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A full discretization of the rough fractional linear heat equation

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Abstract

We study a full discretization scheme for the stochastic linear heat equation

$$\begin{cases} \partial_t \text{⚲} = \Delta \text{⚲} + \dot{B}\,, \quad t \in [0,1],\ x \in \mathbb{R}, \\ \text{⚲}_0 = 0\,, \end{cases}$$

when $\dot{B}$ is a very *rough space-time fractional noise*.

The discretization procedure is divised into three steps: (i) regularization of the noise through a mollifying-type approach; (ii) discretization of the (smoothened) noise as a finite sum of Gaussian variables over rectangles in $[0,1]\times\mathbb{R}$; (iii) discretization of the heat operator on the (non-compact) domain $[0,1]\times\mathbb{R}$, along the principles of Galerkin finite elements method.

We establish the convergence of the resulting approximation to ⚲, which, in such a specific rough framework, can only hold in a space of distributions. We also provide some partial simulations of the algorithm.

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1 Introduction and main result

1.1 Introduction

The main objective of this study is to provide a full discretization scheme for the solution ⚲ of the stochastic heat equation

$$\begin{cases} \partial_t \text{⚲} = \Delta \text{⚲} + \dot{B}\,, \quad t \in [0,1],\ x \in \mathbb{R}, \\ \text{⚲}_0 = 0\,, \end{cases} \tag{1.1}$$

where $\dot{B}$ is a stochastic space-time noise, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In fact, the specificity of our analysis will lie in the consideration of a *rough*

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fractional noise. Namely, given a fractional sheet $\{B_{t,x},\, (t,x) \in \mathbb{R}^2\}$ of Hurst indexes $H_0, H_1 \in (0,1)$ (see Definition 2.1 for details), we set

$$\dot{B} := \frac{\partial^2 B}{\partial t \partial x}\,, \tag{1.2}$$

where the derivatives are understood in the sense of distributions.

Due to its great flexibility, the fractional noise model has now been widely recognized as one of the most relevant alternatives to the standard white noise situation, whether for finite-dimensional systems or in SPDE settings. The fractional setting is also known to provide a convenient framework to study the influence of the noise roughness on the dynamics. In brief, when letting the parameters H_0, H_1 progressively decrease from 1 to 0, the regularity of $\dot{B}$ decreases as well, and the analysis becomes more and more intricate.

In this context, let us recall that the space-time white noise setting precisely corresponds to the case where $H_0 = H_1 = \frac{1}{2}$. In this specific situation, the approximation issue for the stochastic heat model (1.1) or its extensions (whether with a multiplicative noise, or a non-linear drift) has been the source of a huge amount of papers since the late nineties and the pioneering works by Gyöngy, Nualart and others (see for instance [12, 13, 14, 15], or [1] and its bibliography). A few fractional situations (that is, situations where $(H_0, H_1) \neq (\frac{1}{2}, \frac{1}{2})$) have also been recently considered in the approximation literature: let us quote for instance [2] for a white-in-time fractional-in-space noise (that is, $H_0 = \frac{1}{2}, H_1 \neq \frac{1}{2}$), or [29] for a fractional-in-time white-in-space noise (more precisely, $H_0 > \frac{1}{2}$ and $H_1 = \frac{1}{2}$).

Our objective in the present study is to go beyond all these previous studies and consider a *space-time* fractional noise of overall *lower regularity*. Indeed, we will here focus on the case of a fractional noise $\dot{B}$ with indexes H_0, H_1 satisfying the condition

$$0 < 2H_0 + H_1 < 1\,. \tag{1.3}$$

Our essential motivation for considering such a rough situation is actually easy to formulate: one can indeed show that as soon as $2H_0 + H_1 < 1$, the solution of (1.1) is *no longer a function in space*, but only a *general distribution* (see Proposition 2.2 for details). Accordingly, the associated approximation issue cannot be examined through function norms either, and negative-order Sobolev topologies must come into the picture. This strongly contrasts with most of the existing statements in the white-noise literature (typically, convergence is therein established using the L^2-norm in space), and we thus consider our handling of negative-order Sobolev norms in the analysis as a new contribution in the understanding of the stochastic linear heat problem.

In this regard, the assumption (1.3), leading to a distributional-valued , can be compared with the behaviour of the corresponding two-dimensional heat equation driven by a space-time white noise (that is, the equation on $[0,1] \times \mathbb{R}^2$ or $[0,1] \times \mathbb{T}^2$). Indeed, it turns out that the solution of such two-dimensional white-noise equation cannot be treated as a process with values in $L^2(\mathbb{R}^2)$, but only as a process with values in the Sobolev space $H^{-\varepsilon}(\mathbb{R}^2)$ for any $\varepsilon > 0$ (see for instance [5, 6]). The consideration of a rough fractional noise thus allows us to face a similar challenge, but in the one-dimensional setting, for which discretization methods are naturally more convenient to set up.

Another important motivation behind our interest for the linear solution lies in the central role played by this process in many recent developments about the pathwise approach to general stochastic PDEs. For instance, in the study of the celebrated white-noise-driven Φ^4-model

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \xi, \quad t \in [0,T],\ x \in \mathbb{T}^d,\ d \in \{2,3\}, \tag{1.4}$$

the corresponding linear solution Υ (i.e., $\partial_t \Upsilon = \Delta \Upsilon + \xi$) can be regarded as some first-order approximation of Φ, and the analysis then consists in the control of the more regular path $\Psi := \Phi - \Upsilon$ (see [6] for details when $d = 2$, [16, Section 9.2], [17, Section 6] when $d = 3$, and let us also stress the fundamental role of Υ in the renormalization procedures associated with such singular models). Similar phenomena have recently been exhibited in the fractional situation for the quadratic counterpart of (1.4) (see [24] for $d = 2$, and [27] for $d \geq 1$), and the strategy happens to be equally fruitful in wave and Schrödinger settings (see e.g. [8, 9] and [10], respectively).

With these various works in mind, we consider the present investigations about the discretization (and possibly the simulation) of Υ as an important first step toward the discretization of more general singular stochastic PDEs.

Before we present our approximation strategy, let us emphasize the following three major difficulties raised by the model.

(i) First, it is well-known that the flexibility of fractional noises (i.e., the fact that one can control the overall roughness of the noise through the parameters H_0, H_1) comes at a cost: indeed, as soon as $H_i \neq \frac{1}{2}$, sophisticated fractional kernels must be involved in the analysis of any construction related to the field, which rules out the drastic simplifications offered by Itô-type isometry properties. These additional technicalities can be observed right from the proof of Proposition 2.2, that is right from the interpretation phase of the model, and they will also have a major impact on the subsequent steps.

(ii) In the rough regime (1.3), and as we mentioned earlier, the solution Υ is no longer a well-defined Gaussian process on $[0, 1] \times \mathbb{R}$, and it can only be handled as a distribution in space (see Section 2 for more details). We are thus forced to deal with negative-order Sobolev norms (represented by fractional weights in the Fourier mode) throughout the study, which naturally adds another level of technicality to our computations.

(iii) As it can be seen from (1.1), we intend to handle the equation on the whole Euclidean space $\mathbb{R}$, which, as far as we know, is not the most common setting in the approximation literature (bounded domains appear to be much more frequently considered). In fact, our objective in this regard is to make a first possible step toward some of the most recent developments on parabolic models driven by fractional noises, and which are all concerned with equations on the non-compact domain $[0, T] \times \mathbb{R}^d$ (see for instance [3, 4, 7, 18, 20, 21]). Let us also point out that the definition of a space-time fractional noise on the Euclidean space is quite obvious (along (1.2)), whereas there is no consensus about the definition of such an object on a torus.

Of course, the consideration of a non-compact space domain is not costless either. As a particular consequence, our discretization scheme for (1.1) shall appeal to a finite grid *growing to* $\mathbb{R}$ (see details in Section (1.2)), which requires a careful control of the approximation process on the growing boundary (see for instance the bound derived from the Galerkin approximation of the heat operator in Proposition 4.2). Besides, due to the asymptotic behaviour of the fractional sheet, convergence estimates for Υ and its approximation can only be analyzed by means of weighted topologies in space, which eventually echoes in the statement of our main result (Theorem 1.3), as can be seen from the involvement of the arbitrary cut-off function ρ in our final bound (1.20).

Thus, even though the overall linear dynamics of (1.1) *may look quite basic at first sight, we think that the above-described features* (i)-(ii)-(iii) *turn the analysis and discretization of the problem into a highly non-trivial question, and to the best of our knowledge, there exists no previous approximation study taking those specificities into account.*

Let us now briefly describe the successive steps that will punctuate our discretization procedure.

(1) Interpretation of the solution through a smoothening procedure. The high roughness of the noise $\dot{B}$ under consideration (as induced by Assumption (1.3)) immediately gives rise to a first basic question before one can think about discretization, namely: how to interpret the solution $\overset{\circ}{|}$ of (1.1) in this setting? If $\dot{B}$ were to be more regular, then this solution would be explicitly given by the space-time convolution $\overset{\circ}{|} = G * \dot{B}$, where G stands for the heat kernel

$$G_t(x) := \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{x^2}{4t}\Big)\, \mathbf{1}_{\{t>0\}}\,,\ t \in [0,T],\ x \in \mathbb{R}\,.$$

Unfortunately, when switching to the rough regime, the meaning of the convolution of $\dot{B}$ with the singular kernel G is no longer clear.

In order to reach such an interpretation, we will rely on a standard regularization procedure, and thus follow the strategy used in most recent pathwise approaches to SPDEs (regularity structures, paracontrolled calculus). In other words, starting from a smooth approximation B^n of B, we intend to define $\overset{\circ}{|}$ as the (potential) limit, in a suitable Sobolev space, of the sequence $G * \partial_t\partial_x B^n$ of approximated solutions.

For further reference, let us specify right now our choice for the approximation of B (we will comment further on this choice in Section 2, see Remark 2.4 and Remark 2.5): namely, for every parameter $\kappa > 0$, we consider the sequence $(B^{\kappa,n})_{n\geq 1}$ defined for every $n \geq 1$ as

$$B_t^{\kappa,n}(x) = c_{H_0} c_{H_1} \int_{|\xi|\leq 2^{2\kappa n}} \int_{|\eta|\leq 2^{\kappa n}} \widehat{W}(d\xi, d\eta)\, \frac{e^{\imath\xi t}-1}{|\xi|^{H_0+\frac12}}\, \frac{e^{\imath\eta x}-1}{|\eta|^{H_1+\frac12}}\,, \tag{1.5}$$

where $\widehat{W}$ stands for the Fourier transform of a space-time white noise, and

$$c_{H_i} = \frac{1}{2}\left(\int_0^\infty d\xi\, \frac{1-\cos\xi}{|\xi|^{2H_i+1}}\right)^{-1/2}\,,\ i = 0,1.$$

It is easy to check that for every fixed $n \geq 1$, the process $B^{\kappa,n}$ is (a.s.) smooth on $\mathbb{R}^2$, due to the "frequency" cut-off $\{|\xi| \leq 2^{2\kappa n},\ |\eta| \leq 2^{\kappa n}\}$ in the representation (1.5). Besides, using standard results about the harmonizable representation of fractional sheets (see e.g. [26]), it can be shown that $B^{\kappa,n}$ converges (a.s.) to a fractional sheet B of Hurst indexes H_0, H_1.

With this approximation in hand, we will prove (Proposition 2.2) the existence of a threshold value $\alpha_{d,H} \geq 0$ such that for every $\alpha > \alpha_{d,H}$, the sequence $(\overset{\circ}{|}^{\kappa,n})_{n\geq 1}$ of classical solutions to

$$\begin{cases} \partial_t \overset{\circ}{|}^{\kappa,n} = \Delta \overset{\circ}{|}^{\kappa,n} + \partial_t\partial_x B^{\kappa,n}\,, & t \in [0,1],\ x \in \mathbb{R}, \\ \overset{\circ}{|}^{\kappa,n}_0 = 0\,, \end{cases} \tag{1.6}$$

converges in the scale $\mathcal{C}([0,T], \mathcal{W}^{-\alpha,p})$ for every $p \geq 1$, where the notation $\mathcal{W}^{-\alpha,p}$ refers to the fractional Bessel-potential space in $\mathbb{R}$ (see (2.3)). Along the above considerations, we henceforth define $\overset{\circ}{|}$ as the limit of this sequence.

(2) Discretization of the noise. We can now turn to the discretization procedure itself (note indeed that the previous smoothened solution $\overset{\circ}{|}^{\kappa,n}$ is clearly not sufficient in this regard). In fact, our general objective can be loosely summed up as follows: find a way to approximate the solution $\overset{\circ}{|}$ through a discrete iterative algorithm involving (a finite number of) Gaussian increments.

At a basic level, this challenge somehow corresponds to the search for an extension, in the heat setting, of the discretization methods used for the elementary linear standard differential equation

$$d\Psi_t = \dot{B}_t\,,\ \Psi_0 = 0\,, \tag{1.7}$$

where B is a standard one-parameter fractional Brownian motion. The solution of (1.7) is of course given by the process B itself, but when it comes to discretization, the standard linear interpolation of Ψ can be regarded as the result of a two-step scheme: (i) discretize the noise $\dot{B}$ through increments of B

$$\dot{B}^n := n\sum_{i=0}^{n-1}(B_{t_{i+1}}-B_{t_i})\mathbf{1}_{[t_i,t_{i+1})}\,, \quad t_i := \frac{i}{n}\,;$$

(ii) define Ψ^n as the convolution of $\dot{B}^n$ with the Heaviside kernel $\mathbf{1}_{\mathbb{R}_+}$, i.e. set, for $t\in[t_i,t_{i+1})$,

$$\Psi^n_t := \int_0^t ds\,\dot{B}^n_s = B_{t_i} + n(t-t_i)(B_{t_{i+1}}-B_{t_i})\,, \tag{1.8}$$

which indeed leads us to an aproximation of Ψ based on a Gaussian vector

$$(X_0,\ldots,X_{n-1}) := (B_{t_1}-B_0, B_{t_2}-B_{t_1},\ldots,B_{t_n}-B_{t_{n-1}})\,.$$

Let us transpose the above steps in the present heat situation, and more precisely to the equation (1.6) (as a reminiscence of our interpretation of ♀ as the limit of $\text{♀}^{\kappa,n}$). Accordingly, we first discretize the noise in (1.6) by means of rectangular increments of $B^{\kappa,n}$ along the (growing) dyadic grid

$$t_i := \frac{i}{2^n}\ \ (i=0,\ldots,2^n), \quad x_j := \frac{j}{2^n}\ \ (j=-2^{2n},\ldots,2^{2n})\,, \tag{1.9}$$

that is we consider the approximation of $\partial_t\partial_x B^{\kappa,n}$ on $[0,1]\times\mathbb{R}$ given by

$$\partial_t\partial_x\widetilde{B}^{\kappa,n} := \sum_{i=0}^{2^n-1}\sum_{j=-2^{2n}}^{2^{2n}-1}(2^{2n}\square^n_{i,j}B^{\kappa,n})\,\mathbf{1}_{\square^n_{ij}}\,, \tag{1.10}$$

where $\mathbf{1}_{\square^n_{ij}}(s,x) := \mathbf{1}_{[t_i,t_{i+1})}(s)\mathbf{1}_{[x_j,x_{j+1})}(x)$, and for every two-parameter path $b:[0,1]\times\mathbb{R}\to\mathbb{R}$,

$$\square^n_{i,j}b := b_{t_{i+1}}(x_{j+1})-b_{t_{i+1}}(x_j)-b_{t_i}(x_{j+1})+b_{t_i}(x_j)\,. \tag{1.11}$$

Then, following the one-parameter pattern in (1.8), we define the approximation $\widetilde{\text{♀}}^{\kappa,n}$ as the solution related to $\partial_t\partial_x\widetilde{B}^{\kappa,n}$, that is as the (well-defined) convolution

$$\widetilde{\text{♀}}^{\kappa,n}_t(x) := (G*\partial_t\partial_x\widetilde{B}^{\kappa,n})_t(x)\,. \tag{1.12}$$

Just as in (1.8), and for every $(t,x)\in[0,T]\times\mathbb{R}$, the value of $\widetilde{\text{♀}}^{\kappa,n}_t(x)$ can thus be expressed as a combination of the values of the Gaussian vector

$$\{\square^n_{i,j}B^{\kappa,n},\ i=0,\ldots,2^n-1,\ j=-2^{2n},\ldots,2^{2n}-1\}\,.$$

This natural noise-discretization step will be fully justified in Section 3: we will therein prove that, at least if the "frequency" parameter κ in (1.5) is small enough, i.e. if the ratio between the smoothening speed ($2^{\kappa n}$) and the discretization speed (2^n) is small enough, then the sequence $(\widetilde{\text{♀}}^{\kappa,n})_{n\geq 1}$ does converge to the actual solution ♀, as $n\to\infty$.

(3) Space-time discretization of the heat operator. Let us go back to the interpretation of the process $\widetilde{\text{♀}}^{\kappa,n}$ in (1.12) as the solution of the heat equation

$$\begin{cases}\partial_t\widetilde{\text{♀}}^{\kappa,n} = \Delta\widetilde{\text{♀}}^{\kappa,n} + \partial_t\partial_x\widetilde{B}^{\kappa,n}\,, & t\in[0,1],\ x\in\mathbb{R},\\ \widetilde{\text{♀}}^{\kappa,n}_0 = 0\,, & \end{cases} \tag{1.13}$$

and note that, for every fixed $n\geq 1$, $\partial_t\partial_x\widetilde{B}^{\kappa,n}$ now stands for a (random) bounded function on $[0,1]\times\mathbb{R}$, as it can immediately be seen from (1.10).

In order to achieve our full-discretization objective, we still need to propose an approximation scheme for the heat dynamics. In fact, when dealing with a well-defined perturbation function (such as $\partial_t\partial_x\widetilde{B}^{\kappa,n}$, for fixed $n \geq 1$), space-time discretizations of the heat operator can be derived from standard (deterministic) finite element methods, which ultimately generate basic linear iterative systems (see e.g. [22, 28]).

Our purpose in the third (and final) step of the study will thus be to carefully examine how these deterministic methods can be applied in our setting, and above all how the resulting approximation of (1.13) can be controlled in terms of the perturbation $\partial_t\partial_x\widetilde{B}^{\kappa,n}$ (seen as an element of $L^\infty([0,1]\times\mathbb{R})$). To implement this strategy, we will focus on the combination of a – space – Galerkin-type projection and a – time – implicit Euler scheme, a standard choice in the heat-approximation literature (see Section 4 for a complete description).

Of course, the involvement of the L^∞-norm of $\partial_t\partial_x\widetilde{B}^{\kappa,n}$ in the corresponding estimates can only come at a price as far as n is concerned (recall that $\partial_t\partial_x\widetilde{B}^{\kappa,n}$ is only expected to be uniformly bounded in n as a negative-order distribution). In light of our controls (see Proposition 4.6), a possible way to counterbalance this n-loss will consist in the application of the Galerkin procedure on a finer grid than the one used to discretize $\widetilde{B}^{\kappa,n}$. Let us slightly anticipate the next section and point out that this balancing phenomenon can be easily observed on the description (1.18) of our discretization scheme, by comparing the $(2^{-4n}, 2^{-2n})$ space-time mesh in (1.14)–(1.18) with the $(2^{-n}, 2^{-n})$ discretization mesh used for $B^{\kappa,n}$ in (1.15) (see also Remark 1.1).

1.2 Main discretization scheme

Let us now be more explicit about the algorithm resulting from the three above-described discretization steps, and also about our calibration of it (as far as grids are concerned). We recall first that for all $\kappa > 0$ and $n \geq 0$, the notation $B^{\kappa,n}$ refers to the smoothened version of B defined by (1.5), and at the core of our interpretation of (along Proposition 2.2).

From now on, we consider the parabolic-type grid (note the change of scaling with respect to (1.9))

$$t_i = t_i^n := \frac{i}{2^{4n}},\ i = 0,\ldots,2^{4n}, \quad x_j = x_j^n := \frac{j}{2^{2n}},\ j \in \mathbb{Z}. \tag{1.14}$$

Given $i = 0,\ldots,2^{4n}$, we will denote by $\tilde{i}$ the (only) integer such that $\frac{\tilde{i}}{2^n} \leq t_i < \frac{\tilde{i}+1}{2^n}$. In the same way, given $j \in \mathbb{Z}$, we denote by $\tilde{\tilde{j}}$ the (only) integer such that $\frac{\tilde{\tilde{j}}}{2^n} \leq x_j < \frac{\tilde{\tilde{j}}+1}{2^n}$.

With this notation, the noise increments $\delta B^{\kappa,n}$ involved in the scheme are defined as follows: for all $i = 0,\ldots,2^{4n}$ and $j \in \mathbb{Z}$,

$$\delta B^{\kappa,n}_{ij} := \mathbf{1}_{\{x_j > \frac{\tilde{\tilde{j}}}{2^n}\}} \square^n_{\tilde{i},\tilde{\tilde{j}}} B^{\kappa,n} + \mathbf{1}_{\{x_j = \frac{\tilde{\tilde{j}}}{2^n}\}} \left[\frac{1}{2}\square^n_{\tilde{i},\tilde{\tilde{j}}-1} B^{\kappa,n} + \frac{1}{2}\square^n_{\tilde{i},\tilde{\tilde{j}}} B^{\kappa,n}\right], \tag{1.15}$$

where we recall that the notation $\square$ for the rectangular increments has been introduced in (1.11).

Finally, we introduce the set of functions $\Phi_j^n : \mathbb{R} \to \mathbb{R}$ $(j \in \mathbb{Z})$ defined along the formula

$$\Phi_j^n(x) := \begin{cases} 2^{2n}(x - x_{j-1}) & \text{if } x \in [x_{j-1}, x_j] \\ 2^{2n}(x_{j+1} - x) & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise}, \end{cases} \tag{1.16}$$

and consider the related mass, resp. stiffness, matrix

$$\mathcal{M}_n := (\langle \Phi_j^n, \Phi_k^n \rangle)_{-2^{3n+1}+1 \leq j,k \leq 2^{3n+1}-1}, \quad \text{resp. } \mathcal{A}_n := (\langle \nabla\Phi_j^n, \nabla\Phi_k^n \rangle)_{-2^{3n+1}+1 \leq j,k \leq 2^{3n+1}-1}.$$

We are now in a position to describe our approximation process. Namely, for all $i = 0, \ldots, 2^{4n}$ and $x \in \mathbb{R}$, we set

$$\bar{\circ}^{\kappa,n}_{t_i}(x) := \sum_{j=-2^{3n+1}+1}^{2^{3n+1}-1} \bar{\bullet}^{j}_{t_i}\, \Phi^n_j(x)\,, \tag{1.17}$$

where the points $\bar{\bullet}^{j}_{t_i}$ ($i = 0, \ldots, 2^{4n}$, $j = -2^{3n+1}+1, \ldots, 2^{3n+1}-1$) are given by the iteration procedure

$$\left[2^{4n}\mathcal{M}_n + \mathcal{A}_n\right] \bar{\bullet}_{t_{i+1}} = 2^{4n}\mathcal{M}_n\, \bar{\bullet}_{t_i} + \delta B^{\kappa,n}_{i.} \tag{1.18}$$

with $\bar{\bullet}_{t_i} := \big(\bar{\bullet}^{j}_{t_i}\big)_{-2^{3n+1}+1 \le j \le 2^{3n+1}-1}$ and $\delta B^{\kappa,n}_{i.} := \big(\delta B^{\kappa,n}_{ij}\big)_{-2^{3n+1}+1 \le j \le 2^{3n+1}-1}$.

Remark 1.1. Observe that following (1.15) (and recalling the notation $\tilde{i}, \tilde{\tilde{j}}$), the above scheme only involves the rectangular increments of $B^{\kappa,n}$ over the sub-grid $(\frac{i}{2^n}, \frac{j}{2^n})_{i,j}$ of $(t_i, x_j)_{i,j}$.

Remark 1.2. The above specific calibration of the scheme (i.e. the choice of the specific grid (t_i, x_j) in (1.14)) is naturally derived from the subsequent theoretical convergence results. Note however that we do not expect this calibration to be optimal. In other words, the convergence property in the forthcoming Theorem 1.3 certainly remains true for coarser grids $t'_i = \frac{i}{2^{\lambda n}}$, $x'_j = \frac{j}{2^{\beta n}}$, with $1 \le \lambda < 4$ and $1 \le \beta < 2$ (possibly depending on (H_0, H_1)). See Proposition 4.6 and the related Remarks 4.8 and 4.9 for further details about this choice of calibration.

1.3 Main convergence statement

Let us present the main theoretical result of the paper, proving suitability of the discretization scheme (1.17)–(1.18).

Theorem 1.3. *Fix $(H_0, H_1) \in (0,1)^2$ such that $0 < 2H_0 + H_1 < 1$, and set*

$$\alpha_0 := 1 - (2H_0 + H_1) > 0\,. \tag{1.19}$$

Then, for every $\alpha > \alpha_0$, and for every smooth compactly-supported function $\rho : \mathbb{R} \to [0,1]$, there exist a deterministic constant $\nu = \nu((H_0, H_1), \alpha) > 0$, as well as a random constant $C = C(\rho, (H_0, H_1), \alpha) > 0$, such that for all $0 < \kappa \le \frac{\alpha_0}{5}$ and $n \ge 1$, one has almost surely

$$\sup_{i=0,\ldots,2^{4n}} \left\| \rho \cdot \left\{ \bar{\circ}^{\kappa,n}_{t_i} - \circ_{t_i} \right\} \right\|_{\mathcal{H}^{-\alpha}(\mathbb{R})} \le C\, 2^{-n\nu}\,. \tag{1.20}$$

The above result stems from the combination of the estimates obtained in Proposition 2.2 (noise smoothening), Proposition 3.1 (noise discretization) and Proposition 4.6 (space-time discretization of the heat operator). The retriction on κ, namely $0 < \kappa \le \frac{\alpha_0}{5}$, can be seen as a consequence of some balance strategy between the noise-smoothening step ($B \mapsto B^{\kappa,n}$) and the noise-discretization step ($\partial_t\partial_x B^{\kappa,n} \mapsto \partial_t\partial_x \widetilde{B}^{\kappa,n}$), as it will be detailed in Section 3.

Let us complete the statement of Theorem 1.3 with a few additional remarks.

Remark 1.4. The involvement of a (arbitrary) space cut-off function ρ in (1.20) must be understood as a way to express "local convergence" in the space $\mathcal{H}^{-\alpha}(\mathbb{R})$ of negative-order distributions. Observe indeed that if $\alpha = 0$, that is if one considers $\mathcal{H}^{-\alpha}(\mathbb{R}) = L^2(\mathbb{R})$, then by taking $\rho : \mathbb{R} \to [0,1]$ equal to 1 on any given compact set $K \subset \mathbb{R}$, one has of course $\|f\|_{L^2(K)} \le \|\rho \cdot f\|_{L^2(\mathbb{R})}$ for any $f \in L^2_{loc}(\mathbb{R})$, and accordingly estimates such as (1.20) would entail L^2-convergence on compact sets. Note also that the convergence results and its proof would certainly remain true for a more general class of weights ρ on $\mathbb{R}$, with support possibly non compact (e.g., for a Gaussian weight $\rho(x) := e^{-x^2}$).

Remark 1.5. Some possible explicit value for ν can be derived from the statements of Proposition 2.2, Proposition 3.1 and Proposition 4.6. For instance, with the notation α_0 introduced in (1.19), we can take

$$\nu := \min\left(\frac{2\alpha_0 H_0}{5}, \frac{\alpha_0 H_1}{5}, \frac{\alpha_0(\alpha-\alpha_0)}{5}, 1-\alpha_0, \frac{\alpha_0}{5}\right).$$

In any case, we do not expect the subsequent analysis to provide us with an "optimal" speed of convergence for the proposed scheme (1.18) (say for fixed $\alpha > \alpha_0$ in the left-hand side of (1.20)), due to our consideration of a deterministic strategy and a $L^2(\mathbb{R})$-norm in the space-time heat discretization step (see Proposition 4.6 and the related Remark 4.9 for further details).

Remark 1.6. As we evoked it earlier, our results can somehow be seen as an extension of the results in [2, 29] to the rough regime (1.3). The overall discretization method used in [2, 29] is indeed (partially) similar to ours, even if the latter references are concerned with more regular situations, where solutions can be treated as well-defined functions.

Besides, in both [2] and [29], the fact that one of the parameters H_i is assumed to be $\frac{1}{2}$ allows the authors to rely on some Itô-type isometry property (see [2, Theorem 2.1] and [29, Equation (2.13)]), which is not a tool at our disposal in the present *space-time* fractional setting.

Let us complete this brief comparison by mentioning the fact that both works [2] and [29] focus on the equation on a torus. The latter framework slightly deviates from the unbounded situation prevailing in the standard "fractional SPDE" literature (see e.g. [18, 20, 21]), which motivated our additional efforts to handle the problem on the whole Euclidean space.

The rest of the paper is organized as follows. In Section 2, we examine the noise-smoothening procedure toward a proper definition of ⚲. Then, in Section 3, we initiate the discretization scheme through the transition from $\partial_t\partial_x B^{\kappa,n}$ to $\partial_t\partial_x \widetilde{B}^{\kappa,n}$. The theoretical analysis is completed in Section 4, with the space-time discretization of the heat operator. Finally, we have provided, in Section 5, a few results and comments about the (partial) simulation of the algorithm (1.17)–(1.18).

From a technical point of view, the subsequent analysis relies on the combination of fractional calculus with (relatively) standard discretization techniques. For pedagogical purposes, we have endeavored to provide many details at every step of these investigations, which hopefully can make the study accessible to a large audience.

2 Definition of the solution

For the sake of completeness, let us first recall the definition of the fractional sheet, that is the field at the core of this study.

Definition 2.1. *On a complete probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*, we call a fractional sheet of Hurst indexes* $H_0, H_1 \in (0,1)$ *on* $[0,1]\times\mathbb{R}$ *any centered Gaussian field* $B : \Omega\times([0,1]\times\mathbb{R}) \to \mathbb{R}$ *with covariance function given by the formula: for all* $s,t \in [0,1]$ *and* $x,y\in\mathbb{R}$*,*

$$\mathbb{E}\big[B_s(x)B_t(y)\big] = R_{H_0}(s,t)R_{H_1}(x,y)\ , \text{ where } R_H(a,b) := \frac{1}{2}\big\{|a|^{2H}+|b|^{2H}-|a-b|^{2H}\big\}\ .$$

When $H_0 = H_1 = \frac{1}{2}$, the above definition of the fractional sheet is known to coincide with the one of a standard Brownian field. In any case, that is for every $(H_0, H_1) \in (0,1)^2$, it can be shown that B is not a differentiable field, and accordingly the definition of the noise $\dot{B}$ in (1.2) can only be understood as a general distribution. Owing to this lack of regularity, the interpretation of ⚲ as the convolution of $\dot{B}$ with the (singular) heat kernel G is clearly not a trivial issue, and we propose to address this question through a regularization procedure. Thus, for all fixed $\kappa > 0$ and $n \geq 0$, we consider the smooth approximation $B^{\kappa,n}$ of B provided by (1.5). Using the so-called harmonizable representation of B, i.e.

$$B_t(x) = c_{H_0} c_{H_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{W}(d\xi, d\eta) \, \frac{e^{\imath \xi t} - 1}{|\xi|^{H_0 + \frac{1}{2}}} \frac{e^{\imath \eta x} - 1}{|\eta|^{H_1 + \frac{1}{2}}} \,, \tag{2.1}$$

it can indeed be shown that for every $\kappa > 0$, one has almost surely

$$B^{\kappa,n} \stackrel{n\to\infty}{\longrightarrow} B \quad \text{in } \mathcal{C}([0,1] \times \mathbb{R}; \mathbb{R}) \,.$$

Our objective now is to study the convergence of the sequence of (classical) solutions associated with $(B^{\kappa,n})_{n\geq 1}$, that is the sequence of well-defined processes

$$\Psi_t^{\kappa,n}(x) := (G * \partial_t \partial_x B^{\kappa,n})_t(x) \,. \tag{2.2}$$

To do so, we will appeal to the following scale of fractional Sobolev spaces.

Notation. For all $s \in \mathbb{R}$ and $1 \leq p < \infty$, let the notation $\mathcal{W}^{s,p}$ refer the Bessel-potential

$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \ \|f|\mathcal{W}^{s,p}(\mathbb{R}^d)\| = \|\mathcal{F}^{-1}(\{1 + |.|^2\}^{\frac{s}{2}} \mathcal{F} f) | L^p(\mathbb{R})\| < \infty \right\} \,. \tag{2.3}$$

Also we will consider the spaces

$$\mathcal{H}^s := \mathcal{W}^{s,2}, \quad \text{for every } s \in \mathbb{R} \,. \tag{2.4}$$

Using the above notation, the result at the basis of our interpretation of (1.1) can be stated as follows.

Proposition 2.2. *Fix $(H_0, H_1) \in (0,1)^2$ such that $0 < 2H_0 + H_1 < 1$, and set*

$$\alpha_0 := 1 - \big(2H_0 + H_1\big) > 0 \,. \tag{2.5}$$

Then the following assertions hold true:
(i) *For all $\kappa > 0$ and $t > 0$,*

$$\mathbb{E}\Big[\big\|\Psi_t^{\kappa,n}\big\|^2_{L^2([0,1])}\Big] \stackrel{n\to\infty}{\longrightarrow} \infty \,. \tag{2.6}$$

(ii) *For every $\kappa > 0$ and for every cut-off function $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ (i.e., smooth and compactly-supported), the sequence $(\rho \cdot \Psi^{\kappa,n})_{n\geq 0}$ converges in the space $L^{2p}(\Omega; \mathcal{C}([0,1]; \mathcal{W}^{-\alpha,2p}(\mathbb{R})))$, for all $\alpha > \alpha_0$ and $p \geq 1$. Moreover, the limit, that we denote by $\rho \cdot \Psi$, does not depend on κ.*
(iii) *For all $\alpha > \alpha_0$, $\kappa > 0$, $p \geq 1$, $n \geq 1$ and $\varsigma > 0$ such that*

$$0 < \varsigma < \min\big(2H_0, H_1, \alpha - \alpha_0\big), \tag{2.7}$$

one has almost surely

$$\sup_{t\in[0,1]} \big\|\rho \cdot \Psi_t^{\kappa,n} - \rho \cdot \Psi_t\big\|_{\mathcal{H}^{-\alpha}(\mathbb{R})} \lesssim 2^{-\varsigma\kappa n}, \tag{2.8}$$

where the (random) proportional constant does not depend on n.

The proof of Proposition 2.2 will be developed in Sections 2.1, 2.2 and 2.3 below.

Note that even if the result of item (i) will not serve us as such in the sequel, it emphasizes the fact that the object Ψ at the center of this work could not be handled as a function in space (at least not a function in $L^2_{\text{loc}}(\mathbb{R})$), which is certainly the main specificity of our setting.

Besides, observe that the convergence result in item (ii) actually gives birth to a family of processes

$$\left\{\rho \cdot \Psi \in L^{2p}(\Omega; \mathcal{C}([0,1]; \mathcal{W}^{-\alpha,2p}(\mathbb{R}))),\ \rho \in \mathcal{C}_c^\infty(\mathbb{R})\right\}.$$

For the sake of completeness, we can then patch together those local solutions into a single distribution Ψ. The details of this canonical procedure can be found in [10, Section 2.5], and we will only sum up the result in the following Proposition-Definition (to simplify the presentation, we fix $p \geq 2$ and $\alpha > \alpha_0$, and we denote by $\mathcal{F}([0,1]; \mathcal{D}'(\mathbb{R}))$ the set of distributional-valued functions on $[0,1]$).

Proposition-Definition 2.3. *Let $\mathcal{P}$ stand for the set of sequences $\sigma = (\sigma_k)_{k\geq 1}$, where, for each $k \geq 1$, $\sigma_k : \mathbb{R} \to \mathbb{R}$ is a smooth function such that*

$$\sigma_k(x) = \begin{cases} 1 & \text{if } |x| \leq k\,, \\ 0 & \text{if } |x| \geq k+1\,. \end{cases}$$

Then, for every $\sigma \in \mathcal{P}$, there exists a subspace $\Omega^{(\sigma)} \subset \Omega$ of full measure 1 and an element

$$\Psi^{(\sigma)} : \Omega^{(\sigma)} \to \mathcal{F}([0,1]; \mathcal{D}'(\mathbb{R}))$$

such that the following assertions hold true:
(i) *For any (space) cut-off function $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ and for any $\kappa > 0$, one has, on $\Omega^{(\sigma)}$,*

$$\rho \cdot \Psi^{\kappa,n} \underset{n\to\infty}{\to} \rho \cdot \Psi^{(\sigma)} \quad \text{in } \mathcal{C}([0,1]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))\,.$$

(ii) *If $\sigma, \gamma \in \mathcal{P}$, then one has $\Psi^{(\sigma)} = \Psi^{(\gamma)}$ on $\Omega^{(\sigma)} \cap \Omega^{(\gamma)}$.*

Owing to these two properties, we define Ψ as $\Psi := \Psi^{(\sigma)}$ for some fixed (arbitrary) sequence $\sigma \in \mathcal{P}$, and we call this random element in $\mathcal{F}([0,1]; \mathcal{D}'(\mathbb{R}))$ the mild solution of (1.1).

Thanks to the above result, we are now endowed with a *globally* defined solution Ψ, which locally coincides with the limits exhibited in Proposition 2.2. This being said, in our subsequent investigations, we will only focus on *local* convergence to Ψ, as it can be seen from our main estimate (1.20).

Before we turn to the proof of Proposition 2.2, let us complete the statement with two remarks.

Remark 2.4. The consideration of the "Fourier-type" approximation $B^{\kappa,n}$ of B is quite natural in our fractional Sobolev setting, and it will indeed readily provide us with a convenient covariance formula for the process $\Psi^{\kappa,n}$ (see Proposition 2.6).

Another usual choice for the approximation of B is the one derived from a mollifying procedure, that is one takes $B^{\varphi,\kappa,n} := \varphi_{\kappa,n} * B$, where $\varphi_{\kappa,n}(s,x) := 2^{3\kappa n}\varphi(2^{2\kappa n}s, 2^{\kappa n}x)$, for some mollifier $\varphi : \mathbb{R}^2 \to \mathbb{R}$. In fact, our approximation $B^{\kappa,n}$ can somehow be regarded as a particular case of this general mollifying procedure. Indeed, starting from the representation (2.1) of B, one can write (at least formally)

$$\begin{aligned}
(\partial_t\partial_x B^{\varphi,\kappa,n})(t,x) &= (\varphi_{\kappa,n} * \partial_t\partial_x B)(t,x) \\
&= c_{H_0}c_{H_1} \iint ds dy\, \varphi_{\kappa,n}(s,y) \iint \widehat{W}(d\xi, d\eta)\,(-\xi\eta)\frac{e^{\imath\xi(t-s)}}{|\xi|^{H_0+\frac12}}\frac{e^{\imath\eta(x-y)}}{|\eta|^{H_1+\frac12}} \\
&= c_{H_0}c_{H_1} \iint \widehat{W}(d\xi, d\eta)\,\widehat{\varphi_{\kappa,n}}(\xi,\eta)(-\xi\eta)\frac{e^{\imath\xi t}}{|\xi|^{H_0+\frac12}}\frac{e^{\imath\eta x}}{|\eta|^{H_1+\frac12}} \\
&= c_{H_0}c_{H_1} \iint \widehat{W}(d\xi, d\eta)\,\widehat{\varphi}(2^{-2\kappa n}\xi, 2^{-\kappa n}\eta)(-\xi\eta)\frac{e^{\imath\xi t}}{|\xi|^{H_0+\frac12}}\frac{e^{\imath\eta x}}{|\eta|^{H_1+\frac12}}\,,
\end{aligned}$$

and thus, picking φ such that $\widehat{\varphi}(\xi,\eta) = \mathbf{1}_{\{|\xi|\leq 1\}}\mathbf{1}_{\{|\eta|\leq 1\}}$, one recovers the approximation $\partial_t\partial_x B^{\kappa,n}$ of the noise. We think that, at the price of a few technical modifications, it is certainly possible to extend the whole subsequent analysis to a more general class of mollifying approximations $B^{\varphi,\kappa,n}$.

Remark 2.5. The above construction procedure of the solution $\mathcal{T}$, based on the specific approximation $B^{\kappa,n}$ of B, has already been implemented in [8, 9] for the fractional wave equation, and in [10] for the Schrödinger fractional equation. To be more specific, in the three references [8, 9, 10], the authors' analysis only relies on the consideration of $B^{1,n}$, i.e. $B^{\kappa,n}$ with $\kappa = 1$. Letting κ vary (in Section 3) will here give us the possibility to maintain a certain balance within the two-step transformation process of the noise (see Remark 3.2).

2.1 Preliminary considerations

Observe first that the approximated noise $\partial_t\partial_x B^{\kappa,n}$ derived from the representation (1.5) could be equivalently defined as the centered (real) Gaussian field with covariance given by the formula: for all $\kappa,\kappa' > 0$, $n,m \geq 1$, $s,t \geq 0$ and $x,y \in \mathbb{R}^d$,

$$\mathbb{E}\Big[(\partial_t\partial_x B^{\kappa,n})_s(x)(\partial_t\partial_x B^{\kappa',m})_t(y)\Big] = c_H^2 \int_{(\xi,\eta)\in D^{\kappa,n}\cap D^{\kappa',m}} \frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}} e^{\imath\xi(s-t)}e^{\imath\eta(x-y)}$$

where $c_H := c_{H_0}c_{H_1}$ and where we have set, for all $\kappa > 0$ and $n \geq 1$,

$$D^{\kappa,n} := \{(\xi,\eta)\in\mathbb{R}^2 : |\xi| \leq 2^{2\kappa n}, |\eta| \leq 2^{\kappa n}\}. \tag{2.9}$$

Based on this expression, we can readily compute

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{T}^{\kappa,n}_s(x)\mathcal{T}^{\kappa',m}_t(y)\Big]\\
&= \int_0^s du\int_{\mathbb{R}}dz\int_0^t dv\int_{\mathbb{R}}dw\, G_{s-u}(x-z)G_{t-v}(y-w)\mathbb{E}\Big[(\partial_t\partial_x B^{\kappa,n})_u(z)(\partial_t\partial_x B^{\kappa',m})_v(w)\Big]\\
&= c_H^2\int_{(\xi,\eta)\in D^{\kappa,n}\cap D^{\kappa',m}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}\\
&\qquad\qquad \int_0^s du\int_{\mathbb{R}}dz\int_0^t dv\int_{\mathbb{R}}dw\, G_{s-u}(x-z)G_{t-v}(y-w)e^{\imath\xi(u-v)}e^{\imath\eta(z-w)}\\
&= c_H^2\int_{(\xi,\eta)\in D^{\kappa,n}\cap D^{\kappa',m}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}e^{\imath\eta(x-y)}\\
&\qquad\qquad \Big[\int_0^s du\, e^{\imath\xi(s-u)}\int_{\mathbb{R}}dz\, e^{-\imath\eta z}G_u(z)\Big]\Big[\int_0^t dv\, e^{-\imath\xi(t-v)}\int_{\mathbb{R}}dw\, e^{\imath\eta w}G_v(w)\Big],
\end{aligned} \tag{2.10}$$

which leads us to the following assertion.

Proposition 2.6. *The family of random variables*

$$\big\{\mathcal{T}^{\kappa,n}_s(x),\ \kappa > 0,\ n \geq 1,\ s \geq 0,\ x \in \mathbb{R}\big\}$$

defines a centered (real) Gaussian field with covariance given by the formula: for all $\kappa,\kappa' > 0$*,* $n,m \geq 1$*,* $s,t \geq 0$ *and* $x,y \in \mathbb{R}^d$*,*

$$\mathbb{E}\Big[\mathcal{T}^{\kappa,n}_s(x)\mathcal{T}^{\kappa',m}_t(y)\Big] = c_H^2\int_{(\xi,\eta)\in D^{\kappa,n}\cap D^{\kappa',m}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}\gamma_s(\xi,|\eta|)\overline{\gamma_t(\xi,|\eta|)}\, e^{\imath\eta(x-y)}, \tag{2.11}$$

where the notation $D^{\kappa,n}$ *has been introduced in* (2.9)*, and where the quantity* $\gamma_t(\xi,r)$ *is defined for all* $t \geq 0$*,* $\xi \in \mathbb{R}$ *and* $r > 0$ *by*

$$\gamma_t(\xi,r) := \int_0^t ds\, e^{\imath\xi(t-s)}\widehat{G_s}(r) = e^{\imath\xi t}\int_0^t e^{-s\{r^2+\imath\xi\}}ds. \tag{2.12}$$

Notation 2.7. *For any function f on $\mathbb{R}_+$ and for all times $0 \le s \le t$, we set $f_{s,t} := f_t - f_s$.*

The following elementary bound on $\gamma_{s,t}$ will turn out to be the key estimate in the proof of Proposition 2.2.

Lemma 2.8. *Fix $0 < H < 1$. Then for all $\eta \in \mathbb{R}$, $\varepsilon \in (0, H)$ and $0 \le s \le t \le 1$, one has*

$$\int_{\mathbb{R}} d\xi \, \frac{|\gamma_{s,t}(\xi, |\eta|)|^2}{|\xi|^{2H-1}} \lesssim \frac{|t-s|^{\varepsilon}}{1 + |\eta|^{4(H-\varepsilon)}} \,. \tag{2.13}$$

Proof. By the definition (2.12) of $\gamma_t(\xi, |\eta|)$, one has

$$\gamma_{s,t}(\xi, |\eta|) = \left\{ e^{\imath \xi t} - e^{\imath \xi s} \right\} \int_0^t dr \, e^{-r(|\eta|^2 + \imath \xi)} + e^{\imath \xi s} \int_s^t dr \, e^{-r(|\eta|^2 + \imath \xi)} \,,$$

from which we immediately deduce, for all $\varepsilon_1, \varepsilon_2, \lambda \in [0, 1]$,

$$\left| \gamma_{s,t}(\xi, |\eta|) \right| \lesssim \frac{|t-s|^{\varepsilon_1} |\xi|^{\varepsilon_1}}{||\eta|^2 + \imath \xi|^{\lambda}} + \frac{|t-s|^{\varepsilon_2}}{||\eta|^2 + \imath \xi|^{1-\varepsilon_2}} \,. \tag{2.14}$$

Based on this estimate, we have on the one hand, for any $\varepsilon \in (0, H)$,

$$\int_{\mathbb{R}} d\xi \, \frac{|\gamma_{s,t}(\xi, |\eta|)|^2}{|\xi|^{2H-1}} \lesssim |t-s|^{2\varepsilon} \left[\int_{|\xi| \le 1} \frac{d\xi}{|\xi|^{2H-1}} + \int_{|\xi| \ge 1} \frac{d\xi}{|\xi|^{1+2(H-\varepsilon)}} \right] \lesssim |t-s|^{2\varepsilon} \,. \tag{2.15}$$

On the other hand, thanks to (2.14), one has for all $|\eta| \ge 1$ and $\varepsilon \in (0, H/2)$,

$$\begin{aligned} &\int_{\mathbb{R}} d\xi \, \frac{|\gamma_{s,t}(\xi, |\eta|)|^2}{|\xi|^{2H-1}} \\ &\lesssim |t-s|^{2\varepsilon} \left[\frac{1}{|\eta|^{4-4\varepsilon}} \int_{|\xi| \le 1} \frac{d\xi}{|\xi|^{2H-1}} + \frac{1}{|\eta|^{4(H-2\varepsilon)}} \int_{|\xi| \ge 1} \frac{d\xi}{|\xi|^{2H-1} |\xi|^{2(1-H+\varepsilon)}} \right] \lesssim \frac{|t-s|^{2\varepsilon}}{|\eta|^{4(H-2\varepsilon)}} \,. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16) clearly yields (2.13). □

The following – independent – technical lemma, borrowed from [10, Lemma 2.6], will also prove useful in the estimates of the next section.

Lemma 2.9. *Let $\rho : \mathbb{R} \to \mathbb{R}$ be a test function and fix $\sigma \in \mathbb{R}$. Then, for every $p \ge 1$ and for all $\eta_1, \ldots, \eta_p \in \mathbb{R}^d$, it holds that*

$$\left| \int_{\mathbb{R}} dx \prod_{i=1}^{p} \int_{\mathbb{R}^2} \frac{d\lambda_i d\tilde{\lambda}_i}{\{1 + |\lambda_i|^2\}^{\frac{\sigma}{2}} \{1 + |\tilde{\lambda}_i|^2\}^{\frac{\sigma}{2}}} e^{\imath \langle x, \lambda_i - \tilde{\lambda}_i \rangle} \widehat{\rho}(\lambda_i - \eta_i) \overline{\widehat{\rho}(\tilde{\lambda}_i - \eta_i)} \right| \lesssim \prod_{i=1}^{p} \frac{1}{\{1 + |\eta_i|^2\}^{\sigma}} \,,$$

where the proportional constant only depends on ρ and σ.

Remark 2.10. The above preliminary material, as well as the subsequent proof, can be seen as the "heat" counterpart of the analysis carried out in [8, Section 2] for the wave model, and in [10, Section 2.1] for the Schrödinger case.

2.2 Proof of Proposition 2.2, item (i)

Using the covariance formula (2.11), we can immediately write the second moment under consideration as

$$\mathbb{E}\left[\| \mathring{|}_t^{\kappa,n} \|_{L^2([0,1])}^2 \right] = \int_0^1 dx \, \mathbb{E}\left[\left| \mathring{|}_t^{\kappa,n}(x) \right|^2 \right] = c_H^2 \int_{(\xi,\eta) \in D^{\kappa,n}} \frac{d\xi \, d\eta}{|\xi|^{2H_0 - 1} |\eta|^{2H_1 - 1}} \left| \gamma_t(\xi, |\eta|) \right|^2 .$$

Using elementary changes of variable, we obtain that

$$\begin{aligned}
&\int_{(\xi,\eta)\in D^{\kappa,n}} \frac{d\xi\, d\eta}{|\xi|^{2H_0-1}|\eta|^{2H_1-1}} \big|\gamma_t(\xi,|\eta|)\big|^2 \\
&= 2^{2\kappa n(3-(2H_0+H_1))} \int_{|\xi|\leq 1, |\eta|\leq 1} \frac{d\xi\, d\eta}{|\xi|^{2H_0-1}|\eta|^{2H_1-1}} \big|\gamma_t(2^{2\kappa n}\xi, 2^{\kappa n}|\eta|)\big|^2 \\
&\geq c_1\, 2^{2\kappa n(3-(2H_0+H_1))} \int_{\frac12\leq|\xi|\leq 1, \frac12\leq|\eta|\leq 1} d\xi d\eta\, \big|\gamma_t(2^{2\kappa n}\xi, 2^{\kappa n}|\eta|)\big|^2
\end{aligned}$$

for some constant $c_1>0$. Then observe that for all $\frac12 < |\xi| \leq 1$ and $\frac12 < |\eta| < 1$,

$$\big|\gamma_t(2^{2\kappa n}\xi, 2^{\kappa n}|\eta|)\big|^2 = 2^{-4\kappa n} \frac{\big|1-e^{-2^{2\kappa n}t(|\eta|^2+\imath\xi)}\big|^2}{\big||\eta|^2+\imath\xi\big|^2} \geq c_2\, 2^{-4\kappa n}\big|1-e^{-2^{2\kappa n}t(|\eta|^2+\imath\xi)}\big|^2$$

for some constant $c_2>0$. Thus we have shown the existence of a constant $c_3>0$ such that

$$\mathbb{E}\Big[\big\|\Upsilon^{\kappa,n}_t\big\|^2_{L^2([0,1])}\Big] \geq c_3\, 2^{2\kappa n(1-(2H_0+H_1))} \int_{\frac12\leq|\xi|\leq 1, \frac12\leq|\eta|\leq 1} d\xi d\eta\, \big|1-e^{-2^{2\kappa n}t(|\eta|^2+\imath\xi)}\big|^2 .$$

Finally, since $t>0$, we can use the dominated convergence theorem to assert that

$$\int_{\frac12\leq|\xi|\leq 1, \frac12\leq|\eta|\leq 1} d\xi d\eta\, \big|1-e^{-2^{2\kappa n}t(|\eta|^2+\imath\xi)}\big|^2 \overset{n\to\infty}{\longrightarrow} 1,$$

which, due to the condition $2H_0+H_1<1$, leads us to the desired divergence statement in (2.6).

2.3 Proof of Proposition 2.2, items (ii) and (iii)

Following the statement of the proposition, we fix $\alpha>\alpha_0$, where α_0 is the quantity defined by (2.5). Besides, recall that the notation $f_{s,t}$ for time increments has been introduced in Notation 2.7.

Step 1: A moment estimate. We show that for all $p\geq 1$, $1\leq n\leq m$, $0<\kappa\leq\kappa'$, $0\leq s\leq t\leq 1$ and $\varsigma>0$ satisfying (2.7), we can find $\varepsilon>0$ small enough such that

$$\int_{\mathbb{R}} dx\, \mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\Upsilon^{\kappa',m}_{s,t}-\Upsilon^{\kappa,n}_{s,t}\big]\big)\Big)(x)\Big|^{2p}\Big] \lesssim 2^{-2n\kappa\varsigma p}|t-s|^{2\varepsilon p}\,, \tag{2.17}$$

where the proportional constant only depends on p,α and ρ.

One can first notice that the random variable under consideration is clearly Gaussian, and so, for every $p\geq 1$, one has

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\Upsilon^{\kappa',m}_{s,t}-\Upsilon^{\kappa,n}_{s,t}\big]\big)\Big)(x)\Big|^{2p}\Big] \\
&\leq c_p\left(\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\Upsilon^{\kappa',m}_{s,t}-\Upsilon^{\kappa,n}_{s,t}\big]\big)\Big)(x)\Big|^{2}\Big]\right)^p,
\end{aligned} \tag{2.18}$$

where the constant c_p only depends on p. Let us then write

$$\begin{aligned}
&\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\Upsilon^{\kappa',m}_{s,t}-\Upsilon^{\kappa,n}_{s,t}\big]\big)\Big)(x) \\
&= \int_{\mathbb{R}} d\lambda\, e^{\imath x\lambda}\{1+|\lambda|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\Upsilon^{\kappa',m}_{s,t}-\Upsilon^{\kappa,n}_{s,t}\big]\big)(\lambda) \\
&= \int_{\mathbb{R}} d\lambda\, \{1+|\lambda|^2\}^{-\frac{\alpha}{2}} e^{\imath x\lambda}\left(\int_{\mathbb{R}} d\beta\, \widehat{\rho}(\lambda-\beta)\mathcal{F}\big(\big[\Upsilon^{\kappa',m}_{s,t}-\Upsilon^{\kappa,n}_{s,t}\big]\big)(\beta)\right),
\end{aligned}$$

and hence

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\Psi^{\kappa',m}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big)\Big)(x)\Big|^2\Big]\\
&=\int_{\mathbb{R}^2}\frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\frac{\alpha}{2}}\{1+|\tilde{\lambda}|^2\}^{\frac{\alpha}{2}}}e^{\imath x(\lambda-\tilde{\lambda})}\int_{\mathbb{R}^2}d\beta d\tilde{\beta}\,\widehat{\rho}(\lambda-\beta)\overline{\widehat{\rho}(\tilde{\lambda}-\tilde{\beta})}\mathcal{Q}^{\kappa,\kappa'}_{n,m;s,t}(\beta,\tilde{\beta}),
\end{aligned}
\tag{2.19}
$$

where we have set

$$
\mathcal{Q}^{\kappa,\kappa'}_{n,m;s,t}(\beta,\tilde{\beta}):=\mathbb{E}\Big[\mathcal{F}\big(\big[\Psi^{\kappa',m}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big)(\beta)\overline{\mathcal{F}\big(\big[\Psi^{\kappa',m}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big)(\tilde{\beta})}\Big].
$$

Based on the covariance formula (2.11), one has now

$$
\begin{aligned}
&\mathbb{E}\Big[\big[\Psi^{\kappa',m}_{s,t}(y)-\Psi^{\kappa,n}_{s,t}(y)\big]\big[\Psi^{\kappa',m}_{s,t}(\tilde{y})-\Psi^{\kappa,n}_{s,t}(\tilde{y})\big]\Big]\\
&=c_H^2\int_{(\xi,\eta)\in D^{\kappa',m}\setminus D^{\kappa,n}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}|\gamma_{s,t}(\xi,|\eta|)|^2e^{\imath\eta y}e^{-\imath\eta\tilde{y}},
\end{aligned}
$$

which allows us to recast the above quantity into

$$
\mathcal{Q}^{\kappa,\kappa'}_{n,m;s,t}(\beta,\tilde{\beta})=c_H^2\int_{(\xi,\eta)\in D^{\kappa',m}\setminus D^{\kappa,n}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}|\gamma_{s,t}(\xi,|\eta|)|^2\delta_{\beta=\eta}\delta_{\tilde{\beta}=\eta}.
\tag{2.20}
$$

Thanks to (2.18), (2.19) and (2.20), we get that

$$
\begin{aligned}
&\int_{\mathbb{R}}\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\Psi^{\kappa',m}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big)\Big)(x)\Big|^{2p}\Big]dx\\
&\lesssim\int_{\mathbb{R}}dx\Big(\int_{(\xi,\eta)\in D^{\kappa',m}\setminus D^{\kappa,n}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}|\gamma_{s,t}(\xi,|\eta|)|^2\\
&\qquad\qquad\int_{\mathbb{R}^2}\frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\frac{\alpha}{2}}\{1+|\tilde{\lambda}|^2\}^{\frac{\alpha}{2}}}e^{\imath x(\lambda-\tilde{\lambda})}\,\widehat{\rho}(\lambda-\eta)\overline{\widehat{\rho}(\tilde{\lambda}-\eta)}\Big)^p\\
&\lesssim\Big(\int_{(\xi,\eta)\in D^{\kappa',m}\setminus D^{\kappa,n}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\,d\xi d\eta\Big)^p,
\end{aligned}
$$

where the last inequality is a consequence of Lemma 2.9.

Observe now that $\mathbf{1}_{D^{\kappa',m}\setminus D^{\kappa,n}}\leq\mathbf{1}_{\{(\xi,\eta)\in\mathbb{R}^2:\,|\xi|\geq 2^{2\kappa n}\}}+\mathbf{1}_{\{(\xi,\eta)\in\mathbb{R}^2:\,|\eta|\geq 2^{\kappa n}\}}$, and thus

$$
\begin{aligned}
&\Big(\int_{(\xi,\eta)\in D^{\kappa',m}\setminus D^{\kappa,n}}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2H_1-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\,d\xi d\eta\Big)^p\\
&\lesssim\Big(\int_{|\xi|\geq 2^{2\kappa n}}\frac{d\xi}{|\xi|^{2H_0-1}}\int_{\mathbb{R}}\frac{d\eta}{|\eta|^{2H_1-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\,d\xi d\eta\Big)^p\\
&\qquad+\Big(\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H_0-1}}\int_{|\eta|\geq 2^{\kappa n}}\frac{d\eta}{|\eta|^{2H_1-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\,d\xi d\eta\Big)^p\\
&=:(\mathbb{I}^{\kappa,n}(s,t))^p+(\mathbb{II}^{\kappa,n}(s,t))^p.
\end{aligned}
\tag{2.21}
$$

Let us estimate the quantity $\mathbb{I}^{\kappa,n}(s,t)$ first. To do so, pick $\varsigma>0$ satisfying (2.7), which yields

$$
\begin{aligned}
\mathbb{I}^{\kappa,n}(s,t)&\leq 2^{-2n\kappa\varsigma}\int_{\mathbb{R}^2}\frac{d\xi}{|\xi|^{2(H_0-\frac{\varsigma}{2})-1}}\frac{d\eta}{|\eta|^{2H_1-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\\
&\lesssim 2^{-2n\kappa\varsigma}\int_0^\infty\frac{dr}{r^{2H_1-1}\{1+r^2\}^{\alpha}}\Big(\int_{\mathbb{R}}d\xi\,\frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2(H_0-\frac{\varsigma}{2})-1}}\Big).
\end{aligned}
$$

We are here in a position to apply Lemma 2.8 with $H := H_0 - \frac{\varsigma}{2}$, which entails that for all $0 < \varepsilon < H_0 - \frac{\varsigma}{2}$,

$$\begin{aligned}\mathbb{I}^{\kappa,n}(s,t) &\lesssim 2^{-2n\kappa\varsigma}|t-s|^{\varepsilon}\int_0^\infty \frac{dr}{r^{2H_1-1}\{1+r^2\}^{\alpha}}\frac{1}{1+r^{4(H_0-\frac{\varsigma}{2}-\varepsilon)}}\\ &\lesssim 2^{-2n\kappa\varsigma}|t-s|^{\varepsilon}\left(\int_0^1 \frac{dr}{r^{2H_1-1}}+\int_1^\infty \frac{1}{r^{2(\alpha-\alpha_0-\varsigma)+1-4\varepsilon}}dr\right).\end{aligned} \tag{2.22}$$

Owing to Assumption (2.7), we can pick $\varepsilon > 0$ small enough such that $\alpha - \alpha_0 - \varsigma > 2\varepsilon$, and for this choice, it is readily checked that the integrals in brackets in (2.22) are both finite. We have thus established that

$$\mathbb{I}^{\kappa,n}(s,t) \lesssim 2^{-2n\kappa\varsigma}|t-s|^{\varepsilon}.$$

The estimation of $\mathbb{III}^{\kappa,n}(s,t)$ can then be done along similar arguments. Namely, one has, for all $0 < \varepsilon < H_0$,

$$\begin{aligned}\mathbb{III}^{\kappa,n}(s,t) &\le 2^{-2n\kappa\varsigma}\int_{\mathbb{R}^2}\frac{d\xi}{|\xi|^{2H_0-1}}\frac{d\eta}{|\eta|^{2(H_1-\varsigma)-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\\ &\lesssim 2^{-2n\kappa\varsigma}\int_0^\infty \frac{dr}{r^{2(H_1-\varsigma)-1}\{1+r^2\}^{\alpha}}\left(\int_{\mathbb{R}}d\xi\,\frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H_0-1}}\right)\\ &\lesssim 2^{-2n\kappa\varsigma}|t-s|^{\varepsilon}\int_0^\infty \frac{dr}{r^{2(H_1-\varsigma)-1}\{1+r^2\}^{\alpha}}\frac{1}{1+r^{4(H_0-\varepsilon)}}\\ &\lesssim 2^{-2n\kappa\varsigma}|t-s|^{\varepsilon}\left(\int_0^1\frac{dr}{r^{2(H_1-\varsigma)-1}}+\int_1^\infty\frac{1}{r^{2(\alpha-\alpha_0-\varsigma)+1-4\varepsilon}}dr\right).\end{aligned}$$

Again, we can pick $\varepsilon > 0$ so that $\alpha - \alpha_0 - \varsigma > 2\varepsilon$, which leads us to

$$\mathbb{III}^{\kappa,n}(s,t) \lesssim 2^{-2n\kappa\varsigma}|t-s|^{\varepsilon}.$$

Going back to (2.21), we obtain the expected estimate (2.17).

Step 2: Conclusion. Estimate (2.17) can be equivalently formulated as

$$\mathbb{E}\Big[\big\|\rho\cdot\big[\Psi^{\kappa',m}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big] \lesssim 2^{-2n\kappa\varsigma p}|t-s|^{2\varepsilon p}, \tag{2.23}$$

for all $p \ge 1$, $1 \le n \le m$, $0 < \kappa \le \kappa'$, $0 \le s \le t \le 1$, $\varsigma > 0$ satisfying (2.7), and $\varepsilon > 0$ small enough. In particular, it holds that

$$\mathbb{E}\Big[\big\|\rho\cdot\big[\Psi^{\kappa,m}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big] \lesssim 2^{-2n\kappa\varsigma p}|t-s|^{2\varepsilon p}, \tag{2.24}$$

and

$$\mathbb{E}\Big[\big\|\rho\cdot\big[\Psi^{\kappa',n}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big] \lesssim 2^{-2n\kappa\varsigma p}|t-s|^{2\varepsilon p}. \tag{2.25}$$

By picking $p \ge 1$ large enough in (2.24), we get, by the Kolmogorov criterion, that $\rho\cdot\big[\Psi^{\kappa,m}-\Psi^{\kappa,n}\big] \in \mathcal{C}([0,1];\mathcal{W}^{-\alpha,2p}(\mathbb{R}^d))$ almost surely. We can then apply the classical Garsia-Rodemich-Rumsey estimate and assert that a.s, for all $p \ge 1$, $\varepsilon' > 0$, $0 \le t \le 1$, one has

$$\big\|\rho\cdot\big[\Psi^{\kappa,m}_{t}-\Psi^{\kappa,n}_{t}\big]\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}} \lesssim \int_{[0,1]^2}\frac{\big\|\rho\cdot\big[\Psi^{\kappa,m}_{u,v}-\Psi^{\kappa,n}_{u,v}\big]\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}}{|u-v|^{2\varepsilon' p+2}}\,dudv,$$

for some (deterministic) proportional constant that only depends on ε' and p.

As a consequence, using (2.23) again, we obtain that for any $0 < \varepsilon' < \varepsilon$,

$$\mathbb{E}\Big[\big\|\rho\cdot\big[\Psi^{\kappa,m}-\Psi^{\kappa,n}\big]\big\|^{2p}_{\mathcal{C}([0,1];\mathcal{W}^{-\alpha,2p})}\Big] \lesssim 2^{-2n\kappa\varsigma p}\int_{[0,1]^2}\frac{dudv}{|u-v|^{-2(\varepsilon-\varepsilon')p+2}} \;\lesssim\; 2^{-2n\kappa\varsigma p},$$

for any $p \geq p_0$, where $p_0 \geq 1$ is such that $-2(\varepsilon - \varepsilon')p_0 + 2 < 1$.

Note that for $1 \leq p \leq p_0$, one has, since ρ is compactly-supported,

$$\mathbb{E}\Big[\big\|\rho \cdot \big[\Psi^{\kappa,m} - \Psi^{\kappa,n}\big]\big\|^{2p}_{\mathcal{C}([0,1];\mathcal{W}^{-\alpha,2p})}\Big]^{\frac{1}{2p}} \lesssim \mathbb{E}\Big[\big\|\rho \cdot \big[\Psi^{\kappa,m} - \Psi^{\kappa,n}\big]\big\|^{2p_0}_{\mathcal{C}([0,1];\mathcal{W}^{-\alpha,2p_0})}\Big]^{\frac{1}{2p_0}},$$

and so we can conclude that for any $p \geq 1$,

$$\big\|\rho \cdot \big[\Psi^{\kappa,m} - \Psi^{\kappa,n}\big]\big\|_{L^{2p}(\Omega;\mathcal{C}([0,1];\mathcal{W}^{-\alpha,2p}))} \lesssim 2^{-n\kappa\varsigma}. \tag{2.26}$$

In particular, $(\rho \cdot \Psi^{\kappa,n})_{n\geq 1}$ is a Cauchy sequence in $L^{2p}(\Omega;\mathcal{C}([0,1];\mathcal{W}^{-\alpha,2p}(\mathbb{R}^d)))$, and thus it converges in this space. Let us (temporarily) denote by $\rho \cdot \Psi^{\kappa}$ the limit of this sequence, for fixed $\kappa > 0$.

The fact that $\rho \cdot \Psi^{\kappa}$ actually does not depend on κ can be readily deduced from (2.25). Indeed, if $0 < \kappa \leq \kappa'$, one has, for all $t \in [0,1]$ and $n \geq 1$,

$$\begin{aligned}\mathbb{E}\Big[\big\|\rho \cdot \Psi^{\kappa'}_t - \rho \cdot \Psi^{\kappa}_t\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big] &\lesssim \mathbb{E}\Big[\big\|\rho \cdot \Psi^{\kappa'}_t - \rho \cdot \Psi^{\kappa',n}_t\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big] \\ &\quad + \mathbb{E}\Big[\big\|\rho \cdot \Psi^{\kappa',n}_t - \rho \cdot \Psi^{\kappa,n}_t\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big] + \mathbb{E}\Big[\big\|\rho \cdot \Psi^{\kappa,n}_t - \rho \cdot \Psi^{\kappa}_t\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big],\end{aligned}$$

and it is clear that those three quantities tend to 0 as n tends to ∞. Therefore, we can henceforth write $\rho \cdot \Psi$ instead of $\rho \cdot \Psi^{\kappa}$.

The bound (2.8) can finally be derived from an application of the Borel-Cantelli lemma. Indeed, by letting m tend to infinity in (2.26) (for fixed $n \geq 1$ and $p = 2$), we get that

$$\mathbb{E}\Big[\big\|\rho \cdot \Psi^{\kappa,n} - \rho \cdot \Psi\big\|^2_{\mathcal{C}([0,1];\mathcal{H}^{-\alpha})}\Big] \lesssim 2^{-2n\kappa\varsigma},$$

and accordingly, for all $0 < \tilde{\varsigma} < \varsigma$ and $n \geq 1$,

$$\mathbb{P}\Big(\big\|\rho \cdot \Psi^{\kappa,n} - \rho \cdot \Psi\big\|_{\mathcal{C}([0,1];\mathcal{H}^{-\alpha})} > 2^{-n\kappa\tilde{\varsigma}}\Big) \lesssim 2^{-2n\kappa(\varsigma-\tilde{\varsigma})}.$$

3 Noise discretization

Let us now initiate our discretization procedure, starting with the treatment of the noise. To be more specific, and as we announced it in the introduction, we are here interested in the discretization of the smoothened version $\partial_t\partial_x B^{\kappa,n}$ of $\dot{B}$, at the basis of our interpretation of Ψ (along (2.2) and Proposition 2.2).

For this section (and this section only), we will rely on the time-space grid introduced in (1.9), that is we set

$$t_i = t_i^n := \frac{i}{2^n} \;\; (i = 0,\ldots,2^n - 1), \quad x_j = x_j^n := \frac{j}{2^n} \;\; (j = -2^{2n},\ldots,2^{2n} - 1).$$

Note in particular that the set of points $(x_j^n)_{n\geq 1, -2^{2n}\leq j\leq 2^{2n}-1}$ is dense in $\mathbb{R}$. With this grid in hand, we now define the discretized noise $\partial_t\partial_x \widetilde{B}^{\kappa,n}$ as

$$(\partial_t\partial_x \widetilde{B}^{\kappa,n})(t,x) := \begin{cases} 2^{2n}\square^n_{i,j} B^{\kappa,n} & \text{if } t \in [t_i, t_{i+1}) \text{ and } x \in [x_j, x_{j+1}), \\ 0 \text{ otherwise}, \end{cases} \tag{3.1}$$

where we recall the notation $\square^n_{i,j} b := b_{t_{i+1},x_{j+1}} - b_{t_{i+1},x_j} - b_{t_i,x_{j+1}} + b_{t_i,x_j}$ for any two-parameter path b.

Observe that the so-defined approximation $\partial_t\partial_x \widetilde{B}^{\kappa,n}$ indeed corresponds to the space-time derivative of some (piecewise) smooth approximation $\widetilde{B}^{\kappa,n}$ of B: consider for instance the continuous sheet given for all $t \in [t_i, t_{i+1})$ and $x \in [x_j, x_{j+1})$ by

$$\begin{aligned}\widetilde{B}^{\kappa,n}_{t,x} &:= B^{\kappa,n}_{t_i,x_j} + 2^n(t - t_i)(B^{\kappa,n}_{t_{i+1},x_j} - B^{\kappa,n}_{t_i,x_j}) \\ &\quad + 2^n(x - x_j)(B^{\kappa,n}_{t_i,x_{j+1}} - B^{\kappa,n}_{t_i,x_j}) + 2^{2n}(t - t_i)(x - x_j)\square^n_{i,j} B^{\kappa,n}.\end{aligned} \tag{3.2}$$

The corresponding approximation of Υ can then be written, for every $(t,x) \in [0,1] \times \mathbb{R}$, as

$$\widetilde{\Upsilon}_t^{\kappa,n}(x) := (G * \partial_t \partial_x \widetilde{B}^{\kappa,n})_t(x) = 2^{2n} \sum_{i=0}^{2^n-1} \sum_{j=-2^{2n}}^{2^{2n}-1} G^n_{i,j}(t,x)\, \square^n_{i,j} B^{\kappa,n}\,, \tag{3.3}$$

where we have set

$$G^n_{i,j}(t,x) := \left(\int_{t_i}^{t_{i+1}} ds \int_{x_j}^{x_{j+1}} dy\, G_{t-s}(x-y) \right).$$

Our main control regarding the above noise-discretization procedure can now be stated as follows (we recall that the notation $\Upsilon^{\kappa,n}$ refers to the solution associated with $\partial_t \partial_x B^{\kappa,n}$, i.e. to the process considered in Proposition 2.2).

Proposition 3.1. *Fix $(H_0,H_1) \in (0,1)^2$ such that $2H_0 + H_1 < 1$, and let $\alpha_0 > 0$ be defined as in (2.5).*

Then, for all $\alpha > \alpha_0$, $0 < \kappa \leq \frac{\alpha_0}{5}$, $0 < \varsigma \leq \min(1-5\kappa, \alpha_0 - 4\kappa)$, $p \geq 1$ and for every test function $\rho : \mathbb{R} \to \mathbb{R}$ (i.e., smooth and compactly-supported), one has almost surely

$$\sup_{t\in[0,1]} \big\| \rho \cdot \big\{ \Upsilon_t^{\kappa,n} - \widetilde{\Upsilon}_t^{\kappa,n} \big\} \big\|_{\mathcal{H}^{-\alpha}(\mathbb{R})} \lesssim 2^{-\varsigma n}, \tag{3.4}$$

where the proportional constant does not depend on n.

Remark 3.2. The restriction on the "frequency" parameter κ, that is the condition $0 < \kappa \leq \frac{\alpha_0}{5}$, can be interpreted as the result of some interesting competition phenomenon between the $2^{\kappa n}$-approximation scaling in step 1 (see (1.5)) and the size of the discretization grid in the present step 2. From a technical point of view, this competition can be observed through the chain of estimates (3.9), (3.10) and (3.11) in the proof of Proposition 3.1. At this point, it is not completely clear to us whether some finer estimates or the use of alternative Besov topologies could lead to a less stringent condition on κ, or even to the extension of the estimate (3.4) to any $\kappa > 0$. Also, we do not know whether a similar bound could be established when replacing the approximation $B^{\kappa,n}$ with the original sheet B in the right-hand side of (3.1) (this would morally corresponds to taking $\kappa = \infty$).

Remark 3.3. Let us briefly go back to the basic comparison sketched out in the introduction between the one-parameter process Ψ^n (defined by (1.8)) and the above approximation process $\widetilde{\Upsilon}^{\kappa,n}$. Along this analogy, the result of Proposition 3.4 (morally) corresponds to the parabolic counterpart of the convergence of Ψ^n to B. Now recall that a natural strategy to show that $\Psi^n \to B$ in the Sobolev scale $\mathcal{H}^\gamma$ (or more generally $\mathcal{W}^{\gamma,p}$) consists in the use of the continuous embedding (see e.g. [25]): for all $\gamma \in (0,1)$ and $\varepsilon > 0$ small enough,

$$\mathcal{S}^{\gamma+\varepsilon,p} \subset \mathcal{W}^{\gamma,p}$$

where $\mathcal{S}^{\alpha,p}$ ($\alpha \in (0,1)$) refers to the so-called Slobodeckij space

$$\mathcal{S}^{\alpha,p} := \Big\{ f :\ \iint dsdt\, \frac{|f_t - f_s|^p}{|t-s|^{1+\alpha p}} < \infty \Big\}.$$

In this way, the convergence of Ψ^n to B in $\mathcal{W}^{\gamma,p}$ (for $\gamma < H$) can be easily derived from the well-known (pathwise) Hölder regularity of B.

Unfortunately, due to the negative-order Sobolev regularity of the solution process Υ (as seen in Proposition 2.2), such a simplification through an embedding strategy does not seem available in the present rough heat situation, and thus computations cannot be reduced to a pathwise control of spatial increments.

3.1 Proof of Proposition 3.1

We rely on a similar two-step strategy as in the proof of Proposition 2.2.

Step 1: A moment estimate. The first (and main) objective is to establish the following bound:

$$\int_{\mathbb{R}} dx \left(\mathbb{E}\Big[\Big| \mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}} \mathcal{F}\big(\rho \cdot [\widetilde{\tree}^{\kappa,n}_{s,t} - \tree^{\kappa,n}_{s,t}]\big)\Big)(x) \Big|^2 \Big] \right)^p \lesssim 2^{-2n\varsigma p} |t-s|^{2\varepsilon p}\,, \tag{3.5}$$

for all $p \geq 1$ and $\kappa, \varsigma, \varepsilon > 0$ small enough, and where the proportional constant does not depend on n.

With the notation introduced in (3.3) and with representation (1.5) in mind, one has (at least formally)

$$\begin{aligned}
\widetilde{\tree}^{\kappa,n}_{s,t} &= 2^{2n} \sum_{i=0}^{2^n-1} \sum_{j=-2^{2n}}^{2^{2n}-1} G^n_{i,j}(t,x)\, \square^n_{i,j} B^{\kappa,n} \\
&= -c_H \int_{\{|\xi|\leq 2^{2\kappa n}, |\eta| \leq 2^{\kappa n}\}} \widehat{W}(d\xi, d\eta)\, \frac{\xi}{|\xi|^{H_0+\frac12}} \frac{\eta}{|\eta|^{H_1+\frac12}} \\
&\qquad\qquad \sum_{i=0}^{2^n-1} \sum_{j=-2^{2n}}^{2^{2n}-1} G^n_{i,j}(s,t;y) \Big(2^{2n} \int_{t_i}^{t_{i+1}} du \int_{x_j}^{x_{j+1}} dz\, e^{\imath \xi u} e^{\imath \eta z} \Big)
\end{aligned}$$

where we have set $c_H := c_{H_0} c_{H_1}$ and $G^n_{i,j}(s,t;y) := G^n_{i,j}(t,y) - G^n_{i,j}(s,y)$.

As a result, the difference $\tree^{\kappa,n}_{s,t} - \widetilde{\tree}^{\kappa,n}_{s,t}$ can now be (formally) recast into

$$\widetilde{\tree}^{\kappa,n}_{s,t}(y) - \tree^{\kappa,n}_{s,t}(y) = -c_H \int_{\{|\xi|\leq 2^{2\kappa n}, |\eta| \leq 2^{\kappa n}\}} \widehat{W}(d\xi, d\eta)\, \frac{\xi}{|\xi|^{H_0+\frac12}} \frac{\eta}{|\eta|^{H_1+\frac12}} \mathfrak{X}^n_{s,t}(y;\xi,\eta)$$

with

$$\begin{aligned}
&\mathfrak{X}^n_{s,t}(y;\xi,\eta) \\
&:= \Big[\sum_{i=0}^{2^n-1} \sum_{j=-2^{2n}}^{2^{2n}-1} G^n_{i,j}(s,t;y) \Big(2^{2n} \int_{t_i}^{t_{i+1}} du \int_{x_j}^{x_{j+1}} dz\, e^{\imath \xi u} e^{\imath \eta z} \Big) \Big] - e^{\imath \eta y} \gamma_{s,t}(\xi, |\eta|)\,.
\end{aligned}$$

This easily leads us to the following identity (proved along the same lines as in (2.10))

$$\begin{aligned}
&\mathbb{E}\Big[\big[\widetilde{\tree}^{\kappa,n}_{s,t} - \tree^{\kappa,n}_{s,t}\big](y) \big[\widetilde{\tree}^{\kappa,n}_{s,t} - \tree^{\kappa,n}_{s,t}\big](\tilde{y}) \Big] \\
&= c_H^2 \int_{\{|\xi|\leq 2^{2\kappa n}, |\eta| \leq 2^{\kappa n}\}} \frac{d\xi d\eta}{|\xi|^{2H_0-1} |\eta|^{2H_1-1}} \mathfrak{X}^n_{s,t}(y;\xi,\eta) \overline{\mathfrak{X}^n_{s,t}(\tilde{y};\xi,\eta)}.
\end{aligned}$$

Thus, setting

$$\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\lambda) := \int_{\mathbb{R}} dy\, \rho(y) e^{-\imath \lambda y} \mathfrak{X}^n_{s,t}(y;\xi,\eta)\,,$$

one has

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\rho\cdot\Big[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}\Big]\Big)\Big)(x)\Big|^2\Big]\\
&=\int_{\mathbb{R}}d\lambda\int_{\mathbb{R}}d\tilde{\lambda}\,e^{\imath x(\lambda-\tilde{\lambda})}\frac{1}{\{1+|\lambda|^2\}^{\frac{\alpha}{2}}}\frac{1}{\{1+|\tilde{\lambda}|^2\}^{\frac{\alpha}{2}}}\\
&\qquad\qquad\int_{\mathbb{R}}dy\,\rho(y)e^{-\imath\lambda y}\int_{\mathbb{R}}d\tilde{y}\,\rho(\tilde{y})e^{\imath\tilde{\lambda}\tilde{y}}\,\mathbb{E}\Big[\big[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}\big](y)\overline{\big[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}\big](\tilde{y})}\Big]\\
&=\int_{\{|\xi|\leq 2^{2\kappa n},|\eta|\leq 2^{\kappa n}\}}\frac{d\xi d\eta}{|\xi|^{2H_0-1}|\eta|^{2H_1-1}}\int_{\mathbb{R}}d\lambda\int_{\mathbb{R}}d\tilde{\lambda}\,e^{\imath x(\lambda-\tilde{\lambda})}\frac{1}{\{1+|\lambda|^2\}^{\frac{\alpha}{2}}}\frac{1}{\{1+|\tilde{\lambda}|^2\}^{\frac{\alpha}{2}}}\\
&\qquad\qquad\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\lambda)\overline{\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\tilde{\lambda})}\\
&=\int_{\{|\xi|\leq 2^{2\kappa n},|\eta|\leq 2^{\kappa n}\}}\frac{d\xi d\eta}{|\xi|^{2H_0-1}|\eta|^{2H_1-1}}\int_{\mathbb{R}}d\lambda\int_{\mathbb{R}}d\tilde{\lambda}\,e^{\imath x(\lambda-\tilde{\lambda})}\frac{1}{\{1+|\eta+\lambda|^2\}^{\frac{\alpha}{2}}}\frac{1}{\{1+|\eta+\tilde{\lambda}|^2\}^{\frac{\alpha}{2}}}\\
&\qquad\qquad\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\overline{\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\tilde{\lambda})}\,.
\end{aligned}
$$

Based on the latter expression, we get

$$
\begin{aligned}
&\int_{\mathbb{R}}dx\left(\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot\big[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}\big]\big)\Big)(x)\Big|^2\Big]\right)^p\\
&=\prod_{i=1}^{p-1}\int_{\{|\xi_i|\leq 2^{2\kappa n},|\eta_i|\leq 2^{\kappa n}\}}\frac{d\xi_i d\eta_i}{|\xi_i|^{2H_0-1}|\eta_i|^{2H_1-1}}\\
&\quad\int_{\mathbb{R}}\frac{d\lambda_i}{\{1+|\eta_i+\lambda_i|^2\}^{\frac{\alpha}{2}}}\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi_i,\eta_i,\eta_i+\lambda_i)\int_{\mathbb{R}}\frac{d\tilde{\lambda}_i}{\{1+|\eta_i+\tilde{\lambda}_i|^2\}^{\frac{\alpha}{2}}}\overline{\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi_i,\eta_i,\eta_i+\tilde{\lambda}_i)}\\
&\quad\int_{\{|\xi_p|\leq 2^{2\kappa n},|\eta_p|\leq 2^{\kappa n}\}}\frac{d\xi_p d\eta_p}{|\xi_p|^{2H_0-1}|\eta_p|^{2H_1-1}}\int_{\mathbb{R}}\frac{d\lambda_p}{\{1+|\eta_p+\lambda_p|^2\}^{\frac{\alpha}{2}}}\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi_p,\eta_p,\eta_p+\lambda_p)\\
&\quad\frac{1}{\{1+|\eta_p+(\lambda_p-\tilde{\lambda}_{p-1}+\lambda_{p-1}-\ldots+\lambda_1)|^2\}^{\frac{\alpha}{2}}}\overline{\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi_p,\eta_p,\eta_p+(\lambda_p-\tilde{\lambda}_{p-1}+\lambda_{p-1}-\ldots+\lambda_1))}\\
&\leq\Bigg(\prod_{i=1}^{p-1}\int_{\{|\xi_i|\leq 2^{2\kappa n},|\eta_i|\leq 2^{\kappa n}\}}\frac{d\xi_i d\eta_i}{|\xi_i|^{2H_0-1}|\eta_i|^{2H_1-1}}\\
&\quad\int_{\mathbb{R}}\frac{d\lambda_i}{\{1+|\eta_i+\lambda_i|^2\}^{\frac{\alpha}{2}}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi_i,\eta_i,\eta_i+\lambda_i)\big|\int_{\mathbb{R}}\frac{d\tilde{\lambda}_i}{\{1+|\eta_i+\tilde{\lambda}_i|^2\}^{\frac{\alpha}{2}}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi_i,\eta_i,\eta_i+\tilde{\lambda}_i)\big|\Bigg)\\
&\quad\left(\int_{\{|\xi_p|\leq 2^{2\kappa n},|\eta_p|\leq 2^{\kappa n}\}}\frac{d\xi_p d\eta_p}{|\xi_p|^{2H_0-1}|\eta_p|^{2H_1-1}}\int_{\mathbb{R}}\frac{d\lambda_p}{\{1+|\eta_p+\lambda_p|^2\}^{\alpha}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi_p,\eta_p,\eta_p+\lambda_p)\big|^2\right),
\end{aligned}
\tag{3.6}
$$

where the latter inequality is deduced from the Cauchy-Schwarz inequality (applied to the λ_p variable).

We are now in a position to introduce our main technical estimate toward (3.5) (for the sake of clarity, we have postponed the proof of this result to Section 3.2):

Proposition 3.4. *With the above notation, one has, for all* $(\xi,\eta)\in\mathbb{R}^2$ *and* $0<\varepsilon<\min(\frac{1}{4},\frac{\alpha}{2})$,

$$
\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^2\}^{\frac{\alpha}{2}}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\big|\lesssim|t-s|^{\varepsilon}\Big[2^{-n}|\xi|+2^{-n}+2^{-n\alpha_0}|\eta|\Big]\tag{3.7}
$$

and

$$
\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^2\}^{\alpha}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\big|^2\lesssim|t-s|^{2\varepsilon}\Big[2^{-n}|\xi|+2^{-n}+2^{-n\alpha_0}|\eta|\Big]^2.\tag{3.8}
$$

Injecting (3.7) and (3.8) into (3.6), we obtain that

$$\int_{\mathbb{R}} dx \left(\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}]\big)\Big)(x)\Big|^2\Big]\right)^p$$
$$\lesssim |t-s|^{2\varepsilon p}\left(\int_{\{|\xi|\leq 2^{2\kappa n},|\eta|\leq 2^{\kappa n}\}}\frac{d\xi d\eta}{|\xi|^{2H_0-1}|\eta|^{2H_1-1}}\Big|2^{-n}|\xi|+2^{-n}+2^{-n\alpha_0}|\eta|\Big|^2\right)^p$$
$$\lesssim 2^{-2n\varsigma p}|t-s|^{2\varepsilon p}.$$
$$\left(\int_{\{|\xi|\leq 2^{2\kappa n},|\eta|\leq 2^{\kappa n}\}}\frac{d\xi d\eta}{|\xi|^{2H_0-1}|\eta|^{2H_1-1}}\Big|2^{-n(1-\varsigma)}|\xi|+2^{-n(1-\varsigma)}+2^{-n(\alpha_0-\varsigma)}|\eta|\Big|^2\right)^p. \quad (3.9)$$

Now picking ς such that $0<\varsigma\leq\min(1-5\kappa,\alpha_0-4\kappa)$, one has, for any (ξ,η) such that $|\xi|\leq 2^{2\kappa n}$ and $|\eta|\leq 2^{\kappa n}$,

$$\max\big(2^{-n(1-\varsigma)}|\xi|^2|\eta|,2^{-n(1-\varsigma)}|\xi||\eta|,2^{-n(\alpha_0-\varsigma)}|\xi||\eta|^2\big)$$
$$\lesssim \max\big(2^{-n(1-\varsigma-5\kappa)},2^{-n(1-\varsigma-3\kappa)},2^{-n(\alpha_0-\varsigma-4\kappa)}\big)\lesssim 1\,,$$

and so

$$\Big|2^{-n(1-\varsigma)}|\xi|+2^{-n(1-\varsigma)}+2^{-n(\alpha_0-\varsigma)}|\eta|\Big|\lesssim\frac{1}{1+|\xi|}\frac{1}{1+|\eta|}\,, \quad (3.10)$$

which entails that

$$\sup_{n\geq 0}\int_{\{|\xi|\leq 2^{2\kappa n},|\eta|\leq 2^{\kappa n}\}}\frac{d\xi d\eta}{|\xi|^{2H_0-1}|\eta|^{2H_1-1}}\Big|2^{-n(1-\varsigma)}|\xi|+2^{-n(1-\varsigma)}+2^{-n(\alpha_0-\varsigma)}|\eta|\Big|^2$$
$$\lesssim\left(\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H_0-1}\{1+|\xi|^2\}}\right)\left(\int_{\mathbb{R}}\frac{d\eta}{|\eta|^{2H_1-1}\{1+|\eta|^2\}}\right)\ <\ \infty\,. \quad (3.11)$$

Going back to (3.9), this immediately provides us with the final estimate, namely: for all $p\geq 1$, $0<\varsigma\leq\min(1-5\kappa,\alpha_0-4\kappa)$ and $\varepsilon>0$ small enough,

$$\int_{\mathbb{R}} dx \left(\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}]\big)\Big)(x)\Big|^2\Big]\right)^p\lesssim 2^{-2n\varsigma p}|t-s|^{2\varepsilon p}\,, \quad (3.12)$$

for some proportional constant that does not depend on n.

Step 2: Conclusion. The arguments to conclude are essentially the same as those of the proof of Proposition 2.2 (*Step 2*). First, one has, for every $p\geq 1$,

$$\mathbb{E}\Big[\big\|\rho\cdot[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}]\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big]=\int_{\mathbb{R}}dx\,\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}]\big)\Big)(x)\Big|^{2p}\Big]$$
$$\lesssim\int_{\mathbb{R}} dx \left(\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\big(\rho\cdot[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}]\big)\Big)(x)\Big|^2\Big]\right)^p,$$

where the latter inequality is derived from Gaussian hypercontractivity property.

Thus, thanks to (3.12), we get that for all $p\geq 1$, $0<\varsigma\leq\min(1-5\kappa,\alpha_0-4\kappa)$ and $\varepsilon>0$ small enough,

$$\mathbb{E}\Big[\big\|\rho\cdot[\widetilde{\Psi}^{\kappa,n}_{s,t}-\Psi^{\kappa,n}_{s,t}]\big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big]\lesssim 2^{-2n\varsigma p}|t-s|^{2\varepsilon p}\,.$$

We can here apply the Garsia-Rodemich-Rumsey estimate and assert that for $\tilde{\varepsilon}>0$,

$$\mathbb{E}\Big[\big\|\rho\cdot[\widetilde{\Psi}^{\kappa,n}-\Psi^{\kappa,n}]\big\|^{2p}_{\mathcal{C}([0,1];\mathcal{W}^{-\alpha,2p})}\Big]^{\frac{1}{2p}}\lesssim 2^{-n\varsigma}\left(\int_{[0,1]^2}\frac{dudv}{|u-v|^{2-2p(\varepsilon-\tilde{\varepsilon})}}\right)^{\frac{1}{2p}}.$$

The almost sure bound (3.4) can then be deduced from the same Borel-Cantelli argument as in the proof of Proposition 2.2, and this completes the proof of Proposition 3.1.

3.2 Proof of Proposition 3.4

Let us first recast $\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)$ as

$$\begin{aligned}
&\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\\
&=\Big[\sum_{i=0}^{2^n-1}\sum_{j=-2^{2n}}^{2^{2n}-1}\Big(\int_{\mathbb{R}}dy\,\rho(y)e^{-\imath(\eta+\lambda)y}G^n_{i,j}(s,t;y)\Big)\Big(2^{2n}\int_{t_i}^{t_{i+1}}du\int_{x_j}^{x_{j+1}}dz\,e^{\imath\xi u}e^{\imath\eta z}\Big)\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\widehat{\rho}(\lambda)\gamma_{s,t}(\xi,|\eta|)\,.
\end{aligned}$$

Then, since

$$\begin{aligned}
G^n_{i,j}(s,t;y)&=\int_{t_i}^{t_{i+1}}dr\int_{x_j}^{x_{j+1}}dw\,\big\{G_{t-r}(y-w)-G_{s-r}(y-w)\big\}\\
&=\int_{t_i}^{t_{i+1}}dr\int_{x_j}^{x_{j+1}}dw\int_{\mathbb{R}}d\beta\,\big\{\widehat{G_{t-r}}(\beta)-\widehat{G_{s-r}}(\beta)\big\}e^{\imath\beta(y-w)}\,,
\end{aligned}$$

one can write, for all i,j,

$$\begin{aligned}
&\Big(\int_{\mathbb{R}}dy\,\rho(y)e^{-\imath(\eta+\lambda)y}G^n_{i,j}(s,t;y)\Big)\Big(2^{2n}\int_{t_i}^{t_{i+1}}du\int_{x_j}^{x_{j+1}}dz\,e^{\imath\xi u}e^{\imath\eta z}\Big)\\
&=2^{2n}\int_{\mathbb{R}}dy\,\rho(y)e^{-\imath y(\eta+\lambda)}\\
&\quad\int_{t_i}^{t_{i+1}}dr\int_{x_j}^{x_{j+1}}dw\int_{\mathbb{R}}d\beta\,\big\{\widehat{G_{t-r}}(\beta)-\widehat{G_{s-r}}(\beta)\big\}e^{\imath\beta(y-w)}\int_{t_i}^{t_{i+1}}e^{\imath\xi u}\,du\int_{x_j}^{x_{j+1}}e^{\imath\eta z}\,dz\\
&=\int_{\mathbb{R}}d\beta\Big[\int_{\mathbb{R}}dy\,\rho(y)e^{-\imath y(\eta+\lambda-\beta)}\Big]\Big[2^n\int_{t_i}^{t_{i+1}}dr\,\big\{\widehat{G_{t-r}}(\beta)-\widehat{G_{s-r}}(\beta)\big\}\int_{t_i}^{t_{i+1}}e^{\imath\xi u}\,du\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big[2^n\int_{x_j}^{x_{j+1}}dw\int_{x_j}^{x_{j+1}}dz\,e^{-\imath\beta w}e^{\imath\eta z}\Big]\\
&=\int_{\mathbb{R}}d\beta\Big[\int_{\mathbb{R}}dy\,\rho(y)e^{-\imath y(\lambda-\beta)}\Big]\Big[2^n\int_{t_i}^{t_{i+1}}dr\,\big\{\widehat{G_{t-r}}(\eta+\beta)-\widehat{G_{s-r}}(\eta+\beta)\big\}\int_{t_i}^{t_{i+1}}e^{\imath\xi u}\,du\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big[2^n\int_{x_j}^{x_{j+1}}dw\int_{x_j}^{x_{j+1}}dz\,e^{-\imath(\eta+\beta)w}e^{\imath\eta z}\Big]
\end{aligned}$$

and so

$$\begin{aligned}
\sum_{i=0}^{2^n-1}\sum_{j=-2^{2n}}^{2^{2n}-1}&\Big(\int_{\mathbb{R}}dy\,\rho(y)e^{-\imath(\eta+\lambda)y}G^n_{i,j}(s,t;y)\Big)\Big(2^{2n}\int_{t_i}^{t_{i+1}}du\int_{x_j}^{x_{j+1}}dz\,e^{\imath\xi u}e^{\imath\eta z}\Big)\\
&=\int_{\mathbb{R}}d\beta\,\widehat{\rho}(\lambda-\beta)\gamma^n_{s,t}(\xi,\eta+\beta)\delta^n(\eta,\eta+\beta)
\end{aligned}$$

with

$$\gamma^n_t(\xi,\beta):=\int_0^t dr\,\widehat{G_{t-r}}(\beta)\Big(\sum_{i=0}^{2^n-1}\mathbf{1}_{\{t_i<r<t_{i+1}\}}2^n\int_{t_i}^{t_{i+1}}e^{\imath\xi u}\,du\Big) \tag{3.13}$$

and

$$\delta^n(\eta,\beta):=\int_{\mathbb{R}}dw\,e^{-\imath w\beta}\Big(\sum_{j=-2^{2n}}^{2^{2n}-1}\mathbf{1}_{\{x_j<w<x_{j+1}\}}2^n\int_{x_j}^{x_{j+1}}dz\,e^{\imath\eta z}\Big)\,. \tag{3.14}$$

We have thus derived the representation

$$\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)=\Big[\int_{\mathbb{R}}d\beta\,\widehat{\rho}(\lambda-\beta)\gamma^n_{s,t}(\xi,\eta+\beta)\delta^n(\eta,\eta+\beta)\Big]-\widehat{\rho}(\lambda)\gamma_{s,t}(\xi,|\eta|)\,, \tag{3.15}$$

which will help us to prove the following intermediate estimate:

Lemma 3.5. *For all* $(\xi,\eta,\lambda)\in\mathbb{R}^3$*,* $\ell\geq 1$*,* $\sigma_1,\sigma_2,\sigma_3\in[0,1]$ *and* $0<\varepsilon<\frac14$*, it holds that*

$$
\begin{aligned}
&|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)|\\
&\lesssim \big|\widehat{\rho}(\lambda)\big||t-s|^{\sigma_1}\{1+|\eta|^{2\sigma_1}\}2^{-n\sigma_2}|\xi|^{\sigma_2}+\frac{|t-s|^{\varepsilon}}{1+|\lambda|}\Big[2^{-n\ell}+2^{-n}|\eta|+2^{-n\sigma_3}|\eta||\lambda|^{\sigma_3}\Big], \quad (3.16)
\end{aligned}
$$

where the proportional constant does not depend on n,ξ,η,λ*.*

Before we prove this technical lemma, let us see how it allows us to derive the estimates in Proposition 3.4. In fact, applying (3.16) with $\ell=1$, $\sigma_1=\varepsilon$, $\sigma_2=1$ and $\sigma_3=\alpha_0$, we deduce that for every $0<\varepsilon<\frac14$,

$$
\begin{aligned}
\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^2\}^{\frac{\alpha}{2}}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\big| &\lesssim |t-s|^{\varepsilon}\{1+|\eta|^{2\varepsilon}\}2^{-n}|\xi|\int_{\mathbb{R}}d\lambda\,\frac{\big|\widehat{\rho}(\lambda)\big|}{\{1+|\eta+\lambda|^2\}^{\frac{\alpha}{2}}}\\
&+|t-s|^{\varepsilon}\Big[2^{-n}+2^{-n}|\eta|+2^{-n\alpha_0}|\eta|\Big]\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^{\alpha}\}\{1+|\lambda|^{1-\alpha_0}\}}\,. \quad (3.17)
\end{aligned}
$$

At this point, one easily checks that for every $\beta>0$,

$$
\int_{\mathbb{R}}d\lambda\,\frac{\big|\widehat{\rho}(\lambda)\big|}{\{1+|\eta+\lambda|^2\}^{\frac{\beta}{2}}}\lesssim\frac{1}{1+|\eta|^{\beta}}\,. \quad (3.18)
$$

Also, since $0<\alpha_0<\min(1,\alpha)$, it is not hard to see that

$$
\sup_{\eta\in\mathbb{R}}\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^{\alpha}\}\{1+|\lambda|^{1-\alpha_0}\}}\ <\ \infty\,,
$$

and therefore (3.17) leads us to

$$
\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^2\}^{\frac{\alpha}{2}}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\big|\lesssim|t-s|^{\varepsilon}\Big[2^{-n}|\xi|\frac{1+|\eta|^{2\varepsilon}}{1+|\eta|^{\alpha}}+2^{-n}+2^{-n\alpha_0}|\eta|\Big]
$$

which, if $0<\varepsilon<\min(\frac14,\frac{\alpha}{2})$, immediately yields (3.7).

We can then derive (3.8) with similar arguments. Namely, applying again (3.16) with $\ell=1$, $\sigma_1=\varepsilon$, $\sigma_2=1$ and $\sigma_3=\alpha_0$, we get first that for every $0<\varepsilon<\frac14$,

$$
\begin{aligned}
\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^2\}^{\alpha}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\big|^2 &\lesssim|t-s|^{2\varepsilon}\Big[\{1+|\eta|^{2\varepsilon}\}2^{-n}|\xi|\Big]^2\int_{\mathbb{R}}d\lambda\,\frac{\big|\widehat{\rho}(\lambda)\big|^2}{\{1+|\eta+\lambda|^2\}^{\alpha}}\\
&+|t-s|^{2\varepsilon}\Big[2^{-n}+2^{-n}|\eta|+2^{-n\alpha_0}|\eta|\Big]^2\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^{2\alpha}\}\{1+|\lambda|^{2(1-\alpha_0)}\}}\,. \quad (3.19)
\end{aligned}
$$

For the same elementary reasons as in (3.18) (write $\big|\widehat{\rho}(\lambda)\big|^2=\big|(\widehat{\rho*\rho})(\lambda)\big|$), one has

$$
\int_{\mathbb{R}}d\lambda\,\frac{\big|\widehat{\rho}(\lambda)\big|^2}{\{1+|\eta+\lambda|^2\}^{\alpha}}\lesssim\frac{1}{1+|\eta|^{2\alpha}}\,,
$$

and again, since $0<\alpha_0<\min(1,\alpha)$, it is easy to check that

$$
\sup_{\eta\in\mathbb{R}}\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^{2\alpha}\}\{1+|\lambda|^{2(1-\alpha_0)}\}}\ <\ \infty\,.
$$

Therefore, we end up with

$$
\int_{\mathbb{R}}\frac{d\lambda}{\{1+|\eta+\lambda|^2\}^{\alpha}}\big|\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi,\eta,\eta+\lambda)\big|^2\lesssim|t-s|^{2\varepsilon}\Big[2^{-n}|\xi|\frac{1+|\eta|^{2\varepsilon}}{1+|\eta|^{\alpha}}+2^{-n}+2^{-n\alpha_0}|\eta|\Big]^2
$$

and by picking $0 < \varepsilon < \min(\frac{1}{4}, \frac{\alpha}{2})$, we obtain (3.8), which achieves the proof of Proposition 3.4.

Thus, it only remains us to provide the details behind Lemma 3.5.

Proof of Lemma 3.5. Going back to the definition (3.14) of δ^n, one has

$$\delta^n(\eta, \eta+\beta) = \int_{\mathbb{R}} dw\, e^{-\imath w\beta} \Big(\sum_{j=-2^{2n}}^{2^{2n}-1} \mathbf{1}_{\{x_j < w < x_{j+1}\}} 2^n \int_{x_j}^{x_{j+1}} dz\, e^{\imath\eta(z-w)} \Big)$$
$$= \int_{\mathbb{R}} dw\, e^{-\imath w\beta} \Big(\sum_{j=-2^{2n}}^{2^{2n}-1} \mathbf{1}_{\{x_j < w < x_{j+1}\}} 2^n \int_{x_j}^{x_{j+1}} dz\, \{e^{\imath\eta(z-w)} - 1\} \Big) + \int_{\mathbb{R}} dw\, e^{-\imath w\beta} \mathbf{1}_{\{|w| \le 2^n\}}$$

so that, according to (3.15), we can write

$$\begin{aligned}
&\mathcal{J}^{\mathfrak{X},n}_{s,t}(\xi, \eta, \eta+\lambda) \\
&= \Big[\int_{\mathbb{R}} dw \Big(\int_{\mathbb{R}} d\beta\, e^{-\imath w\beta} \widehat{\rho}(\lambda-\beta) \gamma^n_{s,t}(\xi, \eta+\beta) \Big) \\
&\qquad\qquad \Big(\sum_{j=-2^{2n}}^{2^{2n}-1} \mathbf{1}_{\{x_j < w < x_{j+1}\}} 2^n \int_{x_j}^{x_{j+1}} dz\, \{e^{\imath\eta(z-w)} - 1\} \Big) \Big] \\
&\qquad + \int_{\mathbb{R}} dw\, \mathbf{1}_{\{|w| \le 2^n\}} \int_{\mathbb{R}} d\beta\, e^{-\imath w\beta} \widehat{\rho}(\lambda-\beta) \gamma^n_{s,t}(\xi, \eta+\beta) - \widehat{\rho}(\lambda) \gamma_{s,t}(\xi, |\eta|) \\
&= \Big[\int_{\mathbb{R}} dw \Big(\int_{\mathbb{R}} d\beta\, e^{-\imath w\beta} \widehat{\rho}(\lambda-\beta) \gamma^n_{s,t}(\xi, \eta+\beta) \Big) \\
&\qquad\qquad \Big(\sum_{j=-2^{2n}}^{2^{2n}-1} \mathbf{1}_{\{x_j < w < x_{j+1}\}} 2^n \int_{x_j}^{x_{j+1}} dz\, \{e^{\imath\eta(z-w)} - 1\} \Big) \Big] \\
&+ \Big[- \int_{|w| \ge 2^n} dw \int_{\mathbb{R}} d\beta\, e^{-\imath w\beta} \widehat{\rho}(\lambda-\beta) \gamma^n_{s,t}(\xi, \eta+\beta) \Big] + \Big[\widehat{\rho}(\lambda) \gamma^n_{s,t}(\xi, \eta) - \widehat{\rho}(\lambda) \gamma_{s,t}(\xi, |\eta|) \Big] \\
&=: \mathbb{I}^n_{s,t}(\xi, \eta, \lambda) + \mathbb{II}^n_{s,t}(\xi, \eta, \lambda) + \mathbb{III}^n_{s,t}(\xi, \eta, \lambda)\,.
\end{aligned} \tag{3.20}$$

Let us estimate these three quantities separately.

Treatment of $\mathbb{I}^n_{s,t}(\xi, \eta, \lambda)$. We have

$$\begin{aligned}
\int_{\mathbb{R}} d\beta\, e^{-\imath w\beta} \widehat{\rho}(\lambda-\beta) \gamma^n_{s,t}(\xi, \eta+\beta) &= e^{-\imath\lambda w} \int_{\mathbb{R}} d\beta\, e^{\imath w\beta} \widehat{\rho}(\beta) \gamma^n_{s,t}(\xi, \eta+\lambda-\beta) \\
&= e^{-\imath\lambda w} f^n_{s,t}(w; \xi, \eta, \lambda)\,,
\end{aligned} \tag{3.21}$$

with

$$f^n_{s,t}(w; \xi, \eta, \lambda) := \Big[\rho * \mathcal{F}^{-1} \Big(\gamma^n_{s,t}(\xi, \eta+\lambda-.) \Big) \Big](w)\,. \tag{3.22}$$

Using the additional notation

$$h^{n,j}(w; \eta) := 2^n \int_{x_j}^{x_{j+1}} dz\, \big\{ e^{\imath\eta(z-w)} - 1 \big\}\,,$$

we can rewrite $\mathbb{I}^n_{s,t}(\xi, \eta, \lambda)$ as

$$\mathbb{I}^n_{s,t}(\xi, \eta, \lambda) = \sum_{j=-2^{2n}}^{2^{2n}-1} \int_{x_j}^{x_{j+1}} dw\, e^{-\imath\lambda w} f^n_{s,t}(w; \xi, \eta, \lambda) h^{n,j}(w; \eta)\,.$$

On the one hand, it is clear that $|h^{n,j}(w;\eta)| \leq 2^{-n}|\eta|$ for any $w \in [x_j, x_{j+1}]$, which immediately yields

$$\left|\mathbb{I}^n_{s,t}(\xi,\eta,\lambda)\right| \leq 2^{-n}|\eta| \left\|f^n_{s,t}(.;\xi,\eta,\lambda)\right\|_{L^1} . \tag{3.23}$$

On the other hand, one has for every j,

$$\begin{aligned}
&\int_{x_j}^{x_{j+1}} dw\, e^{-\imath\lambda w} f^n_{s,t}(w;\xi,\eta,\lambda) h^{n,j}(w;\eta) \\
&= \frac{-1}{\imath\lambda}\bigg[e^{-\imath\lambda x_{j+1}} f^n_{s,t}(x_{j+1};\xi,\eta,\lambda)h^{n,j}(x_{j+1};\eta) - e^{-\imath\lambda x_j} f^n_{s,t}(x_j;\xi,\eta,\lambda)h^{n,j}(x_j;\eta) \\
&\qquad\qquad - \int_{x_j}^{x_{j+1}} dw\, e^{-\imath\lambda w}(\partial_w f^n_{s,t})(w;\xi,\eta,\lambda)h^{n,j}(w;\eta) \\
&\qquad\qquad\qquad - \int_{x_j}^{x_{j+1}} dw\, e^{-\imath\lambda w} f^n_{s,t}(w;\xi,\eta,\lambda)(\partial_w h^{n,j})(w;\eta)\bigg] \\
&= \frac{-1}{\imath\lambda}\Big[I^{n,j}_{s,t} + II^{n,j}_{s,t} + III^{n,j}_{s,t}\Big] ,
\end{aligned} \tag{3.24}$$

with

$$\begin{aligned}
I^{n,j}_{s,t} &:= \big\{e^{-\imath\lambda x_{j+1}} f^n_{s,t}(x_{j+1};\xi,\eta,\lambda) - e^{-\imath\lambda x_j} f^n_{s,t}(x_j;\xi,\eta,\lambda)\big\} h^{n,j}(x_{j+1};\eta) , \\
II^{n,j}_{s,t} &:= -\int_{x_j}^{x_{j+1}} dw\, e^{-\imath\lambda w}(\partial_w f^n_{s,t})(w;\xi,\eta,\lambda)h^{n,j}(w;\eta) , \\
III^{n,j}_{s,t} &:= \int_{x_j}^{x_{j+1}} dw\, \big\{e^{-\imath\lambda x_j} f^n_{s,t}(x_j;\xi,\eta,\lambda) - e^{-\imath\lambda w} f^n_{s,t}(w;\xi,\eta,\lambda)\big\}(\partial_w h^{n,j})(w;\eta) .
\end{aligned}$$

First, since $h^{n,j}(x_{j+1};\eta) = 2^n \int_0^{2^{-n}} dz\, \big\{e^{-\imath\eta z} - 1\big\}$ does not depend on j, we get that

$$\begin{aligned}
&\Big|\sum_{j=-2^{2n}}^{2^{2n}-1} I^{n,j}_{s,t}\Big| \\
&= \left|\left(2^n \int_0^{2^{-n}} dz\, \big\{e^{-\imath\eta z} - 1\big\}\right)\big\{e^{-\imath\lambda 2^n} f^n_{s,t}(2^n;\xi,\eta,\lambda) - e^{\imath\lambda 2^n} f^n_{s,t}(-2^n;\xi,\eta,\lambda)\big\}\right| \\
&\lesssim 2^{-n}|\eta| \left\|f^n_{s,t}(.;\xi,\eta,\lambda)\right\|_{L^\infty} .
\end{aligned} \tag{3.25}$$

Then

$$\begin{aligned}
\Big|\sum_{j=-2^{2n}}^{2^{2n}-1} II^{n,j}_{s,t}\Big| &\lesssim 2^{-n}|\eta| \sum_{j=-2^{2n}}^{2^{2n}} \int_{x_j}^{x_{j+1}} dw\, \big|(\partial_w f^n_{s,t})(w;\xi,\eta,\lambda)\big| \\
&\lesssim 2^{-n}|\eta| \left\|(\partial_w f^n_{s,t})(.;\xi,\eta,\lambda)\right\|_{L^1} .
\end{aligned} \tag{3.26}$$

Finally, since $|(\partial_w h^{n,j})(w;\eta)| = \left|(-\imath\eta)2^n \int_{x_j}^{x_{j+1}} dz\, e^{\imath\eta(z-w)}\right| \leq |\eta|$, we have for any $\sigma \in [0,1]$

$$\begin{aligned}
&|III^{n,j}_{s,t}| \\
&\leq |\eta| \int_{x_j}^{x_{j+1}} dw\, \Big[\big|f^n_{s,t}(x_j;\xi,\eta,\lambda) - f^n_{s,t}(w;\xi,\eta,\lambda)\big| + \big|e^{-\imath\lambda x_j} - e^{-\imath\lambda w}\big|\big|f^n_{s,t}(w;\xi,\eta,\lambda)\big|\Big] \\
&\leq |\eta|\bigg[\int_{x_j}^{x_{j+1}} dw \int_{x_j}^{w} dv\, \big|(\partial_w f^n_{s,t})(v;\xi,\eta,\lambda)\big| + |\lambda|^\sigma 2^{-n\sigma} \int_{x_j}^{x_{j+1}} dw\, \big|f^n_{s,t}(w;\xi,\eta,\lambda)\big|\bigg] \\
&\leq |\eta|\bigg[2^{-n} \int_{x_j}^{x_{j+1}} dv\, \big|(\partial_w f^n_{s,t})(v;\xi,\eta,\lambda)\big| + |\lambda|^\sigma 2^{-n\sigma} \int_{x_j}^{x_{j+1}} dw\, \big|f^n_{s,t}(w;\xi,\eta,\lambda)\big|\bigg] ,
\end{aligned}$$

which immediately entails

$$\Big|\sum_{j=-2^{2n}}^{2^{2n}-1} III^{n,j}_{s,t}\Big| \leq \Big[2^{-n}|\eta|\,\big\|\big(\partial_w f^n_{s,t}\big)(.;\xi,\eta,\lambda)\big\|_{L^1} + 2^{-n\sigma}|\eta||\lambda|^{\sigma}\,\big\|f^n_{s,t}(.;\xi,\eta,\lambda)\big\|_{L^1}\Big]. \quad (3.27)$$

Injecting (3.25)–(3.26)–(3.27) into (3.24), and then combining the result with (3.23), we obtain that

$$\big|\mathbb{I}^n_{s,t}(\xi,\eta,\lambda)\big| \lesssim \frac{1}{1+|\lambda|}\Big[2^{-n}|\eta| + 2^{-n\sigma}|\eta||\lambda|^{\sigma}\Big]$$
$$\Big[\big\|f^n_{s,t}(.;\xi,\eta,\lambda)\big\|_{L^\infty} + \big\|f^n_{s,t}(.;\xi,\eta,\lambda)\big\|_{L^1} + \big\|\big(\partial_w f^n_{s,t}\big)(.;\xi,\eta,\lambda)\big\|_{L^1}\Big]. \quad (3.28)$$

In order to go further, note that by the definition of $f^n_{s,t}$, and since $\rho \in \mathcal{C}^\infty_c(\mathbb{R})$, one has

$$\big\|f^n_{s,t}(.;\xi,\eta,\lambda)\big\|_{L^\infty} + \big\|f^n_{s,t}(.;\xi,\eta,\lambda)\big\|_{L^1} + \big\|\big(\partial_w f^n_{s,t}\big)(.;\xi,\eta,\lambda)\big\|_{L^1} \lesssim \Big\|\mathcal{F}^{-1}\Big(\gamma^n_{s,t}(\xi,\eta+\lambda-.)\Big)\Big\|_{L^1}. \quad (3.29)$$

Then observe that

$$\mathcal{F}^{-1}\Big(\gamma^n_{s,t}(\xi,\eta+\lambda-.)\Big)(w)$$
$$= e^{\imath w(\lambda+\eta)}\int_0^t dr\,\big\{G_{t-r}(w)-G_{s-r}(w)\big\}\Big[\sum_{i=0}^{2^n-1}\mathbf{1}_{\{t^n_i<r<t^n_{i+1}\}}2^n\int_{t_i}^{t_{i+1}} du\, e^{\imath\xi u}\Big], \quad (3.30)$$

and so, for any $0<\varepsilon<\frac{1}{4}$,

$$\Big\|\mathcal{F}^{-1}\Big(\gamma^n_{s,t}(\xi,\eta+\lambda-.)\Big)\Big\|_{L^1} \leq \int_{\mathbb{R}} dw \int_0^t dr\,\big|G_{t-r}(w)-G_{s-r}(w)\big| \lesssim |t-s|^{\varepsilon}. \quad (3.31)$$

Injecting (3.29)–(3.31) into (3.28), we get that for any $\sigma\in[0,1]$ and $0<\varepsilon<\frac{1}{4}$,

$$\big|\mathbb{I}^n_{s,t}(\xi,\eta,\lambda)\big| \lesssim \frac{|t-s|^{\varepsilon}}{1+|\lambda|}\Big[2^{-n}|\eta| + 2^{-n\sigma}|\eta||\lambda|^{\sigma}\Big], \quad (3.32)$$

for some proportional constant that does not depend on ξ,η,λ.

Treatment of $\mathbb{III}^n_{s,t}(\xi,\eta,\lambda)$. Following (3.21), we can write

$$\mathbb{III}^n_{s,t}(\xi,\eta,\lambda) = -\int_{|w|\geq 2^n} dw\, e^{-\imath\lambda w} f^n_{s,t}(w;\xi,\eta,\lambda)$$

with $f^n_{s,t}$ defined by (3.22), and so

$$\big|\mathbb{III}^n_{s,t}(\xi,\eta,\lambda)\big| = \Big|\int_{|w|\geq 2^n} dw\, e^{-\imath\lambda w} f^n_{s,t}(w;\xi,\eta,\lambda)\Big|$$
$$\lesssim \min\Big(\int_{|w|\geq 2^n} dw\,\big|f^n_{s,t}(w;\xi,\eta,\lambda)\big|,$$
$$\frac{1}{|\lambda|}\Big\{\sup_{|w|\geq 2^n}\big|f^n_{s,t}(w;\xi,\eta,\lambda)\big| + \int_{|w|\geq 2^n} dw\,\big|(\partial_w f^n_{s,t})(w;\xi,\eta,\lambda)\big|\Big\}\Big). \quad (3.33)$$

Remember that $f^n_{s,t} = \rho * \mathcal{F}^{-1}\big(\gamma^n_{s,t}(\xi,\eta+\lambda-.)\big)$, and so, since $\operatorname{supp}\rho \subset [-A,A]$ for some $A>0$, it is readily checked that

$$\int_{|w|\geq 2^n} dw\,\big|f^n_{s,t}(w;\xi,\eta,\lambda)\big| + \sup_{|w|\geq 2^n}\big|f^n_{s,t}(w;\xi,\eta,\lambda)\big| + \int_{|w|\geq 2^n} dw\,\big|(\partial_w f^n_{s,t})(w;\xi,\eta,\lambda)\big|$$
$$\lesssim \int_{|w|\geq 2^n-A} dw\Big|\mathcal{F}^{-1}\Big(\gamma^n_{s,t}(\xi,\eta+\lambda-.)\Big)(w)\Big| \lesssim \int_{|w|\geq 2^{n-1}} dw\Big|\mathcal{F}^{-1}\Big(\gamma^n_{s,t}(\xi,\eta+\lambda-.)\Big)(w)\Big|, \quad (3.34)$$

for any n large enough. Then, using the representation (3.30), we deduce that for all $\ell \geq 1$ and $0 < \varepsilon < \frac{1}{4}$,

$$\int_{|w|\geq 2^{n-1}} dw \Big| \mathcal{F}^{-1}\Big(\gamma^n_{s,t}(\xi, \eta + \lambda - .)\Big)(w)\Big|$$
$$\leq \int_{|w|\geq 2^{n-1}} dw \int_0^t dr \,\big|G_{t-r}(w) - G_{s-r}(w)\big| \lesssim 2^{-n\ell}|t-s|^{\varepsilon}\,. \tag{3.35}$$

Combining (3.34)–(3.35) with (3.33), we obtain that

$$\big|\mathbb{II}^n_{s,t}(\xi,\eta,\lambda)\big| \lesssim \frac{|t-s|^{\varepsilon}}{1+|\lambda|} 2^{-n\ell}\,, \tag{3.36}$$

for all $\ell \geq 1$ and $0 < \varepsilon < \frac{1}{4}$.

Treatment of $\mathbb{III}^n_{s,t}(\xi,\eta,\lambda)$. We need to control the difference $\gamma^n_{s,t}(\xi,\eta) - \gamma_{s,t}(\xi,|\eta|)$. In fact, with expression (2.12) in mind, one has

$$\begin{aligned}
&\gamma^n_{s,t}(\xi,\eta)\\
&= \int_0^t dr\,\big\{\widehat{G_{t-r}}(\eta) - \widehat{G_{s-r}}(\eta)\big\} e^{\imath\xi r}\Bigg[\sum_{i=0}^{2^n-1} \mathbf{1}_{\{t_i<r<t_{i+1}\}} 2^n \int_{t_i}^{t_{i+1}} du\, e^{\imath\xi(u-r)}\Bigg]\\
&= \gamma_{s,t}(\xi,|\eta|)\\
&\qquad + \int_0^t dr\,\big\{\widehat{G_{t-r}}(\eta) - \widehat{G_{s-r}}(\eta)\big\} e^{\imath\xi r}\Bigg[\sum_{i=0}^{2^n-1} \mathbf{1}_{\{t_i<r<t_{i+1}\}} 2^n \int_{t_i}^{t_{i+1}} du\,\big\{e^{\imath\xi(u-r)}-1\big\}\Bigg].
\end{aligned}$$

Now it is clear that for any $\sigma_1, \sigma_2 \in [0,1]$,

$$\begin{aligned}
&\Bigg|\int_0^t dr\,\big\{\widehat{G_{t-r}}(\eta) - \widehat{G_{s-r}}(\eta)\big\} e^{\imath\xi r}\Bigg[\sum_{i=0}^{2^n-1} \mathbf{1}_{\{t_i<r<t_{i+1}\}} 2^n \int_{t_i}^{t_{i+1}} du\,\big\{e^{\imath\xi(u-r)}-1\big\}\Bigg]\Bigg|\\
&\lesssim 2^{-n\sigma_1}|\xi|^{\sigma_1}\Bigg[\int_0^s dr\,\Big|e^{-(t-r)|\eta|^2} - e^{-(s-r)|\eta|^2}\Big| + \int_s^t dr\, e^{-(t-r)|\eta|^2}\Bigg]\\
&\lesssim 2^{-n\sigma_1}|\xi|^{\sigma_1}|t-s|^{\sigma_2}\{1+|\eta|^{2\sigma_2}\}\,.
\end{aligned}$$

Therefore, we can conclude that

$$\big|\mathbb{III}^n_{s,t}(\xi,\eta,\lambda)\big| \lesssim \big|\widehat{\rho}(\lambda)\big| 2^{-n\sigma_1}|\xi|^{\sigma_1}|t-s|^{\sigma_2}\{1+|\eta|^{2\sigma_2}\}\,. \tag{3.37}$$

Injecting (3.32), (3.36) and (3.37) into (3.20), we finally get the desired estimate (3.16). $\square$

4 Space-time discretization of the heat operator

At this point of the analysis, we are endowed with the approximation $\widetilde{\Psi}^{\kappa,n}$ of the solution, derived from the discretization $\partial_t\partial_x \widetilde{B}^{\kappa,n}$ of the noise (see (3.1)). Otherwise stated, for all fixed $\kappa > 0$ and $n \geq 1$, $\widetilde{\Psi}^{\kappa,n}$ corresponds to the classical solution of the standard heat equation

$$\begin{cases} \partial_t \Psi^{\kappa,n} = \Delta \Psi^{\kappa,n} + \partial_t\partial_x \widetilde{B}^{\kappa,n}\,, \quad t \in [0,1],\ x \in \mathbb{R},\\ \Psi^{\kappa,n}_0 = 0\,. \end{cases} \tag{4.1}$$

This section is devoted to the third and final step of our discretization procedure, namely the space-time discretization of the heat operator in (4.1), *for fixed* $\kappa > 0, n \geq 1$ (which implies that $\partial_t\partial_x \widetilde{B}^{\kappa,n}$ is here regarded as a well-defined bounded function).

To this end, we will rely on a Galerkin-type approximation strategy. Let us start with a general description of the method and an associated estimate (Section 4.1), which we will then apply to our stochastic problem (Section 4.2).

4.1 Description of the algorithm and a general estimate

For every mesh size $h>0$, we consider the basis of functions $(\Phi^h_j)_{j\in\mathbb{Z}}$ given by

$$\Phi^h_j(x) := \begin{cases} \frac{1}{h}(x-x_{j-1}) & \text{if } x\in[x_{j-1},x_j] \\ \frac{1}{h}(x_{j+1}-x) & \text{if } x\in[x_j,x_{j+1}] \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.2}$$

where the subdivision points $(x_j)_{j\in\mathbb{Z}}$ are here merely defined as $x_j := jh$. Also, we set

$$\mathcal{S}^{(h,L)} := \text{Span}\big(\Phi^h_j,\, -N+1\leq j\leq N-1\big). \tag{4.3}$$

For all $h,L>0$ such that $N:=\frac{L}{h}\in\mathbb{N}$, and for all $m\geq 1$, let us now introduce the Galerkin approximation operator $\mathcal{G}^{(h,L)}_m$ associated with the basis $\Phi^h := (\Phi^h_j)_{j\in\mathbb{Z}}$, the domain $\Omega_L := [-L,L]$ and the uniform subdivision $D_m := \{t_i = t^m_i := \frac{i}{2^m},\, i=0,\dots,2^m\}$ of $[0,1]$.

Namely, for every $u\in H^{1,2}_{loc}([0,1]\times\mathbb{R})$, we define $\bar{u} := \mathcal{G}^{(h,L)}_m(u)$ as the sequence of functions $(\bar{u}_{t_i})_{i=0,\dots,2^m}$ characterized by the three conditions: (i) $\bar{u}_0\equiv 0$; (ii) $\bar{u}_{t_i}\in\mathcal{S}^{(h,L)}$; (iii) for all $i=1,\dots,2^m$ and $\varphi\in\mathcal{S}^{(h,L)}$,

$$2^m\langle \bar{u}_{t_i}-\bar{u}_{t_{i-1}},\varphi\rangle + \langle\nabla\bar{u}_{t_i},\nabla\varphi\rangle = 2^m\int_{t_{i-1}}^{t_i} ds\,\big[\langle(\partial_t u)_s,\varphi\rangle + \langle\nabla u_s,\nabla\varphi\rangle\big]\,. \tag{4.4}$$

Denoting the mass, resp. stiffness, matrix by

$$\mathcal{M}_{h,L} := \big(\langle\Phi^h_j,\Phi^h_k\rangle\big)_{-N+1\leq j,k\leq N-1},\quad \text{resp. } \mathcal{A}_{h,L} := \big(\langle\nabla\Phi^h_j,\nabla\Phi^h_k\rangle\big)_{-N+1\leq j,k\leq N-1}, \tag{4.5}$$

one can easily rephrase (4.4) as follows: if $\bar{u}_{t_i}(x) = \sum_{j=-N+1}^{N-1}\bar{\mathbf{u}}^j_{t_i}\Phi^h_j(x)$, then

$$2^m\mathcal{M}_{h,L}(\bar{\mathbf{u}}_{t_i}-\bar{\mathbf{u}}_{t_{i-1}}) + \mathcal{A}_{h,L}\bar{\mathbf{u}}_{t_i} = 2^m\int_{t_{i-1}}^{t_i} ds\,\langle(\partial_t u)_s - \Delta u_s,\Phi^h\rangle\,, \tag{4.6}$$

where $\bar{\mathbf{u}}_{t_i} := (\bar{\mathbf{u}}^j_{t_i})_{-N+1\leq j\leq N-1}$ and for every function $f\in L^2_{loc}(\mathbb{R})$,

$$\langle f,\Phi^h\rangle := (\langle f,\Phi^h_j\rangle)_{-N+1\leq j\leq N-1}.$$

Formula (4.6) thus offers a convenient way to compute the matrix $\bar{\mathbf{u}} := (\bar{\mathbf{u}}_{t_i})_{i=0,\dots,2^m}$, and accordingly to compute $\bar{u}$.

Remark 4.1. To be more specific in the terminology, the above-defined operation $\mathcal{G}^{(h,L)}_m$ corresponds in fact to the combination of a (space) Galerkin projection and a (time) implicit Euler scheme.

Let us now state the important estimate (for the difference $u-\mathcal{G}^{(h,L)}_m(u)$) that will serve us in this third discretization step of our problem.

Proposition 4.2. *Fix a smooth compactly-supported function $\rho:\mathbb{R}\to[0,1]$. Then, for all $u\in H^{1,2}_{loc}([0,1]\times\mathbb{R})$ such that $u_0=0$, all $h,L>0$ such that $N:=\frac{L}{h}\in\mathbb{N}$, all $m\geq 1$ and $\varepsilon>0$, it holds that*

$$\begin{aligned}&\sup_{i=1,\dots,2^m}\big\|\rho\cdot\big\{u_{t_i}-\mathcal{G}^{(h,L)}_m(u)_{t_i}\big\}\big\|_{L^2(\mathbb{R})}\\ &\lesssim\Big[2^{-m(1-\varepsilon)}\operatorname*{ess\,sup}_{t\in[0,1]}\|(\partial_t u)_t\|_{L^2(\mathbb{R})} + 2^{m\varepsilon}h^2\operatorname*{ess\,sup}_{t\in[0,1]}\|\Delta u_t\|_{L^2(\mathbb{R})}\Big]\\ &\qquad\qquad + \sup_{t\in[0,1]}\sup_{x\in\partial\Omega_L}\Big\{|u_t(x)| + L^{1/2}\big|(\partial_t u)_t(x)\big|\Big\}\,,\end{aligned} \tag{4.7}$$

for some proportional constant that does not depend on h, L and m.

The above estimate corresponds to a generalization – to arbitrary functions u in the space $H^{1,2}_{loc}([0,1]\times\mathbb{R})$ – of a well-known control for functions u vanishing on the boundary (see for instance [22, Theorem 8.2]).

For the sake of clarity, we have postponed the proof of this (purely deterministic) property to Section A.2 of the appendix, and we will rather focus here on its application to our stochastic problem.

4.2 Application to the stochastic problem

We now intend to apply the result of Proposition 4.2 to our problem (4.1), that is to $u = \widetilde{\rotatebox{90}{$\multimap$}}^{\kappa,n} = G * \partial_t\partial_x \widetilde{B}^{\kappa,n}$. For this application to be relevant, we naturally need, first, to find suitable bounds on the terms involved in the right-hand side of (4.7). This is the purpose of the next three lemmas.

Lemma 4.3. *Let $f \in L^\infty([0,1]\times\mathbb{R})$ be a function of the form*

$$f_s(x) = \sum_{k=-K}^{K-1} a_s^k\, \mathbf{1}_{[x_k,x_{k+1})}(x),$$

for some $K \geq 1$, some coefficients $a^k \in L^\infty([0,1])$ and some points $x_{-K} \leq \ldots \leq x_K$. Setting

$$M := \Big(\sup_{k=-K,\ldots,K-1} |x_{k+1}-x_k|\Big)^{-1}$$

and assuming that $M \geq 1$, one has for every $\varepsilon > 0$

$$\sup_{t\in[0,1]} \big\|\partial_t(G*f)_t\big\|_{L^2(\mathbb{R})} + \sup_{t\in[0,1]} \big\|\Delta(G*f)_t\big\|_{L^2(\mathbb{R})} \lesssim \frac{K}{M^{\frac12-\varepsilon}} \operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})}\,, \tag{4.8}$$

where the proportional constant does not depend on K and M.

Proof. Using the Plancherel theorem, we can write

$$\begin{aligned}
\big\|\Delta(G*f)_t\big\|^2_{L^2(\mathbb{R})} &= c\int_{\mathbb{R}} d\xi\,|\xi|^4 \big|\mathcal{F}\big((G*f)_t\big)(\xi)\big|^2\\
&= c\int_{\mathbb{R}} d\xi\,|\xi|^4 \bigg|\int_0^t ds\, e^{-\xi^2(t-s)}\big(\mathcal{F}f_s\big)(\xi)\bigg|^2\\
&\lesssim \int_{\mathbb{R}} d\xi\,|\xi|^\varepsilon \bigg(\int_0^t \frac{ds}{(t-s)^{1-\frac{\varepsilon}{4}}}\big|\big(\mathcal{F}f_s\big)(\xi)\big|\bigg)^2.
\end{aligned}$$

Then one has, for every $\lambda\in[0,1]$,

$$\begin{aligned}
\big|\big(\mathcal{F}f_s\big)(\xi)\big| &= \bigg|\sum_{k=-K}^{K-1} a_s^k \int_{x_k}^{x_{k+1}} dx\, e^{-\imath x\xi}\bigg|\\
&\leq \sum_{k=-K}^{K-1} |a_s^k| \bigg|\int_{x_k}^{x_{k+1}} dx\, e^{-\imath x\xi}\bigg|^\lambda \bigg|\int_{x_k}^{x_{k+1}} dx\, e^{-\imath x\xi}\bigg|^{1-\lambda}\\
&\lesssim \frac{1}{M^\lambda|\xi|^{1-\lambda}} \sum_{k=-K}^{K-1} |a_s^k|\,,
\end{aligned}$$

and so

$$\big\|\Delta(G*f)_t\big\|^2_{L^2(\mathbb{R})} \lesssim \Big(K \operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})}\Big)^2 \bigg[\frac{1}{M^2}\int_{|\xi|\leq 1} d\xi\,|\xi|^\varepsilon + \frac{1}{M^{1-2\varepsilon}}\int_{|\xi|\geq 1}\frac{d\xi}{|\xi|^{1+\varepsilon}}\bigg],$$

which gives the desired bound for $\sup_{t\in[0,1]} \left\|\Delta(G*f)_t\right\|_{L^2(\mathbb{R})}$.

As for $\partial_t(G*f)_t$, recall that $\partial_t(G*f)_t = \Delta(G*f)_t + f_t$, and

$$\sup_{t\in[0,1]} \|f_t\|_{L^2} \leq \left(\frac{2K}{M}\right)^{1/2} \operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})} \lesssim \frac{K}{M^{\frac{1}{2}-\varepsilon}} \operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})}. \qquad \square$$

Lemma 4.4. *Let $f \in L^\infty([0,1]\times\mathbb{R})$ be a function such that $\bigcup_{t\in[0,1]} \operatorname{supp} f_t \subset [-2^n, 2^n]$, for some $n \geq 1$. Then for all $\beta > 0$ and $L > 2^n$, it holds that*

$$\sup_{t\in[0,1]} \sup_{x\in\partial\Omega_L} \Big\{ \big|(G*f)_t(x)\big| + L^{1/2}\big|(\partial_t(G*f))_t(x)\big| \Big\} \lesssim \frac{L^{1/2}}{|L-2^n|^\beta} \Big(\operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})} \Big), \quad (4.9)$$

where the proportional constant does not depend on n and L.

Proof. For $x \in \{-L, L\}$, and for all $\varepsilon \in (0,1)$, $\beta > 0$, one has

$$\begin{aligned}
\big|(\partial_t(G*f))_t(x)\big| &\lesssim \int_0^t ds \int_{-2^n}^{2^n} dy\, |f_{t-s}(y)| e^{-\frac{(x-y)^2}{4s}} \left[\frac{1}{s^{\frac{3}{2}}} + \frac{(x-y)^2}{s^{\frac{5}{2}}} \right] \\
&\lesssim \Big(\operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})} \Big) \int_0^t \frac{ds}{s^{\frac{3}{2}-\frac{\beta}{2}}} \int_{-2^n}^{2^n} dy\, \frac{e^{-\varepsilon\frac{(x-y)^2}{4s}}}{|x-y|^\beta} \\
&\lesssim \frac{1}{|L-2^n|^\beta} \Big(\operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})} \Big) \int_0^t \frac{ds}{s^{\frac{3}{2}-\frac{\beta}{2}}} \int_{\mathbb{R}} dy\, e^{-\varepsilon\frac{(x-y)^2}{4s}} \\
&\lesssim \frac{1}{|L-2^n|^\beta} \Big(\operatorname*{ess\,sup}_{t\in[0,1]} \|f_t\|_{L^\infty(\mathbb{R})} \Big) \int_0^t \frac{ds}{s^{1-\frac{\beta}{2}}}\,.
\end{aligned}$$

The quantity $|(G*f)_t(x)|$ can then be estimated along similar arguments. $\square$

Based on the estimates (4.7) and (4.8)–(4.9), it remains us to exhibit a bound for the supremum norm of the approximated fractional noise.

Lemma 4.5. *Fix $(H_0, H_1) \in (0,1)^2$ and for all $\kappa > 0$, $n \geq 1$, let $\partial_t\partial_x \widetilde{B}^{\kappa,n}$ be the approximated fractional noise given by* (3.1)*. Then almost surely, and for every $\varepsilon > 0$, it holds that*

$$\sup_{t\in[0,1]} \left\|(\partial_t\partial_x \widetilde{B}^{\kappa,n})_t\right\|_{L^\infty(\mathbb{R})} \lesssim 2^{n(2-H_0-H_1+\varepsilon)}\,, \quad (4.10)$$

for some (random) proportional constant that does not depend on κ and n.

Proof. It is essentially a direct consequence of the (almost sure) regularity of the space-time fractional sheet. To be more specific, observe that for all $(s,x),(t,y) \in [0,1]\times\mathbb{R}$, and for every $p \geq 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\big|B_t^{\kappa,n}(y) - B_s^{\kappa,n}(y) - B_t^{\kappa,n}(x) + B_s^{\kappa,n}(x)\big|^{2p}\Big] \\
&\lesssim \mathbb{E}\Big[\big|B_t^{\kappa,n}(y) - B_s^{\kappa,n}(y) - B_t^{\kappa,n}(x) + B_s^{\kappa,n}(x)\big|^2\Big]^p \\
&\lesssim \mathbb{E}\left[\left| \int_{|\xi|\leq 2^{2\kappa n}} \int_{|\eta|\leq 2^{\kappa n}} \widehat{W}(d\xi, d\eta)\, \frac{e^{\imath\xi t} - e^{\imath\xi s}}{|\xi|^{H_0+\frac{1}{2}}} \frac{e^{\imath\eta y} - e^{\imath\eta x}}{|\eta|^{H_1+\frac{1}{2}}} \right|^2\right]^p \\
&\lesssim \left(\int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\eta\, \frac{|e^{\imath\xi t} - e^{\imath\xi s}|^2}{|\xi|^{2H_0+1}} \frac{|e^{\imath\eta y} - e^{\imath\eta x}|^2}{|\eta|^{2H_1+1}} \right)^p \\
&\lesssim \left(\int_{\mathbb{R}} \int_{\mathbb{R}} d\xi d\eta\, \frac{|e^{\imath\xi(t-s)} - 1|^2}{|\xi|^{2H_0+1}} \frac{|e^{\imath\eta(y-x)} - 1|^2}{|\eta|^{2H_1+1}} \right)^p \lesssim |t-s|^{2H_0 p} |y-x|^{2H_1 p}\,,
\end{aligned}$$

where the proportional constants do not depend on $(s,x),(t,y)$, κ and n. Therefore, we can apply for instance [19, Theorem 3.1] to assert that almost surely, and for all $n \geq 1$, $i,j \in \mathbb{Z}$, $\varepsilon > 0$,

$$\left|\Box^n_{i,j} B^{\kappa,n}\right| \lesssim 2^{-n(H_0+H_1-\varepsilon)},$$

for some (random) proportional constant that does not depend on n and i,j. The claimed estimate (4.10) is then an immediate consequence of the definition (3.1) of $\partial_t\partial_x \widetilde{B}^{\kappa,n}$. $\square$

With the representation

$$(\partial_t\partial_x \widetilde{B}^{\kappa,n})_s(x) = \sum_{\ell=-2^{2n}}^{2^{2n}-1} \left(\sum_{k=0}^{2^n-1} 2^{2n} (\Box^n_{k,\ell} B^{\kappa,n}) \mathbf{1}_{[\frac{k}{2^n},\frac{k+1}{2^n}[}(s) \right) \mathbf{1}_{[\frac{\ell}{2^n},\frac{\ell+1}{2^n}[}(x) \tag{4.11}$$

in mind, we can inject the estimates of Lemma 4.3, Lemma 4.4 and Lemma 4.5 into the result of Proposition 4.2, which gives successively (for (h,L) such that $\frac{L}{h} \in \mathbb{N}$)

$$\begin{aligned}
&\sup_{i=0,\dots,2^m} \left\| \rho \cdot \left\{ \widetilde{\Psi}^{\kappa,n}_{t_i} - \mathcal{G}^{(h,L)}_m\big(\widetilde{\Psi}^{\kappa,n}\big)_{t_i} \right\} \right\|_{L^2(\mathbb{R})} \\
&\lesssim 2^{-m(1-\varepsilon)} \sup_{t\in[0,1]} \|(\partial_t \widetilde{\Psi}^{\kappa,n})_t\|_{L^2(\mathbb{R})} + 2^{m\varepsilon} h^2 \sup_{t\in[0,1]} \|\Delta \widetilde{\Psi}^{\kappa,n}_t\|_{L^2(\mathbb{R})} \\
&\qquad\qquad + \sup_{t\in[0,1]} \sup_{x\in\partial\Omega_L} \left\{ |\widetilde{\Psi}^{\kappa,n}_t(x)| + L^{1/2} |(\partial_t \widetilde{\Psi}^{\kappa,n})_t(x)| \right\} \\
&\lesssim \left[2^{-m(1-\varepsilon)} 2^{n(\frac{3}{2}+\varepsilon)} + 2^{m\varepsilon} h^2 2^{n(\frac{3}{2}+\varepsilon)} + \frac{L^{1/2}}{|L-2^n|^\beta} \right] \cdot \sup_{t\in[0,1]} \|(\partial_t\partial_x \widetilde{B}^{\kappa,n})_t\|_{L^\infty(\mathbb{R})} \\
&\lesssim \left[2^{-m(1-\varepsilon)} 2^{n(\frac{3}{2}+\varepsilon)} + 2^{m\varepsilon} h^2 2^{n(\frac{3}{2}+\varepsilon)} + \frac{L^{1/2}}{|L-2^n|^\beta} \right] \cdot 2^{n(2-H_0-H_1+\varepsilon)},
\end{aligned} \tag{4.12}$$

which corresponds to the main estimate of this section.

Proposition 4.6. *Fix $(H_0,H_1) \in (0,1)^2$ and for all $\kappa > 0$, $n \geq 1$, let $\widetilde{\Psi}^{\kappa,n}$ be the solution to the equation (4.1), driven by the approximated fractional noise $\partial_t\partial_x \widetilde{B}^{\kappa,n}$. Also, fix a smooth compactly-supported function $\rho : \mathbb{R} \to [0,1]$.*

Then for all $h > 0$, $L > 2^n$ such that $\frac{L}{h} \in \mathbb{N}$, for all $\kappa > 0$, $n \geq 1$, $m \geq 1$, and for all $\beta > 0$, $\varepsilon \in (0,1)$, one has almost surely

$$\begin{aligned}
&\sup_{i=0,\dots,2^m} \left\| \rho \cdot \left\{ \widetilde{\Psi}^{\kappa,n}_{t_i} - \mathcal{G}^{(h,L)}_m\big(\widetilde{\Psi}^{\kappa,n}\big)_{t_i} \right\} \right\|_{L^2(\mathbb{R})} \\
&\lesssim 2^{-m(1-\varepsilon)} 2^{n(\frac{7}{2}-H_0-H_1+\varepsilon)} + 2^{m\varepsilon} h^2 2^{n(\frac{7}{2}-H_0-H_1+\varepsilon)} + \frac{L^{1/2}}{|L-2^n|^\beta} 2^{n(2-H_0-H_1+\varepsilon)},
\end{aligned} \tag{4.13}$$

where the proportional constant does not depend on κ, n, h, L and m.

In particular, if $\bar{\Psi}^{\kappa,n}$ stands for the approximation defined by (1.17), then for every $(H_0,H_1) \in (0,1)^2$, one has almost surely

$$\sup_{i=0,\dots,2^m} \left\| \rho \cdot \left\{ \widetilde{\Psi}^{\kappa,n}_{t_i} - \bar{\Psi}^{\kappa,n}_{t_i} \right\} \right\|_{L^2(\mathbb{R})} \lesssim 2^{-\frac{n}{2}}, \tag{4.14}$$

where the proportional constant does not depend on κ and n.

Remark 4.7. The estimate (4.13) emphasizes a standard feature of the finite-element method (when applied in a parabolic setting), namely the fact that the time mesh size (i.e. 2^{-m}) must somehow be considered at the same level as the square of the spatial mesh size (i.e. h^2) in the discretization procedure.

Proof of Proposition 4.6. The estimate (4.13) corresponds to (4.12). As for (4.14), it suffices to observe that $\bar{\mathbb{I}}^{\kappa,n}$ is nothing but $\mathcal{G}_m^{(h,L)}(\widetilde{\mathbb{I}}^{\kappa,n})$ for the specific calibration $h := 2^{-2n}$, $L := 2^{n+1}$ and $m := 4n$ (which yields the rate $2^{-\frac{n}{2}}$ in (4.14)). This identification can be immediately deduced from the comparison between (4.1)–(4.6) and (1.16)–(1.18), combined with the identity

$$2^{4n}\int_{t_i}^{t_{i+1}} ds\,\langle(\partial_t\partial_x\widetilde{B}^{\kappa,n})_s,\Phi_j^n\rangle = \delta B_{ij}^{\kappa,n}\,, \tag{4.15}$$

where $\delta B_{ij}^{\kappa,n}$ is the quantity defined by (1.15).

The proof of (4.15) follows from straightforward computations. Using the convention $\tilde{i},\approx{j}$ introduced in Section 1.2, and with expression (4.11) in mind, one can write, for every $j = -N+1,\dots,N-1$,

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} ds\,\langle(\partial_t\partial_x\widetilde{B}^{\kappa,n})_s,\Phi_j^n\rangle\\
&= \sum_{\ell=-2^{2n}}^{2^{2n}-1}\sum_{k=0}^{2^n-1} 2^{2n}(\square_{k,\ell}^n B^{\kappa,n})\int_{t_i}^{t_{i+1}} ds\,\mathbf{1}_{[\frac{k}{2^n},\frac{k+1}{2^n}[}(s)\int_{x_{j-1}}^{x_{j+1}} dx\,\mathbf{1}_{[\frac{\ell}{2^n},\frac{\ell+1}{2^n}[}(x)\Phi_j^n(x)\\
&= 2^{-2n}\sum_{\ell=-2^{2n}}^{2^{2n}-1}\square_{\tilde{i},\ell}^n B^{\kappa,n}\\
&\qquad\Big[\mathbf{1}_{\{x_j>\frac{\approx{j}}{2^n}\}}\int_{x_{j-1}}^{x_{j+1}} dx\,\mathbf{1}_{[\frac{\ell}{2^n},\frac{\ell+1}{2^n}[}(x)\Phi_j^n(x)+\mathbf{1}_{\{x_j=\frac{\approx{j}}{2^n}\}}\int_{x_{j-1}}^{x_{j+1}} dx\,\mathbf{1}_{[\frac{\ell}{2^n},\frac{\ell+1}{2^n}[}(x)\Phi_j^n(x)\Big]\,.
\end{aligned} \tag{4.16}$$

Now

$$\begin{aligned}
&\sum_{\ell=-2^{2n}}^{2^{2n}-1}\square_{\tilde{i},\ell}^n B^{\kappa,n}\mathbf{1}_{\{x_j>\frac{\approx{j}}{2^n}\}}\int_{x_{j-1}}^{x_{j+1}} dx\,\mathbf{1}_{[\frac{\ell}{2^n},\frac{\ell+1}{2^n}[}(x)\Phi_j^n(x)\\
&= \mathbf{1}_{\{x_j>\frac{\approx{j}}{2^n}\}}\square_{\tilde{i},\approx{j}}^n B^{\kappa,n}\int_{x_{j-1}}^{x_{j+1}} dx\,\Phi_j^n(x)\\
&= \mathbf{1}_{\{x_j>\frac{\approx{j}}{2^n}\}}2^{-2n}\square_{\tilde{i},\approx{j}}^n B^{\kappa,n}\,,
\end{aligned}$$

while

$$\begin{aligned}
&\sum_{\ell=-2^{2n}}^{2^{2n}-1}\square_{\tilde{i},\ell}^n B^{\kappa,n}\mathbf{1}_{\{x_j=\frac{\approx{j}}{2^n}\}}\int_{x_{j-1}}^{x_{j+1}} dx\,\mathbf{1}_{[\frac{\ell}{2^n},\frac{\ell+1}{2^n}[}(x)\Phi_j^n(x)\\
&= \mathbf{1}_{\{x_j=\frac{\approx{j}}{2^n}\}}\Big[\square_{\tilde{i},\approx{j}-1}^n B^{\kappa,n}\int_{x_{j-1}}^{x_j} dx\,\Phi_j^n(x)+\square_{\tilde{i},\approx{j}}^n B^{\kappa,n}\int_{x_j}^{x_{j+1}} dx\,\Phi_j^n(x)\Big]\\
&= \mathbf{1}_{\{x_j=\frac{\approx{j}}{2^n}\}}2^{-2n}\Big[\frac{1}{2}\square_{\tilde{i},\approx{j}-1}^n B^{\kappa,n}+\frac{1}{2}\square_{\tilde{i},\approx{j}}^n B^{\kappa,n}\Big]\,.
\end{aligned}$$

Going back to (4.16), we have obtained that

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} ds\,\langle(\partial_t\partial_x\widetilde{B}^{\kappa,n})_s,\Phi_j^n\rangle\\
&= \mathbf{1}_{\{x_j>\frac{\approx{j}}{2^n}\}}2^{-4n}\square_{\tilde{i},\approx{j}}^n B^{\kappa,n}+\mathbf{1}_{\{x_j=\frac{\approx{j}}{2^n}\}}2^{-4n}\Big[\frac{1}{2}\square_{\tilde{i},\approx{j}-1}^n B^{\kappa,n}+\frac{1}{2}\square_{\tilde{i},\approx{j}}^n B^{\kappa,n}\Big]\,,
\end{aligned}$$

which precisely corresponds to (4.15). □

Let us conclude this theoretical analysis with two remarks about our calibration choice (h, L, m) in (4.14) (and therefore in the scheme of Section 1.2, leading to the definition of $\bar{\text{♀}}^{\kappa,n}$).

Remark 4.8. Observe that in light of (4.13), the calibration $(h, L, m) := (2^{-2n}, 2^{n+1}, 4n)$ ensures the convergence of the scheme for all possible values of $(H_0, H_1) \in (0,1)^2$, which, for the sake of clarity, motivated our choice in the scheme proposed in Section 1.2. However, for *fixed* (H_0, H_1), one could naturally choose more optimal values for (h, L, m), which would possibly reduce the number of computations in the associated system (1.18).

Remark 4.9. Due to our use of a Galerkin-type approximation procedure (and Galerkin-type bounding arguments), the estimate result in Proposition 4.6 is stated in terms of $L^2(\mathbb{R})$-topology in space, in contrast with the weaker $\mathcal{H}^{-\alpha}(\mathbb{R})$-norm used in Proposition 2.2 and Proposition 3.1, and which is more natural in this rough setting (remember that the solution ♀ is a path with values in $\mathcal{H}^{-\alpha}(\mathbb{R})$). By considering the $\mathcal{H}^{-\alpha}(\mathbb{R})$-norm in the left-hand side of (4.13), it might be possible to improve the latter estimate with respect to the parameters h, m or n (without changing the topology in our main control (1.20)). In turn, this could allow us to relax the current (h, L, m)-calibration of the scheme in Section 1.2. Nevertheless, at this point, it is not clear to us how one could adapt the successive arguments of Section 4 in order to – sharply – take negative-order Sobolev norms into account.

5 Numerical results and possible improvements

We devote this last section to a few details and comments related to the numerical implementation of the algorithm (1.17)–(1.18).

5.1 Simulation of the scheme

As described in Section 1.2, the simulation of our discretized process $\text{♀}^{\kappa,n}$ boils down to the computation of the values $\text{♀}^{j}_{t_i}$, along the iterative formula (1.18). As far as randomness is concerned, we are thus left with the implementation of the quantities

$$\delta B^{\kappa,n}_{ij}, \quad i = 0, \ldots, 2^{4n},\ j = -N+1, \ldots, N-1\,,\ N := 2^{3n+1}. \tag{5.1}$$

To this end, let us briefly recall that any Gaussian sheet can be easily simulated through its mean and covariance formulas. To be more specific, let us fix $t_1 < \ldots < t_p$, $x_1 < \ldots < x_q$, $p, q \geq 1$, and consider a centered Gaussian field $\{X_t(x),\, t, x \in \mathbb{R}\}$ with covariance given by

$$\mathbb{E}[X_s(x)X_t(y)] = C_0(s,t)C_1(x,y)\,.$$

Then define the matrices $\mathtt{C}_0, \mathtt{C}_1$ along the formulas

$$\mathtt{C}_0(i,i') := C_0(t_i, t_{i'}), \quad \mathtt{C}_1(j,j') := C_1(x_j, x_{j'}),$$

and consider auxiliary symmetric matrices $\mathtt{D}_0, \mathtt{D}_1$ such that ${\mathtt{D}_0}^2 = \mathtt{C}_0$ and ${\mathtt{D}_1}^2 = \mathtt{C}_1$ (in the subsequent implementations, $\mathtt{D}_0, \mathtt{D}_1$ are computed with the help of the `sqrtm` function). Now, if $\mathtt{W}$ stands for a random matrix in $\mathcal{M}_{p\times q}(\mathbb{R})$ with independent and $\mathcal{N}(0,1)$-distributed entries, we set

$$\mathtt{X} := \mathtt{D}_0 * \mathtt{W} * \mathtt{D}_1, \tag{5.2}$$

so that, for all $1 \leq i, i' \leq p$ and $1 \leq j, j' \leq q$,

$$\begin{aligned}\mathbb{E}\big[\mathtt{X}(i,j)\mathtt{X}(i',j')\big] &= \sum_{k,k'=1}^{p} \sum_{\ell,\ell'=1}^{q} \mathtt{D}_0(i,k)\mathtt{D}_0(i',k')\mathtt{D}_1(\ell,j)\mathtt{D}_0(\ell',j')\mathbb{E}\big[\mathtt{W}(k,\ell)\mathtt{W}(k',\ell')\big] \\ &= \sum_{k=1}^{p} \mathtt{D}_0(i,k)\mathtt{D}_0(k,i') \sum_{\ell=1}^{q} \mathtt{D}_1(j,\ell)\mathtt{D}_0(\ell,j') = \mathtt{C}_0(i,i')\mathtt{C}_1(j,j') = \mathbb{E}\big[X_{t_i}(x_j)X_{t_{i'}}(x_{j'})\big] .\end{aligned}$$

With this general strategy in mind, let us go back to our approximation $B^{\kappa,n}$ of the fractional sheet B, for fixed Hurst indexes $H_0, H_1 \in (0,1)$. According to the representation (1.5), $B^{\kappa,n}$ corresponds indeed to a centered Gaussian field with covariance of the form

$$\mathbb{E}\big[B_s^{\kappa,n}(x)B_t^{\kappa,n}(y)\big] = C_0^{\kappa,n}(s,t)C_1^{\kappa,n}(x,y), \tag{5.3}$$

with

$$C_0^{\kappa,n}(s,t) := c_{H_0}^2 \int_{|\xi|\leq 2^{2\kappa n}} d\xi \, \frac{(e^{i\xi t}-1)(e^{-i\xi s}-1)}{|\xi|^{2H_0+1}},$$
$$C_1^{\kappa,n}(x,y) := c_{H_1}^2 \int_{|\eta|\leq 2^{\kappa n}} d\eta \, \frac{(e^{i\eta x}-1)(e^{-i\eta y}-1)}{|\eta|^{2H_1+1}} .$$

Note that the latter integrals can be more conveniently expanded as

$$\begin{aligned}C_0^{\kappa,n}(s,t) =& c_{H_0}^2 \int_{|\xi|\leq 2^{2\kappa n}} d\xi \, \frac{\cos(\xi(t-s)) - \cos(\xi t) - \cos(\xi s) + 1}{|\xi|^{2H_0+1}} \\ =& c_{H_0}^2 \, 2^{1-4H_0\kappa n} \int_0^1 d\xi \, \frac{\cos(2^{2\kappa n}\xi(t-s)) - \cos(2^{2\kappa n}\xi t) - \cos(2^{2\kappa n}\xi s) + 1}{|\xi|^{2H_0+1}},\end{aligned}$$

with a similar expression for $C_1^{\kappa,n}(x,y)$, which in turn allows us to approximate $C_0^{\kappa,n}, C_1^{\kappa,n}$ through a standard Riemann-sum procedure, i.e. as

$$C_0^{\kappa,n}(s,t) \approx c_{H_0}^2 \frac{2^{1-4H_0\kappa n}}{M_0} \sum_{m=1}^{M_0} \frac{\cos(2^{2\kappa n}\frac{m}{M_0}(t-s)) - \cos(2^{2\kappa n}\frac{m}{M_0}t) - \cos(2^{2\kappa n}\frac{m}{M_0}s) + 1}{|\frac{m}{M_0}|^{2H_0+1}} ,$$

$$C_1^{\kappa,n}(x,y) \approx c_{H_1}^2 \frac{2^{1-2H_1\kappa n}}{M_1} \sum_{m=1}^{M_1} \frac{\cos(2^{\kappa n}\frac{m}{M_1}(x-y)) - \cos(2^{\kappa n}\frac{m}{M_1}x) - \cos(2^{\kappa n}\frac{m}{M_1}y) + 1}{|\frac{m}{M_1}|^{2H_1+1}} ,$$

with M_0, M_1 large enough.

As a consequence of decomposition (5.3), the values of $B^{\kappa,n}_{\frac{i}{2^{2n}}}(\frac{j}{2^n})$ can now be easily simulated through the above-described method, i.e. using (5.2), which immedialy provides us with the set of increments

$$\square_{i,j}^n B^{\kappa,n}, \quad i = 0,\dots,2^n,\ j = -2^{2n},\dots,2^{2n}, \tag{5.4}$$

involved in the scheme. Observe that, following (1.15), each quantity $\delta B_{i,j}^{\kappa,n}$ in (1.18) is in fact computed from the pair $(\square^n_{\tilde{i},\tilde{\tilde{j}}}B^{\kappa,n}, \square^n_{\tilde{i},\tilde{\tilde{j}}-1}B^{\kappa,n})$, where $\tilde{i} := \lfloor i2^{-3n}\rfloor$ and $\tilde{\tilde{j}} := \lfloor j2^{-n}\rfloor$.

Once endowed with the (renormalized) quantities $\beta(j,i) := \frac{3}{2^{2n+1}}\delta B_{i,j}^{\kappa,n}$, the simulation of (1.18) merely relies on the consideration of the two matrices

$$\mathtt{A}_1 := \begin{pmatrix} 4 & -\frac{5}{4} & 0 & \cdots & 0 \\ -\frac{5}{4} & 4 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -\frac{5}{4} \\ 0 & \cdots & 0 & -\frac{5}{4} & 4 \end{pmatrix} \quad \text{and} \quad \mathtt{A}_2 := \begin{pmatrix} 1 & \frac{1}{4} & 0 & \cdots & 0 \\ \frac{1}{4} & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & \frac{1}{4} \\ 0 & \cdots & 0 & \frac{1}{4} & 1 \end{pmatrix}.$$

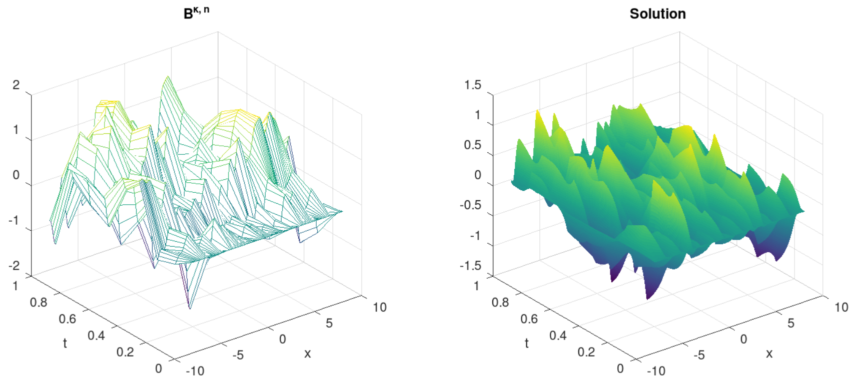

Figure 1: $H_0 = H_1 = \frac{1}{4}$

Namely, setting $\phi(j,i) := \overset{\bullet}{|}{}^{j}_{t_i}$, formula (1.18) can be more efficiently recast into the iterative scheme

$$\mathtt{A}_1 \phi(\cdot, 1) = \beta(\cdot, 1), \quad \mathtt{A}_1 \phi(\cdot, i+1) = \mathtt{A}_2 \phi(\cdot, i) + \beta(\cdot, i) \quad \text{for all } i > 1, \tag{5.5}$$

the implementation of which becomes an easy task (see for instance Figure 1 for a simulation with $n = 3$, $\kappa = 1$, $M_0 = 10000$, $M_1 = 1000$ and $H_0 = H_1 = \frac{1}{4}$).

Remark 5.1. We cannot hide the fact that, due to our consideration of a grid with extremely fine mesh (namely 2^{-4n} in time, 2^{-3n+1} in space) and growing support (in space), the simulation of the above scheme soon reveals to be highly demanding as n increases, and we have actually not been able to implement the algorithm for $n \geq 4$. As a consequence of this computational restriction, the process $B^{\kappa,n}$, with $\kappa > 0$ small enough (following the result of Theorem 1.3) and $1 \leq n \leq 3$, can only be seen as a coarse approximation of B, which explains the relative smoothness of the simulated sheets in Figure 1.

5.2 Open issues

Let us conclude the study with two natural open questions raised by the above simulation procedure, and which could motivate possible future improvements of our theoretical results.

Replacing $B^{\kappa,n}$ with B. As we emphasized it in Remark 3.2, our restriction on the parameter κ (in $B^{\kappa,n}$) plays an important technical role in the proof of the convergence property (3.4), but we cannot firmly assert that the result of Theorem 1.3 (or some similar convergence statement toward $\overset{\bullet}{|}$) would fail for larger values of $\kappa > 0$, or even for $\kappa = \infty$, which corresponds to replacing $B^{\kappa,n}$ with B in the algorithm.

Figure 2 corresponds to a simulation of such a modified scheme (where $\kappa = \infty$, $H_0 = H_1 = \frac{1}{4}$ and $n = 3$), and thus the resulting sheet might represent a more faithful approximation of $\overset{\bullet}{|}$.

Grid synchronization. Another natural question arising from our scheme is to know whether the convergence in Theorem 1.3 (or some similar property) would hold if one discretized the noise and the heat operator *over the same grid* (say $t_i = \frac{i}{2^n}, x_j = \frac{j}{2^n}$).

Recall that the strong discrepancy between the "Galerkin grid" in (5.1) and the "noise grid" in (5.4) is – at least partially – due to our treatment of $B^{\kappa,n}$ as a L^∞-function throughout Section 4. As we mentioned in Remark 4.9, we expect some more direct analysis in $\mathcal{H}^{-\alpha}$ to provide sharper estimates with respect to the stochastic perturbation, which in turn could allow us to – at least partially – fill the gap between the two grids.

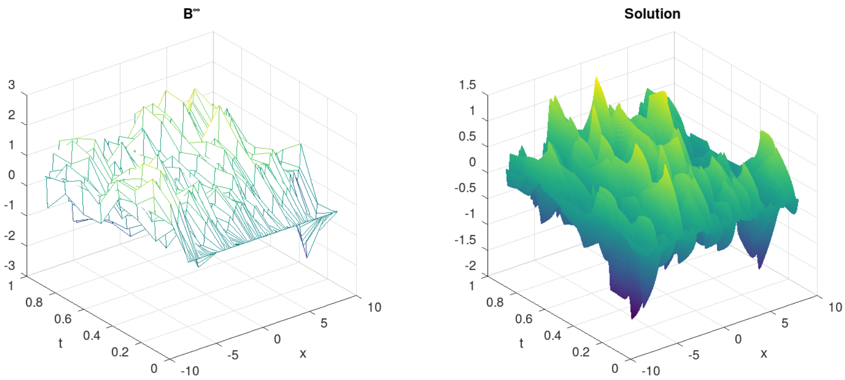

Figure 2: $H_0 = H_1 = \frac{1}{4}$

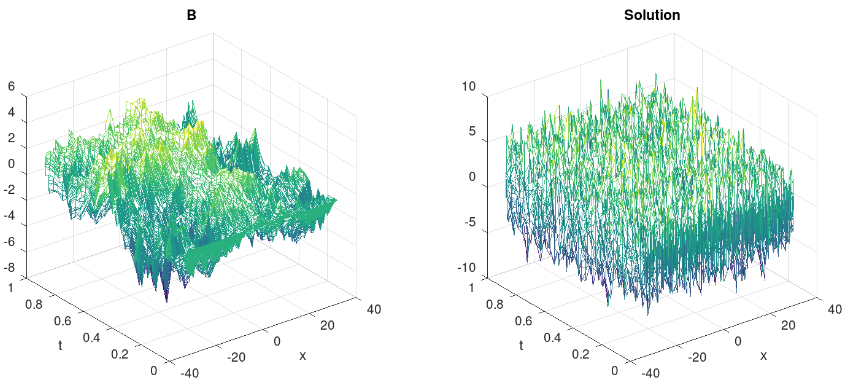

Figure 3: $H_0 = H_1 = \frac{1}{4}$

In this setting, Figure 3 (where $n = 5$, $H_0 = H_1 = \frac{1}{4}$) accounts for the simulation of the corresponding "synchronized" scheme over the common grid $t_i = \frac{i}{2^n}$, $x_j = \frac{j}{2^n}$. We have also provided a simulation of this scheme in a more regular situation for which $2H_0 + H_1 > 1$ (see Figure 4, where $H_0 = H_1 = \frac{3}{4}$ and $n = 5$): the two figures 3 and 4 thus offer a clear contrast between the regular "functional" case, and the rough "distributional" regime.

A Galerkin estimates

The ultimate goal of this section is to provide the details behind the estimate of Proposition 4.2. As we mentioned in Section 4.1, the latter result can in fact be regarded as a generalization of a well-known control for functions vanishing on the boundary (see for instance [22, Theorem 8.2], or [23, Chapter 3] and [28, Chapter 1] for similar bounds).

For the reader's convenience, we propose to briefly review the main preliminary steps toward such Galerkin estimates (Section A.1), before we turn to the extension itself (Section A.2). We will also seize this opportunity to insist on the dependence/independence of each intermediate bound with respect to the length L of the domain under consideration, as this question happens to be crucial in our unbounded setting (in contrast with the situation in [22, 23, 28]).

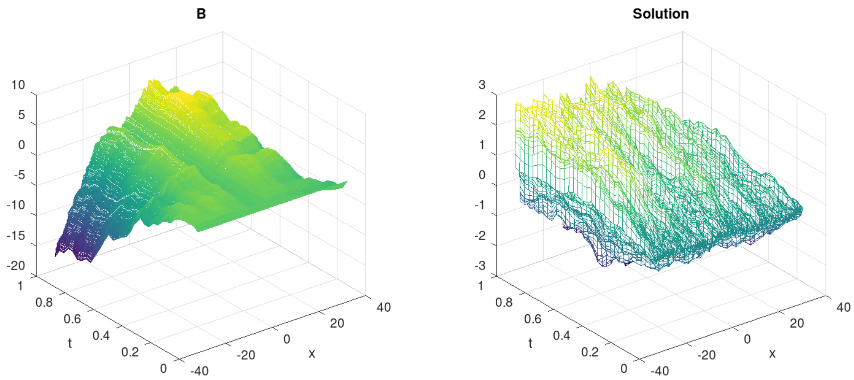

Figure 4: $H_0 = H_1 = \frac{3}{4}$

A.1 Preliminary considerations

We go back here to the setting and notation introduced in Section 4.1. In particular, $\mathcal{S}^{(h,L)}$ is the space defined in (4.3), while $\mathcal{M}_{h,L}$, resp. $\mathcal{A}_{h,L}$, stands for the mass, resp. stiffness, matrix defined in (4.5).

It is easy to check that $\mathcal{M}_{h,L}$ is a symmetric positive definite matrix, and accordingly it can be decomposed as $\mathcal{M}_{h,L} = \mathcal{E}^*_{h,L}\mathcal{E}_{h,L}$, for some lower triangular matrix $\mathcal{E}_{h,L} = \big((\mathcal{E}_{h,L})_{jk}\big)_{-N+1\leq j,k\leq N-1}$ with positive diagonal entries.

With these matrices in hand, we can rely on the following representation result for the elements derived from identities such as (4.4).

Lemma A.1. *Fix (h,L) such that $N := \frac{L}{h} \in \mathbb{N}$. Let $\theta_{t_i} \in \mathcal{S}^{(h,L)}$ ($i = 0,\ldots,2^m$) be such that $\theta_0 = 0$ and for all $i = 1,\ldots,2^m$, $\varphi \in \mathcal{S}^{(h,L)}$,*

$$2^m\langle \theta_{t_i} - \theta_{t_{i-1}}, \varphi\rangle + \langle \nabla\theta_{t_i}, \nabla\varphi\rangle = 2^m\langle f_i, \varphi\rangle\,,$$

for some functions $f_i \in L^2_{loc}(\mathbb{R})$.

Then, if $\theta_{t_i}(x) = \sum_{j=-N+1}^{N-1} \boldsymbol{\theta}^j_{t_i}\Phi^h_j(x)$, the vector $\boldsymbol{\theta}_{t_i} = (\boldsymbol{\theta}^j_{t_i})_{-N+1\leq j\leq N-1}$ is explicitly given by

$$\boldsymbol{\theta}_{t_i} = \mathcal{E}^{-1}_{h,L}\sum_{k=1}^{i}\big[(I+2^{-m}\mathsf{M}_{h,L})^{-1}\big]^{i-k+1}(\mathcal{E}^*_{h,L})^{-1}\langle f_k, \Phi^h\rangle\,, \text{ where } \mathsf{M}_{h,L} := (\mathcal{E}^{-1}_{h,L})^*\mathcal{A}_{h,L}\mathcal{E}^{-1}_{h,L}\,. \tag{A.1}$$

Let us recall a few important bounds related to the matrices $\mathcal{E}_{h,L}, \mathsf{M}_{h,L}$. To this end, for all vectors $\mathbf{u} = (\mathbf{u}^j)_{-N+1\leq j\leq N-1}, \mathbf{v} = (\mathbf{v}^j)_{-N+1\leq j\leq N-1}$, and for every matrix $M = (M_{jk})_{-N+1\leq j,k\leq N-1}$, we set

$$\langle \mathbf{u}, \mathbf{v}\rangle_2 := \sum_{j=-N+1}^{N-1}\mathbf{u}^j\mathbf{v}^j\,, \quad \|\mathbf{v}\|_2^2 := \langle \mathbf{v}, \mathbf{v}\rangle_2\,, \quad \|M\|_{2;2} := \sup_{\mathbf{v}\neq 0}\frac{\|M\mathbf{v}\|_2}{\|\mathbf{v}\|_2}\,.$$

Lemma A.2. *Fix the pair (h,L) in such a way that $N := \frac{L}{h} \in \mathbb{N}$.*
(i) *If $v := \sum_{j=-N+1}^{N-1}\mathbf{v}^j\,\Phi^h_j$ for some real-valued vector $(\mathbf{v}^j)$, then one has*

$$\frac{h}{3}\|\mathbf{v}\|_2^2 \leq \|v\|^2_{L^2(\mathbb{R})} \leq h\|\mathbf{v}\|_2^2. \tag{A.2}$$

(ii) *It holds that*

$$\|\mathcal{E}^{-1}_{h,L}\|^2_{2;2} \leq \frac{3}{h}\,. \tag{A.3}$$

(iii) *The matrix $\mathcal{M}_{h,L}$ introduced in* (A.1) *is positive, and so, for every $h > 0$, one has*

$$\sup_{L>0:\,\frac{L}{h}\in\mathbb{N}}\ \sup_{\ell\geq 0} \big\| \big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{\ell} \big\|_{2;2} \leq 1\,. \tag{A.4}$$

(iv) *For all vectors $\mathbf{v}_{t_1}, \ldots, \mathbf{v}_{t_{2^m}}$, all $i = 1, \ldots, 2^m$ and $\varepsilon > 0$, it holds that*

$$\sup_{L>0:\,\frac{L}{h}\in\mathbb{N}} \Big\| \sum_{k=1}^{i-1} \Big[\big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{k} - \big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{k+1} \Big] \mathbf{v}_{t_{i-k}} \Big\|_2 \lesssim 2^{m\varepsilon} \sup_{k=1,\ldots,2^m} \|\mathbf{v}_{t_k}\|_2\,, \tag{A.5}$$

where the proportional constant does not depend on h and i.

Let us finally evoke an intermediate result for functions vanishing on the boundary $\{-L, L\}$ of Ω_L. For every $w \in H_0^1(\Omega_L)$, we recall that the *Ritz projection* $\mathcal{R}^{(h,L)}(w)$ *of* w is defined as the orthogonal projection of w on $\mathcal{S}^{(h,L)}$ with respect to the product

$$\langle\!\langle w_1, w_2 \rangle\!\rangle := \langle \nabla w_1, \nabla w_2 \rangle\,.$$

Then we define the operator $\mathscr{R}_m^{(h,L)}$ along the formula: for all $u \in L^\infty([0,1]; H_0^1(\Omega_L))$ with $u_0 = 0$,

$$\mathscr{R}_m^{(h,L)}(u)_0 := 0 \quad \text{and} \quad \mathscr{R}_m^{(h,L)}(u)_{t_i} := 2^m \int_{t_{i-1}}^{t_i} ds\, \mathcal{R}^{(h,L)}(u_s) \quad \text{for } i = 1, \ldots, 2^m\,. \tag{A.6}$$

The following related control (the proof of which can be derived for instance from [28, Theorem 1.1]) turns out to be a central ingredient toward the subsequent Galerkin estimates.

Lemma A.3. *Fix $h, L > 0$ such that $N := \frac{L}{h} \in \mathbb{N}$. For all $u \in H^{(1,2)}([0,1] \times \Omega_L) \cap L^\infty([0,1]; H_0^1(\Omega_L) \cap H^2(\Omega_L))$ with $u_0 = 0$, $i = 1, \ldots, 2^m$ and $\varphi \in \mathcal{S}^{(h,L)}$, it holds that*

$$\langle \nabla(\mathscr{R}_m^{(h,L)}(u)_{t_i}), \nabla\varphi \rangle = 2^m \int_{t_{i-1}}^{t_i} ds\, \langle \nabla u_s, \nabla \varphi \rangle \tag{A.7}$$

and

$$\sup_{i=1,\ldots,2^m} \big\| u_{t_i} - \mathscr{R}_m^{(h,L)}(u)_{t_i} \big\| \lesssim 2^{-m} \operatorname*{ess\,sup}_{t\in[0,1]} \|(\partial_s u)_t\| + h^2 \operatorname*{ess\,sup}_{t\in[0,1]} \|\Delta u_t\|\,, \tag{A.8}$$

where the proportional constant does not depend on h, L and m.

A.2 Proof of Proposition 4.2

We now have the tools in hand to estimate the difference between u and its Galerkin approximation $\mathcal{G}_m^{(h,L)}(u)$ (as defined in (4.1)).

For the sake of clarity, let us first consider the situation where u vanishes on the boundary $\{-L, L\}$ of Ω_L. In this case, a possible estimate for $u - \mathcal{G}_m^{(h,L)}(u)$ goes as follows.

Lemma A.4. *For all $h, L > 0$ such that $N := \frac{L}{h} \in \mathbb{N}$, all $u \in H^{(1,2)}([0,1] \times \Omega_L) \cap L^\infty([0,1]; H_0^1(\Omega_L) \cap H^2(\Omega_L))$ with $u_0 = 0$, all $m \geq 1$ and $\varepsilon > 0$, one has*

$$\sup_{i=1,\ldots,2^m} \big\| u_{t_i} - \mathcal{G}_m^{(h,L)}(u)_{t_i} \big\|_{L^2(\Omega_L)} \lesssim 2^{-m(1-\varepsilon)} \operatorname*{ess\,sup}_{t\in[0,1]} \|(\partial_t u)_t\|_{L^2(\Omega_L)} + 2^{m\varepsilon} h^2 \operatorname*{ess\,sup}_{t\in[0,1]} \|\Delta u_t\|_{L^2(\Omega_L)}\,, \tag{A.9}$$

where the proportional constant does not depend on h, L and m.

Proof. For every $i = 0, \ldots, 2^m$, let us decompose the difference $u_{t_i} - \mathcal{G}^{(h,L)}_m(u)_{t_i}$ as

$$\begin{aligned} u_{t_i} - \mathcal{G}^{(h,L)}_m(u)_{t_i} &= \big[u_{t_i} - \mathscr{R}^{(h,L)}_m(u)_{t_i}\big] + \big[\mathscr{R}^{(h,L)}_m(u)_{t_i} - \mathcal{G}^{(h,L)}_m(u)_{t_i}\big] \\ &=: (\gamma^{(h,L)}_m)_{t_i} + (\theta^{(h,L)}_m)_{t_i}\,, \end{aligned} \tag{A.10}$$

where $\mathscr{R}^{(h,L)}_m$ is the operator introduced in (A.6). Thanks to (A.8), we already know that

$$\sup_{i=1,\ldots,2^m} \big\|(\gamma^{(h,L)}_m)_{t_i}\big\|_{L^2(\Omega_L)} \lesssim 2^{-m} \operatorname*{ess\,sup}_{t\in[0,1]} \|(\partial_s u)_t\| + h^2 \operatorname*{ess\,sup}_{t\in[0,1]} \|\Delta u_t\|\,, \tag{A.11}$$

for some proportional constant that does not depend on h, L and m.

Let us now turn to the estimate for $(\theta^{(h,L)}_m)_{t_i}$ ($i = 1, \ldots, 2^m$), and to alleviate the notation, let us write from now on $\gamma, \theta, \mathscr{R}, \mathcal{G}$ instead of $\gamma^{(h,L)}_m, \theta^{(h,L)}_m, \mathscr{R}^{(h,L)}_m, \mathcal{G}^{(h,L)}_m$, respectively.

Given $\varphi \in \mathcal{S}^{(h,L)}$, observe that for every $i = 1, \ldots, 2^m$,

$$\begin{aligned} &2^m\langle \theta_{t_i} - \theta_{t_{i-1}}, \varphi\rangle + \langle \nabla\theta_{t_i}, \nabla\varphi\rangle \\ &= \big[2^m\langle \mathscr{R}(u)_{t_i} - \mathscr{R}(u)_{t_{i-1}}, \varphi\rangle + \langle \nabla(\mathscr{R}(v)_{t_i}), \nabla\varphi\rangle\big] \\ &\qquad\qquad - \big[2^m\langle \mathcal{G}(u)_{t_i} - \mathcal{G}(u)_{t_{i-1}}, \varphi\rangle + \langle \nabla(\mathcal{G}(u)_{t_i}), \nabla\varphi\rangle\big] \\ &= \big[2^m\langle \mathscr{R}(u)_{t_i} - \mathscr{R}(u)_{t_{i-1}}, \varphi\rangle + \langle \nabla(\mathscr{R}(u)_{t_i}), \nabla\varphi\rangle\big] - 2^m \int_{t_{i-1}}^{t_i} ds\, \big[\langle (\partial_t u)_s, \varphi\rangle - \langle \nabla u_s, \nabla\varphi\rangle\big] \\ &= 2^m\big[\langle \mathscr{R}(u)_{t_i} - \mathscr{R}(u)_{t_{i-1}}, \varphi\rangle - \langle u_{t_i} - u_{t_{i-1}}, \varphi\rangle\big] \\ &\qquad\qquad + \left[\langle \nabla(\mathscr{R}(u)_{t_i}), \nabla\varphi\rangle - 2^m \int_{t_{i-1}}^{t_i} ds\, \langle \nabla u_s, \nabla\varphi\rangle\right] \end{aligned}$$

and so, using identity (A.7), we end up with the relation

$$2^m\langle \theta_{t_i} - \theta_{t_{i-1}}, \varphi\rangle + \langle \nabla\theta_{t_i}, \nabla\varphi\rangle = -2^m\langle \gamma_{t_i} - \gamma_{t_{i-1}}, \varphi\rangle\,. \tag{A.12}$$

Now recall that for every $i = 1, \ldots, 2^m$, $\theta_{t_i} \in \mathcal{S}^{(h,L)}$, and therefore this element can be expanded $\theta_{t_i}(x) = \sum_{j=-N+1}^{N-1} \boldsymbol{\theta}^j_{t_i} \Phi^h_j(x)$ for some vector $(\boldsymbol{\theta}^j_{t_i})_{-N+1\leq j\leq N-1}$. With this notation in hand, and since $\theta_0 = 0$, we can apply Lemma A.1 to the relation (A.12) and assert that for every $i = 1, \ldots, 2^m$,

$$\begin{aligned} \boldsymbol{\theta}_{t_i} &= -\mathcal{E}_{h,L}^{-1} \sum_{k=1}^{i} \big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{i-k+1} (\mathcal{E}^*_{h,L})^{-1} \langle \gamma_{t_k} - \gamma_{t_{k-1}}, \Phi^h\rangle \\ &= -\mathcal{E}_{h,L}^{-1}\big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big](\mathcal{E}^*_{h,L})^{-1}\langle \gamma_{t_i}, \Phi^h\rangle \\ &\quad + \mathcal{E}_{h,L}^{-1} \sum_{k=1}^{i-1} \big\{\big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{i-k} - \big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{i-k+1}\big\}(\mathcal{E}^*_{h,L})^{-1}\langle \gamma_{t_k}, \Phi^h\rangle\,, \end{aligned}$$

where the second identity is derived from a discrete integration by parts, and the fact that $\gamma_0 = 0$.

Using (A.3) and (A.4), we can first assert that

$$\big\|\mathcal{E}_{h,L}^{-1}\big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big](\mathcal{E}^*_{h,L})^{-1}\langle \gamma_{t_i}, \Phi^h\rangle\big\|_2 \lesssim \frac{1}{h}\big\|\langle \gamma_{t_k}, \Phi^h\rangle\big\|_2\,,$$

Then, using (A.3) and (A.5), one gets for every $\varepsilon > 0$,

$$\begin{aligned} &\left\|\mathcal{E}_{h,L}^{-1} \sum_{k=1}^{i-1} \big\{\big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{i-k} - \big[(I + 2^{-m}\mathcal{M}_{h,L})^{-1}\big]^{i-k+1}\big\}(\mathcal{E}^*_{h,L})^{-1}\langle \gamma_{t_k}, \Phi^h\rangle\right\|_2 \\ &\lesssim \frac{2^{m\varepsilon}}{h^{1/2}} \sup_{k=1,\ldots,2^m} \big\|(\mathcal{E}^*_{h,L})^{-1}\langle \gamma_{t_k}, \Phi^h\rangle\big\|_2 \lesssim \frac{2^{m\varepsilon}}{h} \sup_{k=1,\ldots,2^m} \big\|\langle \gamma_{t_k}, \Phi^h\rangle\big\|_2\,. \end{aligned}$$

Therefore, we obtain that

$$\|\boldsymbol{\theta}_{t_i}\|_2 \lesssim \frac{2^{m\varepsilon}}{h} \sup_{k=1,\ldots,2^m} \big\|\langle \gamma_{t_k}, \Phi^h \rangle\big\|_2 .$$

It is now readily checked that

$$\big\|\langle \gamma_{t_k}, \Phi^h \rangle\big\|_2^2 = \sum_{j=-N+1}^{N-1} \left| \int_{x_{j-1}}^{x_{j+1}} dx\, \gamma_{t_k}(x) \Phi_j^h(x) \right|^2 \lesssim h\, \|\gamma_{t_k}\|_{L^2(\Omega_L)}^2$$

for proportional constants independent of h and L, and so, using (A.2),

$$\|\theta_{t_i}\|_{L^2(\mathbb{R})} \leq h^{1/2} \|\boldsymbol{\theta}_{t_i}\|_2 \lesssim 2^{m\varepsilon} \sup_{k=1,\ldots,2^m} \|\gamma_{t_k}\|_{L^2(\Omega_L)} . \tag{A.13}$$

Going back to the decomposition (A.10), the desired estimate (A.9) follows from the combination of (A.11) and (A.13). □

We are finally in a position to prove Proposition 4.2, by extending the previous estimate to situations where u does not necessarily vanish on the boundary.

Proof of Proposition 4.2. First, let us introduce a function $u^L \in H^{1,2}_{loc}([0,1] \times \mathbb{R})$ such that $u^L_s(-L) = u^L_s(L) = 0$ and $\Delta u^L_s = \Delta u_s$. To be more specific, we take

$$u^L_s(x) := u_s(x) - \frac{1}{2L}(u_s(L) - u_s(-L))x - \frac{1}{2}(u_s(-L) + u_s(L)) . \tag{A.14}$$

Also, in the sequel, we always consider L large enough so that $\operatorname{supp} \rho \subset \Omega_L$. Then one has

$$\begin{aligned}
&\sup_{i=1,\ldots,2^m} \big\|\rho \cdot \big\{u_{t_i} - \mathcal{G}^{(h,L)}_m(u)_{t_i}\big\}\big\|_{L^2(\mathbb{R})} \\
&\leq \sup_{i=1,\ldots,2^m} \big\|u^L_{t_i} - \mathcal{G}^{(h,L)}_m(u^L)_{t_i}\big\|_{L^2(\Omega_L)} + \sup_{t\in[0,1]} \big\|\rho \cdot \big\{u_t - u^L_t\big\}\big\|_{L^2(\mathbb{R})} \\
&\qquad\qquad + \sup_{i=1,\ldots,2^m} \big\|\mathcal{G}^{(h,L)}_m(u - u^L)_{t_i}\big\|_{L^2(\Omega_L)} ,
\end{aligned} \tag{A.15}$$

and we can estimate each of these three quantities separately.

To handle the first quantity, observe that the function u^L satisfies the conditions of Lemma A.4, and therefore, thanks to this result, one has for every $\varepsilon > 0$ and every $L > 0$ large enough

$$\begin{aligned}
&\sup_{i=1,\ldots,2^m} \big\|u^L_{t_i} - \mathcal{G}^{(h,L)}_m(u^L)_{t_i}\big\|_{L^2(\Omega_L)} \\
&\lesssim 2^{-m(1-\varepsilon)} \operatorname*{ess\,sup}_{t\in[0,1]} \|(\partial_t u^L)_t\|_{L^2(\Omega_L)} + 2^{m\varepsilon} h^2 \operatorname*{ess\,sup}_{t\in[0,1]} \|\Delta u^L_t\|_{L^2(\Omega_L)} \\
&\lesssim \left[2^{-m(1-\varepsilon)} \operatorname*{ess\,sup}_{t\in[0,1]} \|(\partial_t u)_t\|_{L^2(\mathbb{R})} + 2^{m\varepsilon} h^2 \operatorname*{ess\,sup}_{t\in[0,1]} \|\Delta u_t\|_{L^2(\mathbb{R})} \right] \\
&\qquad\qquad + \sup_{t\in[0,1]} \sup_{x\in\partial\Omega_L} L^{1/2} \big|(\partial_t u)_t(x)\big| ,
\end{aligned} \tag{A.16}$$

where the proportional constants do not depend on h, L and m.

For the second quantity in (A.15), and with the explicit expression (A.14) in mind, it is clear that

$$\sup_{t\in[0,T]} \big\|\rho \cdot \big\{u_t - u^L_t\big\}\big\|_{L^2(\mathbb{R})} \lesssim \sup_{t\in[0,T]} \sup_{x\in\partial\Omega_L} |u_t(x)| , \tag{A.17}$$

where the proportional constant only depends on ρ.

As for the third quantity in (A.15), one has, owing to (4.4) and Lemma A.1:

$$\mathcal{G}_m^{(h,L)}(u-u^L)_{t_i} = \sum_{j=-N+1}^{N-1} \boldsymbol{\delta}_{t_i}^j \Phi_j^h,$$

with

$$\boldsymbol{\delta}_{t_i} = \mathcal{E}_{h,L}^{-1} \sum_{k=1}^{i} \left[(I+2^{-m}\mathcal{M}_{h,L})^{-1}\right]^{i-k+1} (\mathcal{E}_{h,L}^*)^{-1} \int_{t_{k-1}}^{t_k} ds \, \langle \partial_t (u-u^L)_s, \Phi^h \rangle \,,$$

where we have used the fact that $\langle \nabla(u_s - u_s^L), \nabla\varphi\rangle = 0$ for all $s \in [0,1]$ and $\varphi \in \mathcal{S}^{(h,L)}$.

Using (A.2), (A.3) and (A.4), we deduce that

$$\big\|\mathcal{G}_m^{(h,L)}(u-u^L)_{t_i}\big\|_{L^2(\Omega_L)} \leq h^{1/2} \|\boldsymbol{\delta}_{t_i}\|_2 \lesssim \frac{1}{h^{1/2}} \sup_{s\in[0,1]} \|\langle \partial_t (u-u^L)_s, \Phi^h\rangle\|_2 \,. \tag{A.18}$$

Then we can observe that

$$\begin{aligned}
\|\langle \partial_t (u-u^L)_s, \Phi^h\rangle\|_2^2 &\lesssim \left|\frac{1}{2L}\big\{(\partial_s u)_s(L) - (\partial_s u)_s(-L)\big\}\right|^2 \sum_{j=-N+1}^{N-1} \left|\int_{\mathbb{R}} dx\, x\Phi_j^h(x)\right|^2 \\
&\qquad + \left|\frac{1}{2}\big\{(\partial_s u)_s(L) + (\partial_s u)_s(-L)\big\}\right|^2 \sum_{j=-N+1}^{N-1} \left|\int_{\mathbb{R}} dx\, \Phi_j^h(x)\right|^2 \\
&\lesssim \left(\sup_{x\in\partial\Omega_L} \big|(\partial_s u)_s(x)\big|^2\right) \sum_{j=-N+1}^{N-1} \left(\int_{\mathbb{R}} dx\, |\Phi_j^h(x)|\right)^2 \\
&\lesssim h^2 N \sup_{x\in\partial\Omega_L} \big|(\partial_s u)_s(x)\big|^2 .
\end{aligned}$$

Now remember that $Nh = L$, and therefore, going back to (A.18), we deduce

$$\sup_{i=1,\ldots,2^m} \big\|\mathcal{G}_m^{(h,L)}(u-u^L)_{t_i}\big\|_{L^2(\Omega_L)} \lesssim \sup_{s\in[0,1]} \sup_{x\in\partial\Omega_L} L^{1/2} \big|(\partial_s u)_s(x)\big| \,. \tag{A.19}$$

□

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