Electron. Commun. Probab. **27** (2022), article no. 17, 1–5.
https://doi.org/10.1214/22-ECP456
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Erratum: The remainder in the renewal theorem*

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Abstract

We point out an error in "The remainder in the renewal theorem", and show that the result is essentially correct in two important special cases.

Keywords: asymtotic stability; random walks; regular variation; renewal theorem.
MSC2020 subject classifications: 60G50; 60K05.
Submitted to ECP on November 5, 2021, final version accepted on February 12, 2022.

The main result in [1] claims that in a renewal process $S = (S_n, n \geq 0)$ whose step distribution F has finite mean m and whose tail $\overline{F}$ is regularly varying with index $-\alpha$ with $\alpha \in (1, 2)$, the renewal function U has the following asymptotic behaviour:

$$W(x) := U(x) - m^{-1} \int_0^x (1 + \overline{\Phi}(y))dy \backsim \frac{|c_\alpha| x \overline{\Phi}(x)^2}{m|2\beta - 1|} \text{ as } x \to \infty. \tag{0.1}$$

Here

$$\beta = \alpha - 1, c_\alpha = \frac{(1 - 2\beta)\Gamma(1 - \beta)^2}{\Gamma(2 - 2\beta)},$$

and

$$\overline{\Phi}(x) = \int_x^\infty \phi(y)dy, \text{ with } \phi(y) = \frac{\overline{F}(y)}{m}, \; y \geq 0. \tag{0.2}$$

Since $\overline{\Phi} \in RV(-\beta)$ the RHS of (0.1) $\in RV(1-2\beta)$, so this is a substantial improvement on the previously known result that $W(x) = o(\int_0^x \overline{\Phi}(y)dy)$, particularly for the case $\beta > 1/2$.

If F is non-lattice, it is natural for ϕ to be involved, since it is the stationary density for the overshoot process of S, which fact is used in [1] to derive the following relation. First write ϕ_2 for the convolution $\phi * \phi$ and define real-valued functions g and $\overline{G}$ on $[0, \infty)$ by

$$\begin{aligned} g(y) &= 2\phi(y) - \phi_2(y), \\ \overline{G}(y) &= \int_y^\infty g(z)dz, \text{ so that } \overline{G}(0) = 1. \end{aligned}$$

Then the relation

$$W(x) = \int_{[0,x)} \overline{G}(x - y)U(dy), \tag{0.3}$$

*Corrected article: https://doi.org/10.1214/20-ECP287.

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which is (2.4) in [1], is key to the results therein. (Note that our W is denoted by $m^{-1}V$ in [1].) The second crucial fact is that although $\overline{G}$ is the difference of two functions which are both in $RV(-\beta)$, it is in $RV(-2\beta)$, and actually

$$\lim_{x\to\infty}\frac{\overline{G}(x)}{\overline{\Phi}(x)^2}=c_\alpha=\frac{(1-2\beta)\Gamma(1-\beta)^2}{\Gamma(2-2\beta)}. \tag{0.4}$$

Unfortunately there are mistakes in the proof of (0.1) for the case $\alpha\in(3/2,2)$. Specifically on P5, L9 of [1], it is claimed that having fixed $x_0>0$ such that $g^*(x):=-g(x)>0$ for $x>x_0$, then given $\varepsilon>0$ we can find $x_1>x_0$ with

$$\int_{x_1}^{x}\overline{G^*}(x-y)dU(y)\le\frac{1+\varepsilon}{m}\int_{x_1}^{x}\overline{G^*}(x-y)dy, \tag{0.5}$$

where $\overline{G^*}(z):=-\overline{G}(z)$. But on $[0,x_0]$ we have no control over the sign of $\overline{G^*}$, so this statement cannot be justified. It is also unclear how Lemma 2.1 can be applied, since the condition $\int_0^\infty Q(y)dy=\infty$ fails for $\alpha>3/2$. A final error is that it is implicitly assumed in [1] that in the lattice case ϕ is a stationary density, but of course this is wrong: actually $\{\phi(n),n\in\boldsymbol{Z}\}$ is a stationary *mass function.*

Nevertheless the claimed result (0.1) is essentially correct in the two most important situations. In the lattice case the last mentioned error necessitates a slight change in the definition of W, (see (1.1) below and compare the LHS of (0.1)), but then we are able to give a simple argument to show that (0.1) holds with this new definition. In the absolutely continuous case, under a minor technical assumption we show that (0.1) follows, after some manipulation and use of (0.3), from a stronger result for the density of U which is established in [2].

1 Lattice case

In this section we assume that F is carried by $\boldsymbol{Z}$ and has period 1, and we specify that the renewal function U and its modification W are given for $x\ge 0$ by

$$U(x)=\sum_{r=0}^{[x]}u(r),\ W(x)=U(x)-m^{-1}(\sum_{s=0}^{[x]}(1+\overline{\Phi}(s)), \tag{1.1}$$

where $u(r)=\sum_0^\infty P(S_n=r)$. We start from the observation that the distribution with *mass function*

$$\phi(n)=\frac{P(X>n)}{m}=\frac{\overline{F}(n)}{m},\ n=0,1,2,\cdots$$

is stationary for the overshoot process. $\overline{\Phi}$ is the tail function of this discrete distribution, so

$$\overline{\Phi}(x)=\overline{\Phi}(n)=\sum_{m=n+1}\phi(m)\text{ for }n\le x<n+1,\ n=0,1,2,\cdots.$$

With this definition it is clear that W is piece-wise constant, and it follows that (0.1) will hold in general if it holds as $x\to\infty$ through the integers, and we will now establish this.

Again the functions g and $\overline{G}$ are defined by $g(n)=2\phi(n)-\phi_2(n)$, $n=0,1,2,\cdots$ where $\phi_2(n)$ is the discrete convolution $\sum_0^n\phi(r)\phi(n-r)$ and for $x\ge 0$

$$\overline{G}(x)=\sum_{m=[x]+1}g(m)=\overline{G}([x]). \tag{1.2}$$

The stationarity of ϕ gives $\sum_0^n U(r)\phi(n-r) = m^{-1}(n+1)$, and then

$$\begin{aligned}\sum_0^n U(r)\phi_2(n-r) &= \sum_{r=0}^n U(r)\sum_{s=0}^{n-r}\phi(s)\phi(n-r-s)\\ &= \sum_{s=0}^n \phi(s)\sum_{r=0}^{n-s}\phi(n-r-s)U(r)\\ &= m^{-1}\sum_{s=0}^n \phi(s)(n+1-s) = m^{-1}(n+1-\sum_{s=0}^n \overline{\Phi}(s)).\end{aligned}$$

So

$$\sum_0^n U(r)g(n-r) = m^{-1}(n+1+\sum_{s=0}^n \overline{\Phi}(s)),$$

then

$$W(n) = U(n) - \sum_0^n U(r)g(n-r),$$

and summation by parts gives

$$W(n) = \sum_0^n u(r)\overline{G}(n-r). \tag{1.3}$$

This is the discrete analogue of (0.3). Next we see that the proof in [1] of (0.4) is also valid in this lattice case, with minor changes. Finally when $\beta > 1/2$ the condition $\int_0^\infty \overline{G}(y)dy = 0$ also holds, but because of (1.2) it is equivalent to

$$\sum_{m=0}^\infty \overline{G}(m) = 0. \tag{1.4}$$

It is straightforward to see that the results in Theorem 1.1 of [1] when $\beta \le 1/2$ hold in this lattice case with x restricted to the integers, and we now show that the same is true when $\beta > 1/2$. Recalling that $c_\alpha < 0$ in this case, so that $\overline{G^*}(n) = -\overline{G}(n)$ is positive for all large n, we assume we know that

$$|\sum_0^n (u_n - u_{n-r})\overline{G}(r)| = o(n\overline{G^*}(n)) \text{ as } n\to\infty. \tag{1.5}$$

Then (1.3) and (1.4) give

$$\begin{aligned}W(n) &= u_n\sum_0^n \overline{G}(r) + o(n\overline{G^*}(n))\\ &= \frac{1}{m}\sum_{n+1}^\infty \overline{G^*}(r) + o(n\overline{G^*}(n)) \backsim \frac{|c_\alpha| n\overline{\Phi}(n)^2}{m(2\beta-1)},\end{aligned}$$

which is the required result. Next suppose that with $\Delta_n := u_n - u_{n-1}$ we have $n\Delta_n \to 0$ as $n\to\infty$. For any fixed $\delta\in(0,1)$ we can bound the LHS of (1.5) by $S_1+S_2+S_3$, where

$$\begin{aligned}S_1 &= \max_{n(1-\delta)\le m\le n}|u_n - u_{n-m}|.\sum_{n(1-\delta)}^n |\overline{G^*}(m)| \le c\sum_{n(1-\delta)}^n \overline{G^*}(m),\\ S_2 &= \max_{n\delta\le m\le n(1-\delta)}|u_n - u_{n-m}|.\sum_{n\delta}^{n(1-\delta)} |\overline{G^*}(m)| = o(1).\sum_{n\delta}^{n(1-\delta)}\overline{G^*}(m),\\ S_3 &= |\sum_0^{n\delta}\overline{G^*}(m)\sum_{n-m+1}^n \Delta_r| = o(1)\sum_0^{n\delta}|\overline{G^*}(m)|\frac{m}{n} = o(n\overline{G^*}(n)).\end{aligned}$$

Then (1.5) follows by letting $n \to \infty$ and then $\delta \to 0$. The fact that $n\Delta_n \to 0$ can be seen by an application of the Riemann-Lebesgue Lemma: we have the inversion formula

$$\begin{aligned}\Delta_n = \sum_0^\infty P(S_m = n) - P(S_m = n-1) &= \\ \sum_0^\infty \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{p}(t)^m e^{-itn}(1-e^{it})dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-e^{it})e^{-itn}dt}{1-\hat{p}(t)}. \qquad (1.6)\end{aligned}$$

Integrating by parts and noting that since everything is periodic with period 2π the contribution from the end points cancel, gives

$$2\pi\Delta_n = \frac{1}{n}\int_{-\pi}^{\pi} e^{-itn} f_1(t)dt, \text{ with} \qquad (1.7)$$

$$f_1(t) = \frac{i(e^{it}-1)\hat{p}'(t) - e^{it}(1-\hat{p}(t))}{(1-\hat{p}(t))^2}. \qquad (1.8)$$

Known results (see e.g. [3]) give the asymptotic behaviour of $\hat{p}(t)$ and $\hat{p}'(t)$ as $|t| \to 0$ and from them we see that $|f_1|$ is regularly varying as $|t| \to 0$ with index $a - 2 > -1$. We deduce that f_1 is integrable over $[-\pi, \pi]$ and the result follows.

Remark 1.1. Alternatively, we could appeal to a stronger result on the asymptotic behaviour of Δ_n in [4], but the proof there uses Banach Algebra techniques.

1.1 The absolutely continuous case

Assuming that F has a density f and the characteristic function $\hat{p}(t) = E(e^{itX})$ is such that $|\hat{p}(t)|^b$ is integrable for some $b \geq 1$, Isozaki [2] has used an inversion theorem to find an asymptotic estimate of the density u of the renewal measure. This estimate, which is actually valid in the random walk case whenever $E|X|^\gamma < \infty$ for some $\gamma \in (3/2, 2)$, when specialised to the renewal case becomes

$$u(x) = \sum_1^N f_j(x) + \frac{1}{m}(1 + \overline{\Phi}(x) + \overline{G}(x)) + \varepsilon(x), \text{ where } \varepsilon(x) = o(x^{-\gamma}) \text{ as } x \to \infty. \qquad (1.9)$$

Here N is the smallest integer $\geq b+1$, f_j is the density of S_j, and it is necessary to check, by integration by parts, that the function denoted by r_1 in [2] agrees with our $m^{-1}\overline{G}$. Integrating (1.9) and noting that $\int_0^x f_j(y)dy = 1 + o(x^{1-\gamma})$ for $1 \leq j \leq N$ gives

$$U(x) = \frac{1}{m}(x + \int_0^x \overline{\Phi}(y)dy + \int_0^x \overline{G}(y)dy) + C + o(x^{1-\gamma}), \text{ where } C = N + \int_0^\infty \varepsilon(y)dy. \quad (1.10)$$

By the same argument as used in [1] the existence of the γ-th moment implies that $\int_0^\infty \overline{G}(y)dy = 0$, so we can replace $\int_0^x \overline{G}(y)dy$ by $\int_x^\infty \overline{G^*}(y)dy$, and we also know, from our relation (0.3) and the key renewal theorem, that

$$W(x) \to \frac{1}{m}\int_0^\infty \overline{G}(y)dy = 0,$$

which means that $C = 0$. Then the estimate (1.10) reduces to

$$W(x) = \frac{1}{m}\int_x^\infty \overline{G^*}(y)dy + o(x^{1-\gamma}), \qquad (1.11)$$

and this is valid whenever the γ-th moment exists, for some $\gamma \in (3/2, 2)$. In particular under our assumption of asymptotic stability with index $\alpha \in (3/2, 2)$, we can choose any $\gamma = \alpha - \delta$ with δ sufficiently small that $1 - \gamma = \delta - \beta < 1 - 2\beta$ and $\gamma > 3/2$. Then (0.1) follows, since we know the first term dominates the RHS of (1.11) and we can read off its asymptotic behaviour from (0.4).

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