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Discrete approximation to Brownian motion with varying dimension in unbounded domains

Shuwen Lou*

Abstract

We establish the discrete approximation to Brownian motion with varying dimension (BMVD in abbreviation) by random walks. The setting is very similar to that in [11], but here we use a different method allowing us to get rid the restrictions in [11] (or [3]) that the underlying state space has to be bounded, and that the initial distribution of the limiting continuous process has to be its invariant distribution. The approach in this paper is that we first obtain heat kernel upper bounds for the approximating random walks that are uniform in their mesh sizes, by establishing an isoperimetric inequality and a Nash-type inequality based on their Dirichlet form characterization. Using the heat kernel upper bound, we then show the tightness of the approximating random walks via delicate analysis.

Keywords: space of varying dimension; Brownian motion; random walk; Dirichlet forms; heat kernel estimates; isoperimetric inequality; Nash-type inequality; tightness; Skorokhod space.
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1 Introduction

Brownian motion on spaces with varying dimension was introduced in [7]. The state space of such a process looks like a plane with a vertical half line installed on it, "embedded" in the following space:

$$\mathbb{R}^2 \cup \mathbb{R}_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0 \text{ or } x_2 = x_3 = 0 \text{ and } x_1 > 0\}.$$

As has been noted in [7], Brownian motion cannot be defined on such a state space in the usual sense because a two-dimensional Brownian motion does not hit a singleton. The BMVD in [7] was constructed by "shorting" a closed disc on $\mathbb{R}^2$ to a singleton, which in other words, makes the resistance on this closed disc zero, so that the process travels on the disc at infinite velocity. The resulting Brownian motion hits the shorted disc in finite time with probability one. Then an infinite half line $\mathbb{R}_+$ is attached to the plane $\mathbb{R}^2$ at this "shorted" disc.

*Loyola University Chicago, United States of America. E-mail: slou1@luc.edu

The state space of BMVD can be rigorously defined as follows: Fix $0 < \varepsilon < 1/64$ and denote by B_ε the closed disk on $\mathbb{R}^2$ centered at $(0,0)$ with radius ε. Let $D_\varepsilon := \mathbb{R}^2 \setminus B_\varepsilon$. By identifying B_ε with a singleton denoted by a^*, we introduce a topological space $E := D_\varepsilon \cup \{a^*\} \cup \mathbb{R}_+$, with the origin of $\mathbb{R}_+$ identified with a^* and a neighborhood of a^* defined as $\{a^*\} \cup (V_1 \cap \mathbb{R}_+) \cup (V_2 \cap D_\varepsilon)$ for some neighborhood V_1 of 0 in $\mathbb{R}^1$ and V_2 of B_ε in $\mathbb{R}^2$. Let m be the measure on E whose restriction on $\mathbb{R}_+$ or D_ε is 1- or 2-dimensional Lebesgue measure, respectively. In particular, we set $m(\{a^*\}) = 0$. Note that the measure m depends on ε, the radius of the "hole" B_ε.

Same as in [7], the state space E is equipped with the geodesic distance ρ. Namely, for $x, y \in E$, $\rho(x, y)$ is the shortest path distance (induced from the Euclidean space) in E between x and y. For notation simplicity, we write $|x|_\rho$ for $\rho(x, a^*)$. We use $|\cdot|$ to denote the usual Euclidean norm. For example, for $x, y \in D_\varepsilon$, $|x - y|$ is the Euclidean distance between x and y in $\mathbb{R}^2$. Note that for $x \in D_\varepsilon$, $|x|_\rho = |x| - \varepsilon$. Clearly,

$$\rho(x, y) = |x - y| \wedge (|x|_\rho + |y|_\rho) \quad \text{for } x, y \in D_\varepsilon \tag{1.1}$$

and $\rho(x, y) = |x| + |y| - \varepsilon$ when $x \in \mathbb{R}_+$ and $y \in D_\varepsilon$ or vice versa. Here and in the rest of this paper, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$.

The following definition for BMVD can be found in [7, Definition 1.1].

Definition 1.1 (Brownian motion with varying dimension)**.** An m-symmetric diffusion process satisfying the following properties is called Brownian motion with varying dimension.

(i) its part process in $\mathbb{R}_+$ or D_ε has the same law as standard Brownian motion in $\mathbb{R}_+$ or D_ε;

(ii) it admits no killings on a^*;

It follows from the definition that BMVD spends zero amount of time under Lebesgue measure (i.e. zero sojourn time) at a^*. The following theorem gives the Dirichlet form characterization of BMVD.

Theorem 1.2 ([7])**.** For every $\varepsilon > 0$, BMVD on E with parameter ε exists and is unique. Its associated Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E; m)$ is given by

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= \left\{ f : f|_{D_\varepsilon} \in W^{1,2}(D_\varepsilon),\, f|_{\mathbb{R}_+} \in W^{1,2}(\mathbb{R}_+),\, \text{and } f(x) = f(0) \text{ q.e. on } \partial D_\varepsilon \right\}, \\ \mathcal{E}(f, g) &= \frac{1}{4} \int_{D_\varepsilon} \nabla f(x) \cdot \nabla g(x) dx + \frac{1}{2} \int_{\mathbb{R}_+} f'(x) g'(x) dx. \end{aligned}$$

Furthermore, such a BMVD is a Feller process with strong Feller property.

Remark 1.3. For the computation convenience in this paper, we let BMVD in Theorem 1.2 corresponds to the BMVD defined in [7, Theorem 2.2] with parameters $(\varepsilon, p = 1)$ but running on D_ε at a speed $1/2$.

It is well-known that Brownian motion on Euclidean spaces is the scaling limit of simple random walks on square lattices. In [3], it was shown that reflected Brownian motion in bounded domains can be approximated by simple random walks. Roughly speaking, the method in [3] consists of two steps: The first step is to prove the tightness of the laws of the random walks by verifying the tightness of the martingale parts of the random walks through the analysis of their quadratic variations and then applying the Lyons-Zheng decomposition. The second step is to show the uniqueness of the subsequential limits of the laws of the random walks by characterizing the limits as the solution to the martingale problem for the infinitesimal generator of the continuous process. Utilizing the same method, it was proved in [11] that BMVD killed upon exiting a bounded domain can be approximated weakly by simple random walks in lattices with varying dimension. However, the method used in both [3] and [11] has the limitation that it only works on bounded domains, and that the initial distribution has to be the invariant measure of the continuous limiting process. In this paper, we use a different approach to get rid of the constrain that the approximation can only be established on bounded domains with initial distribution having to be the invariant measure.

Same as in [11], we use a sequence of approximating random walks indexed by $k \geq 1$ on lattices with with mesh-size 2^{-k}. Notice that in [11], these approximating random walks can be characterized in terms of Dirichlet forms. Therefore carefully applying the combination of Nash-type inequality and Davies method provides some heat kernel upper bound that is "uniform in k" for the entire family of random walks. Intuitively speaking, this gives some level of "equi-continuity" for the transition densities of this family random walks. From there. the C-tightness of the random walks can be established via some delicate analysis. As a result, we show that starting from the darning point a^*, BMVD on E can be weakly approximated by a family of random walks with varying dimension starting from the respective darning point in each of their state spaces. We note that although in this paper, the approximation is only established for BMVD starting from the darning point a^*, with similar computation one can show the same approximation results for BMVD starting from any single point.

The rigorous description of the state spaces of random walks with varying dimension has been given in [11]. Here we repeat it for completion. For $k \in \mathbb{N}$, let $D^k_\varepsilon := D_\varepsilon \cap 2^{-k}\mathbb{Z}^2$. We identify vertices of $2^{-k}\mathbb{Z}^2$ that are contained in the closed disc B_ε as a singleton a^*_k. Let $E^k := 2^{-k}\mathbb{Z}_+ \cup \{a^*_k\} \cup D^k_\varepsilon$, where $\mathbb{Z}_+ = \{1, 2, \dots\}$.

Recall that in general, a graph G can be written as "$G = \{G_v, G_e\}$", where G_v is its collection of vertices, and G_e is its connection of edges. Given any two vertices in $a, b \in G$, if there is unoriented edge with endpoints a and b, we say a and b are adjacent to each other in G, written "$a \leftrightarrow b$ in G". One can always assume that given two vertices a, b on a graph, there is at most one such unoriented edge connecting these two points (otherwise edges with same endpoints can be removed and replaced with one single edge). This unoriented edge is denoted by e_{ab} or e_{ba} (e_{ab} and e_{ba} are viewed as the same elelment in G_e). In this paper, for notational convenience, we denote by $\mathcal{G}_2 := \{2^{-k}\mathbb{Z}^2, \mathcal{V}_2\}$, where $\mathcal{V}_2$ is the collection of the edges of $2^{-k}\mathbb{Z}^2$. Also we denote by $\mathcal{G}_1 := \{2^{-k}\mathbb{Z}_+ \cup \{0\}, \mathcal{V}_1\}$ the 1-dimensional lattice over $2^{-k}\mathbb{Z}_+ \cup \{0\}$, where $\mathcal{V}_1$ is the collection of edges of $2^{-k}\mathbb{Z}^1_+ \cup \{0\}$. Here we emphasize that $\mathcal{G}_i$, $\mathcal{V}_i$, $i = 1, 2$, all refer to the usual 1- or 2-dimensional Euclidean spaces (without darning). We prepare these notations in order to introduce the graph structures on the space with darning in the next paragraph.

Now we introduce the graph structure on E^k. Let $G^k = \{G^k_v, G^k_e\}$ be a graph where $G^k_v = E^k$ is the collection of vertices and G^k_e is the collection of unoriented edges over E^k defined as follows:

$$\begin{aligned} G^k_e :=& \{e_{xy} : x, y \in D^k_\varepsilon, \text{ there exists an } e_{xy} \in \mathcal{V}_2, \text{ such that } e_{xy} \cap B_\varepsilon = \emptyset\} \\ & \cup \{e_{xy} : \exists\, x, y \in 2^{-k}\mathbb{Z}_+ \cup \{0\},\, |x - y| = 2^{-k},\, e_{xy} \in \mathcal{V}_1\} \\ & \cup \{e_{xa^*_k} : x \in D^k_\varepsilon, \text{ there exists an } e_{xy} \in \mathcal{V}_2 \text{ such that } e_{xy} \cap B_\varepsilon \neq \emptyset\}, \end{aligned} \tag{1.2}$$

where $e_{xy} \in \mathcal{V}_2$ is the line segment (in the usual $\mathbb{R}^2$ without darning) connecting x and y, including the two endpoints. In view of (1.2), any $x \in D^k_\varepsilon$ is said to be adjacent to a^*_k in G^k if and only if when being viewed as an element in $2^{-k}\mathbb{Z}^2$, at least one of the following two conditions holds for x:

(i) There exists at least one (at most two) $y \in B^k_\varepsilon$ satisfying $y \leftrightarrow x$ in $2^{-k}\mathbb{Z}^2$;

(ii) There exists $y \in D^k_\varepsilon$ satisfying $y \leftrightarrow x$ in $2^{-k}\mathbb{Z}^2$ with $e_{xy} \cap B_\varepsilon \neq \emptyset$, where $e_{xy} \in \mathcal{V}_2$ is a line segment in $\mathbb{R}^2$.

Note that for $x \in D^k_\varepsilon$, $x \leftrightarrow a^*_k$ in G^k, the Euclidean distance between x and B_ε must be up to 2^{-k}. It follows that $G^k = \{G^k_v, G^k_e\}$ is a connected graph. We emphasize that given any $x \in G^k_v$, $x \neq a^*_k$, there is at most one element in G^k_e with endpoints x and a^*_k. Denote by $v_k(x) = \#\{e_{xy} \in G^k_e\}$, i.e., the number of vertices in G^k_v adjacent to x. E^k is equipped with the following underlying reference measure:

$$m_k(x) := \begin{cases} \frac{2^{-2k}}{4} v_k(x), & x \in D^k_\varepsilon; \\ \frac{2^{-k}}{2} v_k(x), & x \in 2^{-k}\mathbb{Z}_+; \\ \frac{2^{-k}}{2} + \frac{2^{-2k}}{4}\left(v_k(x) - 1\right), & x = a^*_k. \end{cases} \tag{1.3}$$

Next we define the random walks that will be shown to approximate the BMVD. Consider the following Dirichlet form on $L^2(E^k, m_k)$:

$$\begin{cases} \mathcal{D}(\mathcal{E}^k) = L^2(E^k, m_k) \\ \mathcal{E}^k(f,f) = \frac{1}{8} \sum\limits_{\substack{e^o_{xy}:\, e_{xy} \in G^k_e, \\ x,y \in D^k_\varepsilon \cup \{a^*_k\}}} (f(x) - f(y))^2 + \frac{2^k}{4} \sum\limits_{\substack{e^o_{xy}:\, e_{xy} \in G^k_e, \\ x,y \in 2^{-k}\mathbb{Z}_+ \cup \{a^*_k\}}} (f(x) - f(y))^2, \end{cases} \tag{1.4}$$

where e^o_{xy} is an *oriented edge* from x to y. In other words, given any pair of adjacent vertices $x, y \in G^k_v$, the edge with endpoints x and y is represented twice in the sum: e^o_{xy} and e^o_{yx}. One can verify that $(\mathcal{E}^k, \mathcal{D}(\mathcal{E}^k))$ on $L^2(E^k, m_k)$ is a regular symmetric Dirichlet form. We denote the symmetric strong Markov process associated with $(\mathcal{E}^k, \mathcal{D}(\mathcal{E}^k))$ by X^k. The explicit distribution of X^k is presented in Proposition 2.2. In this paper, for every fixed $0 < \varepsilon < 1/64$, we select and then fix some $k_0 \in \mathbb{N}$ only depending on ε such that

$$2^{-k} < \varepsilon/4 \quad \text{for all } k \geq k_0. \tag{1.5}$$

Our main result is the following theorem.

Theorem 1.4. For every $T > 0$, the laws of $\{X^k, \mathbb{P}^{a^*_k}\}_{k \geq k_0}$ are tight in the space $\mathbf{D}([0,T], E, \rho)$ equipped with Skorokhod topology. Furthermore, as $k \to \infty$, $(X^k, \mathbb{P}^{a^*_k})$ converges weakly to BMVD with parameter ε starting from a^*.

Remark 1.5. In the statement of Theorem 1.4, for every $k \in \mathbb{N}$, a^*_k can be identified with a^* when being viewed as an element in E.

The rest of this paper is organized as follows: For Section 2, we first give a brief introduction to continuous-time reversible pure jump proceses and their corresponding symmetric Dirichlet forms in §2.1. Then in §2.2 we present some basics about the approximating random walks $\{X^k, k \geq 1\}$, including their explicit transition probabilities. These results were obtained in [11]. In §2.3, we review the results on isoperimetric inequalities for weighted graphs summarized from [1]. In Section 3, we first establish a Nash-type inequality for $\{X^k, k \geq 1\}$. From there using Davies method we obtain some heat kernel upper bounds for this family of random walks. The tightness of $\{X^k, k \geq 1\}$ is shown in Section 4, which is done with some very delicate analysis based upon the heat kernel upper bounds. Finally, the weak approximation result is completed in Section 5, by identifying the limit of $\{X^k, k \geq 1\}$ as the unique solution to the martingale problem for the infinitesimal generator of BMVD, i.e., the generator associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in Theorem 1.2.

In this paper we follow the convention that in the statements of the theorems or propositions $C, C_1, \cdots$ denote positive constants, whereas in their proofs $c, c_1, \cdots$ denote positive constants whose exact value is unimportant and may change from line to line.

2 Preliminaries

2.1 Continuous-time reversible pure jump processes and symmetric Dirichlet forms

In this section, we give a brief background on continuous-time reversible pure jump processes and symmetric Dirichlet forms. The results in this section can be found in [5, §2.2.1].

Suppose E is a locally compact separable metric space and $\{Q(x,dy)\}$ is a probability kernel on $(\mathsf{E},\mathcal{B}(\mathsf{E}))$ with $Q(x,\{x\})=0$ for every $x\in\mathsf{E}$. Given a constant $\lambda>0$, we can construct a pure jump Markov process X as follows: Starting from $x_0\in\mathsf{E}$, X remains at x_0 for an exponentially distributed holding time T_1 with parameter $\lambda(x_0)$ (i.e., $\mathbb{E}[T_1]=1/\lambda(x_0)$), then it jumps to some $x_1\in\mathsf{E}$ according to distribution $Q(x_0,dy)$; it remains at x_1 for another exponentially distributed holding time T_2 also with parameter $\lambda(x_1)$ before jumping to x_2 according to distribution $Q(x_1,dy)$. T_2 is independent of T_1. X then continues. The probability kernel $Q(x,dy)$ is called the *road map* of X, and the $\lambda(x)$ is its *speed function*. If there is a σ-finite measure m_0 on E with supp $[\mathsf{m}_0]=\mathsf{E}$ such that

$$Q(x,dy)\mathsf{m}_0(dx)=Q(y,dx)\mathsf{m}_0(dy), \tag{2.1}$$

m_0 is called a *symmetrizing measure* of the road map Q. Another way to view (2.1) is that, $Q(x,dy)$ is the one-step transition "probability" distribution, so its density with respect to the symmetrizing measure $Q(x,dy)/\mathsf{m}_0(dy)$ must be symmetric in x and y, i.e.,

$$\frac{Q(x,dy)}{\mathsf{m}_0(dy)}=\frac{Q(y,dx)}{\mathsf{m}_0(dx)}.$$

The following theorem is a restatement of [5, Theorem 2.2.2].

Theorem 2.1 ([5])**.** Given a speed function $\lambda>0$. Suppose (2.1) holds, then the reversible pure jump process X described above can be characterized by the following Dirichlet form $(\mathfrak{E},\mathfrak{F})$ on $L^2(\mathsf{E},\mathsf{m})$ where the underlying reference measure is $\mathsf{m}(dx)=\lambda(x)^{-1}\mathsf{m}_0(dx)$ and

$$\begin{cases}\mathfrak{F}=L^2(\mathsf{E},\ \mathsf{m}(x)),\\ \mathfrak{E}(f,g)=\dfrac{1}{2}\displaystyle\int_{\mathsf{E}\times\mathsf{E}}(f(x)-f(y))(g(x)-g(y))Q(x,dy)\mathsf{m}_0(dx).\end{cases} \tag{2.2}$$

2.2 Continuous-time random walks on lattices with varying dimension

The following proposition follows from the exact same argument for [11, Proposition 2.2], which describes the behavior of X^k in the unbounded space with varying dimension E^k.

Proposition 2.2. For every $k=1,2,\dots$, X^k has constant speed function $\lambda_k=2^{2k}$ and road map

$$J_k(x,dy)=\sum_{z\in E^k,\ z\leftrightarrow x\text{ in }G^k}j_k(x,z)\delta_{\{z\}}(dy),$$

where

(i)

$$j_k(x,y)=\frac{1}{v_k(x)},\quad\text{if } x\in D^k_\varepsilon\cup 2^{-k}\mathbb{Z}_+,\ y\leftrightarrow x\text{ in }G^k \tag{2.3}$$

(ii)

$$j(a^*_k,y)=\begin{cases}\dfrac{1}{v_k(a^*_k)+2^{k+1}-1}, & y\in D^k_\varepsilon,\ y\leftrightarrow x\text{ in }G^k;\\ \dfrac{2^{k+1}}{v_k(a^*_k)+2^{k+1}-1}, & y\in 2^{-k}\mathbb{Z}_+,\ y\leftrightarrow x\text{ in }G^k.\end{cases} \tag{2.4}$$

The next proposition is essentially contained in the proof of [11, Proposition 2.1].

Proposition 2.3. For any fixed $0 < \varepsilon \le 1/64$ and all $k \ge k_0$, where k_0 is specified in (1.5),

$$m_k(a_k^*) < 2^{-k}. \tag{2.5}$$

Proof. It has been shown in the proof of [11, Proposition 2.1] that for all $k = 1, 2, \dots$,

$$v_k(a_k^*) \le 56\varepsilon \cdot 2^k + 28. \tag{2.6}$$

It thus follows from (1.5) that for $k \ge k_0$,

$$\begin{aligned} m_k(a_k^*) &\le \frac{2^{-k}}{2} + \frac{2^{-2k}}{4}\left(56\varepsilon \cdot 2^k + 28\right) = \frac{2^{-k}}{2} + 14\varepsilon \cdot 2^{-k} + 7 \cdot 2^{-k} \cdot 2^{-k} \\ &\le \frac{2^{-k}}{2} + 14\varepsilon \cdot 2^{-k} + \frac{7\varepsilon}{4} \cdot 2^{-k} \le \left(\frac{1}{2} + \frac{63\varepsilon}{4}\right) 2^{-k} < 2^{-k}. \end{aligned}$$ □

2.3 Isoperimetric inequalities for weighted graphs

Before we establish Nash-type inequality for X^k, we give a summary on the isoperimetric inequalities for weighted graphs. Most of the results in this section can be found in [1]. In the following, Γ is a locally finite connected graph, and the collection of vertices of Γ is denoted by $\mathbb{V}$. If two vertices $x, y \in \mathbb{V}$ are adjacent to each other, then the the unoriented edge connecting x and y is assigned a unique weight $\mu_{xy} > 0$. Set $\mu_{xy} = 0$ if x and y are not adjacent in Γ. Denote by $\mu := \{\mu_{xy} : x, y \text{ connected in } \Gamma\}$ the assignment of the weights on all the unoriented edges. (Γ, μ) is called a locally finite connected *weighted graph*. We equip the weighted graph with (Γ, μ) following measure ν on $\mathbb{V}$:

$$\nu(x) := \sum_{y \in \mathbb{V}: y \leftrightarrow x \text{ in } \Gamma} \mu_{xy}, \quad x \in \mathbb{V}. \tag{2.7}$$

Given two sets of vertices A, B in $\mathbb{V}$, we define

$$\mu_\Gamma(A, B) := \sum_{x \in A} \sum_{y \in B} \mu_{xy}. \tag{2.8}$$

The following definition of isoperimetric inequality is taken from [1, Definition 3.1].

Definition 2.4. For $\alpha \in [1, \infty)$, we say that a weighted graph (Γ, μ) satisfies α-isoperimetric inequality (I_α) if there exists $C_0 > 0$ such that

$$\frac{\mu_\Gamma(A, \mathbb{V} \backslash A)}{\nu(A)^{1-1/\alpha}} \ge C_0, \quad \text{for every finite non-empty } A \subset \mathbb{V}. \tag{2.9}$$

Proposition 2.5 ([1])**.** Let (Γ, μ) be a locally finite connected weighted graph satisfying α-isoperimetric inequality with constant C_0. Let ν be the measure defined in (2.7). Then (Γ, μ) satisfies the following Nash-type inequality:

$$\frac{1}{2} \sum_{x \in \mathbb{V}} \sum_{y \in \mathbb{V}, y \leftrightarrow x} \left(f(x) - f(y)\right)^2 \mu_{xy} \ge 4^{-(2+\alpha/2)} C_0^2 \|f\|_{L^2(\nu)}^{2+4/\alpha} \|f\|_{L^1(\nu)}^{-4/\alpha}, \quad f \in L^1(\nu) \cap L^2(\nu)$$

Proof. This can be seen combining [1, Theorem 3.7, Lemma 3.9, Theorem 3.14] and the proofs therein. □

The next proposition follows immediately from [1, Theorem 3.26], which states the isoperimetric inequality on $2^{-k}\mathbb{Z}^2$. As a notation in [1], given a weighted graph (Γ, μ) with collection of vertices $\mathbb{V}$. We denote the counting measure times 2^{-2k} on $2^{-k}\mathbb{Z}^2$ by $\mu_k^{(2)}$, which can be viewed as the measure "ν" in (2.7) corresponding to weighted $2^{-k}\mathbb{Z}^2$ with all edges weighing $2^{-2k}/4$.

Proposition 2.6 ([1]). Let $k \in \mathbb{N}$. Let all edges of $2^{-k}\mathbb{Z}^2$ be assigned with a weight of $2^{-2k}/4$. There exists a constant $C_1 > 0$ independent of k such that for any finite subset A of $2^{-k}\mathbb{Z}^2$,

$$\mu_{2^{-k}\mathbb{Z}^2}(A, 2^{-k}\mathbb{Z}^2 \backslash A) \geq C_1 \cdot 2^{-k} \mu_k^{(2)}(A)^{1/2}. \tag{2.10}$$

3 Nash-type inequality and equicontinuity for random walks on lattices with varying dimension

In the following, for every $k \in \mathbb{N}$, we view $2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}$ as a subgraph of G^k. Let all the edges in the $2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}$ be assigned with a weight $2^{-k}/2$. Then we define a measure on $2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}$:

$$\nu_k^{(1)}(x) := \frac{2^{-k}}{2} \cdot \#\{y \in 2^{-k}\mathbb{Z}_+ \cup \{a_k^*\} : y \leftrightarrow x \text{ in } 2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}\}, \quad \forall x \in 2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}. \tag{3.1}$$

Similarly, let all the edges in the connected graph $D_\varepsilon^k \cup \{a_k^*\}$ be assigned with a weight of $2^{-2k}/4$. Then we define a measure on $D_\varepsilon^k \cup \{a_k^*\}$:

$$\nu_k^{(2)}(x) := \frac{2^{-2k}}{4} \cdot \#\{y \in D_\varepsilon \cup \{a_k^*\} : y \leftrightarrow x \text{ in } D_\varepsilon \cup \{a_k^*\}\}, \quad \forall x \in D_\varepsilon^k \cup \{a_k^*\}. \tag{3.2}$$

The next lemma establishes the isoperimetric inequalities in the form of (2.9) on $D_\varepsilon^k \cup \{a_k^*\}$ and $2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}$ respectively, which gives us the two Nash-type inequalities on these two subspaces separately. Then in Proposition 3.2, we "add up" the two Nash-type inequalities on these two subspaces to get the desired Nash-type inequality for the entire state space E^k.

Lemma 3.1. Let $D_\varepsilon^k \cup \{a_k^*\}$ and $2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}$ be equipped with the weights and measure described in the preceding paragraph. There exists a constant $C_2 > 0$ independent of k such that for all $k \geq k_0$ (see (1.5) for the definition of k_0),

$$\frac{\mu_{D_\varepsilon^k \cup \{a_k^*\}}(A, (D_\varepsilon^k \cup \{a_k^*\}) \backslash A)}{\nu_k^{(2)}(A)^{1/2}} \geq 2^{-k} C_2, \quad \text{for any finite set } A \subset (D_\varepsilon^k \cup \{a_k^*\}), \tag{3.3}$$

and

$$\mu_{2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}}(A, (2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}) \backslash A) \geq 2^{-k} C_2, \quad \text{for any finite set } A \subset (2^{-k}\mathbb{Z}_+ \cup \{a_k^*\}). \tag{3.4}$$

Proof. (3.4) is clear. In the following we prove (3.3). We divide it into two cases depending on whether $a_k^* \in A$ or not. For the remainder of this proof, we denote by

$$B_\varepsilon^k := 2^{-k}\mathbb{Z}^2 \cap B_\varepsilon.$$

Case (i). $a_k^* \notin A$. In view of the definitions of $\mathcal{V}_2$ and (1.2), as well the paragraph of explanation following it,

$$\begin{aligned} & \mu_{D_\varepsilon \cup \{a_k^*\}}(A,\, (D_\varepsilon^k \cup \{a_k^*\}) \backslash A) \\ = \quad & \frac{2^{-2k}}{4} \sum_{x \in A} \#\Big\{ y \in D_\varepsilon^k \backslash A : \exists\, e_{xy} \in \mathcal{V}_2 \text{ such that } e_{xy} \cap B_\varepsilon = \emptyset \Big\} \\ + \quad & \frac{2^{-2k}}{4} \# \{x \in A : x \leftrightarrow a_k^*\} \\ = \quad & \frac{2^{-2k}}{4} \sum_{x \in A} \#\Big\{ y \in D_\varepsilon^k \backslash A : \exists\, e_{xy} \in \mathcal{V}_2 \text{ such that } e_{xy} \cap B_\varepsilon = \emptyset \Big\} \end{aligned}$$

$$
\begin{aligned}
& + \quad 2\cdot\frac{1}{2}\cdot\frac{2^{-2k}}{4}\#\left\{x\in A: x\leftrightarrow a_k^*\right\} \\
& \geq \quad \frac{2^{-2k}}{4}\sum_{x\in A}\#\left\{y\in D_\varepsilon^k\backslash A: \exists\, e_{xy}\in\mathcal{V}_2 \text{ such that } e_{xy}\cap B_\varepsilon=\emptyset\right\} \\
& + \quad \frac{1}{2}\cdot\frac{2^{-2k}}{4}\#\left\{x\in A:\exists\, y\in B_\varepsilon^k \text{ such that } x\leftrightarrow y \text{ in } 2^{-k}\mathbb{Z}^2\right\} \\
& + \quad \frac{1}{2}\cdot\frac{2^{-2k}}{4}\#\left\{x\in A:\ \exists\, y\in D_\varepsilon^k \text{ such that } e_{xy}\in\mathcal{V}_2 \text{ but } e_{xy}\cap B_\varepsilon\neq\emptyset\right\} \\
& \geq \quad \frac{2^{-2k}}{4}\sum_{x\in A}\#\left\{y\in D_\varepsilon^k\backslash A: \exists\, e_{xy}\in\mathcal{V}_2 \text{ such that } e_{xy}\cap B_\varepsilon=\emptyset\right\} \\
& + \quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{x\in A}\#\left\{y\in B_\varepsilon^k \text{ such that } x\leftrightarrow y \text{ in } 2^{-k}\mathbb{Z}^2\right\} \\
& + \quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{x\in A}\#\left\{y\in D_\varepsilon^k:\ \exists e_{xy}\in\mathcal{V}_2 \text{ but } e_{xy}\cap B_\varepsilon\neq\emptyset\right\} \\
& \geq \quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{x\in A}\#\left\{y\in D_\varepsilon^k\backslash A: \exists\, e_{xy}\in\mathcal{V}_2\right\} \\
& + \quad \frac{2^{-2k}}{4}\cdot\frac{1}{4}\sum_{x\in A}\#\left\{y\in B_\varepsilon^k:\ y\leftrightarrow x \text{ in } 2^{-k}\mathbb{Z}^2\right\} \\
& = \quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{x\in A}\#\left\{y\in 2^{-k}\mathbb{Z}^2\backslash A: y\leftrightarrow x \text{ in } 2^{-k}\mathbb{Z}^2\right\} \\
& \overset{(2.8)}{=} \quad \frac{1}{4}\,\mu_{2^{-k}\mathbb{Z}^2}(A,\, 2^{-k}\mathbb{Z}^2\backslash A) \\
& \overset{(2.10)}{\geq} \quad \frac{C_1}{4}\cdot 2^{-k}\mu_k^{(2)}(A)^{1/2}\geq\frac{C_1}{4}\cdot 2^{-k}\nu_k^{(2)}(A)^{1/2},
\end{aligned}
\tag{3.5}
$$

where the first inequality results from the explanation following (1.2), and the second inequality is due to the fact for every $x\in A$, there are at most two $y\in B_\varepsilon^k$ adjacent to it in $2^{-k}\mathbb{Z}^2$, as well as at most two $y\in D_\varepsilon^k$ such that $e_{xy}\in\mathcal{V}_2$ but $e_{xy}\cap B_\varepsilon\neq\emptyset$. The last inequality above is due to the fact that for $A\subset D_\varepsilon^k$, $\mu_k^{(2)}(A)\geq\nu_k^{(2)}(A)$ in view of the definitions for both.

Case (ii). $a_k^*\in A$. Similar to the argument for Case (i), noting that $(D_\varepsilon^k\cup\{a_k^*\})\backslash A = D_\varepsilon^k\backslash A$ in this case, we have

$$
\begin{aligned}
& \mu_{D_\varepsilon\cup\{a_k^*\}}(A,\,(D_\varepsilon^k\cup\{a_k^*\})\backslash A) \\
= \quad & \frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\left\{x\in A\backslash\{a_k^*\}: x\leftrightarrow y \text{ in } D_\varepsilon^k\cup\{a_k^*\}\right\} \\
+ \quad & \frac{2^{-2k}}{4}\#\left\{y\in D_\varepsilon^k\backslash A: y\leftrightarrow a_k^* \text{ in } D_\varepsilon^k\cup\{a_k^*\}\right\} \\
= \quad & \frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\left\{x\in A\backslash\{a_k^*\}: x\leftrightarrow y \text{ in } D_\varepsilon^k\cup\{a_k^*\}\right\} \\
+ \quad & 2\cdot\frac{1}{2}\cdot\frac{2^{-2k}}{4}\#\left\{y\in D_\varepsilon^k\backslash A: y\leftrightarrow a_k^* \text{ in } D_\varepsilon^k\cup\{a_k^*\}\right\} \\
\geq \quad & \frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\left\{x\in A\backslash\{a_k^*\}: \exists\, e_{xy}\in\mathcal{V}_2 \text{ such that } e_{xy}\cap B_\varepsilon=\emptyset\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\quad \frac{1}{2}\cdot\frac{2^{-2k}}{4}\#\Big\{y\in D_\varepsilon^k\backslash A:\ \exists\, x\in B_\varepsilon^k \text{ such that } x\leftrightarrow y \text{ in } 2^{-k}\mathbb{Z}^2\Big\}\\
&+\quad \frac{1}{2}\cdot\frac{2^{-2k}}{4}\#\Big\{y\in D_\varepsilon^k\backslash A:\ \exists\, x\in D_\varepsilon^k \text{ such that } e_{xy}\in\mathcal{V}_2,\ \text{but } e_{xy}\cap B_\varepsilon=\emptyset\Big\}\\
&\geq\quad \frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\Big\{x\in A\backslash\{a_k^*\}:\exists\, e_{xy}\in\mathcal{V}_2 \text{ such that } e_{xy}\cap B_\varepsilon=\emptyset\Big\}\\
&+\quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\Big\{x\in B_\varepsilon^k \text{ such that } x\leftrightarrow y \text{ in } 2^{-k}\mathbb{Z}^2\Big\}\\
&+\quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\Big\{x\in D_\varepsilon^k \text{ such that }\exists\, e_{xy}\in\mathcal{V}_2,\ \text{but } e_{xy}\cap B_\varepsilon=\emptyset\Big\}\\
&\geq\quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\Big\{x\in A\backslash\{a_k^*\} \text{ such that } x\leftrightarrow y\in 2^{-k}\mathbb{Z}^2\Big\}\\
&+\quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\Big\{x\in B_\varepsilon^k \text{ such that } x\leftrightarrow y \text{ in } 2^{-k}\mathbb{Z}^2\Big\}\\
&=\quad \frac{1}{4}\cdot\frac{2^{-2k}}{4}\sum_{y\in D_\varepsilon^k\backslash A}\#\Big\{x\in (A\backslash\{a_k^*\})\cup B_\varepsilon^k: x\leftrightarrow y \text{ in } 2^{-k}\mathbb{Z}^2\Big\}\\
&\geq\quad \frac{1}{4}\mu_{2^{-k}\mathbb{Z}^2}((A\backslash\{a_k^*\})\cup B_\varepsilon^k,\ 2^{-k}\mathbb{Z}^2\backslash A)\\
&\overset{(2.10)}{\geq}\quad \frac{C_1}{2}\cdot 2^{-k}\mu_k^{(2)}((A\backslash\{a_k^*\})\cup B_\varepsilon^k)^{1/2}\\
&\overset{(1.3)}{\geq}\quad \frac{C_1}{2}\cdot 2^{-k}\left(\mu_k^{(2)}((A\backslash\{a_k^*\})+m_k(a_k^*)-\frac{2^{-k}}{2}\right)^{1/2}\geq \frac{C_1}{2}\cdot 2^{-k}\nu_k^{(2)}(A)^{1/2}, \qquad (3.6)
\end{aligned}
$$

where the first inequality results from the explanation following (1.2), and the second inequality is due to the fact for every $y\in D_\varepsilon^k\backslash A$, there are at most two $x\in B_\varepsilon^k$ adjacent to it in $2^{-k}\mathbb{Z}^2$, as well as at most two $x\in D_\varepsilon^k$ such that $e_{xy}\in\mathcal{V}_2$ but $e_{xy}\cap B_\varepsilon\neq\emptyset$. The second last inequality follows from the definition of $mu_k^{(2)}$ as well as the fact that $m_k(a_k^*)-\frac{2^{-k}}{2}$ equals the number of edges in $D_\varepsilon^k\cup\{a_k^*\}$ with an endpoint a_k^*. The last inequality above is due to the fact that for $A\subset D_\varepsilon^k$, $\mu_k^{(2)}(A)\geq\nu_k^{(2)}(A)$, as well as the fact that $\nu_k^{(2)}(a_k^*)=m_k(a_k^*)-\frac{2^{-k}}{2}$. □

Proposition 3.2. For every $k\in\mathbb{N}$, let $(P_t^k)_{t\geq 0}$ be the transition semigroup of X^k with respect to m_k. There exists a constant $C_3>0$ independent of k such that for all $k\geq k_0$,

$$
\|P_t^k\|_{1\to\infty}\leq C_3\left(\frac{1}{t}+\frac{1}{\sqrt{t}}\right),\quad \forall t\in(0,+\infty]. \qquad (3.7)
$$

Proof. We still consider the weighted graph $2^{-k}\mathbb{Z}_+\cup\{a_k^*\}$ with all edges having equal weight $2^{-k}/2$, and all edges of the weighted graph $D_\varepsilon^k\cup\{a_k^*\}$ have an equal weight of $2^{-2k}/4$. Also for now, we let the measures on $2^{-k}\mathbb{Z}_+\cup\{a_k^*\}$ and $D_\varepsilon^k\cup\{a_k^*\}$ be $\nu_k^{(1)}$ and $\nu_k^{(2)}$, respectively. Lemma 3.1 says that the weighted graph $2^{-k}\mathbb{Z}_+\cup\{a_k^*\}$ described above satisfies isoperimetric inequality I_1 with constant $2^{-k}C_2$, and the weighted graph $D_\varepsilon^k\cup\{a_k^*\}$ described above satisfies isoperimetric inequality I_2 also with constant $2^{-k}C_2$, for all $k\geq k_0$. Therefore, by Proposition 2.5, it holds for all $f\in L^1\left(2^{-k}\mathbb{Z}_+\cup\{a_k^*\},\ \nu_k^{(1)}\right)\cap$

$L^2\left(2^{-k}\mathbb{Z}_+ \cup \{a_k^*\},\ \nu_k^{(1)}\right)$ that

$$\begin{aligned}&\frac{2^{-k}}{4}\sum_{x\in 2^{-k}\mathbb{Z}_+\cup\{0\}}\ \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\cup\{0\},\\ y\leftrightarrow x}}(f(x)-f(y))^2\\ \geq\ & 4^{-4}2^{-2k}C_2^2\|f\|^6_{L^2\left(2^{-k}\mathbb{Z}_+\cup\{a_k^*\},\ \nu_k^{(1)}\right)}\cdot\|f\|^{-4}_{L^1\left(2^{-k}\mathbb{Z}_+\cup\{a_k^*\},\ \nu_k^{(1)}\right)}.\end{aligned}$$

Similarly, also by Proposition 2.5, for all $f\in L^1\left(D^k_\varepsilon\cup\{a_k^*\},\ \nu_k^{(2)}\right)\cap L^2\left(D^k_\varepsilon\cup\{a_k^*\},\ \nu_k^{(2)}\right)$,

$$\begin{aligned}&\frac{2^{-2k}}{8}\sum_{x\in D^k_\varepsilon\cup\{a_k^*\}}\ \sum_{\substack{y\in D^k_\varepsilon\cup\{a_k^*\},\\ y\leftrightarrow x}}(f(x)-f(y))^2\\ \geq\ & 4^{-4}2^{-2k}C_2^2\|f\|^4_{L^2\left(D^k_\varepsilon\cup\{a_k^*\},\ \nu_k^{(2)}\right)}\cdot\|f\|^{-2}_{L^1\left(D^k_\varepsilon\cup\{a_k^*\},\ \nu_k^{(2)}\right)}.\end{aligned}$$

The above two inequalities can be rewritten as

$$\begin{aligned}&\left(\frac{2^k}{4}\sum_{x\in 2^{-k}\mathbb{Z}_+\cup\{a_k^*\}}\ \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\cup\{a_k^*\},\\ y\leftrightarrow x}}(f(x)-f(y))^2\right)^{1/3}\|f\|^{4/3}_{L^1\left(2^{-k}\mathbb{Z}_+\cup\{a_k^*\},\nu_k^{(1)}\right)}\\ \geq\ & 4^{-4/3}C_2^{2/3}\|f\|^2_{L^2\left(2^{-k}\mathbb{Z}_+\cup\{a_k^*\},\nu_k^{(1)}\right)}\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&\left(\sum_{x\in D^k_\varepsilon\cup\{a_k^*\}}\ \sum_{\substack{y\in D^k_\varepsilon\cup\{a_k^*\},\\ y\leftrightarrow x}}(f(x)-f(y))^2\right)^{1/2}\cdot\|f\|_{L^1\left(D^k_\varepsilon\cup\{a_k^*\},\ \nu_k^{(2)}\right)}\\ \geq\ & 4^{-2}C_2\|f\|^2_{L^2\left(D^k_\varepsilon\cup\{a_k^*\},\ \nu_k^{(2)}\right)}.\end{aligned}\tag{3.9}$$

Notice that any $f\in L^1(E^k,m_k)\cap L^2(E^k,m_k)$, it holds that

$$\|f\|_{L^p(E^k,m_k)}=\left\|f|_{2^{-k}\cup\{a_k^*\}}\right\|_{L^p\left(2^{-k}\cup\{a_k^*\},\nu_k^{(1)}\right)}+\left\|f|_{D^k_\varepsilon\cup\{a_k^*\}}\right\|_{L^p\left(D^k_\varepsilon\cup\{a_k^*\},\nu_k^{(2)}\right)},\quad p=1,2.$$

For notational convenience, for $f\in L^1(E^k,m_k)\cap L^2(E^k,m_k)$, we set $f_1:=f|_{2^{-k}\cup\{a_k^*\}}$ and $f_2:=f|_{D^k_\varepsilon\cup\{a_k^*\}}$. Adding up (3.9) and (3.8) yields that for all $f\in L^1(E^k,m_k)\cap L^2(E^k,m_k)$,

$$\begin{aligned}&\left(4^{-4/3}C_2^{2/3}\wedge 4^{-2}C_2\right)\|f\|^2_{L^2(E^k,m_k)}\\ \leq\ & \left(\frac{2^k}{4}\sum_{x\in 2^{-k}\mathbb{Z}_+\cup\{0\}}\ \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\cup\{0\},\\ y\leftrightarrow x}}(f_1(x)-f_1(y))^2\right)^{1/3}\cdot\|f_1\|^{4/3}_{L^1\left(2^{-k}\mathbb{Z}_+\cup\{0\},\ \mu_k^{(1)}\right)}\\ +\ & \left(\sum_{x\in D^k_\varepsilon\cup\{a_k^*\}}\ \sum_{\substack{y\in D^k_\varepsilon\cup\{a_k^*\},\\ y\leftrightarrow x}}(f_2(x)-f_2(y))^2\right)^{1/2}\cdot\|f_2\|_{L^1\left(D^k_\varepsilon\cup\{a_k^*\},\ \nu_k^{(2)}\right)}\\ \leq\ & \mathcal{E}^k(f,f)^{1/3}\|f\|^{4/3}_{L^1(E^k,m_k)}+\mathcal{E}^k(f,f)^{1/2}\|f\|_{L^1(E^k,m_k)},\end{aligned}\tag{3.10}$$

The desired conclusion thus follows by selecting $\mu=1$ and $\nu=2$ in [4, Corollary 2.12]. □

With the Nash-type inequality established above, next we use Davies method to get a heat kernel upper bound for $(\mathcal{E}^k, \mathcal{D}(\mathcal{E}^k))$ that can be viewed as "equicontinuity" of the heat kernels of $\{X^k, k \geq 1\}$ in k. For this purpose, we first rewrite (1.4) as follows:

$$\begin{cases} \mathcal{D}(\mathcal{E}^k) = L^2(E^k, m_k) \\ \mathcal{E}^k(f,f) = \dfrac{2^{2k}}{8} \displaystyle\sum_{x \in D^k_\varepsilon} \sum_{y \leftrightarrow x} (f(y)-f(x))^2 \, m_k(x) + \dfrac{2^{2k}}{4} \sum_{x \in 2^{-k}\mathbb{Z}_+} \sum_{y \leftrightarrow x} (f(y)-f(x))^2 \, m_k(x) \\ \qquad + \dfrac{2^{2k}}{8} \displaystyle\sum_{\substack{y \leftrightarrow a^*_k \\ y \in D^k_\varepsilon}} (f(y)-f(a^*_k))^2 \, \dfrac{2^{-2k}}{4} + \dfrac{2^{2k}}{4} \sum_{\substack{y \leftrightarrow a^*_k \\ y \in 2^{-k}\mathbb{Z}_+}} (f(y)-f(a^*_k))^2 \, \dfrac{2^{-k}}{2}. \end{cases} \tag{3.11}$$

Now for every $k \in \mathbb{N}$, we define a metric $d_k(\cdot,\cdot)$ on E^k as follows:

$$d_k(x,y) \;:=\; 2^{-k} \times \text{number of edges along the shortest (geodesic) path between } x \text{ and } y \text{ in } G^k. \tag{3.12}$$

In the next proposition as well as the corollary following it, we establish an heat kernel upper bound estimate for X^k using Davies's method. In the remainder of this paper, for every $k \in \mathbb{N}$, we denote by $p_k(t,x,y)$ the transition density function of $(P^k_t)_{t\geq 0}$ with respect to m_k.

Proposition 3.3. There exist $C_4 > 0$ independent of k, such that for any $k \geq k_0$ fixed in (1.5) and any $\alpha_k \leq 2^{k-1}$,

$$p_k(t,x,y) \leq C_4 \left(\frac{1}{t} \vee \frac{1}{\sqrt{t}} \right) \exp\left(-\alpha_k d_k(x,y) + 2t\alpha_k^2 \right), \quad 0 < t < \infty, \, x, y \in E^k. \tag{3.13}$$

Proof. We prove this result using [4, Corollary 3.28]. Towards this, for each k, we set $\widehat{\mathcal{F}}^k := \{h + c : \; h \in \mathcal{D}(\mathcal{E}^k), \, h \text{ bounded, and } c \in \mathbb{R}^1\}$. It is known that the regular symmetric Dirichlet form $(\mathcal{E}^k, \mathcal{D}(\mathcal{E}^k))$ can be written in terms of an energy measure Γ^k as follows:

$$\mathcal{E}^k(u,u) = \int_{E^k} \Gamma^k(u,u), \quad u \in \widehat{\mathcal{F}}^k.$$

where Γ^k is a positive semidefinite symmetric bilinear form on $\mathcal{F}^k$ with values being signed Radon measures on E^k, which is also called the energy measure. Now we define $\widehat{\mathcal{F}}^k_\infty$ as a subset of $\psi \in \widehat{\mathcal{F}}^k$ satisfying the following conditions:

(i) Both $e^{-2\psi}\Gamma^k(e^\psi, e^\psi)$ and $e^{2\psi}\Gamma^k(e^{-\psi}, e^{-\psi})$ as measures are absolutely continuous with respect to m_k on E^k.

(ii) Furthermore,

$$\Gamma^k(\psi) := \left(\left\| \frac{de^{-2\psi} \; \Gamma^k(e^\psi, e^\psi)}{dm_k} \right\|_\infty \vee \left\| \frac{de^{2\psi} \; \Gamma^k(e^{-\psi}, e^{-\psi})}{dm_k} \right\|_\infty \right)^{1/2} < \infty. \tag{3.14}$$

We have fixed a constant $\alpha_k \leq 2^{k-1}$ and thus denote by

$$\psi_{k,n}(x) := \alpha_k \cdot (d_k(x, a^*_k) \wedge n), \quad \text{for every } n \in \mathbb{N}. \tag{3.15}$$

In order to apply [4, Corollary 3.28], we first check that for every n, $\psi_{k,n} \in \widehat{\mathcal{F}}^k_\infty$. Notice that $\psi_{k,n}$ is a constant outside of a bounded domain of E^k, therefore it is in $\widehat{\mathcal{F}}^k$. Now we compute $e^{-2\psi_{k,n}}\Gamma^k(e^{\psi_{k,n}}, e^{\psi_{k,n}})$ as a measure. Noting that $\psi_{k,n}(a_k^*)=0$, we first rewrite the expression of $\mathcal{E}^k$ in (1.4) as follows: for $f\in\widetilde{\mathcal{F}}^k_\infty$,

$$\begin{aligned}
&\int_{E^k}\Gamma^k(f,f)=\mathcal{E}^k(f,f)\\
&=\frac{1}{8}\sum_{\substack{e^o_{xy}:\, e_{xy}\in G^k_e,\\ x,y\in D^k_\varepsilon\cup\{a^*_k\}}}(f(x)-f(y))^2+\frac{2^k}{4}\sum_{\substack{e^o_{xy}:\, e_{xy}\in G^k_e,\\ x,y\in 2^{-k}\mathbb{Z}_+\cup\{a^*_k\}}}(f(x)-f(y))^2\\
&=\frac{2^{2k}}{8}\sum_{x\in D^k_\varepsilon\cap A}\sum_{y\leftrightarrow x}(f(y)-f(x))^2 m_k(x)+\frac{2^{2k}}{4}\sum_{x\in 2^{-k}\mathbb{Z}_+\cap A}\sum_{y\leftrightarrow x}(f(y)-f(x))^2 m_k(x)\\
&+\mathbf{1}_{\{a^*_k\in A\}}\left(\frac{2^{2k}}{8}\sum_{\substack{y\leftrightarrow a^*_k\\ y\in D^k_\varepsilon}}(f(y)-f(a^*_k))^2\,\frac{2^{-2k}}{4}+\frac{2^{2k}}{4}\sum_{\substack{y\leftrightarrow a^*_k\\ y\in 2^{-k}\mathbb{Z}_+}}(f(y)-f(a^*_k))^2\,\frac{2^{-k}}{2}\right).
\end{aligned}\tag{3.16}$$

On account of (3.16), given any subset $A\subset E^k$, we have

$$\begin{aligned}
&e^{-2\psi_{k,n}}\Gamma^k(e^{\psi_{k,n}}, e^{\psi_{k,n}})(A)\\
=\;&\frac{2^{2k}}{8}\sum_{x\in D^k_\varepsilon\cap A}e^{-2\psi_{k,n}(x)}\sum_{y\leftrightarrow x}\left(e^{\psi_{k,n}(y)}-e^{\psi_{k,n}(x)}\right)^2 m_k(x)\\
+\;&\frac{2^{2k}}{4}\sum_{x\in 2^{-k}\mathbb{Z}_+\cap A}e^{-2\psi_{k,n}(x)}\sum_{y\leftrightarrow x}\left(e^{\psi_{k,n}(y)}-e^{\psi_{k,n}(x)}\right)^2 m_k(x)\\
+\;&\mathbf{1}_{\{a^*_k\in A\}}\left(\frac{2^{2k}}{8}\sum_{\substack{y\leftrightarrow a^*_k\\ y\in D^k_\varepsilon}}\left(e^{\psi_{k,n}(y)}-1\right)^2\frac{2^{-2k}}{4}+\frac{2^{2k}}{4}\sum_{\substack{y\leftrightarrow a^*_k\\ y\in 2^{-k}\mathbb{Z}_+}}\left(e^{\psi_{k,n}(y)}-1\right)^2\frac{2^{-k}}{2}\right).
\end{aligned}$$

In view of the definition of m_k in (1.3), we further have

$$\begin{aligned}
&e^{-2\psi_{k,n}}\Gamma^k(e^{\psi_{k,n}}, e^{\psi_{k,n}})(A)\\
&\le\frac{1}{8}\sum_{x\in D^k_\varepsilon\cap A}\sum_{y\leftrightarrow x}\left(e^{\alpha_k(d_k(y,a^*_k)\wedge n-d_k(x,a^*_k)\wedge n)}-1\right)^2\\
&+\frac{2^k}{4}\sum_{x\in 2^{-k}\mathbb{Z}\cap A}\sum_{y\leftrightarrow x}\left(e^{\alpha_k(d_k(y,a^*_k)\wedge n-d_k(x,a^*_k)\wedge n)}-1\right)^2\\
&+\mathbf{1}_{\{a^*_k\in A\}}\left(\frac{1}{32}\sum_{\substack{y\leftrightarrow a^*_k\\ y\in D^k_\varepsilon}}\left(e^{\alpha_k(d_k(y,a^*_k)\wedge n)}-1\right)^2+\frac{2^k}{8}\sum_{\substack{y\leftrightarrow a^*_k\\ y\in 2^{-k}\mathbb{Z}_+}}\left(e^{\alpha_k(d_k(y,a^*_k)\wedge n)}-1\right)^2\right).
\end{aligned}\tag{3.17}$$

From the above it is clear that $e^{-2\psi_{k,n}}\Gamma^k(e^{\psi_{k,n}}, e^{\psi_{k,n}})$ is absolutely continuous with respect to m_k. By similar compution, one can show that so is $e^{2\psi_{k,n}}\Gamma^k(e^{-\psi_{k,n}}, e^{-\psi_{k,n}})$. Thus $\psi_{k,n}$ satisifies the first condition in the definition of $\widehat{\mathcal{F}}^k_\infty$.

Next we verify that $\psi_{k,n}$ also satisfies the second condition of the definition of $\widehat{\mathcal{F}}^k_\infty$. It is obvious from (1.3) that

$$m_k(x)\ge\begin{cases}\dfrac{2^{-2k}}{4}, & x\in D^k_\varepsilon;\\[2mm] \dfrac{2^{-k}}{2}, & x\in 2^{-k}\mathbb{Z}_+\cup\{a^*_k\}.\end{cases}\tag{3.18}$$

Hence combining (3.17) and (3.18) shows that

$$\begin{aligned}
&\left\|\frac{de^{-2\psi_{k,n}}\ \Gamma^k(e^{\psi_{k,n}},e^{\psi_{k,n}})}{dm_k}\right\|_\infty \\
\le\ & \max_{x\in D^k_\varepsilon}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n-d_k(x,a_k^*)\wedge n)}-1\right)^2\right\} \\
\vee\ & \max_{x\in 2^{-k}\mathbb{Z}}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n-d_k(x,a_k^*)\wedge n)}-1\right)^2\right\} \\
\vee\ & \left(\frac{2^k}{16}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in D^k_\varepsilon}}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2+\frac{2^{2k}}{4}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in 2^{-k}\mathbb{Z}_+}}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2\right).
\end{aligned} \tag{3.19}$$

By similar computation one can show that

$$\begin{aligned}
&\left\|\frac{de^{2\psi_{k,n}}\ \Gamma^k(e^{-\psi_{k,n}},e^{-\psi_{k,n}})}{dm_k}\right\|_\infty \\
\le\ & \max_{x\in D^k_\varepsilon}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k(d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n)}-1\right)^2\right\} \\
\vee\ & \max_{x\in 2^{-k}\mathbb{Z}}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k(d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n)}-1\right)^2\right\} \\
\vee\ & \left(\frac{2^k}{16}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in D^k_\varepsilon}}\left(e^{-\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2+\frac{2^{2k}}{4}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in 2^{-k}\mathbb{Z}_+}}\left(e^{-\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2\right)
\end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) we get

$$\begin{aligned}
\Gamma^k(\psi_{k,n}) &= \left(\left\|\frac{de^{-2\psi_{k,n}}\ \Gamma^k(e^{\psi_{k,n}},e^{\psi_{k,n}})}{dm_k}\right\|_\infty \vee \left\|\frac{de^{2\psi_{k,n}}\ \Gamma^k(e^{-\psi_{k,n}},e^{-\psi_{k,n}})}{dm_k}\right\|_\infty\right)^{1/2} \\
&\le \left(\max_{x\in D^k_\varepsilon}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k|d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n|}-1\right)^2\right\}\right)^{1/2} \\
&\vee \left(\max_{x\in 2^{-k}\mathbb{Z}}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k|d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n|}-1\right)^2\right\}\right)^{1/2} \\
&\vee \left(\frac{2^k}{16}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in D^k_\varepsilon}}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2+\frac{2^{2k}}{4}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in 2^{-k}\mathbb{Z}_+}}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2\right)^{1/2}.
\end{aligned} \tag{3.21}$$

In the following we claim that all three "$(\cdots)^{1/2}$" terms on the right hand side of (3.21) are bounded by $\sqrt{2}\alpha_k$, where α_k is fixed in the line above (3.15). First of all, we note that $e^x-1<2x$, for all $0<x<1/2$. Thus for all $k\in\mathbb{N}$,

$$e^{\alpha_k x}-1\le 2\alpha_k x,\quad \text{for } 0<x\le 2^{-k},\ \alpha_k\le 2^{k-1}. \tag{3.22}$$

In view of the definition of d_k, $d_k(x,y) \le 2^{-k}$ for any $x \leftrightarrow y$. Thus for all $k \ge k_0 \ge 1$,

$$\begin{aligned}
&\left(\max_{x\in D_\varepsilon^k}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k|d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n|}-1\right)^2\right\}\right)^{1/2}\\
\le\ & \frac{2^k}{\sqrt{2}}\max_{x\in D_\varepsilon^k}\left\{\sum_{y\leftrightarrow x}\left(e^{\alpha_k|d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n|}-1\right)\right\}\\
\le\ & \frac{2^k}{\sqrt{2}}\cdot\max_{x\in D_\varepsilon^k}\sum_{y\leftrightarrow x}2\left(\alpha_k\left|d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n\right|\right) \qquad (3.22)\\
\le\ & \frac{2^k}{\sqrt{2}}\cdot 2\alpha_k\cdot 2^{-k}\le\sqrt{2}\alpha_k
\end{aligned}\tag{3.23}$$

By similar computation one can show that for the second "$(\cdots)^{1/2}$",

$$\left(\max_{x\in 2^{-k}\mathbb{Z}}\left\{\frac{2^{2k}}{2}\sum_{y\leftrightarrow x}\left(e^{\alpha_k|d_k(x,a_k^*)\wedge n-d_k(y,a_k^*)\wedge n|}-1\right)^2\right\}\right)^{1/2}\le\sqrt{2}\alpha_k. \tag{3.24}$$

Finally, to bound the last "$(\dots)^{1/2}$" on the right hand side of (3.21), we have for $0<\varepsilon<1/64$ and $k\ge k_0$ that

$$\begin{aligned}
&\left(\frac{2^k}{16}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in D_\varepsilon^k}}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2+\frac{2^{2k}}{4}\sum_{\substack{y\leftrightarrow a_k^*\\ y\in 2^{-k}\mathbb{Z}_+}}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2\right)^{1/2}\\
\le&\left(\frac{2^k}{16}\cdot\left(56\varepsilon\cdot 2^k+28\right)\cdot\max_{\substack{y\leftrightarrow a_k^*\\ y\in D_\varepsilon^k}}\left(e^{\alpha_k(d_k(y,a_k^*)\wedge n)}-1\right)^2+\frac{2^{2k}}{4}\cdot\left(e^{\alpha_k\cdot 2^{-k}}-1\right)^2\right)^{1/2}\\
\le&\left(\left(\frac{7\varepsilon}{2}+\frac{7\cdot 2^{-k}}{4}\right)\cdot 2^{2k}\cdot\left(e^{\alpha_k\cdot 2^{-k}}-1\right)^2+\frac{2^{2k}}{4}\left(e^{\alpha_k\cdot 2^{-k}}-1\right)^2\right)^{1/2}\\
\overset{(1.5)}{<}&\left(\frac{2^{2k}}{2}\left(e^{\alpha_k\cdot 2^{-k}}-1\right)^2\right)^{1/2}\le\frac{2^k}{\sqrt{2}}\left(e^{\alpha_k\cdot 2^{-k}}-1\right)\overset{(3.22)}{\le}\sqrt{2}\alpha_k,
\end{aligned}\tag{3.25}$$

where the first inequality above is due to Proposition 2.3. Combining (3.23), (3.24), and (3.25), we have for $k\ge k_0$,

$$\Gamma^k(\psi_{k,n})\le\sqrt{2}\alpha_k. \tag{3.26}$$

This also shows that $\psi_{k,n}$ is indeed in the class $\widehat{\mathcal{F}}_\infty$ defined in the second paragraph of the proof. Finally, to complete using [4, Corollary 3.28], by Proposition 3.2 and the choice of $\psi_{k,n}$ in (3.15), we have that for every $x\in E^k$ and every n, there exists a constant $C_4>0$ only depending through the C_3 given in Proposition 3.2 but not k that

$$p_k(t,x,y)\le C_4\left(\frac{1}{t}\vee\frac{1}{\sqrt{t}}\right)\exp\left(-\alpha_k\left(d_k(x,y)\wedge n\right)+2t\alpha_k^2\right),\quad \text{for } 0<t<\infty, m_k\text{-a.e. } y\in E^k.$$

Since all singletons in E^k have strictly positive measures, we may drop the "a.e." in the inequality above. Finally, the proof is completed by letting $n\to\infty$. □

Corollary 3.4. There exist $C_5 > 0$ independent of k, such that for any $k \geq k_0$ fixed in (1.5) and any $\alpha_k \leq 2^{k-1}$, it holds for all $x, y \in E^k$ and all $t \geq 0$ that

$$p_k(t,x,y) \leq \begin{cases} C_5 \left(\frac{1}{t} \vee \frac{1}{\sqrt{t}}\right) e^{-d_k(x,y)^2/(32t)}, & \text{when } d_k(x,y) \leq 16 \cdot 2^k t; \\ C_5 \left(\frac{1}{t} \vee \frac{1}{\sqrt{t}}\right) e^{-2^k d_k(x,y)/2}, & \text{when } d_k(x,y) \geq 16 \cdot 2^k t. \end{cases} \tag{3.27}$$

In particular, given any $T > 0$, there exists $C_6 > 0$ such that

$$p_k(t,x,y) \leq \frac{C_6}{t}\left(e^{-d_k(x,y)^2/(32t)} + e^{-2^k d_k(x,y)/2}\right), \text{ for all } (t,x,y) \in (0,T]\times E^k \times E^k. \tag{3.28}$$

Proof. To prove this, in Proposition 3.3, given any $k \geq k_0$, for any fixed $t_0 > 0$ and any pair of $x_0, y_0 \in E^k$, we take

$$\alpha_k := \frac{d_k(x_0,y_0)}{16t_0} \wedge 2^k.$$

Then Proposition 3.3 yields that for all $t > 0$ and $x, y \in E^k$,

$$\begin{aligned} &p_k(t_0,x,y) \\ &\leq C_4 \left(\frac{1}{t_0} \vee \frac{1}{\sqrt{t_0}}\right) \exp\left[-\left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right) d_k(x,y) + 2t_0 \left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right)^2\right], \end{aligned}$$

where C_4 only depends on the C_3 in Proposition 3.2. Taking $x = x_0$ and $y = y_0$ yields that

$$\begin{aligned} &p_k(t_0,x_0,y_0) \\ &\leq C_4 \left(\frac{1}{t_0} \vee \frac{1}{\sqrt{t_0}}\right) \exp\left[-\left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right) d_k(x_0,y_0) + 2t_0 \left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right)^2\right]. \end{aligned} \tag{3.29}$$

Now we divide our discussion into two cases:

Case 1. $d_k(x_0,y_0) \geq 16 \cdot 2^k t_0$. Then for the exponential term on the right hand side of (3.29) it holds

$$\begin{aligned} & -\left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right) d_k(x_0,y_0) + 2t_0 \left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right)^2 \\ \leq\ & -2^k d_k(x_0,y_0) + 2t_0 (2^k)^2 \\ \leq\ & -2^k d_k(x_0,y_0) + 2t_0 \frac{d_k(x_0,y_0)}{16t_0} \cdot 2^k \\ \leq\ & -2^k d_k(x_0,y_0) + \frac{1}{8} \cdot 2^k \cdot d_k(x_0,y_0) \leq -\frac{2^k}{2} d_k(x_0,y_0). \end{aligned} \tag{3.30}$$

Case 2. $d_k(x_0,y_0) \leq 16 \cdot 2^k t_0$. For this case,

$$\begin{aligned} & -\left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right) d_k(x_0,y_0) + 2t_0 \left(\frac{d_k(x_0,y_0)}{16t_0} \wedge 2^{-k}\right)^2 \\ \leq\ & -\frac{d_k(x_0,y_0)^2}{16t_0} + 2t_0 \left(\frac{d_k(x_0,y_0)}{16t_0}\right)^2 \leq -\frac{d_k(x_0,y_0)^2}{32t_0}. \end{aligned} \tag{3.31}$$

The proof is completed by substituting the power of the exponential term on the right hand side of (3.29) with (3.30) and (3.31) for the two cases, respectively. ☐

Remark 3.5. It worths pointing out that we do not expect the heat kernel upper bound obtained in Corollary 3.4 to be sharp (c.f. [7, Theorem 1.3-1.4]). However, this upper bound is sufficient for proving the tightness of $\{X^k\}_{k\geq 1}$ using Proposition 4.1.

4 Tightness of random walks on spaces with varying dimensions

4.1 General fact regarding tightness

In order to use Corollary 3.4 to establish the tightness of $\{X^k\}_{k\geq 1}$, we first record a proposition in [10, Chapter VI Theorem 3.21], which provides a criterion for tightness for càdlàg processes adapted to our setting. As a standard notation, given a metric $d(\cdot,\,\cdot)$, we denote by

$$w_d(x,\,\theta,\,T):=\inf_{\{t_i\}_{1\leq i\leq n}\in\Pi}\max_{1\leq i\leq n}\sup_{s,t\in[t_i,t_{i-1}]}d(x(s),x(t)),$$

where Π is the collection of all possible partitions of the form $0=t_0<t_1<\cdots<t_{n-1}<T\leq t_n$ with $\min_{1\leq i\leq n}(t_i-t_{i-1})\geq\theta$ and $n\geq 1$.

Proposition 4.1 (Chapter VI, Theorem 3.21 in [10])**.** Let $\{Y_k,\mathbb{P}^y\}_{k\geq 1}$ be a a sequence of càdlàg processes on state space E. Given $y\in E$, the laws of $\{Y_k,\mathbb{P}^y\}_{k\geq 1}$ are tight in the Skorokhod space $D([0,T],E,\rho)$ if and only if

(T-i). For any $T>0$, $\delta>0$, there exist $N_1\in\mathbb{N}$ and $M>0$ such that for all $k\geq N_1$,

$$\mathbb{P}^y\left[\sup_{t\in[0,T]}\left|Y_t^k\right|_\rho>M\right]<\delta. \tag{4.1}$$

(T-ii). For any $T>0$, $\delta_1,\delta_2>0$, there exist $\delta_3>0$ and $N_2>0$ such that for all $k\geq N_2$,

$$\mathbb{P}^y\left[w_\rho\left(Y^k,\,\delta_3,\,T\right)>\delta_1\right]<\delta_2. \tag{4.2}$$

In the next two subsections, we establish the tightness of $\{X_n\}_{n\geq 1}$ by verityiing the two conditions (T-i) and (T-ii) in Proposition 4.1 using the heat kernel upper bounds in Corollary 3.4.

4.2 Validation of (T-i) in Proposition 4.1 for $\{X^k;k\geq 1\}$

We first prepare the following simple lemma which will be used later in this subsection.

Lemma 4.2. Given any $T,M>0$, for any sufficiently large $k\geq k_0$ specified in (1.5) such that $2^{-k}<T$, it holds for all $x\in E^k$ that

$$\begin{aligned}\mathbb{P}^x\left[\sup_{t\in[0,T]}|X_t^k|_\rho\geq M\right]\leq\;&\mathbb{P}^x\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho\geq M\right]+\mathbb{P}^x\left[\left|X_T^k\right|_\rho\geq\frac{M}{2}\right]\\&+\mathbb{P}^x\left[T-8^{-k}\leq\tau_M\leq T,\,\left|X_T^k\right|_\rho\leq\frac{M}{2}\right]\\&+\mathbb{P}^x\left[8^{-k}\leq\tau_M\leq T-8^{-k},\,\left|X_T^k\right|_\rho\leq\frac{M}{2}\right],\end{aligned}$$

where $\tau_M:=\inf\{t>0:\;|X_t^k|_\rho\geq M\}$.

Proof. By inclusions of events, we have

$$\begin{aligned}&\mathbb{P}^x\left[\sup_{t\in[0,T]}|X_t^k|_\rho\geq M\right]\\&=\mathbb{P}^x\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho\geq M\right]+\mathbb{P}^x\left[\sup_{t\in[8^{-k},T]}|X_t^k|_\rho\geq M,\;\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho<M\right]\\&=\mathbb{P}^x\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho\geq M\right]\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}^x\left[\sup_{t\in[8^{-k},T]}|X_t^k|_\rho\ge M,\,|X_T^k|_\rho\ge\frac{M}{2},\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho<M\right]\\
&+\mathbb{P}^x\left[\sup_{t\in[8^{-k},T]}|X_t^k|_\rho\ge M,\,|X_T^k|_\rho\le\frac{M}{2},\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho<M\right]\\
&\le\mathbb{P}^x\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho\ge M\right]+\mathbb{P}^x\left[\left|X_T^k\right|\rho\ge\frac{M}{2}\right]+\mathbb{P}^x\left[8^{-k}\le\tau_M\le T,\,\left|X_T^k\right|_\rho\le\frac{M}{2}\right]\\
&\le\mathbb{P}^x\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho\ge M\right]+\mathbb{P}^x\left[\left|X_T^k\right|\rho\ge\frac{M}{2}\right]\\
&+\mathbb{P}^x\left[8^{-k}\le\tau_M\le T-8^{-k},\,\left|X_T^k\right|_\rho\le\frac{M}{2}\right]+\mathbb{P}^x\left[T-8^{-k}\le\tau_M\le T,\,\left|X_T^k\right|_\rho\le\frac{M}{2}\right].
\end{aligned}
\tag{4.3}
$$

□

In the next few propositions, by verifying the two conditions in Proposition 4.1 respectively, we establish the tightness of the laws of $\{X_t^k,\mathbb{P}^{a_k^*}\}_{k\ge1}$.

Proposition 4.3. For any $\delta>0$, any $T>0$, there exists $M_1>0$ such that for all $k\ge k_0$ specified in (1.5):

$$
\sup_{y\in E^k}\mathbb{P}^y\left[\sup_{t\in[0,8^{-k}]}\rho\left(X_0^k,X_t^k\right)\ge M_1\right]<\delta. \tag{4.4}
$$

Proof. We denote by $T_0^k:=0$, and

$$
T_l^k:=\inf\{t>T_{l-1}^k:\ X_t^k\ne X_{t-}^k\},\quad \text{for } l=1,2,\dots,
$$

i.e., T_l^k is the l^{th} holding time of X^k, and $\{T_l^k\}_{l\ge1}$ are i.i.d. exponential random variables, each with mean 2^{-k}. We then denote by $N_t^k:=\sup\{l\ge1: T_l^k\le t\}$. It is clear that N_t^k follows Poisson distribution with parameter $t\cdot2^{2k}$ Recall that $d_k(x,y)$ is the smallest number of edges that a path with endpoints x and y has multiplied by 2^{-k}, which implies that for any $k\in\mathbb{N}$ and $x,y\in E^k$, $d_k(x,y)\ge\rho(x,y)$. Therefore, for any $y\in E^k$, given any $\delta>0$, for any M satisfying $\frac{e}{M}<\frac{\delta\wedge1}{16}$, it holds that

$$
\begin{aligned}
\mathbb{P}^y\left[\sup_{t\in[0,8^{-k}]}\rho\left(X_0^k,X_t^k\right)\ge M\right] &\le \mathbb{P}^y\left[\sup_{t\in[0,8^{-k}]}d_k\left(y,X_t^k\right)\ge M\right]\\
&\le \mathbb{P}^y\left[2^{-k}N_{8^{-k}}^k\ge M\right]\\
(N_t^k\sim\text{Poisson}(t\cdot2^{2k}))\quad &= e^{-8^{-k}\cdot2^{2k}}\sum_{j=M\cdot2^k}^\infty\frac{(8^{-k}\cdot2^{2k})^j}{j!}\\
(\text{Stirling's formula})\quad &\le \sum_{j=M\cdot2^k}^\infty\frac{(8^{-k}\cdot2^{2k})^j}{j^je^{-j}\sqrt{2\pi j}}\\
&\le \sum_{j=M\cdot2^k}^\infty\left(\frac{8^{-k}4^ke}{M\cdot2^k}\right)^j\\
&\le \sum_{j=1}^\infty\left(\frac{e}{M}\right)^j<\frac{\frac{\delta\wedge1}{16}}{1-\frac{\delta\wedge1}{16}}\le\delta\wedge1.\quad\square
\end{aligned}
\tag{4.5}
$$

Proposition 4.4. For any $\delta>0$, any $T\ge1$, there exists $M_2>0$ such that for all $k\ge k_0$ specified in (1.5):

$$
\sup_{8^{-k}\le t\le T}\mathbb{P}^{a_k^*}\left[d_k(X_t^k,a_k^*)\ge M_2\right]<\delta.
$$

Proof. To prove this, we utilize Proposition 3.4. Given any $k \ge k_0$ fixed in (1.5), any $T \ge 1$, any $t \in [8^{-k}, T]$, and any $M > 1$,

$$\begin{aligned}
\mathbb{P}^{a_k^*}\left[d_k(X_t^k, a_k^*) \ge M\right] &\le \sum_{d_k(y,a^*)\ge M} \frac{c_1}{t}\left(e^{-\frac{d_k(a_k^*,y)^2}{32t}} + e^{-\frac{2^k d_k(a_k^*,y)}{2}}\right) m_k(dy). \\
&\le \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(y,a_k^*)\ge M}} \frac{c_1}{t}\left(e^{-\frac{d_k(a_k^*,y)^2}{32t}} + e^{-\frac{2^k d_k(a_k^*,y)}{2}}\right)\cdot 2^{-k} \\
&+ \sum_{\substack{y\in D_\varepsilon^k\\ d_k(y,a_k^*)\ge M}} \frac{c_1}{t}\left(e^{-\frac{d_k(a_k^*,y)^2}{32t}} + e^{-\frac{2^k d_k(a_k^*,y)}{2}}\right)\cdot 2^{-2k}. \qquad (4.6)
\end{aligned}$$

To handle the first term on the right hand side of (4.6), for $x \in E^k$, we let $n_{a_k^*}(x) = 2^k d_k(x, a_k^*)$, i.e., the smallest number of edges a path has connecting x and a_k^*. Thus for the first term on the right hand side of (4.6), it holds for $M > 1$ that

$$\begin{aligned}
\sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(y,a_k^*)\ge M}} \frac{c_1}{t} e^{-\frac{2^k d_k(a_k^*,y)}{2}}\cdot 2^{-k} &\overset{i=n_{a_k^*}(y)}{\le} \sum_{i=2^k[M]}^{\infty} \frac{c_2\cdot 2^{-k}}{t} e^{-i/2} \\
&\le \sum_{i=2^k[M]}^{\infty} \frac{c_2\cdot 2^{-k}}{t} e^{-i/4}\cdot e^{-i/4} \\
(t\in[8^{-k},T]) \qquad &\le c_2\cdot 4^k e^{-2^k/4} \sum_{i=2^k[M]}^{\infty} e^{-i/4}. \qquad (4.7)
\end{aligned}$$

On the right hand side above, it is clear that $\sup_{k\ge 1}\{4^k e^{-2^k/4}\} < \infty$. Also since $\int_0^\infty e^{-x/4}dx < \infty$, by integral comparison theorem we know that the infinite sum $\sum_{i=0}^\infty e^{-i/4}$ converges. There, the right hand side of (4.7) can be made arbitrarily small as long as M is sufficiently large. In addition, to bound the other term in the line above (4.6), noticing that there exists some constant $c(M) > 0$ decreasing in $M > 1$ such that

$$\frac{1}{t} e^{-\frac{M^2}{64t}} < c(M), \quad \text{for all } t \in (0,T], \quad \lim_{M\uparrow\infty} c(M) = 0, \qquad (4.8)$$

we have

$$\begin{aligned}
\sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(y,a_k^*)\ge M}} \frac{c_1}{t} e^{-\frac{d_k(a_k^*,y)^2}{32t}}\cdot 2^{-k} &\le \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(y,a_k^*)\ge M}} \frac{c_1\cdot 2^{-k}}{t} e^{-\frac{d_k(a_k^*,y)^2}{64t}}\cdot e^{-\frac{d_k(a_k^*,y)^2}{64t}} \\
&\overset{(4.8)}{\le} \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(y,a_k^*)\ge M}} c_1\cdot c(M)\cdot 2^{-k}\cdot e^{-\frac{d_k(a_k^*,y)^2}{64t}} \\
(i = n_{a_k^*}(y) = 2^k d_k(a_k^*,y)) \qquad &\le \sum_{i=2^k[M]}^{\infty} c_1\cdot c(M)\cdot 2^{-k} e^{-\frac{i^2 4^{-k}}{64t}} \\
&\le c_1\cdot c(M) \sum_{j=[M]}^{\infty} \sum_{i=2^k j}^{2^k(j+1)-1} 2^{-k} e^{-\frac{i^2 4^{-k}}{64t}} \\
&\le c_1\cdot c(M) \sum_{j=[M]}^{\infty} e^{-\frac{j^2}{64T}}. \qquad (4.9)
\end{aligned}$$

In view of (4.8), the right hand side of (4.9) can be made arbitrarily small with sufficiently large M. Combining the discussion above, we come to the conclusion that for the right hand side of (4.6), given any $\delta > 0$, there exists $c_3 > 0$ sufficiently large, such that for all $k \ge k_0$ and $M > c_3$, it holds that

$$\sum_{\substack{y \in 2^{-k}\mathbb{Z}_+ \\ d_k(y,a_k^*) \ge M \cdot 2^k}} \frac{c_1}{t} \left(e^{-\frac{d_k(a_k^*,y)^2}{32t}} + e^{-\frac{2^k d_k(a_k^*,y)}{2}} \right) \cdot 2^{-k} < \delta. \tag{4.10}$$

To handle the second term on the right hand side of (4.6), we denote by S_i^k the boundary of the square centered at the origin with four vertices $(i2^{-k}, i2^{-k}), (i2^{-k}, -i2^{-k})$, $(-i2^{-k}, -i2^{-k})$, $(-i2^{-k}, i2^{-k})$. S_i contains at most $8i$ points in E^k. In addition, by elementary geomemtry,

$$\{y \in E^k : \; d_k(y, a_k^*) \ge M\} \subset \bigcup_{i=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty} S_i, \quad \text{for } i \ge [\varepsilon 2^k] + 1. \tag{4.11}$$

Noticing that for any $y \in S_i$,

$$d_k(a_k^*, y) \ge \left(i - [\varepsilon 2^k]\right) \cdot 2^{-k} \ge i2^{-k} - [\varepsilon], \tag{4.12}$$

by selecting $M \ge 16\varepsilon$, we first have

$$\begin{aligned}
& \sum_{\substack{y \in D_\varepsilon^k \\ d_k(y,a_k^*) \ge M}} \frac{c_1}{t} e^{-\frac{2^k d_k(a_k^*,y)}{2}} \cdot 4^{-k} \\
\le \;& \sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty} \sum_{y \in S_j} \frac{c_1}{t} e^{-\frac{2^k d_k(a_k^*,y)}{2}} \cdot 4^{-k} \\
\overset{(4.12)}{\le} \;& \sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty} c_1 \cdot 8j \cdot \frac{4^{-k}}{8^{-k}} \exp\left(-\frac{2^k}{2}\left(j2^{-k} - [\varepsilon]\right) \right) \\
= \;& c_4 \cdot 2^k \sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty} j \exp\left(-\frac{2^k}{4}\left(j2^{-k} - [\varepsilon]\right) \right) \exp\left(-\frac{2^k}{4}\left(j2^{-k} - [\varepsilon]\right) \right) \\
\le \;& c_4 \cdot 2^k \exp\left[-\frac{2^k\left(\left[\frac{M+\varepsilon}{\sqrt{2}}\right] - [\varepsilon]\right)}{4} \right] \sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty} j \exp\left[-\frac{2^k}{4}\left(j2^{-k} - [\varepsilon]\right) \right] \\
= \;& c_4 \cdot 2^k \exp\left[-\frac{2^k\left(\left[\frac{M+\varepsilon}{\sqrt{2}}\right] - [\varepsilon]\right)}{4} \right] \sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty} j \exp\left(-\frac{j - 2^k[\varepsilon]}{4} \right) \\
\overset{(M \ge 16\varepsilon)}{\le} \;& c_4 \cdot 2^k e^{-\frac{2^k M}{16}} \sum_{j=2^k\left[\frac{M}{2}\right]}^{\infty} j e^{-\frac{j}{8}}.
\end{aligned} \tag{4.13}$$

We notice that $\sup_{k\ge 1} 2^k e^{-2^k/16} < \infty$, $\sum_{j=1}^{\infty} j e^{-j/8} < \infty$, Therefore, since the right hand side of (4.13) decreases to zero as $M \uparrow \infty$, it can be made arbitrarily small with sufficiently large M. To finish bounding the second term on the right hand side of (4.6), we again first note that for any $M \ge 16\varepsilon > 0$, there exists $c(M) \downarrow 0$ as $M \uparrow +\infty$ such that

$$\sup_{x \ge M, 0<t<T} \frac{x+1}{t} e^{-\frac{(x-\varepsilon)^2}{64t}} < c(M). \tag{4.14}$$

Hence we have

$$\begin{aligned}\sum_{\substack{y\in D^k_\varepsilon\\ d_k(y,a^*_k)\ge M}} \frac{c_1}{t}e^{-\frac{d_k(a^*_k,y)^2}{32t}}\cdot 4^{-k} &\le \sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty}\sum_{y\in S_j}\frac{c_1}{t}e^{-\frac{d_k(a^*_k,y)^2}{32t}}\cdot 4^{-k}\\ &\le c_1\sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty}\sum_{y\in S_j}\frac{4^{-k}}{t}\exp\left[\frac{\left(j2^{-k}-[\varepsilon]\right)^2}{32t}\right]\\ &\le c_1\sum_{j=2^k\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty}\frac{4^{-k}\cdot 8j}{t}\exp\left[\frac{\left(j2^{-k}-[\varepsilon]\right)^2}{32t}\right]\\ &\le c_1\sum_{u=\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty}\sum_{j=2^ku}^{2^k(u+1)-1}\frac{4^{-k}\cdot 8j}{t}\exp\left[\frac{\left(j2^{-k}-[\varepsilon]\right)^2}{32t}\right]\\ &\le 8c_1\sum_{u=\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty}\frac{4^{-k}\cdot 2^k(u+1)}{t}e^{-\frac{(u-\varepsilon)^2}{64t}}e^{-\frac{(u-\varepsilon)^2}{64t}}\\ &\overset{(4.14)}{\le} 8c_1\cdot c(M)\sum_{u=\left[\frac{M+\varepsilon}{\sqrt{2}}\right]}^{\infty}e^{-\frac{(u-\varepsilon)^2}{64T}}. \end{aligned}\tag{4.15}$$

Again, the right hand side of (4.15) can be made arbitrarily small with sufficiently large M. This combined with the earlier discussion shows that for any $\delta>0$, there exist $c_5>0$ sufficiently large, such that for all $k\ge k_0$ and $M>c_5$,

$$\sum_{\substack{y\in D^k_\varepsilon\\ d_k(y,a^*_k)\ge M}} \frac{c_1}{t}\left(e^{-\frac{d_k(a^*_k,y)^2}{64t}}+e^{-\frac{2^kd_k(a^*_k,y)}{4}}\right)\cdot 4^{-k}<\delta. \tag{4.16}$$

Replacing the two terms on the right hand side of (4.6) with the two upper bounds (4.10) and (4.16) respectively, we have shown that given any $\delta>0$, for all $k\ge k_0$ and all $M\ge\max\{c_3,c_5\}$,

$$\mathbb{P}^{a^*_k}\left[d_k(X^k_t,a^*_k)\ge M\right]\le 2\delta.$$

This completes the proof. □

The next proposition justifies the first tightness condition (T-i) in Proposition 4.1 for $\{X^k\}_{k\ge1}$.

Proposition 4.5. For any fixed $T>1$, $\delta>0$, there exist $k_1\in\mathbb{N}$ and $M_3>0$ such that for all $k\ge k_1$,

$$\mathbb{P}^{a^*_k}\left[\sup_{t\in[0,T]}\left|X^k_t\right|_\rho>M\right]<\delta.$$

Proof. We first note Proposition 4.4 implies that: Given $T>1$ fixed, for every $\delta>0$, there is $M_2>0$ such that

$$\mathbb{P}^{a^*_k}\left[d_k(X^k_T,a^*_k)\ge M_2\right]<\delta.$$

Since $\rho(x,a^*)\le d_k(x,a^*_k)$ for all $x\in E^k\subset E$, this further implies that

$$\mathbb{P}^{a^*_k}\left[\left|X^k_T\right|_\rho\ge M_2\right]<\delta. \tag{4.17}$$

In view of Lemma 4.2, it suffices that in the following we show that for every $T>1$ and every $\delta>0$, there exists an $M>0$ and $n_1\in\mathbb{N}$ such that for all $k\ge n_1$,

(i) $\mathbb{P}^{a_k^*}\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho \geq M\right] < \delta$, and $\mathbb{P}^{a_k^*}\left[T-8^{-k}\leq \tau_M \leq T,\, \left|X_T^k\right|_\rho \leq \frac{M}{2}\right] < \delta$;

(ii) $\mathbb{P}^{a_k^*}\left[8^{-k}\leq \tau_M \leq T-8^{-k},\, \left|X_T^k\right|_\rho \leq \frac{M}{2}\right] < \delta$,

where $\tau_M = \inf\{t>0:\ |X_t^k|_\rho \geq M\}$. The first statement of (i) is proved in Proposition 4.3. In order to claim the second statement of (i), for the M_1 specified in Proposition 4.3, when $M > 2M_1$ and $k \geq k_0$, on account of strong Markov property,

$$\begin{aligned}
& \mathbb{P}^{a_k^*}\left[T-8^{-k}\leq \tau_M \leq T,\, \left|X_T^k\right|_\rho \leq \frac{M}{2}\right] \\
\leq\ & \mathbb{E}^{a_k^*}\left[\mathbb{P}^{X^k_{\tau_M}}\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho \leq \frac{M}{2}\right], T-8^{-k}\leq \tau_M \leq M\right] \\
\leq\ & \sup_{|y|_\rho\geq M}\mathbb{P}^y\left[\sup_{t\in[0,8^{-k}]}|X_t^k|_\rho \leq \frac{M}{2}\right] \\
\leq\ & \sup_{|y|_\rho\geq M}\mathbb{P}^y\left[\sup_{t\in[0,8^{-k}]}\rho\left(X_t^k, X_0^k\right) \geq \frac{M}{2}\right] \\
\leq\ & \sup_{|y|_\rho\geq M}\mathbb{P}^y\left[\sup_{t\in[0,8^{-k}]}\rho\left(X_t^k, X_0^k\right) \geq M_1\right] < \delta.
\end{aligned} \tag{4.18}$$

This establishes (i). To verify (ii), again by strong Markov property, we have for any $M>0$ that

$$\begin{aligned}
& \mathbb{P}^{a_k^*}\left[8^{-k}\leq \tau_M \leq T-8^{-k},\, \left|X_T^k\right|_\rho \leq \frac{M}{2}\right] \\
=\ & \int_{8^{-k}}^{T-8^{-k}} \mathbb{E}^{a_k^*}\left[\mathbb{P}^{X_s^k}\left[\left|X_{T-s}^k\right| \leq \frac{M}{2}\right]; \tau_M \in ds\right] \\
\leq\ & \int_{8^{-k}}^{T-8^{-k}} \mathbb{E}^{a_k^*}\left[\sup_{t\in[8^{-k},T-8^{-k}]}\mathbb{P}^{X_s^k}\left[|X_t^k|_\rho \leq \frac{M}{2}\right]; \tau_M \in ds\right] \\
\leq\ & \sup_{\substack{|y|_\rho\geq M \\ t\in[8^{-k},T-8^{-k}]}} \mathbb{P}^y\left[|X_t^k|_\rho \leq \frac{M}{2}\right].
\end{aligned} \tag{4.19}$$

To bound the right hand side of (4.19), we utilize Proposition 3.4. Given any $k\geq k_0$, for any $y\in E^k$ such that $|y|_\rho \geq M$ and any $t\in[8^{-k}, T-8^{-k}]$,

$$\mathbb{P}^y\left[|X_t^k|_\rho \leq \frac{M}{2}\right] = \sum_{\substack{x\in 2^{-k}\mathbb{Z} \\ |x|_\rho\leq \frac{M}{2}}} p_k(t,y,x)m_k(dx) + \sum_{\substack{x\in D_\varepsilon^k \\ |x|_\rho\leq \frac{M}{2}}} p_k(t,y,x)m_k(dx) + p_k(t,y,a_k^*)m_k(a_k^*), \tag{4.20}$$

In the following we give upper bounds to each of the three terms on the right hand side of (4.20) respectively. We note that for $|y|_\rho \geq M$, $d_k(y,x)\geq \rho(y,x)\geq M/2$ when $|x|_\rho \leq M/2$. Also note that $\#\{x: x\in 2^{-k}\mathbb{Z}_+, |x|_\rho \leq M/2\} \leq M2^k$. Therefore, or the first term on the right hand side of (4.20), by Corollary 3.4 we have

$$\begin{aligned}
\sum_{\substack{x\in 2^{-k}\mathbb{Z}_+ \\ |x|_\rho\leq \frac{M}{2}}} p_k(t,y,x)m_k(dx) &\leq \sum_{\substack{x\in 2^{-k}\mathbb{Z} \\ |x|_\rho\leq \frac{M}{2}}} \frac{c_1}{t}\left(e^{-\frac{d_k(y,x)^2}{32t}} + e^{-\frac{2^k d_k(y,x)}{2}}\right)\cdot 2^{-k} \\
(d_k(y,x)\geq M/2) &\leq \sum_{\substack{x\in 2^{-k}\mathbb{Z}_+ \\ |x|_\rho\leq \frac{M}{2}}} \frac{c_1 2^{-k}}{t}\left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}}\right)
\end{aligned}$$

$$\begin{aligned} &\leq 2^k M \cdot \frac{c_1 2^{-k}}{t}\left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}}\right) \\ &= \frac{c_1 M}{t} e^{-\frac{M^2}{128t}} + \frac{c_1 M}{t} e^{-\frac{2^k M}{4}}. \end{aligned} \tag{4.21}$$

To estimate the two terms on the right hand side of (4.21), we note that given $M > 0$ and $T > 1$ fixed, there exists some $c(M) < \infty$ decreasing in M such that

$$\sup_{x \geq M, 0<t<T} \frac{1}{t} e^{-x^2/(256t)} < c(M). \tag{4.22}$$

Thus for $t \in [8^{-k}, T - 8^{-k}]$,

$$\frac{c_1 M}{t} e^{-\frac{M^2}{128t}} = \frac{c_1 M}{t} e^{-\frac{M^2}{256t}} \cdot e^{-\frac{M^2}{256t}} \overset{(4.22)}{\leq} c_1 \cdot M \cdot c(M) e^{-M^2/(256T)}, \tag{4.23}$$

which can be made arbitrarily small by choosing M sufficiently large. For the second term on the right hand side of (4.21), noticing that $t \in [8^{-k}, T - 8^{-k}]$, we have

$$\frac{c_1 M}{t} e^{-\frac{2^k M}{4}} \leq c_1 M 8^k e^{-\frac{2^k M}{4}} = \frac{(2^k M)^3}{M^2} e^{-\frac{2^k M}{4}}, \tag{4.24}$$

which again, regardless of the value of $k \geq k_0$, can be made arbitrarily small as long as M is chosen sufficiently large, regardless of the value of $k \geq k_0$, in view of the fact that $\lim_{x\to+\infty} x^3 e^{-x/4} = 0$. The discussion above shows that both terms on the right hand side of (4.21) can be made arbitrarily small with sufficiently large M, i.e., given any $\delta > 0$, there exists $c_2 > 0$ such that for all $k \geq k_0$ and all $M > c_2$,

$$\sum_{\substack{x \in 2^{-k}\mathbb{Z} \\ |x|_\rho \leq \frac{M}{2}}} p_k(t, y, x) m_k(dx) < \delta. \tag{4.25}$$

Now we take care of the second term on the right hand side of (4.20). We again denote by S_i the boundary of the square centered at the origin with four vertices $(i2^{-k}, i2^{-k})$, $(i2^{-k}, -i2^{-k})$, $(-i2^{-k}, -i2^{-k})$, $(-i2^{-k}, i2^{-k})$. Similarly, since $|y|_\rho > M$ and $|x|_\rho \leq M/2$, provided that $M \geq 4\varepsilon$, it holds

$$\begin{aligned} \sum_{\substack{x \in D^k_\varepsilon \\ |x|_\rho \leq \frac{M}{2}}} p_k(t,y,x) m_k(dx) &\leq \sum_{\substack{x \in D^k_\varepsilon \\ |x|_\rho \leq \frac{M}{2}}} \frac{c_1}{t}\left(e^{-\frac{d_k(y,x)^2}{32t}} + e^{-\frac{2^k d_k(y,x)}{2}}\right) \cdot 4^{-k} \\ &\leq \sum_{\substack{x \in D^k_\varepsilon \\ |x|_\rho \leq \frac{M}{2}}} \frac{c_1}{t}\left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}}\right) \cdot 4^{-k} \\ &\leq \sum_{j=1}^{2^k([M/2+\varepsilon]+1)} \sum_{x \in S_j} \frac{c_1 4^{-k}}{t}\left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}}\right) \\ (\#\{x : x \in S_j\} \leq 8j) \quad &\leq \sum_{j=1}^{2^k([M/2+\varepsilon]+1)} \frac{c_1 4^{-k} 8j}{t}\left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}}\right) \\ &\leq \sum_{u=0}^{([M/2+\varepsilon]+1)} \sum_{j=2^k u}^{2^k(u+1)-1} \frac{c_1 4^{-k} 8j}{t}\left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}}\right) \\ &\leq \sum_{u=0}^{([M/2+\varepsilon]+1)} \frac{c_1 2^{-k} 8(u+1)}{t}\left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}}\right) \end{aligned}$$

$$\leq \quad \frac{c_3 2^{-k} M^2}{t} \left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}} \right). \tag{4.26}$$

To bound the two terms on the right hand of (4.26) respectively, we first notice that there exists some constant $c_4 > 0$ such that $\sup_{x \geq 0} x e^{-x/128} \leq c_4$. Thus

$$\frac{c_3 2^{-k} M^2}{t} e^{-\frac{M^2}{128t}} \leq c_3 c_4 2^{-k}, \tag{4.27}$$

which can be made arbitrarily small with sufficiently large k. For the other term on the right hand side of (4.26), noticing that $t \geq 8^{-k}$,

$$\frac{c_3 2^{-k} M^2}{t} e^{-\frac{2^k M}{4}} \leq c_3 4^k M^2 e^{-\frac{2^k M}{4}} \leq c_3 \left(2^k M\right)^2 e^{-\frac{2^k M}{4}},$$

which again can be made arbitrarily small as long as M is sufficiently large, regardless of the value of $k \geq k_0$, because $\lim_{x \to +\infty} x^2 e^{-x/4} \to 0$. Combining the disccusion above regarding both terms on the right hand side of (4.26), we see that given any $\delta > 0$, there exists an integer $n_1 \geq k_0$ and $c_5 > 0$, such that for all $k \geq n_1$ and all $M > c_5$,

$$\sum_{\substack{x \in D_\varepsilon^k \\ |x|_\rho \leq \frac{M}{2}}} p_k(t, x, y) m_k(dx) < \delta. \tag{4.28}$$

Finally, for the third term on the right hand side of (4.20), since $|y|_\rho \geq M$,

$$p_k(t, y, a_k^*) m_k(a_k^*) \overset{(2.5)}{\leq} \frac{c_1}{t} \left(e^{-\frac{d_k(y, a_k^*)^2}{32t}} + e^{-\frac{2^k d_k(y, a_k^*)}{2}} \right) \cdot 2^{-k} \leq \frac{c_1 2^{-k}}{t} \left(e^{-\frac{M^2}{128t}} + e^{-\frac{2^k M}{4}} \right).$$

From here, using the same argument as that for (4.25), it can be shown that given any $\delta > 0$, there exists there exists $c_6 > 0$ such that for all $k \geq k_0$ and all $M > c_6$,

$$p_k(t, y, a_k^*) m_k(a_k^*) < \delta. \tag{4.29}$$

Combining (4.25), (4.28), and (4.29), in view of (4.20), we have showed that given any $\delta > 0$, for all $k \geq n_1$ and all $M > \max\{c_2, c_5, c_6\}$, it holds that

$$\sup_{\substack{|y|_\rho \geq M \\ t \in [8^{-k}, T - 8^{-k}]}} \mathbb{P}^y \left[|X_t^k|_\rho \leq \frac{M}{2} \right] < 3\delta. \tag{4.30}$$

In view of (4.19), we have verified the condition (ii) stated at the beginning of this proof. Now that both conditions (i) and (ii) have been verified, the proof is complete. □

4.3 Validation of (T-ii) in Proposition 4.1 for $\{X^k; k \geq 1\}$

Before we establish the second tightness condition (T-ii) in Proposition 4.1 for $\{X^k\}_{k \geq 1}$, we need the following lemma.

Lemma 4.6. For any $T > 0$, $\delta_1, \delta_2 > 0$ given, there exist $\delta_3 > 0$ and $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$,

$$\sup_{x \in E} \sup_{t \in [8^{-k}, \delta_3]} \left(\frac{1}{\delta_3} + 1 \right) \mathbb{P}^x \left[\rho\left(X_0^k, X_t^k\right) \geq \delta_1 \right] < \delta_2.$$

Proof. The idea of this proof is similar to that of Proposition 4.4. By the definition of d_k, for any $x \in E$ and any $t \in [8^{-k}, \delta_3]$,

$$\left(\frac{1}{\delta_3} + 1 \right) \mathbb{P}^x \left[\rho\left(X_0^k, X_t^k\right) \geq \delta_1 \right]$$

$$\begin{aligned}
&\leq \left(\frac{1}{\delta_3}+1\right)\mathbb{P}^x\left[d_k\left(X_0^k,X_t^k\right)\geq\delta_1\right] \\
&\leq \left(\frac{1}{\delta_3}+1\right)\sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(x,y)\geq\delta_1}}\frac{c_1}{t}\left(e^{-\frac{d_k(x,y)^2}{32t}}+e^{-\frac{2^kd_k(x,y)}{2}}\right)\cdot 2^{-k} \\
&+ \left(\frac{1}{\delta_3}+1\right)\sum_{\substack{y\in D_\varepsilon^k\\ d_k(x,y)\geq\delta_1}}\frac{c_1}{t}\left(e^{-\frac{d_k(x,y)^2}{32t}}+e^{-\frac{2^kd_k(x,y)}{2}}\right)\cdot 4^{-k} \\
&+ \mathbf{1}_{\{d_k(x,a_k^*)\geq\delta_1\}}(x)\cdot\left(\frac{1}{\delta_3}+1\right)\frac{c_1}{t}\left(e^{-\frac{d_k(x,a_k^*)^2}{32t}}+e^{-\frac{2^kd_k(x,a_k^*)}{2}}\right)m_k(a_k^*) \\
&= (I)+(II)+(III). \qquad (4.31)
\end{aligned}$$

Now we need to claim that given any $\delta_1,\delta_2>0$, there exist $\delta_3>0$ and $n_1\in\mathbb{N}$ such that for all $k\geq n_1$, all three terms (I)-(III) on the right hand side of (4.31) are smaller than δ_2. Towards this purpose, for the first term in (I), we let $n_k(x,y):=d_k(x,y)2^k$, i.e., the smallest number of edges between x and y in E^k. For k sufficiently large such that $\delta_1 2^{k/4}\geq 1$, noticing that there exists some $c_2>0$ such that

$$\sup_{k\geq 1}4^k\cdot\exp\left(-2^{\frac{3k}{16}}\right)\leq c_2, \qquad (4.32)$$

we have

$$\begin{aligned}
\sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(x,y)\geq\delta_1}}\frac{c_12^{-k}}{t}e^{-\frac{2^kd_k(x,y)}{2}} &= \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ n_k(x,y)\geq\delta_12^k}}\frac{c_12^{-k}}{t}e^{-\frac{n_k(x,y)}{2}} \\
(i=n_k(x,y)) &= \sum_{i=[\delta_12^k]+1}^{\infty}\frac{c_12^{-k}}{t}e^{-\frac{i}{2}} \\
(t\geq 8^{-k}) &\leq \sum_{i=[\delta_12^k]}^{\infty}c_14^ke^{-\frac{i}{4}}e^{-\frac{i}{4}} \\
(\delta_12^{k/4}>1) &\leq \sum_{i=[2^{(3k)/4}]}^{\infty}c_14^ke^{-\frac{i}{4}}e^{-\frac{i}{4}} \\
&\leq \sup_{k\geq1}\left(4^k\exp\left(-2^{\frac{3k}{16}}\right)\right)\sum_{i=[2^{(3k)/4}]}^{\infty}c_1e^{-\frac{i}{4}} \\
(4.32) &\leq c_2\sum_{i=[2^{(3k)/4}]}^{\infty}c_1e^{-\frac{i}{4}}. \qquad (4.33)
\end{aligned}$$

For the second term in (I) on the right hand side of (4.31), since there exists some $c_3>0$ only depending on δ_1 such that

$$\sup_{t>0}\frac{1}{t}e^{-\frac{\delta_1^2}{64t}}\leq c_3, \qquad (4.34)$$

we have

$$\begin{aligned}
\sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(x,y)\geq\delta_1}}\frac{c_1}{2}^{-k}te^{-\frac{d_k(x,y)^2}{32t}} &\leq \sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(x,y)\geq\delta_1}}\frac{c_12^{-k}}{t}e^{-\frac{d_k(x,y)^2}{64t}}e^{-\frac{d_k(x,y)^2}{64t}} \\
&\leq \left(\sup_{8^{-k}\leq t\leq\delta_3}\frac{c_1}{t}e^{-\frac{\delta_1^2}{64t}}\right)2^{-k}\sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ n_k(x,y)\geq\delta_12^k}}c_1e^{-\frac{n_k(x,y)^24^{-k}}{64t}}
\end{aligned}$$

$$\begin{aligned}
((4.34), i=n_k(x,y)) &\le c_3\cdot 2^{-k}\sum_{i=[\delta_1 2^k]}^{\infty} c_1 e^{-\frac{i^2 4^{-k}}{64t}} \\
&\le c_3\cdot 2^{-k}\sum_{u=[\delta_1]}^{\infty}\sum_{i=u2^k}^{(u+1)2^k-1} c_1 e^{-\frac{i^2 4^{-k}}{64t}} \\
&\le c_3\cdot 2^{-k}\sum_{u=[\delta_1]}^{\infty}\left(2^k\cdot c_1 e^{-\frac{u^2}{64t}}\right) \\
(t\le\delta_3) &\le c_3\sum_{u=[\delta_1]}^{\infty} e^{-\frac{u^2}{64\delta_3}}, \qquad (4.35)
\end{aligned}$$

Combining (4.33) and (4.35), we have shown that for any given $\delta_1>0$, for all k large enough such that $(2^{1/4})^k\delta_1>1$, there exists $c_4>0$ such that

$$\begin{aligned}
&\left(\frac{1}{\delta_3}+1\right)\sum_{\substack{y\in 2^{-k}\mathbb{Z}_+\\ d_k(x,y)\ge\delta_1}}\frac{c_1}{t}\left(e^{-\frac{d_k(x,y)^2}{32t}}+e^{-\frac{2^k d_k(x,y)}{2}}\right)\cdot 2^{-k} \\
\le\ & c_4\left(\frac{1}{\delta_3}+1\right)\left(\sum_{i=[2^{(3k)/4}]}^{\infty} e^{-\frac{i}{4}}+\sum_{u=[\delta_1]}^{\infty} e^{-\frac{u^2}{64\delta_3}}\right). \qquad (4.36)
\end{aligned}$$

Now we take care of (II) on the right hand side of (4.31). First of all, we claim that for any given $x\in E^k$ any $j\in\mathbb{N}$,

$$\#\left\{y\in D^k_\varepsilon:\ d_k(x,y)\le j\cdot 2^{-k}\right\}\le 256\left(j^2+\varepsilon^2 4^k\right). \qquad (4.37)$$

To see (4.37), it suffices to note that for any $y_1,y_2\in\left\{y\in D^k_\varepsilon,\ d_k(x,y)\le j2^{-k}\right\}$, if we denote by $n_k(y_1,y_2)$ the smallest number of edges between y_1 and y_2, then it must hold that $n_k(y_1,y_2)\le 2j$. Therefore $|y_1-y_2|\le 2j2^{-k}+2\varepsilon$. Thus the Lebesgue measure of the set $\left\{y\in D^k_\varepsilon:\ d_k(x,y)\le j2^{-k}\right\}$ is at most $\pi\left(2j2^{-k}+2\varepsilon\right)^2$. Since any two points in this set is at least 2^{-k} Euclidean distance apart, any two discs with Euclidean radius 2^{-k-2} centered at two distinct points in the set $\left\{y\in D^k_\varepsilon:\ d_k(x,y)\le j\right\}$ must be disjoint. Thus

$$\#\left\{y\in D^k_\varepsilon:\ d_k(x,y)\le j2^{-k}\right\}\le\frac{\pi(2j2^{-k}+2\varepsilon)^2}{\pi 2^{-2k-4}}\le 128\left(j^2+j\varepsilon 2^k+\varepsilon^2 4^k\right)\le 256\left(j^2+\varepsilon^2 4^k\right).$$

This verifies (4.37). Now for the first term in (II) on the right hand side of (4.31), it holds that

$$\begin{aligned}
\sum_{\substack{y\in D^k_\varepsilon\\ d_k(x,y)\ge\delta_1}}\frac{c_1}{t}e^{-\frac{d_k(x,y)^2}{64t}}\cdot 4^{-k} &\le \sum_{j=0}^{\infty}\sum_{\substack{y\in D^k_\varepsilon\\ \delta_1+j\le d_k(x,y)<\delta_1+j+1}}\frac{c_1 4^{-k}}{t}e^{-\frac{d_k(x,y)^2}{64t}} \\
(4.37) &\le \sum_{j=0}^{\infty}\frac{c_1 4^{-k}}{t}\left[\left((\delta_1+j+1)2^k\right)^2+4^k\right]e^{-\frac{(\delta_1+j)^2}{64t}} \\
&\le \sum_{j=0}^{\infty}\frac{c_1 4^{-k}}{t}\left((\delta_1+j+1)^2 4^k+4^k\right)e^{-\frac{(\delta_1+j)^2}{64t}} \\
&\le \sum_{j=0}^{\infty}\frac{2c_1}{t}\left(\delta_1+j+1\right)^2 e^{-\frac{(\delta_1+j)^2}{64t}} \\
&\le 4c_1\left(\delta_1+1\right)^2\sum_{j=0}^{\infty}\frac{1}{t}e^{-\frac{(\delta_1+j)^2}{64t}}+4c_1\sum_{j=0}^{\infty}\frac{j^2}{t}e^{-\frac{(\delta_1+j)^2}{64t}}
\end{aligned}$$

$$\leq 4c_1(\delta_1+1)^2\frac{1}{t}e^{-\frac{\delta_1^2}{64t}}+8c_1(\delta_1+1)^2\sum_{j=0}^{\infty}\frac{j^2}{t}e^{-\frac{(\delta_1+j)^2}{64t}}. \quad (4.38)$$

for any $\delta_1, \delta_2 > 0$ given, we first note that

$$\begin{aligned}\left(\sup_{0<t<\delta_3}\sum_{j=0}^{\infty}\frac{j^2}{t}e^{-\frac{(\delta_1+j)^2}{64t}}\right) &\leq \left(\sup_{0<t<\delta_3}\frac{j^2}{t}e^{-\frac{\delta_1^2}{64t}}\sum_{j=0}^{\infty}j^2e^{-\frac{j^2}{64t}}\right)\\ &\leq \sum_{j=0}^{\infty}j^2e^{-\frac{j^2}{64\delta_3}}\left(\sup_{t>0}\frac{1}{t}e^{-\frac{\delta_1^2}{64t}}\right)\\ (4.34)\quad &\leq c_3\sum_{j=0}^{\infty}j^2e^{-\frac{j^2}{64\delta_3}}. \quad (4.39)\end{aligned}$$

Replacing the last summation term on the right hand side of (4.38) with (4.39), we get

$$\sum_{\substack{y\in D_\varepsilon^k\\ d_k(x,y)\geq\delta_1}}\frac{c_1}{t}e^{-\frac{d_k(x,y)^2}{64t}}\cdot 4^{-k}\leq 4c_1(\delta_1+1)^2\frac{1}{t}e^{-\frac{\delta_1^2}{64t}}+8c_5(\delta_1+1)^2\sum_{j=0}^{\infty}j^2e^{-\frac{j^2}{64\delta_3}}.$$

For the second term in (II) on the right hand side of (4.31), we have

$$\begin{aligned}&\sum_{\substack{y\in D_\varepsilon^k\\ d_k(x,y)\geq\delta_1}}\frac{c_1}{t}e^{-\frac{2^kd_k(x,y)}{4}}\cdot 4^{-k}\\ &\leq \sum_{j=0}^{\infty}\sum_{\substack{y\in D_\varepsilon^k\\ \delta_1+j\leq d_k(x,y)<\delta_1+j+1}}\frac{c_14^{-k}}{t}e^{-\frac{2^kd_k(x,y)}{4}}\\ (4.37)\quad &\leq \sum_{j=0}^{\infty}\frac{c_14^{-k}}{t}\left[\left((\delta_1+j+1)2^k\right)^2+4^k\right]e^{-\frac{\delta_1\cdot 2^k+j\cdot 2^k}{4}}\\ (t\geq 8^{-k})\quad &\leq \sum_{j=0}^{\infty}4c_12^k\left(\delta_1^24^k+j^2\cdot 4^k+4^k\right)e^{-\frac{\delta_1\cdot 2^k+j\cdot 2^k}{4}}\\ &\leq \sum_{j=0}^{\infty}4c_1\left[(\delta_1^2+1)8^k+j^2\cdot 8^k\right]e^{-\frac{(\delta_1+j)2^k}{4}}. \quad (4.40)\end{aligned}$$

Finally, for (III) on the right hand side of (4.31), on account of Proposition 2.3, we have for $k\geq k_0$ and $t\in[8^{-k},\delta_3]$ that

$$\begin{aligned}&\mathbf{1}_{\{d_k(x,a_k^*)\geq\delta_1\}}(x)\cdot\left(\frac{1}{\delta_3}+1\right)\frac{c_1}{t}\left(e^{-\frac{d_k(x,a_k^*)^2}{32t}}+e^{-\frac{2^kd_k(x,a_k^*)}{2}}\right)m_k(a_k^*)\\ (\text{Proposition 2.3})\quad &\leq \left(\frac{1}{\delta_3}+1\right)\frac{c_1\cdot 2^{-k}}{t}\left(e^{-\frac{\delta_1^2}{32t}}+e^{-\frac{2^k\delta_1}{2}}\right)\\ (t\geq 8^{-k})\quad &\leq \left(\frac{1}{\delta_3}+1\right)\left(\frac{c_1}{t}e^{-\frac{\delta_1^2}{32t}}+c_14^ke^{-\frac{2^k\delta_1}{2}}\right). \quad (4.41)\end{aligned}$$

Now combining the discussion for (I)-(III) above, i.e., replacing the right hand side of (4.31) with the right hand side terms of (4.36), (4.38), (4.40), and (4.41), we have

$$\left(\frac{1}{\delta_3}+1\right)\mathbb{P}^x\left[\rho\left(X_0^k,X_t^k\right)\geq\delta_1\right]$$

$$
\begin{aligned}
&\leq \quad c_4\left(\frac{1}{\delta_3}+1\right)\left(\sum_{i=[2^{(3k)/4}]}^{\infty} e^{-\frac{i}{4}}+\sum_{u=[\delta_1]}^{\infty} e^{-\frac{u^2}{64\delta_3}}\right) \\
&+\quad \left(\frac{1}{\delta_3}+1\right)\left\{4c_1(\delta_1+1)^2\frac{1}{t}e^{-\frac{\delta_1^2}{64t}}+8c_5(\delta_1+1)^2\sum_{j=0}^{\infty} j^2 e^{-\frac{j^2}{64\delta_3}}\right. \\
&+\quad \left. 4c_1\sum_{j=0}^{\infty}\left(\delta_1^2+1)8^k+j^2\cdot 8^k\right)\cdot e^{-\frac{(\delta_1+j)2^k}{4}}\right\} \\
&+\quad \left(\frac{1}{\delta_3}+1\right)\left(\frac{c_1}{t}e^{-\frac{\delta_1^2}{32t}}+c_1 4^k e^{-\frac{2^k\delta_1}{2}}\right). \qquad (4.42)
\end{aligned}
$$

To bound the right hand side of (4.42), notice that for any pair of $\delta_1, \delta_2 > 0$ given, we may first select $\delta_3 > 0$ sufficiently small so that

$$
\left(\frac{1}{\delta_3}+1\right)\left(c_4\sum_{u=[\delta_1]}^{\infty} e^{-\frac{u^2}{64\delta_3}}\right) < \delta_2, \quad \left(\frac{1}{\delta_3}+1\right)\sup_{0<t<\delta_3}\frac{c_1}{t}e^{-\frac{\delta_1^2}{32t}} < \delta_2, \qquad (4.43)
$$

as well as

$$
\left(\frac{1}{\delta_3}+1\right)\left\{4c_1(\delta_1+1)^2\sup_{0<t<\delta_3}\left(\frac{1}{t}e^{-\frac{\delta_1^2}{64t}}\right)+\left(\frac{1}{\delta_3}+1\right)8c_5(\delta_1+1)^2\sum_{j=0}^{\infty} j^2 e^{-\frac{j^2}{64\delta_3}}\right\} < \delta_2. \qquad (4.44)
$$

Then with this $\delta_3 > 0$ fixed, we then choose $n_1 \in \mathbb{N}$ big enough so that for all $k \geq n_1$,

$$
c_4\left(\frac{1}{\delta_3}+1\right)\left(\sum_{i=[2^{(3k)/4}]}^{\infty} e^{-\frac{i}{4}}\right) < \delta_2, \quad \left(\frac{1}{\delta_3}+1\right)\left(c_1 4^k e^{-\frac{2^k\delta_1}{2}}\right) < \delta_2, \qquad (4.45)
$$

and

$$
\left(\frac{1}{\delta_3}+1\right)4c_1\sum_{j=0}^{\infty}\left(\left((\delta_1^2+1)8^k+j^2\cdot 8^k\right)e^{-\frac{(\delta_1+j)2^k}{4}}\right) < \delta_2. \qquad (4.46)
$$

Combining (4.42)-(4.46), it has been shown that for any pair $\delta_1, \delta_2 > 0$ given, there exists $\delta_3 > 0$ and $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$,

$$
\left(\frac{1}{\delta_3}+1\right)\mathbb{P}^x\left[\rho\left(X_0^k, X_t^k\right) \geq \delta_1\right] < 6\delta_2, \quad \text{for all } x \in E, t \in [8^{-k}, \delta_3],
$$

which completes the proof. □

The next proposition justifies the second tightness condition in Proposition 4.1 for $\{X^k\}_{k\geq 1}$.

Proposition 4.7. For any $T > 0$, $\delta_1, \delta_2 > 0$, there exist $\delta_3 > 0$ and $k_3 \in \mathbb{N}$ such that for all $k \geq k_3$,

$$
\mathbb{P}^{a_k^*}\left[w_\rho\left(X^k, \delta_3, T\right) > \delta_1\right] < \delta_2, \qquad (4.47)
$$

where

$$
w_\rho(x,\, \delta_3,\, T) := \inf_{\{t_i\}}\max_i \sup_{s,t\in[t_i,t_{i-1}]} \rho(x(s), x(t)),
$$

where $\{t_i\}$ ranges over all possible partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) \geq \delta_3$ and $n \geq 1$.

Proof. Based on Proposition 4.6, we further first claim that given any $T>0$, $\delta_1,\delta_2>0$, there exist $\delta_3>0$ and $n_1\in\mathbb{N}$ such that for all $k\ge n_1$,

$$\sup_{x\in E}\sup_{t\in[0,\delta_3]}\mathbb{P}^x\left[\rho\left(X_0^k,X_t^k\right)\ge\delta_1\right]<\delta_2. \tag{4.48}$$

In view of Proposition 4.6, it suffices to show that given any $T>0$, $\delta_1,\delta_2>0$, there exists $n_1\in\mathbb{N}$ such that for all $k\ge n_1$,

$$\sup_{x\in E}\sup_{t\in[0,8^{-k}]}\mathbb{P}^x\left[\rho\left(X_0^k,X_t^k\right)\ge\delta_1\right]<\delta_2. \tag{4.49}$$

In fact, for any $x\in E$, any $k\in\mathbb{N}$ and any $t\in[0,8^{-k}]$, by the same computation as that in the proof to Proposition 4.3 using Stirling's formula,

$$\begin{aligned}\mathbb{P}^x\left[\rho\left(X_0^k,X_t^k\right)>\delta_1\right] &\le \mathbb{P}^x\left[\sup_{s\in[0,8^{-k}]}\rho\left(X_0^k,X_t^k\right)>\delta_1\right]\\ &\le \sum_{j=\delta_1\cdot 2^k}^{\infty}\left(\frac{8^{-k}4^k e}{\delta_1\cdot 2^k}\right)^j\le\sum_{j=\delta_1\cdot 2^k}^{\infty}\left(\frac{e}{\delta_1 4^k}\right)^j, \end{aligned}\tag{4.50}$$

which proves (4.49). This combined with Proposition 4.6 shows (4.48). For any $\delta_1,\delta_3>0$, in view of the definition of w_ρ, by strong Markov property we have

$$\begin{aligned} &\mathbb{P}^{a_k^*}\left[w_\rho\left(X^k,\delta_3,T\right)>\delta_1\right]\\ \le\ &\mathbb{P}^{a_k^*}\left[\sup_{1\le i\le[T/\delta_3]}\ \sup_{s,t\in[(i-1)\delta_3,i\delta_3\wedge T]}\rho\left(X_s^k,X_t^k\right)>\delta_1\right]\\ \le\ &\mathbb{P}^{a_k^*}\left[\bigcup_{i=1}^{[T/\delta_3]}\left\{\sup_{s,t\in[(i-1)\delta_3,i\delta_3\wedge T]}\rho\left(X_s^k,X_t^k\right)>\delta_1\right\}\right]\\ \text{(strong Markov property)}\quad \le\ &\left(\left[\frac{T}{\delta_3}\right]+1\right)\sup_{x\in E}\mathbb{P}^x\left[\sup_{s,t\in[0,\delta_3]}\rho\left(X_s^k,X_t^k\right)>\delta_1\right]. \end{aligned}\tag{4.51}$$

In order to handle the last display in (4.51), we first denote by $\tau^k_{\delta_1/2}:=\{t>0,\ \rho(X_0^k,X_t^k)\ge\delta_1/2\}$. It then follows by strong Markov property that for any $x\in E$,

$$\begin{aligned} &\mathbb{P}^x\left[\sup_{s,t\in[0,\delta_3]}\rho\left(X_s^k,X_t^k\right)\ge\delta_1\right]\\ \le\ &\mathbb{P}^x\left[\sup_{s\in[0,\delta_3]}\rho\left(X_0^k,X_s^k\right)\ge\frac{\delta_1}{2}\right]\\ \le\ &\mathbb{P}^x\left[\rho\left(X_0^k,X_{\delta_3}^k\right)\ge\frac{\delta_1}{4}\right]+\mathbb{P}^x\left[\tau^k_{\delta_1/2}<\delta_3,\ \rho\left(X_0^k,X_{\delta_3}^k\right)\le\frac{\delta_1}{4}\right]\\ \le\ &\mathbb{P}^x\left[\rho\left(X_0^k,X_{\delta_3}^k\right)\ge\frac{\delta_1}{4}\right]+\int_0^{\delta_3}\mathbb{E}^x\left[\mathbb{P}^{X^k_{\tau_{\delta_1/2}}}\left[\rho\left(X_0^k,X_{\delta_3-s}^k\right)\ge\frac{\delta_1}{4}\right],\ \tau_{\delta_1/2}\in ds\right]\\ \le\ &2\sup_{\substack{y\in E\\ 0\le s\le\delta_3}}\mathbb{P}^y\left[\rho\left(X_0^k,X_s^k\right)\ge\frac{\delta_1}{4}\right]. \end{aligned}\tag{4.52}$$

Replacing the last term on the right hand side of (4.51) with (4.52), we get that for any $\delta_1,\delta_3>0$, and any $k\in\mathbb{N}$,

$$\mathbb{P}^{a_k^*}\left[w_\rho\left(X^k,\delta_3,T\right)>\delta_1\right]\le 2\left(\left[\frac{T}{\delta_3}\right]+1\right)\sup_{\substack{y\in E\\ 0\le s\le\delta_3}}\mathbb{P}^y\left[\rho\left(X_0^k,X_s^k\right)\ge\frac{\delta_1}{4}\right]. \tag{4.53}$$

Now we are ready to apply Proposition 4.6 to finish the proof. Indeed, by Proposition 4.6, for any $T>0$, given any $\delta_1,\delta_2>0$, there exist $\delta_3>0$ and $n_1\in\mathbb{N}$ such that for all $k\ge n_1$,

$$\left(1+\frac{1}{\delta_3}\right)\sup_{\substack{y\in E\\0\le s\le\delta_3}}\mathbb{P}^y\left[\rho\left(X_0^k,X_s^k\right)\ge\frac{\delta_1}{4}\right]<\frac{\delta_2}{4(T+1)}. \tag{4.54}$$

Thus (4.53) yields that

$$\begin{aligned}\mathbb{P}^{a_k^*}\left[w_\rho\left(X^k,\delta_3,T\right)>\delta_1\right] &\le 2\left(\left[\frac{T}{\delta_3}\right]+1\right)\sup_{\substack{y\in E\\0\le s\le\delta_3}}\mathbb{P}^y\left[\rho\left(X_0^k,X_s^k\right)\ge\frac{\delta_1}{4}\right]\\ &\le \frac{2\left(T+\delta_3\right)}{\delta_3}\sup_{\substack{y\in E\\0\le s\le\delta_3}}\mathbb{P}^y\left[\rho\left(X_0^k,X_s^k\right)\ge\frac{\delta_1}{4}\right]\\ \overset{(4.54)}{} &\le \frac{2\left(T+\delta_3\right)}{\delta_3}\cdot\frac{\delta_2}{4(T+1)}\cdot\frac{\delta_3}{\delta_3+1}<\delta_2. \end{aligned}\tag{4.55}$$

This completes the proof. □

Theorem 4.8. For every $T>0$, the laws of $\{X^k,\mathbb{P}^{a_k^*},k\ge1\}$ are C-tight in the Skorokhod space $\mathbf{D}([0,T],E,\rho)$ equipped with the Skorokhod topology.

Proof. This follows immediately from [10, Chapter VI, Proposition 3.26], in view of Proposition 4.5 and Proposition 4.7. □

5 Weak limit of random walks on spaces with varying dimension

We first establish the uniform convergence of the generators of X^k. The method is similar to that in [11]. For notation convenience, we define the following class of functions $\mathcal{G}$:

$$\begin{aligned}\mathcal{G}:=\{&f:\mathbb{R}^2\cup\mathbb{R}_+\to\mathbb{R},\ f|_{B_\varepsilon}=\text{const}=f|_{\mathbb{R}_+}(0),\ f|_{\mathbb{R}^2}\in C^3(\mathbb{R}^2),f|_{\mathbb{R}_+}\in C^3(\mathbb{R}_+),\\ &f\text{ is supported on a compact subset of }(E\backslash\{a^*\})\cup B_\varepsilon\}.\end{aligned}\tag{5.1}$$

Every $f\in\mathcal{G}$ can be uniquely identified as a function mapping E to $\mathbb{R}$. Thus for $f\in\mathcal{G}$, we define

$$\mathcal{L}_kf(x):=2^{2k}\sum_{\substack{y\in E^k,\\y\leftrightarrow x\text{ in }G^k}}\left(f(y)-f(x)\right)J_k(x,dy),\quad\text{for }x\in E^k. \tag{5.2}$$

We also set

$$S^k:=\{x\in D_\varepsilon^k\cap E^k:\ v_k(x)=4\}\cup2^{-k}\mathbb{Z}_+.$$

It is easy to see that $\{S^k\}_{k\ge1}$ is an increasing sequence of sets. Also, it is clear that if a vertex $x\in E^k\backslash S^k$, then it must be that either $x=a_k^*$ or $x\leftrightarrow a_k^*$ in G^k. By a similar argument as that for [11, Lemma 2.7], we have the following lemma.

Lemma 5.1. For every fixed $k_0\in\mathbb{N}$ and every $f\in\mathcal{G}$, $\mathcal{L}_kf$ converges uniformly to

$$\mathcal{L}f:=\frac12\Delta f|_{\mathbb{R}_+}+\frac14\Delta f|_{D_\varepsilon}\quad\text{on }S^{k_0}\text{ as }k\to\infty. \tag{5.3}$$

Also, there exists some constant $C_7>0$ independent of k such that for all $k\ge1$ and all $x\in E^k$,

$$\mathcal{L}_kf(x)\le C_7.$$

Proof. This can be proved by an argument very similar to [11, Lemma 2.7]. Thus it is omitted. □

We prepare the following lemma for the main theorem.

Lemma 5.2. Fix $0 < \varepsilon < 1/64$. Given any $0 < \delta < (1 \wedge T)/4$, there exists $k_\delta \in \mathbb{N}$ such that for all $k \geq k_\delta$,

$$\sup_{t \in [2^{-k}/\delta, T]} \mathbb{P}^{a_k^*}\left[X_t^k \notin S^k\right] \leq 4C_5\delta,$$

where the $C_5 > 0$ on the right hand side above is the same as in Corollary 3.4.

Proof. Given k_0 specified in (1.5), let $k_\delta \geq k_0$ be an integer large enough such that $2^{-k_\delta} < \delta^2$. Recall that for any $x \in E^k \backslash S^k$, it must hold that either $x = a_k^*$, or $x \in D_\varepsilon^k$ and $x \leftrightarrow a_k^*$ in G^k. Notice that for $t \geq 2^{-k}/\delta$ and $y \leftrightarrow a_k^*$, it holds

$$d(a_k^*, y) \leq 1 \leq 16 \cdot 2^k t, \tag{5.4}$$

it then follows that for the C_5 speicified in Corollary 3.4, for any $t \in [2^{-k}/\delta, T]$,

$$\begin{aligned} \mathbb{P}^{a_k^*}\left[X_t^k \notin S^k\right] &\leq \sum_{y \notin S^k} C_5 \left(\frac{1}{t} \vee \frac{1}{\sqrt{t}}\right) e^{-\frac{d_k(a_k^*, y)^2}{32t}} m_k(dy) \\ \text{(1.3)} \quad &\leq \left(\sum_{y \in D_\varepsilon^k, y \leftrightarrow a_k^*} \frac{C_5}{t} \cdot 2^{-2k}\right) + \frac{C_5}{t} \cdot m_k(a_k^*) \\ \text{(2.6)} \quad &\leq C_5 \delta \cdot 2^k \cdot 2^{-2k}\left(56\varepsilon \cdot 2^k + 28\right) + C_5 \cdot \delta \cdot 2^k \cdot 2^{-k} \\ (\varepsilon < 1/64) \quad &\leq 4C_5\delta. \end{aligned} \tag{5.5}$$

The desired conclusion readily follows. □

Theorem 5.3. $\{X^k, \mathbb{P}^{a_k^*}, k \geq 1\}$ converges weakly to the BMVD described in Theorem 1.2 starting from a^*.

Proof. This proof is adapted from that for [11, Proposition 2.9] with some minor changes. We spell out the details for readers' convenience. Since the laws of $\{X^k, \mathbb{P}^{a_k^*}\}_{k \geq 1}$ are C-tight in $\mathbf{D}([0,T], E, \rho)$, any sequence has a weakly convergent subsequence supported on the set of continuous paths. Denote by $\{X^{k_j}, \mathbb{P}^{a_{k_j}^*} : j \geq 1\}$ any such weakly convergent subsequence, whose weak limit must be continuous and starting from a^* almost surely (note Remark 1.5). Denote such a solution by $(Y, \mathbb{P}^{a^*})$. By Skorokhod representation theorem (see, e.g., [8, Chapter 3, Theorem 1.8]), we may assume that $\{X^{k_j}, \mathbb{P}^{a_{k_j}^*};\ j \geq 1\}$ as well as $(Y, \mathbb{P}^{a^*})$ are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so that $\{X^{k_j}, j \geq 1\}$ converges almost surely to $(Y, \mathbb{P}^{a^*})$ in the Skorokhod topology.

For every $t \in [0,T]$, we set $\mathcal{M}_t^{k_j} := \sigma(X_s^{k_j}, s \leq t)$ and $\mathcal{M}_t := \sigma(Y_s, s \leq t)$. It is obvious that $\mathcal{M}_t \subset \sigma\{\mathcal{M}_t^{k_j} : j \geq 1\}$. With the class of functions $\mathcal{G}$ defined in (5.1), in the following we first show that $(Y, \mathbb{P}^{a^*})$ is a solution to the $\mathbf{D}([0,T], E, \rho)$ martingale problem $(\mathcal{L}, \mathcal{G})$ with respect to the filtration $\{\mathcal{M}_t\}_{t \geq 0}$. That is, for every $f \in \mathcal{G}$, we need to show that

$$\left\{f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}f(Y_s)ds\right\}_{t \geq 0}$$

is a martingale with respect to $\{\mathcal{M}_t\}_{t \geq 0}$. By [9, Corollary 5.4.1], we know that for any $k \geq 1$ and any $f \in \mathcal{G}$,

$$\left\{f(X_t^k) - f(X_0^k) - \int_0^t \widetilde{\mathcal{L}}_k f(X_s^k)ds\right\}_{t \geq 0}$$

is a martingale with respect to $\{\mathcal{M}_t^k\}_{t\ge 0}$. Therefore, for any $0\le t_1<t_2\le T$ and any $A\in\mathcal{M}_{t_1}^{k_j}$, it holds for every $j\in\mathbb{N}$ that

$$\mathbb{E}^{a^*_{k_j}}\left[\left(f(X_{t_2}^{k_j})-f(X_{t_1}^{k_j})-\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})ds\right)\mathbf{1}_A\right]=0. \tag{5.6}$$

We first claim that for any $A\in\mathcal{M}_{t_1}$,

$$\lim_{j\to\infty}\mathbb{E}^{a^*_{k_j}}\left[\left(f(X_{t_2}^{k_j})-f(X_{t_1}^{k_j})\right)\mathbf{1}_A\right]=\mathbb{E}^{a^*}\left[\left(f(Y_{t_2})-f(Y_{t_1})\right)\mathbf{1}_A\right]. \tag{5.7}$$

Towards this, we note that it has been claimed at the beginning of this proof that one can assume $\{X^{k_j}, j\ge 1\}$ as well as Y are defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$, so that $\{X^{k_j}, j\ge 1\}$ converges almost surely to Y in the Skorokhod topology, and that Y is a continuous process. From the proof of [8, Chapter 3, Theorem 7.8]), we can tell that: Given a squence $\{\omega_n, n\ge 1\}$ convergent to ω in the Skorokhod topology and ω is continuous at some $t_0>0$, then $\lim_{n\to\infty}\omega_n(t_0)=\omega(t_0)$. This yields that outside of a zero probability subset of $(\Omega,\mathcal{F},\mathbb{P})$, $X_t^{k_j}\to Y_t$ as $j\to\infty$ for all $t\in[0,T]$. Thus (5.7) follows from dominated convergence theorem.

In order to show the convergence of the integral term in (5.6), for $k\ge 1$, we denote by

$$T_0^k=0,\quad\text{and } T_l^k=\inf\{t>T_{l-1}^k:\ X_{T_l}^k\neq X_{T_l-}^k\}\quad\text{for } l=1,2,\dots,$$

i.e., T_l^k is the l^{th} holding time of X^k. $\{T_l^k: l\ge 1\}$ are i.i.d. random variables, each exponentially distributed with mean 2^{-2k}. Similar to the proof for [11, Proposition 2.9], it holds

$$\begin{aligned}
&\left|\mathbb{E}^{a^*_{k_j}}\left[\left(\sum_{l:\ t_1<T_l^{k_j}\le t_2}\mathcal{L}_{k_j}f\left(X^{k_j}_{T_l^{k_j}}\right)\right)\mathbf{1}_A\right]-\mathbb{E}^m\left[\left(\int_{t_1}^{t_2}\mathcal{L}f(Y_s)ds\right)\mathbf{1}_A\right]\right|\\
\le\ &\left|\mathbb{E}^{a^*_{k_j}}\left[\left(\sum_{l:\ t_1<T_1^{k_j}\le t_2}\mathcal{L}_{k_j}f\left(X^{k_j}_{T_l^{k_j}}\right)\left(T_l^{k_j}-T_{l-1}^{k_j}\right)-\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})ds\right)\mathbf{1}_A\right]\right|\\
+\ &\left|\mathbb{E}^{a^*_{k_j}}\left[\left(\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})ds-\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})ds\right)\mathbf{1}_A\right]\right|\\
+\ &\left|\mathbb{E}^{a^*_{k_j}}\left[\left(\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})ds\right)\mathbf{1}_A\right]-\mathbb{E}^{a^*}\left[\left(\int_{t_1}^{t_2}\mathcal{L}f(Y_s)ds\right)\mathbf{1}_A\right]\right|\\
=\ &(I)+(II)+(III).
\end{aligned} \tag{5.8}$$

Again by the same reasoning as in [11, Proposition 2.9], both (I) and (III) on the right hand side of (5.8) converge to zero. To take care of (II), for any $\delta>0$, we let k_δ be chosen as in Lemma 5.2. Hence

$$\begin{aligned}
&\left|\mathbb{E}^{a^*_{k_j}}\left[\left(\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})ds-\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})ds\right)\mathbf{1}_A\right]\right|\\
\le\ &\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})ds-\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})ds\right]\right|\\
\le\ &\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\in S^{k_j}\right\}}ds-\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\in S^{k_j}\right\}})ds\right]\right|\\
+\ &\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds+\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds\right]\right|.
\end{aligned} \tag{5.9}$$

For the first term on the right hand side of (5.9), by Lemma 5.1, we have

$$\begin{aligned}&\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\in S^{k_j}\right\}}ds-\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\in S^{k_j}\right\}})ds\right]\right|\\ =\;&\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{t_1}^{t_2}\left(\mathcal{L}_{k_j}f(X_s^{k_j})-\mathcal{L}f(X_s^{k_j})\right)\mathbf{1}_{\left\{X_s^{k_j}\in S^{k_j}\right\}}ds\right]\right|\overset{j\to\infty}{\to}0. \qquad (5.10)\end{aligned}$$

For the second term on the right hand side of (5.9), given any $\delta\in(0,(1\wedge T)/4$, for $k_j\geq k_\delta$ where k_δ is specified in Lemma 5.2 satisfying $2^{-k_\delta}<\delta^2$,

$$\begin{aligned}&\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds+\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds\right]\right|\\ &\leq\left|\mathbb{E}^{a^*_{k_j}}\left[\int_0^T\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds+\int_0^T\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds\right]\right|\\ &=\left|\mathbb{E}^{a^*_{k_j}}\left[\int_0^{2^{-k}/\delta}\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds+\int_0^{2^{-k}/\delta}\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds\right]\right|\\ &+\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{2^{-k}/\delta}^T\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds+\int_{2^{-k}/\delta}^T\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds\right]\right|\\ \text{(Lemma 5.1)}&\leq(C_5+\|\mathcal{L}f\|_\infty)\cdot\frac{2^{-k}}{\delta}+T\cdot(C_5+\|\mathcal{L}f\|_\infty)\sup_{t\in[2^{-k}/\delta,T]}\mathbb{P}^{a^*_{k_j}}\left[X_t^{k_j}\notin S^{k_j}\right]\\ \text{(Lemma 5.2)}&\leq(C_5+\|\mathcal{L}f\|_\infty)\,\delta+T\cdot(C_5+\|\mathcal{L}f\|_\infty)\cdot4C_1\delta. \qquad (5.11)\end{aligned}$$

Since δ can be made arbitrarily small, we have proved that for the second term on the right hand side of (5.9), it also holds that

$$\lim_{j\to\infty}\left|\mathbb{E}^{a^*_{k_j}}\left[\int_{t_1}^{t_2}\mathcal{L}_{k_j}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds+\int_{t_1}^{t_2}\mathcal{L}f(X_s^{k_j})\mathbf{1}_{\left\{X_s^{k_j}\notin S^{k_j}\right\}}ds\right]\right|=0. \qquad (5.12)$$

Combining (5.10) and (5.12), we have showed that (II) on the right hand side of (5.8), thus the entire right hand side of (5.8) tends to zero as $j\to\infty$. This combined with (5.7) shows that $(Y,\mathbb{P}^{a^*})$ is indeed a solution to the $\mathbf{D}([0,T],E,\rho)$ martingale problem $(\mathcal{L},\mathcal{G})$ with respect to the filtration $\{\mathcal{M}_t\}_{t\geq0}$.

To finish the proof, it remains to show that the $\mathbf{D}([0,T],E,\rho)$ martingale problem $(\mathcal{L},\mathcal{G})$ has a unique solution. Towards this, we denote the infinitesimal generator of BMVD defined in Theorem 1.2 by $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ (see [7, Theorem 2.3] for its description). In view of the definition of $\mathcal{G}$, it is easy to verify that the bp-closure (whose definition can be found, e.g., in [2, Definition 3.4.3]) of the graph of $\mathcal{L}$ restricted on $\mathcal{D}(\mathcal{L})\cap\mathcal{G}$ is the same as the bp-closure of the graph of $\mathcal{L}$ restricted on $\mathcal{D}(\mathcal{L})\cap C_c(E)$. Therefore by [2, Proposition 3.4.19], the $\mathbf{D}([0,T],E,\rho)$ martingale problem $(\mathcal{L},\mathcal{D}(\mathcal{L})\cap\mathcal{G})$ has the same set of solution(s) as the $\mathbf{D}([0,T],E,\rho)$ martingale problem $(\mathcal{L},\mathcal{D}(\mathcal{L})\cap C_c(E))$.

Finally, given any $f\in C_0(E)\cap\mathcal{D}(\mathcal{L})$, there exists $\{f_n\}_{n\geq1}\subset C_c(E)\cap\mathcal{D}(\mathcal{L})$ such that $f_n\to f$ and $\mathcal{L}f_n\to\mathcal{L}f$ both in L^2-norm. This means that the closure of $\mathcal{L}$ restricted on $\mathcal{D}(\mathcal{L})\cap C_c(E)$ coincides with the closure of $\mathcal{L}$ restricted on $\mathcal{D}(\mathcal{L})\cap C_0(E)$, which corresponds to the BMVD defined in Theorem 1.2 which is a Feller process with strong Feller property. By [12, Theorem 3.1, Remark 3.3], the $\mathbf{D}([0,T],E,\rho)$ martingale problem $(\mathcal{L},\mathcal{D}(\mathcal{L})\cap C_c(E))$ has a unique solution, which has to be the BMVD defined in Theorem 1.2. In view of the last sentence of the last paragraph, BMVD defined in Theorem 1.2 also has to be the $\mathbf{D}([0,T],E,\rho)$ martingale problem $(\mathcal{L},\mathcal{D}(\mathcal{L})\cap\mathcal{G})$. Since earlier in thie proof we have claimed that $(Y, \mathbb{P}^{a^*})$ is a solution to the $\mathbf{D}([0,T], E, \rho)$ martingale problem $(\mathcal{L}, \mathcal{G})$ with respect to the filtration $\{\mathcal{M}_t\}_{t\geq 0}$, $(Y, \mathbb{P}^{a^*})$ coincides with the BMVD defined in Theorem 1.2 starting from a^*. Since Y is the sequencial limit of any weakly convergent subsequence of $\{X^k\}_{k\geq 1}$, the proof is complete. □

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