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Existence and properties of connections decay rate for high temperature percolation models*

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Abstract

We consider generic finite range percolation models on $\mathbb{Z}^d$ under a high temperature/low density assumption (exponential decay of connection probabilities and exponential ratio weak mixing). We prove that the rate of decay of point-to-point connections exists in every directions and show that it naturally extends to a norm on $\mathbb{R}^d$. This result is the base input to obtain fine understanding of the high temperature phase (e.g. Ornstein-Zernike asymptotics for point-to-point connexions) and is usually proven using correlation inequalities (such as FKG). The present work makes no use of such model specific properties and is therefore a step towards the universality of Ornstein-Zernike asymptotics.

Keywords: percolation; asymptotics; high-temperature; decay rate; correlation length; convexity; mixing.
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1 Introduction and results

1.1 Decay rate of connections

Let P denote an edge-percolation measure on $\mathbb{Z}^d$. The central object of our investigation is(are) the rate(s) of exponential decay for point-to-point connection probabilities (two point functions):

Definition 1.1 (Inverse correlation length)**.** *Let $s \in \mathbb{S}^{d-1}$. The point-to-point decay rates are*

$$\begin{aligned}\overline{\nu}(s) &= \limsup_{n\to\infty} -\frac{1}{n}\log P(0 \leftrightarrow ns),\\ \underline{\nu}(s) &= \liminf_{n\to\infty} -\frac{1}{n}\log P(0 \leftrightarrow ns).\end{aligned} \tag{1.1}$$

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1.2 Motivation

The main motivation of this work comes from the (supposed) *universal behaviour* of two point functions in high temperature systems: they should decay exponentially with a well-defined rate and the pre-factor to this decay should be the one predicted by the Ornstein-Zernike theory [17, 21]. See [20] for a review on this topic. On the one hand some fairly satisfactory universal statements are available in perturbative regimes (*very high temperature* regime), see [2]. In the other hand, a non-perturbative approach (giving statements about the whole high temperature regime) has been developed over the past decades, proving the expected behaviour in various *specific* models: [1, 9, 14, 6, 7, 8, 18]. A key ingredient in the proofs being the validity of Theorem 1.2, which is usually proven using correlation inequalities.

Let us give more details on this universality conjecture in the case of percolation models. Denote $\mathcal{D}$ the set of translation-invariant percolation measures in the high temperature/low density regime, which definition we now discuss. Comparing the common properties of models (spin models and percolation models) at high temperature or low density, two main ones are recurrent: exponential decay of correlations (connectivities) and exponential mixing (a strong form of the two is also used in [10] as the definition of the *completely analytic regime* for finite range lattice spin models). We propose here to characterize $\mathcal{D}$ as the set of finite range percolation measures satisfying: exponential decay of connectivities (in infinite volume) and ratio weak mixing (Definition 2.1). One should also add some condition excluding pathologies occurring in systems with hard core constraints, but we will ignore this issue for the present discussion. Then, the following universal asymptotic should hold for any $P \in \mathcal{D}$:

$$P(0 \leftrightarrow x) = \frac{\psi(x/\|x\|)}{\|x\|^{(d-1)/2}} e^{-\nu(x)}(1 + o_{\|x\|}(1)),$$

where ν is a norm on $\mathbb{R}^d$ and $\psi : \mathbb{S}^{d-1} \to \mathbb{R}_+$ is some smooth function (both depending on P). The universal quantity is the order of the pre-factor: $\|x\|^{-(d-1)/2}$.

The latest non-perturbative approaches (mainly [8] combined with refinements from [19] and [18]) seem to be robust enough to tackle the problem (with some work...) for *any percolation models* in $\mathcal{D}$, provided one already obtained the estimate

$$P(0 \leftrightarrow x) = e^{-\nu(x)(1+o_{\|x\|}(1))},$$

for ν a norm on $\mathbb{R}^d$. This asymptotic is the goal of the present work. Notice that the main motivation is to obtain a statement about universality, not to obtain the result for a given model. Nevertheless, one can also apply this generic result to obtain information on specific models of interest (see Subsection 1.5 for an application which should also illustrate the difference between working in perturbative regimes and working under a high temperature assumption).

1.3 Results

Our main result is (see Section 2 for missing definitions):

Theorem 1.2. *Let $E \subset \{\{i,j\} \subset \mathbb{Z}^d\}$ be finite range, irreducible, invariant under translations. Let P be a percolation measure on E. Suppose that*

- *P is invariant under translations,*
- *P has the insertion tolerance property (Definition 2.3) with constant $\theta > 0$,*
- *P satisfies the exponential ratio weak mixing property (Definition 2.1) with rate $c_{\mathrm{mix}} > 0$ and constant $C_{\mathrm{mix}} < \infty$ for the set of local connection events,*
- *there exists $c_{\mathrm{co}} > 0$ such that $P(0 \leftrightarrow \Lambda_n^c) \leq e^{-c_{\mathrm{co}} n}$ for any n large enough.*

Then, for any $s \in \mathbb{S}^{d-1}$,

$$\overline{\nu}(s) = \underline{\nu}(s) \equiv \nu(s). \tag{1.2}$$

Moreover, the extension of ν by positive homogeneity of order one defines a norm on $\mathbb{R}^d$.

Remark 1.3. The ratio weak mixing condition demanded can look less natural and more stringent than the weak mixing property (not ratio). However, it has been shown, see [4, Theorem 3.3], that in many cases the two are equivalent. In particular, if $P(\omega) \propto \prod_{C\in \mathrm{cl}(\omega)} f(C)$ (formally, the R.H.S. being infinite, cl denotes the set of connected components), the assumption $P(0 \leftrightarrow \Lambda_N^c \,|\, \mathcal{F}_{E\setminus E(\Lambda_N)}) \leq e^{-cN}$ implies that the model has exponentially bounded controlling regions in the sense of [4].

Remark 1.4. The insertion tolerance property excludes degeneracies occurring in hard-core models. Moreover, it gives lower bounds on local connections implying for example that the decay rates of Definition 1.1 are in $(\epsilon, \epsilon^{-1})$ for some $\epsilon > 0$ (non-degenerate).

Remark 1.5. ν obviously inherit additional symmetries of P.

1.4 Extensions

A first extension is the relaxation of the mixing condition: modulo straightforward changes in the proofs, one can replace the exponential mixing by any power law mixing with power $> d$. But this type of mixing can generally be enhanced to exponential (see for example the discussion on mixing in [15]).

A second is to consider directed percolation models. In this case, one obtains the same result with the change that ν extends to an asymmetric norm instead of a norm.

A third straightforward extension is to replace the percolation model P by some translation invariant weight function on connected sub-graphs of $(\mathbb{Z}^d, E)$ (such as the ones considered in [5]).

1.5 Example of application

A simple example where Theorem 1.2 provides a new result is the Random Cluster model on $\mathbb{Z}^d$ with nearest-neighbour edges for $0 < q < 1$. Indeed, it is well know that the model has finite energy and does not satisfy the FKG inequality [13] (negative correlations are even expected). Define

$$\mathcal{D}_{\mathrm{co}} = \{p \in [0,1] :\ \exists c > 0, C \geq 0, \sup_{\eta} \Phi^{\eta}_{\Lambda_{2N};p,q}(0 \leftrightarrow \Lambda_N^c) \leq Ce^{-cN}\}$$
$$\mathcal{D}_{\mathrm{mix}} = \{p \in [0,1] :\ \Phi_{p,q} \text{ is ratio weak mixing}\}$$

where $\Phi^{\eta}_{\Lambda_N;p,q}$ is the Random Cluster measure in $\Lambda_N = \{-N, \cdots, N\}$ with boundary conditions η (see [13, 11] for definition), and $\Phi_{p,q}$ is any infinite volume measure (if $p \in \mathcal{D}_{\mathrm{mix}}$, there is only one such measure). Let $\mathcal{D} = \mathcal{D}_{\mathrm{co}} \cap \mathcal{D}_{\mathrm{mix}}$. It is the natural high temperature/low density regime of the model and the regime in which one expects Ornstein-Zernike picture of connectivities to hold (see [8]). The condition $p \in \mathcal{D}$ is non-perturbative: it does not assume $p < p_0$ for p_0 given by an argument *à la* Peierls (moreover, one could relax the condition of $\mathcal{D}_{\mathrm{mix}}$ to weak mixing (not ratio) without changing $\mathcal{D}$ by the results of [4]). Theorem 1.2 holds for any $p \in \mathcal{D}$ (even if $\mathcal{D}$ turned out not to be an interval) while one can only obtain the claim for p close to 0 by perturbative arguments (at least, to the best of the author's knowledge).

This example should also clarifies the interest of working under non-perturbative conditions instead of considering a perturbative regime of the parameters.

We quickly discuss how to obtain a "lower bound" on $\mathcal{D}$ (which is better than the one obtained via a naive use of cluster expansion). One has (by a straightforward computation, see [13, section 5.8]) that the model is stochastically dominated by a Bernoulli percolation

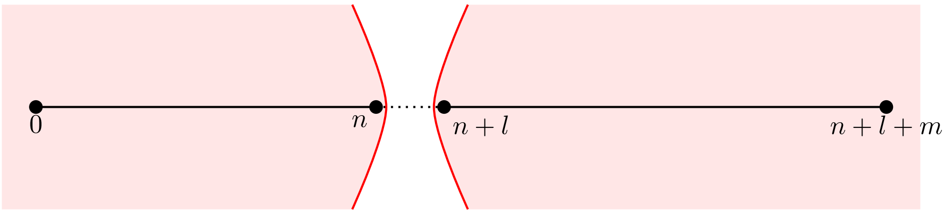


Figure 1: Directed constraint induced by controlling regions.

of parameter $p_0(q) = \frac{p}{p+(1-p)q}$ *independently of the volume considered and of boundary conditions*. When $p_0(q) < p_c(\mathbb{Z}^d)$ (the percolation threshold for Bernoulli percolation), one obtains $\inf_{s\in\mathbb{S}^{d-1}} \underline{\nu}(s) > 0$ via stochastic domination and sharpness of the phase transition (see [16, 3] for the first proofs of the latter and [12] for a more recent one). A simple coupling argument allows then to prove weak mixing: one can monotonically couple the finite volume measures with different boundary conditions to the Bernoulli measure, explore the outermost closed surface in the latter and resample the Random Cluster measures inside the surface (as the surface will be closed in all the Random Cluster measures, they can therefore be taken to be equal inside it). Weak mixing can then be enhanced to ratio weak mixing via [4, Theorem 3.3]. So, $\{p: \ p_0(q) < p_c(\mathbb{Z}^d)\} \subset \mathcal{D}$. We take the occasion to stress that stochastic domination alone *does not* yield existence of ν (nor that it extends to a norm), it only provides a lower bound (which is the basic hypotheses of the present work).

1.6 Strategy of the proof

To give an idea of what one would like to do, let us consider some translation invariant percolation model P. When P satisfies the FKG inequality, one has $P(0 \leftrightarrow x + y) \geq P(0 \leftrightarrow x \leftrightarrow x+y) \geq P(0 \leftrightarrow x)P(0 \leftrightarrow y)$. The equality $\overline{\nu} = \underline{\nu} \equiv \nu$ is then easy consequence of Fekete's Lemma. One can further extend ν by positive homogeneity. The above inequality directly implies that ν satisfies the triangle inequality.

One is therefore tempted to use the following strategy: for n, l, m with l small compare to n, m start with

$$P(0 \leftrightarrow (n+l+m)\mathrm{e}) \geq \theta^{-l} P\big(0 \leftrightarrow n\mathrm{e}, (n+l)\mathrm{e} \leftrightarrow (n+l+m)\mathrm{e}\big)$$

(e denotes a unit vector along a coordinate axis). Then, restrict the two connexions to some large sets and use mixing to factor the probability, then use some variation of Fekete's Lemma. Let us see how this does not work directly. The main problem is that there is *a priori* no reason that the connexions restricted to some subspace have the same decay rate as the free connexions. See Figure 1 for an illustration. Roughly, the problem lies in showing that connexions restricted to half spaces and full space connexions behave the same at the exponential scale... without using Theorem 1.2! The proof is mainly about addressing this issue. Another problem arises when one wants to obtain that ν satisfies the triangular inequality: there, one needs to be able to say that connexions at a macroscopic scale occur arbitrarily close to a straight line without error in the exponential rate.

Let us now summarize the proof. One can indeed recover sub-additivity from mixing *if typical clusters realizing connections are somehow directed* (e.g.: occurring in a cone). We thus introduce various notions of "directed connections" for which we prove existence of an asymptotic decay rate. We then show that all these rates are equal and define a norm $\tilde{\nu}$. To relate the obtained "directed rate" to the "real rates", we do a small detour: we introduce point to hyperplanes decay rates and their directed version. Showing that these two agree is much easier than for point-to-point connections and is done using a suitable coarse-graining argument. We then relate directed point-to-point to directed point-to-hyperplane via convex duality (approximately: directed point-to-hyperplane connections in a direction s are realized by a directed point-to-point connection in an optimal direction s'). Finally, we relate (non directed) point-to-point connections to (non directed) point-to-hyperplane connections via another coarse-graining argument.

2 Definitions and notations

2.1 General notations

Denote $\| \ \|$ the Euclidean norm on $\mathbb{R}^d$ and d the associated distance. Write $\mathbb{S}^{d-1}$ the unit sphere for $\| \ \|$. $\langle \ , \ \rangle$ will denote the scalar product. s will always be an element of $\mathbb{S}^{d-1}$. For a (possibly asymmetric) norm $\mu : \mathbb{R}^d \to \mathbb{R}_+$, define the unite ball of μ and its polar set ("Wulff shape")

$$\mathcal{U}_\mu = \{x \in \mathbb{R}^d : \ \mu(x) \leq 1\}, \quad \mathcal{W}_\mu = \bigcap_{s\in\mathbb{S}^{d-1}} \{x \in \mathbb{R}^d : \ \langle x, s\rangle \leq \mu(s)\}.$$

For $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, write $A + x$ the translate of A by x, ∂A the boundary of A and $\mathring{A} = A \setminus \partial A$ the interior of A.

Denote

$$\Lambda_N = [-N, N]^d, \quad \Lambda_N(x) = x + \Lambda_N.$$

We also denote Λ_N the intersection of Λ_N with $\mathbb{Z}^d$.

Define the half spaces: for $s \in \mathbb{S}^{d-1}$,

$$H_s = \{x \in \mathbb{R}^d : \ \langle x, s\rangle \geq 0\}, \quad H_s(x) = x + H_s. \tag{2.1}$$

Then, for $\delta \in [0, 1]$, define the cones

$$\mathcal{Y}_{s,\delta} = \{x \in \mathbb{R}^d : \ \langle x, s\rangle \geq (1 - \delta)\|x\|\}, \quad \mathcal{Y}_{s,\delta}(x) = x + \mathcal{Y}_{s,\delta}. \tag{2.2}$$

$\delta = 0$ is a line and $\delta = 1$ is the half space H_s.

Also introduce the truncated cones

$$\mathcal{Y}^K_{s,\delta} = \mathcal{Y}_{s,\delta} \setminus H_s(Ks), \quad \mathcal{Y}^K_{s,\delta}(x) = x + \mathcal{Y}^K_{s,\delta}. \tag{2.3}$$

For $x \in \mathbb{R}^d$, we denote $\mathrm{int}(x)$ the point in $\mathbb{Z}^d$ closest to x, with some fixed breaking of draws respecting symmetries/translations of $\mathbb{Z}^d$. We will often omit int from the notation.

We fix a priori some arbitrary total order on $\mathbb{Z}^d$.

We will regularly use the following notation: for $(a_n)_{n\geq1} \in \mathbb{R}^{\mathbb{N}}$ a sequence, we denote $\overline{a} = \limsup_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$, $\underline{a} = \liminf_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$. When $\overline{a} = \underline{a}$, we write the limit a.

A quantity $f(n)$ is $o_n(1)$ if $\lim_{n\to\infty} f(n) = 0$.

2.2 Percolation

We consider edge percolation models, in all this work E will be a subset of $\big\{\{i, j\} \subset \mathbb{Z}^d\big\}$ with the properties:

- *Irreducibility*: $(\mathbb{Z}^d, E)$ is connected.
- *Finite Range*: there exists $r > 0$ such that $\|i - j\| \geq r \implies \{i, j\} \notin E$. The smallest such r is called the *range* of E and is denoted $R \equiv R(E)$.
- *Translation Invariance*: for any $e \in E$ and $x \in \mathbb{Z}^d$, $x + e \in E$.

The graph distance on $(\mathbb{Z}^d, E)$ is denoted d_E. As E is finite range and irreducible, there exists $c_E > 0$ such that

$$c_E^{-1}\mathrm{d}(x,y) \leq \mathrm{d}_E(x,y) \leq c_E \mathrm{d}(x,y).$$

For a set $A \subset \mathbb{Z}^d$, denote $A^c = \mathbb{Z}^d \setminus A$, $\partial^{\mathrm{int}} A = \{x \in A : \ \exists y \in A^c, \ \{x,y\} \in E\}$, $\partial^{\mathrm{ext}} A = \{x \in A^c : \ \exists y \in A, \ \{x,y\} \in E\}$. Also define $E(A) = \{\{x,y\} \in E : \ x,y \in A\}$.

For $\omega \in \{0,1\}^E$, we systematically identify the $\{0,1\}$-valued function and the edge set induced by the set $\{e : \omega_e = 1\}$, the set of open edges. When talking about connectivity properties of ω, it is assumed that the graph $(\mathbb{Z}^d, \omega)$ is considered.

For $F \subset E$ finite, denote $\mathcal{F}_F = \{A \subset \{0,1\}^F\}$ and for $F \subset E$ infinite, denote $\mathcal{F}_F$ the sigma algebra generated by the collection $(\mathcal{F}_{F'})_{F' \subset F \text{ finite}}$. A percolation measure P is a probability measure such that $(P, \mathcal{F}_E, E)$ is a probability space. We write $\{x \leftrightarrow y\}$ for the event that x, y lie in the same connected component (and $\{A \leftrightarrow B\}$ for the event that there exists $x \in A, y \in B$ with $x \leftrightarrow y$). We also will write $\{x \overset{F}{\leftrightarrow} y\}$ for the event that x is connected to y by a path of open edges in F. ω will be a random variable with law P.

2.3 Hypotheses

One of our hypotheses is a mixing condition, called the *exponential ratio weak mixing property* for connections events:

Definition 2.1 (Ratio mixing)**.** *We say that P has the* ratio weak mixing property *with rate $c > 0$ and constant $C \geq 0$ if for any sets $F, F' \subset E$ and events $A \in \mathcal{F}_F, B \in \mathcal{F}_{F'}$ with $P(A)P(B) > 0$,*

$$\left|1 - \frac{P(A \cap B)}{P(A)P(B)}\right| \leq C \sum_{e \in F, e' \in F'} e^{-c\mathrm{d}(e,e')}, \tag{2.4}$$

where d is the Euclidean distance. We say that the property is satisfied for the class $\mathcal{C} \subset \mathcal{F}_E$ if (2.4) *holds whenever, in addition to the hypotheses, $A, B \in \mathcal{C}$.*

Definition 2.2 (Connexion events)**.** *The class of* local connection events *is the set of events of the form*

$$\{A \overset{\Delta}{\leftrightarrow} B\},$$

where $A, B \subset \mathbb{Z}^d$, $\Delta \subset E$ are finite.

Definition 2.3 (Insertion tolerance)**.** *A percolation measure P on E is said to have the* insertion tolerance *property if for any edge $e \in E$ there exists $\theta_e > 0$ such that*

$$P(\omega_e = 1 \,|\, \mathcal{F}_{E \setminus \{e\}}) \geq \theta_e.$$

If P is finite range and translation invariant, it is equivalent to the existence of $\theta > 0$ such that

$$\min_{e \in E} P(\omega_e = 1 \,|\, \mathcal{F}_{E \setminus \{e\}}) \geq \theta.$$

A useful consequence of insertion tolerance is

Lemma 2.4. *Suppose P is a finite range, translation invariant percolation measure on E. Then, for any $x, y \in \mathbb{Z}^d$, and any sets $A, B \subset \mathbb{Z}^d$,*

$$P(x \leftrightarrow A, y \leftrightarrow B, x \leftrightarrow y) \geq \theta^{\mathrm{d}_E(x,y)} P(x \leftrightarrow A, y \leftrightarrow B),$$

where d_E is the graph distance on $(\mathbb{Z}^d, E)$ and $\theta > 0$ is given by Definition 2.3.

Proof. Let γ be a path (seen as set of edges) from x to y realizing $\mathrm{d}_E(x,y)$ (in particular, $|\gamma| = \mathrm{d}_E(x,y)$). Now,

$$\begin{aligned} P(x \leftrightarrow A, y \leftrightarrow B, x \leftrightarrow y) &\geq P\big(P(x \leftrightarrow A, y \leftrightarrow B, \gamma \subset \omega \,|\, \mathcal{F}_{E\setminus\gamma})\big) \\ &= P\big(\mathbb{1}_{\omega\cup\gamma\in\{x\leftrightarrow A\}}\mathbb{1}_{\omega\cup\gamma\in\{y\leftrightarrow B\}}P(\gamma\subset\omega\,|\,\mathcal{F}_{E\setminus\gamma})\big) \\ &\geq P\big(\mathbb{1}_{\omega\in\{x\leftrightarrow A\}}\mathbb{1}_{\omega\in\{y\leftrightarrow B\}}\big)\theta^{|\gamma|}. \end{aligned}$$

□

We will regularly use this kind of argument without explicitly writing down the details.

3 Coarser lattice, restricted connections, preliminary results

In all this Section, we work under the hypotheses of Theorem 1.2.

3.1 Coarse connections

To avoid dealing with trivialities occurring from the discrete structure of $\mathbb{Z}^d$, we will look at a coarser notion of connections. As the model is finite range and $(\mathbb{Z}^d, E)$ is connected and translation invariant, we can find $R_0 < \infty$ such that for any $\Delta \subset \mathbb{R}^d$ connected, any two sites in $\Delta \cap \mathbb{Z}^d$ are connected by a path using only edges between sites in $[\Delta] = \bigcup_{x\in\Delta} \Lambda_{R_0}(x) \cap \mathbb{Z}_d$.

For $x, y \in \mathbb{R}^d, \Delta \subset \mathbb{R}^d$, we write $\{x \overset{\Delta}{\leftrightarrow} y\} = \{[x] \xleftrightarrow{E([\Delta])} [y]\}$. By Lemma 2.4, when $\Delta = \mathbb{R}^d$, these events have the same asymptotic decay rates as the point-to-point rates.

In the same spirit, for $\Delta \subset \mathbb{R}^d$, we say that an event is Δ-measurable if it is in $\mathcal{F}_{E([\Delta])}$.

3.2 A family of coarse graining

We will regularly use coarse-graining of the cluster of 0. We describe here a generic coarse-graining procedure parametrized by the "unit cell" of the coarse graining. These procedures are a general formulation of the coarse-graining procedure applied in [8]. Let $0 \in \Delta \subset \mathbb{Z}^d$ be finite. Let $\Delta_K = \bigcup_{x\in\Delta} \Lambda_K(x)$. Let $\mathcal{T} = \mathcal{T}(\Delta, K)$ be the set of embedded rooted trees defined as follows: $T \in \mathcal{T}$ is the data consisting of a set of vertices $t = \{t_0, \cdots, t_m\}$ where each $t_i \in \mathbb{Z}^d$, and a set of edges $f = \{f_1, \cdots, f_m\}$ with $f_i \subset t, |f_i| = 2$ such that

- The graph (t, f) is a tree.
- A given point in $\mathbb{Z}^d$ can only occur once as element of t.
- $t_0 = 0$, $t_i \in \partial^{\mathrm{ext}}(\Delta_K + t_j)$ where $f_i = \{t_i, t_j\}$.
- The labels and edges can be inductively reconstructed from the set of vertices (without labels) W as follows: t_i is the smallest (for the fixed total order on $\mathbb{Z}^d$) element of $W \setminus \{0, t_1, \cdots, t_{i-1}\}$ belonging to $\bigcup_{j=0}^{i-1} \partial^{\mathrm{ext}}(t_j + \Delta_K)$ and f_i is given by $\{t_i, v^*\}$ where v^* is the smallest element of $\{t_0, \cdots, t_{i-1}\}$ with $t_i \in \partial^{\mathrm{ext}}(v^* + \Delta_K)$.

A fairly direct observation is that the degree of a vertex t_i in (t, f) is less than $d_{\Delta_K} = |\partial^{\mathrm{ext}}\Delta_K|$ and one has a natural inclusion of $\mathcal{T}_l = \{T \in \mathcal{T} : \ |t| = l\}$ in the set of sub-trees of $\mathbb{T}_{d_{\Delta_K}}$ (the d_{Δ_K}-regular tree) containing 0 and having l vertices. In particular, there exists $c > 0$ universal such that

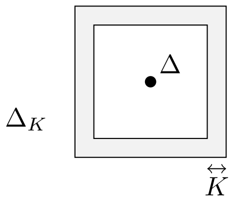
$$|\mathcal{T}_l| \leq e^{c\log(d_{\Delta_K})l}. \tag{3.1}$$

We now define a mapping $\mathrm{CG}_{\Delta,K}$ from the set of clusters containing 0 to $\mathcal{T}(\Delta, K)$. We define it via an algorithm constructing $T \in \mathcal{T}$ from $C \ni 0$. Fix some $C \ni 0$. Consider the graph formed by the vertices of $\mathbb{Z}^d$ and the edges in C. Construct t, f as follows

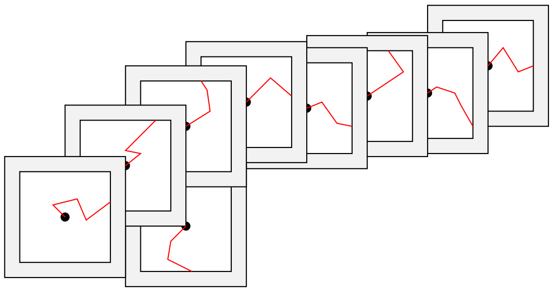
Algorithm 1: Coarse graining of a cluster containing 0.

Set $t_0 = 0$, $t = \{t_0\}$, $f = \varnothing$, $V = \Delta_K$, $i = 1$;
while $A = \{z \in \partial^{\text{ext}} V :\ z \xleftrightarrow{(z+\Delta)\setminus V} \partial^{\text{ext}}(z+\Delta)\} \neq \varnothing$ **do**
  Set $t_i = \min A$;
  Let v_* be the smallest $v \in t$ such that $t_m \in \partial^{\text{ext}}(\Delta + v^*)$;
  Set $f_i = \{v^*, t_i\}$;
  Update $t = t \cup \{t_i\}$, $f = f \cup \{f_i\}$, $V = V \cup (t_i + \Delta_K)$, $i = i + 1$;
end
return (t, f);

Write $\mathrm{CG}_{\Delta,K}(C) = (t(C), f(C))$. One has automatically that C is in a $(K + 2 \times \mathrm{radius}(\Delta))$-neighbourhood of $\mathrm{CG}_{\Delta,K}(C)$.



A possible cell Δ.



A coarse graining using the square cell.

Figure 2: Coarse graining example. Required connections are depicted in red.

The usefulness of such coarse graining is the conjunction of the combinatorial control we mentioned on trees with given number of vertices and the following energy bound.

Lemma 3.1. *Suppose the hypotheses of Theorem 1.2 hold. Then, there exists* $K_0 \geq 0$ *such that for any* $0 \in \Delta \subset \mathbb{Z}^d$ *finite,* $K \geq K_0$*, and* $T = (t, f) \in \mathcal{T}(\Delta, K)$*,*

$$P\big(\mathrm{CG}_{\Delta,K}(C_0) = T\big) \leq \Big(P(0 \leftrightarrow \Delta^c)(1 + |\Delta| e^{-c_{\text{mix}} K/2})\Big)^{|f|}.$$

Proof. Let $T = (t, f)$. The event $\mathrm{CG}_{\Delta,K}(C_0) = T$ implies in particular that

$$\bigcap_{i=1}^{|f|} \{t_i \xleftrightarrow{(t_i+\Delta)\setminus V_i} \partial^{\text{ext}}(t_i + \Delta)\} \equiv \bigcap_{i=1}^{|f|} A_i$$

occurs, where $V_i = \bigcup_{0 \leq j < i} (t_j + \Delta_K)$. Now, let F_i denote the support of A_i. One has that $|F_i| \leq C|\Delta|$ for any i (recall P has finite range) and $\mathrm{d}(F_i, F_j) \geq K$. In particular, by (2.4),

$$\begin{aligned}
P\Big(\bigcap_{i=1}^{|f|} A_i\Big) &\leq P\Big(\bigcap_{i=1}^{|f|-1} A_i\Big) P(A_{|f|}) \Big(1 + C_{\text{mix}} \sum_{e \in F_{|f|}, e' : \mathrm{d}(e,e') \geq K} e^{-c_{\text{mix}} \mathrm{d}(e,e')}\Big) \\
&\leq P\Big(\bigcap_{i=1}^{|f|-1} A_i\Big) P(A_{|f|}) \Big(1 + C|\Delta| K^{d-1} e^{-c_{\text{mix}} K}\Big) \\
&\leq P\Big(\bigcap_{i=1}^{|f|-1} A_i\Big) P(0 \leftrightarrow \Delta^c) \Big(1 + C|\Delta| K^{d-1} e^{-c_{\text{mix}} K}\Big)
\end{aligned}$$

where we used inclusion of events and translation invariance in the last line. Iterating $|f|$ times gives the result. $\square$

4 Proofs

The proof will go by introducing a family of decay rates (rates associated to various connection events). The idea is to prove the wanted properties for convenient rates and then to prove that all rates are in fact the same. Again, we work under the hypotheses of Theorem 1.2 which are implicitly assumed in the statements.

4.1 Constraint point-to-point

First introduce a family of connection events. For $\delta \in (0,1]$ and $s, s' \in \mathbb{S}^{d-1}$ such that $s \in \mathring{\mathcal{Y}}_{s',\delta}$,

$$Q_{s',\delta}(s,N) = \{0 \xleftrightarrow{\mathcal{Y}_{s',\delta}\setminus H_{s'}(Ns)} Ns\}.$$

Lemma 4.1. *For any $\delta \in (0,1]$ and $s, s' \in \mathbb{S}^{d-1}$ such that $s \in \mathring{\mathcal{Y}}_{s',\delta}$, the limit*

$$\tilde{\nu}_{s',\delta}(s) = \lim_{N\to\infty} -\frac{1}{N}\log P(Q_{s',\delta}(s,N))$$

exists.

Proof. Fix $s, s' \in \mathbb{S}^{d-1}$, $\delta \in (0,1]$ such that $s \in \mathring{\mathcal{Y}}_{s',\delta}$. By assumption, $P(0 \leftrightarrow \Lambda_n^c) \leq e^{-c_{\mathrm{co}}n}$. Denote $l = 2\limsup_{N\to\infty} -\frac{1}{N}\log P(Q_{s',\delta}(s,N))$ and set $\alpha = \frac{l}{c_{\mathrm{co}}}$. In particular, there exists N_0 such that for any $N \geq N_0$, $P(Q_{s',\delta}(s,N)) \geq e^{-lN}$. So, $-\frac{1}{N}\log P(Q_{s',\delta}(s,N))$ has the same upper and lower limits as the sequence

$$\frac{a_N}{N} = -\frac{1}{N}\log P\big(0 \xleftrightarrow{(\mathcal{Y}_{s',\delta}\setminus H_{s'}(Ns))\cap\Lambda_{\alpha N}} Ns\big).$$

See Figure 3 for the volume in which the connection is required to occur. This additional

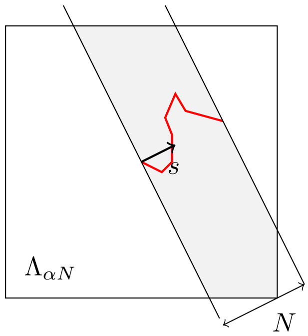


Figure 3: The volume $(\mathcal{Y}_{s,1} \setminus H_s(Ns)) \cap \Lambda_{\alpha N}$ (in grey).

manipulation is only needed to handle $\delta = 1$, see Figure 4. We show that a_N satisfies the hypotheses of Lemma A.1. Let $\Delta_N = (\mathcal{Y}_{s',\delta} \setminus H_{s'}(Ns)) \cap \Lambda_{\alpha N}$. Let $n \geq m$ be large enough, $r = \log(m)^2$, and set $N = n+m+r$. Then, $\Delta_n \subset \Delta_N$, $\big((n+r)s + \Delta_m\big) \subset \Delta_N$, and $\mathrm{d}(\Delta_n, (n+r)s + \Delta_m) \geq r$. Then,

$$P\big(0 \xleftrightarrow{\Delta_N} Ns\big) \geq \theta^{c_E r} P\big(0 \xleftrightarrow{\Delta_n} ns, (n+r)s \xleftrightarrow{(n+r)s+\Delta_m} Ns\big).$$

by inclusion of events and Lemma 2.4. Then, ratio mixing implies

$$P\big(0 \xleftrightarrow{\Delta_n} ns, (n+r)s \xleftrightarrow{(n+r)s+\Delta_m} Ns\big) \geq (1 - |\Delta_m|e^{-c_{\mathrm{mix}}r/2})P\big(0 \xleftrightarrow{\Delta_n} ns\big)P\big(0 \xleftrightarrow{\Delta_m} ms\big)$$

for any m large enough. $|\Delta_m|$ being upper bounded by a degree d polynomial in m, the wanted property (hypotheses of Lemma A.1) follows with $g(m) = \log(m)^2$, and $f(m) = 2 + c_E\log(\theta^{-1})\log(m)^2$. $\square$

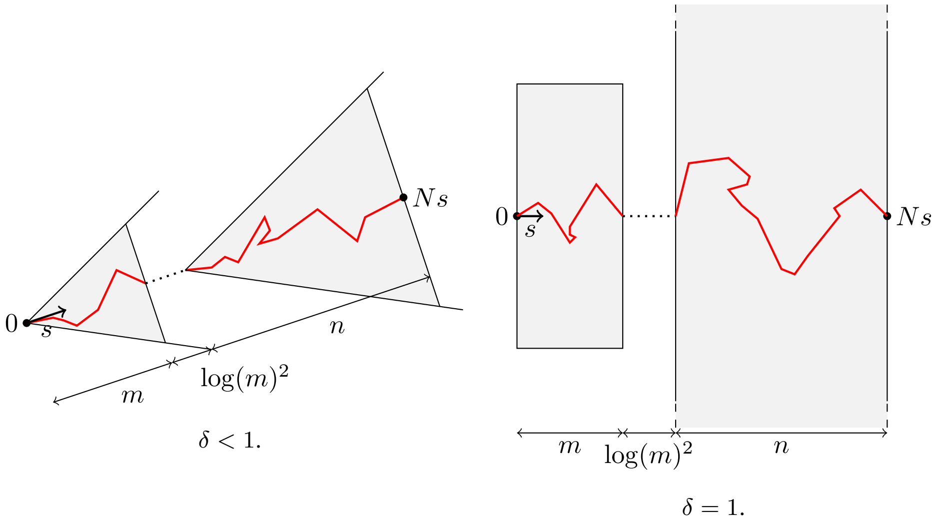


Figure 4: Construction of the local event. Dotted lines denote the use of insertion tolerance.

Lemma 4.2. *For any $s \in \mathbb{S}^{d-1}$, $\tilde{\nu}_{s',\delta}(s)$ does not depend on $\delta \in (0,1]$ and $s' \in \mathbb{S}^{d-1}$ as long as $s \in \mathring{\mathcal{Y}}_{s',\delta}$.*

Proof. Fix $s \in \mathbb{S}^{d-1}$ and omit it from notation. Let $\delta', \delta'' \in (0,1]$ and $s', s'' \in \mathbb{S}^{d-1}$ be such that $s \in \mathring{\mathcal{Y}}_{s',\delta'} \cap \mathring{\mathcal{Y}}_{s'',\delta''}$. To lighten notation, write $r' = \tilde{\nu}_{s',\delta'}$ and $r'' = \tilde{\nu}_{s'',\delta''}$. We first prove $r' \leq r''$. Let $\alpha = \frac{2r''}{c_{\mathrm{co}}}$. In particular, defining $\Delta_n = (\mathcal{Y}_{s'',\delta''} \setminus H_{s''}(ns)) \cap \Lambda_{\alpha n}$ (see Figure 5),

$$P(0 \xleftrightarrow{\Delta_n} ns) = e^{-r'' n(1+o_n(1))}.$$

Then, fix $\epsilon > 0$ small and n large enough. Write $\ell = \log(n)^2$. For any N large, $(1-\epsilon)N =$

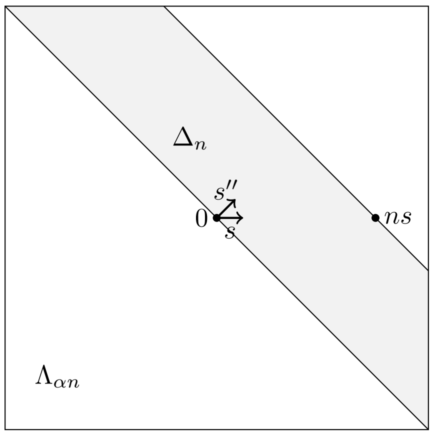


Figure 5: The volume Δ_n when $\delta'' = 1$ (in grey).

$q(n+\ell) + b$ with $b < n + \ell$ (integer parts are implicitly taken). One has

$$P\big(Q_{s',\delta'}(s,N)\big) \geq$$
$$\geq \theta^{c_E \epsilon N + b + q\ell} P\Big(\bigcap_{i=0}^{q-1} \big\{(\frac{\epsilon}{2}N + i(n+\ell))s \xleftrightarrow{(\epsilon N + i(n+\ell))s + \Delta_n} (\frac{\epsilon}{2}N + i(n+\ell)+n)s\big\}\Big),$$

where we used insertion tolerance (Lemma 2.4). See Figure 6.

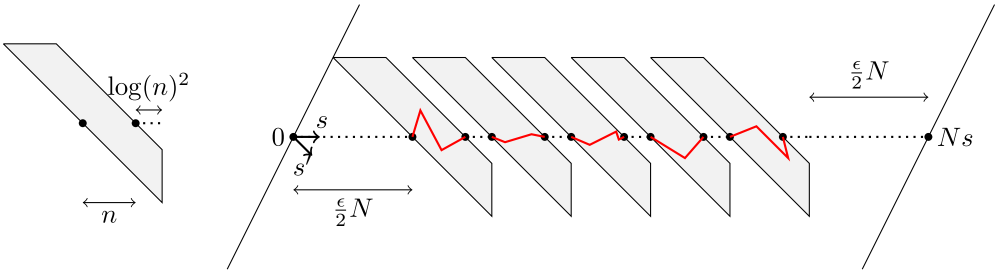


Figure 6: The construction of the lower bound. Dotted lines denote the use of insertion tolerance.

Using ratio mixing and translation invariance, the probability in the RHS is lower bounded by

$$e^{-q}\prod_{i=0}^{q-1} P\big(0 \xleftrightarrow{\Delta_n} ns\big) = e^{-q}e^{-r'' qn(1+o_n(1))}$$

whenever n is larger than some fixed constant. Taking the log, dividing by $-N$ and taking $N \to \infty$, one obtains

$$r' \le c_E \log(\theta^{-1})\epsilon + \frac{(\log(\theta^{-1})\ell + 1)(1-\epsilon)}{n+\ell} + \frac{(1-\epsilon)nr''}{n+\ell}$$

$\epsilon > 0$ is arbitrary and n is arbitrarily large. Take $n \to \infty$ and then $\epsilon \searrow 0$ to obtain the wanted inequality.

Repeating the argument with (s', δ') and (s'', δ'') exchanged yields the reverse inequality and thus the result. □

From Lemma 4.1 and 4.2, it is natural to introduce $\tilde{\nu} : \mathbb{R}^d \to \mathbb{R}_+$ as the extension by positive homogeneity of $\tilde{\nu}_{s',\delta}(s)$.

Lemma 4.3. *$\tilde{\nu}$ defines a norm on $\mathbb{R}^d$.*

Proof. First, point separation follows from the exponential decay assumption ($c_{\mathrm{co}} > 0$). Then, positive homogeneity of order one is a direct consequence of the way we extended $\tilde{\nu}$ to $\mathbb{R}^d$ and of

$$P\big(Q_{s,1}(s,N)\big) = P\big(Q_{-s,1}(-s,N)\big),$$

by translation invariance. Remains the triangle inequality. Fix $x, y \in \mathbb{R}^d$. Let $s_{xy} = \frac{x+y}{\|x+y\|}$, $s_x = \frac{x}{\|x\|}$, $s_y = \frac{y}{\|y\|}$. We can suppose that $x, x+y \in \mathring{H}_{s_{xy}}$ (otherwise, exchange the role of 0 and $x+y$, see Figure 7). Then, for $\epsilon > 0$ fixed, for any $\delta > 0$ small enough and any N

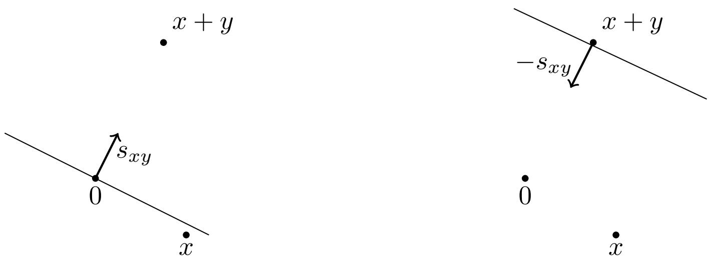


Figure 7

large

$$P\big(Q_{s_{xy},1}(s_{xy}, N\|x+y\|)\big) \geq$$
$$\geq \theta^{\epsilon c(\|x\|+\|y\|)N} P\big(Q_{s_x,\delta}(s_x, \|x\|(1-\epsilon)N), x+y \xleftrightarrow{\mathcal{Y}^{\|y\|(1-\epsilon)N}_{-s_y,\delta}(x+y)} x+\epsilon N\|y\|s_y\big),$$

where we used insertion tolerance. See Figure 8. Now, for $\epsilon > 0$ fixed and $\delta > 0$ small

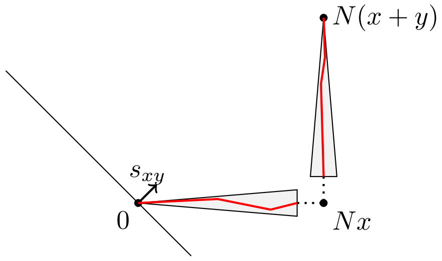


Figure 8: Construction of the forced connection through Nx. Dotted lines denote the use of insertion tolerance.

enough (depending on ϵ), one can use ratio mixing to obtain that the last probability is lower bounded by

$$(1-e^{-c'\epsilon N})P\big(Q_{s_x,\delta}(s_x, \|x\|(1-\epsilon)N)\big)P\big(Q_{s_y,\delta}(s_y, \|y\|(1-\epsilon)N)\big).$$

Taking the log, dividing by $-N$ and sending $N\to\infty$, one obtains

$$\|x+y\|\tilde{\nu}(s_{xy}) \leq \log(\theta^{-1})\epsilon c(\|x\|+\|y\|) + (1-\epsilon)\|x\|\tilde{\nu}(s_x) + (1-\epsilon)\|y\|\tilde{\nu}(s_y).$$

$\epsilon > 0$ was arbitrary, taking $\epsilon \searrow 0$ and using positive homogeneity gives $\tilde{\nu}(x+y) \leq \tilde{\nu}(x) + \tilde{\nu}(y)$. □

4.2 Point-to-half-space

Lemma 4.4. *Let $s \in \mathbb{S}^{d-1}$. The limit*

$$\nu_H(s) = \lim_{N\to\infty} -\frac{1}{N}\log P\big(0 \leftrightarrow H_s(Ns)\big)$$

exists.

Proof. We fix $s \in \mathbb{S}^{d-1}$ and omit it from the notation. Let $(n_k)_{k\geq 1}$ be an increasing sequence of integers such that

$$\lim_{k\to\infty} -\frac{1}{n_k}\log P\big(0 \leftrightarrow H_s(n_k s)\big) = \limsup_{N\to\infty} -\frac{1}{N}\log P\big(0\leftrightarrow H_s(Ns)\big) \equiv \overline{\nu}_H$$

In particular, $P\big(0 \leftrightarrow H_s(n_k s)\big) = e^{-n_k\overline{\nu}_H(1+o_k(1))}$.

By our hypotheses,

$$P(0 \leftrightarrow \Lambda_M) \leq e^{-c_{\mathrm{co}}M}$$

for any M large enough. Let then $\alpha = \frac{\overline{\nu}_H}{c_{\mathrm{co}}}$. Set $\Delta_k = \Lambda_{\alpha n_k}\setminus H_s(n_k s)$, $K_k = \log(n_k)^2$, $\overline{\Delta}_k = \bigcup_{v\in\Delta_k}\Lambda_{K_k}(v)$. See Figure 9. In particular, we have

$$P(0\leftrightarrow \Delta_k^c) \leq P(0\leftrightarrow\Lambda^c_{\alpha n_k}) + P\big(0\leftrightarrow H_s(n_k s)\big) \leq e^{-\overline{\nu}_H n_k(1+o_k(1))} \tag{4.1}$$

where we used a union bound.

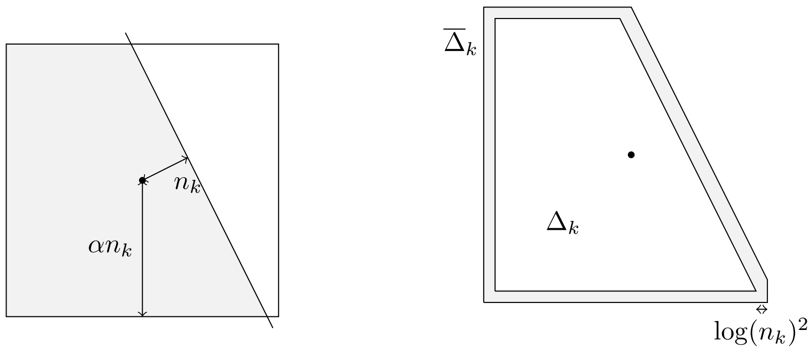


Figure 9: The cell Δ_k.

We now coarse-grain C_0 using $\mathrm{CG}_k \equiv \mathrm{CG}_{\Delta_k,K_k}$ (see Section 3.2). Write $\mathrm{CG}_k(C_0) = (t(C_0), f(C_0))$. One has that C_0 is included in an $3\alpha n_k$-neighbourhood of $t(C_0)$. We have

$$P(0 \leftrightarrow X) = \sum_{T\in\mathcal{T}} P\big(0 \leftrightarrow X, \mathrm{CG}_k(C_0) = T\big) \leq \sum_{T\sim X} P\big(\mathrm{CG}_k(C_0) = T\big) \tag{4.2}$$

where $T \sim X$ means that $\mathrm{d}(X, t(C_0)) \leq 3\alpha n_k$. We can then use Lemma 3.1 and the bound on the number of trees, (3.1), to obtain that for any fixed large enough k, as N goes to infinity,

$$\begin{aligned} P\big(0 \leftrightarrow H_s(Ns)\big) &\leq \sum_{l \geq \frac{N}{n_k+K_k}} \sum_{T\in\mathcal{T}_l} P\big(\mathrm{CG}_k(C_0) = T\big) \\ &\leq \sum_{l \geq \frac{N}{n_k+K_k}} e^{c\log(d_{\overline{\Delta}_k})l} e^{-\overline{\nu}_H n_k l(1+o_k(1))} \\ &= \sum_{l \geq \frac{N}{n_k+K_k}} e^{-\overline{\nu}_H n_k l(1+o_k(1)+o_{n_k}(1))} = e^{-N\overline{\nu}_H(1+o_k(1)+o_{n_k}(1))}\big(1-o_k(1)\big)^{-1} \end{aligned}$$

as $d_{\overline{\Delta}_k}$ is upper bounded by a polynomial of degree d in n_k and any tree T with $T \sim H_s(N_s)$ has $|f| \geq \frac{N}{n_k+K_k}$. In particular, for any k large enough,

$$\underline{\nu}_H \equiv \liminf_{N\to\infty} -\frac{1}{N}\log P(0 \leftrightarrow Ns + H_s) \geq \overline{\nu}_H(1 + o_k(1) + o_{n_k}(1)).$$

Taking $k \to \infty$ yields $\underline{\nu}_H \geq \overline{\nu}_H$. The direction s being arbitrary, $\overline{\nu}_H(s) = \underline{\nu}_H(s)$ for all $s \in \mathbb{S}^{d-1}$. □

4.3 Constrained point-to-half-space

Lemma 4.5. *Let $s \in \mathbb{S}^{d-1}$. The limit*

$$\tilde{\nu}_H(s) = \lim_{N\to\infty} -\frac{1}{N}\log P\big(0 \xleftrightarrow{H_s} H_s(Ns)\big)$$

exists. Moreover,

$$\tilde{\nu}_H(s) = \nu_H(s).$$

Proof. We fix $s \in \mathbb{S}^{d-1}$ and omit it from the notation. By inclusion of events, one has $\liminf_{N\to\infty} -\frac{1}{N}\log P\big(0 \xleftrightarrow{H_s} H_s(Ns)\big) \geq \nu_H$. To obtain the other bound, start with, for any $\epsilon > 0$,

$$P\big(0 \leftrightarrow H_s(Ns)\big) \leq \theta^{-c_E\epsilon N} P(\epsilon Ns \leftrightarrow H_s(Ns)) = e^{\lambda\epsilon N} e^{-\nu_H(1-\epsilon)N(1+o_N(1))}, \tag{4.3}$$

where $\lambda = \log(\theta^{-1})c_E > 0$ and

$$P\big(0 \xleftrightarrow{H_s} H_s(Ns)\big) \geq e^{-\lambda\epsilon N} P(\epsilon Ns \xleftrightarrow{H_s} H_s(Ns)), \tag{4.4}$$

by Lemma 2.4 (insertion tolerance).

We then use a coarse-graining described in Section 3.2 (the same as in the proof of Lemma 4.4 with different sizes). Set $\Delta_n = \Lambda_{\alpha n} \setminus H_s(ns)$, $K_n = \log(n)^2$, and $\overline{\Delta}_n = \bigcup_{v\in\Delta_n} \Lambda_{K_n}(v)$, where α is the same quantity as in the proof of Lemma 4.4. As in Lemma 4.4,

$$P(0 \leftrightarrow \Delta_n^c) \leq e^{-\nu_H n(1+o_n(1))}.$$

We use $\mathrm{CG}_n \equiv \mathrm{CG}_{\Delta_n,K_n}$. Write $\mathrm{CG}_n(C_0) = \big(t(C_0), f(C_0)\big)$.

Now, any cluster contributing to $\{\epsilon Ns \leftrightarrow H_s(Ns)\} \setminus \{\epsilon Ns \xleftrightarrow{H_s} H_s(Ns)\}$ has $|f| \geq \frac{\epsilon N}{\sqrt{d}(\alpha n+\log(n)^2)} + \frac{(1-\epsilon)N}{n+\log(n)^2}$ (see Figure 10). So, applying the same argument as in Lemma 4.4,

$$P(\epsilon Ns \leftrightarrow H_s(Ns)) - P(\epsilon Ns \xleftrightarrow{H_s} H_s(Ns)) \leq e^{-\nu_H N(1-\epsilon+\frac{\epsilon}{\sqrt{d}\alpha}+o_n(1))}.$$

In particular, for any fixed n large enough, and any N large

$$\frac{P(\epsilon Ns \xleftrightarrow{H_s} H_s(Ns))}{P(\epsilon Ns \leftrightarrow H_s(Ns))} \geq 1 - e^{-\nu_H N(\epsilon'+o_n(1)+o_N(1))},$$

where $\epsilon' = \frac{\epsilon}{\sqrt{d}\alpha}$. Plugging this in (4.4), and using (4.3), one obtains

$$\begin{aligned} P\big(0 \xleftrightarrow{H_s} H_s(Ns)\big) &\geq e^{-2\lambda\epsilon N}(1 - e^{-\nu_H N(\epsilon'+o_n(1)+o_N(1))})P(0 \leftrightarrow H_s(Ns)) \\ &= e^{-2\lambda\epsilon N}(1 - e^{-\nu_H N(\epsilon'+o_n(1)+o_N(1))})e^{-\nu_H N(1+o_N(1))}. \end{aligned}$$

In particular $\limsup_{N\to\infty} -\frac{1}{N}\log P(0 \xleftrightarrow{H_s} H_s(Ns)) \leq \nu_H + 2\lambda\epsilon$. $\epsilon > 0$ being arbitrary, taking $\epsilon \searrow 0$ yields the result. $\square$

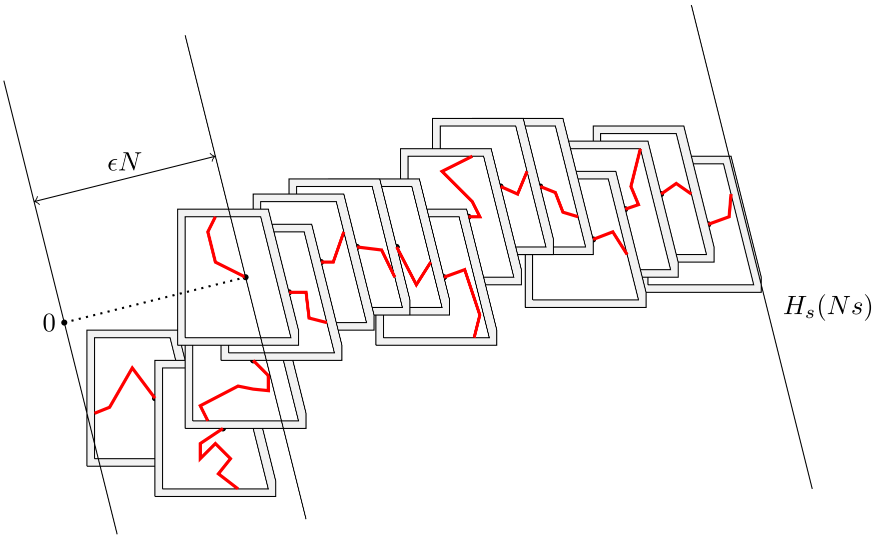


Figure 10: Coarse graining of a cluster contributing to $\{\epsilon Ns \leftrightarrow H_s(Ns)\} \setminus \{\epsilon Ns \xleftrightarrow{H_s} H_s(Ns)\}$.

We highlight at this point that we could easily remove the "directed constraint" for point-to-half-spaces connections, which seems to be much harder to do for point-to-point connections.

4.4 Convex duality

We saw that $\tilde{\nu}$ defines a norm on $\mathbb{R}^d$. In particular, $\mathcal{U}_{\tilde{\nu}}$ (the unit ball for $\tilde{\nu}$) is a convex set. To each $s \in \mathbb{S}^{d-1}$, we associate the set of *dual directions*

$$s^\star = \Big\{ s' \in \mathbb{S}^{d-1} :\ H_{s'}\big(\frac{\langle s, s'\rangle}{\tilde{\nu}(s)} s'\big) \cap \mathcal{U}_{\tilde{\nu}} \subset \partial \mathcal{U}_{\tilde{\nu}} \Big\}.$$

It is the set of directions normal to the boundary of half-spaces tangent to $\mathcal{U}_{\tilde{\nu}}$ at $\frac{s}{\tilde{\nu}(s)}$ (see Figure 11). By abuse of notation, we will write $s^\star$ for an arbitrarily chosen element of the set. It satisfies $\langle s, s^\star\rangle > 0$. Moreover, for a fixed $s^\star$, any s having $s^\star$ as dual is a minimizer of $s' \mapsto \frac{\tilde{\nu}(s')}{\langle s^\star, s'\rangle}$ under the constraint $\langle s^\star, s'\rangle > 0$. Notice that this notion of duality is not the classical convex duality between $\mathcal{U}_{\tilde{\nu}}$ and $\mathcal{W}_{\tilde{\nu}}$ (but it is related via normalization of the dual directions).

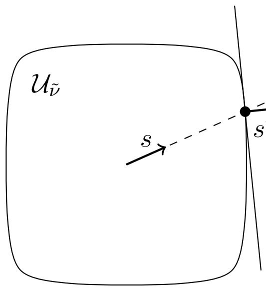


Figure 11: Duality between directions.

The duality statement is

Lemma 4.6. *For any $s \in \mathbb{S}^{d-1}$,*

$$\tilde{\nu}(s) = \nu_H(s^\star)\langle s, s^\star\rangle. \tag{4.5}$$

Proof. Fix $s \in \mathbb{S}^{d-1}$. Let $s^\star$ be a dual direction of s. Start with the easy inequality. By inclusion of events and Lemma 4.2,

$$P\big(0 \leftrightarrow H_{s^\star}(Ns^\star)\big) \geq P\big(0 \xleftrightarrow{H_{s^\star}\setminus H_{s^\star}(aNs)} aNs\big) = e^{-a\tilde{\nu}(s)N(1+o_N(1))}$$

where $a = \langle s, s^\star\rangle^{-1}$. Taking the log, dividing by $-N$ and letting $N \to \infty$, one gets $\nu_H(s^\star) \leq a\tilde{\nu}(s)$.

We now proceed to the harder inequality. We use Lemma 4.5. The idea is illustrated in Figure 12. Then, using the same argument as in the proof of Lemma 4.4, for some α large enough,

$$P\big(0 \xleftrightarrow{H_{s^\star}} H_{s^\star}(Ns^\star)\big) \leq CP\big(0 \xleftrightarrow{H_{s^\star}\cap\Lambda_{\alpha N}} H_{s^\star}(Ns^\star)\big).$$

By a union bound, this is in turn upper bounded by

$$C \sum_{x \in \partial^{\text{int}}[H_{s^\star}(Ns^\star)]\cap\Lambda_{\alpha N}} P(0 \xleftrightarrow{H_{s^\star}\setminus H_{s^\star}(x)} x). \tag{4.6}$$

Let $\delta < 1$ be such that $\partial^{\text{int}}[H_{s^\star}(Ns^\star)] \cap \Lambda_{\alpha N} \subset \mathcal{Y}_{s^\star,\delta}$ for any N large enough. Let $\epsilon > 0$ be small. Choose a finite subset S of $\mathbb{S}^{d-1} \cap \mathcal{Y}_{s^\star,\delta}$ such that $|S| \leq c''\epsilon^{-d+1}$ and $\mathcal{Y}_{s^\star,\delta} \subset \bigcup_{s'\in S} \mathcal{Y}_{s',\epsilon}$. Denote $A_{s'}(N) = \partial^{\text{int}}[Ns^\star + H_{s^\star}] \cap \mathcal{Y}_{s',\epsilon}$. Then, by insertion tolerance, (4.6) is upper bounded by

$$C \sum_{s'\in S} |A_{s'}(N)|\theta^{-c'\epsilon N} P(0 \xleftrightarrow{H_{s^\star}\setminus H_{s^\star}(a_{s'}Ns')} a_{s'}Ns')$$

with $a_{s'} = \langle s', s^\star\rangle^{-1}$. By Lemma 4.2, $P(0 \xleftrightarrow{H_{s^\star}\setminus H_{s^\star}(a_{s'}Ns')} a_{s'}Ns') = e^{-a_{s'}N\tilde{\nu}(s')(1+o_N(1))}$ with the $o_N(1)$ depending on s'. Denote it $o_N^{s'}(1)$. Now, $a_{s'}\tilde{\nu}(s')$ is minimal if $s', s^\star$ are dual directions. So, combining all the previous observations,

$$P\big(0 \xleftrightarrow{H_{s^\star}} H_{s^\star}(Ns^\star)\big) \leq C'N^{d-1}\epsilon^{1-d}\theta^{-c'\epsilon N}e^{a_sN\tilde{\nu}(s)\max_{s'\in S}o_N^{s'}(1)}e^{-a_sN\tilde{\nu}(s)}.$$

Taking the log, dividing by $-N$ and taking $N\to\infty$ gives

$$\nu_H(s^\star) \geq \log(\theta)c'\epsilon + a_s\tilde{\nu}(s).$$

Taking then $\epsilon \searrow 0$ yields the result.

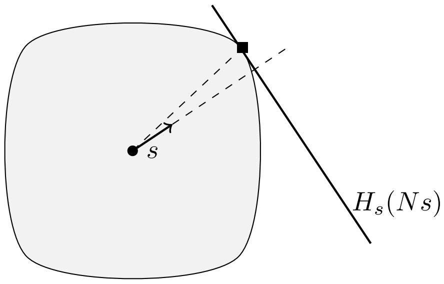


Figure 12: Connection to $H_s(Ns)$ is made at the point minimizing the distance measured with $\tilde{\nu}$ (here the square mark). The grey shape is a dilation of $\mathcal{U}_{\tilde{\nu}}$.

□

4.5 Final coarse-graining

Let us summarize what we did so far. First, we constructed a norm using a directed version of the point-to-point connections (Lemmas 4.1, 4.2, and 4.3). Then, we proved an equivalence (at the level of exponential rates) between directed and un-directed point-to-half-space connections (Lemmas 4.4 and 4.5). Finally, we related these two quantities using convex duality (Lemma 4.6). We can now gather these three results to prove our key estimate

Lemma 4.7. *For any $\epsilon > 0$, there exists $L_0 \geq 0$ such that for any $L \geq L_0$,*

$$P\big(0 \leftrightarrow (L\mathcal{U}_{\tilde{\nu}})^c\big) \leq e^{-L(1-\epsilon)}. \tag{4.7}$$

Proof. Fix $\epsilon > 0$. Take S a finite subset of $\mathbb{S}^{d-1}$ such that $|S| \leq c\delta^{-d+1}$ and $\bigcup_{s\in S}\mathcal{Y}_{s,\delta}\cap\mathcal{U}_{\tilde{\nu}} = \mathcal{U}_{\tilde{\nu}}$. For $s\in S$, denote $A_s = \partial^{\mathrm{ext}}(L\mathcal{U}_{\tilde{\nu}})\cap\mathcal{Y}_{s,\delta}$. Then,

$$\begin{aligned} P\big(0 \leftrightarrow (L\mathcal{U}_{\tilde{\nu}})^c\big) &\leq \sum_{s\in S}\sum_{x\in A_s} P\big(0 \xleftrightarrow{L\mathcal{U}_{\tilde{\nu}}} x\big) \\ &\leq \theta^{-c'\delta L}(c''L^{d-1})\sum_{s\in S} P\Big(0 \xleftrightarrow{L\mathcal{U}_{\tilde{\nu}}} \frac{sL}{\tilde{\nu}(s)}\Big), \end{aligned}$$

where we used insertion tolerance in the second line. Now, for any fixed $s\in S$, let $s^\star$ be dual to s. See Figure 13. One then has (using Lemma 4.6)

$$P\Big(0 \xleftrightarrow{L\mathcal{U}_{\tilde{\nu}}} \frac{sL}{\tilde{\nu}(s)}\Big) \leq P\Big(0 \leftrightarrow \frac{L\langle s, s^\star\rangle}{\tilde{\nu}(s)}s^\star + H_{s^\star}\Big) \leq e^{-\frac{L\langle s,s^\star\rangle}{\tilde{\nu}(s)}\nu_H(s^\star)(1+o_L(1))} = e^{-L(1+o_L(1))}.$$

Now, the $o_L(1)$ depends on s. Write it $o_L^s(1)$. One therefore obtains

$$P\big(0 \leftrightarrow (L\mathcal{U}_{\tilde{\nu}})^c\big) \leq \theta^{-c\delta L}(c'L^{d-1})c''\delta^{-d+1}e^{-L}e^{\max_{s\in S}o_L^s(1)}.$$

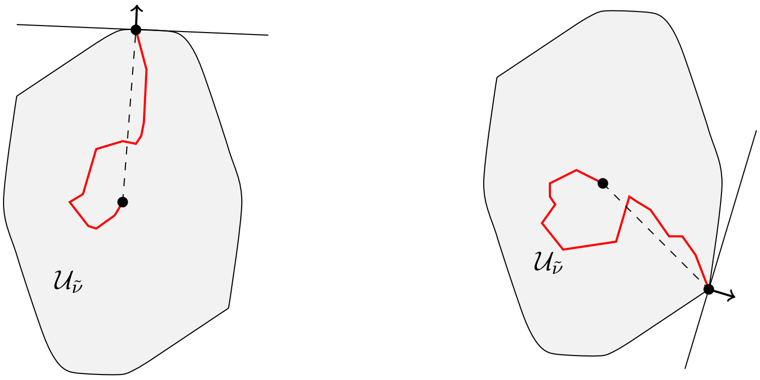


Figure 13: For each direction s, we chose a dual direction for which connecting to a half-spaces is the same as connecting in direction s.

Take δ small enough and then L large enough to have $\theta^{-c\delta L}(c'L^{d-1})c''\delta^{-d+1} \leq e^{\epsilon L/2}$ and $\max_{s\in S} o^s_L(1) \leq \epsilon L/2$. ☐

We then use the coarse graining procedure of Section 3.2 with $\Delta = L\mathcal{U}_{\tilde{\nu}}$ and $K = \log(L)^2$: $\mathrm{CG}_L \equiv \mathrm{CG}_{L\mathcal{U}_{\tilde{\nu}},\log(L)^2}$.

As a corollary of this construction, we obtain

Corollary 4.8. *For any $s \in \mathbb{S}^{d-1}$,*

$$\overline{\nu}(s) \leq \tilde{\nu}(s) \leq \underline{\nu}(s).$$

In particular, ν is well defined and defines a norm on $\mathbb{R}^d$.

Proof. Fix some $s \in \mathbb{S}^{d-1}$. One has the direct lower bound $\tilde{\nu}(s) \geq \overline{\nu}(s)$. To obtain the other bound, we use CG_L. Any cluster $C \ni 0, Ns$ has $|f(C)| \geq \frac{N\tilde{\nu}(s)}{L+\log(L)^2}$ (recall $\mathcal{U}_{\tilde{\nu}}$ is convex). Fix $\epsilon > 0$ small and take $L \geq L_0(\epsilon)$. Using the bound on the combinatoric of trees and Lemmas 3.1 and 4.7, one obtains

$$P(0 \leftrightarrow Ns) \leq e^{(\epsilon+o_L(1))N}e^{-N\tilde{\nu}(s)}.$$

Taking the log, dividing by $-N$ and letting $N \to \infty$ gives $\underline{\nu}(s) \geq \tilde{\nu}(s) - \epsilon + o_L(1)$. Letting $L \to \infty$ and then $\epsilon \searrow 0$ give the result. ☐

A Relaxed Fekete's lemma

We use this Lemma which proof is an easy adaptation of the usual Fekete's Lemma.

Lemma A.1. *Suppose $(a_n)_{n\geq 1}$ is a sequence with $c_- n < a_n < c_+ n$ for some $0 < c_- \leq c_+ < \infty$. Suppose that there exists $N_0 \geq 1$ and functions $f, g : (\mathbb{Z}_{>0}) \to \mathbb{Z}$ such that*

- *$f(n) = o(n)$, $g(n) = o(n)$,*
- *For any $n, m \geq N_0$, $a_{n+m+g(\min(n,m))} \leq a_n + a_m + f(\min(n,m))$.*

Then, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists in $[c_-, c_+]$.

Proof. Let $\underline{l} = \liminf_{n\to\infty} \frac{a_n}{n}$. Let $(n_k)_{k\geq 1}$ be an increasing sequence such that $\lim_{k\to\infty} \frac{a_{n_k}}{n_k} = \underline{l}$. Fix k such that $n_k \geq N_0$. For any N large enough, $N = q(n_k + g(n_k)) + r$ with $r < n_k + g(n_k)$. Then, by $q-1$ iterations of our sub-additivity-type hypotheses

$$\frac{a_N}{N} \leq \frac{(q-1)(a_{n_k} + f(n_k)) + a_{n_k+g(n_k)+r}}{q(n_k + g(n_k)) + r} = \underline{l} + o_k(1) + o_{n_k}(1) + o_N(1).$$

Taking $N \to \infty$, one obtains

$$\limsup_{N\to\infty} \frac{a_N}{N} \leq \underline{l} + o_k(1) + o_{n_k}(1).$$

k being arbitrary, one can now take $k \to \infty$ to obtain the wanted result. □

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