\|_\infty\ell(T)$ using Doob's inequality and $\mathbb{E}\left[\mathcal{W}_{T\ell(T)}(x)\right]\le\|u_0\|_\infty$. Using the fact that $B<x$ implies $A<2x$ or $A-B>x$ for $A\ge B>0$ and $x>0$, by Doob's inequality with (sub-)martingales $\mathcal{W}_s(x)^{-20}$ and $\mathcal{W}_s(x)-\widetilde{\mathcal{W}}_s(x)$, the second term is bounded from above by

$$\begin{aligned}&\mathbb{P}(\mathcal{W}_s(x)<4\ell(T),\,\exists s\in[0,T\ell(T)])+\mathbb{P}(\mathcal{W}_s(x)-\widetilde{\mathcal{W}}_s(x)>2\ell(T),\,\exists s\in[0,T\ell(T)])\\&\le 2^{20}\ell(T)\mathbb{E}[\mathcal{W}_{T\ell(T)}(x)^{-20}]+\ell(T)^{-1}\mathbb{E}[\mathcal{W}_{T\ell(T)}(x)-\widetilde{\mathcal{W}}_{T\ell(T)}(x)]\le C\ell(T).\end{aligned}$$

□

Proof of Lemma 3.19. By Hölder's inequality and Minkowski's inequality, the expectation is bounded from above by

$$\begin{aligned}&C\mathbb{E}\left[\int_{\tau_T}^{T\ell(T)}\left(\frac{1}{\mathcal{W}_s(x)^2}+1\right)\mathrm{d}\langle\mathcal{W}(x)\rangle_s\right]\\&\le C\mathbb{E}\left[\int_0^{T\ell(T)}\left(\frac{1}{\mathcal{W}_s(x)^2}+1\right)\mathrm{d}\langle\mathcal{W}(x)\rangle_s;\ A_T(x)^c\right]\\&\le C\mathbb{E}\left[\left(\int_0^{T\ell(T)}\left(\frac{1}{\mathcal{W}_s(x)^2}+1\right)\mathrm{d}\langle\mathcal{W}(x)\rangle_s\right)^p\right]^{\frac{1}{p}}\mathbb{P}\left(A_T(x)^c\right)^{\frac{1}{q}}\\&\le C\left(\mathbb{E}\left[\left(\int_0^{T\ell(T)}\frac{\mathrm{d}\langle\mathcal{W}(x)\rangle_s}{\mathcal{W}_s(x)^2}\right)^p\right]^{\frac{1}{p}}+\mathbb{E}\left[\left(\int_0^{T\ell(T)}\mathrm{d}\langle\mathcal{W}(x)\rangle_s\right)^p\right]^{\frac{1}{p}}\right)\mathbb{P}\left(A_T(x)^c\right)^{\frac{1}{q}},\end{aligned}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ are constants with $2p<p_{\hat\beta}$. Then, by the Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}\left[\left(\int_0^{T\ell(T)}\mathrm{d}\langle\mathcal{W}(x)\rangle_s\right)^p\right]\le C\mathbb{E}\left[\left(\mathcal{W}_{T\ell(T)}(x)-\mathcal{W}_0(x)\right)^{2p}\right]\le C.$$

Applying Itô's lemma to $\log\mathcal{W}_s(x)$, we have

$$\begin{aligned}\log\mathcal{W}_s(x)&=\log\bar u(t,x)+\int_0^s\frac{\mathrm{d}\mathcal{W}_u(x)}{\mathcal{W}_u(x)}-\frac{1}{2}\int_0^s\frac{\mathrm{d}\langle\mathcal{W}(x)\rangle_u}{\mathcal{W}_u(x)^2}\\&:=\log\bar u(t,x)+G'_s(x)-\frac{1}{2}H'_s(x),\end{aligned}$$

with

$$\langle G'(x)\rangle_s = H'_s(x).$$

In particular, we have

$$\begin{aligned}\mathbb{E}\left[H'_s(x)^2\right] &\le 12\left(\log \bar{u}(t,x)\right)^2 + 12\mathbb{E}\left[G'_s(x)^2\right] + 12\mathbb{E}\left[\left(\log \mathcal{W}_s(x)\right)^2\right] \\ &= 12\left(\log u(t,x)\right)^2 + 24\mathbb{E}\left[H'(x)\right] + 12\mathbb{E}\left[\left(\log \mathcal{W}_s(x)\right)^2\right] \le C,\end{aligned}$$

for some constant $C > 0$. Putting things together with Lemma 3.22, we have

$$\frac{1}{\beta_\varepsilon}\mathbb{E}\left[\int_{\tau_T(x)}^{T\ell(T)} |F''(\mathcal{W}_s(x))|\mathrm{d}\langle \mathcal{W}(x)\rangle_s\right] \le \frac{C}{\beta}\mathbb{P}\left(A_T(x)^c\right)^{\frac{1}{q}} \to 0. \square$$

Proof of Lemma 3.20. Recall the notation $\mathtt{F}_{s,t}(B,x)$ from (3.23). Since $\mathcal{W}_s(x)+\widetilde{\mathcal{W}}_s(x)^{-1} \le \ell(T)^{-\frac{1}{20}}$ for $s \le \tau_T(x)$, we have

$$\begin{aligned}&\left|\int_0^{\tau_T} F''(\mathcal{W}_s(x))\mathrm{d}\langle \mathcal{W}(x)\rangle_s - \int_0^{\tau_T} F''(\widetilde{\mathcal{W}}_s(x))\mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s\right| \\ &\le \left|\int_0^{\tau_T} F''(\mathcal{W}_s(x))\mathrm{d}\langle \mathcal{W}(x)\rangle_s - \int_0^{\tau_T} F''(\mathcal{W}_s(x))\mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s\right| \\ &\qquad + \left|\int_0^{\tau_T} \left(F''(\mathcal{W}_s(x)) - F''(\widetilde{\mathcal{W}}_s(x))\right)\mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s\right| \\ &\le C\beta^2\left(1+\ell(T)^{-\frac{1}{10}}\right) \\ &\quad \times \int_0^{\tau_T} \mathrm{d}s \mathrm{E}_x\otimes\mathrm{E}_x\left[V(B_s-\widetilde{B}_s)\Phi_s(B)\Phi_s(\widetilde{B}) : \mathtt{F}_{T\ell(T),\sqrt{Tn(T)}}(B,x)^c \cup \mathtt{F}_{T\ell(T),\sqrt{Tn(T)}}(\widetilde{B},x)^c\right] \\ &\quad + C\beta^2\int_0^{\tau_T}\mathrm{d}s\left|F''(\mathcal{W}_s(x)) - F''(\widetilde{\mathcal{W}}_s(x))\right|\mathrm{E}_x\otimes\mathrm{E}_x\left[V(B_s-\widetilde{B}_s)\Phi_s(B)\Phi_s(\widetilde{B})\right].\end{aligned}$$

Using Hölder's inequality, there exists $p > 2$ such that the first term is bounded from above by

$$\begin{aligned}&\mathbb{E}\left[\int_0^{T\ell(T)}\mathrm{d}s \mathrm{E}_x\otimes\mathrm{E}_x\left[V(B_s-\widetilde{B}_s)\Phi_s(B)\Phi_s(\widetilde{B}) : \mathtt{F}_{T\ell(T),\sqrt{Tn(T)}}(B,x)^c \cup \mathtt{F}_{T\ell(T),\sqrt{Tn(T)}}(\widetilde{B},x)^c\right]\right] \\ &\le 2\|V\|_\infty\int_0^{T\ell(T)}\mathrm{d}s\mathbb{E}\left[\mathrm{E}_x\left[\Phi_s(B)\Phi_s(\widetilde{B}) : \mathtt{F}_{T\ell(T),\sqrt{Tn(T)}}(B,x)^c\right]\right] \\ &\le C\|V\|_\infty\int_0^{T\ell(T)}\mathrm{d}s\mathrm{P}_x\left(\mathtt{F}_{T\ell(T),\sqrt{Tn(T)}}(B,x)^c\right)^{\frac{1}{p}}.\end{aligned}$$

For the second term, we first note that for each $s \le \tau_T$,

$$|F''(\mathcal{W}_s(x)) - F''(\widetilde{\mathcal{W}}_s(x))| = \left|\int_{\widetilde{\mathcal{W}}_s(x)}^{\mathcal{W}_s(x)} F'''(r)\mathrm{d}r\right| \le C(1+\ell(T)^{-\frac{3}{20}})(\mathcal{W}_s(x) - \widetilde{\mathcal{W}}_s(x)),$$

and $\frac{\mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s}{\mathrm{d}s} \le \beta^2\|V\|_\infty\widetilde{\mathcal{W}}_s(x)^2 \le \|V\|_\infty\ell(T)^{-\frac{1}{10}}$. Hence,

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^{\tau_T}|F''(\mathcal{W}_s(x)) - F''(\widetilde{\mathcal{W}}_s(x))|\mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s\right|\right] \\ &\le C\|V\|_\infty(1+\ell(T)^{-\frac{3}{20}})\ell(T)^{-\frac{1}{10}}\int_0^{T\ell(T)}\mathbb{E}\left[\left|\mathcal{W}_s(x) - \widetilde{\mathcal{W}}_s(x)\right|\right]\mathrm{d}s \\ &= C\|V\|_\infty(1+\ell(T)^{-\frac{3}{10}})\ell(T)^{-\frac{1}{10}}\int_0^{T\ell(T)}\mathrm{P}_x\left(\mathtt{F}_{T\ell(T),\sqrt{Tn(T)}}(B,x)^c\right)\mathrm{d}s.\end{aligned}$$

By (3.34), the statement holds. $\square$

Proof of Lemma 3.21. We define

$$\widetilde{H}_s(x) = \int_0^s F''(\widetilde{\mathcal{W}}_s(x))\mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s.$$

We remark that for $|x-y| \geq 3\sqrt{Tn(T)}$

$$\mathrm{Cov}\left(\widetilde{H}_{T\ell(T)}(x), \widetilde{H}_{T\ell(T)}(y)\right) = 0,$$

so that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^2} \mathrm{d}x f(x)\left(\widetilde{H}_{\tau_T}(x_T) - \mathbb{E}\left[\widetilde{H}_{\tau_T}(x_T)\right]\right)\right)^2\right] \\
&= \int_{|x-y|\leq 3\sqrt{n(T)}} \mathrm{d}x\mathrm{d}y f(x)f(y)\mathrm{Cov}\left(\widetilde{H}_{\tau_T}(x_T), \widetilde{H}_{\tau_T}(y_T)\right) \\
&\leq \int_{|x-y|\leq 3\sqrt{n(T)}} \mathrm{d}x\mathrm{d}y |f(x)f(y)|\, \mathbb{E}\left[\widetilde{H}_{\tau_T}(x_T)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\widetilde{H}_{\tau_T}(y_T)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Since

$$|\widetilde{H}_{\tau_T}(x)| \leq C\int_0^{\tau_T} (1+\widetilde{\mathcal{W}}_s(x)^{-1})^2 \mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s \leq C(1+\ell(T)^{-\frac{1}{10}})\int_0^{\tau_T} \mathrm{d}\langle \widetilde{\mathcal{W}}(x)\rangle_s,$$

by the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}\left[\widetilde{H}_{\tau_T}(x)^2\right] \leq C(1+\ell(T)^{-\frac{1}{10}})^2 \mathbb{E}\left[\sup_{0\leq s\leq \tau_T} (\mathcal{W}_s(x) - \bar{u}(1,x))^4\right] \leq C(1+\ell(T)^{-\frac{1}{10}})^2 \ell(T)^{-\frac{1}{5}}.$$

Putting things together, we have

$$\begin{aligned}
&\frac{1}{\beta^2}\mathbb{E}\left[\left(\int_{\mathbb{R}^2} \mathrm{d}x f(x)\left(\int_0^{\tau_T} F''(\widetilde{\mathcal{W}}_s(x_T))\mathrm{d}\langle\widetilde{\mathcal{W}}(x_T)\rangle_s - \mathbb{E}\left[\int_0^{\tau_T} F''(\widetilde{\mathcal{W}}_s(x_T))\mathrm{d}\langle\widetilde{\mathcal{W}}(x_T)\rangle_s\right]\right)\right)^2\right] \\
&\leq C\frac{n(T)}{\beta^2 \ell(T)^{\frac{2}{5}}}.
\end{aligned}$$

□

3.5 Multidimensional convergence in the EW limits

We have proved the Gaussian fluctuations for $F(u_\varepsilon)$ in one dimensional time t. In this section, we complete the proof of Theorem 1.1 by proving the Gaussian fluctuations for $F(u_\varepsilon)$ in multidimensional times.

To ease the presentation, we restrict ourselves to the case where $F(x) = x$, and $\hat{\beta} \in (0,1)$ is fixed, although a repetition of the argument would lead to the result for the general initial conditions and the functions F that we have been considering.

Also, we note that for all $0 \leq t_1 \leq \cdots \leq t_n = t$, $u_0^{(1)}, \cdots, u_0^{(n)} \in C_b(\mathbb{R}^2)$, and $f_1, \cdots, f_n \in C_c^\infty(\mathbb{R}^2)$

$$\begin{aligned}
&\left(u_\varepsilon^{(u_0^{(1)})}(t_1, f_1), \cdots, u_\varepsilon^{(u_0^{(n)})}(t_n, f_n)\right) \\
&\overset{(d)}{=} \left(\mathcal{W}_{T(t-t_1)}^{\left(t,T,u_0^{(1)}\right)}(Tt, f_1), \cdots, \mathcal{W}_{T(t-t_{n-1})}^{\left(t,T,u_0^{(n-1)}\right)}(Tt, f_{n-1}), \mathcal{W}_0^{\left(t,T,u_0^{(n)}\right)}(Tt, f_n)\right),
\end{aligned}$$

where for fixed $t > 0$ we define that for $u, s \geq 0$, $x \in \mathbb{R}^2$, and $f \in C_c^\infty(\mathbb{R}^2)$,

$$u_\varepsilon^{(u_0)}(s, f) = \int_{\mathbb{R}^2} f(x) u_\varepsilon^{(u_0)}(s, x)\mathrm{d}x$$

$$\mathcal{W}_u(s,x) = \mathcal{W}_u^{(t,T,u_0)}(s,x) = \begin{cases} \mathrm{E}_x\left[\Phi_{u,s}(B) u_0\left(\frac{B_{Tt-u}}{\sqrt{T}}\right)\right], & 0 \le u \le s, \\ u_0(x), & 0 \le s \le u. \end{cases}$$

and

$$\mathcal{W}_u(s,f) = \mathcal{W}_u^{(t,T,u_0)}(s,f) = \int_{\mathbb{R}^2} f(x) \mathcal{W}_u^{(t,T,u_0)}(s,x_T)\mathrm{d}x.$$

Thus, to complete the proof of Theorem 1.1, it suffices to show that jointly for finitely many $u \in [0,t]$, $u_0 \in C_b(\mathbb{R}^2)$, and $f \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, as $\varepsilon \to 0$,

$$\frac{1}{\beta_\varepsilon} \int f(x) \left(\mathcal{W}_{Tu,Tt}^{(t,T,u_0)}(x_T) - \bar{u}(t-u,x)\right) \mathrm{d}x \xrightarrow{(d)} \mathscr{U}_u^{(t,u_0)}(t,f), \tag{3.40}$$

where $\left\{\mathscr{U}_u^{(t,u_0)}(s,f) : f \in C_c^\infty(\mathbb{R}^2), u_0 \in C_b(\mathbb{R}^2), 0 \le u \le s \le t\right\}$ is a family of centered Gaussian fields with covariance

$$\begin{aligned} &\mathrm{Cov}\left(\mathscr{U}_u^{(t,u_0)}(s,f), \mathscr{U}_{u'}^{(t,u_0')}(s,f')\right) \\ &= \frac{1}{1-\hat{\beta}^2} \int_{u\vee u'}^s \mathrm{d}\sigma \int \mathrm{d}x\mathrm{d}y f(x) f'(y) \int \mathrm{d}z \rho_{\sigma-u}(x,z)\rho_{\sigma-u'}(y,z)\bar{u}(t-\sigma,z)\bar{u}'(t-\sigma,z). \end{aligned}$$

Following the same strategy as in Subsection 3.1, we are reduced to showing that

$$\frac{1}{\beta_\varepsilon} \mathcal{M}_u^{(t,T,u_0)}(\tau,f) \xrightarrow{(d)} \mathscr{U}_u^{(t,u_0)}(\tau,f) \quad \text{jointly in } u \in [0,\tau], f \in \mathcal{C}_c^\infty, \tag{3.41}$$

where (see (3.15))

$$\mathcal{M}_u^{(t,T,u_0)}(\tau,f) := \begin{cases} \int_{\mathbb{R}^2} f(x) \int_{Tu+T(t-u)\ell(T)}^{T\tau} \mathrm{d}\mathcal{M}_u^{(t,T,u_0)}(s,x_T)\mathrm{d}x, & \tau \ge u+(t-u)\ell(T), \\ 0, & \tau \le u+(t-u)\ell(T), \end{cases}$$

and

$$\begin{aligned} &\mathrm{d}\mathcal{M}_u^{(t,T,u_0)}(s,x) := \\ &\beta_\varepsilon \mathcal{Z}_{Tu,Tu+(s-Tu)\ell(T)}(x) \int \xi(\mathrm{d}s,\mathrm{d}b) \int \rho_{s-Tu}(x,z)\phi(z-b) \overleftarrow{\mathcal{Z}}_{s,(s-Tu)\ell(T)}(z) \mathrm{E}_z\left[u_0\left(\frac{B_{Tt-s}}{\sqrt{T}}\right)\right]\mathrm{d}z. \end{aligned}$$

Then, for all $u \ge 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $\tau \to \mathcal{M}_u^{(t,T)}(\tau,f)$ is a continuous martingale. In view of the desired convergence (3.40), we have again in mind the functional CLT for martingales Theorem 3.3, so we are interested in the limit of the cross-bracket $\langle \mathcal{M}_{u_1}^{(t,T)}(\cdot,f_1), \mathcal{M}_{u_2}^{(t,T)}(\cdot,f_2)\rangle_\tau$.

Proposition 3.23. *For all test functions f and f' in $\mathcal{C}_c^\infty$, $u_0, u_0' \in C_b(\mathbb{R}^2)$, $0 \le u_2 \le u_1 \le t$, and $\tau \ge 0$,*

$$\begin{aligned} &\frac{1}{\beta_\varepsilon^2} \langle \mathcal{M}_{u_1}^{(t,T,u_0)}(\cdot,f_1), \mathcal{M}_{u_2}^{(t,T,u_0')}(\cdot,f_2)\rangle_\tau \\ &\xrightarrow{L^1} \frac{1}{1-\hat{\beta}^2} \int_{u_1\wedge\tau}^{\tau} \mathrm{d}\sigma \int \mathrm{d}x\mathrm{d}y f_1(x) f_2(y) \int \mathrm{d}z \rho_{\sigma-u_1}(x,z)\rho_{\sigma-u_2}(y,z)\bar{u}(t-\sigma,z)\bar{u}'(t-\sigma,z), \end{aligned} \tag{3.42}$$

as $\varepsilon \to 0$.

Proof. We first remark that for $\tau \le u_1$, the cross variation vanishes. For $\tau > u_1$, we have $\tau \ge u_1 + (t-u_1)\ell(T)$ if $\varepsilon > 0$ is sufficiently small. Hence,

$$\begin{aligned}
&\frac{1}{\beta^2}\langle \mathcal{M}_{u_1}^{(t,T,u_0)}(\cdot, f_1), \mathcal{M}_{u_2}^{(t,T,u_0')}(\cdot, f_2)\rangle_\tau \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}x\mathrm{d}y f_1(x) f_2(y) \int_{Tu_1+T(t-u_1)\ell(T)}^{T\tau} \mathrm{d}s \mathcal{Z}_{Tu_1,Tu_1+(s-Tu_1)\ell(T)}(x_T)\mathcal{Z}_{Tu_2,Tu_2+(s-Tu_2)\ell(T)}(y_T) \\
&\quad\times \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}z_1\mathrm{d}z_2 \rho_{s-Tu_1}(z_1-x_T)\rho_{s-Tu_2}(z_2-y_T)V(z_1-z_2)\overleftarrow{\mathcal{Z}}_{s,(s-Tu_1)\ell(T)}(z_1)\overleftarrow{\mathcal{Z}}_{s,(s-Tu_2)\ell(T)}(z_2) \\
&\quad\times \mathrm{E}_{z_1}\left[u_0\left(\frac{B_{Tt-s}}{\sqrt{T}}\right)\right]\mathrm{E}_{z_2}\left[u_0'\left(\frac{B_{Tt-s}}{\sqrt{T}}\right)\right].
\end{aligned}$$

By a repetition of the arguments that lead to (3.17), we have that

$$\begin{aligned}
&\frac{1}{\beta^2}\langle \mathcal{M}_{u_1}^{(t,T,u_0)}(\cdot, f_1), \mathcal{M}_{u_2}^{(t,T,u_0')}(\cdot, f_2)\rangle_\tau \\
&\approx_{L^1} \int f_1(x) f_2(y)\mathrm{d}x\mathrm{d}y \\
&\qquad \times \int_{u_1+(t-u_1)\ell(T)}^{\tau} \mathrm{d}\sigma \mathbb{E}\left[\mathcal{Z}_{Tu,Tu+T(\sigma-u)\ell(T)}(x_T)\right]\mathbb{E}\left[\mathcal{Z}_{Tu_2,Tu_2+(s-Tu_2)\ell(T)}(y_T)\right]\Theta_T(x,y) \\
&= \int f_1(x) f_2(y)\mathrm{d}x\mathrm{d}y \int_{u_1+(t-u_1)\ell(T)}^{\tau} \mathrm{d}\sigma\Theta_T(x,y),
\end{aligned}$$

where

$$\begin{aligned}
&\Theta_T(x,y) \\
&= \int \mathrm{d}z\mathrm{d}v \rho_{\sigma-u_1}(z-x)\rho_{\sigma-u_2}(z-\frac{v}{\sqrt{T}}-y)V(v)\mathbb{E}\left[\overleftarrow{\mathcal{Z}}_{\sigma,(\sigma-u_1)\ell(T)}(z_T)\overleftarrow{\mathcal{Z}}_{\sigma,(\sigma-u_2)\ell(T)}(z_T+v)\right] \\
&\qquad\qquad \times \mathrm{E}_{z_T}\left[u_0\left(\frac{B_{Tt-T\sigma}}{\sqrt{T}}\right)\right]\mathrm{E}_{z_T+v}\left[u_0'\left(\frac{B_{Tt-T\sigma}}{\sqrt{T}}\right)\right] \\
&\to \frac{1}{1-\hat\beta^2}\int \mathrm{d}z\rho_{\sigma-u_1}(x-z)\rho_{\sigma-u_2}(y-z)\bar u(t-\sigma,z)\bar u'(t-\sigma,z). \qquad \square
\end{aligned}$$

References

[AKQ14] T. Alberts, K. Khanin and J. Quastel, The intermediate disorder regime for directed polymers in dimension 1+1, *Ann. Probab.* **42** (2014) 1212–1256. MR3189070

[AKQ14b] T. Alberts, K. Khanin and J. Quastel, The continuum directed random polymer, *J. Stat. Phys.* **154** (2014) 305–326. MR3162542

[BC20] E. Bates, S. Chatterjee The endpoint distribution of directed polymers, *Ann. Probab.* Volume 48, Number 2 (2020), 817–871. MR4089496

[BT10] Q. Berger and F. Toninelli, On the critical point of the random walk pinning model in dimension $d = 3$, *Elect. J. Prob.* 15, 654–683, (2010). MR2650777

[BC95] L. Bertini and N. Cancrini, The stochastic heat equation: Feynman-Kac formula and intermittence, *J. Statist. Phys.* 78(5-6):1377-1401, (1995). MR1316109

[BG97] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, Vol. 183, No. 3, pp. 571–607, 1997. MR1462228

[B99] P. Billingsley: Convergence of probability measures, Second. New York : John Wiley & Sons Inc., 1999 *Wiley Series in Probability and Statistics: Probability and Statistics*. A Wiley-Interscience Publication. ISBN 0-471-19745-9. MR1700749

[B89] E. Bolthausen, A note on the diffusion of directed polymers in a random environment, *Commun. Math. Phys.* 123(4), 529–534 (1989). MR1006293

[BGH11] M. Birkner, A. Greven and F. den Hollander, Collision local time of transient random walks and intermediate phases in interacting stochastic systems, *Elec. J. Probab.* 16, 552–586, (2011). MR2786642

[BS10] M. Birkner and R. Sun, Annealed vs quenched critical points for a random walk pinning model, *Ann. Henri Poinc.*, Prob. et Stat., Vol. 46, No. 2, pp. 414–441, (2010). MR2667704

[BS11] M. Birkner and R. Sun, Disorder relevance for the random walk pinning model in dimension 3,*Ann Henri Poincaré.* Prob. et Stat., Vol. 47, No. 1, pp. 259–293, (2011). MR2779405

[BM19] Y. Bröker and C. Mukherjee, Localization of the Gaussian multiplicative chaos in the Wiener space and the stochastic heat equation in strong disorder, *Ann. Appl. Probab.* 29, 6 (2019), 3745–3785. MR4047991

[CC22] F. Caravenna and F. Cottini, Gaussian limits for subcritical chaos, *Elec. J. Probab.*, Vol. 27, pp.1–35, (2022). MR4441143

[CSZ17a] F. Caravenna, R. Sun and N. Zygouras, Polynomial chaos and scaling limits of disordered systems, *J. Eur. Math.* Soc. 19 (2017), 1–65. MR3584558

[CSZ17b] F. Caravenna, R. Sun and N. Zygouras, Universality in marginally relevant disordered systems, *Ann. Appl. Prob.* 27 (2017), 3050–3112. MR3719953

[CSZ19a] F. Caravenna, R. Sun and N. Zygouras, The Dickman subordinator, renewal theorems, and disordered systems *Electron. J. Probab.* Volume 24 (2019), paper no. 101, 40 pp. MR4017119

[CSZ19b] F. Caravenna, R. Sun and N. Zygouras, On the Moments of the $(2+1)$-Dimensional Directed Polymer and Stochastic Heat Equation in the Critical Window *Communications in Mathematical Physics*, Volume 372, (2019), No. 2, 385–440. MR4032870

[CSZ20] F. Caravenna, R. Sun and N. Zygouras, The two-dimensional KPZ equation in the entire subcritical regime, *Ann. Prob.* Volume 48, (2020), No. 3, 1086–1127. MR4112709

[CSZ21] F. Caravenna, R. Sun and N. Zygouras, The critical $2d$ stochastic heat flow, *arXiv:2109.03766.* (2021) MR4017119

[CH02] P. Carmona and Y. Hu, On the partition function of a directed polymer in a Gaussian random environment, *Prob. Th. Rel. Fields.*, **124** (2002) 431–457. MR1939654

[CD20] S. Chatterjee and A. Dunlap, Constructing a solution of the $(2+1)$-dimensional KPZ equation, *Ann. Prob.,* 48 (2020), no. 2, 1014–1055. MR4089501

[C17] F. Comets, Directed polymers in random environments, *Lect. Notes Math.* 2175, Springer, 2017. MR3444835

[CC18] F. Comets and C. Cosco, Brownian Polymers in Poissonian Environment: a survey, arXiv:1805.10899. (2018). MR3996766

[CCM20] F. Comets, C. Cosco and C. Mukherjee, Renormalizing the Kardar-Parisi-Zhang equation in weak disorder in $d \geq 3$, *Journal of Statistical Physics.* (2020). MR4099995

[CCM19] F. Comets, C. Cosco and C. Mukherjee, Space-time fluctuation of the Kardar-Parisi-Zhang equation in $d \geq 3$ and the Gaussian free field, *arXiv:1905.03200*. MR4262971

[CL17] F. Comets and Q. Liu, Rate of convergence for polymers in a weak disorder, *J. Math. Anal. Appl.* **455** (2017), 312–335. MR3665102

[CY06] F. Comets and N. Yoshida, Directed polymers in random environment are diffusive at weak disorder, *Ann. Probab.* 34 (2006), no. 5, 1746–1770. MR2271480

[CN21] C. Cosco and S. Nakajima, Gaussian fluctuations for the directed polymer partition function for $d \geq 3$ and in the whole L^2-region, *Ann. Inst. H. Poincaré Probab. Statist.*, 57(2): 872–889 (2021). MR4260488

[CNN22] C. Cosco, S. Nakajima, and M. Nakashima Law of large numbers and fluctuations in the sub-critical and L^2 regions for SHE and KPZ equation in dimension $d \geq 3$, *Stochastic Process. Appl.,* Vol. 151, (2022), pp. 127–173. MR4441505

[CZ21] C. Cosco and O. Zeitouni, Moments of partition functions of 2D Gaussian polymers in the weak disorder regime – I , *arXiv:2112.03767*, (2021).

[DS80] D.A. Dawson and H. Salehi, Spatially homogeneous random evolutions, *J. Multivariate Anal.*, Vol. 10, No. 2, (1980), pp. 141–180. MR0575923

[DG20] A. Dunlap and Y. Gu, A forward-backward SDE from the $2D$ nonlinear stochastic heat equation, (2020), arXiv:2010.03541 MR4413215

[DGRZ18b] A. Dunlap, Y. Gu, Lenya Ryzhik and Ofer Zeitouni, The random heat equation in dimensions three and higher: the homogenization viewpoint, *Archive for Rational Mechanics and Analysis*, 242, pp. 827–873 (2021). MR4331017

[DGRZ20] A. Dunlap, Y. Gu, Lenya Ryzhik and Ofer Zeitouni, Fluctuations of the solutions to the KPZ equation in dimensions three and higher, *Probab. Theory Related Fields* 176 (2020), no. 3-4, 1217–1258. MR4087492

[EK86] S. Ethier and Thomas G. Kurtz, Markov Processes Characterization and Convergence, John Wiley & Sons, (1986). MR0838085

[G21] S. Gabriel, Invariance principle for the $(2+1)$-dimensional directed polymer in the weak disorder limit, https://arxiv.org/abs/2104.07755.

[GIP15] M. Gubinelli, P. Imkeller and N. Perkowski, Paracontrolled distributions and singular PDEs, Forum Math. Pi, 3:e6, 75, 2015. MR3406823

[GP18] M. Gubinelli and N. Perkowski, Energy solutions of KPZ are unique, J. Amer. Math. Soc., 31(2):427–471, (2018). MR3758149

[GL20] Y. Gu and J.Li, Fluctuations of a nonlinear stochastic heat equation in dimensions three and higher, SIAM Journal on Mathematical Analysis, 52, (2020), no. 6, 5422–5440. MR4169750

[GQT19] Y. Gu, J. Quastel and L.C. Tsai, Moments of the 2D SHE at criticality, *Probability and Mathematical Physics*, 2(1), 179–219, (2021). MR4404819

[GRZ18] Y. Gu, L. Ryzhik and O. Zeitouni, The Edwards-Wilkinson limit of the random heat equation in dimensions three and higher, Comm. Math. Phys., **363** (2018), No. 2, pp. 351–388. MR3851818

[G20] Y. Gu, Gaussian fluctuations of the 2D KPZ equation, *Stoch. Partial Differ. Equ. Anal. Comput.* 8 (2020), no. 1, 150–185. MR4058958

[H13] M. Hairer, Solving the KPZ equation, *Annals of Mathematics* **178** (2013), 558–664. MR3071506

[H14] M. Hairer, A theory of regularity structures, *Inventiones mathematicae* **198:2** (2014), 269–504. MR3274562

[IS88] J. Imbrie and T. Spencer, Diffusion of directed polymers in a random environment, *Journal of Statistical Physics.* 52(3-4), 609–626, (1988). MR0968950

[JS87] J. Jacod and A. Shiryaev, *Limit theorems for stochastic processes*, Springer-Verlag, Berlin (1987). MR0959133

[J97] S. Janson, *Gaussian Hilbert Spaces*, Vol.129, Cambridge University Press (1997). MR1474726

[KPZ86] M. Kardar, G. Parisi, Y.C. Zhang, Dynamic scaling of growing interfaces, *Physical Review Letters*, Vol.56, No. 9, pp.889–892, 1986

[KM17] A. Kupiainen and M. Marcozzi, Renormalization of generalized KPZ equation. *Journal of Statistical Physics* **166** (2017) 876–902. MR3607594

[LZ20] D. Lygkonis and N. Zygouras. Edwards-Wilkinson fluctuations for the directed polymer in the full L^2-regime for dimensions $d \geq 3$, *Ann. Inst. H. Poincaré Probab. Statist.*, **58** (1), 65–104, (2022). MR4374673

[LZ21] D. Lygkonis and N. Zygouras. Moments of the 2d directed polymer in the subcritical regime and a generalisation of the Erdös-Taylor theorem, *arXiv:2109.06115*, (2021)

[MU17] J. Magnen and J. Unterberger, The scaling limit of the KPZ equation in space dimension 3 and higher, *Journal of Statistical Physics.* **171:4** (2018), 543–598. MR3790153

[MSZ16] C. Mukherjee, A. Shamov and O. Zeitouni, Weak and strong disorder for the stochastic heat equation and the continuous directed polymer in $d \geq 3$, *Electr. Comm. Prob.* **21** (2016) 12 pp. MR3548773

[S95] Y. Sinai, A remark concerning random walks with random potentials, *Fund. Math.* **147** (1995), 173–180. MR1341729

[T22] R. Tao, Gaussian fluctuations of a nonlinear stochastic heat equation in dimension two, *arXiv:2204.13866*, (2022) MR4169750

[V06] V. Vargas, A local limit theorem for directed polymers in random media: the continuous and the discrete case, *Ann. Inst. H. Poincaré Probab. Stat.* 42(5), 521–534, (2006). MR2259972

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