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Smoothness estimation of nonstationary Gaussian random fields from irregularly spaced data observed along a curve*

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Abstract: This article considers estimating the smoothness parameter of a class of nonstationary Gaussian random fields on $\mathbb{R}^d$ using irregularly spaced data observed along a curve. The set of covariance functions includes a nonstationary version of the Matérn covariance function as well as isotropic Matérn covariance function. Smoothness estimators are constructed via higher-order quadratic variations. Under mild conditions, these estimators are shown to be strongly consistent and convergence rate upper bounds are established with respect to fixed-domain asymptotics. Simulations indicate that the proposed estimators perform well for moderate sample sizes.

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Contents

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1. Introduction

Let $X(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$, be a Gaussian random field with covariance function $K(\mathbf{x},\mathbf{y}) = \mathrm{Cov}\{X(\mathbf{x}),\ X(\mathbf{y})\}$. Motivated by regression and spatial modeling, Paciorek [22], Paciorek and Schervish [23] propose a nonstationary version of the Matérn covariance function given by

$$K_P(\mathbf{x},\mathbf{y}) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)}|\Sigma_{\mathbf{x}}|^{1/4}|\Sigma_{\mathbf{y}}|^{1/4}\Big|\frac{\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}}{2}\Big|^{-1/2} \times(2\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}})^{\nu}\mathcal{K}_\nu(2\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}}), \qquad \forall \mathbf{x},\mathbf{y}\in\mathbb{R}^d, \tag{1}$$

where ν,σ^2 are positive constants, $\mathcal{K}_\nu(.)$ is the modified Bessel function of the second kind ([3], page 222), $\Sigma_{\mathbf{x}},\Sigma_{\mathbf{y}}$ are $d\times d$ symmetric positive definite matrices and

$$Q_{\mathbf{x},\mathbf{y}} = (\mathbf{x}-\mathbf{y})'\Big(\frac{\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}}{2}\Big)^{-1}(\mathbf{x}-\mathbf{y}).$$

If $\Sigma_{\mathbf{x}} = 4\nu\alpha^{-2}I$, $\forall \mathbf{x}\in\mathbb{R}^d$, for some positive constant α, then (1) reduces to the isotropic Matérn covariance function

$$K_M(\mathbf{x},\mathbf{y}) = \frac{\sigma^2(\alpha\|\mathbf{x}-\mathbf{y}\|)^{\nu}}{2^{\nu-1}\Gamma(\nu)}\mathcal{K}_\nu(\alpha\|\mathbf{x}-\mathbf{y}\|), \qquad \forall \mathbf{x},\mathbf{y}\in\mathbb{R}^d,$$

where $\|.\|$ denotes the Euclidean norm, cf. Stein [24]. Let $\mathbb{Z}_+ = \{1,2,\ldots\}$ and $G_\nu : [0,\infty)\to\mathbb{R}$ be such that $G_\nu(0)=0$ and for $s>0$,

$$G_\nu(s) = \begin{cases} s^{2\nu}, & \text{if } \nu\notin\mathbb{Z}_+,\\ s^{2\nu}\log(s), & \text{if } \nu\in\mathbb{Z}_+. \end{cases}$$

Writing $\psi(.)$ as the digamma function and

$$\sigma^2_{\mathbf{x},\mathbf{y}} = \sigma^2|\Sigma_{\mathbf{x}}|^{1/4}|\Sigma_{\mathbf{y}}|^{1/4}\Big|\frac{\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}}{2}\Big|^{-1/2}, \tag{2}$$

we observe that

$$K_P(\mathbf{x},\mathbf{y}) = \sigma^2_{\mathbf{x},\mathbf{y}}\sum_{k=0}^{\infty}\frac{\nu^k Q^k_{\mathbf{x},\mathbf{y}}}{k!\prod_{i=1}^k(i-\nu)} - \frac{\pi\sigma^2_{\mathbf{x},\mathbf{y}}}{\Gamma(\nu)\sin(\nu\pi)}\sum_{k=0}^{\infty}\frac{G_{\nu+k}(\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}})}{k!\Gamma(k+1+\nu)} \qquad \text{if } \nu\notin\mathbb{Z}_+, \tag{3}$$

and

$$K_P(\mathbf{x},\mathbf{y}) = \frac{2\sigma^2_{\mathbf{x},\mathbf{y}}}{(\nu-1)!}\Big\{(-1)^{\nu+1}\sum_{k=0}^{\infty}\frac{G_{\nu+k}(\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}})}{k!(k+\nu)!}$$

$$+\frac{1}{2}\sum_{k=0}^{\nu-1}(-1)^k\frac{(\nu-k-1)!}{k!}(\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}})^{2k} \tag{4}$$

$$+\frac{(-1)^\nu}{2}\sum_{k=0}^{\infty}[\psi(k+1)+\psi(\nu+k+1)]\frac{\nu^{\nu+k}Q_{\mathbf{x},\mathbf{y}}^{\nu+k}}{k!(\nu+k)!}\Big\} \qquad \text{if } \nu\in\mathbb{Z}_+.$$

Motivated by (3) and (4), this article considers the following class of nonstationary Gaussian random fields of which (1) is a special case. Let $d \in \mathbb{Z}_+$ and $X(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$, be a nonstationary Gaussian random field with mean function $m(\mathbf{t}) = \mathbb{E}X(\mathbf{t})$ and covariance function of the form

$$K(\mathbf{x},\mathbf{y}) = \rho_0(\mathbf{x},\mathbf{y}) + \rho_\nu(\mathbf{x},\mathbf{y}), \qquad \forall \mathbf{x},\mathbf{y}\in\mathbb{R}^d, \tag{5}$$

where $\nu > 0$ is a constant, $\rho_0(.,.)$ is a smooth function satisfying Condition 2 in Section 2,

$$\rho_\nu(\mathbf{x},\mathbf{y}) = \beta_\nu(\mathbf{x},\mathbf{y})G_\nu\{\sqrt{(\mathbf{x}-\mathbf{y})'A(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})}\}, \tag{6}$$

$\beta_\nu(\mathbf{x},\mathbf{y}) \neq 0$ and $A(\mathbf{x},\mathbf{y}) = (A_{j,k}(\mathbf{x},\mathbf{y}))_{1\le j,k\le d}$ is a $d\times d$ symmetric positive definite matrix $\forall \mathbf{x},\mathbf{y}\in\mathbb{R}^d$.

If $K = K_P$, then $\rho_\nu(\mathbf{x},\mathbf{y}) = \beta_\nu(\mathbf{x},\mathbf{y})G_\nu(\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}})$ where

$$\beta_\nu(\mathbf{x},\mathbf{y}) = \begin{cases} -\pi\sigma^2_{\mathbf{x},\mathbf{y}}/[\Gamma(\nu)\Gamma(\nu+1)\sin(\nu\pi)], & \text{if } \nu\notin\mathbb{Z}_+,\\ 2(-1)^{\nu+1}\sigma^2_{\mathbf{x},\mathbf{y}}/[\nu!(\nu-1)!], & \text{if } \nu\in\mathbb{Z}_+.\end{cases}$$

If $K = K_M$, then $\rho_\nu(\mathbf{x},\mathbf{y}) = \beta_\nu G_\nu(\alpha\|\mathbf{x}-\mathbf{y}\|/2)$ where

$$\beta_\nu = \begin{cases} -\pi\sigma^2/[\Gamma(\nu)\Gamma(\nu+1)\sin(\nu\pi)], & \text{if } \nu\notin\mathbb{Z}_+,\\ 2(-1)^{\nu+1}\sigma^2/[\nu!(\nu-1)!], & \text{if } \nu\in\mathbb{Z}_+.\end{cases}$$

$\rho_\nu(\mathbf{x},\mathbf{y})$ is analogous to the principal irregular term of the isotropic Matérn covariance function K_M (cf. Stein [24]). $\rho_\nu(\mathbf{x},\mathbf{y})$ in (6) may not be positive definite. The significance of $\rho_\nu(\mathbf{x},\mathbf{y})$ is that it will be the asymptotically dominant term that remains after the 'filtering process' by appropriate quadratic variations. Proposition 3 shows that ν can be regarded as the smoothness parameter.

The aim of this article is to estimate ν in (5) using a sample of observations $X(\mathbf{t}_1),\ldots,X(\mathbf{t}_n)$ where the design sites $\mathbf{t}_i, 1\le i\le n$, are irregularly spaced on a sufficiently smooth curve $\gamma : [0,L] \to \mathbb{R}^d$ for some constant $L > 0$. In particular, $\mathbf{t}_i = \gamma(t_i)$, $0 \le t_i \le L$.

Selecting the design sites on a (1-dimensional) curve in $\mathbb{R}^d$ is commonly called curved line transect sampling. This is a generalization of the usual line transect sampling (cf. Chapter 17 of Thompson [27]). Hiby and Krishna [12] presented convincing arguments for the use of curved line transect sampling by replacing the straight line in line transect sampling by a curve. Constantine and Hall [8] wrote that "in a variety of practical problems often data are only available through one-dimensional line transect 'samples' of the surface". Adler and Pyke [1] considered first-order quadratic variations using design sites on a curve in $[0,1]^2$ when the underlying Gaussian random field is in some sense 'like' Brownian motion on $\mathbb{R}^2$. Loh [20] discussed second-order quadratic variations from a sample of Gaussian random field observations taken along a smooth curve in $\mathbb{R}^2$. However supporting theory behind curved line transect sampling appears to be rather underdeveloped; possibly because the theoretical extension from line transect to curved line transect is non-trivial. We hope to work out a theoretical justification for curved line transect in the setting of this article.

Even though the likelihood is Gaussian, likelihood methods (such as maximum likelihood estimation) appear to be analytically intractable under fixed-domain asymptotics. The latter asymptotics imply that as sample size $n \to \infty$, the n sites get to be increasingly dense in a compact set in $\mathbb{R}^d$. In this article, the dependence among all the observations $X(\mathbf{t}_1), \ldots, X(\mathbf{t}_n)$ remains strong as sample size $n \to \infty$. Furthermore since $|X(\mathbf{t}_i) - X(\mathbf{t}_j)| \to 0$ as $\|\mathbf{t}_i - \mathbf{t}_j\| \to 0$, the $n \times n$ covariance matrix of the observations tends to a singular matrix and its determinant tends to 0. This is exacerbated by the $\mathbf{t}_i$'s being irregularly spaced. All these indicate that any theoretical analysis of the MLE for ν is a formidable (or even intractable) task with respect to fixed-domain asymptotics. Indeed as far as we know, the consistency of the MLE for ν is still an open problem.

This article is motivated by [20] where the idea of using higher-order quadratic variations $V_{\theta,\ell}$, $\theta \in \{1, 2\}, \ell \in \mathbb{Z}_+$, for constructing smoothness estimators is proposed for irregularly spaced data. However [20] considers stationary, isotropic Gaussian random fields whereas this article is concerned with nonstationary Gaussian random fields. Convergence rates are not available in [20] whereas upper bounds to the convergence rate of the proposed smoothness estimators are established here. The results in this article complement those in [21] with regard to the estimation of the smoothness parameter of an isotropic Gaussian random field with a Matérn covariance function. The difference lies in the choice of the design sites. While this article is concerned with design sites chosen on a curve, [21] chooses design sites randomly on $[0, 1]^d$. Consequently, if $d \geq 2$, the results of [21] are not applicable to this article.

Quadratic variations started with [19] and is currently a rather active field; some examples being [1, 2, 5, 17]. However most higher-order quadratic variations in the literature use data on a regular grid in $\mathbb{R}^d$; cf. [7, 15] and references cited therein.

The estimation of the smoothness parameter ν of a stationary Gaussian random field has been addressed in the literature under various conditions by many authors. [14] proposes a semiparametric method of estimating ν using irregularly spaced observations. The estimates in [14] appear to be analytically intractable under fixed-domain asymptotics. Furthermore in the simulations, [14] uses 200 independent realizations of a Gaussian random field, whereas this article is concerned with estimating ν based on observations from one realization of the underlying Gaussian random field.

In the case of equally spaced data on an interval of the real line, [10] considers a box-counting estimator while [8, 16] study estimators based on process increments. [8, 10, 16] all assume that $\nu \in (0, 1)$. Another example of equally spaced data on an interval is [15] where higher-order quadratic variations are used to construct a consistent estimate for ν given that $\nu \in (D, D + 1)$ for some known integer D. In contrast, this article assumes that $\nu > 0$ and a known upper bound for ν is not required.

Finally, [6] considers smoothness estimation for a class of locally stationary Gaussian processes using irregularly spaced data observed on a compact interval in $\mathbb{R}$. Assuming that the smoothness parameter $\nu \notin \mathbb{Z}_+$, [6] proposes estimators for $\lfloor \nu \rfloor, \nu - \lfloor \nu \rfloor$ and proved, in the almost sure sense, a $O(n^{-1/2} \log^a(n))$ convergence rate for estimating $\nu - \lfloor \nu \rfloor$ where $a > 0$ is some constant. The design sites in [6] are deterministic like those in [20] and do not apply to random designs.

The remainder of this article is organized as follows. Section 2 contains remarks on notation and preliminary technical results that are needed in the sequel.

For $\theta \in \{1, 2\}$ and $\ell \in \mathbb{Z}_+$, Section 3 presents the construction of the ℓth-order quadratic variations $V_{\theta,\ell}$ for stratified design sites. Theorem 1 proves a number of fixed-domain asymptotic results on $V_{\theta,\ell}$. These results are needed for the construction of the smoothness estimators for ν in this section. Two estimators $\hat{\nu}_{n,\ell}$ and $\hat{\nu}_n$ are proposed. Theorems 2 and 3 prove the strong consistency of $\hat{\nu}_{n,\ell}$ and $\hat{\nu}_n$ and establish upper bounds to the convergence rate of these estimators. In particular under mild conditions, $\mathbb{E}|\hat{\nu}_n - \nu| = O(n^{-1/3})$ as $n \to \infty$.

Section 4 adapts the results on stratified design in Section 3 to random design, i.e. the $\mathbf{t}_i$'s are i.i.d. random variables on the curve segment $\gamma : [0, L] \to \mathbb{R}^d$.

Section 5 considers the case where the design sites $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are deterministic points on the curve segment $\gamma : [0, L] \to \mathbb{R}^d$ whose relative spacing between each other is governed by a nonrandom strictly increasing mapping $\varphi \in C^2(\mathbb{R})$. Two estimators $\hat{\nu}^D_{n,\ell}$ and $\hat{\nu}^D_n$ for ν are proposed. Theorems 6 and 7 prove the strong consistency of $\hat{\nu}^D_{n,\ell}$ and $\hat{\nu}^D_n$ and establish convergence rate upper bounds of these estimators. In particular under mild conditions, $\mathbb{E}|\hat{\nu}^D_n - \nu| = O(n^{-1/2})$ as $n \to \infty$.

Section 6 presents Monte Carlo simulations to study the finite sample accuracy of the smoothness estimators $\hat{\nu}_n$ and $\hat{\nu}^D_n$. Since the random design can be reduced to a stratified design, the simulations are carried out only for stratified design and deterministic design.

Appendices A to F contain the proofs of all the results in this article.

2. Some preliminary results

For a function $f : \mathbb{R}^k \to \mathbb{R}$, $k \in \mathbb{Z}_+$, we write

$$f^{(u_1,\ldots,u_k)}(x_1, \ldots, x_k) = \frac{\partial^{u_1+\ldots+u_k}}{\partial x_1^{u_1} \ldots \partial x_k^{u_k}} f(x_1, \ldots, x_k)$$

if the latter exists whose value does not depend on the order of differentiation, where $u_1, \ldots, u_k$ are nonnegative integers. For $M \in \mathbb{Z}_+$, let $C^M(S)$ be the set of functions $f : S \to \mathbb{R}$ that are M times continuously differentiable (i.e. all Mth-order partial derivatives of f exist and are continuous). $\lfloor . \rfloor$ and $\lceil . \rceil$ denote the greatest integer function and least integer function respectively. $a_n \asymp b_n$ means $0 < \liminf_{n\to\infty} a_n/b_n \leq \limsup_{n\to\infty} a_n/b_n < \infty$ and $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. If A is a matrix, then A' is its transpose and if A is a square matrix, $|A|$ denotes its determinant. For $x, y \in \mathbb{R}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Conditions 1 and 2 below will be needed in the sequel.

Condition 1. $X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d$, is a Gaussian random field with covariance function $K(.,.)$ as in (5) and mean function $m(\mathbf{t}) = \mathbb{E}X(\mathbf{t})$ where $m(.) \in C^N(\mathbb{R}^d)$ for some integer $N \geq 2\nu + 6$. □

Condition 2. $\beta_\nu(.,.) \in C^N(\mathbb{R}^{2d})$, $A_{j,k}(.,.) \in C^N(\mathbb{R}^{2d})$, $\forall 1 \leq j, k \leq d$, and $\rho_0(\mathbf{x}, \mathbf{y})$ in (5) satisfies $\rho_0(.,.) \in C^{\lceil 2\nu \rceil + 1}(\mathbb{R}^{2d})$ for some integer $N \geq 2\nu + 6$. Let $D \subset \mathbb{R}^d$ be an arbitrary compact set and $u_1, \ldots, u_d, v_1, \ldots, v_d$ be nonnegative integers with $M = \sum_{i=1}^d (u_i + v_i)$. For any $M \leq N$, there exists a constant $C_{M,D} > 0$ such that

$$|\rho_0^{(u_1,\ldots,u_d,v_1,\ldots,v_d)}(\mathbf{x}, \mathbf{y})| \leq C_{M,D}, \text{ if } M/2 < \nu + 1 \text{ or } M/2 = \nu + 1 \notin \mathbb{Z}_+,$$

for all $\mathbf{x}, \mathbf{y} \in D$ and

$$\begin{aligned} &|\rho_0^{(u_1,\ldots,u_d,v_1,\ldots,v_d)}(\mathbf{x}, \mathbf{y})| \\ \leq\ &\begin{cases} C_{M,D}\{|\log(\|\mathbf{x} - \mathbf{y}\|)| + 1\}, & \text{if } M/2 = \nu + 1 \in \mathbb{Z}_+, \\ C_{M,D}\|\mathbf{x} - \mathbf{y}\|^{2\nu+2-M}, & \text{if } M/2 > \nu + 1, \end{cases} \end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in D$ satisfying $\mathbf{x} \neq \mathbf{y}$. □

Proposition 1. *Condition 2 is satisfied for the covariance function $K_P(.,.)$ in (1) if each entry of the matrix $\Sigma_{\mathbf{x}}$, as a function of $\mathbf{x}$, is in $C^N(\mathbb{R}^d)$. In particular, Condition 2 holds for an isotropic Matérn covariance function.*

This article assumes that the observations of X are taken along a fixed curve γ in $\mathbb{R}^d$ and that $\gamma(.)$ satisfies Condition 3 below.

Condition 3. The curve $\gamma : \mathbb{R} \to \mathbb{R}^d$ is a simple C^N-curve parametrized by its arc length for some integer $N \geq 2\nu + 6$. In particular,

(i) $\gamma(s) \neq \gamma(t)$ if $s \neq t$,

(ii) writing $\gamma(t) = (\gamma_1(t), \ldots, \gamma_d(t))'$ and its kth derivative by $\gamma^{(k)}(t) = (\gamma_1^{(k)}(t), \ldots, \gamma_d^{(k)}(t))'$, we have $\|\gamma^{(1)}(t)\| = 1$ for $t \in \mathbb{R}$ and $\gamma_i(.) \in C^N(\mathbb{R}), i = 1, \ldots, d$. □

Condition 3 implies that

$$\sum_{j=1}^d \gamma_j^{(1)}(t)\gamma_j^{(2)}(t) = 0, \qquad \forall t \in \mathbb{R}, \tag{7}$$

and given any constant $L > 0$, there exists another constant $C_L > 0$ such that

$$\|\gamma(s) - \gamma(t)\| \geq C_L|s - t|, \qquad \forall s, t \in [0, L]. \tag{8}$$

We observe that $X(\gamma(t))$, $t \in \mathbb{R}$, is a 1-dimensional Gaussian random field with covariance function

$$\tilde{K}(x, y) \quad = \quad K(\gamma(x), \gamma(y)),$$

$$= \quad \tilde{\rho}_0(x,y) + \tilde{\rho}_\nu(x,y), \qquad \forall x,y \in \mathbb{R}, \tag{9}$$

where $\tilde{\rho}_0(x,y) = \rho_0(\gamma(x),\gamma(y))$ and $\tilde{\rho}_\nu(x,y) = \rho_\nu(\gamma(x),\gamma(y))$. This "dimension reduction" observation motivates many of the proofs in this article. The motivation for $N \geq 2\nu + 6$ in Conditions 1 to 3 is to ensure that the smoothness of the random fields $X(.)$ and $X(\gamma(.))$ are determined by the principal irregular term $\rho_\nu(.,.)$ in (6).

Proposition 2. *Suppose Conditions 1 to 3 are satisfied. Let $u, v \in \mathbb{Z}_+$, $M = u + v$ and $\tilde{\rho}_0(x,y)$, $\tilde{K}(x,y)$ be as in (9). Then we have $\tilde{\rho}_0(x,y) \in C^{\lceil 2\nu \rceil + 1}(\mathbb{R}^2)$ and for $M \leq N$,*

$$|\tilde{K}^{(u,v)}(x,y)| \leq C_{M,L}, \quad \textit{if } M/2 < \nu \textit{ or } M/2 = \nu \notin \mathbb{Z}_+, \tag{10}$$

for all $x, y \in [0, L]$ and

$$|\tilde{K}^{(u,v)}(x,y)| \leq \begin{cases} C_{M,L}\{|\log(\|\gamma(x) - \gamma(y\|)| + 1\}, & \textit{if } M/2 = \nu \in \mathbb{Z}_+, \\ C_{M,L}\|\gamma(x) - \gamma(y)\|^{2\nu - M}, & \textit{if } M/2 > \nu, \end{cases} \tag{11}$$

for all $x, y \in [0, L]$ such that $x \neq y$, where $C_{M,L} > 0$ is a constant.

We end this section with Proposition 3 below on the smoothness of the Gaussian random fields $X(.)$ and $X(\gamma(.))$.

Definition 1. Following Section 3.1 of [9], two random fields $X_{\mathbf{t}}$ and $Y_{\mathbf{t}}$ defined on a common probability space and indexed by a common set T are said to be equivalent versions of each other if for every fixed $\mathbf{t} \in T$, $\mathbb{P}(X_{\mathbf{t}} = Y_{\mathbf{t}}) = 1$.

Proposition 3. *Suppose Conditions 1 to 3 are satisfied. Then the following statements hold.*

(i) *$X(\gamma(t))$, $t \in \mathbb{R}$, has jth-order mean square derivative if and only if $j < \nu$.*
(ii) *For any bounded open interval $T_1 \subset \mathbb{R}$, $X(\gamma(t))$ when restricted to $t \in T_1$ has an equivalent version which possesses, with probability 1, a $C^{\lceil \nu \rceil - 1}(T_1)$ sample path*
(iii) *$X(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$, has all jth-order mean square partial derivatives if and only if $j < \nu$.*
(iv) *For any bounded open set $T \subset \mathbb{R}^d$, $X(\mathbf{t})$ when restricted to $\mathbf{t} \in T$ has an equivalent version which possesses, with probability 1, a $C^{\lceil \nu \rceil - 1}(T)$ sample path.*

3. Stratified design

Sections 3 and 4 assume that the curve segment $\gamma : [0, L] \to \mathbb{R}^d$ is known. Let the observed sample be $X(\gamma(t_{n,1})), \ldots, X(\gamma(t_{n,n}))$ where $0 \leq t_{n,1} < \ldots < t_{n,n} \leq L$. For brevity, we write $t_i = t_{n,i}$ and $X(\gamma(t_{n,i})) = X_i$. Let $d_{i,j} = \|\gamma(t_i) - \gamma(t_j)\|$ be the Euclidean distance between $\gamma(t_i)$ and $\gamma(t_j)$. We consider the following stratified design where t_i satisfies

$$t_i = \frac{(i - 1 + \delta_i)L}{n}, \tag{12}$$

$0 \leq \delta_i < 1$, $i = 1, \ldots, n$. In this section, the δ_i's are assumed to be (nonrandom) constants though they can vary with n.

3.1. Higher-order quadratic variations

This section introduces a new class of higher-order quadratic variations that are needed in the construction of smoothness estimators for ν. This is accomplished by using these higher-order quadratic variations to filter out the asymptotic contributions of ν. Let ω_n be a positive integer such that $\omega_n \asymp n^\xi$ for some constant $\xi \in (0,1)$. Define for $\ell \in \mathbb{Z}_+$ and $\theta \in \{1,2\}$,

$$\begin{aligned} a_{\theta,\ell;i,k} &= \frac{\ell!}{\prod_{0\leq j\leq \ell, j\neq k}(d_{i,i+\omega_n\theta k} - d_{i,i+\omega_n\theta j})}, \qquad \forall k = 0,\ldots,\ell, \\ \nabla_{\theta,\ell} X_i &= \sum_{k=0}^{\ell} a_{\theta,\ell;i,k} X_{i+\omega_n\theta k}, \qquad \forall i = 1,\ldots,n-\omega_n\theta\ell, \end{aligned}$$

and the ℓth-order quadratic variation based on $X_1, \ldots, X_n$ to be

$$V_{\theta,\ell} = \sum_{i=1}^{n-\omega_n\theta\ell} (\nabla_{\theta,\ell} X_i)^2. \tag{13}$$

The properties of $V_{\theta,\ell}$ depend crucially on Lemma 1 and this lemma is the motivation for using the term "ℓth-order" for $V_{\theta,\ell}$.

Lemma 1. *Let $\theta \in \{1,2\}$, $\omega_n, \ell \in \mathbb{Z}_+$ such that $\omega_n\theta\ell \leq n-1$. Then for $i,j \in \{1,\ldots,n-\omega_n\theta\ell\}$*

$$\sum_{k=0}^{\ell} a_{\theta,\ell;i,k} d_{i,i+\omega_n\theta k}^q = \begin{cases} 0, & \forall q = 0,\ldots,\ell-1, \\ \ell!, & \text{if } q = \ell, \end{cases}$$

and

$$\begin{aligned} &\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} (d_{j,j+\omega_n\theta k_2} - d_{i,i+\omega_n\theta k_1})^q \\ = &\begin{cases} 0, & \forall q = 0,\ldots,2\ell-1, \\ (-1)^\ell (2\ell)!, & \text{if } q = 2\ell, \end{cases} \end{aligned}$$

where we use the convention $0^0 = 1$.

Lemma 1 goes back to at least [26]; see also Section 1.2.3, problem 33, of [18]. For $t > 0$, define

$$q_{1,t} = \begin{cases} n/t, & \text{if } \nu < \ell - 1/2, \\ (n/t)\log^{-1}(n/t), & \text{if } \nu = \ell - 1/2, \\ (n/t)^{2\ell-2\nu}, & \text{if } \nu \in (\ell-1/2, \ell), \\ \log(n/t), & \text{if } \nu = \ell, \\ 1, & \text{if } \nu > \ell, \end{cases}$$

$$q_{2,t} = \begin{cases} n/t, & \text{if } \nu < \ell - 1/4, \\ (n/t)\log^{-1}(n/t)\}, & \text{if } \nu = \ell - 1/4, \\ (n/t)^{4\ell-4\nu}, & \text{if } \nu \in (\ell - 1/4, \ell), \\ \log^2(n/t), & \text{if } \nu = \ell, \\ 1, & \text{if } \nu > \ell, \end{cases}$$

$$q_{3,t} = \begin{cases} (n/t)^2, & \text{if } \nu < \ell - 1/2, \\ (n/t)^2\log^{-1}(n/t), & \text{if } \nu = \ell - 1/2, \\ (n/t)^{4\ell-4\nu}, & \text{if } \nu \in (\ell - 1/2, \ell), \\ \log^2(n/t), & \text{if } \nu = \ell, \\ 1, & \text{if } \nu > \ell. \end{cases} \tag{14}$$

Denote the $(n - \omega_n\theta\ell) \times 1$ vector

$$Y = \begin{cases} \left(\frac{\nabla_{\theta,\ell}X_1}{\sqrt{\mathbb{E}V_{\theta,\ell}}}, \dots, \frac{\nabla_{\theta,\ell}X_{n-\omega_n\theta\ell}}{\sqrt{\mathbb{E}V_{\theta,\ell}}}\right)' & \text{if } \nu \le \ell, \\ \left(\frac{\nabla_{\theta,\ell}X_1}{\sqrt{n}}, \dots, \frac{\nabla_{\theta,\ell}X_{n-\omega_n\theta\ell}}{\sqrt{n}}\right)' & \text{if } \nu > \ell. \end{cases}$$

Let $\mu = \mathbb{E}Y$, $\Sigma = (\Sigma_{i,j})_{(n-\omega_n\theta\ell)\times(n-\omega_n\theta\ell)} = \mathbb{E}\{(Y-\mu)(Y-\mu)'\}$ and $\Sigma_{\text{abs}} = (|\Sigma_{i,j}|)_{(n-\omega_n\theta\ell)\times(n-\omega_n\theta\ell)})$. Denote $\|.\|_2$ and $\|.\|_F$ as the spectral norm and Frobenious norm of matrices respectively. Then

$$Y'Y = Z'\Sigma Z + \mu'\Sigma^{1/2}Z + Z'\Sigma^{1/2}\mu + \mu'\mu = \begin{cases} V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell} & \text{if } \nu \le \ell, \\ V_{\theta,\ell}/n & \text{if } \nu > \ell, \end{cases} \tag{15}$$

where $Z \sim N_{n-\omega_n\theta\ell}(0, I)$. Theorem 1 is crucial to the construction of the estimators for ν in Section 3.2.

Theorem 1. *Suppose Conditions 1 to 3 hold. Let $\theta \in \{1, 2\}$, $V_{\theta,\ell}$ be as in (13) and*

$$\begin{aligned} \Psi(t) &= \{\gamma^{(1)}(t)\}'A(\gamma(t),\gamma(t))\gamma^{(1)}(t), \qquad \forall t \in (-\varepsilon, L+\varepsilon), \\ H_\ell(\nu) &= \sum_{0\le k_1<k_2\le\ell} (-1)^{k_1+k_2}\binom{\ell}{k_1}\binom{\ell}{k_2}G_\nu(k_2-k_1), \end{aligned}$$

for some constant $\varepsilon > 0$.

(a) Suppose $\nu < \ell$. Then

$$\begin{aligned} \mathbb{E}V_{\theta,\ell} &= \frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds \\ &\quad + O\big\{n + (\frac{n}{\omega_n})^{2\ell-2\nu+1}\big\} \\ &\quad + \begin{cases} O\big(n(n/\omega_n)^{2\ell-2\nu-1}\big), & \text{if } \nu \notin \mathbb{Z}_+, \\ O\big(n(n/\omega_n)^{2\ell-2\nu-1}\log(n/\omega_n)\big), & \text{if } \nu \in \mathbb{Z}_+, \end{cases} \end{aligned}$$

$$\asymp \quad n(n/\omega_n)^{2\ell-2\nu},$$

and $V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell} \to 1$ *almost surely as* $n \to \infty$ *uniformly over* $\delta_i \in [0,1), 1 \le i \le n$.

(b) Suppose $\nu = \ell$. *Then*

$$\begin{aligned}\mathbb{E}V_{\theta,\ell} &= (-1)^{\ell+1}(2\ell)! n \log(\frac{n}{\omega_n}) \int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds + O(n) \\ &\asymp n\log(n),\end{aligned}$$

$\mathrm{Var}(V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}) = O\{\log^{-2}(n)\}$ *and* $V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell} \to 1$ *in probability as* $n \to \infty$ *uniformly over* $\delta_i \in [0,1), 1 \le i \le n$.

(c) Suppose $\nu > \ell$. *Then*

$$\mathbb{E}V_{\theta,\ell} \asymp n,$$

$\mathrm{Var}(V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}) \asymp 1$ *and* $V_{1,\ell}/V_{2,\ell} \to 1$ *almost surely as* $n \to \infty$ *uniformly over* $\delta_i \in [0,1), 1 \le i \le n$.

(d)

$$\|\Sigma_{\mathrm{abs}}\|_2 = O(q_{1,\omega_n}^{-1}), \qquad \|\Sigma\|_F^2 = O(q_{2,\omega_n}^{-1}), \qquad \mu'\Sigma\mu = O(q_{3,\omega_n}^{-1}),$$

and for all $s > 0$, *there exist constants* C, C_1 *such that*

$$\begin{aligned}\mathbb{P}(|Y'Y - \mathbb{E}(Y'Y)| \ge s) \quad &\le \quad 2\exp\{-C\min(q_{1,\omega_n}s, q_{2,\omega_n}s^2)\} \\ &\quad + \min\{1, C_1 s^{-1} q_{3,\omega_n}^{-1/2}\exp(-Cs^2 q_{3,\omega_n})\},\end{aligned} \tag{16}$$

uniformly over $\delta_i \in [0,1), 1 \le i \le n$.

3.2. Estimating the smoothness parameter ν

Motivated by Theorem 1(a), we shall now proceed to construct a consistent estimator $\hat{\nu}_{n,\ell}$ for the smoothness parameter ν assuming $\nu \le \ell$ for some known $\ell \in \mathbb{Z}_+$. After that, we introduce the estimator $\hat{\nu}_n$ which no longer relies on the known upper bound ℓ of ν. Let $\hat{\nu}_{n,\ell}$ be such that

$$\Big\{\frac{V_{1,\ell}2^{2\hat{\nu}_{n,\ell}-2\ell}}{V_{2,\ell}} - 1\Big\}^2 = \min_{\nu^* \ge 0}\Big\{\frac{V_{1,\ell}2^{2\nu^*-2\ell}}{V_{2,\ell}} - 1\Big\}^2.$$

Then

$$\hat{\nu}_{n,\ell} = \max\Big\{\ell + \frac{\log(V_{2,\ell}/V_{1,\ell})}{2\log(2)}, 0\Big\}. \tag{17}$$

Theorem 2. *Suppose Conditions 1 to 3 hold. Then*

$$\hat{\nu}_{n,\ell} \to \begin{cases}\nu \text{ almost surely} & \text{if } \nu < \ell, \\ \ell \text{ in probability} & \text{if } \nu = \ell, \\ \ell \text{ almost surely} & \text{if } \nu > \ell,\end{cases}$$

and

$$\mathbb{E}\{(\hat{\nu}_{n,\ell}-\nu)^2\} = \begin{cases} O(\omega_n^{-2}) + O(\omega_n/n) & \text{if } \nu < \ell - 1/4, \\ O(\omega_n^{-2}) + O\{(\omega_n/n)\log(n)\} & \text{if } \nu = \ell - 1/4, \\ O(\omega_n^{-2}) + O\{(\omega_n/n)^{4\ell-4\nu}\} & \text{if } \nu \in (\ell - 1/4, \ell), \\ O\{\log^{-2}(n)\} & \text{if } \nu = \ell, \\ O(1) & \text{if } \nu > \ell, \end{cases}$$

as $n \to \infty$ *uniformly over* $\delta_i \in [0,1), 1 \leq i \leq n$.

Theorem 2 shows that $\hat{\nu}_{n,\ell}$ is a strongly consistent estimator for ν provided $\nu < \ell$. We now propose the estimator $\hat{\nu}_n$ which is of more practical interest as its computation does not require a known upper bound for ν. $\hat{\nu}_n$ is motivated by a construction in [6]. Let M_n be a positive integer such that $M_n = o(n/\omega_n)$ and $M_n \to \infty$ as $n \to \infty$. Define for $\ell = 1, \ldots, M_n$, the events

$$\begin{aligned} \Theta_\ell &= \{\hat{\nu}_{n,\ell} \leq \ell - 1/4\} \cap \{n^{-1} V_{1,\ell} \geq (\frac{n}{\omega_n})^{1/2} \log(\frac{n}{\omega_n})\}, \\ \Omega_\ell &= \bigcap_{j=1}^{\ell-1} \Theta_j^c \cap \Theta_\ell, \\ \Omega_0 &= \bigcap_{j=1}^{M_n} \Theta_j^c, \end{aligned} \tag{18}$$

and $\ell_0 = j$ if Ω_j occurs where $j \in \{0, \ldots, M_n\}$. ℓ_0 is well defined integer-valued random variable as $\Omega_0, \ldots, \Omega_{M_n}$ form a partition of the sample space. Define $\hat{\nu}_n$ as $\hat{\nu}_{n,\ell_0}$ where $\hat{\nu}_{n,0} = M_n$.

Theorem 3. *Suppose Conditions 1 to 3 hold. Then* $\hat{\nu}_n \to \nu$ *almost surely and*

$$\mathbb{E}|\hat{\nu}_n - \nu| = O\{(\omega_n/n)^{1/2} + \omega_n^{-1}\},$$

as $n \to \infty$ *uniformly over* $\delta_i \in [0,1), 1 \leq i \leq n$. *By taking* $\omega_n \asymp n^{1/3}$, *we obtain*

$$\mathbb{E}|\hat{\nu}_n - \nu| = O(n^{-1/3})$$

as $n \to \infty$.

The $O(.)$'s in Section 3 are uniform over $\delta_i \in [0,1), 1 \leq i \leq n$. Hence the theorems in this section hold when the δ_i's are random and are independent of $X(.)$.

4. Random design

Let $t_1, \ldots, t_n$ be a sequence of i.i.d. random variables where t_1 has probability density function $p(t)$, $t \in [0, L]$, satisfying $\inf_{t\in[0,L]} p(t) = p_0 > 0$ for some unknown p_0. We assume that $t_1, \ldots, t_n$ are independent of the Gaussian random field $X(.)$. Set $n_\tau = \lfloor \frac{n}{\tau \log^2(n)} \rfloor$ for any constant $\tau \geq 1$. Then

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{1 \leq i \leq n_1} \bigcap_{j=1}^{n} \{t_j \notin [\frac{(i-1)L}{n_1}, \frac{iL}{n_1})\}\Big) &\leq \sum_{i=1}^{n_1} \mathbb{P}\Big(\bigcap_{j=1}^{n} \{t_j \notin [\frac{(i-1)L}{n_1}, \frac{iL}{n_1})\}\Big) \\
&\leq n_1(1 - \frac{p_0 L}{n_1})^n \\
&\leq \frac{n}{\log^2(n)}(1 - \frac{p_0 L \log^2(n)}{n})^n \\
&\leq \frac{n^{1-p_0 L \log(n)}}{\log^2(n)},
\end{aligned}$$

and

$$\sum_{n=2}^{\infty} \mathbb{P}\Big(\bigcup_{1 \leq i \leq n_1} \bigcap_{j=1}^{n} \{t_j \notin [\frac{(i-1)L}{n_1}, \frac{iL}{n_1})\}\Big) \leq \sum_{n=2}^{\infty} \frac{n^{1-p_0 L \log(n)}}{\log^2(n)} < \infty.$$

It follows from the Borel-Cantelli lemma that with probability 1, there exists a (random) integer N_0 such that for $n \geq N_0$, every interval $[(i-1)L/n, iL/n)$ contains at least one $t_j \in \{t_1, \ldots, t_n\}$.

We propose the following estimator for ν. Let $\hat{\tau}$ be the smallest real number greater than or equal to 1 such that

$$\{t_1, \ldots, t_n\} \cap [\frac{(i-1)L}{n_{\hat{\tau}}}, \frac{iL}{n_{\hat{\tau}}}) \neq \emptyset, \qquad \forall 1 \leq i \leq n_{\hat{\tau}}.$$

Denote $t_j \in [(i-1)L/n_{\hat{\tau}}, iL/n_{\hat{\tau}})$ as $t(i)$ such that

$$t(i) = \frac{(i-1+\delta_i)L}{n_{\hat{\tau}}},$$

where $0 \leq \delta_i < 1$ and the δ_i's are random variables independent of $X(.)$. If there are more that one t_j's in $[(i-1)L/n_{\hat{\tau}}, iL/n_{\hat{\tau}})$, we choose any one of these t_j's to be $t(i)$. It follows from the above argument that $\hat{\tau} = 1$ for sufficiently large n almost surely. Now this random design reduces to the stratified design of Section 3 where the observed sample is

$$\Big\{X(\gamma(t(i))) : t(i) \in [\frac{(i-1)L}{n_{\hat{\tau}}}, \frac{iL}{n_{\hat{\tau}}}), 1 \leq i \leq n_{\hat{\tau}}\Big\}.$$

We note that the effective sample size correspondingly reduces from n to $n_{\hat{\tau}} \asymp n/\log^2(n)$. Let $\hat{\nu}^R_{n_{\hat{\tau}}}$ be as $\hat{\nu}_{n_{\hat{\tau}}}$ in Section 3 but based on the sample $X(\gamma(t(1)))$, $\ldots$, $X(\gamma(t(n_{\hat{\tau}})))$.

Theorem 4. *Suppose Conditions 1 to 3 hold. Then $\hat{\nu}^R_{n_{\hat{\tau}}} \to \nu$ as $n \to \infty$ almost surely and*

$$\mathbb{E}|\hat{\nu}^R_{n_{\hat{\tau}}} - \nu| = O\{(\omega_{n_1}/n_1)^{1/2} + \omega_{n_1}^{-1}\},$$

as $n \to \infty$*. By taking* $\omega_{n_{\hat{\tau}}} \asymp n_{\hat{\tau}}^{1/3}$*, we have*

$$\mathbb{E}|\hat{\nu}_{n_{\hat{\tau}}}^{R} - \nu| = O\big(n^{-1/3}\log^{2/3}(n)\big)$$

as $n \to \infty$*.*

5. Deterministic design

This section introduces a class of deterministic designs governed by a mapping $\varphi \in C^2(\mathbb{R})$. Suppose $t_{n,1}, \ldots, t_{n,n}$ satisfies

Condition 4. For $n \geq 2$, $t_{n,i} = \varphi(L(i-1)/(n-1))$, $i = 1, \ldots, n$, where $\varphi \in C^2(\mathbb{R})$ is a nonrandom function and satisfies $\varphi(0) = 0$, $\varphi(L) = L$ and $\min_{0\leq s\leq L} \varphi^{(1)}(s) > 0$. □

Condition 4 implies that $0 = t_{n,1} < \ldots < t_{n,n} = L$ and there exist constants $C_{5,0}$ and $C_{5,1}$ such that

$$0 < C_{5,0}/n \leq \min_{1\leq i\leq n-1}(t_{n,i+1} - t_{n,i}) \leq \max_{1\leq i\leq n-1}(t_{n,i+1} - t_{n,i}) \leq C_{5,1}/n.$$

The observed sample is $X(\gamma(t_{n,1})), \ldots, X(\gamma(t_{n,n}))$. For brevity, we write $t_i = t_{n,i}$ and $X(\gamma(t_{n,i})) = X_i$. For $\ell \in \mathbb{Z}_+$ and $\theta \in \{1, 2\}$, define

$$\begin{aligned} \tilde{a}_{\theta,\ell;i,k} &= \frac{\ell!}{\prod_{0\leq j\leq \ell, j\neq k}(d_{i,i+\theta k} - d_{i,i+\theta j})}, \qquad \forall k = 0, \ldots, \ell, \\ \widetilde{\nabla}_{\theta,\ell} X_i &= \sum_{k=0}^{\ell} \tilde{a}_{\theta,\ell;i,k} X_{i+\theta k}, \qquad \forall i = 1, \ldots, n - \theta\ell, \end{aligned}$$

and the ℓth-order quadratic variation based on $X_1, \ldots, X_n$ to be

$$\widetilde{V}_{\theta,\ell} = \sum_{i=1}^{n-\theta\ell} (\widetilde{\nabla}_{\theta,\ell} X_i)^2. \tag{19}$$

We observe that $\widetilde{V}_{\theta,\ell} = V_{\theta,\ell}$ in (13) when $\omega_n = 1$. Denote the $(n-\theta\ell)\times 1$ vector

$$\widetilde{Y} = \begin{cases} \left(\frac{\widetilde{\nabla}_{\theta,\ell} X_1}{\sqrt{\mathbb{E}\widetilde{V}_{\theta,\ell}}}, \ldots, \frac{\widetilde{\nabla}_{\theta,\ell} X_{n-\theta\ell}}{\sqrt{\mathbb{E}\widetilde{V}_{\theta,\ell}}}\right)' & \text{if } \nu \leq \ell, \\ \left(\frac{\widetilde{\nabla}_{\theta,\ell} X_1}{\sqrt{n}}, \ldots, \frac{\widetilde{\nabla}_{\theta,\ell} X_{n-\theta\ell}}{\sqrt{n}}\right)' & \text{if } \nu > \ell. \end{cases}$$

Let $\widetilde{\mu} = \mathbb{E}\widetilde{Y}$, $\widetilde{\Sigma} = (\widetilde{\Sigma}_{i,j})_{(n-\theta\ell)\times(n-\theta\ell)} = \mathbb{E}\{(\widetilde{Y} - \widetilde{\mu})(\widetilde{Y} - \widetilde{\mu})'\}$ and $\widetilde{\Sigma}_{\text{abs}} = (|\widetilde{\Sigma}_{i,j}|)_{(n-\theta\ell)\times(n-\theta\ell)})$. Then

$$\widetilde{Y}'\widetilde{Y} = \begin{cases} \widetilde{V}_{\theta,\ell}/\mathbb{E}\widetilde{V}_{\theta,\ell} & \text{if } \nu \leq \ell, \\ \widetilde{V}_{\theta,\ell}/n & \text{if } \nu > \ell. \end{cases}$$

Theorems 5, 6 and 7 below are analogous to Theorems 1, 2 and 3, respectively. The proofs of Theorems 5 to 7 are similar to (though simpler than) the proofs of Theorems 1 to 3 and hence are omitted.

Theorem 5. *Suppose Conditions 1 to 4 hold. Let $\theta \in \{1,2\}$, $\widetilde{V}_{\theta,\ell}$ be as in (19).*
(a) *Suppose $\nu < \ell$. Then*

$$\begin{aligned}\mathbb{E}(\widetilde{V}_{\theta,\ell}) \quad \sim \quad & \frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\theta^{2\ell-2\nu}L^{2\ell-2\nu}} \\ & \times \int_0^L \beta_\nu\{\gamma(\varphi(s)),\gamma(\varphi(s))\}[\Psi\{\varphi(s)\}]^\nu\{\varphi^{(1)}(s)\}^{2\nu-2\ell}ds > 0,\end{aligned}$$

and $\widetilde{V}_{\theta,\ell}/\mathbb{E}\widetilde{V}_{\theta,\ell} \to 1$ almost surely as $n \to \infty$.
(b) *Suppose $\nu = \ell$. Then*

$$\mathbb{E}\widetilde{V}_{\theta,\ell} \sim (-1)^{\ell+1}(2\ell)!n\log(n)\int_0^L \beta_\nu\{\gamma(\varphi(s)),\gamma(\varphi(s))\}[\Psi\{\varphi(s)\}]^\nu ds > 0,$$

$\mathrm{Var}(\widetilde{V}_{\theta,\ell}/\mathbb{E}\widetilde{V}_{\theta,\ell}) = O\{\log^{-2}(n)\}$ *and $\widetilde{V}_{1,\ell}/\widetilde{V}_{2,\ell} \to 1$ in probability as $n \to \infty$.*
(c) *Suppose $\nu > \ell$. Then*

$$\mathbb{E}\widetilde{V}_{\theta,\ell} \asymp n,$$

$\mathrm{Var}(\widetilde{V}_{\theta,\ell}/\mathbb{E}\widetilde{V}_{\theta,\ell}) \asymp 1$ *and $\widetilde{V}_{1,\ell}/\widetilde{V}_{2,\ell} \to 1$ almost surely as $n \to \infty$.*
(d) *Let $q_{j,1}$, $j = 1,2,3$, be as in (14). Then*

$$\|\widetilde{\Sigma}_{\mathrm{abs}}\|_2 = O(q_{1,1}^{-1}), \qquad \|\widetilde{\Sigma}\|_F^2 = O(q_{2,1}^{-1}), \qquad \widetilde{\mu}'\widetilde{\Sigma}\widetilde{\mu} = O(q_{3,1}^{-1}),$$

and there exist constants $C, C_1 > 0$ such that for all $s > 0$

$$\begin{aligned}\mathbb{P}(|\widetilde{Y}'\widetilde{Y} - \mathbb{E}(\widetilde{Y}'\widetilde{Y})| \geq s) \quad \leq \quad & 2\exp\{-C\min(q_{1,1}s, q_{2,1}s^2)\} \\ & + \min\{1, C_1 s^{-1}q_{3,1}^{-1/2}\exp(-Cs^2q_{3,1})\}.\end{aligned}$$

Theorem 6. *Suppose Conditions 1 to 4 hold. Let $\hat{\nu}_{n,\ell}^D$ be such that*

$$\left\{\frac{\widetilde{V}_{1,\ell}2^{2\hat{\nu}_{n,\ell}^D-2\ell}}{\widetilde{V}_{2,\ell}} - 1\right\}^2 = \min_{\nu^* \geq 0}\left\{\frac{\widetilde{V}_{1,\ell}2^{2\nu^*-2\ell}}{\widetilde{V}_{2,\ell}} - 1\right\}^2.$$

Then

$$\hat{\nu}_{n,\ell}^D \to \begin{cases}\nu \text{ almost surely} & \text{if } \nu < \ell, \\ \ell \text{ in probability} & \text{if } \nu = \ell, \\ \ell \text{ almost surely} & \text{if } \nu > \ell,\end{cases}$$

and

$$\mathbb{E}(\hat{\nu}_{n,\ell}^D - \nu)^2 = \begin{cases}O(n^{-1}) & \text{if } \nu < \ell - 1/4, \\ O\{n^{-1}\log(n)\} & \text{if } \nu = \ell - 1/4, \\ O(n^{4\nu-4\ell}) & \text{if } \nu \in (\ell - 1/4, \ell), \\ O\{\log^{-2}(n)\} & \text{if } \nu = \ell, \\ O(1) & \text{if } \nu > \ell,\end{cases}$$

as $n \to \infty$.

Let $\widetilde{M}_n = \lfloor \log(n) \rfloor$. Define for $\ell = 1, \ldots, \widetilde{M}_n$, the events

$$\begin{aligned}\widetilde{\Omega}_\ell &= \bigcap_{j=1}^{\ell-1} \{n^{-1}\widetilde{V}_{1,j} < \sqrt{n}\log n\} \cap \{n^{-1}\widetilde{V}_{1,\ell} \geq \sqrt{n}\log n\}, \\ \widetilde{\Omega}_0 &= \bigcap_{j=1}^{\widetilde{M}_n} \{n^{-1}\widetilde{V}_{1,j} < \sqrt{n}\log n\}.\end{aligned}$$

Set $\widetilde{\ell}_0 = j$ if $\widetilde{\Omega}_j$ occurs for $j \in \{0, \ldots, \widetilde{M}_n\}$ and $\hat{\nu}_n^D = \hat{\nu}_{n,\widetilde{\ell}_0}^D$ where $\hat{\nu}_{n,0}^D = \widetilde{M}_n$.

Theorem 7. *Suppose Conditions 1 to 4 are satisfied. Then $\hat{\nu}_n^D \to \nu$ almost surely and $\mathbb{E}|\hat{\nu}_n^D - \nu| = O(n^{-1/2})$ as $n \to \infty$.*

We remark that $\gamma(t)$ and $\varphi(t)$, $0 \leq t \leq L$, need not be known explicitly in the computation of $\widetilde{V}_{\theta,\ell}$, $\hat{\nu}_{n,\ell}^D$ and $\hat{\nu}_n^D$; only the bijection $\Phi : \{1, \ldots, n\} \to \{X_1, \ldots, X_n\}$ is required. [20], page 2775, presents an algorithm for recovering the true $\Phi(.)$ when n is sufficiently large.

6. Simulation study

Following a suggestion by the referee, Figure 1 illustrates what a Gaussian random field on a curve looks like. The left column presents a random field on a quarter circle $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, \pi/4]$ in $\mathbb{R}^2$ and and the right column presents a helix $\gamma(t) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2})$ for $t \in [0, 4\pi\sqrt{2}]$ in $\mathbb{R}^3$. From top to bottom, the random fields are simulated from stratified design, random design and deterministic design respectively as Gaussian random field with mean 0 and Matérn covariance function K_M with $\sigma^2, \alpha = 1$ and $\nu = 0.1$. For stratified design, δ_i's are independent uniform random variables, for the random design, the location sampling distribution is a uniform distribution on the corresponding curve and for the deterministic design, $\varphi(s) = s(s+1)^3/(\pi/2+1)^3$ for the curve in $\mathbb{R}^2$ and $\varphi(s) = s(s+1)^2/(4\pi\sqrt{2}+1)^2$ for the curve in $\mathbb{R}^3$. We observe that as the curve traces from upper left to bottom right for the curve in $\mathbb{R}^2$ and from top to the bottom for the curve in $\mathbb{R}^3$, the points of the deterministically sampled random fields get closer and closer to each other gradually, the points of the stratified design keep neither too close nor too separated from each other and the spacings of the random design are most irregular.

Since Section 4 shows that the random design can be reduced to stratified design, Monte Carlo simulations are carried out only for the stratified design and the deterministic design. This section assumes that

(i) $X(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$, be a Gaussian random field with mean function $m(.)$ and covariance function $K_P(.,.)$ as in (1),

(ii) $\omega_n = 2 \vee \lfloor n^{1/3} \rfloor$ and $M_n = \lfloor \log(n) \rfloor$,

(iii) the observed sample is $X_1, \ldots, X_n$.

That $\omega_n \asymp n^{1/3}$ follows from Theorem 3. Simulations indicate that the estimators are not too affected by the value of M_n as long as M_n is sufficiently

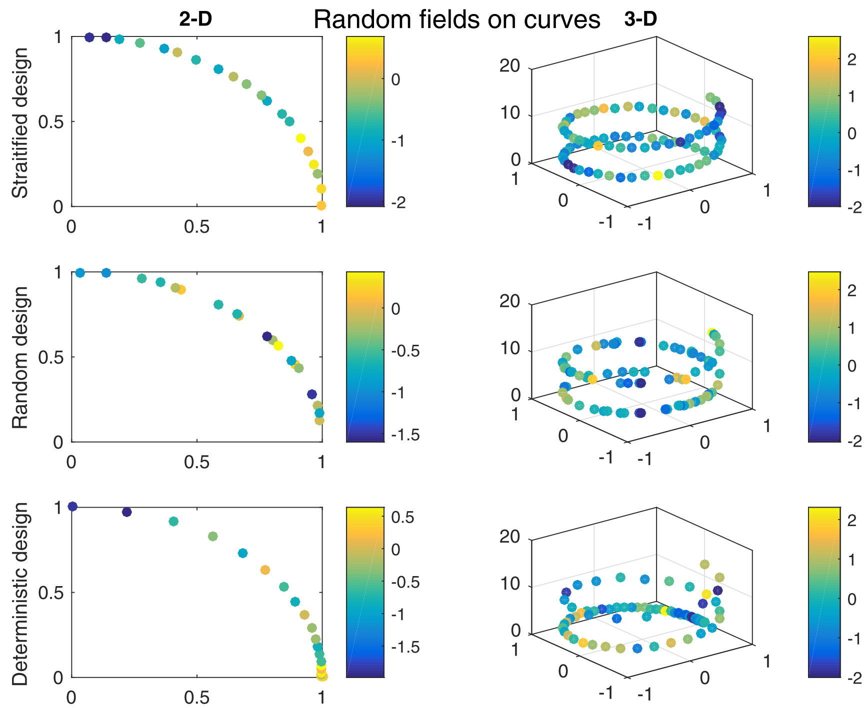

FIG 1. *Random fields on curve of different sampling designs where each dot is a sample point of the random field whose value is displayed via the color.*

large. The optimal choices of ω_n and M_n are not addressed in this article but are left to future work.

Let $\hat{\nu}_n$, $\hat{\nu}_n^D$ be as in Sections 3, 5 respectively. Experiments 1, 2, 3 are conducted to study the finite sample accuracy of $\hat{\nu}_n$ and Experiments 4, 5, 6 are conducted to study the finite sample accuracy of $\hat{\nu}_n^D$. For each experiment, we carry out ten sets of simulations with sample sizes $n = 100$, 200, 300, 400, 500, 600, 700, 800, 900, 1000 with 400 Monte Carlo repetitions each time. The estimated mean absolute errors and their standard errors of the estimators are computed. The results of Experiments 1 to 3 are reported in Tables 1, 2, 3 respectively and the results of Experiments 4 to 6 are reported in Tables 4, 5, 6 respectively.

In summary, $\hat{\nu}_n$ and $\hat{\nu}_n^D$ perform well in this simulation study with $\hat{\nu}_n^D$ significantly more accurate than $\hat{\nu}_n$ if ν is not close to 0. This is consistent with the convergence rates $O(n^{-1/3}), O(n^{-1/2})$ of $\hat{\nu}_n, \hat{\nu}_n^D$ as reported in Theorems 3, 7 respectively.

In Figure 2, we plot logarithm of mean absolute error (MAE) versus $\log(n)$ for Experiments 1-6 when the true smoothness parameter $\nu = 0.5$ and use linear regression to gauge their relationship. We observe that the slopes of the fitted regression lines of the stratified design experiments (Experiments 1-3) are close to $-1/3$ and those of the deterministic design experiments (Experiments 4-6) are close to $-1/2$ which are in line with the theoretical convergence rates.

Experiment 1. Set $d = 1$ and $\nu = 0.1, 0.2, 0.5, 1.0, 1.2, 1.5, 2, 2.2, 2.5$. The other parameters in (1) are $\sigma^2 = \exp(1)$, $\Sigma_t = 1 + \cos^2(t)$ and $m(t) = \sin(t)$. For $i = 1, \ldots, n$, $X_i = X(t_i)$ where $t_i = (i - 1 + \delta_i)/n$ with δ_i's i.i.d. random variables uniformly distributed in $[0, 1)$.

Experiment 2. Set $d = 2$ and $\nu = 0.1, 0.2, 0.5, 1.0, 1.2, 1.5, 2, 2.2, 2.5$. The curve is $\gamma(t) = (\cos t, \sin t)'$ and for $i = 1, \ldots, n$, $X_i = X(\gamma(t_i))$ where $t_i = (i - 1 + \delta_i)L/n$ with δ_i's i.i.d. random variables uniformly distributed in $[0, 1)$ and $L = \pi/2$. The other parameters in (1) are $\sigma^2 = \exp(1)$, $m(\gamma(t)) = \sin[\{\gamma_1(t)\}^2 - \{\gamma_2(t)\}^2]$ and $\Sigma_{\gamma(t)} = H\Lambda_{\gamma(t)}H'$, where

$$H = \begin{pmatrix} \sin(2\pi/5)2/\sqrt{5} & \sin(4\pi/5)2/\sqrt{5} \\ \sin(4\pi/5)2/\sqrt{5} & \sin(8\pi/5)2/\sqrt{5} \end{pmatrix},$$

and

$$\Lambda_{\gamma(t)} = \mathrm{diag}(1 + \cos^2(\gamma_1(t) + \pi/4), 1 + \cos^2(\gamma_2(t) + \pi/2)).$$

Experiment 3. Set $d = 3$ and $\nu = 0.1, 0.2, 0.5, 1.0, 1.2, 1.5, 2, 2.2, 2.5$. The curve is $\gamma(t) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2})'$ and for $i = 1, \ldots, n$, $X_i = X(\gamma(t_i))$ where $t_i = (i-1+\delta_i)L/n$ with δ_i's i.i.d. random variables uniformly distributed in $[0, 1)$ and $L = 4\sqrt{2}\pi$. The other parameters in (1) are $\sigma^2 = \exp(1)$, $m(\gamma(t)) = \sin[\{\gamma_1(t)\}^2 - \{\gamma_2(t)\}^2] \cos\{\gamma_3(t)\}$ and $\Sigma_{\gamma(t)} = H\Lambda_{\gamma(t)}H'$, where

$$\begin{aligned} H &= \begin{pmatrix} \sin(2\pi/7)2/\sqrt{7} & \sin(4\pi/7)2/\sqrt{7} & \sin(6\pi/7)2/\sqrt{7} \\ \sin(4\pi/7)2/\sqrt{7} & \sin(8\pi/7)2/\sqrt{7} & \sin(12\pi/7)2/\sqrt{7} \\ \sin(6\pi/7)2/\sqrt{7} & \sin(12\pi/7)2/\sqrt{7} & \sin(18\pi/7)2/\sqrt{7} \end{pmatrix}, \\ \Lambda_{\gamma(t)} &= \mathrm{diag}\Big(1 + \cos^2(\gamma_1(t) + \frac{\pi}{6}), 1 + \cos^2(\gamma_2(t) + \frac{\pi}{3}), 1 + \cos^2(\gamma_3(t) + \frac{\pi}{2})\Big). \end{aligned}$$

Experiment 4. Set $d = 1$ and $\nu = 0.1, 0.2, 0.5, 1.0, 1.2, 1.5, 2, 2.2, 2.5$. The other parameters in (1) are as in Experiment 1. For $i = 1, \ldots, n$, $X_i = X(t_i)$ where $t_i = \varphi\{(i - 1)/(n - 1)\}$ with $\varphi(s) = s(s + 1)/2$ for $s \in [0, 1]$.

Experiment 5. Set $d = 2$ and $\nu = 0.1, 0.2, 0.5, 1.0, 1.2, 1.5, 2, 2.2, 2.5$. The curve is $\gamma(t) = (\cos t, \sin t)'$ and $\varphi(s) = s(s + 1)/(L + 1)$ for $0 \le s \le L = \pi/2$. The other parameters in (1) are as in Experiment 2. $X_i = X(\gamma(t_i))$ where $t_i = \varphi(L(i - 1)/(n - 1))$, $i = 1, \ldots, n$.

Experiment 6. Set $d = 3$ and $\nu = 0.1, 0.2, 0.5, 1.0, 1.2, 1.5, 2, 2.2, 2.5$. The curve is $\gamma(t) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2})'$ and $\varphi(s) = s(s + 1)/(L + 1)$ for $0 \le s \le L = 4\sqrt{2}\pi$. The other parameters in (1) are as in Experiment 3. $X_i = X(\gamma(t_i))$ where $t_i = \varphi(L(i - 1)/(n - 1))$, $i = 1, \ldots, n$.

As noted in [20], simulating a Gaussian process on $[0, 1]$ accurately when n and ν are large is a difficult problem; cf. [25, 29]. This is especially so when the data are irregularly spaced. This is the reason for setting the upper limit for the value of ν to be 2.5 in Experiments 1 to 6.

Table 1
Mean absolute error of $\hat{\nu}_n$ for Experiment 1 (standard error within parentheses)

ν	$\mathbb{E}\|\hat{\nu}_n - \nu\|$ $n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$
0.1	0.072(0.002)	0.060(0.002)	0.051(0.002)	0.044(0.002)	0.045(0.002)
0.2	0.098(0.003)	0.076(0.003)	0.063(0.002)	0.057(0.002)	0.046(0.002)
0.5	0.098(0.004)	0.083(0.003)	0.070(0.003)	0.063(0.003)	0.057(0.002)
1	0.176(0.007)	0.129(0.005)	0.109(0.005)	0.102(0.004)	0.100(0.004)
1.2	0.169(0.007)	0.125(0.005)	0.106(0.004)	0.095(0.004)	0.087(0.003)
1.5	0.159(0.007)	0.109(0.004)	0.092(0.004)	0.094(0.004)	0.076(0.003)
2	0.275(0.008)	0.193(0.006)	0.155(0.006)	0.151(0.006)	0.115(0.004)
2.2	0.258(0.010)	0.167(0.007)	0.129(0.005)	0.113(0.005)	0.103(0.004)
2.5	0.229(0.011)	0.138(0.006)	0.108(0.004)	0.113(0.004)	0.087(0.003)
ν	$n = 600$	$n = 700$	$n = 800$	$n = 900$	$n = 1000$
0.1	0.039(0.001)	0.038(0.001)	0.038(0.001)	0.037(0.001)	0.031(0.001)
0.2	0.047(0.002)	0.045(0.002)	0.042(0.002)	0.042(0.002)	0.037(0.001)
0.5	0.055(0.002)	0.050(0.002)	0.050(0.002)	0.050(0.002)	0.044(0.002)
1	0.085(0.003)	0.084(0.003)	0.079(0.003)	0.079(0.003)	0.079(0.003)
1.2	0.084(0.003)	0.079(0.003)	0.081(0.003)	0.074(0.003)	0.069(0.003)
1.5	0.079(0.003)	0.069(0.003)	0.064(0.002)	0.058(0.002)	0.060(0.002)
2	0.121(0.005)	0.104(0.004)	0.107(0.004)	0.094(0.004)	0.089(0.003)
2.2	0.102(0.004)	0.086(0.003)	0.088(0.003)	0.084(0.004)	0.078(0.003)
2.5	0.089(0.004)	0.082(0.003)	0.075(0.003)	0.074(0.003)	0.072(0.003)

Table 2
Mean absolute error of $\hat{\nu}_n$ for Experiment 2 (standard error within parentheses)

ν	$\mathbb{E}\|\hat{\nu}_n - \nu\|$ $n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$
0.1	0.072(0.002)	0.059(0.002)	0.051(0.002)	0.045(0.002)	0.044(0.002)
0.2	0.087(0.003)	0.073(0.003)	0.062(0.002)	0.060(0.002)	0.049(0.002)
0.5	0.110(0.005)	0.082(0.003)	0.071(0.003)	0.063(0.003)	0.055(0.002)
1	0.193(0.007)	0.136(0.005)	0.118(0.004)	0.109(0.004)	0.100(0.004)
1.2	0.178(0.007)	0.125(0.005)	0.111(0.004)	0.103(0.004)	0.090(0.004)
1.5	0.187(0.007)	0.131(0.005)	0.107(0.004)	0.095(0.004)	0.082(0.003)
2	0.356(0.008)	0.250(0.007)	0.211(0.006)	0.190(0.006)	0.159(0.005)
2.2	0.373(0.011)	0.216(0.010)	0.171(0.007)	0.142(0.007)	0.111(0.005)
2.5	0.396(0.016)	0.197(0.009)	0.171(0.008)	0.141(0.006)	0.116(0.005)
ν	$n = 600$	$n = 700$	$n = 800$	$n = 900$	$n = 1000$
0.1	0.043(0.002)	0.037(0.001)	0.036(0.001)	0.033(0.001)	0.032(0.001)
0.2	0.048(0.002)	0.044(0.002)	0.045(0.002)	0.043(0.002)	0.041(0.002)
0.5	0.054(0.002)	0.053(0.002)	0.052(0.002)	0.047(0.002)	0.045(0.002)
1	0.091(0.004)	0.090(0.004)	0.084(0.003)	0.083(0.003)	0.077(0.003)
1.2	0.086(0.003)	0.077(0.003)	0.079(0.003)	0.080(0.003)	0.070(0.003)
1.5	0.079(0.003)	0.075(0.003)	0.076(0.003)	0.067(0.003)	0.067(0.002)
2	0.154(0.006)	0.125(0.005)	0.120(0.005)	0.108(0.004)	0.098(0.004)
2.2	0.104(0.004)	0.099(0.004)	0.106(0.004)	0.092(0.003)	0.082(0.003)
2.5	0.107(0.004)	0.099(0.004)	0.098(0.004)	0.084(0.004)	0.083(0.003)

Table 3

Mean absolute error of $\hat{\nu}_n$ for Experiment 3 (standard error within parentheses)

	$\mathbb{E}\|\hat{\nu}_n - \nu\|$				
ν	$n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$
0.1	0.238(0.008)	0.117(0.004)	0.068(0.003)	0.049(0.002)	0.044(0.002)
0.2	0.212(0.008)	0.131(0.005)	0.097(0.003)	0.082(0.003)	0.053(0.002)
0.5	0.174(0.006)	0.134(0.005)	0.111(0.004)	0.118(0.004)	0.094(0.004)
1	0.345(0.011)	0.242(0.008)	0.209(0.007)	0.201(0.006)	0.166(0.006)
1.2	0.468(0.013)	0.374(0.009)	0.324(0.008)	0.284(0.007)	0.223(0.006)
1.5	0.665(0.014)	0.572(0.009)	0.502(0.008)	0.460(0.007)	0.373(0.006)
2	1.022(0.018)	0.945(0.008)	0.851(0.007)	0.785(0.006)	0.682(0.005)
2.2	1.201(0.018)	1.102(0.008)	0.995(0.007)	0.938(0.006)	0.834(0.005)
2.5	1.444(0.019)	1.353(0.008)	1.233(0.006)	1.166(0.006)	1.062(0.004)
ν	$n = 600$	$n = 700$	$n = 800$	$n = 900$	$n = 1000$
0.1	0.039(0.001)	0.038(0.001)	0.037(0.001)	0.032(0.001)	0.032(0.001)
0.2	0.051(0.002)	0.045(0.002)	0.044(0.002)	0.041(0.002)	0.039(0.001)
0.5	0.104(0.004)	0.092(0.004)	0.084(0.003)	0.080(0.003)	0.083(0.003)
1	0.155(0.006)	0.138(0.005)	0.127(0.005)	0.128(0.004)	0.111(0.004)
1.2	0.216(0.006)	0.189(0.005)	0.186(0.005)	0.180(0.005)	0.152(0.005)
1.5	0.368(0.005)	0.322(0.005)	0.321(0.005)	0.283(0.005)	0.253(0.004)
2	0.661(0.005)	0.608(0.004)	0.601(0.004)	0.558(0.004)	0.531(0.003)
2.2	0.816(0.005)	0.747(0.004)	0.738(0.004)	0.702(0.004)	0.655(0.003)
2.5	1.042(0.004)	0.969(0.004)	0.964(0.003)	0.920(0.003)	0.875(0.003)

Table 4

Mean absolute error of $\hat{\nu}_n^D$ for Experiment 4 (standard error within parentheses)

	$\mathbb{E}\|\hat{\nu}_n^D - \nu\|$				
ν	$n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$
0.1	0.071(0.002)	0.059(0.002)	0.048(0.002)	0.046(0.002)	0.041(0.002)
0.2	0.081(0.003)	0.057(0.002)	0.046(0.002)	0.041(0.002)	0.033(0.001)
0.5	0.058(0.002)	0.040(0.002)	0.034(0.001)	0.029(0.001)	0.025(0.001)
1	0.100(0.004)	0.072(0.003)	0.059(0.002)	0.049(0.002)	0.046(0.002)
1.2	0.081(0.003)	0.059(0.002)	0.051(0.002)	0.040(0.002)	0.040(0.002)
1.5	0.071(0.003)	0.049(0.002)	0.037(0.001)	0.033(0.001)	0.028(0.001)
2	0.121(0.005)	0.073(0.003)	0.062(0.002)	0.050(0.002)	0.046(0.002)
2.2	0.096(0.004)	0.059(0.002)	0.052(0.002)	0.044(0.002)	0.046(0.002)
2.5	0.089(0.004)	0.058(0.002)	0.046(0.002)	0.056(0.002)	0.116(0.004)
ν	$n = 600$	$n = 700$	$n = 800$	$n = 900$	$n = 1000$
0.1	0.036(0.001)	0.032(0.001)	0.029(0.001)	0.029(0.001)	0.028(0.001)
0.2	0.034(0.001)	0.030(0.001)	0.030(0.001)	0.028(0.001)	0.026(0.001)
0.5	0.026(0.001)	0.023(0.001)	0.021(0.001)	0.020(0.001)	0.018(0.001)
1	0.041(0.002)	0.040(0.001)	0.036(0.001)	0.032(0.001)	0.030(0.001)
1.2	0.033(0.001)	0.032(0.001)	0.029(0.001)	0.027(0.001)	0.023(0.001)
1.5	0.026(0.001)	0.025(0.001)	0.022(0.001)	0.021(0.001)	0.021(0.001)
2	0.040(0.002)	0.038(0.001)	0.038(0.001)	0.044(0.002)	0.049(0.002)
2.2	0.048(0.002)	0.062(0.002)	0.055(0.002)	0.053(0.002)	0.085(0.003)
2.5	0.106(0.003)	0.248(0.004)	0.425(0.006)	0.791(0.008)	0.980(0.007)

Table 5
Mean absolute error of $\hat{\nu}_n^D$ for Experiment 5 (standard error within parentheses)

	$\mathbb{E}\|\hat{\nu}_n^D - \nu\|$				
ν	$n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$
0.1	0.077(0.002)	0.063(0.002)	0.056(0.002)	0.048(0.002)	0.043(0.002)
0.2	0.083(0.003)	0.066(0.002)	0.049(0.002)	0.039(0.001)	0.039(0.002)
0.5	0.066(0.003)	0.047(0.002)	0.038(0.001)	0.031(0.001)	0.029(0.001)
1	0.117(0.005)	0.089(0.004)	0.071(0.003)	0.064(0.002)	0.056(0.002)
1.2	0.107(0.004)	0.074(0.003)	0.062(0.002)	0.049(0.002)	0.043(0.002)
1.5	0.083(0.003)	0.055(0.002)	0.044(0.002)	0.039(0.001)	0.034(0.001)
2	0.148(0.005)	0.101(0.004)	0.074(0.003)	0.070(0.003)	0.057(0.002)
2.2	0.134(0.005)	0.087(0.003)	0.072(0.003)	0.053(0.002)	0.051(0.002)
2.5	0.105(0.004)	0.070(0.003)	0.050(0.002)	0.050(0.002)	0.058(0.002)
ν	$n = 600$	$n = 700$	$n = 800$	$n = 900$	$n = 1000$
0.1	0.036(0.001)	0.036(0.001)	0.035(0.001)	0.033(0.001)	0.032(0.001)
0.2	0.037(0.001)	0.034(0.001)	0.030(0.001)	0.030(0.001)	0.027(0.001)
0.5	0.027(0.001)	0.023(0.001)	0.023(0.001)	0.021(0.001)	0.021(0.001)
1	0.050(0.002)	0.045(0.002)	0.044(0.002)	0.038(0.001)	0.037(0.001)
1.2	0.042(0.001)	0.040(0.001)	0.037(0.001)	0.034(0.001)	0.031(0.001)
1.5	0.031(0.001)	0.029(0.001)	0.025(0.001)	0.026(0.001)	0.023(0.001)
2	0.054(0.002)	0.050(0.002)	0.047(0.002)	0.044(0.002)	0.046(0.002)
2.2	0.049(0.002)	0.048(0.002)	0.048(0.002)	0.054(0.002)	0.067(0.002)
2.5	0.058(0.002)	0.103(0.004)	0.067(0.002)	0.090(0.002)	0.140(0.003)

Table 6
Mean absolute error of $\hat{\nu}_n^D$ for Experiment 6 (standard error within parentheses)

	$\mathbb{E}\|\hat{\nu}_n^D - \nu\|$				
ν	$n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$
0.1	0.116(0.004)	0.097(0.003)	0.091(0.003)	0.085(0.003)	0.079(0.003)
0.2	0.138(0.004)	0.108(0.004)	0.092(0.003)	0.088(0.003)	0.078(0.003)
0.5	0.115(0.005)	0.075(0.003)	0.060(0.002)	0.048(0.002)	0.042(0.002)
1	0.316(0.013)	0.222(0.010)	0.187(0.007)	0.150(0.006)	0.136(0.005)
1.2	0.258(0.010)	0.154(0.006)	0.122(0.005)	0.111(0.005)	0.100(0.004)
1.5	0.220(0.007)	0.125(0.004)	0.092(0.003)	0.070(0.003)	0.060(0.002)
2	0.447(0.006)	0.313(0.004)	0.240(0.005)	0.184(0.005)	0.152(0.006)
2.2	0.584(0.006)	0.334(0.009)	0.168(0.007)	0.130(0.005)	0.111(0.004)
2.5	0.762(0.011)	0.214(0.008)	0.133(0.005)	0.108(0.004)	0.086(0.003)
ν	$n = 600$	$n = 700$	$n = 800$	$n = 900$	$n = 1000$
0.1	0.076(0.002)	0.071(0.002)	0.069(0.002)	0.069(0.002)	0.064(0.002)
0.2	0.077(0.003)	0.071(0.003)	0.070(0.003)	0.063(0.002)	0.061(0.002)
0.5	0.036(0.001)	0.038(0.002)	0.033(0.001)	0.032(0.001)	0.031(0.001)
1	0.131(0.005)	0.114(0.005)	0.106(0.004)	0.098(0.004)	0.097(0.004)
1.2	0.091(0.004)	0.081(0.003)	0.074(0.003)	0.066(0.003)	0.067(0.002)
1.5	0.054(0.002)	0.048(0.002)	0.046(0.002)	0.043(0.002)	0.038(0.002)
2	0.130(0.005)	0.118(0.005)	0.109(0.004)	0.109(0.004)	0.101(0.004)
2.2	0.103(0.004)	0.091(0.004)	0.085(0.003)	0.077(0.003)	0.072(0.003)
2.5	0.079(0.003)	0.075(0.002)	0.063(0.002)	0.062(0.002)	0.078(0.003)

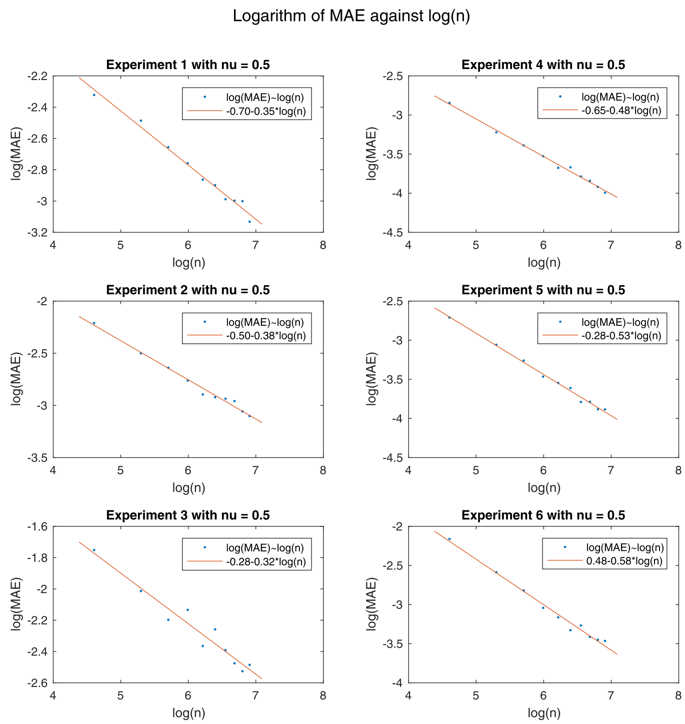

FIG 2. *Blue dots depict Logarithm of MAE versus* $\log(n)$ *for Experiments 1-6 with the true smoothness parameter* $\nu = 0.5$ *and the red lines are the fitted regression lines of the blue dots. The slopes of the fitted regression lines of the stratified design experiments (Experiments 1-3) are close to* $-1/3$ *and those of the deterministic design experiments (Experiments 4-6) are close to* $-1/2$.

Appendix A: Lemma 2 and Lemma 3

Lemma 2. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a* $k+1$ *times continuously differentiable function in a neighborhood of the line segment joining the points* $\mathbf{a}, \mathbf{x} \in \mathbb{R}^2$. *Then*

$$\begin{aligned} f(\mathbf{x}) \quad &= \sum_{u_1+u_2\le k} \frac{f^{(u_1,u_2)}(\mathbf{a})}{u_1!u_2!}(x_1-a_1)^{u_1}(x_2-a_2)^{u_2} \\ &\quad + \sum_{u_1+u_2=k+1} \frac{(k+1)(x_1-a_1)^{u_1}(x_2-a_2)^{u_2}}{u_1!u_2!} \\ &\quad \times \int_0^1 (1-t)^k f^{(u_1,u_2)}(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))dt. \end{aligned}$$

Let $t_1, \ldots, t_n$, be as in (12). Define

$$Q_{i,j} = (\gamma(t_i)-\gamma(t_j))' A(\gamma(t_i),\gamma(t_j))(\gamma(t_i)-\gamma(t_j)), \qquad \forall i,j = 1,\ldots,n, \quad (20)$$

where $A(.,.)$ is the matrix in (6).

Lemma 3. *Let $M \geq 1$ be an arbitrary but fixed integer and k, k_1, k_2 be integers such that $0 \leq k \leq M$, $0 \leq k_1 \leq k_2 \leq M$. Suppose (12) and Condition 3 hold. Then we have the following approximations as $n \to \infty$ uniformly over $1 \leq i \leq n - k_2$ and $\delta_j \in [0,1), 1 \leq j \leq n$.*

$$\begin{aligned} d_{i+\omega_n k_1, i+\omega_n k_2} &= d_{i,i+\omega_n k_2} - d_{i,i+\omega_n k_1} + O(\omega_n^3 n^{-3}), & (21)\\ d_{i+\omega_n k_1, i+\omega_n k_2} &= \frac{\omega_n (k_2 - k_1) L}{n} + O(n^{-1}) + O(\omega_n^3 n^{-3}), & (22)\\ Q_{i+\omega_n k_1, i+\omega_n k_2} &= d^2_{i+\omega_n k_1, i+\omega_n k_2} \{\gamma^{(1)}(t_i)\}' A(\gamma(t_i), \gamma(t_i)) \gamma^{(1)}(t_i) \\ &\quad + O(\omega_n^3 n^{-3}), & (23) \end{aligned}$$

and for any $1 \leq l \leq N$,

$$t_{i+\omega_n k} - t_i = d_{i,i+\omega_n k} + \sum_{j=3}^{l} f_j(t_i) d^j_{i,i+\omega_n k} + o(\omega_n^l n^{-l}), \qquad (24)$$

where $Q_{i+\omega_n k_1, i+\omega_n k_2}$ is as in (20) and $f_j(t_i)$, $j = 3, \ldots, l$, are polynomials of $\gamma^{(u)}(t_i)$, $u = 1, \ldots, l$.

Proof. Using Taylor's expansion, it follows from Condition 3 that

$$\begin{aligned} & d^2_{i+\omega_n k_1, i+\omega_n k_2} \\ = & \sum_{j=1}^{d} \{\gamma_j(t_{i+\omega_n k_1}) - \gamma_j(t_{i+\omega_n k_2})\}^2 \\ = & \sum_{j=1}^{d} \Big\{ \gamma_j^{(1)}(t_i)(t_{i+\omega_n k_1} - t_{i+\omega_n k_2}) + \frac{\gamma_j^{(2)}(t_i)}{2}(t_{i+\omega_n k_1} - t_{i+\omega_n k_2})^2 \\ & + O(\omega_n^3 n^{-3}) \Big\}^2 \\ = & \Big[\sum_{j=1}^{d} \{\gamma_j^{(1)}(t_i)\}^2 (t_{i+\omega_n k_1} - t_{i+\omega_n k_2})^2 \\ & + \sum_{j=1}^{d} \gamma_j^{(1)}(t_i) \gamma_j^{(2)}(t_i)(t_{i+\omega_n k_1} - t_{i+\omega_n k_2})^3 + O(\omega_n^4 n^{-4}) \Big] \\ = & (t_{i+\omega_n k_1} - t_{i+\omega_n k_2})^2 + O(\omega_n^4 n^{-4}), \end{aligned}$$

where the last equality follows from (7). Taking square root, we have

$$d_{i+\omega_n k_1, i+\omega_n k_2} = (t_{i+\omega_n k_2} - t_{i+\omega_n k_1}) + O(\omega_n^3 n^{-3}), \qquad (25)$$

as $n \to \infty$. Setting $k_1 = 0$ in (25), we have

$$\begin{aligned} d_{i,i+\omega_n k_1} &= (t_{i+\omega_n k_1} - t_i) + O(\omega_n^3 n^{-3}), \\ d_{i,i+\omega_n k_2} &= (t_{i+\omega_n k_2} - t_i) + O(\omega_n^3 n^{-3}), \end{aligned}$$

which together with (25) implies (21). (22) follows from (12) and (25) by observing that

$$\begin{aligned} d_{i+\omega_n k_1, i+\omega_n k_2} &= (t_{i+\omega_n k_2} - t_{i+\omega_n k_1}) + O(\omega_n^3 n^{-3}), \\ &= \Big\{\frac{L(i+\omega_n k_2 - 1 + \delta_{i+\omega_n k_2})}{n} - \frac{L(i+\omega_n k_1 - 1 + \delta_{i+\omega_n k_1})}{n}\Big\} \\ &\quad + O(\omega_n^3 n^{-3}) \\ &= \frac{\omega_n (k_2 - k_1) L}{n}\{1 + O(\omega_n^{-1})\} + O(\omega_n^3 n^{-3}). \\ &= \frac{\omega_n (k_2 - k_1) L}{n} + O(n^{-1}) + O(\omega_n^3 n^{-3}), \end{aligned}$$

as $n \to \infty$. (23) follows from the observation that

$$\begin{aligned} & Q_{i+\omega_n k_1, i+\omega_n k_2} \\ = \; & \{\gamma(t_{i+\omega_n k_1}) - \gamma(t_{i+\omega_n k_2})\}' A(\gamma(t_{i+\omega_n k_1}), \gamma(t_{i+\omega_n k_2})) \\ & \times \{\gamma(t_{i+\omega_n k_1}) - \gamma(t_{i+\omega_n k_2})\} \\ = \; & \{\gamma^{(1)}(t_i)(t_{i+\omega_n k_1} - t_{i+\omega_n k_2}) + O(\omega_n^2 n^{-2})\}' \{A(\gamma(t_i), \gamma(t_i)) + O(\omega_n n^{-1})\} \\ & \times \{\gamma^{(1)}(t_i)(t_{i+\omega_n k_1} - t_{i+\omega_n k_2}) + O(\omega_n^2 n^{-2})\} \\ = \; & (t_{i+\omega_n k_1} - t_{i+\omega_n k_2})^2 \{\gamma^{(1)}(t_i)\}' A(\gamma(t_i), \gamma(t_i)) \gamma^{(1)}(t_i) + O(\omega_n^3 n^{-3}), \end{aligned}$$

as $n \to \infty$. To show (24), we observe that

$$\begin{aligned} d^2_{i,i+\omega_n k} &= \|\gamma(t_{i+\omega_n k}) - \gamma(t_i)\|^2 \\ &= \sum_{j=1}^{d} \{\gamma_j(t_{i+\omega_n k}) - \gamma_j(t_i)\}^2 \\ &= \sum_{j=1}^{d} \{\sum_{u=1}^{l} \gamma_j^{(u)}(t_i) \frac{(t_{i+\omega_n k} - t_i)^u}{u!} + o(\omega_n^l n^{-l})\}^2 \\ &= \sum_{u=1}^{l} \sum_{v=1}^{l} \Big\{ \sum_{j=1}^{d} \gamma_j^{(u)}(t_i) \gamma_j^{(v)}(t_i) \frac{(t_{i+\omega_n k} - t_i)^{u+v}}{u!v!} \Big\} + o(\omega_n^{l+1} n^{-l-1}) \\ &= (t_{i+\omega_n k} - t_i)^2 + \sum_{\substack{1 \le u,v \le l \\ u+v \ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i) \rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v}}{u!v!} \\ &\quad + o(\omega_n^{l+1} n^{-l-1}), \end{aligned} \tag{26}$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$. From (26), we have

$$\begin{aligned} & d_{i,i+\omega_n k} \\ = \; & \Big\{ (t_{i+\omega_n k} - t_i)^2 + \sum_{\substack{1 \le u,v \le l \\ u+v \ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i) \rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v}}{u!v!} \end{aligned}$$

$$
\begin{aligned}
& +o(\omega_n^{l+1} n^{-l-1})\Big\}^{1/2} \\
= \;& (t_{i+\omega_n k} - t_i) \\
& \times\Big\{1 + \sum_{\substack{1\le u,v\le l\\ u+v\ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i)\rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v-2}}{u!v!} + o(\omega_n^{l-1} n^{-l+1})\Big\}^{1/2} \\
= \;& (t_{i+\omega_n k} - t_i) + \frac{1}{2} \sum_{\substack{1\le u,v\le l\\ u+v\ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i)\rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v-1}}{u!v!} \\
& +o(\omega_n^{l} n^{-l}) \\
& -\frac{1}{8}(t_{i+\omega_n k} - t_i)\Big\{ \sum_{\substack{1\le u,v\le l\\ u+v\ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i)\rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v-2}}{u!v!} \\
& +o(\omega_n^{l-1} n^{-l+1})\Big\}^{2} \\
& +\frac{1}{16}(t_{i+\omega_n k} - t_i)\Big\{ \sum_{\substack{1\le u,v\le l\\ u+v\ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i)\rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v-2}}{u!v!} \\
& +o(\omega_n^{l-1} n^{-l+1})\Big\}^{3} \\
& +\dots \\
= \;& (t_{i+\omega_n k} - t_i) + o(\omega_n^{l} n^{-l}) \\
& +\sum_{j=1}^{l} c_j (t_{i+\omega_n k} - t_i)\Big\{ \sum_{\substack{1\le u,v\le l\\ u+v\ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i)\rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v-2}}{u!v!} \\
& +o(\omega_n^{l-1} n^{-l+1})\Big\}^{j} \\
= \;& (t_{i+\omega_n k} - t_i) + o(\omega_n^{l} n^{-l}) \\
& +\sum_{j=1}^{l} c_j (t_{i+\omega_n k} - t_i)\Big\{ \sum_{\substack{1\le u,v\le l\\ u+v\ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i)\rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v-2}}{u!v!}\Big\}^{j},
\end{aligned}
$$

where c_j are constants originating from the coefficients of the Taylor expansion of the function $x \mapsto \sqrt{1+x}$. This further yields

$$
\begin{aligned}
& t_{i+\omega_n k} - t_i \\
= \;& d_{i,i+\omega_n k} + o(\omega_n^{l} n^{-l}) \\
& -\sum_{j=1}^{l} c_j (t_{i+\omega_n k} - t_i)\Big\{ \sum_{\substack{1\le u,v\le l\\ u+v\ge 4}} \langle \gamma^{(u)}(t_i), \gamma^{(v)}(t_i)\rangle \frac{(t_{i+\omega_n k} - t_i)^{u+v-2}}{u!v!}\Big\}^{j} \\
= \;& d_{i,i+\omega_n k} + \sum_{j=3}^{l} g_j(t_i)(t_{i+\omega_n k} - t_i)^j + o(\omega_n^{l} n^{-l}), \qquad (27)
\end{aligned}
$$

where $g_j(t_i)$, $j = 3, \ldots, l$, are polynomials of $\gamma^{(u)}(t_i)$, $u = 1, \ldots, l$. By iteratively replacing $t_{i+\omega_n k} - t_i$ by $d_{i,i+\omega_n k} + \sum_{j=3}^{l} g_j(t_i)(t_{i+\omega_n k} - t_i)^j + o(\omega_n^l n^{-l})$ in (27), we obtain

$$t_{i+\omega_n k} - t_i = d_{i,i+\omega_n k} + \sum_{j=3}^{l} f_j(t_i) d_{i,i+\omega_n k}^j + o(\omega_n^l n^{-l}),$$

as $n \to \infty$. This proves Lemma 3. □

Appendix B: Proofs of Propositions 1 to 3

Proof of Proposition 1

Let $\mathbf{x} = (x_1, \ldots, x_d)', \mathbf{y} = (y_1, \ldots, y_d)'$. We observe that when $\nu \notin \mathbb{Z}_+$, $K_P(\mathbf{x}, \mathbf{y})$ in (3) can be written as in (5) with

$$\begin{aligned}
\rho_0(\mathbf{x}, \mathbf{y}) &= \sigma_{\mathbf{x},\mathbf{y}}^2 \sum_{k=0}^{\infty} \frac{\nu^k Q_{\mathbf{x},\mathbf{y}}^k}{k! \prod_{i=1}^{k}(i-\nu)} \\
&\quad - \frac{\pi \sigma_{\mathbf{x},\mathbf{y}}^2}{\Gamma(\nu)\sin(\nu\pi)} \sum_{k=1}^{\infty} \frac{\nu^{k+\nu} Q_{\mathbf{x},\mathbf{y}}^{k+\nu}}{k!\Gamma(k+1+\nu)}, &(28)\\
\beta_\nu(\mathbf{x}, \mathbf{y}) &= -\frac{\pi \sigma_{\mathbf{x},\mathbf{y}}^2}{\Gamma(\nu)\sin(\nu\pi)} \frac{\nu^\nu}{\Gamma(1+\nu)}, &(29)\\
A(\mathbf{x}, \mathbf{y}) &= \left(\frac{\Sigma_{\mathbf{x}} + \Sigma_{\mathbf{y}}}{2}\right)^{-1}, &(30)\\
Q_{\mathbf{x},\mathbf{y}}^\nu &= G_\nu\{\sqrt{(\mathbf{x}-\mathbf{y})' A(\mathbf{x}, \mathbf{y})(\mathbf{x}-\mathbf{y})}\}.
\end{aligned}$$

When $\nu \in \mathbb{Z}_+$, $K_P(\mathbf{x}, \mathbf{y})$ in (4) can be written as in (5) with

$$\begin{aligned}
\rho_0(\mathbf{x}, \mathbf{y}) &= \frac{2\sigma_{\mathbf{x},\mathbf{y}}^2}{(\nu-1)!} \Big\{ (-1)^{\nu+1} \log(\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}}) \sum_{k=1}^{\infty} \frac{\nu^{k+\nu} Q_{\mathbf{x},\mathbf{y}}^{k+\nu}}{k!(k+\nu)!} \\
&\quad + \frac{(-1)^{\nu+1}}{2\nu!} \nu^\nu Q_{\mathbf{x},\mathbf{y}}^\nu \log(\nu) \\
&\quad + \frac{1}{2} \sum_{k=0}^{\nu-1} (-1)^k \frac{(\nu-k-1)!}{k!} (\sqrt{\nu Q_{\mathbf{x},\mathbf{y}}})^{2k} \\
&\quad + \frac{(-1)^\nu}{2} \sum_{k=0}^{\infty} [\psi(k+1) + \psi(\nu+k+1)] \frac{\nu^{k+\nu} Q_{\mathbf{x},\mathbf{y}}^{k+\nu}}{k!(k+\nu)!} \Big\}, \\
\beta_\nu(\mathbf{x}, \mathbf{y}) &= \frac{2\sigma_{\mathbf{x},\mathbf{y}}^2}{(\nu-1)!\nu!} (-1)^{\nu+1} \nu^\nu, &(31)\\
A(\mathbf{x}, \mathbf{y}) &= \left(\frac{\Sigma_{\mathbf{x}} + \Sigma_{\mathbf{y}}}{2}\right)^{-1}, &(32)\\
Q_{\mathbf{x},\mathbf{y}}^\nu \log(\sqrt{Q_{\mathbf{x},\mathbf{y}}}) &= G_\nu\{\sqrt{(\mathbf{x}-\mathbf{y})' A(\mathbf{x}, \mathbf{y})(\mathbf{x}-\mathbf{y})}\}.
\end{aligned}$$

In both of the above cases,

$$\begin{aligned} Q_{\mathbf{x},\mathbf{y}} &= (\mathbf{x}-\mathbf{y})'\Big(\frac{\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}}{2}\Big)^{-1}(\mathbf{x}-\mathbf{y}), \\ \sigma^2_{\mathbf{x},\mathbf{y}} &= |\Sigma_{\mathbf{x}}|^{1/4}|\Sigma_{\mathbf{y}}|^{1/4}\Big|\frac{\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}}{2}\Big|^{-1/2}\sigma^2. \end{aligned} \tag{33}$$

The latter is as in (2). We observe that

$$\Big(\frac{\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}}{2}\Big)^{-1} = \frac{2}{|\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}|}\mathcal{A}(\mathbf{x},\mathbf{y}),$$

where $\mathcal{A}(\mathbf{x},\mathbf{y})$ is the adjugate of $(\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}})$. The (i,j)th entry of $\mathcal{A}(\mathbf{x},\mathbf{y})$ is the (j,i)th cofactor of $(\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}})$ and is therefore a polynomial of the entries of the matrix $(\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}})$. Then $Q_{\mathbf{x},\mathbf{y}} = 2|\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}|^{-1}(\mathbf{x}-\mathbf{y})'\mathcal{A}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})$.

In the following, we let $D \subset \mathbb{R}^{2d}$ be an arbitrary compact set and $D_0 = D\backslash\{(\mathbf{x},\mathbf{y}) : \mathbf{x}=\mathbf{y}\}$. We note that there exist $\psi_{\min,D}, \psi_{\max,D} > 0$ such that

$$\psi_{\min,D}\|\mathbf{x}-\mathbf{y}\|^2 \le (\mathbf{x}-\mathbf{y})'\mathcal{A}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y}) \le \psi_{\max,D}\|\mathbf{x}-\mathbf{y}\|^2$$

for all $(\mathbf{x},\mathbf{y}) \in D$, where $\psi_{\min,D}, \psi_{\max,D}$ can be chosen as the positive lower bound of the smallest eigenvalue of $\mathcal{A}(\mathbf{x},\mathbf{y})$ and upper bound of the largest eigenvalue of $\mathcal{A}(\mathbf{x},\mathbf{y})$ respectively. For $\alpha \in [0,\infty)$, define

$$\begin{aligned} \xi_\alpha(\mathbf{x},\mathbf{y}) &= |\Sigma_{\mathbf{x}}|^{1/4}|\Sigma_{\mathbf{y}}|^{1/4}\Big|\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}\Big|^{-\alpha-1/2}, \\ \tilde{Q}(\mathbf{x},\mathbf{y}) &= (\mathbf{x}-\mathbf{y})'\mathcal{A}(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y}), \\ \tilde{Q}_\alpha(\mathbf{x},\mathbf{y}) &= \{\tilde{Q}(\mathbf{x},\mathbf{y})\}^\alpha, \\ \mathcal{Q}_\alpha(\mathbf{x},\mathbf{y}) &= \xi_\alpha(\mathbf{x},\mathbf{y})\{\tilde{Q}(\mathbf{x},\mathbf{y})\}^\alpha. \end{aligned}$$

We observe that

$$\begin{aligned} \tilde{Q}(\mathbf{x},\mathbf{y}) &= \sum_{1\le l_1,l_2\le d} P_{l_1,l_2}(x_{l_1}-y_{l_1})(x_{l_2}-y_{l_2}), \\ \mathcal{Q}_\alpha(\mathbf{x},\mathbf{y}) &= \xi_\alpha(\mathbf{x},\mathbf{y})\Big\{\sum_{1\le l_1,l_2\le d} P_{l_1,l_2}(x_{l_1}-y_{l_1})(x_{l_2}-y_{l_2})\Big\}^\alpha, \end{aligned}$$

where P_{l_1,l_2} is a polynomial of entries of $\Sigma_{\mathbf{x}}$ and $\Sigma_{\mathbf{y}}$.

Since $\Sigma_{\mathbf{x}}$ is positive definite and N times continuously differentiable over $\mathbb{R}^d$, there exist positive constants C_D and $\tilde{C}_D$ such that $\tilde{C}_D \le |\Sigma_{\mathbf{x}}|, |\Sigma_{\mathbf{y}}|, |\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}| \le C_D$ for all $(\mathbf{x},\mathbf{y}) \in D$. Let $C_{M,D}$ be a positive constant depending on M and D whose value may be different from line to line. Then, we can see that, for any positive integer $M \le N$ such that $\sum_{i=1}^d (u_i+v_i) = M$,

$$|\xi_\alpha^{(u_1,\ldots,u_d,v_1,\ldots,v_d)}(\mathbf{x},\mathbf{y})| \le C_{M,D}^{\alpha+1} \qquad \forall \alpha \in [0,\infty), (\mathbf{x},\mathbf{y}) \in D.$$

Applying similar arguments that prove $\tilde{\rho}_\nu(.,.) \in C^{\lceil 2\nu\rceil-1}(\mathbb{R}^2)$ in Proposition 3, it is not difficult to show that $\rho_0(.,.) \in C^{\lceil 2\nu\rceil+1}(\mathbb{R}^{2d})$. The details are omitted here.

Next we compute appropriate bounds on the partial derivatives of $\rho_0(\mathbf{x},\mathbf{y})$. We observe that the leading term in $\mathcal{Q}_\alpha^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})$ has the same order as that of $\tilde{Q}_\alpha^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})$, which satisfies

$$\begin{aligned}&\tilde{Q}_\alpha^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})\\ \le\ &\begin{cases}C_{M,D}^{\alpha+1}\|\mathbf{x}-\mathbf{y}\|^{2\alpha-M} \quad \forall(\mathbf{x},\mathbf{y})\in D_0 & \text{if } \alpha\notin\mathbb{Z} \text{ and } 2\alpha\le M,\\ C_{M,D}^{\alpha+1} \quad \forall(\mathbf{x},\mathbf{y})\in D & \text{if } \alpha\in\mathbb{Z} \text{ or } 2\alpha> M,\end{cases}\end{aligned} \tag{34}$$

where the constant $C_{M,D}$ is independent of α. Consequently,

$$\begin{aligned}&\mathcal{Q}_\alpha^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})\\ \le\ &\begin{cases}C_{M,D}^{\alpha+1}\|\mathbf{x}-\mathbf{y}\|^{2\alpha-M} \quad \forall(\mathbf{x},\mathbf{y})\in D_0 & \text{if } \alpha\notin\mathbb{Z} \text{ and } 2\alpha\le M,\\ C_{M,D}^{\alpha+1} \quad \forall(\mathbf{x},\mathbf{y})\in D & \text{if } \alpha\in\mathbb{Z} \text{ or } 2\alpha> M.\end{cases}\end{aligned} \tag{35}$$

When $\nu\notin\mathbb{Z}_+$, we observe from (28) that $\rho_0(\mathbf{x},\mathbf{y})$ can be expressed as

$$\rho_0(\mathbf{x},\mathbf{y})=\sum_{k=0}^\infty c_k\mathcal{Q}_k(\mathbf{x},\mathbf{y})+\sum_{k=1}^\infty \tilde{c}_k\mathcal{Q}_{k+\nu}(\mathbf{x},\mathbf{y}),$$

where $c_k,\tilde{c}_k$ are constant coefficients such that for any M and compact D, $\sum_{k=0}^\infty |c_k|C_{M,D}^{k+1}$ and $\sum_{k=1}^\infty|\tilde{c}_k|C_{M,D}^{k+\nu}$ both converge. Taking derivatives, we obtain

$$\begin{aligned}&|\rho_0^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})|\\ \le\ &\sum_{k=0}^\infty |c_k||\mathcal{Q}_k^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})|+\sum_{k=1}^\infty|\tilde{c}_k||\mathcal{Q}_{k+\nu}^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})|\\ \le\ &\begin{cases}C_{M,D} \quad \forall(\mathbf{x},\mathbf{y})\in D & \text{if } M/2\le\nu+1,\\ C_{M,D}\|\mathbf{x}-\mathbf{y}\|^{2\nu+2-M} \quad \forall(\mathbf{x},\mathbf{y})\in D_0 & \text{if } M/2>\nu+1.\end{cases}\end{aligned}$$

When $\nu\in\mathbb{Z}_+$, we have

$$\rho_0(\mathbf{x},\mathbf{y})=\sum_{k=0}^\infty c_k\mathcal{Q}_k(\mathbf{x},\mathbf{y})+\sum_{k=1}^\infty \tilde{c}_k\mathcal{Q}_{k+\nu}(\mathbf{x},\mathbf{y})[\log\{\tilde{Q}(\mathbf{x},\mathbf{y})\}-\log(|\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}|)].$$

Let $\mathscr{Q}_{k+\nu}(\mathbf{x},\mathbf{y})=\mathcal{Q}_{k+\nu}(\mathbf{x},\mathbf{y})\log\{\tilde{Q}(\mathbf{x},\mathbf{y})\}$. By applying similar arguments leading to (34) and (35), it follows that

$$\begin{aligned}&|\mathscr{Q}_{k+\nu}^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})|\\ \le\ &\begin{cases}C_{M,D}^{k+\nu} \quad \forall(\mathbf{x},\mathbf{y})\in D & \text{if } M/2<k+\nu\\ C_{M,D}^{k+\nu}\{\log(\|\mathbf{x}-\mathbf{y}\|)+1\} \quad \forall(\mathbf{x},\mathbf{y})\in D_0 & \text{if } M/2=k+\nu,\\ C_{M,D}^{k+\nu}\|\mathbf{x}-\mathbf{y}\|^{2(k+\nu)-M} \quad \forall(\mathbf{x},\mathbf{y})\in D_0 & \text{if } M/2>k+\nu,\end{cases}\end{aligned} \tag{36}$$

Since $k+\nu$ is an integer and $\Sigma_{\mathbf{x}}$ is positive definite and M times continuously differentiable, the partial derivatives of the term $\mathcal{Q}_k$ and $\mathcal{Q}_{k+\nu}(\mathbf{x},\mathbf{y})\log(|\Sigma_{\mathbf{x}}+\Sigma_{\mathbf{y}}|)$

up to Mth-order are bounded by $C_{M,D}^{k+1}$ and $C_{M,D}^{k+\nu}$ respectively for all $(\mathbf{x},\mathbf{y}) \in D$. Moreover for any M and compact D, $\sum_{k=0}^{\infty} |c_k| C_{M,D}^{k+1}$ and $\sum_{k=1}^{\infty} |\tilde{c}_k| C_{M,D}^{k+\nu}$ both converge when $\nu \in \mathbb{Z}_+$. Thus we have

$$
\begin{aligned}
&|\rho_0^{(u_1,\dots,u_d,v_1,\dots,v_d)}(\mathbf{x},\mathbf{y})| \\
\leq &\begin{cases} C_{M,D} \quad \forall(\mathbf{x},\mathbf{y}) \in D & \text{if } M/2 < \nu+1, \\ C_{M,D}\{\log(\|\mathbf{x}-\mathbf{y}\|)+1\} \quad \forall(\mathbf{x},\mathbf{y}) \in D_0 & \text{if } M/2 = \nu+1, \\ C_{M,D}\|\mathbf{x}-\mathbf{y}\|^{2\nu+2-M} \quad \forall(\mathbf{x},\mathbf{y}) \in D_0 & \text{if } M/2 > \nu+1. \end{cases}
\end{aligned}
$$

This shows that the $\rho_0(\mathbf{x},\mathbf{y})$ part of $K_P(\mathbf{x},\mathbf{y})$ in (1) satisfies the bound in Condition 2. The N times continuously differentiability conditions on $\beta_\nu(.,.)$ and $A(.,.)$ follows from (29), (30), (31), (32) and (33). This proves Proposition 1. □

Proof of Proposition 2

Differentiating $\tilde{\rho}_0(x,y)$ using the chain rule, we observe that $\tilde{\rho}_0^{(u,v)}(x,y)$ can be expressed as a polynomial in the indeterminates $\rho_0^{(u,v)}(\gamma(x),\ \gamma(y))$ and $\gamma^{(k)}(x)$, $\gamma^{(k)}(y)$ for $u+v = 1,\dots,M$, $k = 1,\dots,M$. From the assumption that $\gamma(.)$ is an N times differentiable simple curve, it follows that $\gamma^{(k)}(x)$, $k = 1,\dots,M$, are all bounded uniformly for $x \in [0,L]$. Therefore, the conclusion that $\tilde{\rho}_0(.,.) \in C^{\lceil 2\nu\rceil+1}(\mathbb{R}^2)$ follows directly from Condition 2 and the chain rule.

To show the required bounds on $\tilde{K}^{(u,v)}(x,y)$, it suffices to show the same bounds on $\tilde{\rho}_\nu^{(u,v)}(x,y)$ when $x-y$ is close to 0. When $\nu \notin \mathbb{Z}_+$, the desired bounds follow from (42) and (8) that $|x-y|$ is of the same order as $\|\gamma(x)-\gamma(y)\|$ when $x-y \to 0$. When $\nu \in \mathbb{Z}_+$, the first bound of (12) and the first bound of (13) follow from (44), and the second bound of (13) follows from similar arguments leading to (44) and the observation that when $M/2 > \nu$, the leading term no longer has log terms and is of the same order as $|x-y|^{2\nu-M}$ (and thus is of the same order as $\|\gamma(x)-\gamma(y)\|^{2\nu-M}$). This proves Proposition 2. □

Proof of Proposition 3

We show in the order (i) → (iii) → (iv) → (ii). Since N is large, it suffices to consider the Gaussian random field $X(\mathbf{t}) - m(\mathbf{t})$. Therefore, we assume without loss of generality in the following that $\mathbb{E}\{X(\mathbf{t})\} = 0$ for all $\mathbf{t} \in \mathbb{R}^d$.

Proof of (i). We observe from Lemma 7 that to prove (i), it suffices to show that $\tilde{K}(.,.) \in C^{\lceil 2\nu\rceil-1}(\mathbb{R}^2)$ and

$$
\begin{aligned}
\lim_{(h,l)\to(0,0)} (hl)^{-1}\Big\{&\tilde{K}^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t+h,t+l) - \tilde{K}^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t+h,t) \\
&-\tilde{K}^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t,t+l) + \tilde{K}^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t,t)\Big\} \qquad (37)
\end{aligned}
$$

does not exist for any $t \in \mathbb{R}$. We recall that $\tilde{\rho}_0(x,y) = \rho_0(\gamma(x),\gamma(y))$ and $\tilde{\rho}_\nu(x,y) = \rho_\nu(\gamma(x),\gamma(y))$. According to the differentiability assumption of ρ_0 in Condition 2, to show $\tilde{K}(.,.) \in C^{\lceil 2\nu\rceil-1}(\mathbb{R}^2)$ and (37) does not exist, it suffices to show that $\tilde{\rho}_\nu(.,.) \in C^{\lceil 2\nu\rceil-1}(\mathbb{R}^2)$ and

$$
\lim_{h\to 0} h^{-2}\Big\{\tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t+h,t+h) - \tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t+h,t)
$$

$$-\tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t,t+h)+\tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t,t)\Big\} \tag{38}$$

does not exist. In particular, we shall prove that (38) diverges to either $-\infty$ or $+\infty$. The proof for $\nu \le 1$ is straightforward and we assume $\nu > 1$ in the following. First, we introduce some notation. Let $x, y \in \mathbb{R}$ and define

$$\begin{aligned}
\Psi_x &= \{\gamma^{(1)}(x)\}' A(\gamma(x),\gamma(x))\gamma^{(1)}(x),\\
Q := Q(x,y) &= \{\gamma(x)-\gamma(y)\}' A(\gamma(x),\gamma(y))\{\gamma(x)-\gamma(y)\},\\
\tilde{\beta} := \tilde{\beta}(x,y) &= \beta_\nu(\gamma(x),\gamma(y)).\\
\tilde{\beta}_x &= \beta_\nu(\gamma(x),\gamma(x)).
\end{aligned}$$

We observe that $\tilde{\beta}, Q \in C^N(\mathbb{R}^2)$ by assumption and $\Psi_x > 0$ for all $x \in \mathbb{R}$ since $\gamma^{(1)}(x)$ is a unit length vector and $A(\gamma(x),\gamma(x))$ is a positive definite matrix. In the following, we focus on a bounded open set, say $U \subset \mathbb{R}$, and derive some asymptotic results on the partial derivatives of $\tilde{\rho}_\nu$ when x, y are close together in U. Then, by saying $x - y \to 0$, we mean $x - y$ tends to 0 with x, y restricted to U and $x \ne y$. All the $o(.), O(.)$ terms below are uniform for all x, y in U as $x - y \to 0$. We observe that as $x - y \to 0$,

$$\begin{aligned}
Q &= \Psi_x(x-y)^2 + o(|x-y|^2),\\
Q^{(1,0)} &= 2\{\gamma^{(1)}(x)\}' A(\gamma(x),\gamma(y))\{\gamma(x)-\gamma(y)\}\\
&\quad +\{\gamma(x)-\gamma(y)\}' A^{(1,0)}(\gamma(x),\gamma(y))\{\gamma(x)-\gamma(y)\}\\
&= 2\{\gamma^{(1)}(x)\}' A(\gamma(x),\gamma(x))\gamma^{(1)}(x)(x-y) + o(|x-y|)\\
&= 2\Psi_x(x-y) + o(|x-y|),\\
Q^{(0,1)} &= -2\Psi_x(x-y) + o(|x-y|),\\
Q^{(1,1)} &= -2\Psi_x + o(1),\\
Q^{(2,0)} &= 2\Psi_x + o(1),\\
Q^{(0,2)} &= 2\Psi_x + o(1),\\
Q^{(u,v)} &= O(1), \qquad \forall 3 \le u+v \le N.
\end{aligned} \tag{39}$$

Case 1. Suppose $\nu \notin \mathbb{Z}_+$. We see that $\tilde{\rho}_\nu(.,.)$ is N times continuously differentiable at every point (x,y) for $x \ne y$. Differentiating $\tilde{\rho}_\nu(x,y)$ for $x \ne y$, we obtain

$$\begin{aligned}
\tilde{\rho}_\nu^{(1,0)}(x,y) &= \nu\tilde{\beta}Q^{\nu-1}Q^{(1,0)} + \cdots,\\
\tilde{\rho}_\nu^{(0,1)}(x,y) &= \nu\tilde{\beta}Q^{\nu-1}Q^{(0,1)} + \cdots,\\
\tilde{\rho}_\nu^{(1,1)}(x,y) &= \nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(1,0)}Q^{(0,1)} + \nu\tilde{\beta}Q^{\nu-1}Q^{(1,1)} + \cdots,\\
\tilde{\rho}_\nu^{(2,0)}(x,y) &= \nu(\nu-1)\tilde{\beta}Q^{\nu-2}\{Q^{(1,0)}\}^2 + \nu\tilde{\beta}Q^{\nu-1}Q^{(2,0)} + \cdots,\\
\tilde{\rho}_\nu^{(0,2)}(x,y) &= \nu(\nu-1)\tilde{\beta}Q^{\nu-2}\{Q^{(0,1)}\}^2 + \nu\tilde{\beta}Q^{\nu-1}Q^{(0,2)} + \cdots,\\
\tilde{\rho}_\nu^{(2,1)}(x,y) &= \nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(2,0)}Q^{(0,1)} + 2\nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(1,0)}Q^{(1,1)}\\
&\quad +\nu(\nu-1)(\nu-2)\tilde{\beta}Q^{\nu-3}\{Q^{(1,0)}\}^2Q^{(0,1)} + \cdots\\
\tilde{\rho}_\nu^{(1,2)}(x,y) &= \nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(1,0)}Q^{(0,2)} + 2\nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(0,1)}Q^{(1,1)}
\end{aligned}$$

$$
\begin{aligned}
& +\nu(\nu-1)(\nu-2)\tilde{\beta}Q^{\nu-3}Q^{(1,0)}\{Q^{(0,1)}\}^2+\cdots \\
\tilde{\rho}_\nu^{(2,2)}(x,y) \quad = \quad & \nu(\nu-1)(\nu-2)(\nu-3)\tilde{\beta}Q^{\nu-4}\{Q^{(1,0)}\}^2\{Q^{(0,1)}\}^2 \\
& +4\nu(\nu-1)(\nu-2)\tilde{\beta}Q^{\nu-3}Q^{(1,0)}Q^{(1,1)}Q^{(0,1)} \\
& +\nu(\nu-1)(\nu-2)\tilde{\beta}Q^{\nu-3}\{Q^{(1,0)}\}^2Q^{(0,2)} \\
& +\nu(\nu-1)(\nu-2)\tilde{\beta}Q^{\nu-3}Q^{(2,0)}\{Q^{(0,1)}\}^2 \\
& +\nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(2,0)}Q^{(0,2)} \\
& +2\nu(\nu-1)\tilde{\beta}Q^{\nu-2}\{Q^{(1,1)}\}^2+\ldots \qquad (40)
\end{aligned}
$$

where $\ldots$ represents the negligible terms as $x-y\to 0$.

We observe from (39) that as $x-y\to 0$, $|\tilde{\rho}_\nu(x,y)|\asymp|x-y|^{2\nu}$ since $|\tilde{\beta}|\asymp 1$, $Q\asymp|x-y|^2$, each first-order partial derivative of Q has the same order as $|x-y|$ and each second and higher-order partial derivative has the same order as 1. From this observation and (40), we conclude that when a differentiation with respect to either x or y occurs on one of the terms $Q, Q^{(1,0)}, Q^{(0,1)}$ the order of the resulting term decreases by $|x-y|$ but if the differentiation occurs on $\tilde{\beta}$ or $Q^{(u,v)}$ with $u+v\geq 2$, then the resulting term has the same order as the original term. For example, when we are differentiating $\nu\tilde{\beta}Q^{\nu-1}Q^{(1,1)}$ with respect to x, we have

$$
\begin{aligned}
& \frac{\partial}{\partial x}\{\nu\tilde{\beta}Q^{\nu-1}Q^{(1,1)}\} \\
= \quad & \nu\tilde{\beta}^{(1,0)}Q^{\nu-1}Q^{(1,1)}+\nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(1,0)}Q^{(1,1)}+\nu\tilde{\beta}Q^{\nu-1}Q^{(2,1)}. \qquad (41)
\end{aligned}
$$

The three terms in the right hand side of (41) are of order $|x-y|^{2\nu-2}$, $|x-y|^{2\nu-3}$ and $|x-y|^{2\nu-2}$ respectively. This indicates that as $x-y\to 0$,

$$
\nu\tilde{\beta}^{(1,0)}Q^{\nu-1}Q^{(1,1)} \qquad \text{and} \qquad \nu\tilde{\beta}Q^{\nu-1}Q^{(2,1)}
$$

are negligible compared to

$$
\nu(\nu-1)\tilde{\beta}Q^{\nu-2}Q^{(1,0)}Q^{(1,1)}.
$$

This shows that for any $u,v\in\mathbb{Z}_+$ such that $u+v\leq N$ as $x-y\to 0$, the leading terms of $\tilde{\rho}_\nu^{(u,v)}(x,y)$ are those for which the differentiation only occurred on $Q, Q^{(1,0)}, Q^{(0,1)}$. This implies that for any $u,v\in\mathbb{Z}_+$ such that $u+v\leq N$,

$$
\tilde{\rho}_\nu^{(u,v)}(x,y)=\tilde{\beta}\frac{\partial^{u+v}}{\partial x^u\partial y^v}Q^\nu+\ldots,
$$

as $x-y\to 0$, where $\ldots$ represents the negligible terms. If we continue differentiating $\tilde{\rho}_\nu(x,y)$, $x\neq y$, we observe from (39) that

$$
\begin{aligned}
& \tilde{\rho}_\nu^{(u,v)}(x,y) \\
= \quad & c_0\tilde{\beta}Q^{\nu-u-v}\{2^{u+v}\Psi_x^{u+v}(x-y)^{u+v}+o(|x-y|^{u+v})\} \\
& +c_1\tilde{\beta}Q^{\nu-u-v+1}\{2^{u+v-1}\Psi_x^{u+v-1}(x-y)^{u+v-2}+o(|x-y|^{u+v-1})\} \\
& +c_2\tilde{\beta}Q^{\nu-u-v+2}\{2^{u+v-2}\Psi_x^{u+v-2}(x-y)^{u+v-4}+o(|x-y|^{u+v-2})\}
\end{aligned}
$$

$$
\begin{aligned}
&+\ldots+c_{\lfloor (u+v)/2\rfloor}\tilde\beta Q^{\nu-\lceil (u+v)/2\rceil}\Big\{2^{\lceil (u+v)/2\rceil}\Psi_x^{\lceil (u+v)/2\rceil}\\
&\times(x-y)^{u+v-2\lfloor (u+v)/2\rfloor}+o(|x-y|^{u+v-2\lfloor (u+v)/2\rfloor})\Big\}\\
&+o(|x-y|^{2\nu-u-v})\\
=\;&\tilde\beta\sum_{k=0}^{\lfloor (u+v)/2\rfloor}[c_k 2^{u+v-k}Q^{\nu-u-v+k}\Psi_x^{u+v-k}(x-y)^{u+v-2k}]\\
+\;&o(|x-y|^{2\nu-u-v}) \qquad (42)\\
=\;&\frac{\tilde\beta|x-y|^{2\nu}\Psi_x^{\nu}}{(x-y)^{u+v}}\Big\{\sum_{k=0}^{\lfloor (u+v)/2\rfloor}c_k 2^{u+v-k}+o(1)\Big\},\ \forall u,v\in\mathbb{Z}_+, u+v\le N,
\end{aligned}
$$

as $x-y\to 0$, where $c_0,\ldots,c_{\lfloor (u+v)/2\rfloor}$ are constant coefficients depending on ν only. In particular, the term

$$\sum_{k=0}^{\lfloor (u+v)/2\rfloor}c_k 2^{u+v-k}$$

can be evaluated explicitly by specifying the matrix A and the curve $\gamma(t)$ in Q to special values. Considering the case where the dimension $d=1$, let A be the identity matrix and $\gamma(t)$ be the identity map. Then differentiating Q^ν at (x,y) with $x>y$, we obtain

$$\frac{\partial^{u+v}}{\partial x^u\partial y^v}Q^\nu=\frac{\partial^{u+v}}{\partial x^u\partial y^v}(x-y)^{2\nu}=(-1)^v\Big\{\prod_{k=0}^{u+v-1}(2\nu-k)\Big\}(x-y)^{2\nu-u-v},$$

which implies that

$$\sum_{k=0}^{\lfloor (u+v)/2\rfloor}c_k 2^{u+v-k}=(-1)^v\Big\{\prod_{k=0}^{u+v-1}(2\nu-k)\Big\}. \qquad (43)$$

Next we shall prove that $\tilde\rho_\nu(.,.)\in C^{\lceil 2\nu\rceil-1}(\mathbb{R}^2)$. We observe from (42) that $\tilde\rho_\nu^{(u,v)}(x,y)\to 0$ as $x-y\to 0$ given $u+v\le\lceil 2\nu\rceil-1$. Therefore, to show $\tilde\rho_\nu(.,.)\in C^{\lceil 2\nu\rceil-1}(\mathbb{R}^2)$, we only need to show $\tilde\rho_\nu^{(u,v)}(x,x)$ exists and equals to 0. We use induction to show the result. Observe that

$$
\begin{aligned}
&\lim_{h\to 0}\frac{\tilde\rho_\nu(x,x+h)-\tilde\rho_\nu(x,x)}{h}=\lim_{h\to 0}\frac{\tilde\rho_\nu(x+h,x)-\tilde\rho_\nu(x,x)}{h}\\
&\quad=\lim_{h\to 0}h^{-1}\tilde\beta\{Q(x+h,x)\}^\nu=\lim_{h\to 0}h^{-1}\{\tilde\beta_x+o(1)\}\{\Psi_x h^2+o(h^2)\}^\nu=0.
\end{aligned}
$$

Now suppose $\tilde\rho_\nu^{(u,v)}(x,x)$ exists and equals to 0. We verify that the same holds for $\tilde\rho_\nu^{(u+1,v)}(x,x)$ and $\tilde\rho_\nu^{(u,v+1)}(x,x)$ provided $u+v\le\lceil 2\nu\rceil-2$. We observe from (42) and (43) that

$$\lim_{h\to 0}\frac{\tilde\rho_\nu^{(u,v)}(x+h,x)-\tilde\rho_\nu^{(u,v)}(x,x)}{h}$$

$$= \lim_{h\to 0} h^{-1}\frac{\{\tilde{\beta}_x + o(1)\}|h|^{2\nu}\Psi_x^\nu}{h^{u+v}}\left\{(-1)^v\prod_{k=0}^{u+v-1}(2\nu-k)+o(1)\right\} = 0.$$

For $\tilde{\rho}_\nu^{(u,v+1)}(x,x)$, the result follows similarly. This completes the proof that $\tilde{\rho}_\nu^{(u,v)}(x,x)$ exists and equals to 0 when $u+v \le \lceil 2\nu\rceil - 1$. Since x can be any real number, $\tilde{\rho}_\nu(.,.) \in C^{\lceil 2\nu\rceil-1}(\mathbb{R}^2)$ follows.

Now we show (38) diverges to ∞, where ∞ can be either $+\infty$ or $-\infty$ depending on ν. By using the result $\tilde{\rho}_\nu^{(u,v)}(x,x) = 0$ for all $x \in \mathbb{R}$ and $u+v \le \lceil 2\nu\rceil - 1$, we obtain

$$\begin{aligned}
& h^{-2}\Big\{\tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t+h,t+h) - \tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t+h,t) \\
& -\tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t,t+h) + \tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t,t)\Big\} \\
= & -2h^{-2}\tilde{\rho}_\nu^{(\lceil\nu\rceil-1,\lceil\nu\rceil-1)}(t+h,t) \\
= & -2\frac{\tilde{\beta}_t|h|^{2\nu}\Psi_t^\nu}{h^{2\lceil\nu\rceil}}\Big\{(-1)^{\lceil\nu\rceil-1}\prod_{k=0}^{2\lceil\nu\rceil-3}(2\nu-k)+o(1)\Big\} \to \infty
\end{aligned}$$

as $h \to 0$ since $\nu \notin \mathbb{Z}_+$

Case 2. Suppose $\nu \in \mathbb{Z}_+$. Then for any $u, v \in \mathbb{Z}_+$ such that $u+v \le 2\nu$, we observe that

$$\tilde{\rho}_\nu^{(u,v)}(x,y) = \tilde{\beta}\Big(\frac{\partial^{u+v}}{\partial x^u \partial y^v}Q^\nu\Big)\log Q + \ldots,$$

as $x - y \to 0$, where $\ldots$ represents negligible terms. Using arguments similar to the $\nu \notin \mathbb{Z}_+$ case, we observe that

$$\begin{aligned}
& \tilde{\rho}_\nu^{(u,v)}(x,y) \\
= & \frac{\tilde{\beta}|x-y|^{2\nu}\log(|x-y|)\Psi_x^\nu}{(x-y)^{u+v}}\left\{(-1)^v\prod_{k=0}^{u+v-1}(2\nu-k)+o(1)\right\}, \qquad (44)
\end{aligned}$$

as $x - y \to 0$. The final result follows from the same remaining steps as for the $\nu \notin \mathbb{Z}_+$ case. This completes the proof of (i).

Proof of (iii). The result follows from choosing the curve $\gamma(t)$ in (i) to be each axis of $\mathbb{R}^d$.

Proof of (iv). We observe from Corollary 1.15(b) of [4] that $X(\mathbf{t})$ has an equivalent version with a continuous path on T provided

$$\mathrm{Var}\{X(\mathbf{t}+\mathbf{h}) - X(\mathbf{t})\} \le \frac{C}{\big|\log\|\mathbf{h}\|\big|^r} \qquad (45)$$

for all $\mathbf{t}$ and $\mathbf{h} = (h_1,\ldots,h_d)'$ with $\|\mathbf{h}\|$ sufficiently small. Here $r > 3$ and $C > 0$ are constants. One can see that (45) is equivalent to

$$K(\mathbf{t}+\mathbf{h},\mathbf{t}+\mathbf{h}) - 2K(\mathbf{t}+\mathbf{h},\mathbf{t}) + K(\mathbf{t},\mathbf{t}) \le \frac{C}{\big|\log\|\mathbf{h}\|\big|^r}. \qquad (46)$$

Let $\mathbf{a} = (a_1, \ldots, a_{2d})'$ where the a_i's are nonnegative integers and $|\mathbf{a}| = \sum_{i=1}^{2d} a_i$. We define

$$K^{(\mathbf{a})}(\mathbf{x}, \mathbf{y}) = \frac{\partial^{|\mathbf{a}|}}{\prod_{i=1}^{2d} \partial^{a_i} z_i} K(\mathbf{x}, \mathbf{y}),$$

where $x_j = z_j$ and $y_j = z_{j+d}$, $j = 1, \ldots, d$. Denote $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)'$ as the ith standard basis of $\mathbb{R}^{2d}$ for $i = 1, \ldots, 2d$.

Case 1. Suppose $\nu > 1$. Let $\dot{X}_i(\mathbf{t})$, denote the first-order mean square partial derivative of $X(\mathbf{t})$ in the direction of $\mathbf{e}_i$. First, we check that for any bounded open set $T \subset \mathbb{R}^d$, $\dot{X}_i(\mathbf{t})$ when restricted on T has an equivalent version $\dot{\mathsf{X}}_i(\mathbf{t})$ which possesses, with probability 1, a continuous sample path. From Lemma 7, we observe that the covariance function of $\dot{X}_i(\mathbf{t})$ is $K^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{x}, \mathbf{y})$. It follows from the condition $\rho_0(\mathbf{x}, \mathbf{y}) \in C^{\lceil 2\nu \rceil+1}(\mathbb{R}^{2d})$ that

$$\begin{aligned}
& \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}+\mathbf{h}, \mathbf{t}+\mathbf{h}) - 2K_0^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}+\mathbf{h}, \mathbf{t}) + \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}, \mathbf{t}) \\
= \; & \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}+\mathbf{h}, \mathbf{t}+\mathbf{h}) - \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}, \mathbf{t}+\mathbf{h}) \\
& - \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}+\mathbf{h}, \mathbf{t}) + \rho_0(\mathbf{t}, \mathbf{t}) \\
= \; & \int_0^1 \sum_{u=1}^d \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d}+\mathbf{e}_u)}(\mathbf{t}+s_1\mathbf{h}, \mathbf{t}+\mathbf{h}) h_u ds_1 \\
& - \int_0^1 \sum_{u=1}^d \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d}+\mathbf{e}_u)}(\mathbf{t}+s_1\mathbf{h}, \mathbf{t}) h_u ds_1 \\
= \; & \int_0^1 \sum_{u=1}^d \Big\{ \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d}+\mathbf{e}_u)}(\mathbf{t}+s_1\mathbf{h}, \mathbf{t}+\mathbf{h}) \\
& - \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d}+\mathbf{e}_u)}(\mathbf{t}+s_1\mathbf{h}, \mathbf{t}) \Big\} h_u ds_1 \\
= \; & \int_0^1 \int_0^1 \sum_{u=1}^d \sum_{v=1}^d \rho_0^{(\mathbf{e}_i+\mathbf{e}_{i+d}+\mathbf{e}_u+\mathbf{e}_{v+d})}(\mathbf{t}+s_1\mathbf{h}, \mathbf{t}+s_2\mathbf{h}) h_u h_v ds_1 ds_2 \\
= \; & O(\|\mathbf{h}\|^2),
\end{aligned} \tag{47}$$

as $\|\mathbf{h}\| \to 0$ uniformly for all $\mathbf{t} \in T$.

For $\rho_\nu^{(\mathbf{e}_i+\mathbf{e}_{i+d})}$, we see that the leading term of $\rho_\nu^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{x}, \mathbf{y})$ has the same order as

$$\begin{cases} \|\mathbf{x}-\mathbf{y}\|^{2\nu-2} & \text{if } \nu \notin \mathbb{Z}_+, \\ \|\mathbf{x}-\mathbf{y}\|^{2\nu-2}\{\log(\|\mathbf{x}-\mathbf{y}\|)+1\} & \text{if } \nu \in \mathbb{Z}_+. \end{cases}$$

Consequently,

$$\begin{aligned}
& \rho_\nu^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}+\mathbf{h}, \mathbf{t}+\mathbf{h}) - 2\rho_\nu^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}+\mathbf{h}, \mathbf{t}) + \rho_\nu^{(\mathbf{e}_i+\mathbf{e}_{i+d})}(\mathbf{t}, \mathbf{t}) \\
= \; & \begin{cases} O(1)\|\mathbf{h}\|^{2\nu-2} & \text{if } \nu \notin \mathbb{Z}_+, \\ O(1)\|\mathbf{h}\|^{2\nu-2}\{\log(\|\mathbf{h}\|)+1\} & \text{if } \nu \in \mathbb{Z}_+, \end{cases}
\end{aligned} \tag{48}$$

as $\|\mathbf{h}\| \to 0$ uniformly for all $\mathbf{t} \in T$.

(47) and (48) show that $K^{(\mathbf{e}_i,\mathbf{e}_{i+d})}$ satisfies (46), which means that on T, $\dot{X}_i(\mathbf{t})$ has an equivalent version $\dot{\mathsf{X}}_i(\mathbf{t})$ which possesses, with probability 1, a continuous sample path. Denote $\mathbf{t} = (t_1, \dots, t_d)'$. In the following, $\mathbf{e}_i$ is understood to be the ith standard basis of $\mathbb{R}^{2d}$ or $\mathbb{R}^d$ provided no ambiguity occurs. We have

$$\lim_{h\to 0} \mathbb{E}\Big[\Big\{\dot{\mathsf{X}}_1(\mathbf{t}) - \frac{X(\mathbf{t}+h\mathbf{e}_1) - X(\mathbf{t})}{h}\Big\}^2\Big] = 0.$$

Let $\mathbf{r} \in T$. We have from Cauchy-Schwarz inequality that as $h \to 0$,

$$\begin{aligned}&\mathbb{E}\Big|X(\mathbf{r})\dot{\mathsf{X}}_1(\mathbf{t}) - X(\mathbf{r})\frac{X(\mathbf{t}+h\mathbf{e}_1) - X(\mathbf{t})}{h}\Big| \\ &\qquad \le \sqrt{\mathbb{E}[\{X(\mathbf{r})\}^2]}\sqrt{\mathbb{E}\Big[\Big\{\dot{\mathsf{X}}_1(\mathbf{t}) - \frac{X(\mathbf{t}+h\mathbf{e}_1) - X(\mathbf{t})}{h}\Big\}^2\Big]} \to 0.\end{aligned}$$

which implies

$$\begin{aligned}\mathbb{E}\{X(\mathbf{r})\dot{\mathsf{X}}_1(\mathbf{t})\} &= \lim_{h\to 0}\mathbb{E}\Big\{X(\mathbf{r})\frac{X(\mathbf{t}+h\mathbf{e}_1) - X(\mathbf{t})}{h}\Big\} \\ &= \lim_{h\to 0}\frac{K(\mathbf{t}+h\mathbf{e}_1,\mathbf{r}) - K(\mathbf{t},\mathbf{r})}{h} \\ &= K^{(\mathbf{e}_1)}(\mathbf{t},\mathbf{r}). \qquad (49)\end{aligned}$$

Using the arguments of (47) and (48), one can easily verify that on T, $X(\mathbf{t})$ has an equivalent version, say $\mathsf{X}(\mathbf{t})$, possessing, with probability 1, a continuous sample path.

Let r_1 be such that $(s, t_2, \dots, t_d) \in T$ for all $s \in [r_1, t_1]$. Define

$$Y_1(\mathbf{t}) = \mathsf{X}(r_1, t_2 \dots, t_d) + \int_{r_1}^{t_1} \dot{\mathsf{X}}_1(s, t_2, \dots, t_d)ds.$$

It is clear that on T, with probability 1, the sample path of $Y_1(\mathbf{t})$ has continuous partial derivative in the direction of $\mathbf{e}_1$. Meanwhile, using (49) and Fubini's theorem, we have

$$\begin{aligned}\mathbb{E}\{X(\mathbf{t})Y_1(\mathbf{t})\} &= K(\mathbf{t} - t_1\mathbf{e}_1 + r_1\mathbf{e}_1, \mathbf{t}) + \int_{r_1}^{t_1} K^{(\mathbf{e}_1)}(\mathbf{t} - t_1\mathbf{e}_1 + s\mathbf{e}_1, \mathbf{t})ds \\ &= K(\mathbf{t},\mathbf{t}). \qquad (50)\end{aligned}$$

In the same way,

$$\mathbb{E}[\{Y_1(\mathbf{t})\}^2] = K(\mathbf{t},\mathbf{t}). \qquad (51)$$

It then follows from (50) and (51) that

$$\mathbb{E}[\{X(\mathbf{t}) - Y_1(\mathbf{t})\}^2] = \mathbb{E}[\{X(\mathbf{t})\}^2] + \mathbb{E}[\{Y_1(\mathbf{t})\}^2] - 2\mathbb{E}\{X(\mathbf{t})Y_1(\mathbf{t})\} = 0. \qquad (52)$$

(52) implies for any fixed $\mathbf{t} \in T$,

$$\mathbb{P}\{X(\mathbf{t}) = Y_1(\mathbf{t})\} = 1.$$

If we define analogously $Y_i(\mathbf{t})$ for $i = 2, \dots, d$, using similar arguments we can conclude that for any fixed $\mathbf{t} \in T$

$$\mathbb{P}\{X(\mathbf{t}) = Y_1(\mathbf{t}) = \cdots = Y_d(\mathbf{t})\} = 1.$$

Since $Y_i(\mathbf{t})$, $i = 1, \dots, d$ are continuous with probability 1 in T, it follows that

$$\mathbb{P}\Big[\bigcap_{\mathbf{t}\in T}\{Y_1(\mathbf{t}) = \cdots = Y_d(\mathbf{t})\}\Big] = 1. \tag{53}$$

(53) shows that on T each of $Y_1(\mathbf{t}), \dots, Y_d(\mathbf{t})$ is an equivalent version of $X(\mathbf{t})$ possessing, with probability 1, continuous first-order partial derivatives in all directions $\mathbf{e}_1, \dots, \mathbf{e}_d$.

Applying the arguments above inductively, we conclude that restricted to T, $X(\mathbf{t})$ has an equivalent version possessing, with probability 1, a $C^{\lceil\nu\rceil-1}(T)$ sample path.

Case 2. Suppose that $\nu \le 1$. It is easy to verify that $X(\mathbf{t})$ is mean square continuous and when restricted on T has an equivalent version possessing, with probability 1, a continuous sample path.

Proof of (ii). Since $X(\gamma(t))$ is the Gaussian random field $X(\mathbf{t})$ restricted to the C^N-curve $\gamma(t)$, (ii) follows directly from (iv). This proves Proposition 3. □

Appendix C: Proof of Theorem 1

(a) By using (24), Lemma 1, Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\tilde{\rho}_0(t_{i+\omega_n\theta k_1}, t_{i+\omega_n\theta k_2})\\
= &\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\Big\{\sum_{u_1+u_2\le\lceil 2\nu\rceil}\frac{\tilde{\rho}_0^{(u_1,u_2)}(t_i,t_i)}{u_1!u_2!}\\
&\times(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{i+\omega_n\theta k_2}-t_i)^{u_2} + \sum_{u_1+u_2=\lceil 2\nu\rceil+1}\frac{(\lceil 2\nu\rceil+1)}{u_1!u_2!}\\
&\times(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{i+\omega_n\theta k_2}-t_i)^{u_2}\\
&\times\int_0^1(1-s)^{\lceil 2\nu\rceil}\tilde{\rho}_0^{(u_1,u_2)}\{(t_i,t_i)+s(t_{i+\omega_n\theta k_1}-t_i, t_{i+\omega_n\theta k_2}-t_i)\}ds\Big\}\\
= &\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\Big\{\sum_{u_1+u_2\le\lceil 2\nu\rceil}\frac{\tilde{\rho}_0^{(u_1,u_2)}(t_i,t_i)}{u_1!u_2!}\\
&\times\{d_{i,i+\omega_n\theta k_1}+\sum_{j=3}^{\ell\wedge N} f_j(t_i)d^j_{i,i+\omega_n\theta k_1}+o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})\}^{u_1}
\end{aligned}$$

$$
\begin{aligned}
&\times\{d_{i,i+\omega_n\theta k_2}+\sum_{j=3}^{\ell\wedge N} f_j(t_i)d^j_{i,i+\omega_n\theta k_2}+o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})\}^{u_2}\\
&+\sum_{u_1+u_2=\lceil 2\nu\rceil+1}\frac{(\lceil 2\nu\rceil+1)}{u_1!u_2!}(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{i+\omega_n\theta k_2}-t_i)^{u_2}\\
&\times\int_0^1(1-s)^{\lceil 2\nu\rceil}\tilde\rho_0^{(u_1,u_2)}\{(t_i,t_i)+s(t_{i+\omega_n\theta k_1}-t_i,t_{i+\omega_n\theta k_2}-t_i)\}ds\Big\}\\
=\;& O\big(n(\frac{n}{\omega_n})^{2\ell-\lceil 2\nu\rceil-1}\big)+O(n(\frac{n}{\omega_n})^{2\ell-2(\ell\wedge N)}), \qquad (54)
\end{aligned}
$$

as $n\to\infty$ since in the second last equality, the first $o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})$ is independent of k_2 and the second $o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})$ is independent of k_1. Next, it follows from Lemma 3 and (20) that when $\nu\notin\mathbb{Z}_+$

$$
\begin{aligned}
&\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}\beta_\nu(\gamma(t_{i+\omega_n\theta k_1}),\gamma(t_{i+\omega_n\theta k_2}))\\
&\times G_\nu(\sqrt{Q_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}})\\
=\;&2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{0\le k_1<k_2\le\ell}a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}\{\beta_\nu(\gamma(t_i),\gamma(t_i))+O(\omega_n n^{-1})\}\\
&\times Q^\nu_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\\
=\;&2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{0\le k_1<k_2\le\ell}a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}\beta_\nu(\gamma(t_i),\gamma(t_i))\\
&\{d^2_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\Psi(t_i)+O(\omega_n^3n^{-3})\}^\nu+O(n(\frac{n}{\omega_n})^{2\ell-2\nu-1})\\
=\;&2\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\\
&\times\sum_{0\le k_1<k_2\le\ell}\frac{\ell!}{\prod_{0\le j\le\ell,j\ne k_1}\left\{\frac{\omega_n(k_1-j)\theta L}{n}+O(n^{-1})+O(\omega_n^3n^{-3})\right\}}\\
&\times\frac{\ell!}{\prod_{0\le j\le\ell,j\ne k_2}\left\{\frac{\omega_n(k_2-j)\theta L}{n}+O(n^{-1})+O(\omega_n^3n^{-3})\right\}}\\
&\times\Big[\Big\{\frac{\omega_n(k_2-k_1)\theta L}{n}+O(n^{-1})+O(\omega_n^3n^{-3})\Big\}^2\Psi(t_i)+O(\omega_n^3n^{-3})\Big]^\nu\\
&+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\big)\\
=\;&2\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\Psi^\nu(t_i)(\frac{\omega_n\theta L}{n})^{2\nu-2\ell}\\
&\times\sum_{0\le k_1<k_2\le\ell}\frac{\ell!}{\prod_{0\le j\le\ell,j\ne k_1}\{(k_1-j)+O(\omega_n^{-1})+O(\omega_n^2n^{-2})\}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\ell!}{\prod_{0\le j\le \ell, j\neq k_2}\{(k_2-j)+O(\omega_n^{-1})+O(\omega_n^2 n^{-2})\}} \\
&\times\Big[(k_2-k_1)^2+O(\omega_n^{-1})+O(\omega_n n^{-1})\Big]^{\nu}+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\big) \\
= \;& \frac{2n^{2\ell-2\nu}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\Psi^{\nu}(t_i)+O\big((\frac{n}{\omega_n})^{2\ell-2\nu+1}\big) \\
&+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\big) \\
= \;& \frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^{\nu}(s)ds \\
&+O\big((\frac{n}{\omega_n})^{2\ell-2\nu+1}\big)+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\big), \qquad \text{as } n\to\infty. \qquad (55)
\end{aligned}
$$

When $\nu \in \mathbb{Z}_+$, it follows from Lemma 1 and Lemma 3 that

$$
\begin{aligned}
&\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}\beta_\nu(\gamma(t_{i+\omega_n\theta k_1}),\gamma(t_{i+\omega_n\theta k_2})) \\
&\times G_\nu(\sqrt{Q_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}}) \\
= \;& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{0\le k_1<k_2\le\ell} a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}\{\beta_\nu(\gamma(t_i),\gamma(t_i))+O(\omega_n n^{-1})\} \\
&\times Q^{\nu}_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\log(\sqrt{Q_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}}) \\
= \;& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{0\le k_1<k_2\le\ell} a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}\beta_\nu(\gamma(t_i),\gamma(t_i)) \\
&\times\{d^2_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\Psi(t_i)+O(\omega_n^3 n^{-3})\}^{\nu} \\
&\times\log\{d^2_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\Psi(t_i)+O(\omega_n^3 n^{-3})\}+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\log(\frac{n}{\omega_n})\big) \\
= \;& 2\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\Psi^{\nu}(t_i)\sum_{0\le k_1<k_2\le\ell} a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}d^{2\nu}_{i+\omega_n\theta k_1,i+\omega_n\theta k_2} \\
&\times\log\Big[\{\frac{\omega_n(k_2-k_1)\theta L}{n}+O(n^{-1})+O(\omega_n^3 n^{-3})\}^2\Psi(t_i)+O(\omega_n^3 n^{-3})\Big] \\
&+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\log(\frac{n}{\omega_n})\big) \\
= \;& 2\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\Psi^{\nu}(t_i)\sum_{0\le k_1<k_2\le\ell} a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2} \\
&\times\Big\{\frac{\omega_n(k_2-k_1)\theta L}{n}+O(n^{-1})+O(\omega_n^3 n^{-3})\Big\}^{2\nu}\log(k_2-k_1) \\
&+\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\Psi^{\nu}(t_i)\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i;k_1}a_{\theta,\ell;i;k_2}
\end{aligned}
$$

$$
\begin{aligned}
&\times\{d_{i+\omega_n\theta k_1}-d_{i+\omega_n\theta k_2}+O(\omega_n^3 n^{-3})\}^{2\nu}\log\Big[(\frac{\omega_n\theta L}{n})^2\Psi(t_i)\Big]\\
&+O\big((\frac{n}{\omega_n})^{2\ell-2\nu+1}\big)+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\log(\frac{n}{\omega_n})\big)\\
=\;& 2\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\\
&\times\sum_{0\le k_1<k_2\le\ell}\frac{\ell!}{\prod_{0\le j\le\ell,j\ne k_1}\Big\{\frac{\omega_n(k_1-j)\theta L}{n}+O(n^{-1})+O(\omega_n^3 n^{-3})\Big\}}\\
&\times\frac{\ell!}{\prod_{0\le j\le\ell,j\ne k_2}\Big\{\frac{\omega_n(k_2-j)\theta L}{n}+O(n^{-1})+O(\omega_n^3 n^{-3})\Big\}}\\
&\times\Big\{\frac{\omega_n(k_2-k_1)\theta L}{n}+O(n^{-1})+O(\omega_n^3 n^{-3})\Big\}^{2\nu}\Psi^\nu(t_i)\log(k_2-k_1)\\
&+O\big((\frac{n}{\omega_n})^{2\ell-2\nu+1}\big)+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\log(\frac{n}{\omega_n})\big)\\
=\;& \frac{2n^{2\ell-2\nu}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\Psi^\nu(t_i)\\
&+O\big((\frac{n}{\omega_n})^{2\ell-2\nu+1}\big)+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\log(\frac{n}{\omega_n})\big)\\
=\;& \frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\int_0^L\beta_\nu\big(\gamma(s),\gamma(s)\big)[\Psi(s)]^\nu ds\\
&+O\big((\frac{n}{\omega_n})^{2\ell-2\nu+1}\big)+O\big(n(\frac{n}{\omega_n})^{2\ell-2\nu-1}\log(\frac{n}{\omega_n})\big).
\end{aligned}\tag{56}
$$

as $n\to\infty$. It follows from (54), (55) and (56) that

$$
\begin{aligned}
&\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{i+\omega_n\theta k_2})\\
=\;& \frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\int_0^L\beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds\\
&+O\big\{n(\frac{n}{\omega_n})^{2\ell-2(\ell\wedge N)}+(\frac{n}{\omega_n})^{2\ell-2\nu+1}\big\}\\
&+\begin{cases}O\big(n(n/\omega_n)^{2\ell-2\nu-1}\big), & \text{if } \nu\notin\mathbb{Z}_+,\\ O\big(n(n/\omega_n)^{2\ell-2\nu-1}\log(n/\omega_n)\big), & \text{if } \nu\in\mathbb{Z}_+,\end{cases}
\end{aligned}\tag{57}
$$

as $n\to\infty$. Writing $\tilde{m}(t)=m(\gamma(t))$ and using (24), Lemma 1 and Lemma 3, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}m(\gamma(t_{i+\omega_n\theta k_1}))m(\gamma(t_{i+\omega_n\theta k_2}))\\
=\;& \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\tilde{m}(t_{i+\omega_n\theta k_1})\tilde{m}(t_{i+\omega_n\theta k_2})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\Big\{\sum_{j=0}^{\ell\wedge N-1}\frac{\tilde{m}^{(j)}(t_i)}{j!}(t_{i+\omega_n\theta k_1}-t_i)^j \\
&\quad + \frac{1}{(\ell\wedge N-1)!}\int_{t_i}^{t_{i+\omega_n\theta k_1}}(t_{i+\omega_n\theta k_1}-s)^{\ell\wedge N-1}\tilde{m}^{(\ell\wedge N)}(s)ds\Big\} \\
&\quad \times\Big\{\sum_{j=0}^{\ell\wedge N-1}\frac{\tilde{m}^{(j)}(t_i)}{j!}(t_{i+\omega_n\theta k_2}-t_i)^j \\
&\quad + \frac{1}{(\ell\wedge N-1)!}\int_{t_i}^{t_{i+\omega_n\theta k_2}}(t_{i+\omega_n\theta k_2}-s)^{\ell\wedge N-1}\tilde{m}^{(\ell\wedge N)}(s)ds\Big\} \\
&= \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2} \\
&\quad \times\Big\{\sum_{j=0}^{\ell\wedge N-1}\frac{\tilde{m}^{(j)}(t_i)}{j!}\{d_{i,i+\omega_n\theta k_1}+\sum_{w=3}^{\ell\wedge N}f_w(t_i)d^w_{i,i+\omega_n\theta k_1}+o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})\}^j \\
&\quad + \frac{1}{(\ell\wedge N-1)!}\int_{t_i}^{t_{i+\omega_n\theta k_1}}(t_{i+\omega_n\theta k_1}-s)^{\ell\wedge N-1}\tilde{m}^{(\ell\wedge N)}(s)ds\Big\} \\
&\quad \times\Big\{\sum_{j=0}^{\ell\wedge N-1}\frac{\tilde{m}^{(j)}(t_i)}{j!}\{d_{i,i+\omega_n\theta k_2}+\sum_{w=3}^{\ell\wedge N}f_w(t_i)d^w_{i,i+\omega_n\theta k_2}+o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})\}^j \\
&\quad + \frac{1}{(\ell\wedge N-1)!}\int_{t_i}^{t_{i+\omega_n\theta k_2}}(t_{i+\omega_n\theta k_2}-s)^{\ell\wedge N-1}\tilde{m}^{(\ell\wedge N)}(s)ds\Big\} \\
&= O\{n(\frac{n}{\omega_n})^{2\ell-2(\ell\wedge N)}\}, \qquad (58)
\end{aligned}
$$

as $n\to\infty$. Consequently, it follows from (57) and (58) that

$$
\begin{aligned}
\mathbb{E}V_{\theta,\ell} &= \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\Big\{K(\gamma(t_{i+\omega_n\theta k_1}),\gamma(t_{i+\omega_n\theta k_2})) \\
&\quad + m(\gamma(t_{i+\omega_n\theta k_1}))m(\gamma(t_{i+\omega_n\theta k_2}))\Big\} \\
&= \frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds \\
&\quad + O\big\{n+(\frac{n}{\omega_n})^{2\ell-2\nu+1}\big\} \\
&\quad + \begin{cases} O\big(n(n/\omega_n)^{2\ell-2\nu-1}\big), & \text{if } \nu\notin\mathbb{Z}_+, \\ O\big(n(n/\omega_n)^{2\ell-2\nu-1}\log(n/\omega_n)\big), & \text{if } \nu\in\mathbb{Z}_+, \end{cases} \qquad (59)
\end{aligned}
$$

as $n\to\infty$ since $N\geq 2\nu+6$ (Condition 1). To conclude that

$$
\frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds > 0, \qquad (60)
$$

we need only verify that

$$\frac{2n^{2\ell-2\nu+1}H_\ell(\nu)}{\omega_n^{2\ell-2\nu}(\theta L)^{2\ell-2\nu}}\int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds \neq 0 \tag{61}$$

because $\mathbb{E}V_{\theta,\ell} \geq 0$. We observe that $\beta_\nu(\mathbf{x},\mathbf{y}) \neq 0$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$ implies that $\beta_\nu(\mathbf{x},\mathbf{y})$ does not change sign, namely, it is either positive or negative for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$. From the definition of Ψ and that $A(\mathbf{x},\mathbf{y})$ is positive definite for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$, we conclude that $\Psi(s) > 0$ for all $s \in [0, L]$. All these results together with $H_\ell(\nu) \neq 0$ (see Section E) imply that (61) is true. Recall the notation introduced in equation (15). To prove that $V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell} \to 1$ as $n \to \infty$ almost surely, it suffices to show that

$$Z'\Sigma Z + \mu'\mu \to 1 \tag{62}$$

and

$$\mu'\Sigma^{1/2}Z + Z'\Sigma^{1/2}\mu \to 0 \tag{63}$$

as $n \to \infty$ almost surely. We observe from (15) that

$$\mathbb{E}Z'\Sigma Z = \mathbb{E}V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell} - \mathbb{E}(\mu'\Sigma^{1/2}Z) - \mathbb{E}(Z'\Sigma^{1/2}\mu) - \mu'\mu = 1 - \mu'\mu. \tag{64}$$

Applying the Hanson-Wright inequality [11], it follows from (64) that there exists some constant $C > 0$ such that

$$\begin{aligned}\mathbb{P}\Big(|Z'\Sigma Z - \mathbb{E}(Z'\Sigma Z)| \geq s\Big) &= \mathbb{P}\Big(|Z'\Sigma Z + \mu'\mu - 1| \geq s\Big) \\ &\leq 2\exp\Big\{-C\min\Big(\frac{s}{\|\Sigma_{\mathrm{abs}}\|_2}, \frac{s^2}{\|\Sigma\|_F^2}\Big)\Big\}, \quad \forall s > 0.\end{aligned} \tag{65}$$

We shall now evaluate the orders of $\|\Sigma_{\mathrm{abs}}\|_2$ and $\|\Sigma\|_F^2$. We observe from Lemma 2 that

$$\begin{aligned}&\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}K(\gamma(t_{i+\omega_n\theta k_1}),\gamma(t_{j+\omega_n\theta k_2}))\\ &=\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2})\\ &=\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\Big[\sum_{u_1+u_2\leq(2\ell)\wedge N-1}\frac{(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\\ &\quad\times\tilde{K}^{(u_1,u_2)}(t_i,t_j)\\ &\quad+\sum_{u_1+u_2=(2\ell)\wedge N}\frac{\{(2\ell)\wedge N\}(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\\ &\quad\times\int_0^1(1-s)^{(2\ell)\wedge N-1}\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)\}ds\Big].\end{aligned} \tag{66}$$

Consequently it follows from (7), (8), (24), (66), Condition 3, Lemma 1 and Proposition 2 that

$$
\begin{aligned}
&\frac{1}{\mathbb{E}(V_{\theta,\ell})}\sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1}\Big|\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\\
&\times K(\gamma(t_{i+\omega_n\theta k_1}),\gamma(t_{j+\omega_n\theta k_2}))\Big|\\
=&\frac{1}{\mathbb{E}(V_{\theta,\ell})}\sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1}\Big|\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\\
&\times\Big[\sum_{u_1+u_2\le(2\ell)\wedge N-1}\frac{(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\tilde{K}^{(u_1,u_2)}(t_i,t_j)\\
&+\sum_{u_1+u_2=(2\ell)\wedge N}\frac{\{(2\ell)\wedge N\}(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\\
&\times\int_0^1(1-s)^{(2\ell)\wedge N-1}\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i,t_{j+\omega_n\theta k_2}-t_j)\}ds\Big]\Big|\\
=&\frac{1}{\mathbb{E}(V_{\theta,\ell})}\sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1}\Big|\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\\
&\times\Big[\sum_{u_1+u_2\le(2\ell)\wedge N-1}\frac{\{d_{i,i+\omega_n\theta k_1}+\sum_{w=3}^{\ell\wedge N}f_w(t_i)d^w_{i,i+\omega_n\theta k_1}+o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})\}^{u_1}}{u_1!u_2!}\\
&\times\{d_{j,j+\omega_n\theta k_2}+\sum_{w=3}^{\ell\wedge N}f_w(t_j)d^w_{j,j+\omega_n\theta k_2}+o(\omega_n^{\ell\wedge N}n^{-\ell\wedge N})\}^{u_2}\tilde{K}^{(u_1,u_2)}(t_i,t_j)\\
&+\sum_{u_1+u_2=(2\ell)\wedge N}\frac{\{(2\ell)\wedge N\}(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\\
&\times\int_0^1(1-s)^{(2\ell)\wedge N-1}\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i,t_{j+\omega_n\theta k_2}-t_j)\}ds\Big]\Big|\\
\le&\frac{O(1)}{\mathbb{E}(V_{\theta,\ell})}\sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1}\Big\{\sum_{u_1+u_2\le(2\ell)\wedge N-1}O((\frac{n}{\omega_n})^{2\ell-2(\ell\wedge N)})|\tilde{K}^{(u_1,u_2)}(t_i,t_j)|\\
&+\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}\sum_{u_1+u_2=(2\ell)\wedge N}O\{(\frac{n}{\omega_n})^{2\ell-(2\ell)\wedge N}\}\\
&\times\int_0^1(1-s)^{(2\ell)\wedge N-1}|\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i,t_{j+\omega_n\theta k_2}-t_j)\}|ds\Big\}\\
\le&\frac{O(1)}{\mathbb{E}(V_{\theta,\ell})}\sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1}\Big\{\sum_{u_1+u_2\le(2\ell)\wedge N-1}O((\frac{n}{\omega_n})^{2\ell-2(\ell\wedge N)})
\end{aligned}
$$

$$
\begin{aligned}
&\times \|\gamma(t_i)-\gamma(t_j)\|^{2\nu-u_1-u_2} + O\{(\frac{n}{\omega_n})^{2\ell-(2\ell)\wedge N}\} \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}\int_0^1 (1-s)^{(2\ell)\wedge N-1} \\
&\times \|\gamma(t_i+s(t_{i+\omega_n\theta k_1}-t_i))-\gamma(t_j+s(t_{j+\omega_n\theta k_2}-t_j))\|^{2\nu-(2\ell)\wedge N} ds\Big\} \\
\le & \frac{O\{(n/\omega_n)^{2\ell-2(\ell\wedge N)}\}}{\mathbb{E}(V_{\theta,\ell})} \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} \\
&\times \{(t_{i+\omega_n\theta k_1}-t_{j+\omega_n\theta k_2})\wedge(t_i-t_j)\}^{2\nu-(2\ell)\wedge N} \\
\le & \frac{O\{(n/\omega_n)^{2\ell-(2\ell)\wedge N}\}}{\mathbb{E}(V_{\theta,\ell})} \\
&\times \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} \Big[\frac{(i-j+\omega_n\theta(k_1-k_2)+\delta_{i+\omega_n\theta k_1}-\delta_{j+\omega_n\theta k_2})L}{n} \\
&\wedge \frac{(i-j+\delta_i-\delta_j)L}{n}\Big]^{2\nu-(2\ell)\wedge N} \\
\le & \frac{O\{(n/\omega_n)^{2\ell-(2\ell)\wedge N}\}}{\mathbb{E}(V_{\theta,\ell})} \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \Big(\frac{(i-j-\omega_n\theta\ell-1)L}{n}\Big)^{2\nu-(2\ell)\wedge N} \\
\le & O\{(\frac{\omega_n}{n})^{(2\ell)\wedge N-2\nu}\} \int_{\omega_n/n}^{L} s^{2\nu-(2\ell)\wedge N} ds \\
= & \begin{cases} O(\omega_n/n), & \text{if } 2\nu < (2\ell)\wedge N-1, \\ O\big\{(\omega_n/n)\log(n/\omega_n)\big\}, & \text{if } 2\nu = (2\ell)\wedge N-1, \\ O\big\{(\omega_n/n)^{(2\ell)\wedge N-2\nu}\big\}, & \text{if } 2\nu > (2\ell)\wedge N-1, \end{cases} \qquad (67)
\end{aligned}
$$

as $n\to\infty$ uniformly over $\omega_n\theta\ell+\omega_n+2\le i\le n-\omega_n\theta\ell$. Similarly,

$$
\begin{aligned}
&\sum_{j=i+\omega_n\theta\ell+\omega_n+1}^{n-\omega_n\theta\ell} |\Sigma_{i,j}| \qquad (68) \\
= & \begin{cases} O(\omega_n/n), & \text{if } 2\nu < (2\ell)\wedge N-1, \\ O\big\{(\omega_n/n)\log(n/\omega_n)\big\}, & \text{if } 2\nu = (2\ell)\wedge N-1, \\ O\big\{(\omega_n/n)^{(2\ell)\wedge N-2\nu}\big\}, & \text{if } 2\nu > (2\ell)\wedge N-1, \end{cases}
\end{aligned}
$$

as $n\to\infty$ uniformly over $1\le i\le n-\omega_n\theta\ell-\omega_n-1$. Also, one can show that for any constant $c>0$,

$$
\sum_{1\le j\le n-\theta\ell:|i-j|\le c\omega_n} |\Sigma_{i,j}| = O(\frac{\omega_n}{n}), \qquad (69)
$$

as $n\to\infty$ uniformly over $1\le i\le n-\omega_n\theta\ell$. Consequently using (67), (68) and (69), it follows from [13] that

$$
\|\Sigma_{\text{abs}}\|_2 \quad \le \quad \max_{1\le i\le n-\theta\ell} \Big\{ \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} |\Sigma_{i,j}|
$$

$$
\begin{aligned}
&+ \sum_{j=(i-\omega_n\theta\ell-\omega_n)\vee 1}^{(i+\omega_n\theta\ell+\omega_n)\wedge(n-\omega_n\theta\ell)} |\Sigma_{i,j}| + \sum_{j=i+\omega_n\theta\ell+\omega_n+1}^{n-\omega_n\theta\ell} |\Sigma_{i,j}| \Big\} \\
= &\begin{cases} O(\omega_n/n), & \text{if } 2\nu < (2\ell)\wedge N-1, \\ O\big\{(\omega_n/n)\log(n/\omega_n)\big\}, & \text{if } 2\nu = (2\ell)\wedge N-1, \\ O\big\{(\omega_n/n)^{(2\ell)\wedge N-2\nu}\big\}, & \text{if } 2\nu > (2\ell)\wedge N-1, \end{cases} \\
\leq \;& O(q_{1,\omega_n}^{-1}), \qquad (70)
\end{aligned}
$$

as $n \to \infty$ since $N \geq 2\nu + 6$ (Condition 1). Next we observe that

$$
\begin{aligned}
&\sum_{i=1}^{n-\omega_n\theta\ell} \Sigma_{i,i}^2 + \sum_{j=1}^{\omega_n\theta\ell+\omega_n} \sum_{i=1}^{n-\omega_n\theta\ell-j} \Sigma_{i,i+j}^2 + \sum_{j=1}^{\omega_n\theta\ell+\omega_n} \sum_{i=1+j}^{n-\omega_n\theta\ell} \Sigma_{i,i-j}^2 \\
= \;& O(\omega_n/n), \qquad (71)
\end{aligned}
$$

as $n \to \infty$. Using (24), (66), Proposition 2 and Condition 3, it follows analogously to (67) that

$$
\begin{aligned}
&\sum_{i=\omega_n\theta\ell+\omega_n+2}^{n-\omega_n\theta\ell} \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \Sigma_{i,j}^2 \\
=&\frac{1}{\{\mathbb{E}(V_{\theta,\ell})\}^2} \sum_{i=\omega_n\theta\ell+\omega_n+2}^{n-\omega_n\theta\ell} \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \Big[O((\frac{n}{\omega_n})^{2\ell-2(\ell\wedge N)}) \\
&\times \sum_{u_1+u_2\leq(2\ell)\wedge N-1} |\tilde{K}^{(u_1,u_2)}(t_i,t_j)| + \sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \\
&\times \sum_{u_1+u_2=(2\ell)\wedge N} \frac{\{(2\ell)\wedge N\}(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!} \\
&\times \int_0^1 (1-s)^{(2\ell)\wedge N-1} \tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)\} ds \Big]^2 \\
\leq& \frac{O\{(n/\omega_n)^{4\ell-(4\ell)\wedge(2N)}\}}{\{\mathbb{E}(V_{\theta,\ell})\}^2} \sum_{i=\omega_n\theta\ell+\omega_n+2}^{n-\omega_n\theta\ell} \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \\
&\times \Big[\sum_{u_1+u_2\leq(2\ell)\wedge N-1} (t_i-t_j)^{2\nu-u_1-u_2} \\
&+ \sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} \{(t_{i+\omega_n\theta k_1}-t_{j+\omega_n\theta k_2})\wedge(t_i-t_j)\}^{2\nu-(2\ell)\wedge N} \Big]^2 \\
\leq& \frac{O\{(n/\omega_n)^{4\ell-(4\ell)\wedge(2N)}\}}{\{\mathbb{E}(V_{\theta,\ell})\}^2} \sum_{i=\omega_n\theta\ell+\omega_n+2}^{n-\omega_n\theta\ell} \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \Big(\frac{(i-j-\omega_n\theta\ell)L}{n}\Big)^{4\nu-(4\ell)\wedge(2N)} \\
\leq& O\{(\frac{\omega_n}{n})^{(4\ell)\wedge(2N)-4\nu}\} \int_{\omega_n/n}^{L} s^{4\nu-(4\ell)\wedge(2N)} ds
\end{aligned}
$$

$$= \begin{cases} O(\omega_n/n), & \text{if } 2\nu < (2\ell) \wedge N - 1/2, \\ O\{(\omega_n/n)\log(n/\omega_n)\}, & \text{if } 2\nu = (2\ell) \wedge N - 1/2, \\ O\{(\omega_n/n)^{(4\ell)\wedge(2N)-4\nu}\}, & \text{if } 2\nu > (2\ell) \wedge N - 1/2, \end{cases} \tag{72}$$

as $n \to \infty$. Hence we conclude from (71) and (72) that

$$\begin{aligned} \|\Sigma\|_F^2 &= \sum_{i=1}^{n-\omega_n\theta\ell} \Sigma_{i,i}^2 + \sum_{j=1}^{\omega_n\theta\ell+\omega_n} \sum_{i=1}^{n-\omega_n\theta\ell-j} \Sigma_{i,i+j}^2 + \sum_{j=1}^{\omega_n\theta\ell+\omega_n} \sum_{i=1+j}^{n-\omega_n\theta\ell} \Sigma_{i,i-j}^2 \\ &\quad + \sum_{i=1}^{n-2\omega_n\theta\ell-\omega_n-1} \sum_{j=i+\omega_n\theta\ell+\omega_n+1}^{n-\omega_n\theta\ell} \Sigma_{i,j}^2 + \sum_{i=\omega_n\theta\ell+\omega_n+2}^{n-\omega_n\theta\ell} \sum_{j=1}^{i-\omega_n\theta\ell-\omega_n-1} \Sigma_{i,j}^2 \\ &= \begin{cases} O(\omega_n/n), & \text{if } 2\nu < (2\ell) \wedge N - 1/2, \\ O\{(\omega_n/n)\log(n/\omega_n)\}, & \text{if } 2\nu = (2\ell) \wedge N - 1/2, \\ O\{(\omega_n/n)^{(4\ell)\wedge(2N)-4\nu}\}, & \text{if } 2\nu > (2\ell) \wedge N - 1/2, \end{cases} \\ &\le O(q_{2,\omega_n}^{-1}), \end{aligned} \tag{73}$$

as $n \to \infty$ since $N \ge 2\nu + 6$ (Condition 1). (62) follows from (65), (70), (73) and Borel-Cantelli lemma.

To prove (63), we observe from (58) and (59) that

$$\begin{aligned} \mu'\mu &= \frac{1}{\mathbb{E}V_{\theta,\ell}} \sum_{i=1}^{n-\omega_n\theta\ell} \sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;i,k_2} \tilde{m}(t_{i+\omega_n\theta k_1}) \tilde{m}(t_{i+\omega_n\theta k_2}) \\ &= O\{(\frac{n}{\omega_n})^{2\ell-2(\ell\wedge N)-2\ell+2\nu}\} \\ &= O\{(\frac{\omega_n}{n})^{2(\ell\wedge N)-2\nu}\}, \end{aligned} \tag{74}$$

as $n \to \infty$. Then it follows from (74) that

$$\begin{aligned} &\mu'\Sigma\mu \le \mu'\mu\|\Sigma_{\text{abs}}\|_2 \\ &= \begin{cases} O\{(\omega_n/n)^{1+2(\ell\wedge N)-2\nu}\}, & \text{if } 2\nu < (2\ell) \wedge N - 1, \\ O\{(\omega_n/n)^{2+2(\ell\wedge N)-(2\ell)\wedge N}\log(n/\omega_n)\}, & \text{if } 2\nu = (2\ell) \wedge N - 1, \\ O\{(\omega_n/n)^{2(\ell\wedge N)+(2\ell)\wedge N-4\nu}\}, & \text{if } 2\nu > (2\ell) \wedge N - 1, \end{cases} \\ &\le O(q_{3,\omega_n}^{-1}), \end{aligned} \tag{75}$$

as $n \to \infty$ since $N \ge 2\nu + 6$ (Condition 1). Let $\mathcal{Z}$ be the standard normal random variable. We observe that there exists a constant $C > 0$ such that for large n, we have

$$\begin{aligned} \mathbb{P}(|Z'\Sigma^{1/2}\mu| \ge s) &= \mathbb{P}(|\mu'\Sigma^{1/2}Z| \ge s) \\ &= \mathbb{P}(|\mathcal{Z}| \ge \frac{s}{\sqrt{\mu'\Sigma\mu}}) \\ &\le \min\Big\{1, Cs^{-1}\sqrt{\mu'\Sigma\mu}\exp\Big(-\frac{s^2}{2\mu'\Sigma\mu}\Big)\Big\}, \ \forall s > 0. \end{aligned} \tag{76}$$

Then it follows from (75), (76) that

$$\begin{aligned} & \mathbb{P}(|Y'Y-1|\geq s) \\ \leq\ & \mathbb{P}(|Z'\Sigma Z-1-\mu'\mu|\geq \frac{s}{2})+\mathbb{P}(|\mu'\Sigma^{1/2}Z|\geq \frac{s}{4}) \\ \leq\ & 2\exp\Big\{-C\min\Big(\frac{s}{\|\Sigma_{\mathrm{abs}}\|_2},\frac{s^2}{\|\Sigma\|_F^2}\Big)\Big\} \\ & +\min\Big\{1,Cs^{-1}\sqrt{\mu'\Sigma\mu}\exp\Big(-\frac{s^2}{2\mu'\Sigma\mu}\Big)\Big\} \qquad (77) \\ \leq\ & 2\exp(-C\min(q_{1,\omega_n}s,q_{2,\omega_n}s^2))+\min\{1,C_1 s^{-1}q_{3,\omega_n}^{-1/2}\exp(-Cs^2q_{3,\omega_n})\}. \end{aligned}$$

By applying Borel-Cantelli lemma, we get that

$$V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}=Y'Y\to 1$$

as $n\to\infty$ almost surely. The proof of (a) is complete.

(b) Suppose $\nu=\ell$. First we prove that

$$\mathbb{E}V_{\theta,\ell}=(-1)^{\ell+1}(2\ell)!n\log(\frac{n}{\omega_n})\int_0^L\beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds+O(n), \qquad (78)$$

as $n\to\infty$. Applying (54), (58), Lemma 3 and Proposition 2, we obtain

$$\begin{aligned} \mathbb{E}V_{\theta,\ell} =\ & \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2} \\ & \times\Big\{\tilde{K}(t_{i+\omega_n\theta k_1},t_{i+\omega_n\theta k_2})+\tilde{m}(t_{i+\omega_n\theta k_1})\tilde{m}(t_{i+\omega_n\theta k_2})\Big\} \\ =\ & \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\tilde{\rho}_0(t_{i+\omega_n\theta k_1},t_{i+\omega_n\theta k_2}) \\ & +\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{\substack{0\leq k_1,k_2\leq\ell\\k_1\neq k_2}}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\beta_\nu(\gamma(t_{i+\omega_n\theta k_1}),\gamma(t_{i+\omega_n\theta k_2})) \\ & \times\{d^2_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\Psi(t_i)+O(\omega_n^3n^{-3})\}^\nu \\ & \times\log\{\sqrt{d^2_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\Psi(t_i)+O(\omega_n^3n^{-3})}\}+O(n) \\ =\ & \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{\substack{0\leq k_1,k_2\leq\ell\\k_1\neq k_2}}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\beta_\nu(\gamma(t_i),\gamma(t_i)) \\ & \times d^{2\nu}_{i+\omega_n\theta k_1,i+\omega_n\theta k_2}\Psi^\nu(t_i)\log(d_{i+\omega_n\theta k_1,i+\omega_n\theta k_2})+O(n) \\ =\ & \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{\substack{0\leq k_1,k_2\leq\ell\\k_1\neq k_2}}a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\beta_\nu(\gamma(t_i),\gamma(t_i)) \end{aligned}$$

$$
\begin{aligned}
&\times (d_{i,i+\omega_n\theta k_1} - d_{i,i+\omega_n\theta k_2})^{2\nu}\Psi^\nu(t_i)\log\Big\{\frac{\omega_n|k_2-k_1|\theta L}{n}\Big\} + O(n)\\
= &\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\beta_\nu(\gamma(t_i),\gamma(t_i))\\
&\times (d_{i,i+\omega_n\theta k_1} - d_{i,i+\omega_n\theta k_2})^{2\nu}\Psi^\nu(t_i)\\
&\times \log(\frac{\omega_n\theta L}{n}) + 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{0\le k_1<k_2\le\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\beta_\nu(\gamma(t_i),\gamma(t_i))\\
&\times (d_{i,i+\omega_n\theta k_1} - d_{i,i+\omega_n\theta k_2})^{2\nu}\Psi^\nu(t_i)\log(k_2-k_1) + O(n)\\
= &(-1)^\ell(2\ell)!\log(\frac{\omega_n}{n})\sum_{i=1}^{n-\omega_n\theta\ell}\beta_\nu(\gamma(t_i),\gamma(t_i))\Psi^\nu(t_i) + O(n)\\
= &(-1)^\ell(2\ell)!n\log(\frac{\omega_n}{n})\int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)\Psi^\nu(s)ds + O(n),
\end{aligned}
$$

as $n \to \infty$. We note that

$$
(-1)^{\ell+1}(2\ell)!n\log(\frac{n}{\omega_n})\int_0^L \beta_\nu\big(\gamma(s),\gamma(s)\big)[\Psi(s)]^\nu ds > 0 \tag{79}
$$

follows from similar arguments for the case when $\nu < \ell$; see (60). This proves (78). To evaluate $\mathrm{Var}(V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell})$, we observe that

$$
\begin{aligned}
&\mathrm{Var}\{\sum_{i=1}^{n-\omega_n\theta\ell}(\nabla_{\theta,\ell}X_i)^2\}\\
= &\mathrm{Var}\Big[\sum_{i=1}^{n-\omega_n\theta\ell}\Big\{(\nabla_{\theta,\ell}X_i - \nabla_{\theta,\ell}\tilde m(t_i))^2\\
&+2\nabla_{\theta,\ell}X_i\nabla_{\theta,\ell}\tilde m(t_i) - (\nabla_{\theta,\ell}\tilde m(t_i))^2\Big\}\Big]\\
= &\mathrm{Var}\Big[\sum_{i=1}^{n-\omega_n\theta\ell}\Big\{(\nabla_{\theta,\ell}X_i - \nabla_{\theta,\ell}\tilde m(t_i))^2 + 2\nabla_{\theta,\ell}X_i\nabla_{\theta,\ell}\tilde m(t_i)\Big\}\Big]\\
= &\mathrm{Var}\{\sum_{i=1}^{n-\omega_n\theta\ell}(\nabla_{\theta,\ell}X_i - \nabla_{\theta,\ell}\tilde m(t_i))^2\} + 4\mathrm{Var}\{\sum_{i=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}X_i\nabla_{\theta,\ell}\tilde m(t_i)\}\\
&+4\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big[\mathrm{Cov}\{(\nabla_{\theta,\ell}X_i - \nabla_{\theta,\ell}\tilde m(t_i))^2, \nabla_{\theta,\ell}X_j\nabla_{\theta,\ell}\tilde m(t_j)\}\Big]\\
= &\mathrm{Var}\{\sum_{i=1}^{n-\omega_n\theta\ell}(\nabla_{\theta,\ell}X_i - \nabla_{\theta,\ell}\tilde m(t_i))^2\}\\
&+4\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}\tilde m(t_i)\nabla_{\theta,\ell}\tilde m(t_j)\mathrm{Cov}(\nabla_{\theta,\ell}X_i, \nabla_{\theta,\ell}X_j)
\end{aligned}
$$

$$
\begin{aligned}
&+4\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}\tilde{m}(t_j)\\
&\times\Big[\mathrm{Cov}\{(\nabla_{\theta,\ell}X_i-\nabla_{\theta,\ell}\tilde{m}(t_i))^2,(\nabla_{\theta,\ell}X_j-\nabla_{\theta,\ell}\tilde{m}(t_j))\}\Big]\\
= &\; 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2})\Big\}^2 \qquad (80)\\
&+4\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j)\\
&\times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2}).
\end{aligned}
$$

We shall prove that

$$
\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2})\Big\}^2=O(n^2), \qquad (81)
$$

as $n\to\infty$. It follows from Lemmas 1, 2, Condition 2, Proposition 2 and (24) that

$$
\begin{aligned}
&\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{\rho}_0(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2})\\
= &\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\Big\{\sum_{u_1+u_2\le 2\ell-1}\frac{(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\\
&\times\tilde{\rho}_0^{(u_1,u_2)}(t_i,t_j)+\sum_{u_1+u_2=2\ell}\frac{2\ell(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\\
&\times\int_0^1(1-s)^{2\ell-1}\tilde{\rho}_0^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i,t_{j+\omega_n\theta k_2}-t_j)\}ds\Big\}\\
= &\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\Big\{\sum_{u_1+u_2\le 2\ell-1}\frac{\tilde{\rho}_0^{(u_1,u_2)}(t_i,t_j)}{u_1!u_2!}\\
&\times\{d_{i,i+\omega_n\theta k_1}+\sum_{w=3}^{\ell}f_w(t_i)d^w_{i,i+\omega_n\theta k_1}+o(\omega_n^\ell n^{-\ell})\}^{u_1}\\
&\times\{d_{i,i+\omega_n\theta k_2}+\sum_{w=3}^{\ell}f_w(t_i)d^w_{i,i+\omega_n\theta k_2}+o(\omega_n^\ell n^{-\ell})\}^{u_2}\\
&+\sum_{u_1+u_2=2\ell}\frac{2\ell(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!}\\
&\times\int_0^1(1-s)^{2\ell-1}\tilde{\rho}_0^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i,t_{j+\omega_n\theta k_2}-t_j)\}ds\Big\}
\end{aligned}
$$

$$= \quad O(1), \tag{82}$$

as $n \to \infty$ uniformly over all $i, j \in \{1, \ldots, n - \omega_n\theta\ell\}$. Define

$$\begin{aligned} S_1 &= \{(i,j) : 1 \le i < j \le n - \omega_n\theta\ell, j - i \ge \omega_n\theta\ell + 1\}, \\ S_2 &= \{(i,j) : 1 \le j < i \le n - \omega_n\theta\ell, i - j \ge \omega_n\theta\ell + 1\}, \\ S_3 &= \{(i,j) : 1 \le i, j \le n - \omega_n\theta\ell, |i - j| \le \omega_n\theta\ell\}. \end{aligned}$$

Using Lemma 3, we have

$$\begin{aligned} & \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \beta_\nu(\gamma(t_{i+\omega_n\theta k_1}), \gamma(t_{j+\omega_n\theta k_2})) \\ & \times G_\nu(\sqrt{Q_{i+\omega_n\theta k_1, j+\omega_n\theta k_2}}) \\ = & \sum_{\substack{0\le k_1,k_2\le \ell \\ i+\omega_n\theta k_1 \ne j+\omega_n\theta k_2}} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \{\beta_\nu(\gamma(t_i), \gamma(t_j)) + O(\omega_n n^{-1})\} \\ & \times \{d^2_{i+\omega_n\theta k_1, j+\omega_n\theta k_2}\Psi(t_i) + O(\omega_n^3 n^{-3})\}^\nu \\ & \times \log\{d^2_{i+\omega_n\theta k_1, j+\omega_n\theta k_2}\Psi(t_i) + O(\omega_n^3 n^{-3})\} \\ = & \; O\big(\log(\frac{n}{\omega_n})\big), \end{aligned} \tag{83}$$

as $n \to \infty$ uniformly over all $(i, j) \in S_3$. It follows from (82) and (83) that for $(i, j) \in S_3$

$$\begin{aligned} & \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2}) \\ = & \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \Big\{ \tilde{\rho}_0(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2}) \\ & + \beta_\nu(\gamma(t_{i+\omega_n\theta k_1}), \gamma(t_{j+\omega_n\theta k_2})) G_\nu(\sqrt{Q_{i+\omega_n\theta k_1, j+\omega_n\theta k_2}}) \Big\} \\ = & \; O\big(\log(\frac{n}{\omega_n})\big), \end{aligned} \tag{84}$$

as $n \to \infty$ uniformly over all $(i, j) \in S_3$. Thus it follows from (84) that

$$\sum_{(i,j)\in S_3} \Big\{ \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2}) \Big\}^2 = O\big(n \log^2(\frac{n}{\omega_n})\big), \tag{85}$$

as $n \to \infty$. Now we consider the case when $(i, j) \in S_1$. Using (8), Lemmas 1 to 3, Conditions 2, 3, and Proposition 2, we have

$$\sum_{(i,j)\in S_1} \Big\{ \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2}) \Big\}^2$$

$$
\begin{aligned}
= & \sum_{(i,j)\in S_1} \Big[\sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \\
& \times \Big\{ \sum_{u_1+u_2\le 2\ell-1} \frac{(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!} \\
& \times \tilde{K}^{(u_1,u_2)}(t_i,t_j) + \sum_{u_1+u_2=2\ell} \frac{(2\ell)(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!} \\
& \times \int_0^1 (1-s)^{2\ell-1} \tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)\}ds \Big\} \Big]^2 \\
= & \sum_{(i,j)\in S_1} \Big[\sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \Big\{ \sum_{u_1+u_2\le 2\ell-1} \frac{\tilde{K}^{(u_1,u_2)}(t_i,t_j)}{u_1!u_2!} \\
& \times \{d_{i,i+\omega_n\theta k_1} + \sum_{w=3}^{\ell} f_w(t_i) d^{w}_{i,i+\omega_n\theta k_1} + o(\omega_n^{\ell} n^{-\ell})\}^{u_1} \\
& \times \{d_{i,i+\omega_n\theta k_2} + \sum_{w=3}^{\ell} f_w(t_i) d^{w}_{i,i+\omega_n\theta k_2} + o(\omega_n^{\ell} n^{-\ell})\}^{u_2} \\
& + \sum_{u_1+u_2=2\ell} \frac{2\ell(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!} \\
& \times \int_0^1 (1-s)^{2\ell-1} \tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)\}ds \Big\} \Big]^2 \\
= & \sum_{(i,j)\in S_1} \Big[O(1) + O(1) \sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} \sum_{u_1+u_2=2\ell} \int_0^1 (1-s)^{2\ell-1} \\
& \times |\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)\}| ds \Big]^2 \\
\le & \; O(1) \sum_{(i,j)\in S_1} \Big[\sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} \log(\|\gamma(t_j + s(t_{j+\omega_n\theta k_2}-t_j)) \\
& -\gamma(t_i + s(t_{i+\omega_n\theta k_1}-t_i))\|) \Big]^2 \\
\le & \; O(1) \sum_{(i,j)\in S_1} \Big[\sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} \log\{(t_{j+\omega_n\theta k_2} - t_{i+\omega_n\theta k_1}) \wedge (t_j - t_i)\} \Big]^2 \\
\le & \; O(1) \sum_{(i,j)\in S_1} \log^2 [\frac{(j-i-\omega_n\theta\ell)L}{n}] \\
= & \; O(1) \sum_{i=1}^{n-\omega_n\theta\ell-1} \sum_{j=i+\omega_n\theta\ell+1}^{n-\omega_n\theta\ell} \log^2 [\frac{(j-i-\omega_n\theta\ell)L}{n}]
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{i=1}^{n-\omega_n\theta\ell-1} n\int_{1/n}^{L}\log^2(s)ds \\
&= O(n^2), \qquad (86)
\end{aligned}
$$

as $n \to \infty$. For $(i,j) \in S_2$, by using the same arguments, the same order $O(n^2)$ applies, which combined with (85) and (86) proves (81). For the second term in (80), we observe that

$$
\begin{aligned}
& 4\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j) \\
& \times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2}) \\
= & 4\sum_{(i,j)\in S_1}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j) \\
& \times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2}) \\
& +4\sum_{(i,j)\in S_2}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j) \\
& \times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2}) \\
& +4\sum_{(i,j)\in S_3}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j) \\
& \times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2}). \qquad (87)
\end{aligned}
$$

Using (58) and (84), we obtain

$$
\begin{aligned}
& 4\sum_{(i,j)\in S_3}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j) \\
& \times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2}) \\
\le & 4\sum_{(i,j)\in S_3}|\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j)|\Big| \\
& \times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2})\Big| \\
= & O\big(n\log(\frac{n}{\omega_n})\big), \qquad (88)
\end{aligned}
$$

as $n \to \infty$. When $(i,j) \in S_1$, using Lemmas 1, 2, and Proposition 2, we have

$$
\begin{aligned}
& \Big|\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2})\Big| \\
= & \Big|\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2} \\
& \times\Big[\sum_{u_1+u_2\le 2\ell-1} \frac{(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!} \\
& \times\tilde{K}^{(u_1,u_2)}(t_i,t_j) + \sum_{u_1+u_2=2\ell} \frac{(2\ell)(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!} \\
& \times\int_0^1 (1-s)^{2\ell-1}\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)\}ds\Big]\Big| \\
= & \Big|\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1} a_{\theta,\ell;j,k_2}\Big[\sum_{u_1+u_2\le 2\ell-1}\frac{\tilde{K}^{(u_1,u_2)}(t_i,t_j)}{u_1!u_2!} \\
& \times\{d_{i,i+\omega_n\theta k_1} + \sum_{w=3}^{\ell} f_w(t_i) d^w_{i,i+\omega_n\theta k_1} + o(\omega_n^\ell n^{-\ell})\}^{u_1} \\
& \times\{d_{i,i+\omega_n\theta k_2} + \sum_{w=3}^{\ell} f_w(t_i) d^w_{i,i+\omega_n\theta k_2} + o(\omega_n^\ell n^{-\ell})\}^{u_2} \\
& + \sum_{u_1+u_2=2\ell} \frac{(2\ell)(t_{i+\omega_n\theta k_1}-t_i)^{u_1}(t_{j+\omega_n\theta k_2}-t_j)^{u_2}}{u_1!u_2!} \\
& \times\int_0^1 (1-s)^{2\ell-1}\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)\}ds\Big]\Big| \\
\le & O(1) + \Big|O(1)\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}\sum_{u_1+u_2=2\ell}\int_0^1 (1-s)^{2\ell-1} \\
& \times|\tilde{K}^{(u_1,u_2)}\{(t_i,t_j)+s(t_{i+\omega_n\theta k_1}-t_i, t_{j+\omega_n\theta k_2}-t_j)|ds\Big| \\
\le & O(1)\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}\Big|\log(\|\gamma(t_j+s(t_{j+\omega_n\theta k_2}-t_j)) \\
& -\gamma(t_i+s(t_{i+\omega_n\theta k_1}-t_i))\|)\Big| \\
= & O(1)\Big|\log[\frac{(j-i-\omega_n\theta\ell)L}{n}]\Big|
\end{aligned}
\tag{89}
$$

as $n \to \infty$ uniformly over all $(i,j) \in S_1$. It follows from (89) that

$$
4\sum_{(i,j)\in S_1} \nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j)
$$

$$
\begin{aligned}
&\times \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2})\\
\leq\ & 4\sum_{(i,j)\in S_1}|\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j)|\\
&\times\Big|\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2})\Big|\\
\leq\ & O(1)\sum_{i=1}^{n-\omega_n\theta\ell-1}\ \sum_{j=i+\omega_n\theta\ell+1}^{n-\omega_n\theta\ell}\Big|\log[\frac{(j-i-\omega_n\theta\ell)L}{n}]\Big|\\
=\ & O(1)\sum_{i=1}^{n-\omega_n\theta\ell-1} n\int_{1/n}^{L}|\log(s)|ds\\
=\ & O(n^2), \qquad (90)
\end{aligned}
$$

as $n\to\infty$. Similarly, we have

$$
\begin{aligned}
&4\sum_{(i,j)\in S_2}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j)\\
&\qquad\times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2}) = O(n^2), \qquad (91)
\end{aligned}
$$

as $n\to\infty$. We conclude from (79), (87), (88), (90) and (91) that

$$
\mathrm{Var}(V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}) = \frac{\mathrm{Var}\{\sum_{i=1}^{n-\omega_n\theta\ell}(\nabla_{\theta,\ell}X_i)^2\}}{(\mathbb{E}V_{\theta,\ell})^2} = O\big(\log^{-2}(\frac{n}{\omega_n})\big),
$$

as $n\to\infty$. The convergence of $V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}$ to 1 in probability then follows from Chebyshev's inequality. We observe that

$$
\begin{aligned}
\|\Sigma\|_F^2 &= \frac{1}{(\mathbb{E}V_{\theta,\ell})^2}\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\\
&\quad\times\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1},t_{j+\omega_n\theta k_2})\Big\}^2\\
&= O\big(\log^{-2}(\frac{n}{\omega_n})\big).
\end{aligned}
$$

Hence we have that $\|\Sigma_{\mathrm{abs}}\|_2 \leq \|\Sigma\|_F = O\big(\log^{-1}(n)\big)$. Also,

$$
\begin{aligned}
\mu'\mu &= \frac{1}{\mathbb{E}V_{\theta,\ell}}\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\tilde{m}(t_{i+\omega_n\theta k_1})\tilde{m}(t_{i+\omega_n\theta k_2})\\
&= O\big(\log^{-1}(\frac{n}{\omega_n})\big). \qquad (92)
\end{aligned}
$$

Then it follows similarly to (77) that

$$\begin{aligned}\mathbb{P}(|Y'Y-1|\geq s) \quad &\leq \quad 2\exp\{-C\min(q_{1,\omega_n}s, q_{2,\omega_n}s^2)\} \\ &\quad + \min\{1, C_1 s^{-1}q_{3,\omega_n}^{-1/2}\exp(-Cs^2 q_{3,\omega_n})\}.\end{aligned}$$

This completes the proof of (b).

(c) Suppose $\nu > \ell$. We shall prove that $V_{\theta,\ell}/n$, $\theta \in \{1,2\}$, converge to the same positive (random) limit with probability 1 as $n \to \infty$. Consider the interval $(-\varepsilon, L+\varepsilon)$ for some small constant $\varepsilon > 0$. It follows from Proposition 3 that $X(\gamma(.))$ has an equivalent version possessing, with probability 1, a $\lceil \nu \rceil - 1$ times continuously differentiable sample path on $(-\varepsilon, L+\varepsilon)$. Let $\mathsf{X}^{(j)}(t)$ denote the jth derivative of such an equivalent version of $X(\gamma(t))$ for $j = 1, \ldots, \ell$. We shall first prove a Lipschitz property of $\mathsf{X}^{(\ell)}(t)$ for $t \in (-\varepsilon, L+\varepsilon)$. Since the mean function $m(\mathbf{t}) = \mathbb{E}X(\mathbf{t})$ is assumed to be in $C^N(\mathbb{R}^d)$, it suffices to work with $X(\mathbf{t}) - m(\mathbf{t})$. Hence we shall, without loss of generality, assume that $m(\mathbf{t}) = 0$.

We observe from the proof of Proposition 3(iii) that $\mathsf{X}^{(\ell)}(t)$ is continuous on $t \in (-\varepsilon, L+\varepsilon)$ and is an equivalent version of the ℓth-order mean square derivative of $X(\gamma(t))$. Consequently, it follows from Lemma 7 in Section F that the covariance function of $\mathsf{X}^{(\ell)}(t)$ is $\tilde{K}^{(\ell,\ell)}(x,y)$. Thus

$$\begin{aligned}&\left|\tilde{\rho}_0^{(\ell,\ell)}(t+h,t+h) - 2\tilde{\rho}_0^{(\ell,\ell)}(t+h,t) + \tilde{\rho}_0^{(\ell,\ell)}(t,t)\right| \\ \leq\quad & h^2 \int_0^1\int_0^1 \left|\tilde{\rho}_0^{(\ell+1,\ell+1)}(t+s_1h, t+s_2h)\right| ds_1 ds_2 \\ \leq\quad & Ch^2, \end{aligned} \tag{93}$$

for all h such that $|h| > 0$ is small and $t \in [0, L]$, where $C > 0$ is a constant.

For $\tilde{\rho}_\nu(.,.)$, we observe in the proof of Proposition 3(i) that the leading term of $\tilde{\rho}_\nu^{(\ell,\ell)}(x,y)$ has the same order as

$$\begin{cases} |x-y|^{2\nu-2\ell} & \text{if } \nu \notin \mathbb{Z}_+, \\ |x-y|^{2\nu-2\ell}\{|\log(|x-y|)|+1\} & \text{if } \nu \in \mathbb{Z}_+. \end{cases}$$

Thus we conclude that

$$\begin{aligned}&\left|\tilde{\rho}_\nu^{(\ell,\ell)}(t+h,t+h) - 2\tilde{\rho}_\nu^{(\ell,\ell)}(t+h,t) + \tilde{\rho}_\nu^{(\ell,\ell)}(t,t)\right| \\ \leq\quad & \begin{cases} C|h|^{2\nu-2\ell} & \text{if } \nu \notin \mathbb{Z}_+, \\ C|h|^{2\nu-2\ell}\{|\log(|h|)|+1\} & \text{if } \nu \in \mathbb{Z}_+, \end{cases}\end{aligned} \tag{94}$$

for all h such that $|h| > 0$ is small and $t \in [0, L]$, where $C > 0$ is a constant.

Applying the result on page 186 of [9], (93) and (94) show that for any $\epsilon < (\nu - \ell) \wedge 1$, $\mathsf{X}^{(\ell)}(t)$ can be chosen such that its sample path is Lipschitz continuous of order ϵ. In particular, with probability 1, there exist constants $C, \delta > 0$ (may be random) such that if $|h| < \delta$,

$$|\mathsf{X}^{(\ell)}(t+h) - \mathsf{X}^{(\ell)}(t)| < C|h|^\epsilon,$$

for all $t, t+h \in [0, L]$. We extend the definitions of $f_w(t)$ in (24) by $f_1(t) = 1$ and $f_2(t) = 0$ for any $t \in (-\varepsilon, L+\varepsilon)$. Using Taylor expansion, (24) and Lemma 1, we have with probability 1,

$$
\begin{aligned}
&\nabla_{\theta,\ell} X_i \\
= &\sum_{k=0}^{\ell} a_{\theta,\ell;i,k} X(\gamma(t_{i+\omega_n\theta k})) \\
= &\sum_{k=0}^{\ell} a_{\theta,\ell;i,k} \bigg\{ \sum_{j=0}^{\ell-1} \frac{\mathsf{X}^{(j)}(t_i)}{j!} (t_{i+\omega_n\theta k} - t_i)^j \\
&+ \frac{1}{(\ell-1)!} \int_{t_i}^{t_{i+\omega_n\theta k}} (t_{i+\omega_n\theta k} - s)^{\ell-1} \mathsf{X}^{(\ell)}(s) ds \bigg\} \\
= &\sum_{k=0}^{\ell} a_{\theta,\ell;i,k} \bigg\{ \sum_{j=0}^{\ell-1} \frac{\mathsf{X}^{(j)}(t_i)}{j!} \{ d_{i,i+\omega_n\theta k} + \sum_{w=3}^{\ell} f_w(t_i) d^w_{i,i+\omega_n\theta k} + o(\omega_n^\ell n^{-\ell}) \}^j \\
&+ \frac{1}{(\ell-1)!} \int_{t_i}^{t_{i+\omega_n\theta k}} (t_{i+\omega_n\theta k} - s)^{\ell-1} \mathsf{X}^{(\ell)}(s) ds \bigg\} \\
= &\sum_{k=0}^{\ell} a_{\theta,\ell;i,k} \bigg\{ \sum_{j=1}^{\ell-1} \frac{\mathsf{X}^{(j)}(t_i)}{j!} \\
&\times \bigg(\sum_{\substack{w_1+\cdots+w_j=\ell \\ w_1,\ldots,w_j \geq 1}} f_{w_1}(t_i) \cdots f_{w_j}(t_i) d^\ell_{i,i+\omega_n\theta k} + o(\omega_n^\ell n^{-\ell}) \bigg) \\
&+ \frac{1}{(\ell-1)!} \int_{t_i}^{t_{i+\omega_n\theta k}} (t_{i+\omega_n\theta k} - s)^{\ell-1} \mathsf{X}^{(\ell)}(s) ds \bigg\} \\
= &\ell! \sum_{j=1}^{\ell-1} \frac{\mathsf{X}^{(j)}(t_i)}{j!} \sum_{\substack{w_1+\cdots+w_j=\ell \\ w_1,\ldots,w_j \geq 1}} f_{w_1}(t_i) \cdots f_{w_j}(t_i) \\
&+ \mathsf{X}^{(\ell)}(t_i) \sum_{k=0}^{\ell} \frac{a_{\theta,\ell;i,k}}{\ell!} (t_{i+\omega_n\theta k} - t_i)^\ell \\
&+ \sum_{k=0}^{\ell} a_{\theta,\ell;i,k} \frac{1}{(\ell-1)!} \\
&\times \int_{t_i}^{t_{i+\omega_n\theta k}} (t_{i+\omega_n\theta k} - s)^{\ell-1} \{ \mathsf{X}^{(\ell)}(s) - \mathsf{X}^{(\ell)}(t_i) \} ds + o(1) \\
= &\sum_{j=1}^{\ell-1} \mathsf{X}^{(j)}(t_i) \sum_{\substack{w_1+\cdots+w_j=\ell \\ w_1,\ldots,w_j \geq 1}} \frac{\ell!}{j!} f_{w_1}(t_i) \cdots f_{w_j}(t_i) \\
&+ \mathsf{X}^{(\ell)}(t_i) \sum_{k=0}^{\ell} a_{\theta,\ell;i,k} \frac{1}{\ell!} \{ d_{i,i+\omega_n\theta k} + O(\omega_n^3 n^{-3}) \}^\ell
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{\ell} a_{\theta,\ell;i,k}\frac{1}{(\ell-1)!}\\
&\times\int_{t_i}^{t_{i+\omega_n\theta k}}(t_{i+\omega_n\theta k}-s)^{\ell-1}\{\mathsf{X}^{(\ell)}(s)-\mathsf{X}^{(\ell)}(t_i)\}ds+o(1)\\
=\;&\ell!\sum_{j=1}^{\ell}\frac{\mathsf{X}^{(j)}(t_i)}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\ge 1}} f_{w_1}(t_i)\cdots f_{w_j}(t_i)+o(1),
\end{aligned}
\tag{95}
$$

as $n\to\infty$. Hence it follows from (95) that with probability 1,

$$
\begin{aligned}
&n^{-1}V_{\theta,\ell}\\
=\;&n^{-1}\sum_{i=1}^{n-\omega_n\theta\ell}\{\nabla_{\theta,\ell}X(\gamma(t_i))\}^2\\
=\;&n^{-1}\sum_{i=1}^{n-\omega_n\theta\ell}\Big\{\ell!\sum_{j=1}^{\ell}\frac{\mathsf{X}^{(j)}(t_i)}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\ge 1}} f_{w_1}(t_i)\cdots f_{w_j}(t_i)+o(1)\Big\}^2\\
\to\;&\int_0^L\Big[\ell!\sum_{j=1}^{\ell}\frac{\mathsf{X}^{(j)}(s)}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\ge 1}} f_{w_1}(s)\cdots f_{w_j}(s)\Big]^2ds,
\end{aligned}
$$

as $n\to\infty$. Since $f_1(t),\ldots,f_\ell(t)$ are all bounded, using the arguments in the proof of Proposition 3 (see Section B), it is not difficult to verify that

$$
\ell!\sum_{j=1}^{\ell}\frac{\mathsf{X}^{(j)}(t)}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\ge 1}} f_{w_1}(t)\cdots f_{w_j}(t)
$$

is a Gaussian process with mean 0 and bounded variance for all $t\in[0,L]$. Since the limit is the same for both $\theta=1,2$, it follows that

$$
V_{1,\ell}/V_{2,\ell}\to 1
$$

almost surely as $n\to\infty$ provided

$$
\int_0^L\Big[\ell!\sum_{j=1}^{\ell}\frac{\mathsf{X}^{(j)}(s)}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\ge 1}} f_{w_1}(s)\cdots f_{w_j}(s)\Big]^2ds>0,
\tag{96}
$$

with probability 1. Now, the final task is to verify (96). Define

$$
Y(s)=\ell!\sum_{j=1}^{\ell}\frac{\mathsf{X}^{(j)}(s)}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\ge 1}} f_{w_1}(s)\cdots f_{w_j}(s).
$$

Suppose the probability that (96) is true is strictly less than 1. Then from the sample path continuity of $Y(s)$ on $[0,L]$, it follows that

$$\mathbb{P}\Big(\bigcap_{s\in[0,L]}\{Y(s)=0\}\Big)=\mathbb{P}\Big(\int_0^L\{Y(s)\}^2ds=0\Big)>0.$$

Consequently it follows that for every fixed $s_0\in[0,L]$

$$\mathbb{P}\Big(Y(s_0)=0\Big)\geq\mathbb{P}\Big(\bigcap_{s\in[0,L]}\{Y(s)=0\}\Big)>0,$$

which implies that $\mathbb{P}(Y(s_0)=0)=1$ because $Y(s_0)$ is a Gaussian random variable. Since this holds for every $s_0\in[0,L]$ and that the sample path of $Y(s)$ is continuous on $[0,L]$, we have

$$\mathbb{P}\Big(\bigcap_{s\in[0,L]}\{Y(s)=0\}\Big)=1.$$

Consequently with probability 1, we obtain

$$\mathsf{X}^{(\ell)}(t)=-\ell!\sum_{j=1}^{\ell-1}\frac{\mathsf{X}^{(j)}(t)}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\geq1}}f_{w_1}(t)\cdots f_{w_j}(t)\qquad\forall t\in[0,L].\tag{97}$$

For simplicity of notation, we denote

$$c_j(t)=-\frac{\ell!}{j!}\sum_{\substack{w_1+\cdots+w_j=\ell\\ w_1,\ldots,w_j\geq1}}f_{w_1}(t)\cdots f_{w_j}(t)$$

It follows from the definitions of $f_{w_1},\ldots,f_{w_\ell}$ that $c_j(t)\in C^{N-\ell}(0,L)$. From the proof of Proposition 3 in Section B, the mean square derivatives of $\mathsf{X}^{(1)}(t)$, $\ldots$, $\mathsf{X}^{(\ell-1)}(t)$ coincide with their sample path derivatives which are $\mathsf{X}^{(2)}(t)$, $\ldots,\mathsf{X}^{(\ell)}(t)$ respectively. Using (97), it follows that $\mathsf{X}^{(\ell)}(t)$ is mean square differentiable on $(0,L)$. Denote $\dot{\mathsf{X}}^{(\ell)}(t)$ be the mean square derivative of $\mathsf{X}^{(\ell)}(t)$. One can verify that the product rule in mean square sense applies and yields

$$\dot{\mathsf{X}}^{(\ell)}(t)=\sum_{j=1}^{\ell-1}\{c_j^{(1)}(t)\mathsf{X}^{(j)}(t)+c_j(t)\mathsf{X}^{(j+1)}(t)\}.\tag{98}$$

Plugging (97) into (98), we can inductively differentiate $\mathsf{X}^{(\ell)}(t)$ in the mean square sense $N-\ell$ times. However this contradicts Proposition 3 which asserts that $\mathsf{X}(t)$ can be differentiated in mean square at most $\lceil\nu\rceil-1$ times. This proves that (96) must be true with probability 1. Now we show that

$$\mathbb{E}V_{\theta,\ell}\asymp n.\tag{99}$$

By Fatou's Lemma, it follows that

$$\liminf_{n\to\infty} n^{-1}\mathbb{E}V_{\theta,\ell} \geq \mathbb{E}\liminf_{n\to\infty}(n^{-1}V_{\theta,\ell}) > 0. \tag{100}$$

Also, we observe that

$$\begin{aligned}\mathbb{E}V_{\theta,\ell} &= \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2} \\ &\quad\times\Big\{\tilde{K}(t_{i+\omega_n\theta k_1}, t_{i+\omega_n\theta k_2}) + \tilde{m}(t_{i+\omega_n\theta k_1})\tilde{m}(t_{i+\omega_n\theta k_2})\Big\} \\ &= O(n), \end{aligned} \tag{101}$$

as $n\to\infty$. Thus (99) follows from (100) and (101). Next we prove the result

$$\mathrm{Var}(V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}) \asymp 1 \tag{102}$$

by contradiction. We observe that

$$\begin{aligned}&\mathrm{Var}(V_{\theta,\ell}) \\ = {}& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2})\Big\}^2 \\ & +4\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j) \\ & \times\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2}) \\ = {}& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2})\Big\}^2 \\ & +4\mathbb{E}\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}\tilde{m}(t_i)\nabla_{\theta,\ell}\tilde{m}(t_j)(\nabla_{\theta,\ell}X_i\nabla_{\theta,\ell}X_j) \\ = {}& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2})\Big\}^2 \\ & +4\mathbb{E}\Big(\sum_{i=1}^{n-\omega_n\theta\ell}\nabla_{\theta,\ell}X_i\nabla_{\theta,\ell}\tilde{m}(t_i)\Big)^2 \\ \geq {}& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2})\Big\}^2 \\ = {}& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\sum_{\substack{u_1,u_2\geq 1\\ u_1+u_2\leq 2\ell}}\frac{\tilde{K}^{(u_1,u_2)}(t_i,t_j)}{u_1!u_2!}\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\sum_{w=1}^{\ell} f_w(t_i)d^w_{i,i+\omega_n\theta k_1}\Big)^{u_1}\Big(\sum_{w=1}^{\ell} f_w(t_j)d^w_{j,j+\omega_n\theta k_2}\Big)^{u_2}+o(1)\Big\}^2\\
= \;& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\sum_{\substack{u_1,u_2\geq 1\\ u_1+u_2\leq 2\ell}}\frac{\tilde{K}^{(u_1,u_2)}(t_i,t_j)}{u_1!u_2!}\\
&\times\sum_{w_1=1}^{\ell}\cdots\sum_{w_{u_1}=1}^{\ell}\sum_{w_{u_1+1}=1}^{\ell}\cdots\sum_{w_{u_1+u_2}=1}^{\ell} f_{w_1}(t_i)\cdots f_{w_{u_1}}(t_i)\\
&\times f_{w_{u_1+1}}(t_i)\cdots f_{w_{u_1+u_2}}(t_i)d^{w_1+\cdots+w_{u_1}}_{i,i+\omega_n\theta k_1}d^{w_{u_1+1}+\cdots+w_{u_1+u_2}}_{j,j+\omega_n\theta k_2}+o(1)\Big\}^2\\
= \;& 2\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{\substack{u_1,u_2\geq 1\\ u_1+u_2\leq 2\ell}}\frac{(\ell!)^2\tilde{K}^{(u_1,u_2)}(t_i,t_j)}{u_1!u_2!}\\
&\times\sum_{\substack{w_1+\cdots+w_{u_1}=\ell\\ w_{u_1+1}+\cdots+w_{u_1+u_2}=\ell\\ w_1,\ldots,w_{u_1+u_2}\geq 1}} f_{w_1}(t_i)\cdots f_{w_{u_1}}(t_i)f_{w_{u_1+1}}(t_j)\cdots f_{w_{u_1+u_2}}(t_j)\Big\}^2\\
&+o(n^2)\\
= \;& 2n^2\int_0^L\int_0^L\Big\{\sum_{\substack{u_1,u_2\geq 1\\ u_1+u_2\leq 2\ell}}\frac{(\ell!)^2\tilde{K}^{(u_1,u_2)}(s_1,s_2)}{u_1!u_2!}\\
&\times\sum_{\substack{w_1+\cdots+w_{u_1}=\ell\\ w_{u_1+1}+\cdots+w_{u_1+u_2}=\ell\\ w_1,\ldots,w_{u_1+u_2}\geq 1}} f_{w_1}(s_1)\cdots f_{w_{u_1}}(s_1)\\
&\times f_{w_{u_1+1}}(s_2)\cdots f_{w_{u_1+u_2}}(s_2)\Big\}^2 ds_1ds_2+o(n^2).
\end{aligned}
\tag{103}
$$

It is straightforward to see that $\mathrm{Var}(V_{\theta,\ell})=O(n^2)$ and thus it follows from (99) that $\mathrm{Var}(V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell})=O(1)$. To show $\liminf_{n\to\infty}\mathrm{Var}(V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell})>0$, we show that the integral in (103) is strictly positive. It suffices to prove that the integrand is not identically 0 since the integrand is a continuous function. We note that if the integrand is identically 0,

$$
\begin{aligned}
&\sum_{\substack{u_1,u_2\geq 1\\ u_1+u_2=2\ell}}\frac{(\ell!)^2\tilde{K}^{(u_1,u_2)}(s_1,s_2)}{u_1!u_2!}\\
&\times\sum_{\substack{w_1+\cdots+w_{u_1}=\ell\\ w_{u_1+1}+\cdots+w_{u_1+u_2}=\ell\\ w_1,\ldots,w_{u_1+u_2}\geq 1}} f_{w_1}(s_1)\cdots f_{w_{u_1}}(s_1)f_{w_{u_1+1}}(s_2)\cdots f_{w_{u_1+u_2}}(s_2)\\
= \;& -\sum_{\substack{u_1,u_2\geq 1\\ u_1+u_2<2\ell}}\frac{(\ell!)^2\tilde{K}^{(u_1,u_2)}(s_1,s_2)}{u_1!u_2!}
\end{aligned}
$$

$$\times \sum_{\substack{w_1+\cdots+w_{u_1}=\ell \\ w_{u_1+1}+\cdots+w_{u_1+u_2}=\ell \\ w_1,\ldots,w_{u_1+u_2}\geq 1}} f_{w_1}(s_1)\cdots f_{w_{u_1}}(s_1) f_{w_{u_1+1}}(s_2)\cdots f_{w_{u_1+u_2}}(s_2).$$

This implies that the 2ℓ-th partial derivative of $\tilde{K}(.,.)$ can be expressed as a linear combination of its lower order partial derivatives. Hence due to the differentiability of $\tilde{K}(.,.)$, it can be further differentiated iteratively. This contradicts to the fact that $\tilde{K}(.,.)$ is at most $\lfloor 2\nu \rfloor$ times differentiable. Therefore, the integral in (103) is strictly positive and the proof of (102) is done. We further observe that

$$\mu'\mu = \frac{1}{n}\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;i,k_2}\tilde{m}(t_{i+\omega_n\theta k_1})\tilde{m}(t_{i+\omega_n\theta k_2}) = O(1).$$

To evaluate $\|\Sigma\|_F^2$, we observe that

$$\begin{aligned}
\|\Sigma\|_F^2 &= \sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Sigma_{ij}^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell}\Big\{\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} a_{\theta,\ell;i,k_1}a_{\theta,\ell;j,k_2}\tilde{K}(t_{i+\omega_n\theta k_1}, t_{j+\omega_n\theta k_2})\Big\}^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n-\omega_n\theta\ell}\sum_{j=1}^{n-\omega_n\theta\ell} O(1) \\
&= O(1).
\end{aligned}$$

It thus follows that

$$\|\Sigma_{\text{abs}}\|_2 \leq \|\Sigma\|_F = O(1), \qquad \mu'\Sigma\mu \leq \mu'\mu\|\Sigma\|_2 \leq \mu'\mu\|\Sigma\|_F = O(1).$$

Using Hanson-Wright inequality, we have for sufficiently large n,

$$\begin{aligned}
& \mathbb{P}(|Y'Y - \mathbb{E}Y'Y| \geq s) \\
= \;& \mathbb{P}(|Z'\Sigma Z + Z'\Sigma^{1/2}\mu + \mu'\Sigma Z - \mathbb{E}Z'\Sigma Z| \geq s) \\
\leq \;& \mathbb{P}(|Z'\Sigma Z - \mathbb{E}Z'\Sigma Z| \geq \frac{s}{2}) + \mathbb{P}(|Z'\Sigma^{1/2}\mu| \geq \frac{s}{4}) \\
\leq \;& 2\exp\Big\{-C\min\Big(\frac{s}{\|\Sigma_{\text{abs}}\|_2}, \frac{s^2}{\|\Sigma\|_F^2}\Big)\Big\} \\
& + \min\Big\{1, C_1 s^{-1}\sqrt{\mu'\Sigma\mu}\exp\Big(-\frac{s^2}{2\mu'\Sigma\mu}\Big)\Big\} \\
\leq \;& 2\exp\{-C\min(q_{1,\omega_n}s, q_{2,\omega_n}s^2)\} + \min\{1, C_1 s^{-1} q_{3,\omega_n}^{-1/2}\exp(-Cs^2 q_{3,\omega_n})\},
\end{aligned}$$

for all $s > 0$. The proof of (c) is complete.

The result in (d) can be directly read from (a), (b) and (c). This completes the proof of Theorem 1. □

Appendix D: Proofs of Lemmas 4 to 6 and Theorems 2 to 4

Define $W_\theta = V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}$.

Lemma 4. *Suppose $\nu \le \ell$ and Conditions 1 to 3 hold. For any $j \in \mathbb{Z}_+$, there exists a constant $C_j > 0$ such that*

$$\mathbb{E}\{(\frac{V_{\theta,\ell}}{\mathbb{E}V_{\theta,\ell}} - 1)^{2j}\} \le C_j(q_{1,\omega_n}^{-2j} + q_{2,\omega_n}^{-j} + q_{3,\omega_n}^{-j}), \qquad \forall \theta \in \{1,2\}.$$

Proof. We recall the notation from (15) that for $\nu \le \ell$

$$W_\theta = Y'Y = Z'\Sigma Z + \mu'\Sigma^{1/2}Z + Z'\Sigma^{1/2}\mu + \mu'\mu$$

with

$$Y = \Big(\frac{\nabla_{\theta,\ell}X_1}{\sqrt{\mathbb{E}V_{\theta,\ell}}}, \dots, \frac{\nabla_{\theta,\ell}X_{n-\omega_n\theta\ell}}{\sqrt{\mathbb{E}V_{\theta,\ell}}}\Big)',$$

$Z \sim N_{n-\omega_n\theta\ell}(0, I)$, $\mu = \mathbb{E}Y$ and $\Sigma = (\Sigma_{i,j})_{(n-\omega_n\theta\ell)\times(n-\omega_n\theta\ell)} = \mathbb{E}\{(Y-\mu)(Y-\mu)'\}$. Let C_j be a generic constant that depends on j which can take different values at different locations. We observe from the proof of Theorem 1(a) and (b) that for $j \in \mathbb{Z}_+$,

$$\begin{aligned}
& \mathbb{E}\{(W_\theta - 1)^{2j}\} \\
= & \int_0^\infty \mathbb{P}(|W_\theta - 1|^{2j} \ge s)ds \\
= & \int_0^\infty \mathbb{P}(|W_\theta - 1| \ge s^{1/(2j)})ds \\
\le & \int_0^\infty \Big\{\mathbb{P}(|Z'\Sigma Z + \mu'\mu - 1| \ge s^{1/(2j)}/2) + \mathbb{P}(|\mu'\Sigma^{1/2}Z| \ge s^{1/(2j)}/4)\Big\}ds \\
\le & 2\int_0^\infty \exp\Big\{-C\min(q_{1,\omega_n}s^{1/(2j)}, q_{2,\omega_n}s^{1/j})\Big\}ds \\
& +2\int_0^\infty\int_{s^{1/(2j)}/4}^\infty \frac{1}{\sqrt{2\pi\mu'\Sigma\mu}}\exp(-\frac{t^2}{2\mu'\Sigma\mu})dtds \\
= & C_j\int_0^\infty s^{2j-1}\exp\Big\{-C\min(q_{1,\omega_n}s, q_{2,\omega_n}s^2)\Big\}ds \\
& +2\int_0^\infty\int_0^{(4t)^{2j}} \frac{1}{\sqrt{2\pi\mu'\Sigma\mu}}\exp(-\frac{t^2}{2\mu'\Sigma\mu})dsdt \\
= & C_j\int_0^\infty s^{2j-1}\exp\Big\{-C(q_{1,\omega_n}s)\Big\}ds \\
& +C_j\int_0^\infty s^{2j-1}\exp\Big\{-C(q_{2,\omega_n}s^2)\Big\}ds \\
& +2\int_0^\infty \frac{4^{2j}t^{2j}}{\sqrt{2\pi\mu'\Sigma\mu}}\exp(-\frac{t^2}{2\mu'\Sigma\mu})dt \\
= & C_j(q_{1,\omega_n}^{-2j} + q_{2,\omega_n}^{-j} + q_{3,\omega_n}^{-j}).
\end{aligned}$$

This proves Lemma 4. □

Lemma 5. *Suppose $\nu \leq \ell$ and Conditions 1 to 3 hold. For any $j \in \mathbb{Z}_+$, there exists a constant $C_j > 0$ such that*

$$\mathbb{E}\{\log^{2j}(\frac{V_{\theta,\ell}}{\mathbb{E}V_{\theta,\ell}})\} \leq C_j(q_{1,\omega_n}^{-2j} + q_{2,\omega_n}^{-j} + q_{3,\omega_n}^{-j}), \qquad \forall\theta \in \{1,2\}.$$

Proof. We observe that

$$\begin{aligned} & \mathbb{E}\log^{2j}(W_\theta) \\ = \;& \mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(|W_\theta - 1| < \frac{1}{2})\} + \mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(|W_\theta - 1| \geq \frac{1}{2})\} \\ \leq \;& \sup_{x\in[1/2,3/2]} \log^{2j}(x) + \mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(W_\theta \geq \frac{3}{2})\} \\ & + \mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(0 < W_\theta \leq \frac{1}{2})\} \\ \leq \;& \sup_{x\in[1/2,3/2]} \log^{2j}(x) + \mathbb{E}\{(W_\theta - 1)^{2j}\} + \mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(0 < W_\theta \leq \frac{1}{2})\}. \end{aligned} \tag{104}$$

Recall from (15) that

$$W_\theta = (\Sigma^{1/2}Z + \mu)'(\Sigma^{1/2}Z + \mu).$$

We write $\Sigma = H\Lambda H'$ where Λ is a diagonal matrix and H is an orthogonal matrix. Then it follows that W_θ is identically distributed as

$$\tilde{W}_\theta = (\Lambda^{1/2}Z + H'\mu)'(\Lambda^{1/2}Z + H'\mu).$$

Hence we have

$$\mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(0 < W_\theta \leq \frac{1}{2})\} = \mathbb{E}\{\log^{2j}(\tilde{W}_\theta)\mathcal{I}(0 < \tilde{W}_\theta \leq \frac{1}{2})\}.$$

Let $\lambda_1 \geq \cdots \geq \lambda_{n-\omega_n\theta\ell}$ denote the descending eigenvalues of Σ, write $Z = (Z_1, \ldots, Z_{n-\omega_n\theta\ell})'$ and define

$$\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{n-\omega_n\theta\ell})' = H'\mu.$$

Then we have

$$\tilde{W}_\theta = \sum_{i=1}^{n-\omega_n\theta\ell} (\lambda_i^{1/2}Z_i + \tilde{\mu}_i)^2 \geq (\lambda_1^{1/2}Z_1 + \tilde{\mu}_1)^2.$$

We observe that when $0 < \tilde{W}_\theta \leq 1/2$, $\log(\tilde{W}_\theta) \leq 0$ and hence

$$0 \geq \log(\tilde{W}_\theta)\mathcal{I}(0 < \tilde{W}_\theta \leq \frac{1}{2}) \geq \log\{(\lambda_1^{1/2}Z_1 + \tilde{\mu}_1)^2\}\mathcal{I}(0 < \tilde{W}_\theta \leq \frac{1}{2}).$$

Consequently it follows that for any $j \in \mathbb{Z}_+$,

$$\begin{aligned} \mathbb{E}\log^{2j}(\tilde{W}_\theta)\mathcal{I}(0<\tilde{W}_\theta\leq\frac{1}{2}) &\leq \mathbb{E}\Big[\log^{2j}\{(\lambda_1^{1/2}Z_1+\tilde{\mu}_1)^2\}\mathcal{I}(0<\tilde{W}_\theta\leq\frac{1}{2})\Big] \\ &\leq \mathbb{E}\log^{2j}\{(\lambda_1^{1/2}Z_1+\tilde{\mu}_1)^2\}. \end{aligned}$$

Now we establish lower and upper bounds for λ_1 and an upper bound for $\tilde{\mu}_1$. Since $\sum_{i=1}^{n-\omega_n\theta\ell}\lambda_i = \mathbb{E}\sum_{i=1}^{n-\omega_n\theta\ell}\lambda_i Z_i^2 = \mathbb{E}Z'\Sigma Z = 1-\mu'\mu$, it follows from (74) and (92) that

$$\begin{aligned} \frac{1}{2n} &\leq \frac{1-\mu'\mu}{n-\omega_n\theta\ell}\leq\lambda_1\leq 1, \\ \tilde{\mu}_1^2 &\leq \tilde{\mu}'\tilde{\mu}=\mu'\mu=o(1), \end{aligned} \tag{105}$$

for all large n. Let C_j denote a generic constant that can take different values at different locations. Consequently using (105), we obtain

$$\begin{aligned} &\mathbb{E}\log^{2j}\{(\lambda_1^{1/2}Z_1+\tilde{\mu}_1)^2\} \\ = &\int_{-\infty}^{\infty}\log^{2j}(x^2)\frac{1}{\sqrt{2\pi\lambda_1}}\exp\Big\{-\frac{(x-\tilde{\mu}_1)^2}{2\lambda_1}\Big\}dx \\ = &\int_{0}^{\infty}\log^{2j}(x^2)\frac{1}{\sqrt{2\pi\lambda_1}}\exp\Big\{-\frac{(x-\tilde{\mu}_1)^2}{2\lambda_1}\Big\}dx \\ &+\int_{-\infty}^{0}\log^{2j}(x^2)\frac{1}{\sqrt{2\pi\lambda_1}}\exp\Big\{-\frac{(x-\tilde{\mu}_1)^2}{2\lambda_1}\Big\}dx \\ = &\int_{0}^{\infty}\log^{2j}(x^2)\frac{1}{\sqrt{2\pi\lambda_1}}\Big[\exp\Big\{-\frac{(x-\tilde{\mu}_1)^2}{2\lambda_1}\Big\}+\exp\Big\{-\frac{(x+\tilde{\mu}_1)^2}{2\lambda_1}\Big\}\Big]dx \\ \leq &\ 2^{2j}\int_0^1\log^{2j}(x)\frac{2}{\sqrt{2\pi\lambda_1}}dx+2^{2j}\int_1^\infty\log^{2j}(x)\frac{1}{\sqrt{2\pi\lambda_1}} \\ &\qquad\times\Big[\exp\Big\{-\frac{(x-\tilde{\mu}_1)^2}{2\lambda_1}\Big\}+\exp\Big\{-\frac{(x+\tilde{\mu}_1)^2}{2\lambda_1}\Big\}\Big]dx \\ \leq &\ C_j\sqrt{n}+C_j\int_0^\infty(x-1)^{2j}\frac{1}{\sqrt{2\pi\lambda_1}} \\ &\qquad\times\Big[\exp\Big\{-\frac{(x-\tilde{\mu}_1)^2}{2\lambda_1}\Big\}+\exp\Big\{-\frac{(x+\tilde{\mu}_1)^2}{2\lambda_1}\Big\}\Big]dx \\ \leq &\ C_j\sqrt{n}+C_j\int_{-\infty}^\infty(\lambda_1^{1/2}x+\tilde{\mu}_1-1)^{2j}\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{x^2}{2}\Big\}dx \\ &+C_j\int_{-\infty}^\infty(\lambda_1^{1/2}x-\tilde{\mu}_1-1)^{2j}\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{x^2}{2}\Big\}dx \\ \leq &\ C_j\sqrt{n}+C_j. \end{aligned}$$

Then it follows from (104) and Lemma 4 that for all large n,

$$\mathbb{E}\log^{2j}(W_\theta)\leq C_j\sqrt{n}. \tag{106}$$

Next, we improve the bound (106). Recall that

$$\mathbb{E}\log^{2j}(W_\theta) = \mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(|W_\theta - 1| < \frac{1}{2})\} + \mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(|W_\theta - 1| \geq \frac{1}{2})\}.$$

We observe from (16) that there exists some constant $C, C_1 > 0$ such that for all large n

$$\mathbb{P}(|W_\theta - 1| \geq \frac{1}{2}) \leq 2\exp(-C\min(q_{1,\omega_n}, q_{2,\omega_n})) + \min\{1, C_1 q_{3,\omega_n}^{-1/2}\exp(-Cq_{3,\omega_n})\}.$$

Suppose $\nu < \ell$. Applying Cauchy-Schwartz inequality and (106), it follows that

$$\begin{aligned}&\mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(|W_\theta - 1| \geq \frac{1}{2})\} \leq \sqrt{\{\mathbb{E}\log^{4j}(W_\theta)\}\{\mathbb{P}(|W_\theta - 1| \geq \frac{1}{2})\}}\\ &\leq C_j n^{1/4}\exp(-C\min(q_{1,\omega_n}, q_{2,\omega_n})) + C_j\min\{n^{1/4}, n^{1/4}q_{3,\omega_n}^{-1/4}\exp(-Cq_{3,\omega_n})\}.\end{aligned} \tag{107}$$

Suppose $\nu = \ell$. Applying Hölder's inequality and (106), we can choose a sufficiently large number $r > 1$ such that for some constant $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(|W_\theta - 1| \geq \frac{1}{2})\}\\ \leq\ & \{\mathbb{E}\log^{2jr}(W_\theta)\}^{1/r}\{\mathbb{P}(|W_\theta - 1| \geq \frac{1}{2})\}^{(r-1)/r}\\ \leq\ & C_j n^{1/(2r)}\exp\Big(-\frac{C(r-1)}{r}\min(q_{1,\omega_n}, q_{2,\omega_n})\Big)\\ & + C_j\min\{n^{1/(2r)}, n^{1/(2r)}q_{3,\omega_n}^{-(r-1)/(2r)}\exp\Big(-\frac{C(r-1)}{r}q_{3,\omega_n}\Big)\}\\ =\ & C_j n^{1/(2r)}\exp\Big(-\frac{C(r-1)}{r}\log(\frac{n}{\omega_n})\Big)\\ & + C_j n^{1/(2r)}\{\log(\frac{n}{\omega_n})\}^{-(r-1)/r}\exp\Big(-\frac{C(r-1)}{r}\log^2(\frac{n}{\omega_n})\Big)\\ =\ & O(n^{-\varepsilon}),\end{aligned} \tag{108}$$

for all large n.

By mean value theorem, there exists some $c \in (1/2, 3/2)$ such that

$$\begin{aligned}&\mathbb{E}\{\log^{2j}(W_\theta)\mathcal{I}(|W_\theta - 1| < \frac{1}{2})\}\\ \leq\ & \mathbb{E}\{[(W_\theta - 1) - \frac{1}{2c^2}(W_\theta - 1)^2]^{2j}\mathcal{I}(|W_\theta - 1| < \frac{1}{2})\}\\ \leq\ & \mathbb{E}\{[|W_\theta - 1| + 2(W_\theta - 1)^2]^{2j}\mathcal{I}(|W_\theta - 1| < \frac{1}{2})\}\\ \leq\ & \mathbb{E}\{(W_\theta - 1)^{2j}[1 + 2|W_\theta - 1|]^{2j}\mathcal{I}(|W_\theta - 1| < \frac{1}{2})\}\end{aligned}$$

$$\begin{aligned} &\le 2^{2j}\mathbb{E}(W_\theta - 1)^{2j} \\ &\le C_j(q_{1,\omega_n}^{-2j} + q_{2,\omega_n}^{-j} + q_{3,\omega_n}^{-j}). \end{aligned} \tag{109}$$

The last inequality follows from Lemma 4. Then the desired result follows from (107), (108), (109) and the fact that (107) (or (108) if $\nu = \ell$) is negligible compared to (109) as $n \to \infty$. This proves Lemma 5. □

Lemma 6. *Let Ω_ℓ be as in (18). There exist constants $C_1, C_2 > 0$ such that for n sufficiently large,*

$$\mathbb{P}(\Omega_\ell) \le \begin{cases} C_1 \exp\{-C_2(n/\omega_n)^{1/2}\log(n/\omega_n)\} & \text{if } 1 \le \ell \le \nu, \\ C_1 \exp\{-C_2(n/\omega_n)^{2\lceil\nu\rceil-2\nu}\log(n/\omega_n)\} & \text{if } \ell = \lceil\nu\rceil > \nu \ge \lceil\nu\rceil - 1/4, \\ C_1 \exp\{-C_2(n/\omega_n)\} & \text{if } \ell = \lceil\nu\rceil + 1, \nu < \lceil\nu\rceil - 1/4, \\ C_1 \exp\{-C_2(n/\omega_n)\} & \text{if } \ell \in \{0, \lceil\nu\rceil + 2, \ldots, M_n\}. \end{cases}$$

Proof. Case 1. Suppose that $1 \le \ell < \nu$. We observe from Theorem 1(c) and (16) that for sufficiently large n,

$$\begin{aligned} \mathbb{P}(\Omega_\ell) &\le \mathbb{P}\big(n^{-1}V_{1,\ell} \ge (\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\big) \\ &\le \mathbb{P}\big(n^{-1}|V_{1,\ell} - \mathbb{E}V_{1,\ell}| \ge (\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n}) - n^{-1}\mathbb{E}V_{1,\ell}\big) \\ &\le \mathbb{P}\big(n^{-1}|V_{1,\ell} - \mathbb{E}V_{1,\ell}| \ge \frac{1}{2}(\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\big) \\ &\le C_1\exp\{-C_2(\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\}. \end{aligned} \tag{110}$$

Case 2. Suppose $\ell = \nu$. We observe from Theorem 1(b) and (d) that for sufficiently large n,

$$\begin{aligned} \mathbb{P}(\Omega_\ell) &\le \mathbb{P}\Big\{V_{1,\ell} \ge n(\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\Big\} \\ &\le \mathbb{P}\Big\{\Big|\frac{V_{1,\ell}}{\mathbb{E}V_{1,\ell}} - 1\Big| \ge \frac{n(n/\omega_n)^{1/2}\log(n/\omega_n)}{\mathbb{E}V_{1,\ell}} - 1\Big\} \\ &= C_1\exp\{-C_2(\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\}. \end{aligned} \tag{111}$$

Case 3. Suppose $\ell \ge \lceil\nu\rceil + 2$. We observe from Theorem 1(a) that

$$\frac{\mathbb{E}V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} \to 2^{2\lceil\nu\rceil+2-2\nu},$$

as $n \to \infty$. This implies that for sufficiently large n,

$$\frac{\mathbb{E}V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} \ge \frac{2^{2\lceil\nu\rceil+2-2\nu}}{2^{1/2}}.$$

Consequently, it follows from (17) and Theorem 1(d) that there exist constants $C, C_1 > 0$ such that for sufficiently large n,

$$\begin{aligned}
& \mathbb{P}(\hat{\nu}_{n,\lceil\nu\rceil+1} > \lceil\nu\rceil + 3/4) \\
= \;& \mathbb{P}(\frac{\log(V_{2,\lceil\nu\rceil+1}/V_{1,\lceil\nu\rceil+1})}{2\log(2)} > -\frac{1}{4}) \\
= \;& \mathbb{P}(\frac{V_{2,\lceil\nu\rceil+1}}{V_{1,\lceil\nu\rceil+1}} > 2^{-1/2}) \\
= \;& \mathbb{P}(\frac{V_{2,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} > 2^{-1/2}\frac{\mathbb{E}V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}}\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}}) \\
\leq \;& \mathbb{P}(\frac{V_{2,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} > 2^{-1/2}\frac{\mathbb{E}V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}}\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}}, \frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} > \frac{2}{3}) \\
& + \mathbb{P}(\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} \leq \frac{2}{3}) \\
\leq \;& \mathbb{P}(\frac{V_{2,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} > \frac{2}{3}2^{-1/2}\frac{\mathbb{E}V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}}) + \mathbb{P}(\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} \leq \frac{2}{3}) \\
\leq \;& \mathbb{P}(\frac{V_{2,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} > \frac{2}{3}2^{2\lceil\nu\rceil+1-2\nu}) + \mathbb{P}(\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} \leq \frac{2}{3}) \\
\leq \;& \mathbb{P}(\frac{V_{2,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} > \frac{4}{3}) + \mathbb{P}(\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} \leq \frac{2}{3}) \\
\leq \;& \mathbb{P}(\Big|\frac{V_{2,\lceil\nu\rceil+1}}{\mathbb{E}V_{2,\lceil\nu\rceil+1}} - 1\Big| > \frac{1}{3}) + \mathbb{P}(\Big|\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} - 1\Big| \geq \frac{1}{3}) \\
\leq \;& C\exp\{-C_1(n/\omega_n)\}.
\end{aligned} \tag{112}$$

It follows from (18), (112) and Theorem 1(a) that there exist constants $C, C_1, C_2 > 0$ such that for sufficiently large n,

$$\begin{aligned}
\mathbb{P}(\Omega_\ell) \leq \;& \mathbb{P}(\Theta^c_{\lceil\nu\rceil+1}) \\
\leq \;& \mathbb{P}(\hat{\nu}_{n,\lceil\nu\rceil+1} > \lceil\nu\rceil + 3/4) + \mathbb{P}\big(n^{-1}V_{1,\lceil\nu\rceil+1} < (\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\big) \\
= \;& \mathbb{P}(\hat{\nu}_{n,\lceil\nu\rceil+1} > \lceil\nu\rceil + 3/4) \\
& + \mathbb{P}\Big\{\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} - 1 < \frac{n(n/\omega_n)^{1/2}\log(n/\omega_n)}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} - 1\Big\} \\
\leq \;& C\exp\{-C_1(\frac{n}{\omega_n})\} \\
& + \mathbb{P}\Big\{\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} - 1 < C_2\frac{n(n/\omega_n)^{1/2}\log(n/\omega_n)}{n(n/\omega_n)^{2\lceil\nu\rceil+2-2\nu}} - 1\Big\} \\
\leq \;& C\exp\{-C_1(\frac{n}{\omega_n})\} \\
& + \mathbb{P}\Big\{\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} - 1 < C_2(\frac{n}{\omega_n})^{2\nu-2\lceil\nu\rceil-3/2}\log(\frac{n}{\omega_n}) - 1\Big\}
\end{aligned}$$

$$\begin{aligned} &\leq C\exp\{-C_1(\frac{n}{\omega_n})\} + \mathbb{P}(|\frac{V_{1,\lceil\nu\rceil+1}}{\mathbb{E}V_{1,\lceil\nu\rceil+1}} - 1| > \frac{1}{2}) \\ &\leq C_1\exp\{-C_2(\frac{n}{\omega_n})\}. \end{aligned} \tag{113}$$

Case 4. Suppose $\ell = 0$. The bound (113) also applies to $\mathbb{P}(\Omega_0)$ because

$$\mathbb{P}(\Omega_0) = \mathbb{P}(\bigcap_{l=1}^{M_n}\Theta_l^c) \leq \mathbb{P}(\Theta^c_{\lceil\nu\rceil+1}). \tag{114}$$

Case 5. Suppose $\ell = \lceil\nu\rceil + 1$ and $\nu < \lceil\nu\rceil - 1/4$. We observe that

$$\begin{aligned} \mathbb{P}(\Omega_{\lceil\nu\rceil+1}) &\leq \mathbb{P}(\Theta^c_{\lceil\nu\rceil}) \qquad (115) \\ &\leq \mathbb{P}(\hat{\nu}_{n,\lceil\nu\rceil} > \lceil\nu\rceil - \frac{1}{4}) + \mathbb{P}\big(n^{-1}V_{1,\lceil\nu\rceil} < (\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\big). \end{aligned}$$

It follows from (16) and Theorem 1(a) that there exist constants $C, C_1, C_2 > 0$ such that for sufficiently large n,

$$\begin{aligned} &\mathbb{P}\big(n^{-1}V_{1,\lceil\nu\rceil} < (\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\big) \\ \leq\ &\mathbb{P}(\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1 < \frac{n(n/\omega_n)^{1/2}\log(n/\omega_n)}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1) \\ \leq\ &\mathbb{P}(|\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1| > 1 - \frac{n^{3/2}\log n}{\omega_n^{1/2}\mathbb{E}V_{1,\lceil\nu\rceil}}) \\ \leq\ &\mathbb{P}\big(|\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1| > 1 - C(\frac{n}{\omega_n})^{2\nu-2\lceil\nu\rceil+1/2}\log(\frac{n}{\omega_n})\big) \\ \leq\ &\mathbb{P}(|\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1| \geq \frac{1}{2}) \\ \leq\ &C_1\exp\big(-C_2(n/\omega_n)\big). \end{aligned} \tag{116}$$

We observe from Theorem 1(a) that

$$\frac{\mathbb{E}V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} \to 2^{2\lceil\nu\rceil-2\nu},$$

as $n \to \infty$. Writing $\delta = 2\lceil\nu\rceil - 2\nu - 1/2$, we have $\delta > 0$ and

$$\frac{\mathbb{E}V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} \geq 2^{\delta/2+1/2},$$

for sufficiently large n. Next we observe from (17) and Theorem 1(d) that there exist constants $C, C_1 > 0$ such that for sufficiently large n,

$$\mathbb{P}(\hat{\nu}_{n,\lceil\nu\rceil} > \lceil\nu\rceil - \frac{1}{4})$$

$$
\begin{aligned}
&= \mathbb{P}(\frac{\log(V_{2,\lceil\nu\rceil}/V_{1,\lceil\nu\rceil})}{2\log(2)} > -\frac{1}{4}) \\
&= \mathbb{P}(\frac{V_{2,\lceil\nu\rceil}}{V_{1,\lceil\nu\rceil}} > 2^{-1/2}) \\
&= \mathbb{P}(\frac{V_{2,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} > 2^{-1/2}\frac{\mathbb{E}V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}}\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}}) \\
&\le \mathbb{P}(\frac{V_{2,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} > 2^{-1/2}\frac{\mathbb{E}V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}}\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}}, \frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} > 2^{-\delta/4}) \\
&\quad +\mathbb{P}(\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} \le 2^{-\delta/4}) \\
&\le \mathbb{P}(\frac{V_{2,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} > 2^{-\delta/4}2^{-1/2}\frac{\mathbb{E}V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}}) + \mathbb{P}(\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} \le 2^{-\delta/4}) \\
&\le \mathbb{P}(\frac{V_{2,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} > 2^{-\delta/4}2^{\delta/2}) + \mathbb{P}(\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} \le 2^{-\delta/4}) \\
&\le \mathbb{P}(\frac{V_{2,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} > 2^{\delta/4}) + \mathbb{P}(\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} \le 2^{-\delta/4}) \\
&\le \mathbb{P}(\Big|\frac{V_{2,\lceil\nu\rceil}}{\mathbb{E}V_{2,\lceil\nu\rceil}} - 1\Big| > 2^{\delta/4} - 1) + \mathbb{P}(\Big|\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1\Big| \ge 1 - 2^{-\delta/4}) \\
&\le C\exp(-C_1(n/\omega_n)). \qquad (117)
\end{aligned}
$$

It follows from (115), (116) and (117) that for sufficiently large n,

$$\mathbb{P}(\Omega_{\lceil\nu\rceil+1}) \le C\exp(-C_1(n/\omega_n)). \qquad (118)$$

Case 6. Suppose $\ell = \lceil\nu\rceil > \nu \ge \lceil\nu\rceil - 1/4$. We have from (18) and Theorem 1(b) that there exist constants $C, C_1, C_2 > 0$ such that for sufficiently large n,

$$
\begin{aligned}
\mathbb{P}(\Omega_{\lceil\nu\rceil}) &\le \mathbb{P}\Big\{n^{-1}V_{1,\lceil\nu\rceil} \ge (\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\Big\} \\
&= \mathbb{P}\Big\{\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1 \ge \frac{n(n/\omega_n)^{1/2}\log(n/\omega_n)}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1\Big\} \\
&\le \mathbb{P}\Big\{|\frac{V_{1,\lceil\nu\rceil}}{\mathbb{E}V_{1,\lceil\nu\rceil}} - 1| \ge C\log(\frac{n}{\omega_n})\Big\} \\
&\le C_1\exp\{-C_2(n/\omega_n)^{2\lceil\nu\rceil-2\nu}\log(n/\omega_n)\}. \qquad (119)
\end{aligned}
$$

Lemma 6 follows immediately from (110), (111), (113), (114), (118) and (119). □

Proof of Theorem 2

Define

$$T(\nu^*) = \frac{V_{1,\ell}2^{2\nu^*-2\ell}}{V_{2,\ell}} - 1, \qquad \forall\nu^* \ge 0.$$

Then $T(\nu^*) > -1$ almost surely for every n and $\hat{\nu}_{n,\ell}$ satisfies that

$$\hat{\nu}_{n,\ell} = \frac{1}{2}\log_2\Big[\frac{V_{2,\ell}}{V_{1,\ell}}\{T(\hat{\nu}_{n,\ell})+1\}\Big] + \ell. \tag{120}$$

We observe from Theorem 1 and the definition of $\hat{\nu}_{n,\ell}$ that as $n \to \infty$,

$$V_{2,\ell}/V_{1,\ell} \to \begin{cases} 2^{2\nu-2\ell} \text{ almost surely} & \text{if } \nu < \ell, \\ 1 \text{ in probability} & \text{if } \nu = \ell, \\ 1 \text{ almost surely} & \text{if } \nu > \ell, \end{cases}$$

and

$$|T(\hat{\nu}_{n,\ell})| \le \min\{|T(\nu)|, |T(\ell)|\} \to \begin{cases} 0 \text{ almost surely} & \text{if } \nu \neq \ell, \\ 0 \text{ in probability} & \text{if } \nu = \ell. \end{cases}$$

It then follows from (120) and continuous mapping theorem (cf. Theorem 2.3 of [28]) that as $n \to \infty$

$$\hat{\nu}_{n,\ell} \to \begin{cases} \nu \text{ almost surely} & \text{if } \nu < \ell, \\ \ell \text{ in probability} & \text{if } \nu = \ell, \\ \ell \text{ almost surely} & \text{if } \nu > \ell. \end{cases}$$

Now we proceed to evaluate the convergence rates. Let

$$\hat{\nu}^*_{n,\ell} = \ell + \frac{\log(V_{2,\ell}/V_{1,\ell})}{2\log(2)}.$$

We observe that

$$\hat{\nu}_{n,\ell} = \begin{cases} 0 & \text{if } \hat{\nu}^*_{n,\ell} < 0, \\ \hat{\nu}^*_{n,\ell} & \text{if } \hat{\nu}^*_{n,\ell} \ge 0, \end{cases}$$

and hence

$$\mathbb{E}\{(\hat{\nu}_{n,\ell} - \nu)^2\} \le \mathbb{E}\{(\hat{\nu}^*_{n,\ell} - \nu)^2\}.$$

Define

$$\delta = \log(\frac{\mathbb{E}V_{2,\ell}}{\mathbb{E}V_{1,\ell}}) - 2(\nu - \ell)\log(2).$$

CASE 1. Suppose $\nu \le \ell$. It follows from Theorem 1(a) and (b) that

$$\delta = \begin{cases} O(\omega_n^{-1}) + O\big((\omega_n/n)^{1\wedge(2\ell-2\nu)}\big) & \text{if } \ell > \nu \notin \mathbb{Z}_+, \\ O(\omega_n^{-1}) + O\big((\omega_n/n)\log(n)\big) & \text{if } \ell > \nu \in \mathbb{Z}_+, \\ O\big(\log^{-1}(n)\big) & \text{if } \ell = \nu, \end{cases} \tag{121}$$

as $n \to \infty$. Writing $W_\theta = V_{\theta,\ell}/\mathbb{E}V_{\theta,\ell}$, it follows from (121) and Lemma 5 that

$$\mathbb{E}\{(\hat{\nu}^*_{n,\ell} - \nu)^2\} \quad = \quad \mathbb{E}\Big(\ell + \frac{\log(V_{2,\ell}) - \log(V_{1,\ell})}{2\log 2} - \nu\Big)^2$$

$$
\begin{aligned}
&= \mathbb{E}\Big(\ell-\nu+\frac{\log(W_2)-\log(W_1)}{2\log 2}+\frac{1}{2\log 2}\log(\frac{\mathbb{E}V_{2,\ell}}{\mathbb{E}V_{1,\ell}})\Big)^2\\
&= \mathbb{E}\Big(\frac{\log(W_2)-\log(W_1)}{2\log 2}+\frac{\delta}{2\log 2}\Big)^2\\
&\le \frac{\mathbb{E}\{\log(W_2)+\log(W_1)\}^2}{2\log^2(2)}+\frac{\delta^2}{2\log^2(2)}\\
&\le \frac{\mathbb{E}\log^2(W_2)+\mathbb{E}\log^2(W_1)}{\log^2(2)}+\frac{\delta^2}{2\log^2(2)}\\
&\le C_1(q_{1,\omega_n}^{-2}+q_{2,\omega_n}^{-1}+q_{3,\omega_n}^{-1})+\frac{\delta^2}{2\log^2(2)}\\
&= \begin{cases} O(\omega_n^{-2})+O(\omega_n/n) & \text{if } \nu<\ell-1/4,\\ O(\omega_n^{-2})+O\big(\omega_n n^{-1}\log(\omega_n/n)\big) & \text{if } \nu=\ell-1/4,\\ O(\omega_n^{-2})+O\big((\omega_n/n)^{4\ell-4\nu}\big) & \text{if } \nu\in(\ell-1/4,\ell),\\ O\big(\log^{-2}(n)\big) & \text{if } \nu=\ell,\end{cases}
\end{aligned}
$$

as $n\to\infty$ since $N\ge 2\nu+6$ (Condition 1).

CASE 2. Suppose $\nu>\ell$. We observe that for $t>0$

$$
\begin{aligned}
&\mathbb{P}(|\log(\frac{V_{2,\ell}}{V_{1,\ell}})|\ge t)\\
=\;& \mathbb{P}(\log(\frac{V_{2,\ell}}{V_{1,\ell}})\ge t)+\mathbb{P}(\log(\frac{V_{2,\ell}}{V_{1,\ell}})\le -t)\\
=\;& \mathbb{P}(\frac{V_{2,\ell}}{V_{1,\ell}}\ge e^t)+\mathbb{P}(\frac{V_{2,\ell}}{V_{1,\ell}}\le e^{-t})\\
=\;& \mathbb{P}(\frac{V_{2,\ell}}{V_{1,\ell}}\ge e^t,\frac{V_{1,\ell}}{n}\le e^{-t/2})+\mathbb{P}(\frac{V_{2,\ell}}{V_{1,\ell}}\ge e^t,\frac{V_{1,\ell}}{n}> e^{-t/2})\\
&+\mathbb{P}(\frac{V_{1,\ell}}{V_{2,\ell}}\ge e^t,\frac{V_{2,\ell}}{n}\le e^{-t/2})+\mathbb{P}(\frac{V_{1,\ell}}{V_{2,\ell}}\ge e^t,\frac{V_{2,\ell}}{n}> e^{-t/2})\\
\le\;& \mathbb{P}(\frac{V_{1,\ell}}{n}\le e^{-t/2})+\mathbb{P}(\frac{V_{2,\ell}}{n}\ge e^{t/2})\\
&+\mathbb{P}(\frac{V_{2,\ell}}{n}\le e^{-t/2})+\mathbb{P}(\frac{V_{1,\ell}}{n}\ge e^{t/2}).
\end{aligned}
\tag{122}
$$

It follows from (16) that there exist constants $c, C_1, C_2>0$ such that for $\theta\in\{1,2\}$, $\ell<\nu$ and any $t>0$

$$
\begin{aligned}
&\mathbb{P}(n^{-1}V_{\theta,\ell}\ge e^{t/2})\\
=\;& \mathbb{P}(n^{-1}(V_{\theta,\ell}-\mathbb{E}V_{\theta,\ell})\ge e^{t/2}-n^{-1}\mathbb{E}V_{\theta,\ell})\\
\le\;& \mathbb{P}(n^{-1}|V_{\theta,\ell}-\mathbb{E}V_{\theta,\ell}|\ge e^{t/2}-c)\\
\le\;& C_1\exp(-C_2\min\{0\vee(e^{t/2}-c),(0\vee(e^{t/2}-c))^2\}).
\end{aligned}
\tag{123}
$$

Next, we derive the bound for $\mathbb{P}(n^{-1}V_{\theta,\ell} \le e^{-t/2})$. Recall the notation

$$Y = \Big(\frac{\nabla_{\theta,\ell}X_1}{\sqrt{n}}, \dots, \frac{\nabla_{\theta,\ell}X_{n-\omega_n\theta\ell}}{\sqrt{n}}\Big), \tag{124}$$

$$n^{-1}V_{\theta,\ell} = Y'Y = Z'\Sigma Z + Z'\Sigma^{1/2}\mu + \mu'\Sigma Z + \mu'\mu, \tag{125}$$

where $\mu = \mathbb{E}Y$, $\Sigma = (\Sigma_{ij}) = \mathbb{E}\{(Y-\mu)(Y-\mu)'\}$ and $Z \sim N_{n-\omega_n\theta\ell}(0, I)$. From (124) and (125), we obtain

$$\mathbb{P}(n^{-1}V_{\theta,\ell} \le e^{-t/2}) = \mathbb{P}((\Sigma^{1/2}Z+\mu)'(\Sigma^{1/2}Z+\mu) \le e^{-t/2}).$$

Suppose the orthogonal decomposition of Σ is given as $H\Lambda H'$ where H and Λ are the matrices of eigenvectors and eigenvalues respectively. Then we have

$$(\Sigma^{1/2}Z+\mu)'(\Sigma^{1/2}Z+\mu) = (\Lambda^{1/2}H'Z+H'\mu)'(\Lambda^{1/2}H'Z+H'\mu).$$

Since $H'Z$ and Z have the same distribution, we may rewrite $H'Z$, $H'\mu$ as Z, μ respectively. Denoting the set of eigenvalues of Σ in descending order as $(\lambda_1, \dots, \lambda_{n-\omega_n\theta\ell})$, we have that

$$(\Lambda^{1/2}Z+\mu)'(\Lambda^{1/2}Z+\mu) = \sum_{i=1}^{n-\omega_n\theta\ell} (\lambda_i^{1/2}Z_i + \mu_i)^2,$$

where Z_i and μ_i are the i-th component of the vectors Z and μ respectively for $i = 1, \dots, n-\omega_n\theta\ell$. We observe from Theorem 1(c) that

$$\sum_{i=1}^{n-\omega_n\theta\ell} (\lambda_i + \mu_i^2) \asymp 1,$$

$$2\sum_{i=1}^{n-\omega_n\theta\ell} (\lambda_i^2 + 2\lambda_i\mu_i^2) \asymp 1. \tag{126}$$

Next we shall show that

$$\liminf_{n\to\infty} \lambda_1 > 0. \tag{127}$$

Suppose (127) does not hold. In the following, we write λ_i as $\lambda_{i,n}$ and μ_i as $\mu_{i,n}$ to exhibit the explicit dependence of λ_i and μ_i on n. We let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \lambda_{1,n_k} \to 0$. Then it follows that

$$\sum_{i=1}^{n_k-\theta\ell} (\lambda_{i,n_k}^2 + 2\lambda_{i,n_k}\mu_{i,n_k}^2) \le 2\lambda_{1,n_k} \sum_{i=1}^{n_k-\theta\ell} (\lambda_{i,n_k} + \mu_{i,n_k}^2) \to 0$$

as $k \to \infty$, which contradicts with (126). Using (127), we observe that for large n,

$$\mathbb{P}(n^{-1}V_{\theta,\ell} \le e^{-t/2}) = \mathbb{P}\{(\Lambda^{1/2}Z+\mu)'(\Lambda^{1/2}Z+\mu) \le e^{-t/2}\}$$

$$\begin{aligned} &\leq \mathbb{P}\{(\lambda_1^{1/2} Z_1 + \mu_1)^2 \leq e^{-t/2}\} \\ &\leq \mathbb{P}(-e^{-t/4} \leq \lambda_1^{1/2} Z_1 + \mu_1 \leq e^{-t/4}) \\ &\leq Ce^{-t/4}. \end{aligned} \tag{128}$$

Consequently, it follows from (122), (123) and (128) that

$$\begin{aligned} &\mathbb{E}\log^2(\frac{V_{2,\ell}}{V_{1,\ell}}) \\ = \;& \int_0^\infty \mathbb{P}(\log^2(\frac{V_{2,\ell}}{V_{1,\ell}}) \geq t)dt \\ \leq \;& C\int_0^\infty \Big\{e^{-\sqrt{t}/4} \\ &+C_1 \exp\Big(-C_2\min\{0\vee(e^{\sqrt{t}/2}-c),(0\vee(e^{\sqrt{t}/2}-c))^2\}\Big)\Big\}dt = O(1). \end{aligned} \tag{129}$$

Finally it follows from (129) that

$$\mathbb{E}(\hat{\nu}_{n,\ell}-\nu)^2 \leq \mathbb{E}(\hat{\nu}^*_{n,\ell}-\nu)^2 = \mathbb{E}\Big(\ell-\nu+\frac{1}{2\log 2}\log(\frac{V_{2,\ell}}{V_{1,\ell}})\Big)^2 = O(1),$$

and the proof of Theorem 2 is complete. □

Proof of Theorem 3

CASE 1. Suppose $\nu \geq \lceil\nu\rceil - 1/4$. We observe that

$$\begin{aligned} &\mathbb{E}|\hat{\nu}_n-\nu| \\ = \;& \sum_{\ell=0}^{M_n}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \\ = \;& \mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_0\}) + \sum_{\ell=1}^{\lceil\nu\rceil}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \\ &+ \sum_{\ell=\lceil\nu\rceil+1}^{M_n}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \\ \leq \;& \mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_0\}) + \mathbb{E}(|\hat{\nu}_{n,\lceil\nu\rceil+1}-\nu|) \\ &+ \sum_{\ell=1}^{\lceil\nu\rceil}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) + \sum_{\ell=\lceil\nu\rceil+2}^{M_n}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \end{aligned} \tag{130}$$

It follows from Lemma 6 that there exist some constants $C_1, C_2, C_3 > 0$ such that

$$\sum_{\ell=\lceil\nu\rceil+2}^{M_n}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \leq \sum_{\ell=\lceil\nu\rceil+2}^{M_n}\sqrt{\mathbb{E}[(\hat{\nu}_{n,\ell_0}-\nu)^2\mathcal{I}\{\Omega_\ell\}]}\sqrt{\mathbb{P}(\Omega_\ell)}$$

$$
\begin{aligned}
&\leq \sum_{\ell=\lceil\nu\rceil+2}^{M_n} C_1[(\ell-\frac{1}{4}-\nu)\vee\nu]\exp\{-C_2(n/\omega_n)\}. \\
&\leq C_3M_n^2\exp\{-C_2(n/\omega_n)\}. \qquad (131)
\end{aligned}
$$

Using Lemma 6 and Theorem 2, we have

$$
\begin{aligned}
&\sum_{\ell=1}^{\lceil\nu\rceil}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \\
\leq\ &\sum_{\ell=1}^{\lceil\nu\rceil}\sqrt{\mathbb{E}(\hat{\nu}_{n,\ell}-\nu)^2}\sqrt{\mathbb{P}(\Omega_\ell)} \\
\leq\ &C_1\exp(-C_2(\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n}))+C_1(\frac{\omega_n}{n})^{1/2}\log^{-1}(\frac{n}{\omega_n})\mathcal{I}\{\lceil\nu\rceil=\nu\} \\
&+C_1\exp(-C_2(\frac{n}{\omega_n})^{2\lceil\nu\rceil-2\nu}\log(\frac{n}{\omega_n}))\mathcal{I}\{\lceil\nu\rceil>\nu\}. \qquad (132)
\end{aligned}
$$

Using Lemma 6 again, we have for sufficiently large n,

$$
\begin{aligned}
\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_0\}) &\leq \sqrt{\mathbb{E}[(\hat{\nu}_{n,\ell_0}-\nu)^2\mathcal{I}\{\Omega_0\}]}\sqrt{\mathbb{P}(\Omega_0)} \\
&\leq (M_n+\nu)C_1\exp\{-C_2(n/\omega_n)\}. \qquad (133)
\end{aligned}
$$

We conclude from (130), (131), (132), (133) and Theorem 2 that as $n\to\infty$,

$$
\mathbb{E}|\hat{\nu}_n-\nu| = O\big((\frac{\omega_n}{n})^{1/2}+O(\omega_n^{-1})\big).
$$

Case 2. Suppose $\nu<\lceil\nu\rceil-1/4$. We observe that

$$
\begin{aligned}
&\mathbb{E}|\hat{\nu}_n-\nu| \\
=\ &\sum_{\ell=0}^{M_n}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \\
=\ &\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_0\})+\sum_{\ell=1}^{\lceil\nu\rceil}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \\
&+\sum_{\ell=\lceil\nu\rceil+1}^{M_n}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}) \\
\leq\ &\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_0\})+\mathbb{E}(|\hat{\nu}_{n,\lceil\nu\rceil}-\nu|) \\
&+\sum_{\ell=1}^{\lceil\nu\rceil-1}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\})+\sum_{\ell=\lceil\nu\rceil+1}^{M_n}\mathbb{E}(|\hat{\nu}_{n,\ell_0}-\nu|\mathcal{I}\{\Omega_\ell\}),
\end{aligned}
$$

and the result follows similarly. Finally, we shall prove the almost sure convergence of $\hat{\nu}_n$. We claim that as $n\to\infty$, almost surely, the index ℓ_0 equals either

$\lceil\nu\rceil$ when $\nu < \lceil\nu\rceil - 1/4$ or $\lceil\nu\rceil + 1$ when $\nu \geq \lceil\nu\rceil - 1/4$. Then the desired result follows from Theorem 2. We consider the case when $\nu \geq \lceil\nu\rceil - 1/4$. To verify the latter, it suffices to prove that

$$\mathbb{P}\Big(\bigcup_{j=1}^{\infty}\bigcap_{n=j}^{\infty}\{\ell_0 = \lceil\nu\rceil + 1\}\Big) = 1,$$

or equivalently,

$$\mathbb{P}\Big(\bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{\ell\neq\lceil\nu\rceil+1}\Omega_\ell\Big) = \mathbb{P}\Big(\bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty}\{\ell_0 \neq \lceil\nu\rceil + 1\}\Big) = 0,$$

since $\{\ell_0 \neq \lceil\nu\rceil + 1\} = \Omega^c_{\lceil\nu\rceil+1} = \bigcup_{\ell\neq\lceil\nu\rceil+1}\Omega_\ell$. We observe that $\bigcup_{n=j}^{\infty}\bigcup_{\ell\neq\lceil\nu\rceil+1}\Omega_\ell$ is a decreasing sequence of sets as j increases. Consequently using Lemma 6, we obtain

$$\begin{aligned}
\mathbb{P}\Big(\bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{\ell\neq\lceil\nu\rceil+1}\Omega_\ell\Big) &\leq \lim_{j\to\infty}\sum_{n=r}^{\infty}\sum_{\substack{0\leq\ell\leq M_n\\ \ell\neq\lceil\nu\rceil+1}}\mathbb{P}(\Omega_\ell) \\
&\leq \lim_{j\to\infty}\sum_{n=j}^{\infty}\Big\{C_1\exp\{-C_2(\frac{n}{\omega_n})^{1/2}\log(\frac{n}{\omega_n})\} \\
&\quad + C_1\log(\frac{n}{\omega_n})\exp\{-C_2(\frac{n}{\omega_n})\}\Big\} = 0.
\end{aligned}$$

The case $\nu < \lceil\nu\rceil - 1/4$ follows in a similar manner. □

Proof of Theorem 4

We observe that

$$\begin{aligned}
\mathbb{E}|\hat{\nu}^R_{n_{\hat\tau}} - \nu| &= \mathbb{E}[|\hat{\nu}^R_{n_{\hat\tau}} - \nu|\mathcal{I}(\hat\tau = 1)] + \mathbb{E}[|\hat{\nu}^R_{n_{\hat\tau}} - \nu|\mathcal{I}(\hat\tau \neq 1)] \\
&= \mathbb{E}[|\hat{\nu}^R_{n_{\hat\tau}} - \nu|\big|\hat\tau = 1]\mathbb{P}(\hat\tau = 1) + \mathbb{E}[|\hat{\nu}^R_{n_{\hat\tau}} - \nu|\big|\hat\tau \neq 1]\mathbb{P}(\hat\tau \neq 1) \\
&\leq O\{(\frac{\omega_{n_1}}{n_1})^{1/2} + \omega_{n_1}^{-1}\} + (M_n + \nu)\mathbb{P}(\hat\tau \neq 1) \\
&\leq O\{(\frac{\omega_{n_1}}{n_1})^{1/2} + \omega_{n_1}^{-1}\} + (M_n + \nu)\frac{n^{1-p_0L\log(n)}}{\log^2(n)} \\
&= O\{(\frac{\omega_{n_1}}{n_1})^{1/2} + \omega_{n_1}^{-1}\},
\end{aligned}$$

as $n \to \infty$. □

Appendix E: Proof that $H_\ell(\nu)$ is non-zero

Let $H_\ell(\nu)$, with $\nu < \ell$, be as in Part (a) of Theorem 1. This section proves that $H_\ell(\nu) \neq 0$. Consider the function $F : (0,\infty) \to \mathbb{R}$ where

$$F(x) = (-1)^{\lfloor\nu\rfloor+1}G_\nu(\sqrt{x}) = \begin{cases}(-1)^{\lfloor\nu\rfloor+1}x^\nu & \text{if } \nu \notin \mathbb{Z}_+,\\ \frac{1}{2}(-1)^{\nu+1}x^\nu\log(x) & \text{if } \nu \in \mathbb{Z}_+.\end{cases}$$

Suppose $\nu \notin \mathbb{Z}_+$. We observe that

$$(-1)^{\lfloor\nu\rfloor+1}F^{(\lfloor\nu\rfloor+1)}(x) = \Big\{\prod_{u=0}^{\lfloor\nu\rfloor}(\nu-u)\Big\}x^{\nu-\lfloor\nu\rfloor-1}.$$

Suppose $\nu \in \mathbb{Z}_+$. We observe that

$$\begin{aligned}(-1)^{\nu+1}F^{(\nu+1)}(x) &= \frac{1}{2}(-1)^{2\nu+2}\sum_{j=0}^{\nu+1}\binom{\nu+1}{j}\frac{d^{\nu+1-j}}{dx^{\nu+1-j}}(x^\nu)\frac{d^j}{dx^j}(\log x)\\ &= \frac{1}{2}(-1)^{2\nu+2}\sum_{j=1}^{\nu+1}\binom{\nu+1}{j}\frac{\nu!}{(j-1)!}x^{j-1}(-1)^{j-1}(j-1)!x^{-j}\\ &= \frac{1}{2}(-1)^{2\nu+1}\nu!x^{-1}\sum_{j=1}^{\nu+1}\binom{\nu+1}{j}(-1)^j\\ &= \frac{1}{2}(-1)^{2\nu+1}\nu!x^{-1}\Big\{\sum_{j=0}^{\nu+1}\binom{\nu+1}{j}(-1)^j-1\Big\}\\ &= \frac{\nu!}{2x}.\end{aligned}$$

We define

$$\rho(t) = \begin{cases}\frac{\prod_{u=0}^{\lfloor\nu\rfloor}(\nu-u)}{\Gamma(\lfloor\nu\rfloor-\nu+1)}t^{\lfloor\nu\rfloor-\nu} & \text{if } \nu\notin\mathbb{Z}_+,\\ \frac{\nu!}{2} & \text{if } \nu\in\mathbb{Z}_+.\end{cases}$$

It can be readily verified by elementary calculus that for all $x > 0$

$$(-1)^{\lfloor\nu\rfloor+1}F^{(\lfloor\nu\rfloor+1)}(x) = \int_0^\infty e^{-xt}\rho(t)dt. \tag{134}$$

For some small number $\varepsilon > 0$, denote $F_\varepsilon(x) = F(x+\varepsilon)$. Then applying Taylor expansion, integration by parts and (134), we obtain

$$\begin{aligned}&F_\varepsilon(x) - \sum_{l=0}^{\lfloor\nu\rfloor}\frac{F_\varepsilon^{(l)}(0)}{l!}x^l\\ &= \frac{1}{\lfloor\nu\rfloor!}\int_0^x(x-s)^{\lfloor\nu\rfloor}F_\varepsilon^{(\lfloor\nu\rfloor+1)}(s)ds\\ &= \frac{1}{\lfloor\nu\rfloor!}\int_0^x(x-s)^{\lfloor\nu\rfloor}F^{(\lfloor\nu\rfloor+1)}(s+\varepsilon)ds\\ &= \frac{1}{\lfloor\nu\rfloor!}\int_0^x(x-s)^{\lfloor\nu\rfloor}(-1)^{\lfloor\nu\rfloor+1}\int_0^\infty e^{-(s+\varepsilon)t}\rho(t)dtds\\ &= \frac{(-1)^{\lfloor\nu\rfloor+1}}{\lfloor\nu\rfloor!}\int_0^\infty e^{-(x+\varepsilon)t}\int_0^x s^{\lfloor\nu\rfloor}e^{st}\rho(t)dsdt\\ &= \frac{(-1)^{\lfloor\nu\rfloor+1}}{\lfloor\nu\rfloor!}\int_0^\infty\frac{e^{-(x+\varepsilon)t}}{t^{\lfloor\nu\rfloor+1}}\int_0^{xt}s^{\lfloor\nu\rfloor}e^s\rho(t)dsdt\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{\lfloor\nu\rfloor+1}}{\lfloor\nu\rfloor!}\int_0^\infty \frac{e^{-(x+\varepsilon)t}}{t^{\lfloor\nu\rfloor+1}} \\
&\quad \times\Big\{(-1)^{\lfloor\nu\rfloor+1}\lfloor\nu\rfloor! + e^{xt}\sum_{j=0}^{\lfloor\nu\rfloor}\frac{(-1)^j\lfloor\nu\rfloor!}{(\lfloor\nu\rfloor-j)!}(xt)^{\lfloor\nu\rfloor-j}\Big\}\rho(t)dt \\
&= \int_0^\infty \frac{e^{-(x+\varepsilon)t}}{t^{\lfloor\nu\rfloor+1}}\Big\{1 + e^{xt}\sum_{j=0}^{\lfloor\nu\rfloor}\frac{(-1)^{\lfloor\nu\rfloor+1-j}}{(\lfloor\nu\rfloor-j)!}(xt)^{\lfloor\nu\rfloor-j}\Big\}\rho(t)dt \\
&= \int_0^\infty \frac{e^{-\varepsilon t}}{t^{\lfloor\nu\rfloor+1}}\Big\{e^{-xt} - \sum_{j=0}^{\lfloor\nu\rfloor}\frac{(-xt)^{\lfloor\nu\rfloor-j}}{(\lfloor\nu\rfloor-j)!}\Big\}\rho(t)dt \\
&= \int_0^\infty \frac{e^{-\varepsilon t}}{t^{\lfloor\nu\rfloor+1}}\Big\{e^{-xt} - \sum_{j=0}^{\lfloor\nu\rfloor}\frac{(-xt)^j}{j!}\Big\}\rho(t)dt. \qquad (135)
\end{aligned}
$$

Define $\imath = \sqrt{-1}$ and

$$
c_k = \frac{\ell!}{\prod_{0\le j\le\ell; j\ne k}(k-j)}, \qquad \forall k = 0,\dots,\ell.
$$

Using (135), Fourier representation of $e^{-(k_2-k_1)^2 t}$ and Lemma 1, we have

$$
\begin{aligned}
&\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} c_{k_1}c_{k_2}F_\varepsilon\{(k_2-k_1)^2\} \\
&= \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} c_{k_1}c_{k_2}\sum_{j=0}^{\lfloor\nu\rfloor}\frac{F_\varepsilon^{(j)}(0)}{j!}(k_2-k_1)^{2j} \\
&\quad + \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} c_{k_1}c_{k_2}\int_0^\infty \frac{e^{-\varepsilon t}}{t^{\lfloor\nu\rfloor+1}}\Big\{e^{-(k_2-k_1)^2 t} - \sum_{j=0}^{\lfloor\nu\rfloor}\frac{\{-(k_2-k_1)^2 t\}^j}{j!}\Big\}\rho(t)dt \\
&= \int_0^\infty \frac{e^{-\varepsilon t}}{t^{\lfloor\nu\rfloor+1}}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} c_{k_1}c_{k_2}e^{-(k_2-k_1)^2 t}\rho(t)dt \\
&= \int_0^\infty \frac{e^{-\varepsilon t}}{t^{\lfloor\nu\rfloor+1}}\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} c_{k_1}c_{k_2}e^{\imath k_1 y\sqrt{2t}}e^{-\imath k_2 y\sqrt{2t}}e^{-y^2/2}dy\Big)\rho(t)dt \\
&= \int_0^\infty \frac{e^{-\varepsilon t}}{t^{\lfloor\nu\rfloor+1}}\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty |\sum_{k=0}^{\ell} c_k e^{\imath k y\sqrt{2t}}|^2 e^{-y^2/2}dy\Big)\rho(t)dt > 0.
\end{aligned}
$$

The last inequality follows from the observation for any $t > 0$ the integrand of the inner integral is nonnegative and not constantly 0 which implies that the inner integral, as a continuous function of t, is always strictly positive for all $t > 0$.

Letting $\varepsilon \to 0^+$, it follows from monotone convergence theorem that

$$\begin{aligned}&\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} c_{k_1}c_{k_2}F\{(k_2-k_1)^2\}\\ &= \int_0^\infty \frac{1}{t^{\lfloor\nu\rfloor+1}}\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\sum_{k=0}^{\ell} c_k e^{\imath ky\sqrt{2t}}|^2 e^{-y^2/2}dy\Big)\rho(t)dt > 0. \quad (136)\end{aligned}$$

Since

$$H_\ell(\nu) = C_\nu \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} c_{k_1}c_{k_2}F\{(k_2-k_1)^2\},$$

where

$$C_\nu = \begin{cases}(-1)^{\lfloor\nu\rfloor+1}\frac{1}{2} & \text{if } \nu \notin \mathbb{Z}_+,\\ (-1)^{\nu+1} & \text{if } \nu \in \mathbb{Z}_+,\end{cases}$$

that $H_\ell(\nu)$ is non-zero follows from (136). □

Appendix F: Lemma 7 (if and only if condition for mean square differentiability)

This section investigates conditions on the covariance function which gives rise to mean square differentiability of random fields.

Definition 2. A sequence of random variables $X_1, X_2, \ldots$ in $\mathcal{L}_2$ ($\mathbb{E}|X_i|^2 < \infty$ for all $i = 1, 2, \ldots$) is said to converge in mean square sense to some random variable X if

$$\mathbb{E}(|X_n - X|^2) \to 0$$

as $n \to \infty$.

Lemma 7. *Let $X(t)$ be a real-valued random field on $\mathbb{R}$ with constant mean 0 and covariance function $K(x, y)$. Then $X(t)$ is mean square differentiable at $t \in \mathbb{R}$ if and only if*

$$\lim_{(h,l)\to(0,0)} \frac{K(t+h, t+l) - K(t, t+l) - K(t+h, t) + K(t, t)}{hl} \quad (137)$$

exists. A sufficient condition for (137) to exist is that either $\frac{\partial^2}{\partial x \partial y}K(x, y)$ or $\frac{\partial^2}{\partial y \partial x}K(x, y)$ exists in a neighborhood of (t, t) and is continuous at (t, t). Moreover, let $\dot{X}(t)$ denote the mean square derivative of $X(t)$ provided it exists. Then $\mathbb{E}\dot{X}(t) = 0$ for all $t \in \mathbb{R}$ and the covariance function of $\dot{X}(t)$ is $K^{(1,1)}(x, y)$.

Proof. For simplicity of notation, let $X_h(t) = \{X(t + h) - X(t)\}/h$. Suppose $X_h(t)$ converges in mean square sense as $h \to 0$ to a random field $\dot{X}(t)$, i.e.

$$\lim_{h\to 0} \mathbb{E}[\{X_h(t) - \dot{X}(t)\}^2] = 0.$$

Then it follows from Cauchy-Schwartz inequality that

$$\begin{aligned}
&|\mathbb{E}\{X_h(t)X_l(t)\} - \mathbb{E}[\{\dot{X}(t)\}^2]| \\
\leq &\mathbb{E}|X_h(t)X_l(t) - X_h(t)\dot{X}(t) + X_h(t)\dot{X}(t) - \{\dot{X}(t)\}^2| \\
\leq &\mathbb{E}|X_h(t)\{X_l(t) - \dot{X}(t)\}| + \mathbb{E}|\{X_h(t) - \dot{X}(t)\}\dot{X}(t)| \\
\leq &\sqrt{\mathbb{E}[\{X_h(t)\}^2]}\sqrt{\mathbb{E}[\{X_l(t) - \dot{X}(t)\}^2]} + \sqrt{\mathbb{E}[\{X_h(t) - \dot{X}(t)\}^2]}\sqrt{\mathbb{E}[\{\dot{X}(t)\}^2]} \\
\to &0 \qquad (138)
\end{aligned}$$

as $(h, l) \to (0, 0)$. (138) and

$$\mathbb{E}\{X_h(t)X_l(t)\} = \frac{K(t+h, t+l) - K(t+h, t) - K(t, t+l) + K(t, t)}{hl}$$

imply that (137) exists. This completes the proof of the "only if" part.

To prove the "if" part, that is, $X_h(t)$ converges in mean square sense as $h \to 0$, from the completeness of the $\mathcal{L}_2$ space, it suffices to show that

$$\lim_{(h,l)\to(0,0)} \mathbb{E}[\{X_h(t) - X_l(t)\}^2] = 0. \qquad (139)$$

We observe that

$$\begin{aligned}
&\mathbb{E}[\{X_h(t) - X_l(t)\}^2] \\
= \quad &\frac{K(t+h, t+h) - 2K(t+h, t) + K(t, t)}{h^2} \\
&+\frac{K(t+l, t+l) - 2K(t+l, t) + K(t, t)}{l^2} \\
&-2\frac{K(t+h, t+l) - K(t+h, t) - K(t, t+l) + K(t, t)}{hl}. \qquad (140)
\end{aligned}$$

Then (139) follows from (140) and the assumption that (137) exists.

Suppose h, l are small enough and $\frac{\partial^2}{\partial x \partial y}K(x, y)$ exists in a neighborhood of (t, t) and is continuous at (t, t). Define $A(x) = K(x, t+l) - K(x, t)$. Since $\frac{\partial K}{\partial x}(x, y)$ exists in a neighborhood of (t, t), $A(x)$ is differentiable in a neighborhood of t. Hence by the mean value theorem, there exists some $\theta_1 \in (0, 1)$ such that

$$\begin{aligned}
&\frac{K(t+h, t+l) - K(t+h, t) - K(t, t+l) + K(t, t)}{hl} \\
= \quad &\frac{1}{l}\{\frac{A(t+h) - A(t)}{h}\} \\
= \quad &\frac{1}{l}\frac{dA}{dx}(t + \theta_1 h) \\
= \quad &\frac{1}{l}\{\frac{\partial K}{\partial x}(t + \theta_1 h, t+l) - \frac{\partial K}{\partial x}(t + \theta_1 h, t)\}. \qquad (141)
\end{aligned}$$

Define $B(y) = \frac{\partial K}{\partial x}(t + \theta_1 h, y)$. Since $\frac{\partial^2 K}{\partial x \partial y}(x, y)$ exists in a neighborhood of (t, t), $B(y)$ is differentiable in a neighborhood of t. It follows from the mean value theorem and (141) that there exists some $\theta_2 \in (0, 1)$ such that

$$\begin{aligned} & \frac{K(t+h,t+l) - K(t+h,t) - K(t,t+l) + K(t,t)}{hl} \\ = \; & \frac{B(t+l) - B(t)}{l} \\ = \; & \frac{dB}{dy}(t + \theta_2 l) \\ = \; & \frac{\partial^2 K}{\partial x \partial y}(t + \theta_1 h, t + \theta_2 l). \end{aligned} \tag{142}$$

The existence of (137) follows from the (142) and the continuity of $\frac{\partial^2}{\partial x \partial y} K(x, y)$ at (t, t). If $\frac{\partial^2}{\partial y \partial x} K(x, y)$ exists in a neighborhood of (t, t) and is continuous at (t, t), the result follows similarly.

Next, we calculate the mean function and covariance function of $\dot{X}(t)$. It follows that for $t, x, y \in \mathbb{R}$

$$|\mathbb{E}\dot{X}(t)| \le \lim_{h \to 0} \left\{ |\mathbb{E}\{\dot{X}(t) - X_h(t)\}| + |\mathbb{E}X_h(t)| \right\} = 0, \tag{143}$$

and

$$\begin{aligned} & |\mathbb{E}\{X_h(x)X_l(y)\} - \mathbb{E}\{\dot{X}(x)\dot{X}(y)\}| \\ = \; & |\mathbb{E}\{X_h(x)X_l(y) - X_h(x)\dot{X}(y) + X_h(x)\dot{X}(y) - \dot{X}(x)\dot{X}(y)\}| \\ \le \; & \mathbb{E}|X_h(x)\{X_l(y) - \dot{X}(y)\}| + \mathbb{E}|\{X_h(x) - \dot{X}(x)\}\dot{X}(y)| \\ \le \; & \sqrt{\mathbb{E}[\{X_h(x)\}^2]}\sqrt{\mathbb{E}[\{X_l(y) - \dot{X}(y)\}^2]} \\ & + \sqrt{\mathbb{E}[\{X_h(x) - \dot{X}(x)\}^2]}\sqrt{\mathbb{E}[\{\dot{X}(y)\}^2]} \\ \to \; & 0 \end{aligned} \tag{144}$$

as $(h, l) \to (0, 0)$. We conclude from (143) and (144) that

$$\mathrm{Cov}\{\dot{X}(x), \dot{X}(y)\} = \lim_{(h,l) \to (0,0)} \mathbb{E}\{X_h(x)X_l(y)\} = K^{(1,1)}(x, y).$$

This proves Lemma 7. □

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References

[1] Adler, R. J. and Pyke, R. (1993). Uniform quadratic variation for Gaussian processes. *Stochastic Process. Appl.* **48** 191-209. MR1244542

[2] Anderes, E. B. and Chatterjee, S. (2009). Consistent estimation of deformed isotropic Gaussian random fields on the plane. *Ann. Statist.* **37** 2324-2350. MR2543694

[3] Andrews, G. E., Askey, R. and Roy, R. (1999). *Special Functions.* Cambridge Univ. Press, Cambridge. MR1688958

[4] Azais, J. M. and Wschebor, M. (2009). *Level Sets and Extrema of Random Processes and Fields*. Wiley, New York. MR2478201

[5] Benassi, A., Cohen, S., Istas, J. and Jaffard, S. (1998). Identification of filtered white noises. *Stochastic Process. Appl.* **75** 31-49. MR1629014

[6] Blanke, D. and Vial, C. (2014). Global smoothness estimation of a Gaussian process from general sequence designs. *Electron. J. Statist.* **8** 1152–1187. MR3263116

[7] Chan, G. and Wood, T. A. (2000). Increment-based estimators of fractal dimension for two-dimensional surface data. *Statist. Sinica* **10** 343-376. MR1769748

[8] Constantine, A. G. and Hall, P. (1994). Characterizing surface smoothness via estimation of effective fractal dimension. *J. Roy. Statist. Soc. Ser. B* **56** 97-113. MR1257799

[9] Cramér, H. and Leadbetter, M. R. (1967). *Stationary and Related Stochastic Processes.* Wiley, New York. MR2108670

[10] Hall, P. and Wood, A. (1993). On the performance of box-counting estimators of fractal dimension. *Biometrika* **80** 246-252. MR1225230

[11] Hanson, D. L. and Wright, F. T. (1971). A bound on tail probabilities for quadratic forms in independent random variables. *Ann. Math. Statist.* **42** 1079-1083. MR0279864

[12] Hiby, L. and Krishna, M. B. (2001). Line transact sampling from a curving path. *Biometrics* **57** 727-731. MR1863445

[13] Horn, R. A. and Johnson, C.R. (1985). *Matrix Analysis.* Cambridge Univ. Press, Cambridge. MR0832183

[14] Im, H. K., Stein, M. L. and Zhu, Z. (2007). Semiparametric estimation of spectral density with irregular observations. *J. Amer. Statist. Assoc.* **102** 726-735. MR2381049

[15] Istas, P. and Lang, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. Henri Poincaré Probab. Statist.* **33** 407-735. MR1465796

[16] Kent, J. T. and Wood, A. T. A. (1997). Estimating the fractal dimension of a locally self-similar Gaussian process by using increments. *J. Roy. Statist. Soc. Ser. B* **59** 679-699. MR1452033

[17] Klein, R. and Giné, E. (1975). On quadratic variation of processes with Gaussian increments. *Ann. Probab.* **3** 716-723. MR0378070

[18] Knuth, D. E. (1973). *The Art of Computer Programming Vol. 1: Fundamental Algorithms*, 2nd ed. Addison-Wesley, Reading. MR0445948

[19] Lévy, P. (1940). Le mouvement Brownien plan. *Amer. J. Math.* **62** 487-550. MR0002734

[20] Loh, W.-L. (2015). Estimating the smoothness of a Gaussian random field from irregularly spaced data via higher-order quadratic variations. *Ann. Statist.* **43** 2766-2794. MR3405611

[21] Loh, W.-L., Sun, S. and Wen, J. (2021). On fixed-domain asymptotics, parameter estimation and isotropic Gaussian random fields with Matérn covariance functions. *Ann. Statist.* **49** 3127-3152.

[22] Paciorek, C. J. (2003). *Nonstationary Gaussian Processes for Regression and Spatial Modelling*. PhD thesis, Department of Statistics, Carnegie Mellon University, USA. MR2704401

[23] Paciorek, C. J. and Schervish, M. J. (2006). Spatial modelling using a new class of nonstationary covariance functions. *Environmetrics* **17** 483-506. MR2240939

[24] Stein, M. L. (1999). *Interpolation of Spatial Data: Some Theory for Kriging*. Springer, New York. MR1697409

[25] Stein, M. L. (2012). Simulation of Gaussian random fields with one derivative. *J. Comp. Graph. Statist.* **21** 155-173. MR2913361

[26] Sylvester, J. J. (1857). On the partition of numbers. *Quart. J. Math.* **1** 141-152. MR1003160

[27] Thompson, S. K. (2012). *Sampling*, 3rd edition. Wiley, New York. MR2894042

[28] van der Vaart, A.W. (2000). *Asymptotic Statistics*. Cambridge Univ. Press, New York. MR1652247

[29] Wood, A. T. A. and Chan, G. (1994). Simulation of stationary Gaussian processes in $[0,1]^d$. *J. Comp. Graph. Statist.* **3** 409-432. MR1323050