Electronic Journal of Probability

Electron. J. Probab. **27** (2022), article no. 156, 1–14.
ISSN: 1083-6489 https://doi.org/10.1214/22-EJP871

The topology of SLE$_\kappa$ is random for $\kappa > 4$*

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Abstract

We study the topology of SLE curves for $\kappa > 4$. More precisely, we show that, a.s., there is no homeomorphism $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$, taking the range of one independent SLE curve to another for $\kappa \in (4, 8)$. Furthermore, we extend the result to $\kappa \geq 8$ by showing that there is no homeomorphism taking one SLE curve to another, when viewed as curves modulo parametrization.

Keywords: SLE.
MSC2020 subject classifications: 60J67.
Submitted to EJP on November 19, 2021, final version accepted on October 21, 2022.

1 Introduction

1.1 Initial overview

The Schramm-Loewner Evolutions (SLE$_\kappa$), introduced by Oded Schramm [16], describes a family of probability distributions, parameterized by $\kappa > 0$ on non-self traversing curves connecting two boundary points in a planar, simply connected domain. They are characterized by a conformal invariance condition and a domain Markov property. They were initially observed as possible candidates for the scaling limits of various discrete lattice models in statistical physics; we now know that some of these convergences do hold, and so SLE exhibits a universality in its definition.

SLE$_\kappa$ is the random growth of a set K_t, as described through a conformal map $g_t(z)$ on the complement of K_t. This map $g_t(z)$ is the solution of the Loewner differential equation driven by a Brownian motion, whose 'speed' is determined by a single parameter κ. Rhodes and Schramm in [15] showed that for $\kappa \neq 8$, a.s. there is a (unique) continuous path $\eta : [0, \infty) \to \overline{\mathbb{H}}$ such that for each $t > 0$ the set K_t is the union of $\eta[0, t]$ and the bounded connected components of $\mathbb{H} \setminus \eta[0, t]$. This has also been shown for $\kappa = 8$, but was dealt with separately [8]. We call the path η the SLE trace or SLE curve. It was shown as well in [15] that $\lim_{t\to\infty} |\eta(t)| = \infty$ a.s. We will need the following facts about the curve which are proven in [15]:

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- If $\kappa \leq 4$, then η is simple with $\eta(0,\infty) \subset \mathbb{H}$.
- If $4 < \kappa < 8$, then $\eta(0,\infty)$ has double points and intersects $\mathbb{R}$.
- If $\kappa \geq 8$, the curve is space-filling.

There are three (well studied) variants of SLE: chordal SLE, which connects two boundary points (prime ends) in a given domain, radial SLE, which connects a boundary point to an interior point, and whole-plane SLE, which connects two points on the Riemann sphere. In this paper, we will focus on chordal SLE, but we expect that our results can be generalized to other cases. One can also see [6, 2, 18] for some expository work on SLE which go into details beyond the scope of this paper.

Most works on SLE have focused on its geometric and probabilistic properties, e.g., Hausdorff dimensions of various subsets of the curves, formulas for the probabilities of various events, and connections to other random objects. In this work, we will address a very basic question about the topology of SLE: namely, is the topology of the curve deterministic? Said differently, if we have two independent chordal SLE_κ curves η^1 and η^2 (viewed as curves modulo time parametrization), does there a.s. exist a homeomorphism $\overline{\mathbb{H}} \to \overline{\mathbb{H}}$ taking η^1 to η^2?

Since SLE_κ is a simple curve for $\kappa \leq 4$, the answer to the above question is clearly affirmative in this case. For $\kappa > 4$, however, the answer is less obvious. On the one hand, many events for SLE_κ occur with probability strictly between 0 and 1 (see Section 2 of [12]) so there are many opportunities for one of η^1 or η^2 to do something that the other does not. On the other hand, it is common for seemingly very different fractal sets to be homeomorphic. For example, if K_1 and K_2 are compact, non-empty, totally disconnected subsets of $\mathbb{C}$ without isolated points (e.g., Cantor-type sets), then there is a homeomorphism from $\mathbb{C}$ to $\mathbb{C}$ which takes K_1 to K_2 [13]. The main results of this paper show that the topology of SLE_κ is random for $\kappa > 4$. For $\kappa \in (4,8)$, we prove the stronger statement that the topology of the range is random. The results of this paper are in a similar vein to those of [11], which shows that an SLE_κ curve for $\kappa \in (4,8)$ is not determined by its range. Both this paper and [11] answer a seemingly simple question about SLE whose answer is much less obvious than one might initially expect.

1.2 Summary of results

Theorem 1.1. *The topology of chordal SLE_κ is not deterministic in the following sense: Fix $\kappa \in (4,8)$, and consider two independent SLE_κ curves, η^1 and η^2 in $\mathbb{H}$. Then a.s. there is no homeomorphism on $\overline{\mathbb{H}}$ taking the range of η^1 to the range of η^2.*

We consider the left and right boundaries of an SLE curve η (which are boundary-touching $\mathrm{SLE}_{16/\kappa}(\bar{\rho})$ curves, to be defined later). These curves form 'bubbles' in $\mathbb{H}$ (which we characterize explicitly in a later section) which we use as the primary observable to prove Theorem 1.1.

The result also holds for $\kappa \geq 8$. The proof is similar, though a bit more work is needed in the setup.

Theorem 1.2. *The topology of chordal SLE_κ is not deterministic for $\kappa \geq 8$ in the following sense: Consider two independent SLE_κ curves, η^1 and η^2 in $\mathbb{H}$. Then a.s. there is no homeomorphism $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ such that $\Phi(\eta^1) = \eta^2$ viewed as curves modulo time parametrization.*

Remark 1.3. Notice that in Theorem 1.2, we care about parametrized curves, because preservation of ranges in this setting makes less sense. Recall SLE in this instance is plane filling.

As a natural extension, one can think about the behavior of these curves for varying κ.

Conjecture 1.4. *Let $\kappa_1, \kappa_2 > 4$ be distinct. Let (η^1, η^2) be any coupling of a chordal SLE$_{\kappa_1}$ and a chordal SLE$_{\kappa_2}$. Almost surely, there is no homeomorphism $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ such that $\Phi(\eta^1) = \eta^2$ viewed as curves modulo time parametrization. If one of κ_1 or κ_2 is in $(4, 8)$, a.s., there is no such homeomorphism which takes the range of η^1 to the range of η^2.*

The conjecture says, roughly speaking, that the topology of SLE$_{\kappa_1}$ and SLE$_{\kappa_2}$ should be mutually singular. We expect that this conjecture can be proved using similar ideas to the ones in this paper, but one would have to explicitly compute some of the quantities involved to show that they are κ-dependent.

2 Preliminaries

Here we discuss a few SLE basics as well as how one defines the more general SLE$_\kappa(\bar{\rho})$ processes. We write $\mathbb{H} := \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$. If K is a bounded closed subset of $\mathbb{H}$ such that $\mathbb{H} \setminus K$ is simply connected, then we call K a hull in $\mathbb{H}$ w.r.t. ∞. For such K, there is a unique g_K that maps $\mathbb{H} \setminus K$ conformally onto $\mathbb{H}$ such that $g_K(z) = z + \dfrac{a}{z} + O\left(\dfrac{1}{z^2}\right)$ as $z \to \infty$, for some real a. The quantity a is known as the ***half plane capacity*** of K, and is denoted by hcap$K$. It can be shown that $a \geq 0$. The map g_K is said to satisfy the hydrodynamic normalization at infinity. For a real interval I, let $\mathcal{C}(I)$ denote the real-valued continuous functions on I. Suppose $U \in \mathcal{C}([0,T])$ for some $T \in (0, \infty]$. For each $z \in \overline{\mathbb{H}} \setminus \{0\}$, let $g_t(z)$ be the solution of the ordinary differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z. \tag{2.1}$$

Note that for $z \in \mathbb{C} \setminus 0$, the solution to (2.1) holds $\forall\, t < T_z$ where

$$T_z = \sup_t \{\min\{|g_s(z) - U_s| : 0 \leq s \leq t\} > 0\}.$$

Set

$$K_t := \{z \in \overline{\mathbb{H}} : T_z \leq t\}.$$

The sets K_t are the chordal Loewner hulls, and the collection of maps $\{g_t : t \geq 0\}$ are called the chordal Loewner maps driven by U_t. Suppose that for every $t \in [0, T)$,

$$\eta_t := \lim_{z \in \mathbb{H}, z \to U_t} g_t^{-1}(z) \in \mathbb{H} \cup \mathbb{R}$$

exists, and $\eta[0, T)$ is a continuous curve. Then for every $t \in [0, T)$, K_t is the complement of the unbounded component of $\mathbb{H} \setminus \eta((0, t])$. We call η the chordal Loewner trace driven by U_t. In general, however, such a curve may not exist depending on the choice of driving function.

An SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ is defined by the random family of conformal maps g_t obtained by solving the Loewner ODE driven by Brownian motion. In particular, we let $U_t = \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion. An SLE$_\kappa$ connecting boundary points x and y of an arbitrary simply connected Jordan domain can be constructed as the image of an SLE$_\kappa$ on $\mathbb{H}$ under a conformal transformation $\Psi \colon \mathbb{H} \to D$ sending 0 to x and ∞ to y. SLE curves are characterized by scale invariance and the domain Markov property, and are viewed modulo reparametrization. It is shown in [15, 8] that the SLE$_\kappa$ processes are generated by curves.

SLE$(\kappa; \bar{\rho})$, which is often written as SLE$_\kappa(\bar{\rho}_L; \bar{\rho}_R)$, is the stochastic process one obtains by solving (2.1) with a modification on the driving process U_t, which we now discuss. It is a natural generalization of SLE$_\kappa$ in which one keeps track of additional marked points which are called force points. In this paper, we need only the two force point regime, but the following definitions are easily extended to the multiple force point setting. Fix $x_1 < 0 < x_2$. We associate with each x_i for $i \in \{1,2\}$ a weight $\rho_i \in \mathbb{R}$. An $\text{SLE}_\kappa(\rho_1;\rho_2)$ process with force points $(x_1;x_2)$ is the measure on continuously growing compact hulls K_t generated by the Loewner chain with U_t given by the solution to the system of SDEs given by

$$dU_t = \frac{\rho_1}{U_t - V_t^1}dt + \frac{\rho_2}{U_t - V_t^2}dt + \sqrt{\kappa}\, dB_t, \tag{2.2}$$

$$dV_t^i = \frac{2}{V_t^i - U_t}dt; \quad V_0^i = x_i, \quad i \in \{1,2\}. \tag{2.3}$$

The existence and uniqueness of solutions to this SDE is discussed in [17], and follows from results in [14]. These results are extended to the more general setting of multiple force points in [9].

For $\kappa > 4$, there is also significant interest in the hulls that are generated by the SLE_κ curves. Duplantier conjectured in [4, 5] the duality between SLE_κ and $\text{SLE}_{16/\kappa}$, which says that the boundary of an SLE_κ hull behaves like an $\text{SLE}_{16/\kappa}$ curve, for $\kappa > 4$. Many versions of this duality have been shown in [19, 20, 3, 9, 10].

Lemma 4.9 in [9] asserts that, for $\kappa > 4$, the outer boundary η' of an SLE_κ curve is an $\text{SLE}_\kappa(\bar\rho)$ process. This is done in the setting of imaginary geometry, in which the SLE curves (for $\kappa \in (0,4)$), are realized as ***flow lines*** of the Gaussian free field (i.e $\text{SLE}_\kappa(\bar\rho)$ curves coupled with the Gaussian free field in $\mathbb{H}$), with the outer boundaries, themselves SLE curves (for $\kappa \in [4,\infty)$) described as ***counterflow lines*** (in which the coupling is done with the negation of the Gaussian field). Though we do not need this machinery as presented in [9] and [12], it serves as an excellent framework for proving some general properties of $\text{SLE}_\kappa(\bar\rho)$, some of which we rely on to prove the main results. We state one such fact as follows:

Lemma 2.1. *Fix $\kappa > 0$. Suppose that η is an $\text{SLE}_\kappa(\bar\rho_L;\bar\rho_R)$ process in $\mathbb{H}$ from 0 to ∞ with force points located at $(\bar x_L;\bar x_R)$ with $x_{1,L} = 0^-$ and $x_{1,R} = 0^+$ (possibly by taking $\rho_{1,q} = 0$ for $q \in \{L,R\}$). Assume that $\rho_{1,L}$, $\rho_{1,R} > -2$. Fix $k \in \mathbb{N}$ such that $\rho = \sum_{j=1}^k \rho_{j,R} \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$ and $\epsilon > 0$. There exists $p_1 > 0$ depending only on κ, $\max_{i,q}|\rho_{i,q}|$, ρ, and ϵ such that if $|x_{2,q}| \geq \epsilon$ for $q \in \{L,R\}, x_{k+1,R} - x_{k,R} \geq \epsilon$, and $x_{k,R} \leq \epsilon^{-1}$ then the following is true. Suppose that $\gamma\colon [0,T] \to \mathbb{H}$ is a simple curve starting from 0, terminating in $[x_{k,R}, x_{k+1,R}]$, and otherwise does not hit $\partial\mathbb{H}$, for some $T \in [0,\infty)$. Let $A(\epsilon)$ be the ϵ-neighborhood of $\gamma([0,T])$ and let*

$$\sigma_1 = \inf\{t \geq 0 : \eta(t) \in (x_{k,R}, x_{k+1,R})\} \quad \text{and} \quad \sigma_2 = \inf\{t \geq 0 : \eta(t) \notin A(\epsilon)\}.$$

Then $\mathbb{P}[\sigma_1 < \sigma_2] \geq p_1$.

Intuitively, Theorem 2.1 tells us that an $\text{SLE}_\kappa(\bar\rho_L;\bar\rho_R)$ process has a positive chance to stay close to any fixed deterministic curve for a positive amount of time.

Proof. This is Lemma 2.5 in [12]. □

3 Proof of Theorem 1.1

Consider the left and right boundaries of the SLE curve η, which are boundary-touching $\text{SLE}_{\frac{16}{\kappa}}(\rho)$ curves, with force points starting at 0. In fact, the left boundary of SLE_κ turns out to be $\text{SLE}_{16/\kappa}(\frac{16}{\kappa} - 4; \frac{8}{\kappa} - 2)$ and by symmetry, the right boundary is $\text{SLE}_{16/\kappa}(\frac{8}{\kappa} - 2; \frac{16}{\kappa} - 4)$. This can be deduced from Theorem 5.3 in [19]. These curves are shown in Figure 1. The open region between the left and right boundaries has countably many connected components, which are separated by the intersection points of the left and right boundaries, i.e., the cut points of η. These connected components have a total ordering, and come in four types:

- Type 0: Neither the left nor the right boundary of the component intersects the real line.
- Type 1: Only the right boundary intersects the real line.
- Type 2: Only the left boundary intersects the real line.
- Type 3: The left and right boundaries both intersect the real line.

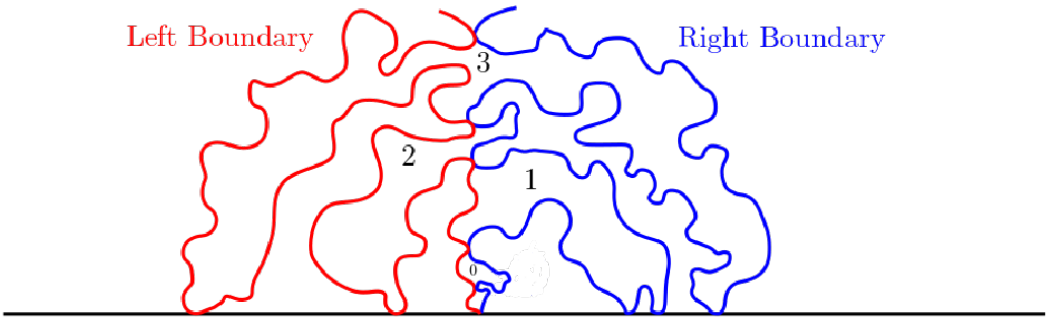

Figure 1: We view the complement of the SLE curve as the union of two boundary-touching $\text{SLE}_\kappa(\bar{\rho})$ processes. We observe 'bubbles' of four types, which we use in constructing the observable invariant.

Note that η is a continuous curve that travels between the positive and negative real axes between any two consecutive components of type 3. This shows that the components of type 3 form a discrete set, to which we may assign a labeling by the integers – written as

$$(\dots U_{-1}, U_0, U_1, U_2, \dots)$$

uniquely, modulo index shift. For concreteness, we choose the indexing for the sequence so that U_0 is the first type 3 bubble which has Euclidean diameter at least 1. We remark here that our construction relies on a few tail triviality arguments, and so we require the following:

Lemma 3.1. *Suppose $t > 0$ and let a_t (resp. b_t) be the last time before t at which η hits the left (resp. right) boundary. Then $\eta|_{[0,t]}$ determines the set of bubbles (i.e. connected components of the region between the left and right boundaries) which are formed before time $\min\{a_t, b_t\}$ as well as their types.*

Proof. This follows trivially from the fact that η cannot cross itself and $\eta([\min\{a_t, b_t\}, t])$ disconnects all of the bubbles formed before time $\min\{a_t, b_t\}$ from $\eta(t)$. $\square$

Between pairs of consecutive type 3 bubbles, U_i and U_{i+1}, we may either observe a type 1 or 2 bubble, or we may not. Let E_i be the event that there is a type 1 or type 2 bubble between U_i and U_{i+1}, and define

$$X := (\dots \mathbb{1}_{E_{-1}}, \mathbb{1}_{E_0}, \mathbb{1}_{E_1}, \mathbb{1}_{E_2}, \dots)$$

the bi-infinite sequence of 0's and 1's consisting of the indicators of the E_i's.

Lemma 3.2. *For any fixed deterministic bi-infinite sequence of 0's and 1's x, we have $\mathbb{P}[X = x] = 0$.*

We delay the proof of Theorem 3.2 to introduce some notation. The proof requires a few key observations which we discuss below. Consider a left-infinite sequence $y = (\dots y_{-2}, y_{-1}, y_0)$. For $k \in \mathbb{N}$, let A_k be the event that $\{\dots X_{-k-1}, X_{-k} = y\}$. We wish to show that $\mathbb{P}[A_0] = 0$. We will argue this by contradiction, but we first require a bit of setup. For $r \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$, let $K_r^{(n)}$ be the n^{th} smallest k such that the Euclidean diameter of U_k is at least r. Now, we claim that $\mathbb{P}[A_{K_1^{(n)}}] = 0$ for all n. We argue to the contrary, and so we assume that there exists some n such that $\mathbb{P}\left[A_{K_1^{(n)}}\right] > 0$. Note that by scale invariance, $\mathbb{P}\left[A_{K_r^{(n)}}\right]$ is independent of r, and so depends only on n. Consider the event $\bigcap_{i=0}^{\infty} \bigcup_{m \geq i} A_{K_{\frac{1}{m}}^{(n)}}$, which is a tail event for the Brownian motion that drives the SLE, for every choice of n. To see this, note that Theorem 3.1 implies that for each t, $\mathcal{F}_t$ determines $A_{K_r^{(n)}}$ for each r which is small enough so that the bubble $U_{K_r^{(n)}}$ is formed before time $\min\{a_t, b_t\}$. Thus, by continuity from above, we note that

$$\mathbb{P}\left[\bigcap_{i=0}^{\infty} \bigcup_{m \geq i} A_{K_{\frac{1}{m}}^{(n)}}\right] \geq \mathbb{P}\left[A_{K_1^{(n)}}\right] > 0$$

and so the Blumenthal $0-1$ law implies that, a.s., there exists a sequence $\{r_j\} \to 0$ such that the events $A_{K_{r_j}^{(n)}}$ occur for all j. This implies that there exist infinitely many k such that A_k occurs. Thus, it follows that a.s., $\exists$ infinitely many k such that

$$(\dots X_{-k-1}, X_{-k}) = y$$

forcing the sequence y to be periodic. We claim that this implies that the sequence $\{X_k\}$ is periodic. Indeed, let m be the period of y. Since there are arbitrarily large k for which $(...X_{-k-1}, X_{-k}) = y$ and y is periodic, it follows that with probability tending to 1 as $r \to 0$, the sequence

$(...X_{-K_r^{(n)}-1}, X_{-K_r^{(n)}})$ is equal to $(...y_{-j-1}, y_{-j})$ for some $j = 1, ..., m$. By scale invariance, the probability that this is the case for all values of r is equal to 1. Thus, as $r \to \infty$, we see that the entire sequence $\{X_k\}$ is equal to y, shifted by some $j = 1, ..., m$. This means that if we observe $(...X_{-k-1}, X_{-k})$ for some k, we can determine the rest of the sequence $\{X_k\}$, forcing this sequence to be itself periodic.

For $t > 0$, we have that by Theorem 3.1 $\mathcal{F}_t$ determines the sequence $(\dots X_{-l-1}, X_{-l})$ for some l, which by periodicity is enough to determine the sequence $\{X_k\}$. Thus, by Theorem 3.1, $\mathcal{F}_t$ determines $\{X_k\}$ modulo an index shift for each $t > 0$, and hence the sequence $\{X_k\}$ is deterministic modulo an index shift. The goal now is to recursively apply Theorem 2.1 to arrive at a contradiction.

Proposition 3.3. *Let $\mathcal{Z}$ be a finite sequence of 0's and 1's which does not appear in y, with $|\mathcal{Z}| = m$. Then it must hold that*

$$\mathbb{P}\left[\{X_1, X_2, \dots, X_m\} = \{\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_m\}\right] > 0.$$

Note that the existence of such a $\mathcal{Z}$ follows from the periodicity of y. With this result, we can conclude that the sequence $\{X_k\}$ can contain any finite sequence of 0's and 1's with positive probability, and hence cannot be periodic and deterministic modulo index shift. We delay the proof of the proposition to state the following key lemma, which uses the fact that the outer boundaries of the curve are $\text{SLE}_\kappa(\rho_L; \rho_R)$ processes, and more specifically the right boundary, η^R, conditioned on the left boundary, η^L, has distribution of $\text{SLE}_{\frac{16}{\kappa}}(-\frac{8}{\kappa}; \frac{16}{\kappa} - 4)$ (see Lemma 7.1 in [9]):

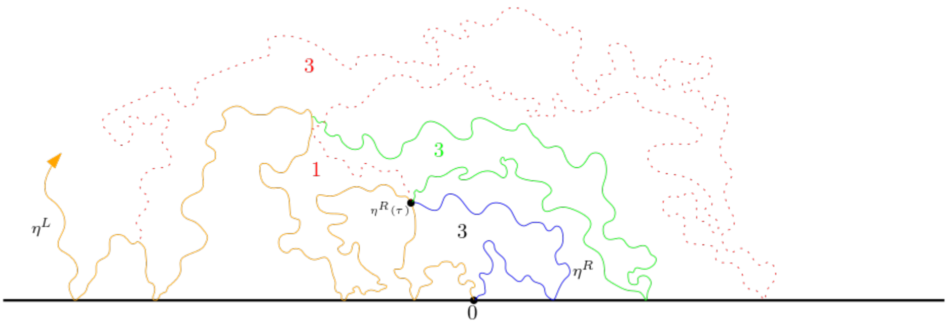

Figure 2: We condition on the left boundary (pictured as the orange curve) and run the right boundary until we first form a type 3 bubble of diameter at least 1 (blue). At this time (denoted $\eta^R(\tau)$), we have two options: either the right boundary hits $[0, \infty)$ before hitting the left boundary again (green), thus forming a type 3 bubble, or it hits the left boundary first (red), forming a type 1 bubble before forming the next type 3 bubble. These events each occur with positive probability.

Lemma 3.4. *Let τ be a stopping time for η^R given η^L, at which η^R forms a type 3 bubble denoted U_{k_τ}. Let E_{k_τ} be the event that there is a type 1 or type 2 bubble between U_{k_τ} and $U_{k_\tau+1}$, as defined previously. Then,*

$$0 < \mathbb{P}\left[E_{k_\tau} \middle| \eta^L, \eta^R_{|[0,\tau]}\right] < 1.$$

Proof. With some setup, this is a straightforward application of Theorem 2.1. Indeed, let $z_\tau := \eta^R(\tau)$ and define C_{z_τ} to be the connected component of $\eta^L \setminus \mathbb{R}$ containing z_τ. Set

$$s^1 := \inf\{t > \tau : \eta^R \cap [0, \infty) \neq \emptyset\}, \qquad s^2 := \inf\{t > \tau : \eta^R \cap \eta^L \setminus (C_{z_\tau} \cup (-\infty, 0]) \neq \emptyset.\}$$

By Theorem 2.1, we have that

$$\mathbb{P}\left[s^2 > s^1 \middle| \eta^L, \eta^R_{|[0,\tau]}\right] > 0; \qquad \mathbb{P}\left[s^2 \leq s^1 \middle| \eta^L, \eta^R_{|[0,\tau]}\right] > 0$$

where the second inequality follows from symmetry considerations. Indeed, we can simply apply Theorem 2.1 to the curve η^R, under the conditional law given η^L. In this case, an interval on the left boundary corresponds to a segment of η^L. Note that these probabilities are strictly less than 1 as they are both positive and complementary. With this, and appealing to the setting of Fig. 2, we have that $\eta^R[\tau, \infty)$, conditioned on $\eta^L, \eta^R_{|[0,\tau]}$, will either first intersect the left boundary and form a type 1 bubble before forming another type 3 bubble, or it will intersect $[0, \infty)$ before hitting the left boundary again, forming another type 3 bubble. In particular, the event that a type 1 bubble is formed after U_{k_τ} occurs with probability strictly between 0 and 1 as desired. □

Proof of Theorem 3.3. We define a sequence of stopping times as follows: For a given bubble U_i, let τ_i be the corresponding time at which U_i is formed. By our choice of indexing of the type 3 bubbles, we have that

$$\tau_0 := 1^{\text{st}} \text{ time we form a type 3 bubble of Euclidean diameter at least 1}$$

$$\tau_1 := 1^{\text{st}} \text{ time after } \tau_0 \text{ we form a type 3 bubble}$$

$$\vdots$$

$$\tau_m := 1^{\text{st}} \text{ time after } \tau_{m-1} \text{ we form a type 3 bubble.}$$

Note that $E_{k_{\tau_i}}$ is measurable with respect to η^L and $\eta^R|_{[0,\tau_{i+1}]}$, and for each $i \in \{1, 2, \ldots, m\}$, we have that by Theorem 3.4,

$$0 < \mathbb{P}\left[E_{k_{\tau_i}} \,\middle|\, \eta^L, \eta^R_{|_{[0,\tau_i]}}\right] < 1.$$

Thus, it follows that

$$\mathbb{P}[X_i = \mathcal{Z}_i | X_1 = \mathcal{Z}_1, ..., X_{i-1} = \mathcal{Z}_{i-1}] > 0.$$

To finish the proof, we note that since $\{X_j = Z_j\}$ is determined by η^L and $\eta^R|_{[0,\tau_i]}$ for $i < j$, so

$$\mathbb{P}[X_1 = \mathcal{Z}_1, ..., X_i = \mathcal{Z}_i] = \mathbb{E}\left[\mathbb{P}[X_i = \mathcal{Z}_i \,|\, \eta^L, \eta^R|_{[0,\tau_i]}] \mathbb{1}_{X_1 = \mathcal{Z}_1, ..., X_{i-1} = \mathcal{Z}_{i-1}}\right].$$

The probability within the expectation on the right hand side is always positive, and so inducting on i (and setting $i = m$ as a final step) yields the desired result. □

Proof of Lemma 3.2. By Theorem 3.3, we see that $\{X_k\}$ can contain any finite sequence of 0's and 1's not contained in y, implying that $\{X_k\}$ cannot be deterministic modulo index shift. This is a contradiction. Thus, $\mathbb{P}[A_{K_1^{(n)}}] = 0$ for every n.

Thus, by scale invariance we see that $\mathbb{P}[A_{K_r^{(n)}}] = 0$ for every r and n. Note that every k is equal to $K_r^{(n)}$ for some rational r and some n. Indeed, every k^{th} bubble has some positive diameter, and there are at most finitely many bubbles before it of larger diameter. Thus, we can set n to be the number of bubbles before the k^{th} bubble with diameter exceeding that of the k^{th} bubble, and simply let r be any rational number slightly smaller than this diameter. From this, it follows that

$$\mathbb{P}[\exists\, k \text{ such that } A_k \text{ occurs }] \le \mathbb{P}\left[\bigcup_{n \in \mathbb{N}} \bigcup_{r \in \mathbb{Q}_{>0}} A^{(n)}_{K_r}\right] = 0.$$

In particular, we have that $\mathbb{P}[A_0] = 0$. □

Proof of Theorem 1.1. Now let η^1 and η^2 be two independent SLE's. In order for $\eta^1 \cup \mathbb{R}$ and $\eta^2 \cup \mathbb{R}$ to be homeomorphic via a homeomorphism that takes $\mathbb{R}$ to $\mathbb{R}$, it must be the case that the corresponding bi-infinite sequences X^1 and X^2 differ by at most an index shift. Indeed, any homeomorphism has to preserve the bi-infinite sequence of connected components lying between the left and right boundaries of the curve, as well as the types of these components. Thus, by the above argument, the probability that X^1 is equal to any of the countably many possible index shifted versions of X^2 is zero. Hence the probability that $\eta^1 \cup \mathbb{R}$ and $\eta^2 \cup \mathbb{R}$ are homeomorphic, via a homeomorphism that takes $\mathbb{R}$ to $\mathbb{R}$, is 0. □

4 Proof of Theorem 1.2

Here, we require a more subtle argument that relies on a less obvious observable. In this section, we fix $\kappa \ge 8$. Let η be an instance of SLE_κ in $\overline{\mathbb{H}}$. We are interested in the successive crossing times (about the origin) of the curve η, i.e., the times at which η hits the real line again, just after having hit it on the opposite side of the origin. Consider one such crossing time, i.e., a single left right crossing about the origin. The SLE goes back and forth between the left and right boundaries of this crossing at some times, and the set of times when it does so has to be a discrete set since the SLE is continuous. As pictured below in Fig. 3, these left and right crossings (within the curve) define a sequence of marked points $\{X_k\}$ along the boundary, which accumulate only at the tip of the curve. Via the corresponding Loewner map g_t^η, we may conformally map this configuration as shown in Fig. 3, so that the tip goes to 0, and we obtain a sequence of marked points along the left boundary. Notice these marked points are determined by the past, so we can condition on (all of) their locations, and the future will still be an SLE by the Markov property.

A bit more care is needed in defining these quantities. Let $\tau(t)$ be the last time before t such that $\eta(t) \in \mathbb{R}$. Define the sets

$$T_- := \{t : \eta(\tau(t)) < 0\} \qquad T_+ := \{t : \eta(\tau(t)) > 0\}$$

and set $\mathcal{S} = \bar{T}_- \cap \bar{T}_+$. Notice that $\mathcal{S}$ is a discrete set since η is continuous, and so it cannot cross back and forth between $(-\infty, 0)$ and $(0, \infty)$ infinitely many times during any compact time interval contained in $(0, \infty)$. Thus, we may index the elements of $\mathcal{S}$ as a countable sequence of well defined crossing times $\{\tau_j\}$.

Notice that these are not stopping times (which poses a problem in applying the strong Markov property), but this can be addressed by adopting some notation from the previous section as follows. Let $\eta_j := \eta|_{[\tau_{j-1}, \tau_j]}$, which is the j^{th} left-right crossing around 0 that we observe, i.e., the crossing of index $j \in \mathbb{Z}$. For $r > 0$, let $J_r^{(n)}$ be the n^{th} smallest j for which the Euclidean diameter of η_j is at least r. It is not difficult to see that the set of times $\{\tau_{J_r^{(n)}}\}$ is indeed a set of stopping times. To see this, let $t > 0$. If one sees $\eta|_{[0,t]}$, then one can determine the set $\{\tau_j : \tau_j \leq t\}$. This follows from the definition of the times $\{\tau_j\}$ as the intersection points of T_- and T_+, as shown previously. Hence $\eta|_{[0,t]}$ determines the set of excursions $\{\eta_j : \tau_j \leq t\}$. We have $\tau_{J_r^{(n)}} \leq t$ if and only if this set of excursions includes at least n elements which have Euclidean diameter at least r. Hence $\{\tau_{J_r^{(n)}} \leq t\}$ is determined by $\eta|_{[0,t]}$, which holds for any choice of t.

We fix some r and some n, and set $J := J_r^{(n)}$. Between the outer boundaries of the crossing η_J, we can keep track of the times at which $\{\eta_t : t < \tau_J\}$ sequentially hits these boundaries. More precisely, we let L_J be the outer boundary of $\eta[0, \tau_J]$. We define our sequence of crossing times inductively as follows:

$$\sigma_{J,1} := \min\{t > \tau_{J-1} : \eta_t \cap L_J \neq \emptyset\}$$

$$\tilde{\sigma}_{J,1} := \min\{t > \sigma_{J,1} : \eta_t \cap L_{J-1} \neq \emptyset\}$$

$$\vdots$$

$$\tilde{\sigma}_{J,k} := \min\{t > \sigma_{J,k} : \eta_t \cap L_{J-1} \neq \emptyset\}$$

$$\sigma_{J,k+1} := \min\{t > \tilde{\sigma}_{J,k} : \eta_t \cap L_J \neq \emptyset\}$$

and so on. The sequences $\{\sigma_{J,k}\}_{k \geq 1}$ and $\{\tilde{\sigma}_{J,k}\}_{k \geq 1}$ define two discrete sets of times that our curve successively hits the outer boundaries L_J and L_{J-1} respectively. We assume without loss of generality that the J^{th} excursion goes from left to right. By considering only the outer boundary L_J (as a priori τ_J is a well-defined stopping time), we can construct a sequence of marked points $\{X_{J,k}\}_{k \geq 1}$ along the negative real axis, via the (shifted) Loewner map which sends $\eta(\tau_J)$ to 0. That is to say, $X_{J,k} := g_{\tau_J}(\eta(\sigma_{J,k})) - U_{\tau_J}$. As we are considering a fixed J, we may write $X_{J,k} := X_k$ for ease.

Consider the points where the future of the SLE process, $\eta|_{[\tau_J, \infty)}$, hits the negative real axis after having hit the real line to the right of 0, which we call crossing endpoints. More precisely, we define these crossing endpoint times as follows:

$$\sigma_1^* := \min\{t > \tau_J : \eta_t \cap \mathbb{R}_{>0} \neq \emptyset\}$$

$$\tilde{\sigma}_1^* := \min\{t > \sigma_1^* : \eta_t \cap \mathbb{R}_{<0} \neq \emptyset\}$$

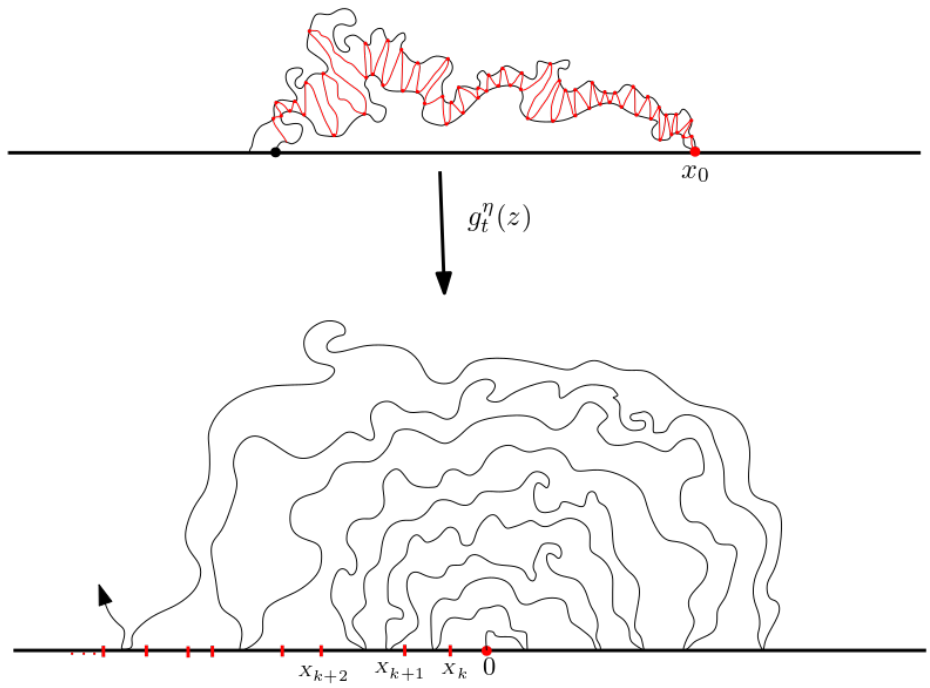

Figure 3: The top picture illustrates a single left-right crossing around 0, with $x_0 = \eta(\tau_J)$ and the corresponding triangulation in red, determined by the (past) piece of the curve making boundary crossings. The marked points X_k define the locations of the tips of the triangles in the triangulation, after conformally mapping to the real line via g_t^η. We thus consider intervals $[X_{k+1}, X_k]$ in which the tips of future triangles, obtained by left right crossings about 0, may lie. Some intervals may have multiple, while some may have none.

$$\vdots$$
$$\tilde{\sigma}_i^* := \min\{t > \sigma_i^* : \eta_t \cap \mathbb{R}_{<0} \neq \emptyset\}$$
$$\sigma_{i+1}^* := \min\{t > \tilde{\sigma}_i^* : \eta_t \cap \mathbb{R}_{>0} \neq \emptyset\}$$

and so on. We let

$$N_k = \#\left\{i : \tilde{\sigma}_i^* \in [X_{k+1}, X_k]\right\}.$$

In other words, we are looking at $\eta' := g_{\tau_J}(\eta|_{[\tau_J,\infty)}) - U_{\tau_J}$ as it successively makes left-right crossings about 0, conditioned on the past, and for each interval we are keeping track of how many crossing endpoints it contains. We wish to show that for every deterministic sequence of integers $\{n_k\}_{k\in\mathbb{N}}$, we have that

$$\mathbb{P}[N_k = n_k;\, \forall\, k] = 0. \tag{4.1}$$

It suffices to show that there are arbitrarily large k such that $\mathbb{P}[N_k = n_k]$ is bounded away from 1. Indeed, the event $\{N_k = n_k$ for all sufficiently large $k\}$ is a tail event for the Brownian motion driving the SLE, and the Blumenthal $0-1$ law implies that this has probability 0 or 1. Thus, being bounded away from 1 guarantees that we have (4.1). We do this in cases as follows:

Case 1: Assume there exist arbitrarily large k such that $n_k \neq 0$. We claim that there exists $q > 0$ such that

$$\mathbb{P}[N_k = n] \leq 1 - q \quad \forall\, n \geq 1.$$

To see this, we consider the segment of the curve η', just after the $(n-1)^{th}$ crossing about 0 is completed. Let $\mathcal{T}_n$ denote the n^{th} time we have a crossing in the interval $[X_{k+1}, X_k]$. Thus $\mathcal{T}_n$ is a stopping time, and conditioned on what we have seen up until this time, the future of the curve is still SLE. The goal is to have an upper bound on the probability that there are exactly n crossings, and we do so by comparing the harmonic measure (from ∞) of the interval $[X_{k+1}, \eta'(\mathcal{T}_{n-1})]$, to that of the outer boundary of the curve $\eta'[0, \mathcal{T}_{n-1}]$(and more precisely, this is the harmonic measure from ∞ in $\mathbb{H} \setminus \eta'[0, \mathcal{T}_{n-1}]$). These quantities are denoted a and b respectively, as shown in Fig. 4.

The proof relies on the following intuitive argument which we formalize later: If a is larger than b, then with positive probability we observe 2 further crossings, hence $n+1$ total crossings. If a is smaller than b, then, with positive probability, we expect the interval $[X_{k+1}, \eta'(\mathcal{T}_{n-1})]$ to be covered before we observe the next crossing. In other words, there is always a positive chance that we observe either $n-1$ crossings or $n+1$ crossings, and so

$$\mathbb{P}[N_k \neq n] > 0.$$

Proposition 4.1. *Let η be an SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$ with $\kappa > 4$. For marked points $a < 0 < c$ along the real line, let $E_{a,c}$ be the event that the chordal SLE$_\kappa$ trace visits $[c, \infty)$ before $(-\infty, a]$. Then*

$$\mathbb{P}\left[E_{a,c}\right] = F\left(\frac{-a}{c-a}\right) \qquad \text{where } F(x) = \frac{1}{Z_\kappa}\int_0^x \frac{du}{u^{\frac{4}{\kappa}}(1-u)^{\frac{4}{\kappa}}}$$

and Z_κ is chosen so that $F(1) = 1$.

Proof. This is Theorem 10 in [1], which is a generalized restatement of Theorem 3.2 in [7]. □

Remark 4.2. It is possible to get an estimate which is weaker than Theorem 3 above, but which is still sufficient for our purposes, via the following elementary argument. For $x \in \mathbb{R}$, let $t_x := \inf[t \geq 0 : \eta(t) = x]$. If we let $P(n) = \mathbb{P}[t_n < t_{-1}]$, a bit of thought shows that

$$P(n) \geq P(n-1)[1 - P(n)]$$

which thus implies that

$$P(n) \geq \frac{P(n-1)}{1 + P(n-1)}.$$

The equality case can be realized as $P(n) = \dfrac{1}{n+1}$, the details of which we omit. By considering $f(x) = \dfrac{x}{x+1}$, which is increasing on $\mathbb{R}_{\geq 0}$, we find that

$$P(n) \geq f(P(n-1)) \geq f^{(2)}(P(n-2) \cdots \geq f^{(n)}\left(\frac{1}{2}\right) = \frac{1}{n+1}$$

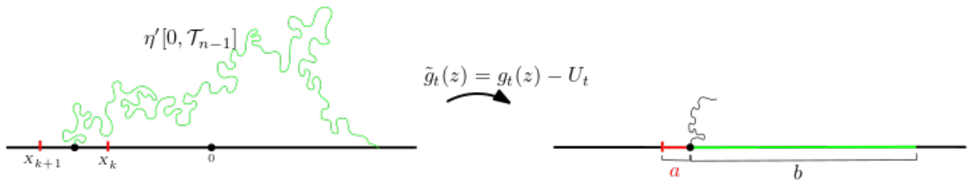

Figure 4: We stop the SLE after it has made its $(n-1)^{\text{th}}$ crossing in the interval shown. Under the map $\tilde{g}$, we send the tip of the curve to the origin, and analyze the likelihood of either observing two more crossings in the red interval of length a, or no more crossings, in which case the interval is swallowed.

which gives a rough (yet easy to compute) estimate. Note, for our purposes, we only require a positive probability.

We return to the notation introduced in Fig. 4, and we consider the the behavior of the SLE curve given the relative quantities a and b. In particular, we require the following two key lemmas to prove the original claim:

Lemma 4.3. *If $a \leq b$, it holds with conditional probability at least $\frac{1}{2}$, given $\eta'|_{[0,\mathcal{T}_{n-1}]}$, that $\eta'|_{[\mathcal{T}_{n-1},\infty)}$ hits X_{k+1} before $[b,\infty)$.*

Proof. Notice that by symmetry, there is a positive chance that we disconnect $[X_{k+1}, \eta'(\mathcal{T}_{n-1})]$ before hitting b. Indeed, this follows from the fact that $\mathbb{P}[t_{-1} < t_1] = \frac{1}{2}$. □

Lemma 4.4. *There exists a deterministic κ-dependent constant $c > 0$ such that if $a > b$, it holds with conditional probability at least c given $\eta'|_{[0,\mathcal{T}_{n-1}]}$ that $\eta'|_{[\mathcal{T}_{n-1},\infty)}$ crosses between $(-\infty, 0)$ and $(0,\infty)$ at least twice before hitting X_{k+1}.*

Proof. If $a > b$, then we can apply the estimate given in Theorem 4.1 via a two step process. We retain the notation from Theorem 4.2, and define t_{-a} and t_b as discussed, after having mapped $\eta'([0,\mathcal{T}_{n-1}])$ to the real line via the map $\tilde{g}$. Note that Theorem 4.1 implies that $\exists\, p > 0$ such that $\mathbb{P}[t_b < t_{-a/2}] \geq p$. In fact, we have assumed $a > b$, so p in this instance can be thought of as a universal bound. We condition on this event occurring, and we look at the harmonic measure of the outer boundary curve $\tilde{\eta}$ of this most recent crossing. Note that $\mathrm{hm}_{\mathbb{H}\setminus\tilde{\eta}}(\infty, \tilde{\eta})$ is bounded above by the harmonic measure of the outer boundary at the time we hit $-\dfrac{a}{2}$. This follows from the fact that the harmonic measure can only increase, as we observe more of the curve. Moreover, the law of this harmonic measure, divided by a, is independent of a by scale invariance, and is almost surely finite. This implies that $\exists\, C = C(p) > 0$ such that

$$\mathbb{P}\left[\mathrm{hm}_{\mathbb{H}\setminus\tilde{\eta}}(\infty, \tilde{\eta}) \leq Ca\right] \geq 1 - \frac{p}{2}$$

from which it follows that

$$\mathbb{P}\left[\mathrm{hm}_{\mathbb{H}\setminus\tilde{\eta}}(\infty, \tilde{\eta}) \leq Ca,\, t_b < t_{-a/2}\right] \geq \frac{p}{2}.$$

This bound guarantees a positive probability that, after we have observed the first crossing, the harmonic measure of the outer boundary is not too large. Now we condition on this event, and we apply Proposition 4.1 to the quantities Ca and $\dfrac{a}{2}$. In particular, This yields a positive κ-dependent constant lower bound for the probability that $\eta'|_{[\mathcal{T}_{n-1},\infty)}$ has at least two crossings before hitting X_{k+1}. □

Case 2: $n_k = 0$ for all but finitely many k.

This condition implies that the SLE travels a positive distance of time without any left-right crossings, which happens with probability 0. This shows that for any fixed deterministic sequence $\{n_k\}_{k\in\mathbb{N}}$ with only finitely many non-zero elements, we have that

$$\mathbb{P}[\{N_k\} = \{n_k\}] = 0.$$

Proof of Theorem 1.2. Consider two instances of SLE_κ in $\mathbb{H}$, η^1 and η^2, with corresponding sequences of points $\{X^1_{m_1,k}\}_{k\in\mathbb{N}}$ and $\{X^2_{m_2,k}\}_{k\in\mathbb{N}}$ respectively, for fixed indicies $m_1, m_2 \in \mathcal{S}$, corresponding to the ${m_1}^{\text{th}}$ crossing of η^1 and ${m_2}^{\text{th}}$ crossing of η^2 respectively. Here, we indicate objects associated with η^j for $j \in \{1,2\}$ by a superscript j. Note that

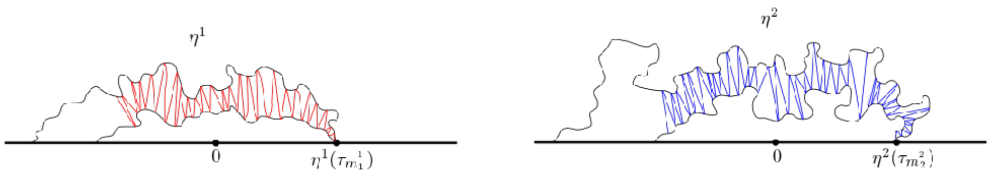

Figure 5: We observe two instances of SLE, η^1 and η^2, stopped after the $m_1{}^{\text{th}}$ and $m_2{}^{\text{th}}$ crossings respectively. Any homeomorphism between the two should send one tip to the other, and retain the structure of the future crossings (i.e., preserve the corresponding sequences $\{N^j_k\}$).

by construction, $m_1 = J^{(n_1),1}_{r_1}$ and $m_2 = J^{(n_2),2}_{r_2}$ for some n_1, n_2 and (rational) r_1, r_2. Each sequence of points $\{X^j_k\}_{k\in\mathbb{N}}$ generates a sequence $\{N^j_k\}_{k\in\mathbb{N}}$ for $j \in \{1,2\}$ and so by the independence of η^1 and η^2, as well as (4.1), we have that for any choice of m_1, m_2 and number l

$$\mathbb{P}\left[N^1_k = N^2_{k+l};\ \forall\, k\right] = \mathbb{P}\left[N^1_k = N^2_{k+l};\ \forall\, k \,|\, \eta^2\right] = 0.$$

This implies that

$$\mathbb{P}\left[\exists\, l \text{ s.t } N^1_k = N^2_{k+l};\, \forall\, k\right] = 0 \tag{4.2}$$

as there are countably many possible choices of l, meaning we can apply this very argument for each fixed choice of l, and apply the union bound.

Observe that a homeomorphism from $\overline{\mathbb{H}}$ to itself taking η^1 to η^2, modulo time parametrization, must preserve the number of left right crossings of the 'future' curves, which correspond to the sequences $\{N^j_k\}$, and it must take $\eta^1(\tau^1_{m_1})$ to $\eta^2(\tau^2_{m_2})$ for some m_2. In particular, as in the setting of Figure 4, for any fixed m_1 and m_2 there is no homeomorphism which takes η^1 to η^2 and $\eta^1(\tau^1_{m_1})$ to $\eta^2(\tau^2_{m_2})$ by (4.2). As the set $\mathcal{S}$ of crossing times is discrete, this holds for any choice of indices $m_1$and m_2, where there are only countably many choices. Thus, it must hold that,

$$\mathbb{P}\left[\exists \text{ a homeomorphism } \Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}} \text{ taking } \eta^1 \text{ to } \eta^2\right] = 0. \qquad \square$$

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Acknowledgments. I would like to thank Prof. Ewain Gwynne for suggesting this problem, and for answering my many questions about the material presented here, as well as SLE in general. I would also like to thank Prof. Gregory Lawler for suggesting readings and proof techniques to supplement this paper.