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An application of the Gaussian correlation inequality to the small deviations for a Kolmogorov diffusion*

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Abstract

We consider an iterated Kolmogorov diffusion X_t of step n. The small ball problem for X_t is solved by means of the Gaussian correlation inequality. We also prove Chung's laws of iterated logarithm for X_t both at time zero and infinity.

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Contents

1 Introduction

Let $\{X_t\}_{0 \leqslant t \leqslant T}$ be an $\mathbb{R}^n$-valued stochastic process with continuous paths such that $X_0 = 0$ a.s. where $T > 0$ is fixed. Denote by $W_0(\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued continuous functions on $[0, T]$ starting at zero. Given a norm $\| \cdot \|$ on $W_0(\mathbb{R}^n)$, the small ball problem for X_t consists in finding the rate of explosion of

$$-\log \mathbb{P}\left(\|X\| < \varepsilon\right)$$

as $\varepsilon \to 0$. More precisely, a process X_t is said to satisfy a *small deviation principle* with rates α and β if there exist a constant $c > 0$ such that

$$\lim_{\varepsilon \to 0} -\varepsilon^{\alpha} |\log \varepsilon|^{\beta} \log \mathbb{P}\left(\|X\| < \varepsilon\right) = c. \tag{1.1}$$

The values of α, β and c depend on the process X_t and on the chosen norm on $W_0\left(\mathbb{R}^n\right)$. Small deviation principles have many applications including metric entropy estimates

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and Chung's law of the iterated logarithm. We refer to the survey paper [13] for more details.

We say that a process X_t satisfies *Chung's law of the iterated logarithm* (LIL) as $t \to \infty$ (resp. as $t \to 0$) with rate $a \in \mathbb{R}_+$ if there exists a constant C such that

$$\liminf_{t\to\infty} \left(\frac{\log\log t}{t}\right)^a \max_{0\leqslant s\leqslant t} |X_s| = C, \quad a.s. \tag{1.2}$$

(resp. $\liminf_{t\to 0} \left(\frac{\log|\log t|}{t}\right)^a \max_{0\leqslant s\leqslant t} |X_s| = C$ a.s.). When X_t is a Brownian motion, it was proven in a famous paper by K.-L. Chung in 1948 that (1.2) holds with $a = \frac{1}{2}$ and $C = \frac{\pi}{\sqrt{8}}$. To find the rates α and β such that the limit in (1.1) exists, and then findind the constant c is an extremely hard problem in general. Even the estimation of the rate of explosion of (1.1) is usually a difficult problem. Indeed, as can be surmised in [10, 15], the small ball problem for Gaussian processes is equivalent to metric entropy problems in functional analysis. In [11] and [20] a Brownian sheet in Hölder and uniform norm is considered, and the integrated Brownian motion in the uniform norm is the content of [8], and the m-fold integrated Brownian motion in both the uniform and L^2-norm is considered in [4]. In [17] and [3] a small deviation principle and Chung's LIL are proved for a class of stochastic integrals and for a hypoelliptic Brownian motion on the Heisenberg group. When X_t is a Gaussian process with stationary increments, upper and lower bounds on (1.1) can be found in [19, 16].

In this paper we consider the Kolmogorov diffusion of step n.

Definition 1.1. *Let $T > 0$ and b_t be a one-dimensional standard Brownian motion. The stochastic process $\{X_t\}_{0\leqslant t\leqslant T}$ on $\mathbb{R}^n$ defined by*

$$X_t := \left(b_t, \int_0^t b_{t_2} dt_2, \int_0^t \int_0^{t_2} b_{t_3} dt_3 dt_2, \ \dots \ , \int_0^t \int_0^{t_2} \cdots \int_0^{t_{n-1}} b_{t_n} dt_n \dots dt_2\right)$$

is the Kolmogorov diffusion of step n.

$\{X_t\}_{0\leqslant t\leqslant T}$ is a Markov process with generator given by $L = \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \sum_{d=2}^n x_{d-1}\frac{\partial}{\partial x_d}$. In particular, when $n = 2$ X_t is the Markov process associated to the differential operator $L = \frac{1}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}$ and it was first introduced by A. N. Kolmogorov in [9], where he obtained an explicit expression for its transition density. Later, L. Hörmander in [6] used L as the simplest example of a hypoelliptic second order differential operator. More precisely, the operator L satisfies the weak Hörmander condition. $\{X_t\}_{0\leqslant t\leqslant T}$ is a Gaussian process and its law μ is a Gaussian measure on the Banach space $(W_0(\mathbb{R}^n), \|\cdot\|)$, where

$$\|f\| := \max_{0\leqslant t\leqslant T} |f(t)|, \quad \forall f \in W_0(\mathbb{R}^n).$$

The main result of this paper is Theorem 2.6, where we prove the small deviation principle (1.1) for X_t with rates $\alpha = 2$, $\beta = 0$, and constant $c = \frac{\pi}{\sqrt{8}}$. Our proof relies on the Gaussian correlation inequality (GCI), see e.g. [18, 12], applied to the Gaussian measure μ on $W_0(\mathbb{R}^n)$. A different application of the GCI to estimate small balls probabilities is given in [14]. In Theorem 2.6 we also state Chung's LIL at time zero and infinity for X_t with rates given by $a = \frac{1}{2}$ and $a = \frac{2n-1}{2}$ respectively.

The stochastic processes considered in [8, 17, 3] all satisfy a scaling property, that is, there exists a scaling constant $\delta \in (0,\infty)$ such that $X_{\varepsilon t} \overset{(d)}{=} \varepsilon^\delta X_t$. Properties of Gaussian measures on Banach spaces and scaling properties have been used to show the existence of a small deviation principle for some processes such as a Brownian motion with values in a finite dimensional Banach space in [5] and an integrated Brownian motion in [8]. Moreover, in [8, 17, 3] the scaling rate δ coincides with the rate of Chung's LIL at infinity, and the small deviations' rates are given by $\beta = 0$, $\alpha = \frac{1}{\delta}$. The Kolmogorov diffusion does not satisfy a scaling property with respect to the standard Euclidean norm, and the small deviations rate α is not related to the Chung's LIL rate.

Lastly, large deviations and Chung's LIL at time zero for the limsup of the Kolmogorov diffusion are discussed in Section [1, Section 4.2] and [2, Example 3.5] respectively.

The paper is organized as follows. In Section 2 we collect some examples and state the main result of this paper, namely, small deviation principle and Chung's LIL at time zero and infinity for a step n Kolmogorov diffusion. Section 3 contains the proof of the main result.

2 The setting and main results

Notation 2.1. 4 Let X_t be an $\mathbb{R}^n$-valued stochastic process with $X_0 = 0$ a.s. Then X_t^* denotes the process defined by

$$X_t^* := \max_{0 \leqslant s \leqslant t} |X_s|,$$

where $|\cdot|$ denotes the Euclidean norm.

Notation 2.2. [Dirichlet eigenvalues in $\mathbb{R}^n$] We denote by $\lambda_1^{(n)}$ the lowest Dirichlet eigenvalue of $-\frac{1}{2}\Delta_{\mathbb{R}^n}$ on the unit ball in $\mathbb{R}^n$.

Let us collect some examples of Chung's LIL and small deviation principle.

Example 2.3. [Brownian motion] Let X_t be a standard Brownian motion. Then $X_{\varepsilon t} \overset{(d)}{=} \varepsilon^{\frac{1}{2}} X_t$, and it satisfies the small deviation principle

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(X_T^* < \varepsilon\right) = \lambda_1^{(1)} T, \tag{2.1}$$

where $\lambda_1^{(1)}$ is defined in Notation 2.2, see e.g. [7, Lemma 8.1]. Moreover, in a famous paper by K.-L. Chung in 1948 it was proven that

$$\liminf_{t \to \infty} \left(\frac{\log \log t}{t}\right)^{\frac{1}{2}} \max_{0 \leqslant s \leqslant t} |X_t| = \sqrt{\lambda_1^{(1)}} \quad a.s. \tag{2.2}$$

Example 2.4. [Integrated Brownian motion]. Let $X_t := \int_0^t b_s ds$, where b_s is a one-dimensional standard Brownian motion. It is easy to see that $X_{\varepsilon t} \overset{(d)}{=} \varepsilon^{\frac{3}{2}} X_t$. In [8] it is shown that there exists a finite constant $c_0 > 0$ such that

$$\liminf_{t \to \infty} \left(\frac{\log \log t}{t}\right)^{\frac{3}{2}} \max_{0 \leqslant s \leqslant t} |X_t| = c_0 \quad a.s. \tag{2.3}$$

and (2.3) was used to prove that

$$\lim_{\varepsilon \to 0} -\varepsilon^{\frac{2}{3}} \log \mathbb{P}\left(X_1^* < \varepsilon\right) = c_0^{\frac{2}{3}}.$$

Example 2.5. [Iterated integrated Brownian motion] Let b_t be a one-dimensional Brownian motion starting at zero. Denote by $X_1(t) := b_t$ and

$$X_d(t) := \int_0^t X_{d-1}(s) ds, \quad t \geqslant 0, d \geqslant 2,$$

the d-fold integrated Brownian motion for a positive integer d. Note that $X_d(\varepsilon t) \overset{(d)}{=} \varepsilon^{\frac{2d-1}{2}} X_d(t)$. In [4] it was shown that for any integer d there exists a constant $\gamma_d > 0$ such that

$$\lim_{\varepsilon \to 0} -\varepsilon^{\frac{2}{2d-1}} \log \mathbb{P}\left(\max_{0 \leqslant t \leqslant 1} |X_d(t)| < \varepsilon \right) = \gamma_d^{\frac{2}{2d-1}},$$

$$\liminf_{t \to \infty} \left(\frac{\log \log t}{t} \right)^{\frac{2d-1}{2}} \max_{0 \leqslant s \leqslant t} |X_d(s)| = \gamma_d \quad a.s. \tag{2.4}$$

Our main object is the Kolmogorov diffusion on $\mathbb{R}^n$ defined by

$$X_t := (X_1(t), \dots, X_n(t)),$$

where

$$X_d(t) := \int_0^t \int_0^{t_2} \cdots \int_0^{t_{d-1}} b_{t_d} dt_d \cdots dt_2, \ \text{ for } \ d = 3, \dots, n,$$

and $X_2(t) := \int_0^t b_s ds$, $X_1(t) := b_t$, where b_t is a one-dimensional standard Brownian motion. Note that $X_d(\varepsilon t) \stackrel{(d)}{=} \varepsilon^{\frac{2d-1}{2}} X_d(t)$ for all $d = 1, \dots, n$, and hence the process X_t does not have a scaling property with respect to the Euclidean norm $|\cdot|$ in $\mathbb{R}^n$.

Theorem 2.6. *Let $T > 0$ and X_t be the Kolmogorov diffusion on $\mathbb{R}^n$. Then*

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(X_T^* < \varepsilon \right) = \lambda_1^{(1)} T, \tag{2.5}$$

$$\liminf_{t \to 0} \sqrt{\frac{\log |\log t|}{t}} \max_{0 \leqslant s \leqslant t} |X_s| = \sqrt{\lambda_1^{(1)}} \quad a.s. \tag{2.6}$$

$$\liminf_{t \to \infty} \left(\frac{\log \log t}{t} \right)^{\frac{2n-1}{2}} \max_{0 \leqslant s \leqslant t} |X_s| = \gamma_n \quad a.s. \tag{2.7}$$

where $\lambda_1^{(1)}$ is defined in Notation 2.2, and γ_n is given by (2.4)

Remark 2.7. By (2.2) and Brownian inversion, it follows that a standard Brownian motion satisfies Chung's LIL at time zero and infinity with rate $a = \frac{1}{2}$, and it satisfies a small deviation principle with rate $\alpha = 2$. By (2.5), the n-step Kolmogorov diffusion X_t satisfies the same small deviation principle as a one-dimensional standard Brownian motion. As far as Chung's LIL for X_t is concerned, the first component dominates when $t \to 0$ with rate $a = \frac{1}{2}$, and the n-th component dominates as $t \to \infty$ with rate $a = \frac{2n-1}{2}$.

3 Proofs

Proof of Theorem 2.6. Let us first prove the small deviation principle (2.5). One has that $\mathbb{P}\left(X_T^* < \varepsilon \right) \leqslant \mathbb{P}\left(b_T^* < \varepsilon \right)$, and hence by (2.1) it follows that

$$\lambda_1^{(1)} T \leqslant \liminf_{\varepsilon \to 0} -\varepsilon^2 \mathbb{P}\left(X_T^* < \varepsilon \right).$$

Let us now show that

$$\limsup_{\varepsilon \to 0} -\varepsilon^2 \mathbb{P}\left(X_T^* < \varepsilon \right) \leqslant \lambda_1^{(1)} T.$$

For any $x_1, \dots, x_n \in (0, 1)$ such that $x_1 + \cdots + x_n = 1$ we have that

$$\mathbb{P}\left(X_T^* < \varepsilon \right) \geqslant \mathbb{P}\left(\max_{0 \leqslant t \leqslant T} |X_1(t)| < x_1 \varepsilon, \dots, \max_{0 \leqslant t \leqslant T} |X_n(t)| < x_n \varepsilon, \right)$$
$$\geqslant \mathbb{P}\left(\max_{0 \leqslant t \leqslant T} |X_1(t)| < x_1 \varepsilon \right) \cdots \mathbb{P}\left(\max_{0 \leqslant t \leqslant T} |X_n(t)| < x_n \varepsilon \right),$$

where in the second line we used the Gaussian correlation inequality for the law of the process $\{X_t\}_{0\leqslant t\leqslant T}$ which is a Gaussian measure on $W_0(\mathbb{R}^n)$. Thus,

$$-\varepsilon^2 \log \mathbb{P}\left(X_T^* < \varepsilon\right) \leqslant -\sum_{d=1}^{n} \varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant t\leqslant T} |X_d(t)| < x_d \varepsilon\right). \tag{3.1}$$

Note that, for any $d = 2, \dots, n$

$$\max_{0\leqslant t\leqslant T} |X_d(t)| \leqslant \int_0^T \int_0^{t_2} \cdots \int_0^{t_{d-2}} \max_{0\leqslant t\leqslant T} |X_2(t)| dt_{d-1} \cdots dt_2 = \frac{T^{d-2}}{(d-2)!} \max_{0\leqslant t\leqslant T} |X_2(t)|,$$

and hence

$$\mathbb{P}\left(\max_{0\leqslant t\leqslant T} |X_2(s)| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon\right) \leqslant \mathbb{P}\left(\max_{0\leqslant t\leqslant T} |X_d(s)| < x_d \varepsilon\right), \tag{3.2}$$

and by [8, Theorem 1.1] we have that, for any $d = 2, \dots, n$

$$\begin{aligned} 0 &\leqslant \limsup_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant t\leqslant T} |X_d(t)| < x_d \varepsilon\right) \\ &\leqslant \lim_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant t\leqslant T} |X_2(t)| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon\right) \\ &= \lim_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant t\leqslant T} \left|\int_0^t b_s ds\right| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon\right) = 0. \end{aligned} \tag{3.3}$$

Thus, by (3.1) and (3.2)

$$-\varepsilon^2 \log \mathbb{P}\left(X_T^* < \varepsilon\right) \leqslant -\varepsilon^2 \log \mathbb{P}\left(b_T^* < x_1 \varepsilon\right) - \sum_{d=2}^{n} \varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant t\leqslant T} |X_2(t)| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon\right),$$

and by (3.3) and (2.1) it follows that

$$\limsup_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}\left(X_T^* < \varepsilon\right) \leqslant \frac{\lambda_1^{(1)}}{x_1^2} T.$$

The result follows by letting x_1 go to one.

Let us now prove (2.6). By (2.2) and Brownian time inversion one has that

$$\liminf_{t\to 0} \sqrt{\frac{\log|\log t|}{t}} \max_{0\leqslant s\leqslant t} |b_s| = \sqrt{\lambda_1} \quad a.s. \tag{3.4}$$

Note that

$$\begin{aligned} |b_s|^2 \leqslant |X_s|^2 &= |b_s|^2 + \sum_{d=2}^{n} |X_d(s)|^2 \\ &\leqslant |b_s|^2 + \max_{0\leqslant u\leqslant s} |b_u|^2 \sum_{d=2}^{n} \frac{s^{2d-2}}{(d-1)!^2}, \end{aligned}$$

and hence

$$\begin{aligned} \frac{\log|\log t|}{t} \max_{0\leqslant s\leqslant t} |b_s|^2 &\leqslant \frac{\log|\log t|}{t} \max_{0\leqslant s\leqslant t} |X_s|^2 \\ &\leqslant \frac{\log|\log t|}{t} \max_{0\leqslant s\leqslant t} |b_s|^2 \left(1 + \sum_{d=2}^{n} \frac{t^{2d-2}}{(d-1)!^2},\right) \end{aligned}$$

By (3.4) it follows that, for any $d = 2, \dots, n$

$$\lim_{t\to 0} t^{2d-2} \frac{\log|\log t|}{t} \max_{0\leqslant s\leqslant t} |b_s|^2 = 0 \quad a.s.$$

and thus

$$\liminf_{t\to 0} \frac{\log|\log t|}{t} \max_{0\leqslant s\leqslant t} |X_s|^2 = \liminf_{t\to 0} \frac{\log|\log t|}{t} \max_{0\leqslant s\leqslant t} |b_s|^2 = \lambda_1 \quad a.s.$$

which completes the proof of (2.6). Let us now prove (2.7). Set

$$\phi(t) := \frac{\log\log t}{t}.$$

By (2.4) we have that, for any $d = 1, \dots, n-1$

$$\liminf_{t\to\infty} \phi(t)^{\frac{2n-1}{2}} \max_{0\leqslant s\leqslant t} |X_d(s)| = \liminf_{t\to\infty} \phi(t)^{n-d} \phi(t)^{\frac{2d-1}{2}} \max_{0\leqslant s\leqslant t} |X_d(s)| = 0 \;\; a.s. \tag{3.5}$$

since $\phi(t) \to 0$ as $t \to \infty$. Note that

$$|X_n(s)|^2 \leqslant |X_s|^2 = \sum_{d=1}^{n-1} |X_d(s)|^2 + |X_n(s)|^2,$$

and hence

$$\begin{aligned} \phi(t)^{2n-1} \max_{0\leqslant s\leqslant t} |X_n(s)|^2 &\leqslant \phi(t)^{2n-1} \max_{0\leqslant s\leqslant t} |X_s|^2 \\ &\leqslant \sum_{d=1}^{n-1} \phi(t)^{2n-1} \max_{0\leqslant s\leqslant t} |X_d(s)|^2 + \phi(t)^{2n-1} \max_{0\leqslant s\leqslant t} |X_n(s)|^2. \end{aligned}$$

Thus, by (2.4) and (3.5) it follows that

$$\liminf_{t\to\infty} \phi(t)^{2n-1} \max_{0\leqslant s\leqslant t} |X_s|^2 = \liminf_{t\to\infty} \phi(t)^{2n-1} \max_{0\leqslant s\leqslant t} |X_n(s)|^2 = \gamma_n^2 \quad a.s.$$

and (2.7) is proven. □

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