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On the central limit theorem for stationary random fields under $\mathbb{L}^1$-projective condition

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Abstract

The first aim of this paper is to wonder to what extent we can generalize the central limit theorem of Gordin [6] under the so-called $\mathbb{L}^1$-projective criteria to ergodic stationary random fields when completely commuting filtrations are considered. Surprisingly it appears that this result cannot be extended to its full generality and that an additional condition is needed.

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1 Introduction and main results

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be an *ergodic* bijective bimeasurable transformation preserving the probability $\mathbb{P}$. Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{A}$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$ and f be an $\mathbb{L}^1(\mathbb{P})$ real-valued centered random variable adapted to $\mathcal{F}_0$. By U we denote the operator U: $f \mapsto f \circ T$. The notation I will denote the identity operator. Define then the stationary sequence $(f_i)_{i\in\mathbb{Z}}$ by $f_i = f \circ T^i = U^i f$, its associated stationary filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = \mathcal{F}_0 \circ T^{-i}$ and let $S_n(f) = \sum_{i=0}^{n-1} U^i f$.

The following theorem is essentially due to Gordin [6] and gives sufficient conditions for $(U^i f)_{i\in\mathbb{Z}}$ to satisfy the central limit theorem.

Theorem 1.1 (Gordin)**.** *Assume that the series*

$$\sum_{i\geq 0} \mathbb{E}(U^i f|\mathcal{F}_0) \text{ converges in } \mathbb{L}^1(\mathbb{P}) \tag{1.1}$$

and

$$\liminf_{n\to\infty} \frac{\mathbb{E}(|S_n(f)|)}{\sqrt{n}} < \infty. \tag{1.2}$$

Then $n^{-1/2}S_n(f)$ converges in distribution to a centered normal variable (that can be degenerate).

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The proof of this result is based on the following coboundary martingale decomposition (see [11] for more details concerning necessary and sufficient conditions for the existence of such a decomposition): Under (1.1),

$$f = m + (I - U)g \tag{1.3}$$

where m and g are in $\mathbb{L}^1(\mathbb{P})$ and $(U^i m)_{i\geq 0}$ is a stationary sequence of martingale differences, and on the following theorem whose complete proof can be found in [4].

Theorem 1.2 (Esseen-Janson)**.** *If $(U^i m)_{i\geq 0}$ is a stationary and ergodic sequence of martingale differences in $\mathbb{L}^1(\mathbb{P})$ satisfying*

$$\liminf_{n\to\infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i m|)}{\sqrt{n}} < \infty, \tag{1.4}$$

then $m \in \mathbb{L}^2(\mathbb{P})$.

Clearly, using the coboundary martingale decomposition (1.3), (1.2) implies (1.4).

The aim of this paper is to prove that Theorem 1.1 can be extended to random fields when the underlying filtrations are completely commuting (see [7, Chap. 1] for a definition of this notion). To fix the idea, let us first state the result in case of multidimensional index of dimension $d = 2$ (the general case will be stated in Section 4). Then, in complement to the previous notation, let S be an ergodic bimeasurable and measure preserving bijection of Ω. By V we denote the operator $V\colon f \mapsto f \circ S$.

In what follows we shall assume that the ergodic transformations T and S are commuting. Note that $T_{i,j} = T^i S^j$ is an ergodic $\mathbb{Z}^2$ action on $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{F}_{0,0}$ be a sub-sigma field of $\mathcal{A}$ and for all $(i,j) \in \mathbb{Z}^2$ define $\mathcal{F}_{i,j} = T^{-i}S^{-j}(\mathcal{F}_{0,0})$. Suppose that the filtration $(\mathcal{F}_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is increasing in i for every j fixed and increasing in j for every i fixed, and is *completely commuting* in the sense that, for any integrable f,

$$\mathbb{E}(\mathbb{E}(f|\mathcal{F}_{i,j})|\mathcal{F}_{u,v}) = \mathbb{E}(f|\mathcal{F}_{i\wedge u, j\wedge v}).$$

To extend Theorem 1.1 to random fields indexed by the lattice $\mathbb{Z}^2$, the first tool is a suitable coboundary orthomartingale decomposition. In what follows, f is an $\mathcal{F}_{0,0}$-measurable centered $\mathbb{L}^1(\mathbb{P})$ function. According to Volný [10], the condition:

$$\text{the series } \sum_{i,j\geq 0} \mathbb{E}(U^i V^j f|\mathcal{F}_{0,0}) \text{ converges in } \mathbb{L}^1(\mathbb{P}) \tag{1.5}$$

implies the existence of the following decomposition:

$$f = m + (I - U)g_1 + (I - V)g_2 + (I - U)(I - V)g_3, \tag{1.6}$$

where $m, g_1, g_2, g_3 \in \mathbb{L}^1(\mathbb{P})$, $(U^i V^j m)$ is a stationary field of orthomartingale differences, $(V^j g_1)_j$ is a stationary martingale differences sequence with respect to the filtration $(\mathcal{F}_{\infty,j})_j$, and $(U^i g_2)_i$ is a stationary martingale differences sequence with respect to the filtration $(\mathcal{F}_{i,\infty})_i$. To fix the ideas, setting $\mathbb{E}_{a,b}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{a,b})$, we have

$$m = \sum_{i,j\geq 0} \big(\mathbb{E}_{0,0}(U^iV^jf) - \mathbb{E}_{-1,0}(U^iV^jf) - \mathbb{E}_{0,-1}(U^iV^jf) + \mathbb{E}_{-1,-1}(U^iV^jf)\big),$$

$$g_1 = \sum_{i,j\geq 0} \big(\mathbb{E}_{-1,0}(U^iV^jf) - \mathbb{E}_{-1,-1}(U^iV^jf)\big),\ g_2 = \sum_{i,j\geq 0} \big(\mathbb{E}_{0,-1}(U^iV^jf) - \mathbb{E}_{-1,-1}(U^iV^jf)\big),$$

and $g_3 = \sum_{i,j\geq 0} \mathbb{E}_{-1,-1}(U^iV^jf)$. Recall also that (U^iV^jm) is said to be a field of orthomartingale differences w.r.t. $(\mathcal{F}_{i,j})$ if

$$\mathbb{E}_{i-1,j}(U^iV^jm) = \mathbb{E}_{i,j-1}(U^iV^jm) = \mathbb{E}_{i-1,j-1}(U^iV^jm) = 0 \text{ a.s.}$$

Note that if f is additionally assumed to be regular in the sense that f is $\mathcal{F}_{\infty,\infty}$-measurable and $\mathbb{E}(f|\mathcal{F}_{0,-\infty}) = \mathbb{E}(f|\mathcal{F}_{-\infty,0}) = 0$ then, it is proved in Volný [10] that the converse is true, meaning that if f satisfies the decomposition (1.6) then (1.5) holds. We also refer to [3] where the existence of the decomposition (1.6) is proved under a reinforcement of (1.5) (they assume that the series of the $\mathbb{L}^1$-norm is convergent). We also mention [5, Theorem 2.2] where a necessary and sufficient condition for an orthomartingale-coboundary decomposition is established when all the underlying random elements are square integrable.

Our first result is the following:

Theorem 1.3. *Let f be an $\mathcal{F}_{0,0}$-measurable centered $\mathbb{L}^1(\mathbb{P})$ random variable. Let $S_{n_1,n_2}(f) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} U^iV^jf$. Assume that condition* (1.5) *is satisfied and that*

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}}\Big\|\sum_{i=0}^{n-1} U^i f\Big\|_1 < \infty,\ \liminf_{N\to\infty} \frac{1}{\sqrt{N}}\Big\|\sum_{j=0}^{N-1} V^j f\Big\|_1 < \infty \tag{1.7}$$

and

$$\liminf_{n_1\to\infty}\liminf_{n_2\to\infty} \frac{\mathbb{E}(|S_{n_1,n_2}(f)|)}{\sqrt{n_1n_2}} < \infty,\ \liminf_{n_2\to\infty}\liminf_{n_1\to\infty} \frac{\mathbb{E}(|S_{n_1,n_2}(f)|)}{\sqrt{n_1n_2}} < \infty. \tag{1.8}$$

Then the random variables m, $(I-U)g_1$ and $(I-V)g_2$ defined in (1.6) *are in $\mathbb{L}^2(\mathbb{P})$.*

Compared to the case of random sequences a natural question is then to wonder if condition (1.5) together with conditions (1.7) and (1.8) are sufficient to ensure that, when $\min(n_1,n_2)\to\infty$, the limiting distribution behavior of $(n_1n_2)^{-1/2}S_{n_1,n_2}(f)$ is the same as that of the orthomartingale part $(n_1n_2)^{-1/2}S_{n_1,n_2}(m)$. In other terms one can wonder if assuming the conditions of Theorem 1.3 is enough to ensure that the coboundaries' behavior, i.e. $(n_1n_2)^{-1/2}(S_{n_1,n_2}(f) - S_{n_1,n_2}(m))$ is negligible for the convergence in distribution. Surprisingly the answer to this question is negative as shown by the next counterexample.

Theorem 1.4. *There exist a probability space $(\Omega,\mathcal{A},\mu)$, a function $g\in\mathbb{L}^1(\mu)$, measurable with respect to a σ-algebra $\mathcal{F}_{0,0}\subset\mathcal{A}$ and bijective bimeasurable ergodic transformations T and S such that $f=(I-U)g$ is in $\mathbb{L}^2(\mu)$, satisfies the conditions* (1.5), (1.7) *and* (1.8) *but such that $(n_1n_2)^{-1/2}S_{n_1,n_2}(f)$ does not converge in distribution to zero as $\min(n_1,n_2)\to\infty$.*

This result proves a drastically different behavior for the case of random fields with dimension $d\geq 2$ compared to the case of random sequences ($d=1$) for which the coboundary is negligible for the convergence in distribution as soon as (1.1) is assumed. Let us also indicate that even if (1.1) is reinforced to a convergence in $\mathbb{L}^p(\mu)$ for some $p\in[1,2)$ this is still not enough for $(n_1n_2)^{-1/2}S_{n_1,n_2}(f)$ and $(n_1n_2)^{-1/2}S_{n_1,n_2}(m)$ to have the same limiting behavior (to see this it suffices to take $n_k=[2^{k/2}]$ and $m_k\sim(n_k/k)^{p/(2-p)}$ in the construction of the counterexample of Theorem 1.4).

However, reinforcing the conditions of Theorem 1.3, we can prove the following CLT.

Theorem 1.5. *In addition to the conditions of Theorem 1.3, assume that*

$$\lim_{\min(n_1,n_2)\to\infty} \frac{\mathbb{E}(|S_{n_1,n_2}(f)|)}{\sqrt{n_1n_2}} \text{ exists.} \tag{1.9}$$

Then, as $\min(n_1,n_2)\to\infty$, $(n_1n_2)^{-1/2}S_{n_1,n_2}(f)$ converges in distribution to a centered normal variable (that can be degenerate).

Remark 1.6. Under the conditions of Theorem 1.3, condition (1.9) is equivalent to

$$\limsup_{n\to\infty}\limsup_{k\to\infty}\frac{1}{\sqrt{nk}}\|S_{n,k}(f)\|_1 \le \lim_{n\to\infty}\frac{1}{n}\|S_{n,n}(f)\|_1 \text{ and}$$
$$\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{1}{\sqrt{nk}}\|S_{n,k}(f)\|_1 \le \lim_{n\to\infty}\frac{1}{n}\|S_{n,n}(f)\|_1 .$$

This is consequence of the proof of Theorem 1.5. The existence of $\lim_{n\to\infty} n^{-1}\|S_{n,n}(f)\|_1$ has been mentioned there.

It is noteworthy to indicate that f does not need to be in $\mathbb{L}^2$ but only in $\mathbb{L}^1$ to apply Theorem 1.5 (see Example 2.1 given below). Theorem 1.5 then gives alternative projective conditions compared to those required in [12, Th. 5.1] or in [8, Th. 1] for the central limit theorem under the normalization $\sqrt{n_1 n_2}$ to hold. Note that the proofs of the two above mentioned results are also based on an orthomartingale approximation. We refer also to [13] where the notion of orthomartingales and completely commuting filtrations have been previously used in the particular case of functions of iid random fields. Let us also indicate that when filtrations in the lexicographic order rather than completely commuting filtrations are considered, [2, Th. 1] provides a projective type condition in the spirit of the $\mathbb{L}^1$-projective condition (1.5) (but still requiring f to be in $\mathbb{L}^2$) for the normalized partial sums associated with a stationary random field to satisfy the central limit theorem. His proof is based on the so-called Lindeberg method.

2 Examples

2.1 An example when f is in $\mathbb{L}^1$ but not in $\mathbb{L}^2$

For $k \in \mathbb{N}^*$ and $i, j \in \mathbb{Z}$, let $e_{k,i,j}$ be mutually independent zero mean random variables with $Ue_{k,i,j} = e_{k,i+1,j}$, $Ve_{k,i,j} = e_{k,i,j+1}$. Let $\mathcal{F}_{a,b} = \sigma(e_{k,i,j}, k \in \mathbb{Z}, i \le a, j \le b)$. We denote $e_k = e_{k,0,0}$. Let $(v_k)_{k\ge1}$ be a sequence of nonnegative reals and $(p_k)_{k\ge1}$ a sequence of reals in $[0,1]$. Assume that for any $i,j \in \mathbb{Z}^2$, $\mathcal{L}(e_{k,i,j}) = \mathcal{L}(e_k)$ and that e_k takes value v_k with probability p_k, $-v_k$ with probability p_k and 0 with probability $1-2p_k$. It follows that $\|e_k\|_1 = 2v_kp_k$ and $\|e_k\|_2^2 = 2v_k^2p_k$. We use the following selection of $(v_k)_{k\ge1}$ and $(p_k)_{k\ge1}$: $v_k = k^2(\log(k+1))^2$ and $p_k = \frac{1}{2k^2(\log(k+1))^4}$. For any $k \ge 1$ and $i \ge 0$, let $a_{k,i} = (k+i)^{-2}$, and define

$$g = \sum_{k\ge1}\sum_{i\ge0} a_{k,i}U^{-k-i}e_k, \quad h = \sum_{k\ge1}\frac{1}{k}U^{-1}V^{-1}e_k, \quad m = \sum_{k\ge1}\frac{1}{k^2}e_k,$$

and $f = m + (I-U)g + (I-U)(I-V)h$. By simple computations we have $\|g\|_1 < \infty$ and $\|h\|_1 < \infty$ but $\|g\|_2 = \infty$ and $\|h\|_2 = \infty$. In addition m is in $\mathbb{L}^2(\mathbb{P})$. On another hand, f is in $\mathbb{L}^1(\mathbb{P})$ but not in $\mathbb{L}^2(\mathbb{P})$, and one can verify that $\sum_{i,j\ge0}\|\mathbb{E}(U^iV^jf|\mathcal{F}_{0,0})\|_1 < \infty$. Moreover, for any positive integer ℓ, by independence of the r.v.'s $e_{k,i,j}$, we infer that

$$\|(I-U^\ell)g\|_2^2 = \sum_{k\ge1}\sum_{i=0}^{\ell-1} a_{k,i}^2\|e_k\|_2^2 + \sum_{k\ge1}\sum_{i\ge0}(a_{k,i}-a_{k,i+\ell})^2\|e_k\|_2^2 .$$

Hence, by simple algebra, there exists a positive constant C such that

$$\|(I-U^\ell)g\|_2^2 \le C\log(\ell+1) \text{ for any positive integer } \ell. \tag{2.1}$$

In particular, we get $\|(I-U)g\|_2 < \infty$. In addition, by (2.1),

$$\frac{\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^j(I-U)g\|_2^2}{nN} = \frac{\|(I-U^n)g\|_2^2}{n} \to_{n\to\infty} 0,$$

which combined with $\frac{\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^j(I-U)(I-V)h\|_1}{\sqrt{nN}}\to 0$, as $\max(n,N)\to\infty$, implies that $\frac{\|S_{n,N}(f-m)\|_1}{\sqrt{nN}}\to 0$, as $\min(n,N)\to\infty$. Next, since, $\lim_{\min(n,N)\to\infty}\frac{\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^jm\|_1}{\sqrt{nN}}$ exists (it is equal to $\sqrt{\frac{2}{\pi}}\|m\|_2$), we can deduce that $\lim_{\min(n,N)\to\infty}\frac{\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^jf\|_1}{\sqrt{nN}}$ also exists. Hence all the conditions of Theorem 1.5 are satisfied. So, as $\min(n_1,n_2)\to\infty$, $(n_1n_2)^{-1/2}S_{n_1,n_2}(f)$ converges in distribution to a centered normal variable.

2.2 An example where f does not satisfy Hannan's $\mathbb{L}^2$-condition

We exhibit an example where f satisfies all the conditions of Theorem 1.5 but not the Hannan's $\mathbb{L}^2$-condition required in [12, Th. 5.1]. We consider the random field $(e_{k,i,j})_{k,i,j}$ of mutually independent random variables as in Example 2.1 with the following choices of $(v_k)_{k\geq 1}$ and $(p_k)_{k\geq 1}$. Let $\alpha>4$. Then for any $k\geq 1$, define

$$v_k=k^\alpha \text{ and } p_k=\frac{1}{2k^5\log(k+1)^2}.$$

For any $k\geq 1$ and $i,j\geq 0$, let $a_{k,i,j}=(k+i+j)^{-\alpha}$. Then, define

$$f=\sum_{k\geq 1}\sum_{u,v\geq 0}a_{k,u,v}U^{-u}V^{-v}e_k.$$

$(U^iV^jf)_{i,j}$ is usually called a super linear random field. Let $\mathcal{F}_{a,b}=\sigma(e_{k,i,j},k\in\mathbb{Z},i\leq a,j\leq b)$. Note that f is an $\mathcal{F}_{0,0}$-measurable random variable, centered and in $\mathbb{L}^2(\mathbb{P})$. In addition, one can check that condition (1.5) is satisfied implying that the orthomartingale coboundary decomposition (1.6) holds. Moreover, by simple algebra, one can verify that $m\in\mathbb{L}^2(\mathbb{P})$ and that there exists a positive constant K (depending on α) such that $\|(I-U^\ell)g_1\|_2^2+\|(I-U^\ell)g_2\|_2^2\leq K\ell(\log(\ell+1))^{-1}$. Proceeding as in Example 2.1, one can verify that conditions (1.7), (1.8) and (1.9) are satisfied. Hence, Theorem 1.5 applies to $(n_1n_2)^{-1/2}S_{n_1,n_2}(f)$.

On another hand, defining $P_{0,0}(\cdot)=\mathbb{E}_{0,0}(\cdot)-\mathbb{E}_{-1,0}(\cdot)-\mathbb{E}_{0,-1}(\cdot)+\mathbb{E}_{-1,-1}(\cdot)$, we get, for any $i,j\geq 0$,

$$\|P_{0,0}(U^iV^jf)\|_2^2=\Big\|\sum_{k\geq 1}a_{k,i,j}e_k\Big\|_2^2=\sum_{k\geq 1}a_{k,i,j}^2\|e_k\|_2^2\geq\sum_{k\geq i+j+1}\frac{k^{2\alpha-5}}{(k+i+j)^{2\alpha}(\log(k+1))^2},$$

implying that

$$\|P_{0,0}(U^iV^jf)\|_2^2\geq\frac{C}{(i+j+1)^4(\log(i+j+2))^2}.$$

Hence $\sum_{i,j}\|P_{0,0}(U^iV^jf)\|_2$ diverges. So the Hannan's $\mathbb{L}^2$-condition in the random fields setting does not hold, and [12, Th. 5.1] does not apply. Note also that for this example, [2, Th. 1] that involves filtrations in the lexicographic order, cannot be applied.

3 Proofs

3.1 Proof of Theorem 1.3

Recall the decomposition (1.6) and let

$$m'=m+(I-V)g_2. \tag{3.1}$$

It follows that $(U^im')_i$ is a stationary sequence of $\mathbb{L}^1(\mathbb{P})$ martingale differences with respect to $(\mathcal{F}_{i,\infty})_i$. Since T is ergodic, according to Theorem 1.2, if

$$\liminf_{n\to\infty}\frac{\mathbb{E}(|\sum_{i=0}^{n-1}U^im'|)}{\sqrt{n}}<\infty, \tag{3.2}$$

then $m' \in \mathbb{L}^2(\mathbb{P})$. By (1.6), $\left\| \sum_{i=0}^{n-1} U^i m' \right\|_1 \le \left\| \sum_{i=0}^{n-1} U^i f \right\|_1 + 2\|g_1\|_1 + 4\|g_3\|_1$. Hence, since g_1 and g_3 are in $\mathbb{L}^1(\mathbb{P})$, under the first part of (1.7)

$$\liminf_{n\to\infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i m'|)}{\sqrt{n}} \le \liminf_{n\to\infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i f|)}{\sqrt{n}} < \infty,$$

Therefore (3.2) holds and $m' \in \mathbb{L}^2(\mathbb{P})$. Next recall that $m' = m + (I - V)g_2$ and recall that $(V^j m)_j$ is a stationary sequence of $\mathbb{L}^1(\mathbb{P})$ martingale differences with respect to $(\mathcal{F}_{\infty,j})_j$. Since S is ergodic, according again to Theorem 1.2, to prove that $m \in \mathbb{L}^2(\mathbb{P})$, it suffices to prove that

$$\liminf_{n\to\infty} \frac{\mathbb{E}(|\sum_{j=0}^{n-1} V^j m|)}{\sqrt{n}} < \infty. \tag{3.3}$$

But since $m' = m + (I - V)g_2$ and g_2 is in $\mathbb{L}^1(\mathbb{P})$, proving (3.3) is reduced to show that

$$\liminf_{n\to\infty} \frac{\mathbb{E}(|\sum_{j=0}^{n-1} V^j m'|)}{\sqrt{n}} < \infty. \tag{3.4}$$

With this aim, recall first that $m' \in \mathbb{L}^2(\mathbb{P})$ and note that

$$\frac{1}{\sqrt{n}} \Big\| \sum_{j=0}^{n-1} V^j m' \Big\|_1 \le \frac{1}{\sqrt{n}} \Big\| \sum_{j=0}^{n-1} V^j m' \Big\|_2 := \sigma_n. \tag{3.5}$$

For any fixed positive integer n, let $d := n^{-1/2} \sum_{j=0}^{n-1} V^j m'$. Since $(U^i m')_i$ is a stationary and ergodic sequence of martingale differences in $\mathbb{L}^2(\mathbb{P})$, so is $(U^i d)_i$. By the CLT for stationary and ergodic martingales in $\mathbb{L}^2(\mathbb{P})$, as $N \to \infty$, $N^{-1/2} \sum_{i=0}^{N-1} U^i d$ converges in distribution to a centered Gaussian random variable G_n with variance σ_n^2. Hence, by [1, Th. 3.4] and noticing that $\mathbb{E}|G_n| = \sigma_n \sqrt{2/\pi}$, for any fixed positive integer n, we get

$$\sigma_n \le \sqrt{\frac{\pi}{2}} \liminf_{N\to\infty} \frac{1}{\sqrt{N}} \Big\| \sum_{i=0}^{N-1} U^i d \Big\|_1 = \sqrt{\frac{\pi}{2}} \liminf_{N\to\infty} \frac{1}{\sqrt{nN}} \Big\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j m' \Big\|_1. \tag{3.6}$$

But, recalling (1.6) and that $m' = m + (I - V)g_2$, we derive

$$\frac{1}{\sqrt{N}} \Big\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j m' \Big\|_1 \le \frac{1}{\sqrt{N}} \Big\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j f \Big\|_1 + \frac{2n}{\sqrt{N}} \|g_1\|_1 + \frac{4}{\sqrt{N}} \|g_3\|_1.$$

Since g_1 and g_3 are in $\mathbb{L}^1(\mathbb{P})$, the two last terms of the right-hand side are converging to zero as N tends to infinity. Hence, taking into account (3.5) and (3.6), we get

$$\liminf_{n\to\infty} \frac{\mathbb{E}(|\sum_{j=0}^{n-1} V^j m'|)}{\sqrt{n}} \le \sqrt{\frac{\pi}{2}} \liminf_{n\to\infty} \liminf_{N\to\infty} \frac{1}{\sqrt{nN}} \Big\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j f \Big\|_1,$$

which is finite by the second part of condition (1.8). This ends the proof of (3.4) (and then of (3.3)) and leads to the fact that m is in $\mathbb{L}^2(\mathbb{P})$. Next recall that we have proved that m' defined in (3.1) is in $\mathbb{L}^2(\mathbb{P})$ which combined with the fact that m is in $\mathbb{L}^2(\mathbb{P})$ implies that $(I - V)g_2$ is in $\mathbb{L}^2(\mathbb{P})$. On another hand setting $m'' = m + (I - U)g_1$, we can use previous arguments to infer that m'' is in $\mathbb{L}^2(\mathbb{P})$. Hence taking into account that $m \in \mathbb{L}^2(\mathbb{P})$, we get that $(I - U)g_1$ is in $\mathbb{L}^2(\mathbb{P})$. This ends the proof of the theorem.

3.2 Proof of Theorem 1.4

Let $(X_{i,j})_{(i,j)\in\mathbb{Z}^2}$ be an iid random field on $(\Omega,\mathcal{B},\mu)$. Let $\mathcal{A}=\sigma\{X_{i,j},(i,j)\in\mathbb{Z}^2\}$. Then, there exist transformations T and S such that $X_{i,j}\circ T=X_{i+1,j}$ and $X_{i,j}\circ S=X_{i,j+1}$. These transformations on $(\Omega,\mathcal{A},\mu)$ are bijective, commuting, probability preserving and ergodic. Let $Uf=f\circ T$ and $Vf=f\circ S$. Denote $e=X_{0,0}$ and $\mathcal{C}=\sigma\{U^ie: i\in\mathbb{Z}\}$. $(\mathcal{C},T)$ is thus a Bernoulli dynamical system and the sigma algebras $S^j\mathcal{C}$, $j\in\mathbb{Z}$, are mutually independent. Let us recall the so-called Rokhlin lemma.

Lemma 3.1 (Rokhlin lemma). *Let $(\Omega,\mathcal{A},\mu,T)$ be an ergodic dynamical system, n a positive integer, and $\epsilon>0$. Then there exists a set $F\in\mathcal{A}$ such that $F,T^{-1}F,\dots,T^{-n+1}F$ are disjoint and $\mu(\cup_{i=0}^{n-1}T^{-i}F)>1-\epsilon$.*

For any integer $k\geq 1$, we set $n_k=[2^{k/2}]$ and $m_k=2^k$. Let $\epsilon>0$. By Lemma 3.1 there exists a Rokhlin tower $F,T^{-1}F,\dots,T^{-N_k+1}F$ with $N_k=m_kn_k\sim 2^{3k/2}$. Note that $1/N_k\geq\mu(T^{-i}F)>(1-\epsilon)/N_k$, for any $i=0,\dots,N_k-1$. We define

$$g_k(\omega)=\begin{cases}(j+1)\sqrt{m_k/n_k} & \text{on } T^{-j}F,\ \ j=0\dots,n_k-1,\\ (2n_k-j-1)\sqrt{m_k/n_k} & \text{on } T^{-j}F,\ \ j=n_k,\dots,2n_k-1,\\ 0 & \text{on the rest of } \Omega.\end{cases}$$

We can notice that g_k-Ug_k is equal to $\sqrt{m_k/n_k}$ on $T^{-j}F$, $j=0\dots,n_k-1$, to $-\sqrt{m_k/n_k}$ on $T^{-j}F$, $j=n_k\dots,2n_k-1$, and to 0 on the rest of Ω. Moreover, we have

$$\|g_k-Ug_k\|_2^2\leq\frac{2}{n_km_k}\sim\frac{2}{2^{k/2}},\ \|g_k\|_1\leq 2\sqrt{\frac{n_k}{m_k}}\leq 2^{1-k/4}\text{ and}$$

$$\Big\|\sum_{i=0}^{2n_k-1}U^i(g_k-Ug_k)\Big\|_2=\|g_k-U^{2n_k}g_k\|_2\leq 2\sqrt{n_k}.$$

Notice that all sums of $U^i(g_k-Ug_k)$ are $\mathcal{C}$-measurable hence the random variables $(V^j(g_k-U^{2n_k}g_k))_j$ are iid. In addition, the support of $g_k-U^{2n_k}g_k$ is included in the union B_k of $F,\dots,T^{-4n_k+1}F$ hence is of measure $\leq 4/m_k=2^{-k+2}$. Next, for any $k\geq 1$, let $A_k=\{|g_k-U^{2n_k}g_k|\geq(1/2)\sqrt{m_kn_k}\}$. The set A_k is included in B_k and is of measure $2/m_k=2^{-k+1}$. Because the sigma algebras $S^i\mathcal{C}$, $i\in\mathbb{Z}$, are mutually independent, the sets $S^{-j}A_k$, $j=0,\dots,m_k-1$, are independent. For $h=1_{A_k}$ using that $\mathrm{e}^{2\ln(1-x)/x}\geq\mathrm{e}^{-4}$ for any $x\in]0,1/2]$ and that $m_k\mu(A_k)=2$, we thus have, for any $k\geq 2$,

$$\mu\Big(\sum_{j=0}^{m_k-1}V^jh=1\Big)=m_k\mu(A_k)(1-\mu(A_k))^{m_k-1}\geq 2/e^4.$$

We then conclude that

$$\mu\Big(\frac{1}{\sqrt{n_k}}\frac{1}{\sqrt{m_k}}\Big|\sum_{i=0}^{2n_k-1}\sum_{j=0}^{m_k-1}U^iV^j(g_k-Ug_k)\Big|\geq 1/2\Big)\geq 2/e^4. \tag{3.7}$$

By recursion we shall define a strictly increasing sequence $k_\ell\nearrow\infty$ and then set

$$g=\sum_{\ell=1}^{\infty}g_{k_\ell}\text{ and } f=g-Ug.$$

For $\ell=1$ we put $k_\ell=1$. Suppose now that for $1\leq\ell'<\ell$ the $k_{\ell'}$ have been defined.

All the functions $g_{k_{\ell'}}$ are bounded ($0\leq g_{k_{\ell'}}\leq\sqrt{n_{k_{\ell'}}m_{k_{\ell'}}}$) hence the sums

$$\sum_{i=0}^{n-1}\sum_{\ell'=1}^{\ell-1}U^i(g_{k_{\ell'}}-Ug_{k_{\ell'}})=\sum_{\ell'=1}^{\ell-1}(g_{k_{\ell'}}-U^ng_{k_{\ell'}}),$$

$n \geq 1$, are uniformly bounded. If k_ℓ is sufficiently large we thus get

$$\frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=1}^{\ell-1} \Big| \sum_{i=0}^{2n_{k_\ell}-1} U^i (g_{k_{\ell'}} - U g_{k_{\ell'}}) \Big| < 1/2^\ell .$$

Next note that $\Big(V^j \Big(\frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=1}^{\ell-1} \sum_{i=0}^{2n_{k_\ell}-1} U^i (g_{k_{\ell'}} - U g_{k_{\ell'}}) \Big) \Big)_{j \geq 0}$ are martingale differences. Hence

$$\Big\| \frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m_{k_\ell}}} \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m_{k_\ell}-1} U^i V^j \sum_{\ell'=1}^{\ell-1} (g_{k_{\ell'}} - U g_{k_{\ell'}}) \Big\|_2 \leq \frac{1}{2^\ell} . \tag{3.8}$$

On another hand, recall that $\|g_k - U g_k\|_2 \leq \sqrt{2}/2^{k/4}$. Hence choosing k_ℓ sufficiently large we get $\big\| \sum_{i=0}^{2n_{k_{\ell'}}-1} U^i (g_{k_\ell} - U g_{k_\ell}) \big\|_2 \leq 4^{-\ell}$, for all $1 \leq \ell' < \ell$. Having constructed the sequence of k_ℓ this way we thus have $\Big\| \frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=\ell+1}^{\infty} \sum_{i=0}^{2n_{k_\ell}-1} U^i (g_{k_{\ell'}} - U g_{k_{\ell'}}) \Big\|_2 < 2^{-\ell}$. Since $\Big(V^j \Big(\frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=\ell+1}^{\infty} \sum_{i=0}^{2n_{k_\ell}-1} U^i (g_{k_{\ell'}} - U g_{k_{\ell'}}) \Big) \Big)_{j \geq 0}$ are martingale differences, we get

$$\Big\| \frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m_{k_\ell}}} \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m_{k_\ell}-1} U^i V^j \sum_{\ell'=\ell+1}^{\infty} (g_{k_{\ell'}} - U g_{k_{\ell'}}) \Big\|_2 \leq \frac{1}{2^\ell} . \tag{3.9}$$

Then, the upper bounds (3.8) and (3.9) entail that

$$\Big\| \frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m_{k_\ell}}} \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m_{k_\ell}-1} U^i V^j (I - U)(g - g_{k_\ell}) \Big\|_2 \leq \frac{2}{2^\ell} . \tag{3.10}$$

Hence taking into account (3.7) and (3.10), it follows that, for $f = g - Ug$, the sequence $(n_1 n_2)^{-1/2} \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} U^i V^j f$ cannot converge in distribution to zero.

On another hand, for any p and q fixed, by independence, $\sum_{i=0}^{p} \sum_{j=0}^{q} \mathbb{E}(U^i V^j f | \mathcal{F}_{0,0}) = g - \mathbb{E}(U^{p+1} g | \mathcal{F}_{0,0})$. But, by construction, $\lim_{p \to \infty} \|U^{p+1} g\|_1 = 0$. Hence $\|\mathbb{E}(U^{p+1} g | \mathcal{F}_{0,0})\|_1$ is going to zero as $p \to \infty$. Therefore condition (1.5) is satisfied.

It remains to prove that the conditions (1.7) and (1.8) are satisfied with $f = g - Ug$. With this aim, note first that $n^{-1/2} \sum_{i=0}^{n-1} U^i f = n^{-1/2} (g - U^n g) \to_{n \to \infty} 0$ in $\mathbb{L}^1$ (recall that $g \in \mathbb{L}^1$). Next, since the random variables $(V^j f)_{j \geq 0}$ are independent and square integrable, $m^{-1/2} \Big\| \sum_{j=0}^{m-1} V^j f \Big\|_2 = \|g - Ug\|_2 < \infty$. Hence both conditions in (1.7) are satisfied. On another hand, for every m fixed,

$$\frac{1}{\sqrt{n}} \frac{1}{\sqrt{m}} \Big\| \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} U^i V^j (g - Ug) \Big\|_1 \leq \frac{2\sqrt{m}}{\sqrt{n}} \|g\|_1 \to_{n \to \infty} 0,$$

proving the second part of condition (1.8). It remains to prove its first part. Here we use particular properties of g constructed above. We have found a sequence of n_{k_ℓ} for which there exists a positive constant $c > 0$ such that $(n_{k_\ell})^{-1/2} \big\| \sum_{i=0}^{2n_{k_\ell}-1} U^i (g - Ug) \big\|_2 \leq c\big(2^{-\ell} + 1\big)$. Since $\big(V^j \sum_{i=0}^{2n_{k_\ell}-1} U^i (g - Ug) \big)_{j \geq 0}$ is a stationary sequence of martingale differences in $\mathbb{L}^2$, it follows that

$$\frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m}} \Big\| \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m-1} U^i V^j (g - Ug) \Big\|_2 \leq c\big(2^{-\ell} + 1\big) \leq 2c.$$

Since the upper bound is uniform for all n_{k_ℓ}, the first part of condition (1.8) holds true.

3.3 Proof of Theorem 1.5

We shall use the coboundary decomposition (1.6). Note first that $(U^iV^jm)_{i,j}$ is an ergodic and stationary $\mathbb{L}^2(\mathbb{P})$ orthomartingale field. Then, according to the CLT for ergodic fields of martingale differences as obtained in [9], as $\min(n_1,n_2) \to \infty$, the sequence $(n_1n_2)^{-1/2}\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}U^iV^jm$ converges in distribution to a centered Gaussian random variable with variance $\|m\|_2^2$. Theorem 1.5 then follows if one can prove that under the conditions of Theorem 1.3 and if condition (1.9) is satisfied, then

$$\lim_{\min(n_1,n_2)\to\infty} \frac{\|S_{n_1,n_2}(f) - S_{n_1,n_2}(m)\|_1}{\sqrt{n_1n_2}} = 0. \tag{3.11}$$

Clearly, since $\Big\|\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}U^iV^j(I-U)(I-V)g_3\Big\|_1 \le 4\|g_3\|_1$, the convergence (3.11) will follow if one can prove that, as $\min(n_1,n_2)\to\infty$,

$$(n_1n_2)^{-1/2}\Big(\Big\|\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}U^iV^j(I-U)g_1\Big\|_1 + \Big\|\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}U^iV^j(I-V)g_2\Big\|_1\Big) \to 0. \tag{3.12}$$

Since $(V^j(I-U)g_1)_{j\ge0}$ and $(U^i(I-V)g_2)_{j\ge0}$ are sequences of martingale differences in $\mathbb{L}^2(\mathbb{P})$, we shall rather prove (3.12) in $\mathbb{L}^2(\mathbb{P})$ and show that

$$\lim_{n\to\infty}\frac{\|(I-U^n)g_1\|_2}{\sqrt{n}} = 0 \text{ and } \lim_{n\to\infty}\frac{\|(I-V^n)g_2\|_2}{\sqrt{n}} = 0. \tag{3.13}$$

With this aim, we start by noticing that, for any n fixed, $d_{1,n} := \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}U^i\big(m + (I-U)g_1\big)$ is such that $(V^jd_{1,n})_{j\ge0}$ is an ergodic and stationary sequence of $\mathbb{L}^2(\mathbb{P})$ martingale differences with respect to the filtration $(\mathcal{F}_{\infty,j})_j$. Hence, by the CLT for ergodic and stationary martingales, as $N\to\infty$, $N^{-1/2}\sum_{j=1}^N V^jd_{1,n} \to^{\mathcal{D}} G_{1,n}$ where $G_{1,n}$ is a centered random Gaussian with standard deviation $C_n = \|d_{1,n}\|_2$. Since $\big(N^{-1/2}\big|\sum_{j=1}^N V^jd_{1,n}\big|\big)_{N\ge1}$ is uniformly integrable, by the convergence of moments theorem (see [1, Th. 3.5]) we have in particular that $\lim_{N\to\infty}\frac{1}{\sqrt{N}}\Big\|\sum_{j=1}^N V^jd_{1,n}\Big\|_1 = \|G_{1,n}\|_1 = \sqrt{\frac{2}{\pi}}C_n$. But $\lim_{N\to\infty}\frac{1}{\sqrt{N}}\big\|\sum_{j=0}^{N-1}V^jd_{1,n}\big\|_1 = \lim_{N\to\infty}\frac{1}{\sqrt{nN}}\big\|S_{n,N}(f)\big\|_1$. So, overall, for any n fixed, $\lim_{N\to\infty}\frac{1}{\sqrt{nN}}\big\|S_{n,N}(f)\big\|_1 = \sqrt{\frac{2}{\pi}}C_n$, implying by standard arguments that there exists an increasing subsequence (n_k) tending to infinity such that

$$\lim_{k\to\infty}\Big|\frac{1}{\sqrt{n_kk}}\big\|S_{n_k,k}(f)\big\|_1 - \sqrt{\frac{2}{\pi}}C_{n_k}\Big| = 0. \tag{3.14}$$

Next, note that $C_n^2 = \|m\|_2^2 + \frac{2}{n}\mathbb{E}\Big((I-U^n)g_1\sum_{i=0}^{n-1}U^im\Big) + \frac{\|(I-U^n)g_1\|_2^2}{n}$. But, since $(V^j(I-U^n)g_1)_{j\ge0}$ is an ergodic and stationary sequence of $\mathbb{L}^2(\mathbb{P})$ martingale differences with respect to the filtration $(\mathcal{F}_{\infty,j})_{j\ge0}$, we have, by using [1, Th. 3.4] and arguments used to get (3.6), $\|(I-U^n)g_1\|_2 \le \sqrt{\frac{\pi}{2}}\liminf_{N\to\infty}\frac{1}{\sqrt{N}}\Big\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^j(I-U)g_1\Big\|_1$. Moreover, according to the coboundary decomposition (1.6), for any n fixed,

$$\liminf_{N\to\infty}\frac{1}{\sqrt{N}}\Big\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^j(I-U)g_1\Big\|_1 = \liminf_{N\to\infty}\frac{1}{\sqrt{N}}\Big\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^j(f-m)\Big\|_1.$$

In addition, $\Big\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^j(f-m)\Big\|_1 \le \Big\|\sum_{i=0}^{n-1}\sum_{j=0}^{N-1}U^iV^jf\Big\|_1 + \sqrt{nN}\|m\|_2$. So, overall, taking into account condition (1.9), we get

$$\kappa := \limsup_{n\to\infty}\frac{\|(I-U^n)g_1\|_2}{\sqrt{n}} \le \sqrt{\frac{\pi}{2}}\Big(\|m\|_2 + \lim_{n,N\to\infty}\frac{1}{\sqrt{nN}}\big\|S_{n,N}(f)\big\|_1\Big) < \infty. \tag{3.15}$$

Now, for any positive real A, write

$$\Big|C_n^2 - \|m\|_2^2 - \frac{\|(I-U^n)g_1\|_2^2}{n}\Big| \le \frac{2A}{\sqrt{n}}\|(I-U^n)g_1\|_1 + 2\frac{\|(I-U^n)g_1\|_2}{\sqrt{n}} \times \Big(\frac{1}{n}\mathbb{E}\Big(\Big|\sum_{i=1}^n U^i m\Big|^2 \mathbf{1}_{\{|\sum_{i=1}^n U^i m|>A\sqrt{n}\}}\Big)\Big)^{1/2}. \quad (3.16)$$

Hence, using that $n^{-1/2}\|(I-U^n)g_1\|_1 \to_{n\to\infty} 0$ and taking into account (3.15) and the fact that $\big(n^{-1}(\sum_{i=0}^{n-1} U^i m)^2\big)_{n\ge 1}$ is uniformly integrable, we derive that the terms in the right-hand side of (3.16) tend to zero by letting first n go to infinity and after A. Therefore $\lim_{n\to\infty}\Big|C_n^2 - \|m\|_2^2 - \frac{\|(I-U^n)g_1\|_2^2}{n}\Big| = 0$. Assume now that

$$\kappa = \limsup_{n\to\infty} \frac{1}{\sqrt{n}}\|(I-U^n)g_1\|_2 > 0. \quad (3.17)$$

Then, there exists an increasing subsequence $(n'_\ell)_{\ell\ge 1}$ tending to infinity such that

$$\lim_{\ell\to\infty} \frac{1}{\sqrt{n'_\ell}}\|(I-U^{n'_\ell})g_1\|_2 = \kappa \text{ and then } \lim_{\ell\to\infty} C_{n'_\ell}^2 = \|m\|_2^2 + \kappa^2. \quad (3.18)$$

According to (3.14) and (3.18), we then infer that if (3.17) holds true then there exist two increasing subsequences $(n''_\ell)_{\ell\ge 1}$ and $(k''_\ell)_{\ell\ge 1}$ tending to infinity such that

$$\lim_{\ell\to\infty} \frac{1}{\sqrt{n''_\ell k''_\ell}}\Big\|S_{n''_\ell,k''_\ell}(f)\Big\|_1 > \sqrt{\frac{2}{\pi}}\|m\|_2. \quad (3.19)$$

But, using once again the coboundary decomposition (1.6), note that

$$\frac{1}{n}S_{n,n}(f) = \frac{1}{n}S_{n,n}(m) + \frac{1}{n}(I-U^n)\sum_{j=1}^n V^j g_1 + \frac{1}{n}(I-V^n)\sum_{i=1}^n U^i g_2 + \frac{1}{n}(I-V^n)(I-V^n)g_3.$$

Birkhoff's theorem in $\mathbb{L}^1(\mathbb{P})$ implies that $\lim_{n\to\infty}\frac{1}{n}\|(I-U^n)\sum_{j=1}^n V^j g_1\|_1 = 0$ and $\lim_{n\to\infty}\frac{1}{n}\|(I-V^n)\sum_{i=1}^n U^i g_2\|_1 = 0$. Using in addition that g_3 is in $\mathbb{L}^1(\mathbb{P})$, we get

$$\lim_{n\to\infty}\frac{1}{n}\|S_{n,n}(f)\|_1 = \lim_{n\to\infty}\frac{1}{n}\|S_{n,n}(m)\|_1. \quad (3.20)$$

But since $n^{-1}S_{n,n}(m)$ converges in distribution to a centered Gaussian random variable with variance $\|m\|_2^2$ and $(\frac{1}{n}|S_{n,n}(m)|)_{n\ge 1}$ is uniformly integrable, we derive, by the convergence of moments theorem, that $\lim_{n\to\infty}\frac{1}{n}\|S_{n,n}(m)\|_1 = \sqrt{\frac{2}{\pi}}\|m\|_2$. This result together with (3.20) imply that

$$\lim_{n\to\infty}\frac{1}{n}\|S_{n,n}(f)\|_1 = \sqrt{\frac{2}{\pi}}\|m\|_2. \quad (3.21)$$

Clearly, under condition (1.9), if (3.17) is supposed to be true, (3.19) and (3.21) are not compatible. This proves that (3.17) cannot be true and then that the first part of (3.13) is satisfied. With similar arguments, one can prove that, provided the additional condition (1.9) is assumed, the second part of (3.13) is also satisfied. This ends the proof of the theorem.

4 Extension to multidimensional index of higher dimension

To state the extension of Theorems 1.3 and 1.5 to higher dimensions, some additional notations are needed. Let $d \geq 1$ and $(T_{\underline{i}})_{\underline{i}\in\mathbb{Z}^d}$ be $\mathbb{Z}^d$ actions on $(\Omega, \mathcal{A}, \mathbb{P})$ generated by commuting invertible and measure-preserving transformations T_{ε_q}, $1 \leq q \leq d$. Here ε_q is the vector of $\mathbb{Z}^d$ which has 1 at the q-th place and 0 elsewhere. By $U_{\underline{i}}$ we denote the operator in $\mathbb{L}^p$ ($1 \leq p \leq \infty$) defined by $U_{\underline{i}} f = f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$. By $\underline{i} \leq \underline{j}$, we understand $i_k \leq j_k$ for all $1 \leq k \leq d$. Let $<d> := \{1, \dots, d\}$. For any subset J of $<d>$, let $U_J = \prod_{\ell\in J} U_{\varepsilon_\ell}$ and $U_J^{\underline{s}} = \prod_{\ell \in J} U_{\varepsilon_\ell}^{s_\ell}$ for any $\underline{s} = (s_\ell)_{\ell\in J}$ in $\mathbb{Z}^{|J|}$.

We suppose that there is a completely commuting filtration $(\mathcal{F}_{\underline{j}})_{\underline{j}\in\mathbb{Z}^d}$, i.e. there is a σ-algebra $\mathcal{F}_{\underline{0}}$ such that $\mathcal{F}_{\underline{i}} = T_{-\underline{i}}\mathcal{F}_{\underline{0}}$, for $\underline{i} \leq \underline{j}$ we have $\mathcal{F}_{\underline{i}} \subset \mathcal{F}_{\underline{j}}$ and for an integrable f,

$$\mathbb{E}(\mathbb{E}(f|\mathcal{F}_{i_1,\dots,i_d})|\mathcal{F}_{j_1,\dots,j_d}) = \mathbb{E}(f|\mathcal{F}_{i_1\wedge j_1,\dots,i_d\wedge j_d}).$$

By $\mathcal{F}_\ell^{(k)}$ we denote the σ-algebra generated by all $\mathcal{F}_{\underline{i}}$ with $\underline{i} = (i_1, \dots, i_d)$ with $i_k \leq \ell$ and $i_j \in \mathbb{Z}$ for $1 \leq j \leq d$, $j \neq \ell$. For σ-algebras $\mathcal{G} \subset \mathcal{F} \subset \mathcal{A}$, by $\mathbb{L}^p(\mathcal{F}) \ominus \mathbb{L}^p(\mathcal{G})$ we denote the space of $f \in \mathbb{L}^p(\mathcal{F})$ for which $\mathbb{E}(f|\mathcal{G}) = 0$ a.s.

For f an $\mathcal{F}_{\underline{0}}$-measurable centered $\mathbb{L}^1(\mathbb{P})$ random variable, it has been proved in Volný [10, Th. 4] that the condition

$$\text{the series} \sum_{i_1,\dots,i_d \geq 0} \mathbb{E}(U_{i_1,\dots,i_d} f|\mathcal{F}_{\underline{0}}) \text{ converges in } \mathbb{L}^1(\mathbb{P}) \tag{4.1}$$

ensures the existence of the following orthomartingale-coboundary decomposition:

$$f = m + \sum_{\emptyset \neq J \subsetneq <d>} \prod_{s\in J}(I - U_{\varepsilon_s}) m_J + \prod_{s=1}^{d}(I - U_{\varepsilon_s}) g \tag{4.2}$$

where m, g and m_J belong to $\mathbb{L}^1(\mathcal{F}_{\underline{0}}, \mathbb{P})$, $\mathbb{L}^1(\prod_{s=1}^d T_{\varepsilon_s}\mathcal{F}_{\underline{0}}, \mathbb{P})$ and $\mathbb{L}^1(\prod_{s\in J} T_{\varepsilon_s}\mathcal{F}_{\underline{0}}, \mathbb{P})$ respectively and $(U^{\underline{i}}_{<d>} m)_{\underline{i}\in\mathbb{Z}^d}$ and $(U^{\underline{i}}_{J^c} m_J)_{\underline{i}\in\mathbb{Z}^{d-|J|}}$ are orthomartingale differences random fields for $\emptyset \neq J \subsetneq <d>$. Above $J^c = <d> \setminus J$.

For any positive integer k less than d, define $\mathcal{S}_{k,d}$ the set of all the injections from $\{1, \dots, k\}$ to $\{1, \dots, d\}$. We state now the extension of Theorems 1.3 and 1.5.

Theorem 4.1. *Let $d \geq 1$ and f a $\mathcal{F}_{\underline{0}}$-measurable centered $\mathbb{L}^1(\mathbb{P})$ random variable. Let $\underline{n}_d = (n_1, \dots, n_d)$ and $S_{\underline{n}_d}(f) = \sum_{i_1=0}^{n_1-1} \dots \sum_{i_d=0}^{n_d-1} U_{i_1,\dots,i_d} f$. Assume that each of the transformations T_{ε_q}, $1 \leq q \leq d$, is ergodic. Suppose also that condition* (4.1) *holds and that for any integer $k \in \{1, \dots, d\}$ and any σ in $\mathcal{S}_{k,d}$,*

$$\liminf_{n_{\sigma(1)}\to\infty} \cdots \liminf_{n_{\sigma(k)}\to\infty} \frac{\mathbb{E}\big(\big|\sum_{i_1=0}^{n_{\sigma(1)}-1} \dots \sum_{i_k=0}^{n_{\sigma(k)}-1} U_{i_1,\dots,i_k} f\big|\big)}{(\prod_{i=1}^{k} n_{\sigma(i)})^{1/2}} < \infty. \tag{4.3}$$

Then $m \in \mathbb{L}^2(\mathbb{P})$ and for any set J such that $\emptyset \neq J \subsetneq <d>$, $\prod_{s\in J}(I - U_{\varepsilon_s}) m_J \in \mathbb{L}^2(\mathbb{P})$ (m and m_J are defined in (4.2)*). If in addition,*

$$\lim_{\min(n_1,n_2,\dots,n_d)\to\infty} \frac{\mathbb{E}(|S_{\underline{n}_d}(f)|)}{(\prod_{i=1}^{d} n_i)^{1/2}} \text{ exists,} \tag{4.4}$$

then $(\prod_{i=1}^d n_i)^{-1/2} S_{\underline{n}_d}(f)$ converges in distribution to a centered normal variable (that can be degenerate) as $\min(n_1, n_2, \dots, n_d) \to \infty$.

Proof of Theorem 4.1. The result will follow by recurrence. Note that it holds for $d = 1$ and also for $d = 2$ as shown in the previous section. Assume that it holds for $d - 1$ and let us prove it for d. Recall the decomposition (4.2) and let

$$m' = m + \sum_{\emptyset \neq J \subseteq <d>_1} \prod_{s\in J}(I - U_{\varepsilon_s}) m_J, \tag{4.5}$$

where $<d>_1=<d> \setminus\{1\} = \{2,\dots,d\}$. Note that $(U^i_{\varepsilon_1} m')_{i\in\mathbb{Z}}$ is a stationary sequence of $\mathbb{L}^1(\mathbb{P})$ martingale differences w.r.t. $(\mathcal{F}_{i,0,\dots,0})_{i\in\mathbb{Z}}$. Since T_{ε_1} is ergodic, by Theorem 1.2, if $\liminf_{n\to\infty} n^{-1/2}\|\sum_{i=0}^{n-1} U^i_{\varepsilon_1} m'\|_1 < \infty$, then $m' \in \mathbb{L}^2(\mathbb{P})$. This follows from (4.2) and the fact that, by condition (4.3), $\liminf_{n\to\infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i_{\varepsilon_1} f|)}{\sqrt{n}} < \infty$. Next, starting from (4.5), and taking into account the induction hypothesis, namely: Theorem 4.1 holds for $d-1$, we infer that if for any integer k in $[2,d]$ and any injection σ from $\{2,\dots,k\}$ to $<d>_1$,

$$\liminf_{n_{\sigma(2)}\to\infty} \cdots \liminf_{n_{\sigma(k)}\to\infty} \frac{\mathbb{E}\big(\big|\sum_{i_2=0}^{n_{\sigma(2)}-1} \dots \sum_{i_k=0}^{n_{\sigma(k)}-1} U_{0,i_2,\dots,i_k} m'\big|\big)}{(\prod_{i=2}^k n_{\sigma(i)})^{1/2}} < \infty, \tag{4.6}$$

then $m \in \mathbb{L}^2(\mathbb{P})$ and, for any set J such that $\emptyset \neq J \subsetneq <d>_1$, $\prod_{s\in J}(I - U_{\varepsilon_s}) m_J \in \mathbb{L}^2(\mathbb{P})$. By using similar arguments as those developed in the proof of Theorem 1.3, we infer that (4.6) is satisfied under condition (4.3). Hence $m \in \mathbb{L}^2(\mathbb{P})$. Then, using in addition that $m' \in \mathbb{L}^2(\mathbb{P})$, we conclude that, for any set J such that $\emptyset \neq J \subseteq <d>_1$, $\prod_{s\in J}(I - U_{\varepsilon_s}) m_J \in \mathbb{L}^2(\mathbb{P})$. The first part of Theorem 4.1 follows by using $d-1$ times the same arguments and replacing $<d>_1$ by $<d>_i$ for $i = 2,\dots,d$. The second part of the theorem follows by applying the CLT for ergodic orthomartingales as proved in Volný [9] for $(\prod_{i=1}^d n_i)^{-1/2} S_{\underline{n}_d}(m)$ and by using similar arguments as those developed in the proof of Theorem 1.5. □

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