Second order cubic corrections of large deviations for perturbed random walks[*]

Giancarlos Oviedo[†] Gonzalo Panizo[‡] Alejandro F. Ramírez[§¶]

Abstract

We prove that the Beta random walk, introduced in [BC17] 2017, has cubic fluctuations from the large deviation principle of the GUE Tracy-Widom type for arbitrary values $\alpha > 0$ and $\beta > 0$ of the parameters of the Beta distribution, removing previous restrictions on their values. Furthermore, we prove that the GUE Tracy-Widom fluctuations still hold in the intermediate disorder regime. We also show that any random walk in space-time random environment that matches certain moments with the Beta random walk also has GUE Tracy-Widom fluctuations in the intermediate disorder regime. As a corollary we show the emergence of GUE Tracy-Widom fluctuations from the large deviation principle for trajectories ending at boundary points for random walks in space (time-independent) i.i.d. Dirichlet random environment in dimension $d = 2$ for a class of asymptotic behavior of the parameters.

Keywords: random walk in random environment; beta random walk; GUE Tracy-Widom distribution.
MSC2020 subject classifications: 60K37; 82D30; 82C23; 82C41.
Submitted to EJP on August 16, 2021, final version accepted on April 27, 2022.

1 Introduction

The Beta-random walk, introduced in [BC17], is a random walk in space-time i.i.d. random environment in $\mathbb{Z}$ which is exactly solvable. Given $\alpha, \beta \in (0, \infty)$, for each space time point $(t, x) \in \mathbb{N} \times \mathbb{Z}$, a Beta-random variable $B_{t,x}$ of parameters α and β is associated

to the space-time point (t, x), so that $(B_{t,x})_{(t,x) \in \mathbb{N} \times \mathbb{Z}}$ are i.i.d. The Beta-random walk $(X_t)_{t \in \mathbb{N}}$ in the environment $(B_{t,x})_{(t,x) \in \mathbb{N} \times \mathbb{Z}}$ is then defined by its transition probabilities

$$P(X_{t+1} = x + 1 | X_t = x) = B_{t,x} \qquad \text{for } x \in \mathbb{Z}.$$

A general quenched large deviation principle for the position of the walk, which includes the Beta-random walk, was proved in [RSY13]. Subsequently, Barraquand and Corwin in [BC17], obtained an explicit formula for the quenched large deviation rate function and also proved for the case $\alpha = \beta = 1$, second order cube-root scale corrections to this large deviation principle, with convergence to the GUE Tracy-Widom distribution. This was recently extended for the case $\alpha = 1$ and $\beta > 0$ in [K21].

Directed polymers in random environment in dimension $1 + 1$ are in strong disorder for all inverse temperatures $\beta > 0$, and it is conjectured that for general distributions, there are second order cube-root fluctuations for the rescaled free energy with GUE Tracy-Widom statistics. Random walks in space-time i.i.d. random environments share several features with directed polymers in random environment. It is hence natural to conjecture that a similar behavior would occur: second order corrections to the quenched large deviation principle for random walk in space-time i.i.d. random environment for a large class of distributions of the random environment, still of scale cube-root, with convergence to the GUE Tracy-Widom distribution. Here, the strength of the disorder given by the random environment, plays the role of the inverse temperature (for dimensions $d \geq 3 + 1$, regimes analogous to the strong and weak disorder regimes of directed polymers exist for the random walk in space-time i.i.d. environment [BMRS19]). The intermediate disorder regime probes the strong-weak disorder transition of directed polymers in dimensions $1 + 1$, by taking the limit $\beta \to 0$ as the length of the polymer is increased. For the special case $\beta_N = \hat{\beta} N^{-1/4}$, it was shown in [AKQ14], that the fluctuations crossover between the Edwards-Willkinson regime to the GUE Tracy-Widom regime as $\hat{\beta} \to \infty$, and the Kardar-Parisi-Zhang equation (KPZ) appears in the limit. Subsequently Krishnan and Quastel in [KQ18], showed through a perturbation argument involving matching a certain number of moments with the log-gamma polymer [S12], the universality of the GUE Tracy-Widom distribution when $1 \gg \beta_N \gg O(N^{-1/4})$ (see similar ideas used within the context of chaos phenomena and ultrametricity in the mixed p-spin model in [AC16]).

The intermediate disorder has also been studied within the context of random walks in space-time i.i.d. random environment in $\mathbb{Z}$ which are perturbations of the simple symmetric random walk. Indeed, in [CG17], the intermediate disorder regime was proven for a random walk in space-time i.i.d. random environment defined by its probability to jump to the right at a given time t from a site x as $\frac{1}{2} + t^{-1/4}\xi(x, t)$ with $(\xi(x, t))_{x \in \mathbb{Z}, t \in \mathbb{N}}$ i.i.d. with values in $[0, 1]$. This corresponds to the case $\beta_N = \hat{\beta} N^{-1/4}$ of directed polymers. A second result in [CG17] gives an intermediate disorder regime limit of the same kind for the logarithmic fluctuations of the transition probability $P_{0,\omega}(X_t = y)$ of the Beta random walk, with scaling $y = \gamma t + x t^{1/2}$, $\gamma \in (0, 1/2)$, $x \in \mathbb{R}$ with time-dependent parameters $\alpha_t = \beta_t = t^{1/2}$, as $t \to \infty$ is time.

In this article we first prove convergence to the GUE Tracy-Widom distribution of the second order fluctuations of the Beta random walk for any $\alpha > 0$ and $\beta > 0$. This result removes the restrictions of [BC17], where the convergence was proven for $\alpha = \beta = 1$ and of [K21] where it was extended to the case $\alpha = 1$ and $\beta > 0$. Furthermore, our result shows that we can keep the same range $\theta \in (0, 0.5)$ parametrizing the target points $x(\theta)$ (which is expressed as a rational function involving polygamma functions) of the large deviation event $\{X_t \geq x(\theta)t\}$ as in [BC17], as long as $\alpha \geq 0.73$ and $\beta > 0$. As a second result we prove convergence to the GUE Tracy-Widom distribution for the logarithmic fluctuations of the probability $P_{0,\omega}(X_t \geq x_t t)$, for $\alpha_t \to \infty$, $\beta_t \to \infty$ and $x_t \to 1$, under

an appropriate condition on the growth of α_t and β_t. For the case $\alpha_t = t^r$, $\beta_t = t^s$, this condition reduces to $r + \max(r - s, 0) < 1$. In the third result of this article we show that the corresponding result for random walks in space-time i.i.d. environments which are perturbations of the Beta random walk still holds, along the lines of [KQ18]. An interesting corollary of our perturbation results, is the appearance of the GUE Tracy-Widom fluctuations for random walks in space i.i.d. (static) Dirichlet environment in dimension $d = 2$, for certain asymptotic behaviour of the parameters, corresponding to events which force the random walk to move through directed trajectories. The main challenge of the proofs is doing a sophisticated steep-descent analysis involving polygamma functions.

In what follows we give a precise formulation of the main results of this article.

2 Main results

We first define the model of random walk in random environment on $\mathbb{Z}^d \times \mathbb{N}$, where the factor $\mathbb{Z}^d$ will represent the space where it moves, while $\mathbb{N}$ is the time. Let $| \cdot |_1$ denote the l_1 norm. Let $U := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$ and $\mathcal{P} := \{(p(e))_{e \in U} \in [0, 1]^{2d} : \sum_{e \in U} p(e) \leq 1\}$. Let $\Omega = \mathcal{P}^{\mathbb{Z}^d \times \mathbb{N}}$. We call Ω the *environmental space* and each element $\omega := (\omega(x, t))_{x \in \mathbb{Z}^d, t \in \mathbb{N}} \in \Omega$, with $\omega(x, t) = (\omega_e(x, t))_{e \in U} \in \mathcal{P}$, an *environment*. Note that we do not assume necessarily that $\sum_{e \in U} \omega_e(x, t) = 1$, which can be interpreted as jump probabilities having a non-vanishing probability of absorption at each step. Given $\omega \in \Omega$, consider the (sub)-Markov chain $(X_t)_{t \geq 0}$ with state space $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$, defined through its transition probabilities

$$P_{x,\omega}(X_{t+1} = y + e | X_t = y) = \omega_e(y, t),$$

for $y \in \mathbb{Z}^d$ and $t \geq 0$, with $P_{x,\omega}(X_0 = x) = 1$. We call the (sub)-Markov process $(X_t)_{t \geq 0}$ a random walk in the space-time environment ω on $\mathbb{Z}^d$ and denote by $P_{x,\omega}$ its law. If $\mathbb{P}$ is a probability measure defined on Ω, we denote the law $P_{x,\omega}$ the *quenched law* of the *random walk in random environment*, and by $E_{x,\omega}$ the expectation corresponding to $P_{x,\omega}$.

A special case of a random walk in random environment is the Beta random walk, defined for $d = 1$, and where the environment is space-time i.i.d. Recall that a random variable B is a Beta random variable of parameters $\alpha > 0$ and $\beta > 0$ if for every $r \in [0, 1]$ we have that

$$P(B \leq r) = \int_0^r x^{\alpha-1}(1-x)^{\beta-1} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} dx$$

Let $(B_{x,t})_{x \in \mathbb{Z}, t \geq 0}$ be an i.i.d. collection of Beta random variables of parameter $\alpha > 0$ and $\beta > 0$. Let $\mathbb{P}_{\alpha,\beta}$ be the joint law of ω with $\omega_1(x, t) = B_{x,t}$ and $\omega_{-1}(x, t) = 1 - B_{x,t}$. Then, the random walk in random environment on $\mathbb{Z}$ with law $\mathbb{P}_{\alpha,\beta}$ is called a *Beta random walk*. We will denote by $\mathbb{E}_{\alpha,\beta}$ the corresponding expectation.

An important class of a random walks in random environment corresponds to the case in which the law $\mathbb{P}$ of the environment is concentrated on environments ω which are time-independent, so that $\omega(x) := \omega(t, x)$ for all $t \geq 0$ and $x \in \mathbb{Z}^d$. In this case we will use the notation $\omega(x) = (\omega(x, e))_{e \in U}$. A particular example of a random walk in (time-independent) random environment on $\mathbb{Z}^d$, is the random walk in Dirichlet environment (RWDE). To define the RWDE, let us use the notation $U = \{e_1, e_2, \ldots, e_{2d}\}$ with the convention $e_{d+i} = -e_i$ for $1 \leq i \leq d$. For each $1 \leq i \leq 2d$, let $\alpha_i > 0$. The Dirichlet distribution with parameters $\alpha := (\alpha_i)_{i \in \{1, \ldots, k\}}$ is the distribution on $\mathcal{P}_1 = \{(p(e))_{e \in U} \in [0, 1]^{2d} : \sum_{e \in U} p(e) \leq 1\}$ which has a density with respect to the Lebesgue measure in $\mathcal{P}_1$ given by

$$\frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} u_1^{\alpha_1 - 1} \cdots u_k^{\alpha_k - 1} \prod_{i \neq i_0} du_i,$$

where i_0 is an irrelevant choice of index among $\{1, \ldots, k\}$ and $u_{i_0} = 1 - \sum_{i \neq i_0} u_i$. The random walk in Dirichlet environment of parameter $\alpha := (\alpha_i)_{i \in \{1, \ldots, 2d\}}$ is defined as the random walk on $\mathbb{Z}^d$ whose environment ω is time-independent and has a law $\mathbb{P}_\alpha$ under which $(\omega(x))_{x \in \mathbb{Z}^d}$ are i.i.d. and have Dirichlet distribution of parameter α.

In what follows we will use the notation $P_\omega := P_{0,\omega}$ for the quenched law of the Beta random walk. Define $P_\omega(t, x) := P_\omega(X_t \geq x)$. In [RS14] and [RSY13], it was shown that $\mathbb{P}_{\alpha, \beta}$-a.s.

$$\lim_{t \to \infty} \frac{1}{t} \log P_\omega(t, xt) = -I(x),$$

where I is the Legendre transform of

$$\lambda(s) := \lim_{t \to \infty} \frac{1}{t} \log \left(E_\omega \left[e^{s X_t} \right] \right) \quad s \in \mathbb{R},$$

where the right-hand side limit exists $\mathbb{P}_{\alpha, \beta}$-a.s. In [BC17] a closed formula for I was obtained using critical Fredholm determinant asymptotics, so that I is implicitely defined by

$$x(\theta) = \frac{\Psi_1(\theta + \alpha + \beta) + \Psi_1(\theta) - 2\Psi_1(\theta + \alpha)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)} \tag{2.1}$$

and

$$\begin{aligned} I(x(\theta)) = {} &\frac{\Psi_1(\theta + \alpha + \beta) - \Psi_1(\theta + \alpha)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)} \left(\Psi(\theta + \alpha + \beta) - \Psi(\theta) \right) \\ &+ \Psi(\theta + \alpha + \beta) - \Psi(\theta + \alpha), \end{aligned}$$

where Ψ is the digamma function defined as $\Psi(z) = \Gamma'(z)/\Gamma(z)$, $\Psi_1(z) = \Psi'(z)$ is the trigamma function, $z \in \mathbb{C}$, $\theta \in (0, \infty)$, and as θ ranges from 0 to ∞, $x(\theta)$ ranges from 1 to $(\alpha - \beta)/(\alpha + \beta)$. Define $\sigma(\theta)$ by the relation

$$\begin{aligned} 2\sigma(\theta)^3 = {} &\Psi_2(\theta + \alpha) - \Psi_2(\theta + \alpha + \beta) \\ &+ \frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)} \left(\Psi_2(\theta + \alpha + \beta) - \Psi_2(\theta) \right), \end{aligned} \tag{2.2}$$

where $\Psi_2(z) := \Psi_1'(z)$. By Lemma 5.3 of [BC17], the right hand side of (2.2) is positive so that $\sigma(\theta) > 0$. Our first result is an extension of Theorem 1.15 of [BC17] and Theorem 1.2 of [K21] which includes the case $\alpha \geq 0.7$, $\beta > 0$ and $\theta \in (0, 0.5)$. Recall that the GUE Tracy-Widom distribution is defined by $F_{GUE}(x) = \det(I - K_{Ai})_{L^2(x, +\infty)}$, $x \in \mathbb{R}$, where $\det(I - K_{A_i})_{L^2(x, +\infty)}$ is the Fredholm determinant of the Airy kernel K_{Ai} kernel defined as

$$K_{Ai}(u, v) = \frac{1}{(2\pi i)^2} \int_{e^{-2\pi i/3}\infty}^{e^{2\pi i/3}\infty} \int_{e^{-\pi i/3}\infty}^{e^{\pi i/3}\infty} \frac{e^{z^3/3 - zu}}{e^{w^3/3 - wv}} \frac{1}{z - w} dz dw,$$

where the contours for z and w do not intersect.

Our first result is an extension of Theorem 1.15 of [BC17] and the implication of Theorem 1.2 of [K21] for the Beta random walk.

Theorem 2.1. *For all $\alpha > 0$, $\beta > 0$ and $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$, we have that*

$$\lim_{t \to \infty} \mathbb{P}_{\alpha, \beta} \left(\frac{\log\left(P_\omega(t, x(\theta)t)\right) + I(x(\theta))t}{t^{1/3}\sigma(\theta)} \leq y \right) = F_{GUE}(y).$$

It should be noted that the range of target points $x(\theta)$ varies from $x(\theta) = 1$ for $\theta = 0$ to $\frac{\alpha - \beta}{\alpha + \beta}$, which is the speed of the random walk, as $\theta \to \infty$. Since the convergence in the above theorem is for values of θ not larger than $\min\{0.5, 0.69 \times \alpha\}$, and $x(\theta)$ is decreasing with θ, the range of target points is far from the velocity $\frac{\alpha - \beta}{\alpha + \beta}$.

Also, Theorem 2.1 shows in particular that the convergence to the GUE Tracy-Widom distribution occurs for all $\theta \in (0, 0.5)$ and $\beta > 0$ as long as $\alpha \geq 0.73$. The case $\alpha = \beta = 1$, was proven for all $\theta \in (0, 0.5)$ in [BC17] and extended to $\alpha = 1$ and $\beta > 0$ in [K21]. Removing these restrictions to obtain Theorem 2.1 is technically challenging and requires sophisticated estimates involving the polygamma functions. Our bounds on θ and α are not optimal, and the methods presented here could give better estimates.

Our second result shows that the intermediate disorder regime holds for the Beta random walk with parameters tending to ∞. It should be noted that since θ will be fixed, by (2.1), we will also have that $x(\theta) \to 1$. To state the theorem, we introduce the following function which will play a key role in the rate at which α and β can tend to infinity,

$$\mathfrak{g}(x, y) := \frac{y}{x(x + y)} \qquad \text{for } x > 0, y > 0.$$

This function will appear from the difference between the trigamma functions $\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)$. We will assume that the parameters α and β depend on the terminal time t, and we will denote them by (α_t, β_t): this means that the random environment is given by space-time i.i.d. beta distributed random variables of parameters (α_t, β_t). We will assume that

$$\lim_{t \to \infty} \alpha_t = \infty, \qquad \lim_{t \to \infty} \beta_t = \infty \tag{2.3}$$

and that

$$\lim_{t \to \infty} t\mathfrak{g}(\alpha_t, \beta_t) = \infty. \tag{2.4}$$

In the following theorem, the quantities $x(\theta)$ and $\sigma(\theta)$ will be time dependent since they will be evaluated using time-dependent parameters (α_t, β_t) for the beta random walk. It will turn out that under conditions (2.3) and (2.4), we will have that

$$tx(\theta) \sim t - C(\theta)t\mathfrak{g}(\alpha_t, \beta_t), \tag{2.5}$$

where $C(\theta) > 0$ is a constant depending on θ, with the correction term $t\mathfrak{g}(\alpha_t, \beta_t) = o(t)$ but $t\mathfrak{g}(\alpha_t, \beta_t) \to \infty$ as $t \to \infty$. This ensures that the entropy tends to ∞ in the sense that the number of trajectories involved in the probability $P_\omega(t, x(\theta)t)$ will tend to ∞ as $t \to \infty$. Furthermore, we will also have under (2.4) that $\sigma(\theta) \sim C'(\theta)\mathfrak{g}(\alpha_t, \beta_t)$, for some constant $C'(\theta) > 0$, so that

$$t^{1/3}\sigma(\theta) \to \infty.$$

Theorem 2.2. *Consider a family of Beta random walks of parameters (α_t, β_t). Assume that conditions (2.3) and (2.4) are satisfied. Then, for all $\theta \in (0, 0.5)$ we have that*

$$\lim_{t \to \infty} \mathbb{P}_{\alpha_t, \beta_t} \left(\frac{\log\left(P_\omega(t, x(\theta)t)\right) + I(x(\theta))t}{t^{1/3}\sigma(\theta)} \leq y \right) = F_{GUE}(y).$$

In the above theorem, the target point $x(\theta)$ is approaching 1 as $t \to \infty$, through its dependence on time from α_t and β_t (see (2.5)). If we want to achieve a result where $\alpha_t \to \infty$, $\beta_t \to \infty$, but the target point $x(\theta)$ is fixed, we would need to choose also the parameter θ as a function of time, so that $\theta_t \to \infty$ as $t \to \infty$. For the moment, this regime is out of the reach of the methods used in this article because the deformations of contours used are justified only for $\theta < \min\{0.5, 0.69 \times \alpha\}$.

In the particular case in which $\alpha_t = c_1 t^r$, $\beta_t = c_2 t^s$, for some constants $c_1 > 0$ and $c_2 > 0$, and the following condition is satisfied,

$$r + \max(r - s, 0) < 1, \tag{2.6}$$

then assumptions (2.3) and (2.4) are satisfied. If also $r = s$, the law of the Beta distribution with parameter (α_t, β_t) converges to the atom at $c_1/(c_1 + c_2)$.

The proof of Theorems 2.1 and 2.2 is based on an exact formula for the Laplace transform of the probabilities $P_\omega(t, x)$ from where an asymptotic analysis through steep-descent methods of integrals on complex contours is carried out. The asymptotic analysis approach is within the spririt of what is carried out in the context of the log-gamma polymer in [BCR13], the O'Connell-Yor semi-discrete polymer and the continuum random polymer in [BCF14] and for the Beta random walk (or Beta polymer) in [BC17] and [K21] for $\alpha = 1$ and $\beta > 0$. For the proof of Theorem 2.1, we have to do some challenging computations involving the polygamma functions, leading to the required steep-descent estimates and involving the following complex function

$$h(z) := I(x(\theta))z + \frac{1 - x(\theta)}{2} \log\left(\frac{\Gamma(\alpha + z)}{\Gamma(z)}\right) + \frac{1 + x(\theta)}{2} \log\left(\frac{\Gamma(\alpha + z)}{\Gamma(\alpha + \beta + z)}\right), \tag{2.7}$$

which turns out to have a critical point at $z = \theta$. We then do a steep-descent analysis near this critical point, so that the choice of contours near this point will be important. Here we choose them in a way similar to the choice of [K21]. It appears that this approach is robust enough to extend the validity of the convergence to the GUE Tracy-Widom of the second order correction of the large deviation probabilities to the range of parameters stated in Theorem 2.1. Nevertheless, the first limitation of this approach is the impossibility of deforming contours to achieve a circular contour centered at the origin of radius larger than $1/2$, restricting the range of parameters to $\theta \in (0, 1/2)$. The second limitation shows up in the necessity of proving two key inequalities involving h and its derivatives, which ensure the steep-descent properties with the chosen contours, which actually turn out to be false for some values of $\alpha > 0$, $\beta > 0$ and $\theta \in (0, 0.5)$. At any rate, we expect that an approach which enables us to go to values of $\theta \geq 0.5$, might be possible with a clever choice of contours, and could give an extension of Theorem 2.1 to all $\theta > 0$, $\alpha > 0$ and $\beta > 0$.

For Theorem 2.2, we have to keep track of the dependence of several estimates on α_t and β_t, and essentially the time scale changes from t to σt^3, but now σ depends on time through α_t and β_t.

Our final result extends Theorem 2.2 to random walks in space time i.i.d. environments which are close to the Beta random walk. To state it, we need to define the concept of matching moments between two families of environments, one of which has Beta distributions. Given an environment ω, define $\xi = (\xi(x, t))_{x \in \mathbb{Z}, t \geq 0}$ by $\xi_+(x, t) = -\log \omega_1(x, t)$ and $\xi_-(x, t) = -\log \omega_{-1}(x, t)$, so that

$$e^{-\xi_+(x,t)} = \omega_1(x, t) \quad \text{and} \quad e^{-\xi_-(x,t)} = \omega_{-1}(x, t).$$

We will also use the multi-index notation for $t = (t_1, t_2) \in \mathbb{R}^2$,

$$|\alpha| = \alpha_1 + \alpha_2, \qquad t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}.$$

Definition 2.3 (Moment matching condition). *Consider a family of Beta probability measures $(\mathbb{P}_{\alpha_t, \beta_t})_{t \geq 0}$ such that (2.3), (2.4) are satisfied. Assume also that*

$$M_1 := \lim_{t \to \infty} \mathbb{E}_{\alpha_t, \beta_t}[\xi_+(0, t)] = -\lim_{t \to \infty} (\Psi(\alpha_t) - \Psi(\alpha_t + \beta_t))$$

and

$$M_2 := \lim_{t \to \infty} \mathbb{E}_{\alpha_t, \beta_t}[\xi_-(0, t)] = -\lim_{t \to \infty} (\Psi(\beta_t) - \Psi(\alpha_t + \beta_t))$$

exist. Define

$$\bar{\xi}_+(x, t) = \bar{\xi}_1(x, t) := \xi_+(x, t) - M_1 \quad \text{and} \quad \bar{\xi}_-(x, t) = \bar{\xi}_2(x, t) := \xi_-(x, t) - M_2.$$

Let $f(t) : [0, \infty) \to [0, \infty)$. Given a family of probability measures $(\mathbb{P}_t)_{t \geq 0}$ defined on the environmental space Ω (with corresponding expectations $(\mathbb{E}_t)_{t \geq 0}$), we say that it matches moments up to order k at rate f with the family $(\mathbb{P}_{\alpha_t, \beta_t})_{t \geq 0}$, if we have that for all $x \in \mathbb{Z}$ and $t \geq 1$,

$$\left| \mathbb{E}_t[\bar{\xi}^\alpha(x, t)] - \mathbb{E}_{\alpha_t, \beta_t}[\bar{\xi}^\alpha(x, t)] \right| \leq f(t), \qquad \text{for } |\alpha| \leq k,$$

and

$$\left| \mathbb{E}_t[\bar{\xi}^\alpha(x, t)] \right| \leq f(t), \qquad \text{for } |\alpha| = k.$$

We can now state the third result of this article. Again, the quantities $x(\theta)$ and $\sigma(\theta)$ will be time dependent through the time-dependent parameters (α_t, β_t) for the beta random walk. To state this theorem, we define for time $t \geq 0$, the set

$$D_t := \{z \in \mathbb{Z}^2 : |z|_1 \leq t\}.$$

Note that the convex hull of the range of the Dirichlet random walk at time t in $\mathbb{Z}^2$ is D_t. Define its boundary

$$\partial D_t := \{z \in \mathbb{Z}^2 : |z|_1 = t\}.$$

We then can define $\partial D_t^{+,+} := \{z \in \partial D_t : z_1 \geq 0, z_2 \geq 0\}$, $\partial D_t^{+,-} := \{z \in \partial D_t : z_1 \geq 0, z_2 \leq 0\}$, $\partial D_t^{-,+} := \{z \in \partial D_t : z_1 \leq 0, z_2 \geq 0\}$ and $\partial D_t^{-,-} := \{z \in \partial D_t : z_1 \leq 0, z_2 \leq 0\}$. We have that $\partial D_t = \partial D_t^{+,+} \cup \partial D_t^{+,-} \cup \partial D_t^{-,+} \cup \partial D_t^{-,-}$. For $t \geq 0$ and $y \in (0, 1)$ consider the set

$$A_{t,y} := \{z \in \mathbb{Z}^2 : |z|_1 = t, z_1 \geq (t + y)/2, z_2 \geq 0\} \subset \partial D_t^{+,+}.$$

See Figure 1 to visualize the sets D_t, ∂D_t, $\partial D_t^{+,+}$, $\partial D_t^{+,-}$, $\partial D_t^{-,+}$, $\partial D_t^{-,-}$ and $A_{t, x(\theta)t}$. We will adopt the notation

$$P_{0,\omega}(t, y) = P_{0,\omega}(X_t \in A_{t,y}). \tag{2.8}$$

Theorem 2.4. *Consider a family of parameters (α_t, β_t) that satisfy (2.3) and (2.4). Let $\theta \in (0, 0.5)$. Let $(\mathbb{P}_t)_{t \geq 0}$ be a family of environmental laws which matches moments up to order k at rate $\alpha_t^{-\lceil \frac{k}{2} \rceil}$ with $(\mathbb{P}_{\alpha_t, \beta_t})_{t \geq 0}$ and such that*

$$\lim_{t \to \infty} \frac{t^2 \alpha_t^{-\lceil \frac{k}{2} \rceil}}{\sigma(\theta) t^{1/3}} = 0. \tag{2.9}$$

Then

$$\lim_{t \to \infty} \mathbb{P}_t \left(\frac{\log(P_{0,\omega}(t, x(\theta)t)) + I(x(\theta))t}{t^{1/3} \sigma(\theta)} \leq y \right) = F_{GUE}(y).$$

Our approach to prove Theorem 2.4 is to express the weak convergence in terms of the convergence of a large enough family of expectations involving smooth enough functions, and then making a Taylor expansion, and apply Theorem 2.2. This is similar to what is presented in [KQ18], although here we have to deal with perturbations which involve two parameters instead of one.

Theorem 2.4 has the following corollary for random walks in Dirichlet random environment in dimension $d = 2$. As in Theorem 2.4, the quantities $x(\theta)$ and $\sigma(\theta)$ will be time dependent through time-dependent parameters (α_t, β_t) and $P_{0,\omega}(t, x(\theta)t)$ is defined in (2.8).

Corollary 2.5. *Consider a family of random walks in Dirichlet environment on $\mathbb{Z}^2$ of parameters $(\alpha_t)_{t \geq 0}$ with $\alpha_t = (\alpha_{t,i})_{i \in \{1,\ldots,4\}}$ and $\alpha_{t,1} = \alpha_{t,2} = t^r$, $\alpha_{t,3} \leq t^{-p}$ and $\alpha_{t,4} \leq t^{-p}$ for $t \geq 1$, for some $r \in (0, 1)$ and*

$$p \geq r\left\lceil \frac{5}{3r} - \frac{1}{3}\right\rceil - r. \tag{2.10}$$

Then for all $\theta \in (0, 0.5)$ we have that

$$\lim_{t \to \infty} \mathbb{P}_{\alpha_t}\left(\frac{\log\left(P_{0,\omega}(t, x(\theta)t)\right) + I(x(\theta))t}{t^{1/3}\sigma(\theta)} \leq y\right) = F_{GUE}(y).$$
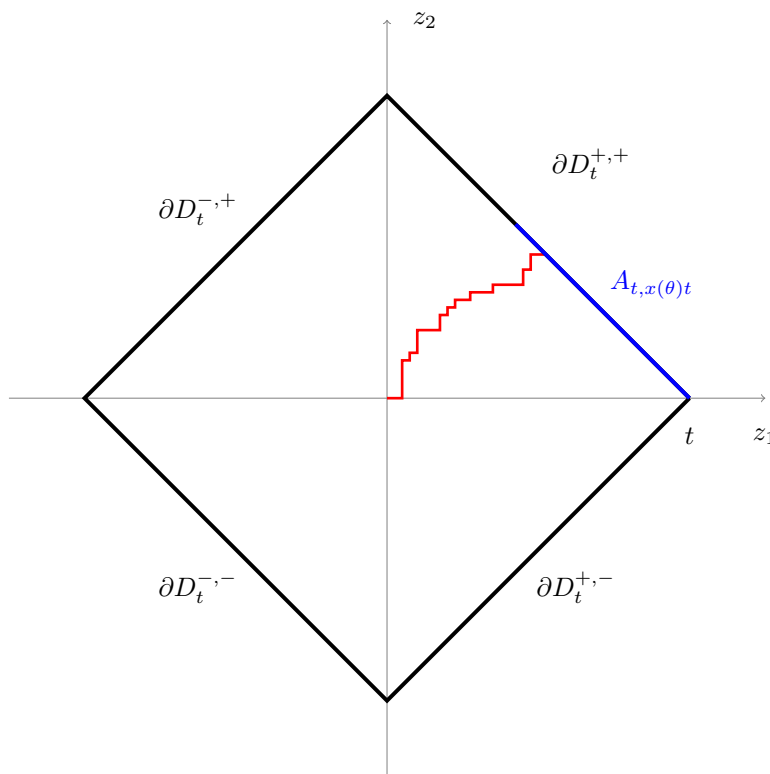

Figure 1: The set D_t and its boundary ∂D_t divided into the four quadrants, the large deviation event $A_{t,x(\theta)t}$ (in blue) of Theorem 2.4 and Corollary 2.5 and a typical directed trajectory (in red) of a random walk starting from 0. In the case of Corollary 2.5, the length of the blue segment $A_{t,x(\theta)t}$ grows like $O(t^{1-r})$ as $t \to \infty$.

Corollary 2.5 is a particular case of perturbations of random walks in i.i.d. random environment in dimension $d = 2$ which are perturbations of the Beta random walk (see

Figure 1), and can be derived from Theorem 2.4 comparing the law of the Dirichlet random walk (for directed trajectories) with the law of a Beta random walk with specific parameters (α_t, β_t) (see Lemma 4.3 for the choice of these parameters). Its proof requires obtaining good enough estimates for the moments of Dirichlet random variables. It should be emphasized that even though the result concerns a random walk in a static i.i.d. environment, the random walk is not allowed to backtrack simply because the event $\{X_t \in A_{t,x(\theta)t}\}$ contains only directed trajectories. On the other hand, if $\alpha_{t,3}$ and $\alpha_{t,4}$ decay to 0 exponentially fast, it would be possible to state a variation of this corollary where the large deviation event includes trajectories which could backtrack. Nevertheless, in a sense, this situation would correspond to a case in which these trajectories do not contribute at all to the large deviations, nor to its correction. It might be possible to extend the corollary to large deviation events which genuinely allow backtracking to some extent, but this would be a task for a separate article and it would require different methods besides the moment matching technique.

In Section 3, we present the proof of Theorems 2.1 and 2.2 in a unified way. In Section 4, we will prove Theorem 2.4. Throughout the rest of this article we will adopt the notation $C_1, C_2, \ldots$, and $c_1, c_2, \ldots$, to denote constants, which in general might depend on some of the parameters involved. In particular, they might depend on α and β, and it will be important to keep track of this dependence in order to prove Theorem 2.2.

3 Proof of Theorems 2.1 and 2.2

To prove Theorems 2.1 and 2.2 we will do a careful steep-descent analysis, whose starting point is the connection between the Laplace transform and the distribution functions. This connection is explained in Section 3.1. In Section 3.2, we present an exact determinantal formula derived in [BC17], which will then lend itself to do an asymptotic analysis. In Section 3.3 we explain the general strategy to do the aymptotic analysis of the determinantal formula to prove Theorems 2.1 and 2.2, summarizing the main step as Proposition 3.4. In Section 3.5, several estimates will be proven showing that the contours involved have the steep-descent property, starting from Lemmas 3.7 and 3.8. This will be applied in Section 3.5 to obtain the necessary bound on the integrands over these complex contours. In Section 3.7, Proposition 3.4 will be proven. Finally in Section 3.9, Lemmas 3.7 and 3.8 are proven.

3.1 Connection to the Laplace transform

The main conclusion of this section will be the following lemma showing how to obtain the limiting distribution function of the fluctuations of $\log P_\omega(t, x(\theta)t)$ through its Laplace transform. Part (i) of the Lemma is stated in [BC17], although it is a standard method already used in similar contexts by other authors (se for example [BCR13] or [BCF14]).

For $y \in \mathbb{R}$ define

$$u(y) := -e^{tI(x(\theta))-t^{1/3}\sigma(\theta)y}. \tag{3.1}$$

Our question is under what assumptions on the parameters $(\alpha_t, \beta_t)_{t \geq 0}$ is the following equation valid:

$$\lim_{t\to\infty} \mathbb{E}_{\alpha_t, \beta_t}\left[e^{u(y)P_\omega(t, x(\theta)t)}\right]$$
$$= \lim_{t\to\infty} \mathbb{P}_{\alpha_t, \beta_t}\left(\frac{\log\left(P_\omega(t, x(\theta)t)\right) + I(x(\theta))t}{t^{1/3}\sigma(\theta)} \leq y\right), \tag{3.2}$$

The statement (3.2) says that also the right-hand side limit exists. But we would also like to ensure that this limit corresponds to some convergence in distribution. We will adopt the convention that for the case of a Beta distribution with fixed parameters (α, β), $\alpha_t = \alpha$ and $\beta_t = \beta$ for all $t \geq 0$.

Let us explain how we will prove (3.2). We will need the following lemma whose proof we omit (see for example [BC14]).

Lemma 3.1. *Consider a sequence of functions $(f_t)_{t \in \mathbb{N}}$ mapping $\mathbb{R} \to [0,1]$ such that*

- *(i) For each t, $f_t(x)$ is strictly decreasing in x.*
- *(ii) For each t, $\lim_{x \to -\infty} f_t(x) = 1$ and $\lim_{x \to \infty} f_t(x) = 0$.*
- *(iii) For each $\delta > 0$, on $\mathbb{R} \backslash [-\delta, \delta]$, f_t converges uniformly to $1(x \leq 0)$.*

Define the r-shift of f_t as $f_t^r(x) = f_t(x - r)$. Consider a sequence of random variables X_t such that for each $r \in \mathbb{R}$,

$$\lim_{t \to \infty} E[f_t^r(X_t)] = p(r),$$

and assume that $p(r)$ is a continuous probability distribution function. Then X_t converges weakly in distribution to a random variable X which is distributed according to $P(X \leq r) = p(r)$.

We will apply Lemma 3.1 to prove (3.2) making a specific choice of functions and random variables. Let $\theta > 0$. Consider the family of functions $(f_t)_{t \in \mathbb{N}}$ defined by

$$f_t(w) := e^{-e^{wt^{1/3}\sigma(\theta)}}. \tag{3.3}$$

In the case in which α, β and θ are fixed and satisfy the assumptions $\alpha > 0$, $\beta > 0$ and $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$, it is obvious that this family satisfies conditions (i), (ii) and (iii) of Lemma 3.1. For the case in which $(\alpha_t)_{t \geq 0}$ and $(\beta_t)_{t \geq 0}$ satisfy (2.3) and (2.4), by parts (iv) and (v) of Corollary 3.6 (which is stated and proven in Section 3.4) we have that

$$\lim_{t \to \infty} t\sigma(\theta)^3 = \infty, \tag{3.4}$$

so that (3.3) still satisfies (i), (ii) and (iii) of Lemma 3.1. Let $(X_t)_{t \in \mathbb{N}}$ be defined by

$$X_t = \frac{\log P_\omega(t, x(\theta)t) + I(x(\theta))t}{t^{1/3}\sigma(\theta)}. \tag{3.5}$$

Consider now the statement

$$\lim_{t \to \infty} E[f_t^r(X_t)] = \lim_{t \to \infty} \mathbb{E}_{\alpha_t, \beta_t} \left[e^{u(r)P_\omega(t, x(\theta)t)} \right] = \lim_{t \to \infty} \mathbb{P}_{\alpha_t, \beta_t}(X_t \leq r). \tag{3.6}$$

The following proposition summarizes what we have explained in relation to (3.2), and will be proven below.

Proposition 3.2. *Consider the family of functions $(f_t)_{t \in \mathbb{N}}$ defined in (3.3) and the random variables $(X_t)_{t \in \mathbb{N}}$ defined in (3.5).*

- *(i) For all $\alpha > 0$, $\beta > 0$ and $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$, we have that (3.6) is satisfied, and the limit exists for all r and $(X_t)_{t \in \mathbb{N}}$ converges in distribution to some random variable X.*
- *(ii) Let $(\alpha_t)_{t \geq 0}$ and $(\beta_t)_{t \geq 0}$ satisfy (2.3) and (2.4). Then, (3.6) is satisfied, the limit exists for all r and $(X_t)_{t \in \mathbb{N}}$ converges in distribution to some random variable X.*

The first equality in (3.6) under the assumption of parts (i) or (ii) of Proposition 3.2 is immediate from the definitions. The second statement of parts (i) and (ii) Proposition 3.2 will be proven in Section 3.7.

3.2 Determinantal formula for the Laplace transform

The second step in the proof of Theorem 2.1 will be the following exact formula for the Laplace transform of $P(t,x)$ proved in [BC17]. One way of deriving this formula is to obtain integral formulas for the moments of $P(t,x)$ using a variant of the Bethe ansatz and a non-commutative binomial identity, throug a recurrent system of equations for the moments of $P(t,x)$.

Theorem 3.3 (Barraquand-Corwin, 2017). *For $u \in \mathbb{C}\backslash\mathbb{R}_{>0}$, fix $t \in \mathbb{Z}_{\geq 0}$, $x \in \{-t,\ldots,t\}$ with the same parity, and $\alpha, \beta > 0$. Then one has*

$$E\left[e^{uP(t,x)}\right] = \det\left(I - K_u^{RW}\right)_{L^2(C_0)},$$

where C_0 is a small positively oriented circle containing 0 but not $-\alpha-\beta$ nor -1, and $K_u^{RW}: L^2(C_0) \to L^2(C_0)$ is defined by its integral kernel

$$K_u^{RW}(v,v') = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)}(-u)^s \frac{g^{RW}(v)}{g^{RW}(v+s)}\frac{ds}{s+v-v'}, \qquad (3.7)$$

where

$$g^{RW}(v) = \left(\frac{\Gamma(v)}{\Gamma(\alpha+v)}\right)^{(t-x)/2}\left(\frac{\Gamma(\alpha+\beta+v)}{\Gamma(\alpha+v)}\right)^{(t+x)/2}\Gamma(v).$$

3.3 Asymptotic analysis of the Laplace transform

Here we will show how to finish the proof of Theorems 2.1 and 2.2, proving the convergence of the Laplace transform of the normalized fluctuations of $\log P_\omega(t,x(\theta)t)$ to the GUE Tracy-Widom distribution. From Proposition 3.2, we can see that to prove these theorems, we can use the determinantal formula of Theorem 3.3. The asymptotic steep-descent analysis that will be later used will be similar to the proof of Theorem 5.2 of [BC17] or Theorem 2.1 of [K21]. Nevertheless, to prove the necessary steep-descent properties, a technical analysis of high complexity involving the polygamma functions must be executed, both to deal with arbitrary values of $\alpha > 0$ and $\beta > 0$ in Theorem 2.1 (specially in order to achieve the range $\theta \in (0,0.5)$ of validity for all $\alpha \geq 0.75$ and $\beta > 0$), and to deal with values of α and β tending to ∞ in Theorem 2.2.

We first rewrite the kernel $K_u^{RW}(v,v')$ choosing $u = u(y)$ as in (3.1) so that

$$K_u^{RW}(v,v') = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{\pi}{\sin(\pi(z-v))}e^{t(h(z)-h(v))-t^{1/3}\sigma(\theta)y(z-v)}\frac{\Gamma(v)}{\Gamma(z)}\frac{dz}{z-v'}. \qquad (3.8)$$

where we recall the definition of h in (2.7). It can be checked that θ is a critical point for the function h, so that $h'(\theta) = h''(\theta) = 0$. We hence follow the steepest-descent method deforming the integration contours so that they go across this critical point. As in [BC17], we deform the contour C_0 in Theorem 3.3 (the Fredholm contour) to the contour

$$C_\theta := \{z \in \mathbb{C} : |z| = \theta\}, \qquad (3.9)$$

defined as the circle centered at 0 with radius θ, and the vertical line in $\mathbb{C}$ passing through $1/2$ in the definition of the kernel (3.7) to

$$D_\theta = \{\theta + iy : y \in \mathbb{R}\}. \qquad (3.10)$$

There is no problem in doing this as long as we avoid the poles, which happens if

$$\theta < \min\{\alpha + \beta, 1/2\}. \tag{3.11}$$

Let also for $\epsilon > 0$, $B(\theta, \epsilon)$ be the ball of radius ϵ centered at θ in $\mathbb{C}$,

$$C_\theta^\epsilon := C_\theta \cap B(\theta, \epsilon), \tag{3.12}$$

the part of the contour C_θ inside $B(\theta, \epsilon)$, while

$$D_\theta^\epsilon := D_\theta \cap B(\theta, \epsilon), \tag{3.13}$$

the part of the contour D_θ inside $B(\theta, \epsilon)$. As it is standard in several results proving convergence to the GUE Tracy-Widom distribution (see for example [BC17] or [K21]), the proof will be based in steep-descent methods for the complex contours. We summarize this in a proposition that we state below. To state it, we will modify the the contour C_θ^ϵ, to a contour which is a piece of a wedge, which in the limit becomes a full wedge defined by a certain angle ϕ. Let us explain how to choose ϕ. For $L > 0$ define the contour for arbitrary $\phi \in (\pi/6, \pi/2)$,

$$W_\theta^L := \left\{ \theta + |y| e^{i(\pi - \phi)\mathrm{sgn}(y)} : y \in [-L, L] \right\}.$$

Note that C_θ^ϵ is an arc of circle which crosses θ vertically. We now choose L and ϕ so that the endpoints of W_θ^L and of C_θ^ϵ coincide. Then note that for ϵ small enough we can replace C_θ^ϵ by W_θ^L. We will also introduce the extension of the piece of wedge W_θ^L outside of the ball $B(\theta, \epsilon)$ through the contour

$$C_\theta^{\epsilon,+} := C_\theta - B(\theta, \epsilon),$$

defined as (see Figure 2)

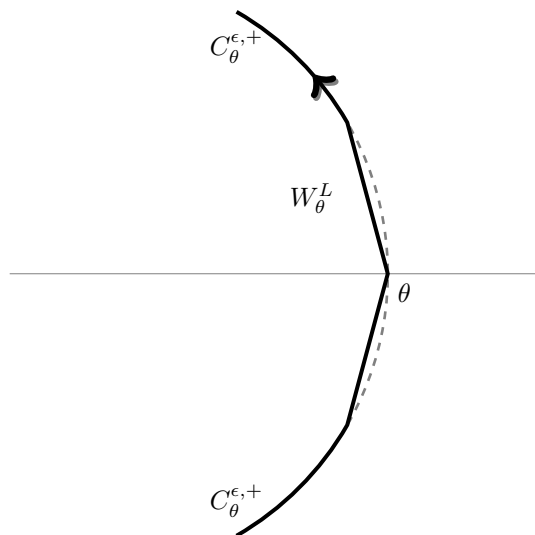$$V_\theta^\epsilon := W_\theta^L \cup C_\theta^{\epsilon,+}.$$

Figure 2: The contour V_θ^ϵ composed by the union of the contours $C_\theta^{\epsilon,+}$ and W_θ^L, and in dashed line the contour C_θ^ϵ.

With the choice of ϕ explained above, we also define

$$W_\theta^\infty := \left\{ \theta + |y|e^{i(\pi - \phi)\mathrm{sgn}(y)} : y \in \mathbb{R} \right\}. \tag{3.14}$$

We will need to define for $y \in \mathbb{R}$ the kernel K_y,

$$K_y(w, w') := \frac{1}{2\pi i} \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \frac{1}{(z - w')(w - z)} \frac{e^{z^3/3 - yz}}{e^{w^3/3 - yw}} dz,$$

where the contour for z is a wedge-shaped contour constituted of two rays going to infinity in the directions $e^{-i\pi/3}$ and $e^{i\pi/3}$ that do not intersect W_θ^∞.

Proposition 3.4. *Consider the equality*

$$\lim_{t \to \infty} \det \left(I - K_u^{RW} \right)_{L^2(C_\theta)} = \det(I + K_y)_{L^2(W_\theta^\infty)}. \tag{3.15}$$

  (i) For all $\alpha > 0$, $\beta > 0$ and $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$, we have that (3.15) is satisfied.

  (ii) Assume that $(\alpha_t)_{t \geq 0}$ and $(\beta_t)_{t \geq 0}$ satisfy (2.3) and (2.4). Then for all $\theta \in (0, 0.5)$, (3.15) is satisfied.

The proof of Proposition 3.4 will proceed in two steps. Firstly, we will truncate all the integrations to contours at a distance ϵ of θ for some $\epsilon \in (0, \theta/2)$, showing that

$$\lim_{t \to \infty} \det \left(I - K_u^{RW} \right)_{L^2(C_\theta)} = \lim_{t \to \infty} \det \left(I - K_{y,\epsilon}^{RW} \right)_{L^2(C_\theta^\epsilon)},$$

where

$$
\begin{aligned}
&K_{y,\epsilon}^{RW}(v, v') \\
&= \frac{1}{2\pi i} \int_{D_\theta^\epsilon} \frac{\pi}{\sin(\pi(z - v))} e^{t(h(z) - h(v)) - t^{1/3}\sigma(\theta)y(z - v)} \frac{\Gamma(v)}{\Gamma(z)} \frac{1}{z - v'} dz.
\end{aligned}
\tag{3.16}
$$

Secondly, we will prove that

$$\lim_{t \to \infty} \det \left(I - K_{y,\epsilon}^{RW} \right)_{L^2(C_\theta^\epsilon)} = \det(I + K_y)_{L^2(C)}.$$

The details of this proof will be given in Section 3.7. From Proposition 3.4, Theorem 2.1 now follows as in [BC17] from the identity

$$\det(I + K_y)_{L^2(C)} = \det(I - K_{Ai})_{L^2(y, +\infty)},$$

valid for $y \in \mathbb{R}$. It remains to prove Proposition 3.4. We will first prove steep-descent estimates needed for the proof of Theorem 2.1 and quantitative versions of them needed for the proof of Theorem 2.2.

3.4 Preliminary estimates involving the polygamma functions

Here we will derive several estimates which will eventually prove the necessary steep-descent properties of the integrals along the complex contours in the Fredholm determinants and its kernel.

Recall that the polygamma function of order k, for $k \geq 1$, is defined as

$$\Psi_k(z) := \frac{d^k}{dz^k} \Psi(z).$$

We start with several estimates involving the polygamma functions.

Lemma 3.5. *The following inequalities are satisfied.*

(i) *For all $k \geq 1$ and $x > 0$,*

$$k! \left(\frac{1}{x^{k+1}} + \frac{1}{k} \frac{1}{(x+1)^k} \right) \leq (-1)^{k+1} \Psi_k(x) \leq k! \left(\frac{1}{x^{k+1}} + \frac{1}{k} \frac{1}{x^k} \right). \qquad (3.17)$$

(ii) *For every $x > 0$, $y > 0$ and $k \geq 1$ we have that*

$$
\begin{aligned}
&k! \mathfrak{g}(x+1,y) \left(\frac{1}{x^k} + \frac{1}{k} \frac{1}{(x+1)^{k-1}} \right) \\
&\leq k! \mathfrak{g}(x,y) \frac{1}{x^k} + (k-1)! \frac{1}{(x+1)^{k-1}} \mathfrak{g}(x+1,y) \\
&\leq (-1)^{k+1} (\Psi_k(x) - \Psi_k(x+y)) \\
&\leq (k+1)! \mathfrak{g}(x,y) \left(\frac{1}{x^k} + \frac{1}{k+1} \frac{1}{x^{k-1}} \right).
\end{aligned}
\qquad (3.18)
$$

Proof. First note that for all $k \geq 1$,

$$\Psi_k(z) = (-1)^{k+1} k! \sum_{j=0}^{\infty} \frac{1}{(z+j)^{k+1}}. \qquad (3.19)$$

Proof of part (i). From (3.19) we have that

$$(-1)^{k+1} \Psi_k(x) = k! \left(\frac{1}{x^{k+1}} + \sum_{j=1}^{\infty} \frac{1}{(x+j)^{k+1}} \right) \leq k! \left(\frac{1}{x^{k+1}} + \frac{1}{k} \frac{1}{x^k} \right).$$

Similarly,

$$(-1)^{k+1} \Psi_k(x) = k! \left(\frac{1}{x^{k+1}} + \sum_{j=1}^{\infty} \frac{1}{(x+j)^{k+1}} \right) \geq k! \left(\frac{1}{x^{k+1}} + \frac{1}{k} \frac{1}{(x+1)^k} \right)$$

Proof of part (ii). From the upper bound of part (i) note that

$$
\begin{aligned}
(-1)^{k+1} (\Psi_k(x) - \Psi_k(x+y)) &= \int_x^{x+y} (-1)^{k+2} \Psi_{k+1}(u) du \\
&\leq (k+1)! \int_x^{x+y} \left(\frac{1}{u^{k+2}} + \frac{1}{k+1} \frac{1}{u^{k+1}} \right) du \\
&\leq (k+1)! \mathfrak{g}(x,y) \left(\frac{1}{x^k} + \frac{1}{k+1} \frac{1}{x^{k-1}} \right).
\end{aligned}
$$

For the lower bound, first note that

$$\int_x^{x+y} u^{-(k+2)} du = \frac{1}{k+1} \left(\frac{1}{x^{k+1}} - \frac{1}{(x+y)^{k+1}} \right) \geq \frac{1}{k+1} \frac{1}{x^k} \mathfrak{g}(x,y).$$

Hence,

$$(-1)^{k+1}\left(\Psi_k(x) - \Psi_k(x+y)\right) \geq (k+1)! \int_x^{x+y} \left(\frac{1}{u^{k+2}} + \frac{1}{k+1}\frac{1}{(u+1)^{k+1}}\right) du$$

$$\geq (k+1)! \left(\frac{1}{k+1}\frac{1}{x^k}\mathfrak{g}(x,y) + \frac{1}{k(k+1)}\frac{1}{(x+1)^{k-1}}\mathfrak{g}(x+1,y)\right)$$

$$\geq k!\mathfrak{g}(x+1,y)\left(\frac{1}{x^k} + \frac{1}{k}\frac{1}{(x+1)^{k-1}}\right). \qquad \Box$$

We will now apply Lemma 3.5 to derive several crucial properties involving the function h [cf. (2.7)], which plays a central role in the steep-descent analysis that will be made. For the record, we write the following expression for $h^{(k)}(\theta)$, valid when $k \geq 2$,

$$h^{(k)}(\theta) = \Psi_{k-1}(\theta+\alpha) - \Psi_{k-1}(\theta+\alpha+\beta)$$
$$+ \frac{\Psi_1(\theta+\alpha) - \Psi_1(\theta+\alpha+\beta)}{\Psi_1(\theta) - \Psi_1(\theta+\alpha+\beta)}\left(\Psi_{k-1}(\theta+\alpha+\beta) - \Psi_{k-1}(\theta)\right). \qquad (3.20)$$

Corollary 3.6. *The following estimates are satisfied,*

(i) *For all $\alpha > 0$ and $\beta > 0$,*

$$\Psi_1(\alpha) - \Psi_1(\alpha+\beta) \geq \mathfrak{g}(\alpha+1,\beta).$$

(ii) *For all $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $k \geq 1$, we have that*

$$|\Psi_k(\theta+\alpha) - \Psi_k(\theta+\alpha+\beta)| \leq (k+1)!\mathfrak{g}(\alpha,\beta)\frac{1+\theta^{-1}}{\theta^{k-1}}.$$

(iii) *For all $\alpha > 0$, $\beta > 0$, $\theta \in (0,0.5)$ and $k \geq 3$ we have that*

$$|h^{(k)}(\theta)| \leq 64\mathfrak{g}(\alpha,\beta)\frac{k!}{\theta^{k+2}}.$$

(iv) *For all $\alpha_0 > 0$ there is a $\theta_0 > 0$ such that for $\alpha \geq \alpha_0$, $\beta > 0$ and $\theta \in (0,\theta_0)$ one has that*

$$\frac{\sigma^3(\theta)}{2\theta} + \frac{h^{(4)}(\theta)}{4!} \geq C_1(\theta,\alpha_0)\mathfrak{g}(\alpha,\beta).$$

where $C_1(\theta,\alpha_0) > 0$ depends only on θ and α_0. Furthermore $\theta_0(\alpha_0)$ can be chosen so that $\lim_{\alpha_0 \to \infty}\theta_0 = \infty$.

(v) *For all $\alpha > 0$, $\beta > 0$ and $\theta > 0$ we have that*

$$\sigma^3(\theta) = \frac{1}{2}h^{(3)}(\theta) > 0 \quad \text{and} \quad h^{(4)}(\theta) < 0. \qquad (3.21)$$

Proof. Proof of part (i). This is immediate from the lower bound of part (ii) of Lemma 3.5.

Proof of part (ii). From the upper bound of part (ii) of Lemma 3.5 note that

$$|\Psi_k(\theta+\alpha) - \Psi_k(\theta+\alpha+\beta)| \leq (k+1)!\mathfrak{g}(\theta+\alpha,\beta)\frac{1+\theta^{-1}}{\theta^{k-1}}.$$

Proof of part (iii). From part (ii) of Lemma 3.5 note that

$$g(\theta + 1, \alpha + \beta)\left(\frac{1}{\theta} + 1\right) \le \Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta) \le g(\theta, \alpha + \beta)\left(\frac{2}{\theta} + 1\right). \qquad (3.22)$$

Combining this with part (ii), we have that for $\theta \in (0, 0.5)$,

$$\frac{|\Psi_{k-1}(\theta + \alpha + \beta) - \Psi_{k-1}(\theta)|}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)} \le (k!)\frac{1}{\theta^{k-1}}\frac{\mathfrak{g}(\theta, \alpha + \beta)}{\mathfrak{g}(\theta + 1, \alpha + \beta)}$$
$$\le (k!)\theta^{-(k-1)}4\theta^{-2} = 4(k!)\theta^{-(k+1)}. \qquad (3.23)$$

On the other hand, using this bound and (3.20), we get that

$$|h^{(k)}(\theta)| = |\Psi_{k-1}(\theta + \alpha) - \Psi_{k-1}(\theta + \alpha + \beta)|$$
$$+ \frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)}|\Psi_{k-1}(\theta + \alpha + \beta) - \Psi_{k-1}(\theta)|$$
$$\le |\Psi_{k-1}(\theta + \alpha) - \Psi_{k-1}(\theta + \alpha + \beta)|$$
$$+ \frac{\Psi_1(\alpha) - \Psi_1(\alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)}|\Psi_{k-1}(\theta + \alpha + \beta) - \Psi_{k-1}(\theta)|$$
$$\le (k-1)!\mathfrak{g}(\alpha, \beta)\frac{1}{\theta^{k-1}} + 32\mathfrak{g}(\alpha, \beta)\frac{k!}{\theta^{k+2}} \le 64\mathfrak{g}(\alpha, \beta)\frac{k!}{\theta^{k+2}}, \qquad (3.24)$$

where in the second to last inequality we have used (3.23) and part (ii) again. This proves part (iii).

Proof of part (iv). Dropping out the positive terms of the first two lines of the above display we obtain that,

$$\frac{\sigma^3(\theta)}{2\theta} + \frac{h^{(4)}(\theta)}{4!} \ge \frac{1}{2\theta}\left(\Psi_2(\theta + \alpha) - \Psi_2(\theta + \alpha + \beta)\right)$$
$$+ \frac{1}{4!}\frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)}\Psi_3(\theta + \alpha + \beta)$$
$$- \frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)}\Xi(\theta)$$
$$\ge \frac{1}{2\theta}\left(\Psi_2(\alpha) - \Psi_2(\alpha + \beta)\right) - \frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)}\Xi(\theta)$$
$$\ge \frac{1}{2\theta}\left(\Psi_2(\alpha) - \Psi_2(\alpha + \beta)\right) + \frac{\Psi_1(1 + \alpha) - \Psi_1(1 + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)}\left(-\Xi(\theta)\right)$$
$$\ge -2\mathfrak{g}(\alpha, \beta)\frac{1}{\alpha} + \frac{\mathfrak{g}(2 + \alpha, \beta)((1 + \alpha)^{-1}) + 1)}{\mathfrak{g}(1, \alpha + \beta)(\theta^{-1} + 1)}\left(-\Xi(\theta)\right),$$

where for the last inequality we have used parts (i) and (ii) of Lemma 3.5, and we have used the fact that

$$\Xi(\theta) = -\sum_{n=0}^{\infty}\frac{2\theta + 4n}{4\theta(\theta + n)^4} < 0.$$

Now assume that $\alpha_0 > 0$ is fixed and $\alpha \ge \alpha_0$, $\beta > 0$ and $\theta > 0$. In this case

$$\frac{\mathfrak{g}(2 + \alpha, \beta)}{\mathfrak{g}(\alpha, \beta)} \ge \left(\frac{\alpha_0}{2 + \alpha_0}\right)^2 =: C_2.$$

We then have that

$$\frac{\sigma^3(\theta)}{2\theta} + \frac{h^{(4)}(\theta)}{4!} \geq \mathfrak{g}(\alpha, \beta)\left(-2\frac{1}{\alpha_0} + 2C_2\frac{1}{1+\theta}\left(-\theta\Xi(\theta)\right)\right).$$

Since $-\theta\Xi(\theta) \to \infty$ as $\theta \to 0_+$, from this inequality it is clear that there is a $\theta_0(\alpha_0) > 0$ (depending only on α_0), such that for all $\alpha \geq \alpha_0$, $\beta > 0$ and $\theta \in (0, \theta_0)$, the right-hand side is positive. Furthermore, we can also easily check $\theta_0(\alpha_0)$ can be chosen so that $\lim_{\alpha_0 \to \infty} \theta_0 = \infty$.

Proof of part (v). The first inequality of (3.21) follows from part (iv) after we prove that $h^{(4)}(\theta) < 0$. So we will just prove that $h^{(4)}(\theta) < 0$. Recall the expression (3.20) for $h^{(4)}(z)$. From this expression, we see that it is enough to show that

$$\frac{\Psi_3(\theta + \alpha) - \Psi_3(\theta + \alpha + \beta)}{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)} < \frac{\Psi_3(\theta) - \Psi_3(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)}.$$

This is equivalent to $G = \frac{\Psi_3}{\Psi_1}$ being strictly convex on $(0, \infty)$, that is,

$$G'' = \frac{\Psi_3''\Psi_1' - \Psi_3'\Psi_1''}{(\Psi_1')^3} \circ \Psi_1^{-1} > 0$$

or, as $\Psi_1 < 0$,

$$(\Psi_3''\Psi_1' - \Psi_3'\Psi_1'')(x) = 48\left(3\sum_{n=0}^{\infty}\frac{1}{x_n^5}\sum_{n=0}^{\infty}\frac{1}{x_n^4} - 5\sum_{n=0}^{\infty}\frac{1}{x_n^6}\sum_{n=0}^{\infty}\frac{1}{x_n^3}\right) < 0,$$

where we used the notation $x_n = x + n$. But this is true, since

$$\sum_{n=0}^{\infty}\frac{1}{x_n^6}\sum_{n=0}^{\infty}\frac{1}{x_n^3} - \sum_{n=0}^{\infty}\frac{1}{x_n^5}\sum_{n=0}^{\infty}\frac{1}{x_n^4} > 0,$$

as can be seen by writing the m-n products (the $m = n$ terms are cero),

$$\frac{1}{x_m^6}\frac{1}{x_n^3} + \frac{1}{x_n^6}\frac{1}{x_m^3} - \frac{1}{x_m^5}\frac{1}{x_n^4} - \frac{1}{x_n^5}\frac{1}{x_m^4} = \frac{1}{x_n^6 x_m^6}(x_m - x_n)(x_m^2 - x_n^2).$$

3.5 Steep-descent estimates

Let us now state three lemmas which will give the steepest-descent properties of h on C_θ and D_θ. The first one, is a quantitative version of Lemma 5.5 of [BC17] which had the restriction $\alpha = \beta = 1$, extending it for all values of α and β, at the cost of having to choose θ small enough and is one of the main challenges in the proof of Theorem 2.1.

Lemma 3.7. *For all $\alpha > 0$, $\beta > 0$, $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$ and $\phi \in [0, 2\pi]$, one has that*

$$\mathrm{Re}\left(i\theta e^{i\phi}h'\left(\theta e^{i\phi}\right)\right) > 0. \tag{3.25}$$

Furthermore, there is an $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$, $\beta > 0$, $\phi \in [0, 2\pi]$ and $\theta \in (0, 0.5)$,

$$\mathrm{Re}\left(i\theta e^{i\phi}h'\left(\theta e^{i\phi}\right)\right) \geq C_3\theta^2 \sin\phi(1 - \cos\phi)\mathfrak{g}(\alpha + 1, \beta),$$

for some constant $C_3 > 0$ depending only on α, but increasing on α.

Numerical computations suggest (3.25) of Lemma 3.7 is false for some values of $\alpha > 0$, $\beta > 0$ and $\theta > 0$ (even $\theta \in (0, 0.5)$). Nevertheless, we expect Theorem 2.1 to be valid for all $\alpha > 0$, $\beta > 0$ and $\theta > 0$. This would imply that different steep descent contours would have to be chosen.

The second lemma we present is a quantitative version of Lemma 5.4 of [BC17]. Define for $x, y \in \mathbb{R}$,

$$\Phi(x, y) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2 + y^2}. \tag{3.26}$$

Lemma 3.8. *For all $\alpha > 0$, $\beta > 0$ and $\theta > 0$ we have that*

(i) $\operatorname{Im} h'(\theta + iy) > 0$ *for $y > 0$ and $\operatorname{Im} h'(\theta + iy) < 0$ for $y < 0$.*

(ii) $\operatorname{Im} h'(\theta + iy) = yH(\theta, y, \alpha, \beta)$ *where*

$$H(\theta, y, \alpha, \beta) \geq 8 \frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)} \int_{\theta}^{\theta+\alpha} \sum_{n \geq 0} \frac{y^2(x - \theta)dx}{((x+n)^2 + y^2)^3}. \tag{3.27}$$

Finally, the third lemma, is a requirement that has been used in the latest version of [BC17], and which in our case gives the biggest limitation on the range of values of α, β and θ for which we can prove Theorem 2.1.

Lemma 3.9. *For all $\alpha > 0$, $\beta > 0$ and $\theta \in (0, \min\{0.5, \alpha\})$, we have that*

$$\frac{\sigma^3(\theta)}{2\theta} + \frac{h^{(4)}(\theta)}{4!} > 0.$$

Numerical computations also suggest that Lemma 2.1 is false for some values of $\alpha > 0$, $\beta > 0$ and $\theta \in (0, 0.5)$. Again, we expect Theorem 2.1 to hold despite this, but different steep descent contours would have to be chosen. We will present the proof of Lemma 2.1 in Section 3.8.

We will present the proofs of Lemmas 3.7 and 3.8 in Section 3.9. Now we will continue developing several consequences of these lemmas. From Lemma 3.7 we obtain the following corollary which extends Lemma 5.5 of [BC17] for arbitrary $\alpha > 0$ and $\beta > 0$.

In what follows we will say that the contour C_θ is steep-descent for the function $-\operatorname{Re}(h)$ if $\operatorname{Re}(h(\theta e^{i\phi}))$ is strictly increasing for $\phi \in (0, \pi)$ and strictly decreasing for $(-\pi, 0)$.

Corollary 3.10. *For all $\alpha > 0$, $\beta > 0$ and $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$ the contour C_θ is steep-descent for $-\operatorname{Re}(h)$.*

On the other hand, exactly as in [BC17], from part (i) of Lemma 3.8, we obtain Lemma 5.4 of [BC17], which we state as the following corollary for convenience.

Corollary 3.11. *For all $\alpha > 0$, $\beta > 0$ and $\theta > 0$, the contour D_θ is steep-descent for the function $\operatorname{Re}(h)$ in the sense that $\operatorname{Re}(h(\theta + iy))$ is strictly decreasing for y positive and strictly increasing for y negative.*

We will now need to modify the vertical line contour D_θ so that it avoids the singularity at θ. This will be a time dependent modification. For each $r > 0$ define,

$$D_\theta^{r,+} := D_\theta - B(\theta, r),$$

$$V_\theta^{r,+} := V_\theta^\epsilon - B(\theta, r),$$

the semicircle

$$S_\theta^t := \left\{ z \in \mathbb{C} : |z - \theta| = \frac{1}{\sigma(\theta)t^{1/3}}, \operatorname{Re}(z - \theta) \geq 0 \right\}$$

and the contour

$$D_\theta^t := D_\theta^{\sigma^{-1}t^{-1/3},+} \cup S_\theta^t.$$

See Figure 3 for a representation of these contours together with the contour V_θ^ϵ. The presence of the factor σ^{-1} in the definition of D_θ^t will be irrelevant in the proof of Theorem 2.1, since it will only change the computations by a fixed constant. Nevertheless, it will be important in the proof of Theorem 2.2, where $\sigma(\theta) \to 0$ as $t \to \infty$. In what follows we adopt the notation $\sigma_t = \sigma_t(\theta)$ to admit the possibility that σ is time dependent because it is a function of time dependent parameters (α_t, β_t).
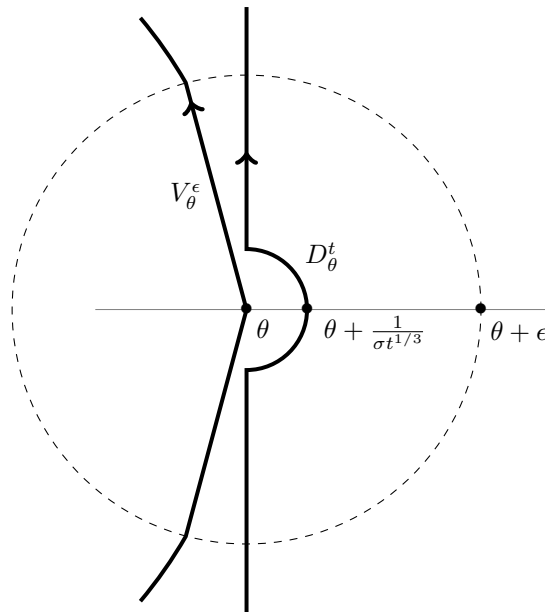

Figure 3: The contours V_θ^ϵ and D_θ^t with its t dependent deformation to avoid the singularity at θ.

Lemma 3.12. *For all $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $t \geq t_0$, where $\sigma_{t_0} t_0^{1/3} \theta \geq 2$, we have that for all $z \in D_\theta^t$,*

$$Re(h(z) - h(\theta)) \leq 128\mathfrak{g}(\alpha, \beta) \frac{1}{\theta^5 \sigma^3 t}.$$

Proof. For $z \in D_\theta - B(\theta, \sigma^{-1}t^{-1/3})$, the lemma follows from Corollary 3.11. For $z \in D_\theta^t \cap B(\theta, \sigma^{-1}t^{-1/3})$ we have that, for $t \geq t_0$ where t_0 is such that $\sigma t_0^{1/3} \theta \geq 2$,

$$|h(z) - h(\theta)| \leq \sum_{k=3}^\infty \frac{1}{k!} \frac{1}{\sigma^k t^{k/3}} |h^{(k)}(\theta)| \leq 64\mathfrak{g}(\alpha, \beta) \sum_{k=3}^\infty \frac{1}{\sigma^k t^{k/3} \theta^{k+2}}$$

$$\leq 64\mathfrak{g}(\alpha, \beta) \frac{1}{\theta^5} \frac{1}{\sigma^3 t} \sum_{k=0}^\infty \frac{1}{\sigma^k t^{k/3} \theta^k} = 128\mathfrak{g}(\alpha, \beta) \frac{1}{\theta^5 \sigma^3 t},$$

where we have used part (iii) of Corollary 3.6. $\qquad\qquad\square$

Corollary 3.13. *For all $\alpha > 0$, $\beta > 0$ and $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$ the following are satisfied.*

(i) *There is a constant $C_4 > 0$ such that for every $z \in D_\theta^{\epsilon,+}$ and $v \in C_\theta$ we have that*

$$\mathrm{Re}(h(z) - h(v)) \leq -C_4 \mathfrak{h}(\alpha, \beta, \theta)\epsilon^4,$$

where $\mathfrak{h}(\alpha, \beta, \theta) > 0$ and for $\alpha \geq 1$ and $\beta \geq 1$ we have that

$$\mathfrak{h}(\alpha, \beta, \theta) \geq C_5(\theta)\mathfrak{g}(\alpha, \beta), \tag{3.28}$$

for some $C_5(\theta) > 0$ which does not depend on α or β.

(ii) *There is a constant $C_6 > 0$ such that for every $z \in D_\theta^t$ and $v \in C_\theta^{\epsilon,+} = V_\theta^{\epsilon,+}$ we have that*

$$\mathrm{Re}(h(z) - h(v)) \leq -C_6 \epsilon^4 \mathfrak{g}(\alpha + 1, \beta) + 128\mathfrak{g}(\alpha, \beta)\frac{1}{\theta^5 \sigma^3 t}.$$

Proof. Proof of part (i). From Corollary 3.10 we can see that $\mathrm{Re}(h(\theta) - h(v)) \leq 0$ which implies that $\mathrm{Re}(h(z) - h(v)) \leq \mathrm{Re}(h(z) - h(\theta))$. On the other hand, by Corollary 3.11, $\mathrm{Re}(h(z))$ is decreasing in y for $z = \theta + iy$, $y \geq \epsilon$ and increasing in y for $z = \theta + iy$, $y \leq -\epsilon$, so that $\mathrm{Re}(h(z) - h(\theta)) \leq \mathrm{Re}(h(\theta + i\epsilon) - h(\theta))$ for $z \in D_\theta^{\epsilon,+}$. It follows that

$$\mathrm{Re}(h(z) - h(v)) = \mathrm{Re}(h(z) - h(\theta)) + \mathrm{Re}(h(\theta) - h(v))$$
$$\leq \mathrm{Re}(h(\theta + i\epsilon) - h(\theta)) = \int_0^\epsilon \frac{d\,\mathrm{Re}(h(\theta + iy))}{dy}dy = -\int_0^\epsilon \mathrm{Im}(h'(\theta + iy))dy$$
$$\leq -\frac{H(\theta, \epsilon, \alpha, \beta)}{4}\epsilon^4,$$

where in the last inequality we have used part (ii) of Lemma 3.8. Now,

$$H(\theta, y, \alpha, \beta) \geq 8y^2 \frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)} \int_\theta^{\theta+\alpha} \sum_{n \geq 0} \frac{(x - \theta)dx}{((x+n)^2 + 1)^3}.$$

Defining,

$$\mathfrak{h}(\alpha, \beta, \theta) := 8y^2 \frac{\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)}{\Psi_1(\theta) - \Psi_1(\theta + \alpha + \beta)} \int_\theta^{\theta+\alpha} \sum_{n \geq 0} \frac{(x - \theta)dx}{((x+n)^2 + 1)^3},$$

we finish the proof.

Proof of part (ii). By Corollary 3.10 and the fact that $v \in C_\theta^{\epsilon,+}$, we have that $\mathrm{Re}(h(\theta) - h(v)) \leq \mathrm{Re}(h(\theta) - h(\theta e^{i\phi_*}))$, where ϕ_* is such that $|\theta e^{i\phi_*} - \theta| = \epsilon$. Hence, by Lemma 3.12, we have that

$$\mathrm{Re}(h(z) - h(v))$$
$$= \mathrm{Re}(h(z) - h(\theta)) + \mathrm{Re}(h(\theta) - h(v)) \leq \mathrm{Re}(h(\theta) - h(\theta e^{i\phi_*}))$$
$$= -\int_0^{\phi_*} \mathrm{Re}(i\theta e^{i\phi} h'(\theta e^{i\phi}))d\phi \leq -\epsilon^4 c_1 \mathfrak{g}(\alpha + 1, \beta) + 128\mathfrak{g}(\alpha, \beta)\frac{1}{\theta^5 \sigma^3 t},$$

for some constant c_1, where in the last inequality we have used Lemma 3.7. $\qquad\square$

3.6 Steep-descent properties

Here we will apply the steep-descent estimates of the previous section to derive important steep descent properties of the Fredholm determinant. We will start proving a couple of properties about the functions involved.

Lemma 3.14. *For all $\alpha > 0$, $\beta > 0$, $\theta \in (0, \min\{0.5, 0.69 \times \alpha\})$ and $\epsilon \in (0, \theta/2)$ the following bounds are satisfied.*

(i) *For all $v \in C_\theta$, $z \in D_\theta^{\epsilon,+}$ and $t \geq 0$ we have that*

$$\left| e^{t(h(z)-h(v))-t^{1/3}\sigma(\theta)y(z-v)} \right| \leq e^{-tC_4\epsilon^4\mathfrak{h}(\alpha,\beta,\theta)+t^{1/3}\sigma(\theta)|y|\epsilon}. \tag{3.29}$$

(ii) *For all $v \in C_\theta^{\epsilon,+} = V_\theta^{\epsilon,+}$, $z \in D_\theta^t$ and $t \geq 0$ we have that*

$$\left| e^{t(h(z)-h(v))-t^{1/3}\sigma(\theta)y(z-v)} \right| \leq e^{-tC_6\epsilon^4\mathfrak{g}(\alpha+1,\beta)+128\mathfrak{g}(\alpha,\beta)\frac{1}{\theta^5\sigma^3}+2t^{1/3}\sigma(\theta)|y|}. \tag{3.30}$$

(iii) *There is a constant $C_7(\theta, \epsilon, \alpha, \beta) > 0$, which is independent of α and β when both $\alpha \geq 1$ and $\beta \geq 1$, such that for all $v \in W_{\theta,\epsilon}^L$, $z \in D_\theta^t$ or $v \in C_\theta^\epsilon$, $z \in D_\theta^t$, and $t \geq 1$, we have that*

$$\left| e^{t(h(z)-h(v))-t^{1/3}\sigma(\theta)y(z-v)} \right| \leq \frac{1}{C_7(\theta, \epsilon, \alpha, \beta)} e^{-C_7(\theta,\epsilon,\alpha,\beta)t\sigma(\theta)^3|v-\theta|^3}. \tag{3.31}$$

Proof. Proof of parts (i) and (ii). Part (i) follows from part (i) of Corollary 3.13 and the inequalities,

$$\left| e^{t(h(z)-h(v))-t^{1/3}\sigma(\theta)y(z-v)} \right| = e^{t\operatorname{Re}((h(z)-h(v)))-t^{1/3}\sigma(\theta)y\operatorname{Re}((z-v))}$$

$$\leq e^{-tC_4\epsilon^4\mathfrak{h}(\alpha,\beta,\theta)+t^{1/3}\sigma(\theta)|y|}.$$

Similarly part (ii) follows from part (ii) of Corollary 3.13.
Proof of part (iii). Let us use the inequality $|z - v| \leq |v - \theta| + \sigma^{-1}t^{-1/3}$ and Lemma 3.12 to conclude that

$$\operatorname{Re}\left(t(h(z) - h(v)) - t^{1/3}\sigma(\theta)y(z - v) \right)$$

$$= \operatorname{Re}(t(h(z) - h(\theta))) + \operatorname{Re}\left(t(h(\theta) - h(v)) - t^{1/3}\sigma(\theta)y(z - v) \right)$$

$$\leq \operatorname{Re}\left(t(h(\theta) - h(v)) \right) + 128\mathfrak{g}(\alpha, \beta)\frac{1}{\theta^5\sigma^3} + t^{1/3}\sigma(\theta)|y||v - \theta| + |y|. \tag{3.32}$$

To estimate the first term of the right-hand side in display (3.32) we will make a Taylor series expansion around θ of $h(v)$. Recall that $v \in W_{\theta,\epsilon}^L$ or $v \in C_\theta^\epsilon$. Define $\bar{v}$ by

$$v = \theta + \frac{1}{\sigma(\theta)t^{1/3}}\bar{v},$$

and $\bar{h}(\bar{v}) := h(v)$. Then,

$$t\bar{h}(\bar{v})$$

$$= th(\theta) + \frac{\bar{v}^3}{3!}\frac{1}{\sigma^3}h^{(3)}(\theta) + \frac{\bar{v}^4}{4!}\frac{1}{\sigma^4 t^{1/3}}h^{(4)}(\theta) + \sum_{k=5}^{\infty}\frac{\bar{v}^k}{k!}\frac{1}{\sigma^k t^{(k-3)/3}}h^{(k)}(\theta).$$

Thus,

$$\left| t\bar{h}(\bar{v}) - th(\theta) - \frac{\bar{v}^3}{3!}\frac{1}{\sigma^3}h^{(3)}(\theta) - \frac{\bar{v}^4}{4!}\frac{1}{\sigma^4 t^{1/3}}h^{(4)}(\theta) \right|$$

$$\leq \frac{|\bar{v}^3|}{\sigma^3}\sum_{k=5}^{\infty}\frac{\epsilon^{k-3}}{k!}h^{(k)}(\theta). \tag{3.33}$$

Now, from part (iii) of Corollary 3.6, we have that

$$\left| \frac{|\bar{v}^3|}{\sigma^3}\sum_{k=5}^{\infty}\frac{\epsilon^{k-3}}{k!}h^{(k)}(\theta) \right| \leq 64\frac{\mathfrak{g}(\alpha,\beta)}{\sigma^3}|\bar{v}^3|\sum_{k=5}^{\infty}\frac{\epsilon^{k-3}}{\theta^{k+2}} \leq 128\frac{\mathfrak{g}(\alpha,\beta)}{\sigma^3}\epsilon^2\frac{1}{\theta^7}|\bar{v}^3|,$$

where in the last inequality we have used $\epsilon < \theta/2$. It follows that

$$\left| t\,\mathrm{Re}(\bar{h}(\bar{v}) - h(\theta)) - \frac{\mathrm{Re}(\bar{v}^3)}{3} - \frac{\mathrm{Re}(\bar{v}^4)}{4!}\frac{1}{\sigma^4 t^{1/3}}h^{(4)}(\theta) \right| \leq 128\frac{\mathfrak{g}(\alpha,\beta)}{\sigma^3}\epsilon^2\frac{1}{\theta^7}|\bar{v}^3|. \tag{3.34}$$

Now, since $v \in W_{\theta,\epsilon}^L$ or $v \in C_\theta^\epsilon$, the argument of $\bar{v}$ is $\pm\left(\frac{\pi}{2} + \frac{\epsilon}{2\theta} + o(\epsilon)\right)$. Then, from the fact that $h^{(4)}(\theta) < 0$ (see part (v) of Corollary 3.6) and that $(\sigma t^{1/3})^{-1}|\bar{v}| < \epsilon$ (because $v \in W_{\theta,\epsilon}^L$), we have

$$-\frac{\mathrm{Re}(\bar{v}^3)}{3} - \frac{\mathrm{Re}(\bar{v}^4)}{4!}\frac{1}{\sigma^4 t^{1/3}}h^{(4)}(\theta)$$

$$= -\sin\left(\frac{3\epsilon}{2\theta} + o(\epsilon)\right)\frac{|\bar{v}|^3}{3} - \cos\left(\frac{2\epsilon}{\theta} + o(\epsilon)\right)\frac{h^{(4)}(\theta)}{\sigma^4 t^{1/3}}\frac{|\bar{v}|^4}{4!}$$

$$\leq -\epsilon|\bar{v}|^3\frac{1}{\sigma^3}\left(\frac{\sigma^3}{2\theta} + \frac{h^{(4)}(\theta)}{4!} + o_1(\epsilon)\frac{\sigma^3}{\epsilon} + o_2(\epsilon)h^{(4)}(\theta)\right)$$

$$= -\epsilon|\bar{v}|^3\frac{1}{\sigma^3}\left(\frac{\sigma^3}{2\theta} + \frac{h^{(4)}(\theta)}{4!}\right) + o_1(\epsilon)|\bar{v}|^3 + o_2(\epsilon)|\bar{v}|^3\frac{h^{(4)}(\theta)}{\sigma^3}$$

$$\leq -\epsilon|\bar{v}|^3 c_2(\theta,\alpha,\beta), \tag{3.35}$$

where in the last inequality we have assumed that ϵ is small enough, and where by parts (iii) and (vi) of Corollary 3.6 and by Lemma 3.9, we have that $c_2(\theta,\alpha,\beta)$ is a constant independent of α and β when both $\alpha \geq 1$ and $\beta \geq 1$, and $c_2(\theta,\alpha,\beta) > 0$. We then have from (3.34) and (3.35), using again part (iv) of Corollary 3.6, that for ϵ small enough,

$$-t\,\mathrm{Re}(\bar{h}(\bar{v}) - h(\theta)) \leq -c_3(\theta,\alpha,\beta)\epsilon|\bar{v}|^3,$$

where $c_3(\theta,\alpha,\beta)$ does not depend on α and β for $\alpha \geq 1$ and $\beta \geq 1$. Hence,

$$-t\,\mathrm{Re}(h(v) - h(\theta)) \leq -c_3(\theta,\alpha,\beta)\epsilon\sigma(\theta)^3 t|v - \theta|^3$$

and from (3.32) and Lemma 3.12, we get that

$$\left| e^{t(h(z)-h(\theta))-t^{1/3}\sigma(\theta)y(z-v)} \right| \leq \left| e^{t(h(z)-h(\theta))+t(h(\theta)-h(v))-t^{1/3}\sigma(\theta)y(z-v)} \right|$$

$$\leq e^{128\mathfrak{g}(\alpha,\beta)\frac{1}{\theta^5\sigma^3}}e^{t(\mathrm{Re}(h(\theta)-h(v)))+t^{1/3}\sigma(\theta)|y(z-v)|}$$

$$\leq e^{-c_3(\theta,\alpha,\beta)\epsilon t\sigma(\theta)^3|v-\theta|^3+128\mathfrak{g}(\alpha,\beta)\frac{1}{\theta^5\sigma^3}+c_4 t^{1/3}\sigma(\theta)|y||v-\theta|},$$

which proves part (iii). $\square$

3.7 Proof of Proposition 3.4

We will now present the proof of Proposition 3.4 (see also Proposition 5.6 of [BC17] and Section 8 of [K21]) using the steep descent properties of Section 3.6 along appropriate contours. At several steps, in order to deal with the singularity of the kernel $K_u^{RW}(v, v')$ for $v = v' = \theta$, we will deform the contours C_θ to V_θ^ϵ (see Figure 2) and we will also have to deform the contour D_θ^ϵ so that it avoids θ. We will then show that the integration outside the ball $B(\theta, \epsilon)$ has a negligible contribution in the limit. In what follows we will assume that either α and β are fixed (time-independent) or that these parameters are time-dependent, $(\alpha_t)_{t \geq 0}$ and $(\beta_t)_{t \geq 0}$, and satisfy (2.3) and (2.4).

Step 0. As explained above, we first deform the contour of integration in the definition of the kernel $K_u^{RW}(v, v')$ from $D_{1/2}$ to D_θ^t (see Figure 3), so that

$$K_u^{RW}(v, v') = \frac{1}{2\pi i} \int_{D_\theta^t} \frac{\pi}{\sin(\pi(z - v))} e^{t(h(z) - h(v)) - t^{1/3}\sigma(\theta)y(z - v)} \frac{\Gamma(v)}{\Gamma(z)} \frac{dz}{z - v'}. \tag{3.36}$$

Step 1. We will now show that whenever $v, v' \in C_\theta$ we have that

$$|K_u^{RW}(v, v') - K_{y,\epsilon}^{RW}(v, v')| \leq \frac{1}{c_5} e^{-\mathfrak{a}_1 t}, \tag{3.37}$$

for some constant $c_5 = c_5(\theta, \epsilon, y) > 0$ independent of α and β and $\mathfrak{a}_1 = \mathfrak{a}_1(\theta, \epsilon, \alpha, \beta, y) > 0$, such that

$$\mathfrak{a}_1(\theta, \epsilon, \alpha, \beta, y) \geq c_5 \mathfrak{g}(\alpha, \beta) \quad \text{for } \alpha \geq 1 \text{ and } \beta \geq 1,$$

where the kernel $K_{y,\epsilon}$ was defined in (3.16). Note that

$$K_u^{RW}(v, v') - K_{y,\epsilon}^{RW}(v, v') =$$
$$\int_{D_\theta^{\epsilon,+}} \frac{\pi}{\sin(\pi(z - v))} e^{t(h(z) - h(v)) - t^{1/3}\sigma(\theta)y(z - v)} \frac{\Gamma(v)}{\Gamma(z)} \frac{dz}{z - v'}. \tag{3.38}$$

From part (i) of Lemma 3.14, we have that since $v \in C_\theta$ and $z \in D_\theta^{\epsilon,+}$,

$$\left| e^{t(h(z) - h(v)) - t^{1/3}\sigma(\theta)y(z - v)} \right| \leq e^{-tC_4\epsilon^4 \mathfrak{h}(\alpha, \beta, \theta) + t^{1/3}\sigma|y|\epsilon}. \tag{3.39}$$

Now, recall the asymptotics $|\Gamma(x + iy)| e^{\frac{\pi}{2}|y|} |y|^{1/2 - x} \to \sqrt{2\pi}$ as $y \to \pm\infty$ for x and y real. This implies that

$$|\Gamma(z)| \geq \frac{1}{c_6} e^{-\frac{\pi}{2}|\operatorname{Im}(z)|}, \tag{3.40}$$

for $z \in D_\theta$ for some $c_6(\epsilon) > 0$. Furthermore, we have that

$$|\sin(\pi z)| \geq c_7 e^{\pi|\operatorname{Im}(z)|}, \tag{3.41}$$

for some $c_7 > 0$. From this and (3.39) we get now using that $v, v' \in C_\theta$ and $z \in D_\theta^{\epsilon,+}$ that

$$\left| \frac{\pi}{\sin(\pi(z - v))} e^{t(h(z) - h(v)) - t^{1/3}\sigma(\theta)y(z - v)} \frac{\Gamma(v)}{\Gamma(z)} \frac{1}{z - v'} \right|$$
$$\leq c_8 |\Gamma(v)| e^{-\frac{\pi}{2}|\operatorname{Im}(z)|} e^{-tC_4\epsilon^4 \mathfrak{h}(\alpha, \beta, \theta) + t^{1/3}\sigma(\theta)|y|\epsilon}, \tag{3.42}$$

for some constant $c_8 = c_8(\theta, \epsilon) > 0$. Note that since $v \in C_\theta$, $|\Gamma(v)| \leq c_9(\theta)$. Hence, from (3.42), using the fact that

$$\mathfrak{h}(\alpha, \beta, \theta) \geq C_5(\theta)\mathfrak{g}(\alpha, \beta) \quad \text{for } \alpha \geq 1 \text{ and } \beta \geq 1,$$

(see (3.28) of part (i) of Corollary 3.13 and part (i) of Lemma 3.14), and part (iii) of Corollary 3.6 and Lemma 3.9 to obtain an upper bound for σ, we obtain (3.37).

Step 2. Here we will show that for $v \in C_\theta^{\epsilon,+}$ and $v' \in C_\theta$ one has that

$$\left|K_{y,\epsilon}^{RW}(v,v')\right| \leq \left|K_u^{RW}(v,v')\right| \leq \frac{1}{c_{10}}e^{-\mathfrak{a}_2 t}, \tag{3.43}$$

for a pair of constants $\mathfrak{a}_2(\theta, \alpha, \beta, y) > 0$ and $c_{10} = c_{10}(\theta, \epsilon, y) > 0$ such that

$$\mathfrak{a}_2 = \mathfrak{a}_2(\theta, \epsilon, \alpha, \beta, y) \geq c_{10}\mathfrak{g}(\alpha, \beta) \quad \text{for } \alpha \geq 1 \text{ and } \beta \geq 1.$$

In fact, using again the estimates (3.40) and (3.41) and part (ii) of Lemma (3.14) we have that (here $z \in D_\theta^t$)

$$\left| \frac{\pi}{\sin(\pi(z-v))} e^{t(h(z)-h(v))-t^{1/3}\sigma(\theta)y(z-v)} \frac{\Gamma(v)}{\Gamma(z)} \frac{1}{z-v'} \right|$$
$$\leq c_{11}|\Gamma(v)|e^{-\frac{\pi}{2}|\operatorname{Im}(z)|}e^{-tC_6\epsilon^4\mathfrak{g}(\alpha+1,\beta)+128\mathfrak{g}(\alpha,\beta)\frac{1}{\theta^5\sigma^3}+2t^{1/3}\sigma(\theta)|y|},$$

for some constant $c_{11} = c_{11}(\theta, \epsilon) > 0$. Integrating over z and using part (iii) of Corollary 3.6 to bound σ, we obtain (3.43).

Step 3. Here we will show that for all $v \in C_\theta^\epsilon$ and $v' \in C_\theta$, we have that

$$\left|K_{y,\epsilon}^{RW}(v,v')\right| \leq \left|K_u^{RW}(v,v')\right| \leq \frac{1}{c_{12}}\sigma t^{1/3}e^{-c_{12}t\sigma^3|v-\theta|^3}, \tag{3.44}$$

for some constant $c_{12} = c_{12}(\theta, \epsilon, \alpha, \beta) > 0$ such that

$$c_{12}(\theta, \epsilon, \alpha, \beta) \geq c_{13} \quad \text{for } \alpha \geq 1, \beta \geq 1,$$

for some constant $c_{13} > 0$ independent of α and β. Indeed, by part (iii) of Lemma 3.14, (3.40), (3.41), the inequality

$$\left| \frac{(z-v)}{\sin(\pi(z-v))} \right| \leq c_{14},$$

and the fact that $|z-v'| \geq c_{15}(\sigma t^{1/3})^{-1}$, we have that

$$\left| \frac{\pi}{\sin(\pi(z-v))} e^{t(h(z)-h(v))-t^{1/3}\sigma(\theta)y(z-v)} \frac{\Gamma(v)}{\Gamma(z)} \frac{1}{z-v'} \right|$$
$$\leq c_{16}\sigma t^{1/3}|\Gamma(v)|e^{-\frac{\pi}{2}|\operatorname{Im}(z)|}\frac{1}{C_7}e^{-C_7 t\sigma(\theta)^3|v-\theta|^3},$$

for some constant $c_{16} > 0$, where we recall that $C_7(\theta, \epsilon, \alpha, \beta) > 0$ does not depend on α and β for $\alpha \geq 1$ and $\beta \geq 1$. which proves (3.44).

Step 4. Here we will prove that

$$\lim_{t\to\infty} \det\left(I - K_u^{RW}\right)_{L^2(C_\theta)} = \lim_{t\to\infty} \det\left(I - K_u^{RW}\right)_{L^2(C_\theta^\epsilon)}, \tag{3.45}$$

as long as the limit in the right-hand side exists. Consider the Fredholm determinantal expansion

$$\det\left(I - K_u^{RW}\right)_{L^2(C_\theta)}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{(C_\theta)^n} \det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n dw_1 \ldots dw_n. \tag{3.46}$$

Now note that

$$\int_{(C_\theta)^n} \det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n dw_1 \ldots dw_n$$
$$= \int_{(C_\theta^\epsilon)^n} \det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n dw_1 \ldots dw_n$$
$$+ \int_{(C_\theta^\epsilon)^n \setminus (C_\theta)^n} \det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n dw_1 \ldots dw_n.$$

Let us now show that the limit as $t \to \infty$ of the second term above vanishes.

Combining (3.43) and (3.44) with Hadamard's inequality, and using part (iii) of Corollary 3.6, we have that

$$\left|\det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n\right| \le n^{n/2}(c_{17}\sigma^3 t)^{n/3} e^{-c_{18}\sigma^3 t}, \tag{3.47}$$

for some constants $c_{17} = c_{17}(\theta, \epsilon, \alpha, \beta) > 0$ and $c_{18} = c_{18}(\theta, \epsilon, \alpha, \beta) > 0$ with the property that both constants are independent of α and β for $\alpha \ge 1$ and $\beta \ge 1$. But note that the right-hand side of (3.47) defines a series

$$\sum_{n=1}^{\infty} \frac{n^{n/2}}{n!}(c_{17}\sigma^3 t)^{n/3} e^{-c_{18}\sigma^3 t} \le e^{-c_{18}\sigma^3 t + 2(c_{17})^2(\sigma^3 t)^{2/3}}. \tag{3.48}$$

Now, in the case in which α and β are fixed it is obvious that the right-hand side of (3.48) converges to zero as $t \to \infty$. On the other hand, under the assumption that $(\alpha_t)_{t \ge 0}$ and $(\beta_t)_{t \ge 0}$ satisfy (2.3) and (2.4), we have that $\lim_{t \to \infty} \sigma^3 t = \infty$ (see (3.4)), so that we also have that the right-hand side of (3.48) tends to 0 as $t \to \infty$. It follows that

$$\lim_{t \to \infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{(C_\theta)^n \setminus (C_\theta^\epsilon)^n} \det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n dw_1 \ldots dw_n = 0,$$

which proves (3.45).

Step 5. Here we will show that

$$\lim_{t \to \infty} \det\left(I - K_u^{RW}\right)_{L^2(C_\theta^\epsilon)} = \lim_{t \to \infty} \det\left(I - K_{y,\epsilon}^{RW}\right)_{L^2(C_\theta^\epsilon)}. \tag{3.49}$$

Now we will use the following inequality for the difference between the determinants of two $n \times n$ matrices $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$, where $(A_i)_{1 \le i \le n}$ and $(B_i)_{1 \le i \le n}$ are the columns of A and B respectively,

$$|\det(A) - \det(B)| \le \sum_{j=1}^n |\det(B_1, \ldots, B_{j-1}, A_j - B_j, A_{j+1}, \ldots, A_n)|. \tag{3.50}$$

Now choose $A = (K_u^{RW}(w_i, w_j))_{i,j=1}^n$ and $B = (K_{y,\epsilon}^{RW}(w_i, w_j))_{i,j=1}^n$, and apply (3.50), the bounds (3.37) of Step 1 and (3.44) of Step 4, and Hadamard's inequality to conclude that

$$\left| \det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n - \det\left(K_{y,\epsilon}^{RW}(w_i, w_j)\right)_{i,j=1}^n \right|$$

$$\leq n\left(n\frac{1}{c_5}e^{-2a_1 t}\right)^{1/2}(nc_{19}(\sigma^3 t)^{2/3})^{n/2} = (c_{20})^{n/2}n^{(n+1)/2}(\sigma^3 t)^{n/2}e^{-a_1 t},$$

for some constants $c_{19} = c_{19}(\theta, \epsilon, y, \alpha, \beta) > 0$ and $c_{20} = c_{20}(\theta, \epsilon, y, \alpha, \beta) > 0$, which are independent of α and β for $\alpha \geq 1$ and $\beta \geq 1$. As in Step 4, this implies that

$$\sum_{n=1}^{\infty}\frac{1}{n!}\left|\int_{(C_\theta^\epsilon)^n}\det\left(K_u^{RW}(w_i, w_j)\right)_{i,j=1}^n dw_1\cdots dw_n\right. \tag{3.51}$$

$$\left.-\int_{(C_\theta^\epsilon)^n}\det\left(K_{y,\epsilon}^{RW}(w_i, w_j)\right)_{i,j=1}^n dw_1\cdots dw_n\right| \leq e^{-c_{21}\sigma^3 t + 2(c_{22})^2(\sigma^3 t)^{2/3}},$$

for a pair of constants $c_{21} = c_{21}(\theta, \epsilon, y, \alpha, \beta) > 0$ and $c_{22} = c_{22}(\theta, \epsilon, y, \alpha, \beta) > 0$, which are independent of α and β for $\alpha \geq 1$ and $\beta \geq 1$, As in Step 3, we conclude that either in the case in which α and β are constant, or in the case in which $(\alpha_t)_{t\geq 0}$ and $(\beta_t)_{t\geq 0}$ satisfy (2.3) and (2.4), the right-hand side of (3.51) tends to 0 as $t \to \infty$ which combined with Step 4, by the Fredholm determinant expansion implies (3.49).

Step 6. We will now show that

$$\lim_{t\to\infty}\det\left(I - K_{y,\epsilon}^{RW}\right)_{L^2(C_\theta^\epsilon)} = \det(I + K_y)_{L^2(C)}.$$

First note that there is no problem in deforming the contour C_θ^ϵ to W_θ^L, so that it is enough to prove that

$$\lim_{t\to\infty}\det\left(I - K_{y,\epsilon}^{RW}\right)_{L^2(W_\theta^L)} = \det(I + K_y)_{L^2(C)}.$$

To prove this, we will first do a change of coordinates in the integration variables of the Fredholm determinant expansion and in the integral defining the kernel $K_{y,\epsilon}^{RW}$, introducing $\bar{z}, \bar{v}$ and $\bar{v}'$ defined by

$$z = \theta + \frac{1}{\sigma(\theta)t^{1/3}}\bar{z}, v = \theta + \frac{1}{\sigma(\theta)t^{1/3}}\bar{v} \quad \text{and} \quad v' = \theta + \frac{1}{\sigma(\theta)t^{1/3}}\bar{v}'. \tag{3.52}$$

We then have that

$$\det\left(I - K_{y,\epsilon}^{RW}\right)_{L^2(W_\theta^L)} = \det\left(I - \bar{K}_\epsilon^t\right)_{L^2(W_\theta^\infty)},$$

where

$$\bar{K}_\epsilon^t(\bar{v}, \bar{v}') := \mathbf{1}_{|\bar{v}|, |\bar{v}'|\leq\epsilon\sigma t^{1/3}}\frac{1}{\sigma(\theta)t^{1/3}}K_{y,\epsilon}^{RW}\left(\theta + \frac{1}{\sigma(\theta)t^{1/3}}\bar{v}, \theta + \frac{1}{\sigma(\theta)t^{1/3}}\bar{v}'\right).$$

We will first prove the pointwise convergence of the kernel $\bar{K}_\epsilon^t$. Consider the contour $L_\epsilon := D_0^{1,\epsilon\sigma t^{1/3},+} \cup S_0^1$ formed by the two vertical lines $D_0^{1,\epsilon\sigma t^{1/3},+} := \{yi : y \in [1, \epsilon\sigma t^{1/3}]\} \cup \{yi : y \in [-1, -\epsilon\sigma t^{1/3}]\}$ and the semicircle $S_0^1 := \{z : |z| = 1, \text{Re}\,z \geq 0\}$. We adopt the convention L_∞ as the contour L_ϵ with $\epsilon = \infty$. We then have

$$\bar{K}_\epsilon^t(\bar{v}, \bar{v}')$$

$$= \frac{\mathbf{1}_{|\bar{v}|, |\bar{v}'|\leq\epsilon\sigma t^{1/3}}}{2\pi i}\int_{L_\epsilon}\frac{\sigma^{-1}t^{-1/3}\pi}{\sin\left(\sigma^{-1}t^{-1/3}\pi(\bar{z} - \bar{v})\right)}e^{t(\bar{h}(\bar{z}) - \bar{h}(\bar{v})) - y(\bar{z} - \bar{v})}\frac{\bar{\Gamma}(\bar{v})}{\bar{\Gamma}(\bar{z})}\frac{1}{\bar{z} - \bar{v}'}d\bar{z},$$

where

$$\bar{h}(w) = h\left(\theta + \sigma^{-1} t^{-1/3} w\right) \quad \text{and} \quad \bar{\Gamma}(w) = \Gamma(\theta + \sigma^{-1} t^{-1/3} w).$$

Now, from the limits (where again we use that is (3.4) satisfied both in the case α and β constant or $(\alpha_t)_{t \geq 0}$ and $(\beta_t)_{t \geq 0}$ satisfying (2.3) and (2.4)),

$$\lim_{t \to \infty} \frac{\sigma(\theta)^{-1} t^{-1/3} \pi}{\sin\left(\sigma^{-1} t^{-1/3} \pi (\bar{z} - \bar{v})\right)} = \frac{1}{\bar{z} - \bar{v}}$$

$$\lim_{t \to \infty} \frac{\bar{\Gamma}(\bar{v})}{\bar{\Gamma}(\bar{z})} = 1$$

$$\lim_{t \to \infty} t(\bar{h}(\bar{z}) - \bar{h}(\bar{v})) = \frac{1}{3}(\bar{z}^3 - \bar{v}^3),$$

it follows that

$$\lim_{t \to \infty} \frac{\sigma^{-1} t^{-1/3} \pi}{\sin\left(\sigma^{-1} t^{-1/3} \pi (\bar{z} - \bar{v})\right)} e^{t(\bar{h}(\bar{z}) - \bar{h}(\bar{v})) - y(\bar{z} - \bar{v})} \frac{\bar{\Gamma}(\bar{v})}{\bar{\Gamma}(\bar{z})} \frac{1}{\bar{z} - \bar{v}'}$$

$$= \frac{1}{(\bar{z} - \bar{v})(\bar{z} - \bar{v}')} e^{\frac{1}{3}(\bar{z}^3 - \bar{v}^3) - y(\bar{z} - \bar{v})}.$$

We want to justify next that the above limit can be commuted with the integration over $\bar{z}$. Note that $v \in W_{\theta,\epsilon}^L$ and $z \in D_\theta^t$ is implies that $\bar{v} \in W^\infty$ and $\bar{z} \in L_\infty$, so we can apply part (iii) of Lemma 3.14 to bound the exponential and conclude that

$$\left| \frac{\sigma^{-1} t^{-1/3} \pi}{\sin\left(\sigma^{-1} t^{-1/3} \pi (\bar{z} - \bar{v})\right)} e^{t(\bar{h}(\bar{z}) - \bar{h}(\bar{v})) - y(\bar{z} - \bar{v})} \frac{\bar{\Gamma}(\bar{v})}{\bar{\Gamma}(\bar{z})} \frac{1}{\bar{z} - \bar{v}'} \right|$$

$$\leq c_{23} \frac{|\Gamma(\bar{v})|}{|\bar{z} - \bar{v}||\bar{z} - \bar{v}'|} e^{-C_7 |\bar{v}|^3}. \tag{3.53}$$

for some constant $c_{23} > 0$, which is integrable in $\bar{z}$ at ∞ because the right-hand side decays quadratically. We conclude then by the dominated convergence theorem that

$$\lim_{t \to \infty} K_\epsilon^t(\bar{v}, \bar{v}') = \int_{L_\infty} \frac{1}{(\bar{z} - \bar{v})(\bar{z} - \bar{v}')} e^{\frac{1}{3}(\bar{z}^3 - \bar{v}^3) - y(\bar{z} - \bar{v})} d\bar{z}. \tag{3.54}$$

Now, from (3.53), we can conclude that

$$K_\epsilon^t(\bar{v}, \bar{v}') \leq c_{24} e^{-C_7 |\bar{v}|^3},$$

for some constant $c_{24} > 0$. By Hadamard's inequality for the determinant, this implies that

$$\det(K_\epsilon^t(\bar{w}_i, \bar{w}_j)_{i,j=1}^n \leq n^{n/2} \prod_{i=1}^n c_{24} e^{-\frac{C_7}{2} |\bar{w}_i|^3}.$$

It follows from this bound, the convergence in (3.54) and the dominated convergence theorem that

$$\lim_{t \to \infty} \int_{(W_\theta^L)^n} \det\left(K_{y,\epsilon}^{RW}(w_i, w_j)\right) dw_1 \cdots dw_n$$

$$= \int_{(W_\theta^\infty)^n} \det(K_y, (\bar{w}_i, \bar{w}_j)) d\bar{w}_1 \cdots d\bar{w}_n.$$

Applying a second time the dominated convergence theorem to interchange the summation of the Fredholm determinant with the limit, we finish the proofs of both parts (i) and (ii) of Proposition 3.4.

3.8 Proof of Lemma 3.9

Here we will prove Lemma 3.9. Define

$$\Xi(x) = \frac{\psi_2(\gamma)}{4\theta} + \frac{\psi_3(\gamma)}{4!} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{x_n^4} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\theta}\frac{1}{x_n^3}.$$

As

$$\frac{\sigma^3(\theta)}{2\theta} + \frac{h^{(4)}(\theta)}{4!} = \frac{h^{(3)}(\theta)}{4\theta} + \frac{h^{(4)}(\theta)}{4!}$$
$$= \Xi(\theta + \alpha) - \Xi(\theta) + \frac{\psi_1(\theta) - \psi_1(\theta + \alpha)}{\psi_1(\theta) - \psi_1(\theta + \alpha + \beta)}\big(\Xi(\theta) - \Xi(\theta + \alpha + \beta)\big)$$

goes to 0 when $\beta \to 0$, it is enough to prove that its derivative with respect to β is positive or, making the change $y = \theta + \alpha + \beta$, that

$$\frac{\psi_1(\theta) - \psi_1(\theta + \alpha)}{(\psi_1(\theta) - \psi_1(y))^2}\left[-\Xi'(y)\big(\psi_1(\theta) - \psi_1(y)\big) + \psi_2(y)\big(\Xi(\theta) - \Xi(y)\big)\right] > 0,$$

for $y > 2\theta$.

The first factor is positive and the second one equals

$$-\sum_{m\geq 0}\left(\frac{3}{2\theta} - \frac{1}{y_m}\right)\frac{1}{y_m^4}\sum_{n\geq 0}\left(\frac{1}{\theta_n^2} - \frac{1}{y_n^2}\right)$$
$$+\sum_{m\geq 0}\frac{1}{y_m^3}\left[\sum_{n\geq 0}\left(\frac{1}{\theta} - \frac{1}{2\theta_n}\right)\frac{1}{\theta_n^3} - \sum_{n\geq 0}\left(\frac{1}{\theta} - \frac{1}{2y_n}\right)\frac{1}{y_n^3}\right]$$
$$= -\sum_{m\geq 0}\frac{3y_m - 2\theta}{2\theta y_m^5}\sum_{n\geq 0}\frac{y_n^2 - \theta_n^2}{\theta_n^2 y_n^2} + \sum_{m\geq 0}\frac{1}{y_m^3}\left[\sum_{n\geq 0}\frac{2\theta_n - \theta}{2\theta\theta_n^4} - \sum_{n\geq 0}\frac{2y_n - \theta}{2\theta y_n^4}\right].$$

We will prove the positivity of the resulting products of terms in the sums. First products of same index terms $(m = n)$

$$-\frac{3y_n - 2\theta}{2\theta y_n^5}\frac{y_n^2 - \theta_n^2}{\theta_n^2 y_n^2} + \frac{1}{y_n^3}\frac{2\theta_n - \theta}{2\theta\theta_n^4} - \frac{1}{y_n^3}\frac{2y_n - \theta}{2\theta y_n^4}.$$

After factoring the common denominator we are left with

$$-(3y_n - 2\theta)(y_n^2 - \theta_n^2)\theta_n^2 + (2\theta_n - \theta)y_n^4 - (2y_n - \theta)\theta_n^4$$
$$= -(3a + 3t - 2\theta)((a+t)^2 - a^2)a^2 + (2a - \theta)(a+t)^4 - (2a + 2t - \theta)a^4,$$

where we have abbreviated θ_n as a and $\alpha + \beta$ as t, so that $y_n = a + t$ and $t > \theta$. After expanding the fourth power binomial we see that the constant and linear in t terms cancel out

$$-(3a + 3t - 2\theta)(2a + t)ta^2 + (2a - \theta)(4a^3t + 6a^2t^2 + 4at^3 + t^4) - 2ta^4$$
$$= -(9at + 3t^2 - 2t\theta)ta^2 + (2a - \theta)(6a^2t^2 + 4at^3 + t^4)$$
$$= t^2\big[3a^3 - 4a^2\theta + (5a^2 - 4a\theta)t + (2a - \theta)t^2\big].$$

As $a \geq \theta$ it is clear that this expresion is positive for $t > \theta$.

Now the products of terms with different index in the above sums

$$-\frac{3y_m - 2\theta}{2\theta y_m^5}\frac{y_n^2 - \theta_n^2}{\theta_n^2 y_n^2} + \frac{1}{y_m^3}\frac{2\theta_n - \theta}{2\theta\theta_n^4} - \frac{1}{y_m^3}\frac{2y_n - \theta}{2\theta y_n^4}$$
$$-\frac{3y_n - 2\theta}{2\theta y_n^5}\frac{y_m^2 - \theta_m^2}{\theta_m^2 y_m^2} + \frac{1}{y_n^3}\frac{2\theta_m - \theta}{2\theta\theta_m^4} - \frac{1}{y_n^3}\frac{2y_m - \theta}{2\theta y_m^4}.$$

The first half can be written as

$$-\frac{(3y_m - 2\theta)(y_n^2 - \theta_n^2)y_n^2}{2\theta y_m^5 \theta_n^2 y_n^4} + \frac{(2\theta_n - \theta)y_n^4 - (2y_n - \theta)\theta_n^4}{2\theta y_m^3 \theta_n^4 y_n^4}$$
$$= \frac{-(3y_m - 2\theta)(y_n^2 - \theta_n^2)y_n^2 + (3y_n - 2\theta)(y_n^2 - \theta_n^2)y_m^2}{2\theta y_m^5 \theta_n^2 y_n^4}$$
$$+ \frac{-(3y_n - 2\theta)(y_n^2 - \theta_n^2)\theta_n^2 + (2\theta_n - \theta)y_n^4 - (2y_n - \theta)\theta_n^4}{2\theta y_m^3 \theta_n^4 y_n^4}.$$

The last numerator was already shown to be positive. By a similar reasoning for the second half of the above expression, all that is left to finish the proof is to show that

$$\frac{-(3y_m - 2\theta)(y_n^2 - \theta_n^2)y_n^2 + (3y_n - 2\theta)(y_n^2 - \theta_n^2)y_m^2}{2\theta y_m^5 \theta_n^2 y_n^4}$$
$$+ \frac{-(3y_n - 2\theta)(y_m^2 - \theta_m^2)y_m^2 + (3y_m - 2\theta)(y_m^2 - \theta_m^2)y_n^2}{2\theta y_n^5 \theta_m^2 y_m^4}$$

is positive for $y > 2\theta$.

This is equivalent to showing the positivity of

$$-(3y_m - 2\theta)(y_n^2 - \theta_n^2)y_n^3\theta_m^2 + (3y_n - 2\theta)(y_n^2 - \theta_n^2)y_m^2 y_n \theta_m^2$$
$$-(3y_n - 2\theta)(y_m^2 - \theta_m^2)y_m^3\theta_n^2 + (3y_m - 2\theta)(y_m^2 - \theta_m^2)y_n^2 y_m \theta_n^2$$

Factorizing and using the abreviations $a = \theta_n$ and $b = \theta_m$,

$$\left[(y_n^2 - \theta_n^2)\theta_m^2 y_n - (y_m^2 - \theta_m^2)\theta_n^2 y_m\right]\left[(3y_n - 2\theta)y_m^2 - (3y_m - 2\theta)y_n^2\right]$$
$$= \left[(2a + t)tb^2(a + t) - (2b + t)ta^2(b + t)\right]$$
$$\times \left[3(a + t)(b + t)(b - a) - 2\theta(a + b + 2t)(b - a)\right].$$

A further algebraic manipulation leads us to the following expresion, which is positive under the stated conditions.

$$\left[(a + b)t^3 + 3abt^2\right]\left[3ab + 3t^2 + (a + b)(3t - 2\theta) - 4\theta t\right](b - a)^2$$

3.9 Proof of Lemmas 3.7 and 3.8

In what follows we present the proofs of Lemma 3.7 in Section 3.9.1 and of Lemma 3.8 in Section 3.9.2.

3.9.1 Proof of Lemma 3.7

The main technical ingredient in the proof of Lemma 3.7 will be to extract the zero of $\mathrm{Re}(izh'(z))$ for $z = e^{i\phi}$ at $\phi = 0$, through a subtraction of appropriate functions. This enables us to extract two key factors ($\sin\phi$ and $(1 - \cos\phi)$) from $\mathrm{Re}(izh'(z))$. We will start deducing several properties of a resulting function in these computations, defined for $x > 0$,

$$\mathcal{P}(x) = -\sum_{n=0}^{\infty}\frac{\theta^2 + 2\theta x_n\cos\phi}{(\theta^2 + 2\theta x_n\cos\phi + x_n^2)(\theta + x_n)^2}, \tag{3.55}$$

where we adopt the convention $x_n = n + x$.

Lemma 3.15. *Let $\theta > 0$. Then, the following are satisfied,*

   (i) *For all $\phi \in (0, \pi)$ such that $\cos \phi \geq 0$, the function $-\mathcal{P}(x)$ is positive and decreasing in $x > 0$.*

  (ii) *For all $\phi \in (0, \pi)$ and $x > 0$ such that $\cos \phi \leq -\frac{\theta}{2x}$, $-\mathcal{P}(x)$ is negative.*

 (iii) *For all $\phi \in (0, \pi)$ and $x > 0$ such that $\cos \phi \leq -\frac{\theta}{x}$, $-\mathcal{P}(x + y)$ is increasing in $y > 0$.*

 (iv) *For all $\phi \in (0, \pi)$, $x \geq \theta$ and $y > 0$ we have that*

$$- (\mathcal{P}(x) - \mathcal{P}(x + y))$$
$$\leq v(\rho, \phi) \mathbf{1}(\cos \phi > -\rho) \left(\Psi_1(\theta + x) - \Psi_1(\theta + x + y) \right),$$

where

$$v(\rho, \phi) = \frac{\rho^2 + 2\rho \cos \phi + \rho}{\rho^2 + 2\rho \cos \phi + 1} \leq \frac{\rho^2 + 3\rho}{(\rho + 1)^2},$$

and $\rho = \frac{\theta}{x}$.

Proof. The positivity of $-\mathcal{P}(x)$ for $\cos \phi \geq 0$ of part (i) is immediate. Also the negativity of $-\mathcal{P}(x)$ for $\cos \phi \leq -\frac{\theta}{2x}$, which proves part (ii). We will now prove that $-\mathcal{P}(x)$ is decreasing in x for $\cos \phi \geq 0$ (part (i)), part (iii) and part (iv). To simplify notation define

$$C(u) := \theta + 2u \cos \phi,$$

$$A(u) := \theta^2 + 2\theta u \cos \phi + u^2$$

and

$$B(u) = (\theta + u)^2.$$

Then note that

$$\frac{C(x_n)}{A(x_n)B(x_n)} - \frac{C((x + y)_n)}{A((x + y)_n)B((x + y)_n)}$$
$$= \frac{\theta + 2x_n \cos \phi}{A(x_n)} \left(\frac{1}{B(x_n)} - \frac{1}{B((x + y)_n)} \right)$$
$$- \frac{2y \cos \phi}{A(x_n)} \frac{1}{B((x + y)_n)} + \frac{\theta + 2(x + y)_n \cos \phi}{A((x + y)_n)} \frac{2y(\theta \cos \phi + x_n) + y^2}{A(x_n)B((x + y)_n)}$$
$$= \frac{\theta + 2x_n \cos \phi}{A(x_n)} \left(\frac{1}{B(x_n)} - \frac{1}{B((x + y)_n)} \right)$$
$$- \frac{2 \cos \phi}{2\theta + 2x_n + y} \frac{B(x_n)}{A(x_n)} \frac{2y\theta + 2x_n y + y^2}{B(x_n)B((x + y)_n)}$$
$$+ \frac{\theta + 2(x + y)_n \cos \phi}{A((x + y)_n)} \frac{B(x_n)}{A(x_n)} \frac{2(\theta \cos \phi + x_n) + y}{2\theta + 2x_n + y} \frac{2\theta y + 2x_n y + y^2}{B(x_n)B((x + y)_n)}$$
$$= (\mathfrak{b} + \mathfrak{c} + \mathfrak{a}) \left(\frac{1}{B(x_n)} - \frac{1}{B((x + y)_n)} \right),$$

where

$$\mathfrak{b} = \frac{\theta + 2x_n \cos \phi}{\theta^2 + 2x_n \theta \cos \phi + x_n^2},$$

$$\mathfrak{c} = -\frac{2\cos\phi}{2\theta + 2x_n + y}\frac{(\theta + x_n)^2}{\theta^2 + 2\theta x_n \cos\phi + x_n^2}$$

and

$$\mathfrak{a}$$
$$= \frac{\theta + 2(x_n + y)\cos\phi}{\theta^2 + 2\theta(x_n + y)\cos\phi + (x_n + y)^2}\frac{(x_n + \theta)^2}{\theta^2 + 2\theta x_n \cos\phi + x_n^2}\frac{2(\theta\cos\phi + x_n) + y}{2(\theta + x_n) + y}.$$

Note that

$$\mathfrak{a} + \mathfrak{c}$$
$$= \frac{(2(x_n + y)(\theta + x_n \cos\phi) - \theta y)(x_n + \theta)^2}{(\theta^2 + 2\theta x_n \cos\phi + x_n^2)(\theta^2 + 2\theta(x_n + y)\cos\phi + (x_n + y)^2)(2\theta + 2x_n + y)}. \tag{3.56}$$

From (3.56) we can easily check that $\mathfrak{a} + \mathfrak{c} \geq 0$ whenever $\cos\phi \geq 0$, which combined with the fact that the condition on ϕ also implies that $\mathfrak{b} \geq 0$, proves that $-\mathcal{P}(x)$ is decreasing when $\cos\phi \geq 0$, and proves part (i). On the other hand, (3.56) also shows that $\mathfrak{a} + \mathfrak{c} \leq 0$ when $\cos\phi \leq -\frac{\theta}{x}$, which combined with the fact that this condition also implies that $\mathfrak{b} \leq 0$, implies that $-\mathcal{P}(x + y)$ is increasing in y when $\cos\phi \leq -\frac{\theta}{x}$ (part (iii)).

Let us now prove part (iv). Note that

$$\mathfrak{a} + \mathfrak{c} \leq \frac{2(x_n + y)(\theta + x_n \cos\phi)(x_n + \theta)^2}{(\theta^2 + 2\theta x_n \cos\phi + x_n^2)(\theta^2 + 2\theta(x_n + y)\cos\phi + (x_n + y)^2)(2\theta + 2x_n + y)}.$$

Now note that for all $\phi \in (0, \pi)$, the function

$$f_1(v) = \frac{v}{\theta^2 + 2\theta v \cos\phi + v^2},$$

is decreasing in v as long as $v \geq \theta$. Therefore, since by assumption we have that $x \geq \theta$, it follows that $x_n + y \geq \theta$, so that for all $n \geq 0$,

$$(\mathfrak{a} + \mathfrak{c})\theta \leq \frac{\theta x_n(\theta + x_n \cos\phi)(x_n + \theta)}{(\theta^2 + 2\theta x_n \cos\phi + x_n^2)^2}1(\cos\phi > -\theta/x_n)$$
$$\leq \frac{\rho_n(\rho_n + \cos\phi)(\rho_n + 1)}{(\rho_n^2 + 2\rho_n \cos\phi + 1)^2}1(\cos\phi > -\rho_n),$$

where $\rho_n := \frac{\theta}{x_n}$. Now, consider the function

$$f_2(u, a) := \frac{u + a}{u^2 + 2ua + 1}.$$

Note that for $u \in (0, 1)$,

$$\frac{\partial f_2(u, a)}{\partial a} = \frac{1 - u^2}{(u^2 + 2ua + 1)^2} > 0,$$

which shows that for fixed u, $f_2(u, a)$ is increasing in a. Hence, for all $n \geq 0$,

$$\frac{\rho_n(\rho_n + \cos\phi)(\rho_n + 1)}{(\rho_n^2 + 2\rho_n \cos\phi + 1)^2} \leq \frac{\rho_n}{\rho_n^2 + 2\rho_n \cos\phi + 1}.$$

It follows that

$$(\mathfrak{a} + \mathfrak{b} + \mathfrak{c})\theta \le f_3(\rho_n, \cos\phi)\mathbf{1}(\cos\phi > -\rho_n),$$

where we define for $u \in [0, 1]$ and $a \in (-1, 1)$,

$$f_3(u, a) = \frac{u^2 + 2ua + u}{u^2 + 2ua + 1}.$$

Now note that for $u \in (0, 1)$ and a arbitrary,

$$\frac{\partial f_3(u, a)}{\partial a} = \frac{2u(1 - u)}{(u^2 + 2ua + 1)^2} > 0.$$

Also for $a \in (-u, 1)$ and $u \in (0, 1)$,

$$\frac{\partial f_3(u, a)}{\partial u} = \frac{1 + 2u + 2a - u^2}{(u^2 + 2ua + 1)^2} > 0.$$

Hence,

$$f_3(\rho_n, a) \le f_3(\rho_0, a).$$

It also follows that $f_3(u, a)$ is increasing in a for fixed u and increasing in u for fixed a as long as $u, a \in (0, 1)$. Therefore, since $\rho_n \le \rho_0$, we conclude that

$$(\mathfrak{a} + \mathfrak{b} + \mathfrak{c})\theta \le \frac{\rho_0^2 + 2\rho_0 \cos\phi + \rho_0}{\rho_0^2 + 2\rho_0 \cos\phi + 1}\mathbf{1}(\cos\phi > -\rho_0) \le \frac{\rho_0^2 + 3\rho_0}{(\rho_0 + 1)^2}\mathbf{1}(\cos\phi > -\rho_0). \qquad \square$$

Let us now proceed to prove Lemma 3.7. Let $z = \theta e^{i\phi}$. Note that

$$\mathrm{Re}(izh'(z)) = \Theta(\alpha) - \Theta(\alpha + \beta)$$
$$+ \frac{\Psi_1(\alpha + \theta) - \Psi_1(\alpha + \beta + \theta)}{\Psi_1(\theta) - \Psi_1(\alpha + \beta + \theta)} \left(\Theta(\alpha + \beta) - \Theta(0)\right),$$

where for γ real we define

$$\Theta(\gamma) := \mathrm{Re}(iz(\Psi(z + \gamma) - \Psi(\theta + \gamma)).$$

Now, note that for $\gamma > 0$, we have the following expansion valid for any $z \notin \{0, -1, -2, \dots\}$,

$$\Psi(z + \gamma) - \Psi(\theta + \gamma) = \sum_{n \ge 0} \frac{z - \theta}{(z + \gamma + n)(\theta + \gamma + n)}. \tag{3.57}$$

Also,

$$Re\left(iz\frac{z - \theta}{\gamma_n + z}\right)$$
$$= Re\left(i\theta^2 \frac{(\cos\phi + i\sin\phi)(\cos\phi - 1 + i\sin\phi)(\theta\cos\phi + \gamma_n - i\theta\sin\phi)}{(\theta\cos\phi + \gamma_n)^2 + \theta^2\sin^2\phi}\right)$$
$$= Re\left(i\theta^2 \frac{(\cos^2\phi - \sin^2\phi - \cos\phi + i\sin\phi(2\cos\phi - 1))(\theta\cos\phi + \gamma_n - i\theta\sin\phi)}{(\theta\cos\phi + \gamma_n)^2 + \theta^2\sin^2\phi}\right)$$
$$= \theta^2 \frac{\theta\sin\phi(\cos^2\phi - \sin^2\phi - \cos\phi) - \sin\phi(2\cos\phi - 1)(\theta\cos\phi + \gamma_n)}{(\theta\cos\phi + \gamma_n)^2 + \theta^2\sin^2\phi}$$
$$= \theta^2 \sin\phi \frac{-\theta - \gamma_n(2\cos\phi - 1)}{\theta^2 + 2\theta\gamma_n\cos\phi + \gamma_n^2}.$$

This implies that

$$\Theta(\gamma) = \theta^2 \sin \phi \mathcal{R}(\gamma), \tag{3.58}$$

where

$$\mathcal{R}(\gamma) := -\sum_{n=0}^{\infty} \frac{\theta + \gamma_n(2\cos\phi - 1)}{(\theta^2 + 2\theta\gamma_n \cos\phi + \gamma_n^2)(\theta + \gamma_n)}.$$

Hence it is enough to prove that for $\phi \in (0, \pi)$,

$$\mathcal{R}(\alpha) - \mathcal{R}(\alpha + \beta) + \frac{\Psi_1(\alpha + \theta) - \Psi_1(\alpha + \beta + \theta)}{\Psi_1(\theta) - \Psi_1(\alpha + \beta + \theta)} (\mathcal{R}(\alpha + \beta) - \mathcal{R}(0)) > 0.$$

On the other hand note that

$$\begin{aligned}
&\mathcal{R}(\gamma) + \Psi_1(\theta + \gamma) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{(\theta + \gamma_n)^2} - \frac{\theta + \gamma_n(2\cos\phi - 1)}{(\theta^2 + 2\theta\gamma_n \cos\phi + \gamma_n^2)(\theta + \gamma_n)} \right) \\
&= 2(1 - \cos\phi)\mathcal{Q}(\gamma), \tag{3.59}
\end{aligned}$$

where we define

$$\mathcal{Q}(\gamma) = \sum_{n=0}^{\infty} \frac{\gamma_n^2}{(\theta^2 + 2\theta\gamma_n \cos\phi + \gamma_n^2)(\theta + \gamma_n)^2}.$$

Therefore it is now enough to prove that for $\phi \in (0, \pi)$,

$$\mathcal{Q}(\alpha) - \mathcal{Q}(\alpha + \beta) + \frac{\Psi_1(\alpha + \theta) - \Psi_1(\alpha + \beta + \theta)}{\Psi_1(\theta) - \Psi_1(\alpha + \beta + \theta)} (\mathcal{Q}(\alpha + \beta) - \mathcal{Q}(0)) > 0.$$

Now note that

$$\begin{aligned}
&\mathcal{Q}(\gamma) - \Psi_1(\theta + \gamma) \\
&= \sum_{n=0}^{\infty} \left(\frac{\gamma_n^2}{(\theta^2 + 2\theta\gamma_n \cos\phi + \gamma_n^2)(\theta + \gamma_n)^2} - \frac{1}{(\theta + \gamma_n)^2} \right) \\
&= \mathcal{P}(\gamma), \tag{3.60}
\end{aligned}$$

where we recall the definition of $\mathcal{P}$ in (3.55) above. So at this point it is enough to prove that for $\phi \in (0, \pi)$,

$$\mathcal{P}(\alpha) - \mathcal{P}(\alpha + \beta) + \frac{\Psi_1(\alpha + \theta) - \Psi_1(\alpha + \beta + \theta)}{\Psi_1(\theta) - \Psi_1(\alpha + \beta + \theta)} (\mathcal{P}(\alpha + \beta) - \mathcal{P}(0)) > 0. \tag{3.61}$$

Indeed, the lower bound in (3.25) now follows from the fact that (3.61), (3.58), (3.59) and (3.60) imply that

$$\begin{aligned}
&\mathrm{Re}(izh'(z)) = \\
&2\theta^2 \sin\phi(1 - \cos\phi)\bigg((\mathcal{P}(\alpha) - \mathcal{P}(\alpha + \beta) + \\
&\frac{\Psi_1(\alpha + \theta) - \Psi_1(\alpha + \beta + \theta)}{\Psi_1(\theta) - \Psi_1(\alpha + \beta + \theta)} (\mathcal{P}(\alpha + \beta) - \mathcal{P}(0)) \bigg).
\end{aligned}$$

From here we see that for $\theta \in [0, 0.5]$ ($\theta = 0$ interpreted as the limit when $\lim_{\theta \to 0+}$), $\alpha \leq 1$ and $\beta \leq 1$, the factor multiplying $2\theta^2 \sin \phi(1 - \cos \phi)$ in the above expression, is positive, while as $\alpha \to \infty$ and $\beta \to \infty$, the dominant term in this factor is $-\frac{\mathcal{P}(0)}{\Psi_1(\theta)}(\Psi_1(\alpha + \theta) - \Psi_1(\alpha + \beta + \theta))$, which gives by compactness the lower bound of (3.25). Let us now introduce the parameter $\rho := \frac{\theta}{\alpha}$.

We will now continue with the proof of (3.61) which will be divided in four cases: Case 1, when $\cos \phi \leq -\rho$; Case 2, when $-\rho \leq \cos \phi \leq -\rho/2$; Case 3, when $-\rho/2 \leq \cos \phi \leq 0$; and Case 4 when $0 \leq \cos \phi \leq 1$. To do this, first note that

$$- \mathcal{P}(0) = \sum_{n=0}^{\infty} \frac{\theta^2 + 2\theta n \cos \phi}{(\theta^2 + 2\theta n \cos \phi + n^2)(\theta + n)^2}. \tag{3.62}$$

In what follows we will assume that $\rho \leq 1$. We will use the following consequence of part (iv) of Lemma 3.15, valid for all $\theta < \min(0.5, \alpha)$ and $\beta > 0$,

$$- (\mathcal{P}(\alpha) - \mathcal{P}(\alpha + \beta))$$
$$\leq \frac{\rho^2 + 2\rho \cos \phi + \rho}{\rho^2 + 2\rho \cos \phi + 1} \mathbf{1}(\cos \phi > -\rho) (\Psi_1(\theta + \alpha) - \Psi_1(\theta + \alpha + \beta)). \tag{3.63}$$

Case 1 ($\cos \phi \leq -\rho$). By part (ii) of Lemma 3.15 we have that $\mathcal{P}(\alpha + \beta) > 0$ while by (3.63), the term $-(\mathcal{P}(\alpha) - \mathcal{P}(\alpha + \beta)) \leq 0$. Therefore it is enough to prove that $-\mathcal{P}(0) > 0$ (whenever $\theta \in (0, 0.5)$). Note that the series (3.62) achieves it's minimum value at $\phi = -\pi$, so we have

$$- \mathcal{P}(0) \geq \sum_{n=0}^{\infty} \frac{\theta^2 - 2\theta n}{(n^2 - \theta^2)^2}$$
$$= \frac{1}{\theta^2} + \frac{\theta^2 - 2\theta}{(1 - \theta^2)^2} - \sum_{n=2}^{\infty} \frac{2\theta n - \theta^2}{(n^2 - \theta^2)^2}$$
$$\geq \frac{1}{\theta^2} + \frac{\theta^2 - 2\theta}{(1 - \theta^2)^2} - \sum_{n=2}^{\infty} \frac{n - \frac{1}{4}}{\left(n^2 - \frac{1}{4}\right)^2} \geq \frac{1}{\theta^2} + \frac{\theta^2 - 2\theta}{(1 - \theta^2)^2} - 0.2$$
$$\geq 4 - \frac{4}{3} - \frac{1}{5} > 0, \tag{3.64}$$

where we used in the last inequality the fact that the terms in the series are decreasing in θ and the minimum value is achieved for $\theta = 0.5$.

Case 2 ($-\rho \leq \cos \phi \leq -\rho/2$). By the assumption $\cos \phi \leq -\frac{\rho}{2}$ and part (ii) Lemma 3.15, we see that $-\mathcal{P}(\alpha + \beta) \leq 0$. This time, the series (3.62) achieves it's minimum value for $\cos \phi = -\rho$. But we will assume that $\rho \leq 0.69$, so we see from (3.64), that it is enough to prove that

$$\frac{1}{\Psi_1(\theta) - \Psi_1(\alpha + \beta + \theta)} \left(\frac{1}{\theta^2} + \frac{\theta^2 - 2\theta \times 0.69}{(\theta^2 - 2\theta \times 0.69 + 1)(1 + \theta)^2} - 0.2 \right)$$
$$\geq \rho \geq \frac{\rho^2 + 2\rho \cos \phi + \rho}{\rho^2 + 2\rho \cos \phi + 1}.$$

Now, $\Psi_1(\theta) - \Psi_1(\theta + \alpha) \leq \Psi_1(\theta)$. Hence, it would be enough to prove that

$$\frac{1}{\theta^2 \Psi_1(\theta)} \left(1 - \theta^2 \frac{2\theta \times 0.69 - \theta^2}{(\theta^2 - 2\theta \times 0.69 + 1)(1 + \theta)^2} - 0.2 \times \theta^2 \right) \geq \rho.$$

Now, since $\theta^2 \Psi_1(\theta)$ is increasing in θ, we see that the inequality is satisfied for all $\rho \leq 0.69$ such that

$$\frac{1}{\frac{1}{4}\Psi_1\left(\frac{1}{2}\right)}\left(1 - \frac{1}{4}\frac{0.69 - \frac{1}{4}}{\left(\frac{1}{4} - 0.69 + 1\right)\left(1 + \frac{1}{2}\right)^2} - 0.2 \times \frac{1}{4}\right) \geq \rho,$$

or

$$\rho \leq 0.6994\cdots.$$

Case 3 $(-\rho/2 \leq \cos\phi \leq 0)$. Let us first bound,

$$-\mathcal{P}(\alpha + \beta) \leq \frac{\theta^2}{(\theta^2 + (\alpha+\beta)^2)(\alpha+\beta+\theta)^2}$$
$$+ \sum_{n=1}^{\infty}\frac{\theta^2}{(\theta^2 + (\alpha+\beta)_n^2)((\alpha+\beta)_n + \theta)^2}$$
$$\leq \frac{1}{\theta^2}\frac{\rho^4}{(1+\rho^2)(1+\rho)^2} + \frac{\theta^2}{\theta^2 + 1}\Psi_1(1). \tag{3.65}$$

On the other hand,

$$-\mathcal{P}(0) \geq \sum_{n=0}^{\infty}\frac{\theta^2 - \theta n}{(\theta^2 - \theta n + n^2)(\theta + n)^2}$$
$$= \frac{1}{\theta^2} - \frac{\theta - \theta^2}{(1 - (\theta - \theta^2))(\theta+1)^2} - \sum_{n=2}^{\infty}\frac{n\theta - \theta^2}{(n^2 - (n\theta - \theta^2))(\theta + n)^2}$$
$$\geq \frac{1}{\theta^2} - \frac{1}{3} - \sum_{n=2}^{\infty}\frac{2n - 1}{(4n^2 - 2n + 1)n^2} \geq \frac{1}{\theta^2} - \frac{1}{3} - 0.65.$$

As in case 2, we now see that it would be enough to prove that

$$\frac{1}{\frac{1}{4}\Psi_1\left(\frac{1}{2}\right)}\left(\frac{1}{\theta^2}\left(1 - \frac{\rho^4}{(1+\rho^2)(1+\rho)^2}\right) - \frac{1}{3} - 0.65 - \frac{\theta^2}{\theta^2 + 1}\Psi_1(1)\right)$$
$$\geq \frac{\rho^2 + \rho}{\rho^2 + 1}.$$

But the left-hand side is bounded from below by the case in which $\theta = 0.5$, from where we see that we have to show that

$$\frac{1}{\frac{1}{4}\Psi_1\left(\frac{1}{2}\right)}\left(4\left(1 - \frac{\rho^4}{(1+\rho^2)(1+\rho)^2}\right) - \frac{1}{3} - 0.65 - \frac{1}{5}\Psi_1(1)\right)$$
$$\geq \frac{\rho^2 + \rho}{\rho^2 + 1},$$

which is satisfied for all $\rho > 0$.

Case 4 $(0 \leq \cos\phi \leq 1)$. Since by part (i) of Corollary 3.6 the difference $\Psi_1(\alpha + \theta) - \Psi_1(\alpha + \beta + \theta)$ is positive, dividing (3.61) by this quantity, we see that now it is enough to show that

$$\frac{\mathcal{P}(\alpha+\beta)-\mathcal{P}(0)}{\Psi_1(\theta)-\Psi_1(\alpha+\beta+\theta)} > \frac{\rho^2+2\rho\cos\phi+\rho}{\rho^2+2\rho\cos\phi+1}\mathbf{1}(\cos\phi>-\rho). \tag{3.66}$$

To do this, first note that

$$-\mathcal{P}(0) = \sum_{n=0}^{\infty} \frac{\theta^2+2\theta n\cos\phi}{(\theta^2+2\theta n\cos\phi+n^2)(\theta+n)^2}.$$

Therefore, the minimum value of $-\mathcal{P}(0)$ is achieved for $\cos\phi=0$, so that

$$-\mathcal{P}(0) \geq \sum_{n=0}^{\infty} \frac{\theta^2}{(\theta^2+n^2)(\theta+n)^2} = \frac{1}{\theta^2}-\mathcal{P}(1),$$

where $-\mathcal{P}(1)$ in the left-hand side is evaluated at $\phi=\pi/2$. On the other hand, for $-\mathcal{P}(\alpha+\beta)$ we have that

$$-\mathcal{P}(\alpha+\beta) = \frac{\theta^2+2\theta(\alpha+\beta)\cos\phi}{(\theta^2+2\theta(\alpha+\beta)\cos\phi+(\alpha+\beta)^2)(\alpha+\beta+\theta)^2}$$
$$+\sum_{n=1}^{\infty} \frac{\theta^2+2\theta(\alpha+\beta)_n\cos\phi}{(\theta^2+2\theta(\alpha+\beta)_n\cos\phi+(\alpha+\beta)_n^2)((\alpha+\beta)_n+\theta)^2}$$
$$\leq \frac{\theta^2+2\theta\alpha\cos\phi}{(\theta^2+2\theta\alpha\cos\phi+\alpha^2)(\alpha+\theta)^2}-\mathcal{P}(\alpha+1)$$
$$\leq \frac{1}{\theta^2}\frac{\rho^2+2\rho}{(\rho+1)^4}-\mathcal{P}(\alpha+1), \tag{3.67}$$

where we have used part (i) of Lemma 3.15. Using again part (i) of Lemma 3.15, we then conclude that

$$\frac{\mathcal{P}(\alpha+\beta)-\mathcal{P}(0)}{\Psi_1(\theta)-\Psi_1(\alpha+\beta+\theta)} \geq \frac{1}{\theta^2\Psi_1(\theta)}\left(1-\frac{\rho^2+2\rho}{(\rho+1)^4}\right)$$
$$\geq \frac{1}{\frac{1}{4}\Psi_1\left(\frac{1}{2}\right)}\left(1-\frac{\rho^2+2\rho}{(\rho+1)^4}\right).$$

Therefore, by part (iv) of Lemma 3.15, it is enough to show that

$$\frac{1}{\frac{1}{4}\Psi_1\left(\frac{1}{2}\right)}\left(1-\frac{\rho^2+2\rho}{(\rho+1)^4}\right) \geq \frac{\rho^2+3\rho}{(\rho+1)^4},$$

which is satisfied for all $\rho>0$.

3.9.2 Proof of Lemma 3.8

The following lemma will be useful to prove Lemma 3.8.

Lemma 3.16. *Let f and g be twice continuously differentiable real functions defined on an interval containing $u < v < w$. If f is convex and strictly decreasing, and $(g''f'-g'f'')(x) \geq \rho(x)$, with $\rho \geq 0$ measurable, then*

$$g(v)-g(w)-\frac{f(v)-f(w)}{f(u)-f(w)}(g(u)-g(w)) \geq \frac{f(v)-f(w)}{f(u)-f(w)}\int_u^v (u-x)\frac{\rho(x)}{f'(x)}dx.$$

Proof. By the chain rule and the inverse function theorem $G = g \circ f^{-1}$ is continuously differentiable on an open interval containing $a < b < c$, the images under f of w, v, and u, respectively. Also, as $G' = (g'/f') \circ f^{-1}$, G' is also continuously differentiable there, with

$$G'' = \frac{g''f' - g'f''}{(f')^3} \circ f^{-1}.$$

By hypothesis f' is negative, so that $G'' \le \rho/(f')^3 \le 0$, that is, we are assuming G is concave. Now let us integrate from b to $y \ge b$ and apply the change of variables $u = f(x)$ to get

$$G'(y) - G'(b) \le \int_b^y \frac{\rho}{(f')^3} \circ f^{-1}(u) du = \int_{f^{-1}(b)}^{f^{-1}(y)} \frac{\rho(x)}{f'(x)^2} dx.$$

Integrating from b to c we have

$$G(c) - G(b) - (c - b)G'(b) \le \int_b^c \int_{f^{-1}(b)}^{f^{-1}(y)} \frac{\rho(x)}{f'(x)^2} \, dx \, dy.$$

Reversing the order of integration –remember f is decreasing– last integral equals

$$\int_{f^{-1}(b)}^{f^{-1}(c)} \int_{f(x)}^c \frac{\rho(x)}{f'(x)^2} \, dy \, dx = \int_u^v (f(x) - f(u)) \frac{\rho(x)}{f'(x)^2} \, dy \, dx.$$

Now, let $t \in [a, b]$ be such that $G'(t) = (G(b) - G(a))/(b - a)$. As G' is decreasing, $G'(t) \ge G'(b)$. In a similar way we have $f(x) - f(u) = f'(\eta)(x - u)$ for an appropriate $\eta \in [u, x]$, and $f'(\eta) \le f'(x)$. With both these inequalities we can write

$$G(c) - G(b) - \frac{c - b}{b - a}(G(b) - G(a)) \le \int_u^v (x - u) \frac{\rho(x)}{f'(x)} \, dx \, dy.$$

Finally, as

$$G(b) - G(a) - \frac{b - a}{c - a}(G(c) - G(a))$$
$$= (G(b) - G(a)) \frac{c - b}{c - a} - (G(c) - G(b)) \frac{b - a}{c - a},$$

multiplying last inequality by $-(b - a)/(c - a)$, we obtain the desired result. $\square$

Let us now prove Lemma 3.8. We have, for $x > 0$ and y real,

$$\operatorname{Im} \Psi(x + iy) = y \, \Phi(x, y),$$

where Φ is defined in (3.26). Note that $\operatorname{Im} h'(\theta + iy) = yH(\theta, y, \alpha, \beta)$, and

$$H(\theta, y, \alpha, \beta) = \Phi(\theta + \alpha, y) - \Phi(\theta + \alpha + \beta, y) - K_1(\Phi(\theta, y) - \Phi(\theta + \alpha + \beta, y)).$$

Applying again Lemma 3.16 we just need to show

$$(\Phi'' \Psi_1' - \Phi' \Psi_1'')(x) \ge -8y^2 \Psi_1'(x) \sum_{n \ge 0} \frac{1}{(x_n^2 + y^2)^3},$$

where the derivatives of Φ are taken with respect to its first variable and the second variable is set as y. Calculating the derivatives and replacing, in particular

$$\Phi''(x) = \sum_{n \ge 0} \frac{6}{(x_n^2 + y^2)^2} - \sum_{n \ge 0} \frac{8y^2}{(x_n^2 + y^2)^3},$$

the above inequality is equivalent to

$$\left(\sum_{n\geq 0}\frac{1}{x_n^4}\right)\left(\sum_{n\geq 0}\frac{x_n}{(x_n^2+y^2)^2}\right)-\left(\sum_{n\geq 0}\frac{x_n}{x_n^4}\right)\left(\sum_{n\geq 0}\frac{1}{(x_n^2+y^2)^2}\right)\geq 0,$$

and this follows by looking at the m-n products (the $m=n$ terms are cero),

$$\frac{1}{x_m^4}\frac{x_n}{(x_n^2+y^2)^2}+\frac{1}{x_n^4}\frac{x_m}{(x_m^2+y^2)^2}-\frac{x_m}{x_m^4}\frac{1}{(x_n^2+y^2)^2}-\frac{x_n}{x_n^4}\frac{1}{(x_m^2+y^2)^2}$$

$$=(x_m-x_n)\left(\frac{1}{x_n^4}\frac{1}{(x_m^2+y^2)^2}-\frac{1}{x_m^4}\frac{1}{(x_n^2+y^2)^2}\right)$$

$$=y^2(x_m+x_n)(x_m-x_n)^2\frac{2x_m^2x_n^2+y^2(x_m^2+x_n^2)}{x_n^4(x_m^2+y^2)^2x_m^4(x_n^2+y^2)^2}\geq 0.$$

4 Perturbative results

Here we will prove Theorem 2.4 and Corollary 2.5. We first need to derive several estimates about Dirichlet and Beta random variables.

Consider a random vector $X=(X_1,\ldots,X_k)$ having a Dirichlet distribution of parameters $\alpha=(\alpha_1,\ldots,\alpha_k)$. Let $\bar{\xi}_i=\log X_i-M_i$ for $1\leq i\leq k$, where M_i are constants whose value will be given later. We want to obtain estimates for the moments,

$$L_{i_1,\ldots,i_k}(\alpha):=\mathbb{E}_\alpha[\bar{\xi}_1^{i_1}\cdots\bar{\xi}_k^{i_k}], \tag{4.1}$$

for the shifted logarithmic moments of the random vector X, with $k\geq 0$, and $i_1,\ldots,i_k\geq 0$. We will call $i_1+\cdots+i_k$ the *degree* of the shifted moment. Consider the following function, which we will call the logarithmic partition function, defined by

$$A(\alpha):=\sum_{i=1}^k\left(\log(\Gamma(\alpha_i))-\alpha_i M_i\right)-\log\Gamma\left(\sum_{i=1}^k\alpha_i\right).$$

Let also

$$B(\alpha)=\frac{\prod_{i=1}^k\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^k\alpha_i\right)}=e^{A(\alpha)}.$$

Note that

$$L_{i_1,\ldots,i_k}(\alpha):=\frac{1}{B(\alpha)}\frac{\partial^{i_1+\cdots+i_k}}{\partial^{i_1}\alpha_1\cdots\partial^{i_k}\alpha_k}B(\alpha). \tag{4.2}$$

From this identity we can recursively compute the shifted moments $L_{i_1,\ldots,i_k}(\alpha)$. Nevertheless, we want a statement giving us a sharp asymptotic bound on the decay of these moments as the parameters tend to ∞ or to 0. We will now define the families of functions corresponding to the higher order derivatives of the logarithmic partition function as,

$$A_{i_1,\ldots,i_k}(\alpha):=\frac{\partial^{i_1+\cdots+i_k}}{\partial^{i_1}\alpha_1\cdots\partial^{i_k}\alpha_k}A(\alpha).$$

We will call $i_1+\cdots+i_k$ the *degree* of $A_{i_1,\ldots,i_k}$ and we will use the notation

$$\mathcal{A}_n:=\{A_{i_1,\ldots,i_k}:i_1+\cdots+i_k=n\},$$

for the set of functions of degree $n\geq 1$. Furthermore, we define the *degree* of a product of functions,

$$\prod_{j=1}^{l} f_j,$$

where $f_j \in \mathcal{A}_{n_j}$ for some $n_j \geq 1$, as the sum $n_1 + \ldots + n_l$. Now, note from (4.2), that we have that

$$L_{1,0,\ldots,0}(\alpha) = A_{1,0,\ldots,0},$$

with analogous equalities for $L_{0,1,0,\ldots,0}$ up to $L_{0,\ldots,0,1}$. Furthermore we have the following recursion formulas,

$$L_{i_1+1,i_2,\ldots,i_k}(\alpha) = L_{1,0,\ldots,0}(\alpha)L_{i_1,\ldots,i_k}(\alpha) + \frac{\partial}{\partial \alpha_1}L_{i_1,\ldots,i_k}(\alpha),$$

$$L_{i_1,i_2+1,i_3,\ldots,i_k}(\alpha) = L_{0,1,0\ldots,0}(\alpha)L_{i_1,\ldots,i_k}(\alpha) + \frac{\partial}{\partial \alpha_2}L_{i_1,\ldots,i_k}(\alpha),$$

$$\vdots$$

$$L_{i_1,\ldots,i_{k-1},i_k+1}(\alpha) = L_{0,\ldots,0,1}(\alpha)L_{i_1,\ldots,i_k}(\alpha) + \frac{\partial}{\partial \alpha_k}L_{i_1,\ldots,i_k}(\alpha). \tag{4.3}$$

From these recursion formulas, we can prove the following lemma.

Lemma 4.1. *Consider the logarithmic moments $L(i_1,\ldots,i_k)$, $i_j \geq 0$, $1 \leq j \leq k$, of a Dirichlet random variable X of parameters $(\alpha_1,\ldots,\alpha_k)$. We have the following representation,*

$$L_{i_1,\ldots,i_k}(\alpha) = \sum_{i=1}^{n} a_i \prod_{j=1}^{l_i} f_j,$$

where $n = i_1 + \cdots + i_k$ is the degree of $L_{i_1,\ldots,i_k}$, $l_1,\ldots,l_n \geq 1$ and $a_1,\ldots,a_n$ are real constants and each term of the sum has the same degree n. Furthermore, one of the terms of this expansion is a product of n functions of degree 1 each.

Proof. It is easy to see that it is true for shifted moments of degree 1. Now, by induction on $i_1 + \cdots + i_k$ and the recursion (4.3), one can check that if the statement is true for all moments of degree $i_1 + \cdots + i_k$, it must also be true for moments of degree $i_1 + \cdots + i_k + 1$. □

We now have the following corollary.

Corollary 4.2. *Consider a family of Dirichlet random variables of parameters $\alpha_{t,1} = \alpha_{t,2} \to \infty$ as $t \to \infty$, while $\alpha_{t,i} \leq 1$ for $3 \leq i \leq 4$. Then, for all $i_1, i_2 \geq 0$ there is a constant $C_8 > 0$ such that*

$$|\mathbb{E}_{\alpha_t}[\bar{\xi}_1^{i_1} \bar{\xi}_2^{i_2}]| \leq \frac{C_8}{\alpha_{t,1}^{\lceil \frac{k}{2} \rceil}},$$

where $i_1 + i_2 = k$.

Proof. Note that

$$\mathbb{E}_{\alpha_t}[\xi_j] = \Psi(\alpha_{t,j}) - \Psi(\sum_{i=1}^{4} \alpha_{t,i}),$$

for $j = 1, 2$. Hence, from the fact that the digamma function has the asymptotics

$$\Psi(x) = \log x - \frac{1}{2x} + o\left(\frac{1}{x}\right), \tag{4.4}$$

when $x \to \infty$, we see that $M_1 = \log(1/2)$. Similarly $M_2 = \log(1/2)$. From (4.4) we also see that

$$L_{1,0,0,0}(\alpha_t) = \frac{c_{25}}{\alpha_{t,1}} + o\left(\frac{1}{\alpha_{t,1}}\right),$$

for some constant $c_{25} > 0$. We can deduce a similar identity for $L_{0,1,0,0}$. On the other hand, note that when $i_1 + i_2 = 2$, we have that

$$L_{i_1,i_2,0,0} = g + f_1 f_2,$$

where $g \in \mathcal{A}_2$ and $f_1, f_2 \in \mathcal{A}_1$. But for functions in $\mathcal{A}_2$ we have the asymptotics

$$g(\alpha) = \frac{c_{26}}{\alpha_{t,1}} + o\left(\frac{1}{\alpha_{t,1}}\right), \tag{4.5}$$

for some constant $c_{26} > 0$ depending on g. In general, for $i_1 + i_2 = 2m$ even, we have from Lemma 4.1 the expansion

$$L_{i_1,i_2,0,0}(\alpha) = \sum_{i=1}^{2m} a_i \prod_{j=1}^{l_i} f_j.$$

Call n_j the degree of f_j, so that $n_1 + \cdots + n_{l_i} = 2m$. Note that

$$f_j = \frac{\partial^\alpha}{\partial^{\alpha_1}\alpha_{t,1}\partial^{\alpha_2}\alpha_{t,2}} A(\alpha_t),$$

for some multi-index $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 + \alpha_2 = n_j$. For $n_j = 1$, this function is of the form

$$\Psi(\alpha_{t,i}) - \Psi(2\alpha_{t,1}),$$

for $i = 1, 2$, so the decay of f_j in this case is bounded by

$$\frac{c_{25}}{\alpha_{t,i}}.$$

For the case $n_j \geq 2$, the function f_j is of the following two possible forms

$$c_{27}(\Psi_{n_j-1}(\alpha_{t,1}) - \Psi_{n_j-1}(2\alpha_{t,1})) \qquad \text{or} \quad c_{28}\Psi_{n_j-1}(\alpha_{t,i}).$$

From part (ii) of Lemma 3.5, in both cases this gives a decay bounded by

$$\frac{c_{29}}{\alpha_{t,i}^{n_j-1}}.$$

This means that the decay of

$$\prod_{i=1}^{l_i} f_j, \tag{4.6}$$

as $t \to \infty$ is

$$\frac{c_{30}}{\alpha_{t,i}^{2m-d}},$$

where d is the number of factors in (4.6) with $n_j \geq 2$. This mean that the dominating term of the form (4.6) is the one which has the highest number of factors of degree larger than 1. It is not difficult to check that this term is

$$\prod_{i=1}^{m} f_i,$$

where $f_i \in \mathcal{A}_2$ for $1 \leq i \leq 2$, which gives, since in this case $d = m$, the decay

$$\frac{c_{31}}{\alpha_{t,i}^m}.$$

For the case in which $i_1 + i_2 = 2m - 1$, there will be several dominant terms of the same degree, inluding one of the form

$$f \prod_{i=1}^{m-1} g_i,$$

where $g_1, \ldots, g_{m-1} \in \mathcal{A}_2$ and $f \in \mathcal{A}_1$, which gives the same decay as the one for the case of degree $2m$. $\qquad\square$

Let us now continue with the proof of Theorem 2.4.

4.1 Proof of Theorem 2.4

We will first need to show that a natural family of Beta random walks matches moments with itself.

Lemma 4.3. *Consider a family of Beta probability measures* $(\mathbb{P}_{\alpha_t, \beta_t})_{t \geq 0}$ *such that (2.3), (2.4) are satisfied. Assume also that*

$$M_1 := \lim_{t \to \infty} \mathbb{E}_{\alpha_t, \beta_t}[\xi_+(0, t)] = \lim_{t \to \infty} (\Psi(\alpha_t) - \Psi(\alpha_t + \beta_t))$$

and

$$M_2 := \lim_{t \to \infty} \mathbb{E}_{\alpha_t, \beta_t}[\xi_-(0, t)] = \lim_{t \to \infty} (\Psi(\beta_t) - \Psi(\alpha_t + \beta_t))$$

exist. Then, for every $k \geq 1$, *and* α *such that* $|\alpha| = k$, *we have that*

$$\left| \mathbb{E}_{\alpha_t, \beta_t}[\bar{\xi}^\alpha(x, t)] \right| \leq \alpha_t^{-\left\lceil \frac{k}{2} \right\rceil}.$$

Hence, for every $k \geq 0$, $(\mathbb{P}_{\alpha_t, \beta_t})_{t \geq 0}$ *matches moments up to order* k *at rate* $\alpha_t^{-\left\lceil \frac{k}{2} \right\rceil}$ *with itself.*

Proof. The proof follows immediately from Corollary 4.2 and the definition of matching moments. $\qquad\square$

Let

$$h_t = \frac{\log\left(P_{0,\omega}(t, x(\theta)t)\right) - I(x(\theta))}{\sigma(\theta)t^{1/3}},$$

where $P_{0,\omega}(t, y)$ is defined in (2.8).

Let $C^k(\mathbb{R})$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ whose derivatives up to order k are uniformly bounded. From the fact that the set $C^k(\mathbb{R})$ is a convergence determining set of functions (see for example [EK86]), Theorem 2.4 follows immediately from the following lemma.

Lemma 4.4. *Consider a family of parameters (α_t, β_t) that satisfy (2.3), (2.4). Let $\theta > 0$. $(\mathbb{P}_t)_{t \geq 0}$ be a family of environmental laws which matches moments up to order k at rate $\alpha_t^{-\left\lceil \frac{k}{2} \right\rceil}$ with $(\mathbb{P}_{\alpha_t, \beta_t})_{t \geq 0}$. Let $\varphi \in C^k(\mathbb{R})$. Then there is a $C_9 > 0$ such that*

$$\left| \mathbb{E}_t[\varphi(h_t)] - \mathbb{E}_{\alpha_t, \beta_t}[\varphi(h_t)] \right| \leq C_9 \frac{t^2 \alpha_t^{-\left\lceil \frac{k}{2} \right\rceil}}{\sigma(\theta) t^{1/3}}.$$

Proof. Let $z \in \mathbb{Z} \times \mathbb{N}$ be a vertex with $z = (v, s)$ for some $v \in \mathbb{Z}$ and $0 \leq s \leq t$. For each $y_1, y_2 \in \mathbb{R}$, define

$$P_\omega(y_1, y_2) = P_\omega^c + e^{y_1 + M_1} P_\omega^+ + e^{y_2 + M_2} P_\omega^-,$$

where

(i) P_ω^c is the probability that $X_t \in A_{t,x(\theta)}$ and that X does not pass through the vertex z,

(ii) P_ω^- is the probability that $X_t = A_{t,x(\theta)}$ and X passes through the edge z, so that $X_s = v$, $X_{s+1} = v + 1$, but with $\omega_+(z) = 1$.

(ii) P_ω^+ is the probability that $X_t = A_{t,x(\theta)}$ and X passes through the edge z, so that $X_s = v$, $X_{s+1} = v - 1$, but with $\omega_-(z) = 1$.

Let

$$h(y_1, y_2) := \frac{\log P_\omega(y_1, y_2) + I(x(\theta))t}{t^{1/3} \sigma(\theta)}.$$

Fixing all weights in the disorder at edges different from f and defining $g(y_1, y_2) := \varphi(h(y_1, y_2))$ we now use Taylor's theorem for $g(y_1, y_2)$ expanding it at $(y_1, y_2) = (0, 0)$ to conclude that

$$\varphi(h(\bar{\xi}_+(z), \bar{\xi}_-(z))) = g(\bar{\xi}_+(z), \bar{\xi}_-(z)) = g(0, 0) + \sum_{i=1}^{2} g_i(0, 0) \bar{\xi}_i$$

$$+ \frac{1}{2} \sum_{i,j=1}^{2} g_{i,j}(0, 0) \bar{\xi}_i \bar{\xi}_j + \cdots + \sum_{i_1, \ldots, i_{k-1}=1}^{2} \frac{1}{(k-1)!} g_{i_1, \ldots, i_{k-1}}(0) \bar{\xi}_{i_1} \cdots \bar{\xi}_{i_{k-1}}$$

$$+ \sum_{i_1, \ldots, i_k=1}^{2} \frac{1}{k!} g_{i_1, \ldots, i_k}(s\bar{\xi}_+(z), s\bar{\xi}_-(z)) \bar{\xi}_{i_1} \cdots \bar{\xi}_{i_k},$$

for some $s \in (0, 1)$, where for $i_1, \ldots, i_j \in \{0, 1\}$,

$$g_{i_1, \ldots, i_j} = \frac{\partial}{\partial y_{i_1}} \cdots \frac{\partial}{\partial y_{i_j}} g,$$

and $\bar{\xi}_1 := \xi_+(z)$ and $\bar{\xi}_2 := \xi_-(z)$. Taking expectation and using the independence of $(\xi_{x,t})_{(x,t) \in \mathbb{Z} \times \mathbb{N}}$, we get that

$$\mathbb{E}_t[\varphi(h(\bar{\xi}_+(z), \bar{\xi}_-(z)))] = a + \sum_{i=1}^{2} a_i \mathbb{E}_t[\bar{\xi}_i]$$

$$+ \frac{1}{2} \sum_{i,j=1}^{2} a_{i,j} \mathbb{E}_t \left[\bar{\xi}_i \bar{\xi}_j \right] + \cdots + \sum_{i_1, \ldots, i_{k-1}=1}^{2} \frac{1}{(k-1)!} a_{i_1, \ldots, i_{k-1}} \mathbb{E}_t[\bar{\xi}_{i_1} \cdots \bar{\xi}_{i_{k-1}}]$$

$$+ \sum_{i_1, \ldots, i_k=1}^{2} \frac{1}{k!} \mathbb{E}_t[g_{i_1, \ldots, i_k}(s\bar{\xi}_+(z), s\bar{\xi}_-(z)) \bar{\xi}_{i_1} \cdots \bar{\xi}_{i_k}], \tag{4.7}$$

where

$$a = \mathbb{E}_t[g(0,0)]$$

and

$$a_{i_1,\ldots,i_j} = \mathbb{E}_t[g_{i_1,\ldots,i_j}(0,\ldots,0)].$$

Now, we will prove that

$$|a_{i_1,\ldots,i_j}| = |g_{i_1,\ldots,i_j}(0,0)| \leq \frac{c_{32}}{t^{1/3}\sigma(\theta)} \tag{4.8}$$

for all $0 \leq j \leq k-1$ and

$$|\bar{a}_{i_1,\ldots,i_k}| := |g_{i_1,\ldots,i_k}(y_1,y_2)| \leq \frac{c_{33}}{t^{1/3}\sigma(\theta)}. \tag{4.9}$$

We have an expansion analogous to (4.7) for $\mathbb{E}_{\alpha_t,\beta_t}[\phi(h(\bar{\xi}_+(z),\bar{\xi}_-(z)))]$, with the same coeffiecients a_j for $0 \leq j \leq k$. We use Faà di Bruno's formula for the chain rule of a composition (here we use the multivariate version, see [KQ18]),

$$g_{i_1,\ldots,i_j}(y) = \sum_{\pi \in \Pi} \varphi^{(|\pi|)}(y) \prod_{B \in \pi} \frac{\partial^{|B|} h}{\prod_{j \in B} \partial y_j},$$

where Π is the set of partitions of $\{1,\ldots,j\}$, π is an arbitrary partition of Π, $|\pi|$ is the number of blocks in the partition π, $B \in \pi$ means that the variable B runs through all the blocks of π, $|B|$ is the size of block B and the variables $y_1,\ldots,y_j$ take only the values y_1 and y_2 (with some abuse of notation). Since by assumption $\phi \in C^k(\mathbb{R})$, it is enough to bound the derivatives $\frac{\partial^{|B|} h}{\prod_{j \in B} \partial y_j}$. Now,

$$\frac{\partial P_\omega(y_1,y_2)}{\partial y_i} = \frac{e^{y_i + M_i} P_\omega^+}{P_\omega^c + e^{y_1+M_1} P_\omega^+ + e^{y_2+M_2} P_\omega^-} =: p_i(y_1,y_2),$$

for $i = 1,2$. We then obtain for the higher order derivatives with $k_1 + k_2 = k$,

$$\frac{\partial^k P_\omega(y_1,y_2)}{\partial^{k_1} y_1 \partial^{k_2} y_2} = \mathcal{P}_{k_1,k_2}(p_1(y),p_2(y)),$$

where the following recursion formula holds,

$$\mathcal{P}_{k_1+1,k_2}(p_1(y),p_2(y)) = \mathcal{P}_{k_1,k_2}^{(1,0)}(p_1(y),p_2(y))p_1^{(1)}(y) + \mathcal{P}_{k_1,k_2}^{(0,1)}(p_1(y),p_2(y))p_2^{(1)}(y),$$

and a similar recursion formula for $\mathcal{P}_{k_1,k_2+1}(p_1(y),p_2(y))$, where $\mathcal{P}^{(1,0)}$ and $\mathcal{P}^{(0,1)}$ are the partial derivatives of $\mathcal{P}$ with respect to its first and second variable respectively, and $p_j^{(i)}$ is the partial derivative of p_j with respect to y_i, $1 \leq i,j \leq 2$. This proves (4.8) and (4.9).

Now, from (4.7), (4.8) and (4.9) (and the corresponding expansion for $\mathbb{E}_{\alpha_t,\beta_t}[\varphi(h(\bar{\xi}_+(z),\bar{\xi}_-(z)))]$, we get that

$$\left| \mathbb{E}_t[\varphi(h(\bar{\xi}_+(z),\bar{\xi}_-(z)))] - \mathbb{E}_{\alpha_t,\beta_t}[\varphi(h(\bar{\xi}_+(z),\bar{\xi}_-(z)))] \right|$$

$$\leq \left(|a| + \sum_{i=1}^2 |a_i| + \sum_{i,j=1}^2 a_{i,j} + \sum_{i_1,\ldots,i_{k-1}} |a_{i_1,\ldots,i_{k-1}}| \right.$$

$$\left. + \sum_{i_1,\ldots,i_{k-1}} \bar{a}_{i_1,\ldots,i_{k-1}} \right) \alpha_t^{-\lceil \frac{k}{2} \rceil}$$

$$\leq \frac{c_{34}\alpha_t^{-\left\lceil \frac{k}{2}\right\rceil}}{t^{1/3}\sigma(\theta)}.$$

Summing over all z that can be reached up to time t starting from $(0,0)$ we finish the proof of the lemma. $\square$

4.2 Proof of Corollary 2.5

Consider a family of Beta random walks with parameters $(\alpha_{t,1}, \alpha_{t,2})_{t\geq 0}$, with $\alpha_{t,1} = \alpha_{t,2} = t^r$. Note that since $r \in (0,1)$ conditions (2.3) and (2.4) are satisfied. Now, using the fact that

$$\sigma(\theta) \sim t^{-r} \qquad \mathfrak{g}(\alpha, \beta) = \frac{1}{2}t^{-r},$$

as $t \to \infty$ (see part (iii) of Corollary 3.6),note that an integer k satisfies (2.9) if and only if

$$\left\lceil \frac{k}{2}\right\rceil > \frac{5}{3r} + 1.$$

Let us denote by $\mathbb{E}'_{\alpha_t}$ the expectation with respect to the environment of this Beta random walk. Recall that the parameters of the Dirichlet environment are $\alpha_{t,1} = \alpha_{t,2} = t^r$, $|\alpha_{t,3}| \leq t^{-p}$ and $\alpha_{t,4} \leq t^{-p}$. On the other hand, note that

$$\begin{aligned}
\left|\mathbb{E}_{\alpha_t}[\bar{\xi}_+] - \mathbb{E}'_{\alpha_t}[\bar{\xi}_+]\right| &= |\Psi(\alpha_{t,1}) - \Psi(2\alpha_{t,1} + 2\alpha_{t,3}) - \Psi(\alpha_{t,1}) + \Psi(2\alpha_{t,1})| \\
&\leq c_{35}\frac{\alpha_{t,3}}{\alpha_{t,1}} \leq c_{35}t^{-(r+p)} \leq c_{36}t^{-r\left\lceil \frac{k}{2}\right\rceil},
\end{aligned} \tag{4.10}$$

for some constants $c_{35} > 0$, $c_{36} > 0$, and the last inequality is satisfied only when

$$p \geq r\left\lceil \frac{k}{2}\right\rceil - r.$$

For higher order moments, using the recursions (4.3), note that the shifted logarithmic moments of the random walk in Dirichlet environment can be obtained from the shifted logarithmic moments of the Beta random walk by changing in all the expressions involving the polygamma functions the sum $\sum_{i=1}^{2}\alpha_{t,i}$ by $\sum_{i=1}^{4}\alpha_{t,i}$. Hence, the same bound (4.10) will be satisfied for the differences between higher order moments. Now choose $k \geq 1$ so that

$$\left\lceil \frac{k}{2}\right\rceil = \left\lceil \frac{5}{3r} - \frac{1}{3}\right\rceil.$$

Note that this is possible because $\frac{5}{3r} - \frac{1}{3} > 0$ for all $r \in (0,1)$. Hence, it is enough to choose p so that

$$p \geq r\left\lceil \frac{5}{3r} - \frac{1}{3}\right\rceil - r.$$

References

[AKQ14] Alberts, T., Khanin, K. and Quastel, J.: The intermediate disorder regime for directed polymers khaninkhanin in dimension 1+1. *Ann. Probab.* **42**, (2014), no. 3, 1212–1256. MR3189070

[AC16] Auffinger, A. and Chen, W.-K.: Universality of chaos and ultrametricity in mixed p-spin models. *Comm. Pure Appl. Math.* **69** (2016), no. 11, 2107–2130. MR3552010

[BC17] Barraquand, G. and Corwin, I.: Random walk in Beta-distributed random environment. *Probab. Theory Related Fields* **167** (2017), no. 3–4, 1057–1116. MR3627433

[BMRS19] Bazaes, R., Mukherjee, C., Ramírez, A.F. and Saglietti, S.: Quenched and averaged large deviation rate functions for random walks in random environments: the impact of disorder, arXiv:1906.05328.

[BC14] Borodin, A. and Corwin, I.: Macdonald processes. *Probab. Theory Related Fields* **158** (2014), no. 1–2, 225–400. MR3152785

[BCF14] Borodin, A., Corwin, I. and Ferrari, P.: Free energy fluctuations for directed polymers in random media in $1 + 1$ dimension. *Comm. Pure Applied Math.* **68** (2014), no. 7, 1129–1214. MR3207195

[BCR13] Borodin, A., Corwin, I. and Remenik, D.: Log-Gamma polymer free energy fluctuations via a Fredholm determinantal identity. *Comm. Math. Phys.* **324** (2013), no. 1, 215–232. MR3116323

[CG17] Corwin, I. and Gu, Y.: Kardar-Parisi-Zhang equation and large deviations for random walks in weak random environments. *J. Stat. Phys.* **166** (2017), no. 1, 150–168. MR3592855

[EK86] Ethier, S. and Kurtz, T.: Markov Processes, Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Physics. *John Wiley & Sons, Inc., New York*, 1986. MR0838085

[K21] Korotkikh, S.: Hidden diagonal integrability of q-hahn vertex model and beta polymer model, arXiv:2105.05058.

[KQ18] Krishnan, A. and Quastel, J.: Tracy-Widom fluctuations for perturbations of the log-gamma polymer in intermediate disorder. *Ann. Appl. Probab.* **28** (2018), no. 6, 3736–3764. MR3861825

[RS14] Rassoul-Agha, F. and Seppäläinen, T.: Quenched point-to-point free energy for random walks in random potentials. *Probab. Theory Related Fields* **158** (2014), no. 3–4, 711–750. MR3176363

[RSY13] Rassoul-Agha, F., Seppäläinen, T. and Yilmaz, A.: Quenched free energy and large deviations for random walks in random potentials. *Comm. Pure Appl. Math.* **66** (2013), no. 2, 202–244. MR2999296

[S12] Seppäläinen, T.: Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.* **40** (2012), no. 1, 19–73. MR2917766

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards

- Free for authors, free for readers

- Quick publication (no backlog)

- Secure publication (LOCKSS[1])

- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4] , ProjectEuclid[5]

- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)

- Submit your best articles to EJP-ECP

- Choose EJP-ECP over for-profit journals

---

[1] LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2] EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
[3] IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4] BS: Bernoulli Society http://www.bernoulli-society.org/
[5] Project Euclid: https://projecteuclid.org/
[6] IMS Open Access Fund: http://www.imstat.org/publications/open.htm