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Averaging 2d stochastic wave equation*

Raul Bolaños Guerrero† David Nualart‡ Guangqu Zheng§

Abstract

We consider a 2D stochastic wave equation driven by a Gaussian noise, which is temporally white and spatially colored described by the Riesz kernel. Our first main result is the functional central limit theorem for the spatial average of the solution. And we also establish a quantitative central limit theorem for the marginal and the rate of convergence is described by the total-variation distance. A fundamental ingredient in our proofs is the pointwise L^p-estimate of Malliavin derivative, which is of independent interest.

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1 Introduction

We consider the 2D stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \sigma(u)\dot{W}, \tag{1.1}$$

on $\mathbb{R}_+ \times \mathbb{R}^2$, where Δ is Laplacian in the space variables and $\dot{W}$ is a Gaussian centered noise with covariance given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)\|x-y\|^{-\beta}$$

for any given $\beta \in (0,2)$. In other words, the driving noise $\dot{W}$ is white in time and it has an homogeneous spatial covariance described by the Riesz kernel. Here $\dot{W}$ is a distribution-valued field and is a notation for $\frac{\partial^3 W}{\partial t \partial x_1 \partial x_2}$, where the noise W will be formally introduced later.

*David Nualart is supported by the NSF Grant DMS 1811181.
†University of Kansas, United States of America and University of Costa Rica, Republic of Costa Rica. E-mail: rbolanosg@ku.edu, raulesteban.bolanos@ucr.ac.cr
‡University of Kansas, United States of America.E-mail: nualart@ku.edu
§University of Kansas, United States of America.E-mail: zhengguangqu@gmail.com

Throughout this article, we fix the boundary conditions

$$u(0,x)=1, \quad \frac{\partial}{\partial t}u(0,x)=0 \tag{1.2}$$

and assume $\sigma:\mathbb{R}\to\mathbb{R}$ is Lipschitz with Lipschitz constant $L\in(0,\infty)$ such that $\sigma(1)\neq 0$. It is well-known (see *e.g.* [6]) that equation (1.1) has a unique *mild solution*, which is adapted to the filtration generated by W, such that $\sup\big\{\mathbb{E}\big[|u(t,x)|^2\big]:(t,x)\in[0,T]\times\mathbb{R}^2\big\}<\infty$ for any finite T and

$$u(t,x)=1+\int_0^t\int_{\mathbb{R}^2}G_{t-s}(x-y)\sigma(u(s,y))W(ds,dy), \tag{1.3}$$

where the above stochastic integral is defined in the sense of Dalang-Walsh (see [5, 23]) and $G_{t-s}(x-y)$ denotes the fundamental solution to the corresponding deterministic 2D wave equation, *i.e.*

$$G_t(x)=\frac{1}{2\pi\sqrt{t^2-\|x\|^2}}\mathbf{1}_{\{\|x\|<t\}}.$$

Because of the choice of boundary conditions (1.2), $\{u(t,x):x\in\mathbb{R}^2\}$ is strictly stationary for any fixed $t>0$, meaning that the finite-dimensional distributions of $\{u(t,x+y):x\in\mathbb{R}^2\}$ do not depend on y; see *e.g.* [7, Footnote 1]. Then it is natural to view the solution $u(t,x)$ as a functional over the homogeneous Gaussian random field W. Such Gaussian functional has been a recurrent topic in probability theory, for example, the celebrated Breuer-Major theorem (see *e.g.* [1, 2, 19]) provides the Gaussian fluctuation for the average of a functional subordinated to a stationary Gaussian random field. Therefore, one may wonder whether or not the spatial average of $u(t,x)$ admits Gaussian fluctuation, that is, as $R\to+\infty$

does $\displaystyle\int_{\{\|x\|\leq R\}}(u(t,x)-1)\,dx$ *converge to* $\mathcal{N}(0,1)$*, after proper normalization?*

Here $t>0$ is fixed, $u(t,x)$ solves (1.1) and $\mathcal{N}(0,1)$ denotes the standard normal law.

Recently, the above question has been investigated for stochastic heat equations (see [4, 9, 10, 20]) and for the 1D stochastic wave equation (see [7]). Our work can be seen as an extension of the work [7] to the two-dimensional case. In Theorem 1.1 below we provide an affirmative answer to the above question and we will provide more literature overview in Remark 1.5.

Let us first fix some notation that will be used *throughout this article*.

Notation. (1) The expression $a\lesssim b$ means $a\leq Kb$ for some immaterial constant K that may vary from line to line.

(2) $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$ and we write $B_R=\{x:\|x\|\leq R\}$. We define for each $t\in\mathbb{R}_+:=[0,\infty)$,

$$F_R(t)=\int_{B_R}(u(t,x)-1)\,dx. \tag{1.4}$$

(3) We fix $\beta\in(0,2)$ throughout this article and there are two relevant constants[1] c_β,κ_β defined by

$$c_\beta=\frac{\Gamma(1-\frac{\beta}{2})}{\pi 4^{\beta/2}\Gamma(\beta/2)}, \qquad \kappa_\beta=\int_{\mathbb{R}^2}d\xi\|\xi\|^{\beta-4}J_1(\|\xi\|)^2, \tag{1.5}$$

[1]Note that the quantity κ_β is finite, since $J_1(\rho)$ is uniformly bounded on $\mathbb{R}_+$ and equivalent to constant times ρ as $\rho\downarrow 0$; see *e.g.* [20, Lemma 2.1].

where $J_1(\cdot)$ is the Bessel function of first kind with order 1, given by (see, for instance, [13, (5.10.4)])

$$J_1(x) = \frac{x}{\pi} \int_0^\pi \sin^2\theta \cos(x\cos\theta) d\theta. \tag{1.6}$$

Note that $4\pi^2 c_\beta \kappa_\beta = \int_{B_1^2} \|y-z\|^{-\beta} dydz$; see Remark 2.3 below.

(4) We write $\|X\|_p$ for the $L^p(\Omega)$-norm of a real random variable X.

Now we are in a position to state our main result.

Theorem 1.1. *Recall $F_R(t)$ defined in (1.4). As $R \to \infty$, the process $\{R^{\frac{\beta}{2}-2} F_R(t) : t \in \mathbb{R}_+\}$ converges in law to a centered Gaussian process $\mathcal{G}$ in the space $C(\mathbb{R}_+;\mathbb{R})$ of continuous functions*[2]*, where*

$$\mathbb{E}\big[\mathcal{G}_{t_1}\mathcal{G}_{t_2}\big] = 4\pi^2 c_\beta \kappa_\beta \int_0^{t_1\wedge t_2} (t_1-s)(t_2-s)\xi^2(s) ds,$$

with $\xi(s) = \mathbb{E}[\sigma(u(s,0))]$ and c_β, κ_β being the two constants given in (1.5). For any fixed $t > 0$,

$$d_{\rm TV}\big(F_R(t)/\sigma_R, Z\big) \lesssim R^{-\beta/2}, \tag{1.7}$$

where $Z \sim \mathcal{N}(0,1)$ and $\sigma_R := \sqrt{\mathrm{Var}(F_R(t))} > 0$ for every $R > 0$.

Remark 1.2. (1) The limiting process $\mathcal{G}$ has the following stochastic integral representation:

$$\{\mathcal{G}_t : t\in\mathbb{R}_+\} \overset{(d)}{=} \left\{ 2\pi\sqrt{c_\beta\kappa_\beta}\int_0^t (t-s)\xi(s) dY_s : t \in \mathbb{R}_+ \right\},$$

where $\{Y_t : t\in\mathbb{R}_+\}$ is a standard Brownian motion.

(2) We point out that $\sigma_R > 0$ is part of our main result. Indeed, it is a consequence of our standing assumption $\sigma(1) \neq 0$. In fact, we have the following equivalences:

$$\sigma_R = 0,\ \forall R>0 \Leftrightarrow \exists R>0,\ s.t.\ \sigma_R = 0 \Leftrightarrow \sigma(1) = 0 \Leftrightarrow \lim_{R\to\infty} \sigma_R^2 R^{\beta-4} = 0.$$

The proof can be done similarly as in [7, Lemma 3.4] and by using Proposition 3.1.

(3) The total-variation distance $d_{\rm TV}$ induces a much stronger topology than that induced by the Fortet-Mourier distance $d_{\rm FM}$, where the latter is equivalent to that of convergence in law. For real random variables X, Y,

$$d_{\rm TV}(X,Y) := \sup_A \big|\mathbb{P}(X\in A) - \mathbb{P}(Y\in A)\big|, \quad d_{\rm FM}(X,Y) := \sup_h \big|\mathbb{E}[h(X) - h(Y)]\big|,$$

where the first supremum runs over all Borel subsets of $\mathbb{R}$ and the second supremum runs overs all bounded Lipschitz functions h with $\|h\|_\infty + \|h'\|_\infty \le 1$. Our quantitative CLT (1.7) is obtained by the Malliavin-Stein approach that combines Stein's method of normal approximation with Malliavin's differential calculus on a Gaussian space; see the monograph [15] for a comprehensive treatment. One can also obtain the rate of convergence in other frequently used distances, such as the 1-Wassertein distance and Kolmogorov distance, and the corresponding bounds are of the same order as in (1.7).

Now let us sketch a few paragraphs to briefly illustrate our methodology in proving Theorem 1.1. The main ingredient is the following fundamental estimate on the p-norm of the Malliavin derivative $Du(t,x)$ of the solution $u(t,x)$. It is well-known (see *e.g.* [14])

[2] The space $C(\mathbb{R}_+;\mathbb{R})$ is equipped with the topology of uniform convergence on compact sets.

that $Du(t,x) \in L^p(\Omega;\mathfrak{H})$ for any $p \in [1,\infty)$, where $\mathfrak{H}$ is the Hilbert space associated to the noise W, defined as the completion of $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ under the inner product

$$\langle f,g\rangle_{\mathfrak{H}} := \int_{\mathbb{R}_+\times\mathbb{R}^4} f(s,y)g(s,z)\|y-z\|^{-\beta}dydzds \tag{1.8}$$

$$= c_\beta \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathscr{F}f(s,\xi)\mathscr{F}g(s,-\xi)\|\xi\|^{\beta-2}d\xi ds, \tag{1.9}$$

where c_β is given in (1.5) and $\mathscr{F}f(s,\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi}f(s,x)dx$.

Theorem 1.3. *The Malliavin derivative $Du(t,x)$ is a random function denoted by $(s,y) \mapsto D_{s,y}u(t,x)$ and for any $p \in [2,\infty)$ and any $t > 0$, the following estimates hold for almost all $(s,y) \in [0,t]\times\mathbb{R}^2$:*

$$G_{t-s}(x-y)\|\sigma(u_{s,y})\|_p \le \big\|D_{s,y}u(t,x)\big\|_p \le C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y), \tag{1.10}$$

where the constants $C_{\beta,p,t,L}$ and $\kappa_{p,t}$ are given in (4.6) and (4.4), respectively.

Remark 1.4. Theorem 1.3 echoes the comment after [10, Lemma 2.1] and generalizes [7, Lemma 2.2] to the solution of a 2D stochastic wave equation. Although the expression in (1.10) looks the same as in [7, Lemma 2.2], *i.e.* L^p-norm of the Malliavin derivative is bounded by the fundamental solution to the corresponding deterministic wave equation, we would like to emphasize that the proof in the 2D setting is much more involved and requires new techniques in dealing with the singularity of $G_{t-s}(x-y)$ while in the 1D case the fundamental solution is the bounded function $\frac{1}{2}\mathbf{1}_{\{|x-y|<t-s\}}$. Modulo sophisticated integral estimates, our proof of Theorem 1.3 is treated through a harmonious combination of tools from Gaussian analysis (Clark-Ocone formula, Burkholder inequality) and Hardy-Littlewood-Sobolev's lemma.

Before we proceed to explaining our proof strategy, let us provide a brief literature overview.

Remark 1.5. It was the paper [9] by Huang, Nualart and Viitasaari that first studied spatial averages of stochastic heat equation with 1 spatial dimension driven by space-time white noise. Soon later, the same authors and Zheng investigated the same equation in higher dimension; in their paper [10], the spatial correlation is described by the Riesz kernel as in the present work. The above two references considered the noise that is white in time, leading to the natural martingale structure. This enables one to take advantage of Itô calculus mentioned in previous remark. However, when the noise is colored in time, these tools are not available any more and we should restrict ourselves to the linear equation (that is, when $\sigma(u) = u$). The linear equation, also known as the *parabolic Anderson model*, admits the explicit Wiener chaos expansions, and in the work [20] by Nualart and Zheng, similar central limit theorems are established at qualitative level by using the so-called chaotic central limit theorem (see *e.g.* [15, Section 6.3]). The authors of [7] first considered the same problem for the stochastic wave equations where spatial dimension is one and the driving Gaussian noise is white in time and fractional in space. Unlike in the heat setting, the fundamental wave solution differs in different dimensions and as we will see shortly, the analysis in our work is quite different from that in [7]. Here we also remark that it is natural to study the same problem for wave equations when the noise is colored in time, and it may be a hard problem to get a quantitative central limit theorem in this setting.

Now let us first sketch the main steps for the proof of Theorem 1.1 and then we will present the key steps in proving (1.10).

The typical proof of the functional CLT consists in three steps:

(**S1**) We establish the limiting covariance structure, this is the content of Section 3.1. In particular, the variance of the spatial average $F_R(t)$ is of order $R^{4-\beta}$, as $R\to\infty$. As one will see shortly, the important part of this step is the proof of the limit (3.3): $\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big]\to 0$ as $\|y-z\|\to\infty$. This limit is straightforward when $\sigma(u)=u$ and in the general case, we will apply the Clark-Ocone formula (see Lemma 2.5) to first represent $\sigma(u(s,y))$ as a stochastic integral and then apply the Itô's isometry in order to break the nonlinearity for further estimations.

(**S2**) From (**S1**), we have the covariance structure of the limiting Gaussian process $\mathcal{G}$. Then we will prove the convergence of $\big\{R^{\frac{\beta}{2}-2}F_R(t):t\in\mathbb{R}_+\big\}$ to $\big\{\mathcal{G}_t:t\in\mathbb{R}_+\big\}$ in finite-dimensional distributions. This is made possible by the following multivariate Malliavin-Stein bound that we borrow from [9, Proposition 2.3] (see also [15, Theorem 6.1.2]). We denote by D the Malliavin derivative and by δ the adjoint operator of D that is characterized by the integration-by-parts formula (2.6). Moreover, $\mathbb{D}^{1,2}$ is the Sobolev space of Malliavin differentiable random variables $X\in L^2(\Omega)$ with $\mathbb{E}\big[\|DX\|_{\mathfrak{H}}^2\big]<\infty$ and $\mathrm{Dom}\delta$ is the domain of δ; see Section 2 for more details.

Proposition 1.6. *Let $F=(F^{(1)},\dots,F^{(m)})$ be a random vector such that $F^{(i)}=\delta(v^{(i)})$ for $v^{(i)}\in\mathrm{Dom}\,\delta$ and $F^{(i)}\in\mathbb{D}^{1,2}$, $i=1,\dots,m$. Let Z be an m-dimensional centered Gaussian vector with covariance matrix $(C_{i,j})_{1\le i,j\le m}$. For any C^2 function $h:\mathbb{R}^m\to\mathbb{R}$ with bounded second partial derivatives, we have*

$$\big|\mathbb{E}[h(F)]-\mathbb{E}[h(Z)]\big|\le\frac{m}{2}\|h''\|_\infty\sqrt{\sum_{i,j=1}^m\mathbb{E}\Big[\big(C_{i,j}-\langle DF^{(i)},v^{(j)}\rangle_{\mathfrak{H}}\big)^2\Big]}\,,\tag{1.11}$$

where $\|h''\|_\infty:=\sup\big\{\big|\frac{\partial^2}{\partial x_i\partial x_j}h(x)\big|:x\in\mathbb{R}^m,\,i,j=1,\dots,m\big\}$.

In view of (1.3), we write $u(t,x)-1=\delta\big(G_{t-\bullet}(x-*)\sigma(u(\bullet,*))\big)$ so that $F_R(t)$ can be represented as

$$F_R(t)=\int_{B_R}\delta\big(G_{t-\bullet}(x-*)\sigma(u(\bullet,*))\big)dx=\delta\big(\varphi_{t,R}(\bullet,*)\sigma(u(\bullet,*))\big)\tag{1.12}$$

by Fubini's theorem, with

$$\varphi_{t,R}(r,y)=\int_{B_R}G_{t-r}(x-y)dx;\tag{1.13}$$

see Section 2.2. Putting $V_{t,R}(s,y)=\varphi_{t,R}(s,y)\sigma(u(s,y))$, and applying the fundamental estimate (1.10), we will establish that, for any $t_1,t_2\in(0,\infty)$,

$$R^{2\beta-8}\mathrm{Var}\big(\langle DF_R(t_1),V_{t_2,R}\rangle_{\mathfrak{H}}\big)\lesssim R^{-\beta}\text{ for }R\ge t_1+t_2.\tag{1.14}$$

Then, we will show that Proposition 1.6 together with the estimate (1.14) imply the convergence in law of the finite-dimensional distributions.

The bound (1.14) for $t_1=t_2=t$ together with the following 1D Malliavin-Stein bound (see, *e.g.* [9, 17, 21]) will lead to the quantitative result (1.7).

Proposition 1.7. *Let $F=\delta(v)$ for some $\mathfrak{H}$-valued random variable $v\in\mathrm{Dom}\,\delta$. Assume $F\in\mathbb{D}^{1,2}$ and $\mathbb{E}[F^2]=1$ and let $Z\sim\mathcal{N}(0,1)$. Then,*

$$d_{\mathrm{TV}}(F,Z)\le2\sqrt{\mathrm{Var}\big[\langle DF,v\rangle_{\mathfrak{H}}\big]}\,.\tag{1.15}$$

(**S3**) The last step is to show tightness, which follows from the tightness of the processes restricted to $[0,T]$ for any finite T. To show the tightness of $\big\{R^{\frac{\beta}{2}-2}F_R(t):$

$t \in [0,T]\big\}$, in view of the well-known criterion of Kolmogorov-Chentsov (see *e.g.* [11, Corollary 16.9]), it is enough to show that for any $p \in [2,\infty)$,

$$\|F_R(t) - F_R(s)\|_p \lesssim R^{2-\frac{\beta}{2}}|t-s|^{1/2} \text{ for } s,t \in [0,T], \tag{1.16}$$

where the implicit constant does not depend on t, s or R. This will proves Theorem 1.1.

Finally let us pave the plan of proving the fundamental estimate (1.10). The story begins with the usual *Picard iteration*: We define $u_0(t,x) = 1$ and for $n \geq 0$,

$$u_{n+1}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma\big(u_n(s,y)\big)W(ds,dy). \tag{1.17}$$

It is a classic result that $u_n(t,x)$ converges in $L^p(\Omega)$ to $u(t,x)$ uniformly in $x \in \mathbb{R}^2$ for any $p \geq 2$; see *e.g.* [6, Theorem 4.3]. Now it has become clear that if we assume $\sigma(1) = 0$, we will end up in the trivial case where $u(t,x) \equiv 1$, in view of the above iteration.

For each $n \geq 0$, $u_{n+1}(t,x)$ is Malliavin differentiable, as one can show by induction on n. Our strategy is to first obtain the uniform estimate of $\sup\big\{\|D_{s,y}u_n(t,x)\|_p : n \geq 0\big\}$ and then one can hope to transfer this estimate to $\|D_{s,y}u(t,x)\|_p$. As mentioned before, $Du(t,x)$ lives in the space $\mathfrak{H}$ that contains generalized functions. To overcome this, we will carefully apply the following inequality of Hardy-Littlewood-Sobolev to show $Du(t,x)$ is a random variable in $L^{\frac{4}{4-\beta}}(\mathbb{R}_+ \times \mathbb{R}^2)$, with $\beta \in (0,2)$ fixed throughout this paper.

Lemma 1.8 (**Hardy-Littlewood-Sobolev**). *If $1 < p < p_0 < \infty$ with $p_0^{-1} = p^{-1} - \alpha n^{-1}$, then there is some constant C that only depends on p, α and n, such that*

$$\|I^\alpha g\|_{L^{p_0}(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)},$$

for any locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$, where with $\alpha \in (0,n)$,

$$\big(I^\alpha g\big)(x) := \int_{\mathbb{R}^n} \|x-y\|^{\alpha-n} g(y)dy.$$

For our purpose, with $n = 2$, $\alpha = 2 - \beta$, $p = 2q = 4/(4-\beta)$ and $p_0 = 4/\beta$, we deduce from Hölder's inequality that

$$\begin{aligned}\langle f, g\rangle_{\mathfrak{H}_0} &:= \int_{\mathbb{R}^2} f(x)g(y)\|x-y\|^{-\beta}dxdy \qquad (1.18)\\ &\leq \|f\|_{L^{2q}(\mathbb{R}^2)}\|I^{2-\beta}g\|_{L^{4/\beta}(\mathbb{R}^2)}\\ &\leq C_\beta\|f\|_{L^{2q}(\mathbb{R}^2)}\|g\|_{L^{2q}(\mathbb{R}^2)}, \qquad (1.19)\end{aligned}$$

for any $f, g \in L^{2q}(\mathbb{R}^2)$; see *e.g.* [22, pages 119-120].

Once we obtain the uniform estimate of $\sup\big\{\|D_{s,y}u_n(t,x)\|_p : n \geq 0\big\}$ and prove $Du(t,x) \in L^{\frac{4}{4-\beta}}(\mathbb{R}_+ \times \mathbb{R}^2)$, that is, $(s,y) \longmapsto D_{s,y}u(t,x)$ is indeed a random function, we proceed to the proof of (1.10). In view of the Clark-Ocone formula (see Lemma 2.5), we have $\mathbb{E}[D_{s,y}u_{t,x}|\mathscr{F}_s] = G_{t-s}(x-y)\sigma(u(s,y))$ almost surely, where $\{\mathscr{F}_s : s \in \mathbb{R}_+\}$ is the filtration generated by the noise; see Section 2.2. Then, the lower bound in (1.10) follows immediately from the conditional Jensen inequality. The upper bound follows from the uniform estimates of $\|D_{s,y}u_n(t,x)\|_p$ by a standard argument.

Before we end this introduction, let us point out another technical difficulty in this paper. After the application of Lemma 1.8 during the process of estimating $\|D_{s,y}u_n(t,x)\|_p$, we will encounter integrals of the form

$$\int_s^t \left(\int_{\mathbb{R}^2} G_{t-r}^{2q}(x-z)G_{r-s}^{2q}(z)dz\right)^\delta dr$$

where $q \in (1/2, 1)$ and $\delta \in \{1, 1/q\}$. In the case of stochastic heat equation, the estimation of the above integrals is straightforward due to the semi-group property. However, for the wave equation the kernel G_t *does not* satisfy the semi-group property and the estimation of the above integrals is quite involved. For the case of the 1D stochastic wave equation, as one can see from the paper [7], the computations take advantage of the simple form of the fundamental solution (*i.e.* $\frac{1}{2}\mathbf{1}_{\{|x-y|<t-s\}}$). For our 2D case, the singularity within the fundamental solution $G_{t-s}(x-y)$ puts the technicality to another level and we have to estimate the convolution $G_{t-r}^{2q} * G_{r-s}^{2q}$ by exact computations. A basic technical tool used in this problem is the following lemma.

Lemma 1.9. *For $0 \le s < t < \infty$, with $\|z\| = \mathbf{w} > 0$ and $q \in (1/2, 1)$, we have*

$$G_t^{2q} * G_s^{2q}(z) \lesssim \mathbf{1}_{\{\mathbf{w}<s\}}\big[t^2 - (s-\mathbf{w})^2\big]^{1-2q} + \big[t^2 - (s+\mathbf{w})^2\big]^{1-2q}\mathbf{1}_{\{t>s+\mathbf{w}\}} + \mathbf{1}_{\{|s-\mathbf{w}|<t<s+\mathbf{w}\}}\big[(\mathbf{w}+s)^2 - t^2\big]^{-q+\frac{1}{2}}\big[t^2 - (s-\mathbf{w})^2\big]^{-q+\frac{1}{2}}, \tag{1.20}$$

where the implicit constant only depends on q.

The rest of this article is organized as follows: Section 2 collects some preliminary facts for our proofs, Section 3 contains the proof of Theorem 1.1 and Section 4 is devoted to proving the fundamental estimate (1.10).

2 Preliminaries

This section provides some preliminary results that are required for further sections. It consists of two subsections: Section 2.1 contains several important facts on the function $G_{t-s}(x-y)$ and Section 2.2 is devoted to a minimal set of results from stochastic analysis, notably the tools from Malliavin calculus.

2.1 Basic facts on the fundamental solution

Let us fix some more notation here.

Notation. For $p \in \mathbb{R}$, we write $(v)_+^p = v^p$ if $v > 0$ and $(v)_+^p = 0$ if $v \le 0$. Then, we can write

$$G_t(x) = \frac{1}{2\pi}(t^2 - \|x\|^2)_+^{-1/2}.$$

Recall the function $\varphi_{t,R}(r, y)$ introduced in (1.13):

$$\varphi_{t,R}(s, y) = \int_{B_R} G_{t-r}(x-y)dx.$$

In what follows, we put together several useful facts on the function $G_t(z)$.

Lemma 2.1. (1) *For any $p \in (0, 1)$ and $t > 0$.*

$$\int_{\mathbb{R}^2} G_t^{2p}(z)dz = \frac{(2\pi)^{1-2p}}{2-2p}t^{2-2p}. \tag{2.1}$$

(2) *For $t > s$, we have $\varphi_{t,R}(s, y) \le (t-s)\mathbf{1}_{\{\|y\| \le R+t\}}$ and $\int_{\mathbb{R}^2} \varphi_{t,R}(s, y)dy = (t-s)\pi R^2$.*

The proof of Lemma 2.1 is omitted, as it follows from simple and exact computations. As a consequence of Lemma 2.1-(2), we have

$$\int_{\mathbb{R}^2} \varphi_{t,R}(s, z+\xi)\varphi_{t,R}(s, z)dz \le \pi(t-s)^2R^2. \tag{2.2}$$

The following lemma is also a consequence of Lemma 2.1.

Lemma 2.2. *For $t_1, t_2 \in (0,\infty)$, we put*

$$\Psi_R(t_1,t_2;s) := R^{\beta-4}\int_{\mathbb{R}^4}\varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\|y-z\|^{-\beta}dydz.$$

Then

(i) $\Psi_R(t_1,t_2;s)$ is uniformly bounded over $s\in[0,t_2\wedge t_1]$ and $R>0$;

(ii) For any $s\in[0,t_2\wedge t_1]$, $\Psi_R(t_1,t_2;s)$ converges to $4\pi^2 c_\beta\kappa_\beta(t_1-s)(t_2-s)$, as $R\to\infty$.

Here the quantities c_β and κ_β are given in (1.5).

Proof. By using Fourier transform as in (1.9), we can write

$$\begin{aligned}\Psi_R(t_1,t_2;s) &= R^{\beta-4}\int_{B_R^2}dxdx'\int_{\mathbb{R}^4}G_{t_1-s}(x-y)G_{t_2-s}(x'-z)\|y-z\|^{-\beta}dydz\\ &= c_\beta R^{\beta-4}\int_{B_R^2}dxdx'\int_{\mathbb{R}^2}d\xi e^{-i(x-x')\cdot\xi}\left(\frac{\sin((t_1-s)\|\xi\|)}{\|\xi\|}\frac{\sin((t_2-s)\|\xi\|)}{\|\xi\|}\right)\|\xi\|^{\beta-2}\\ &= c_\beta\int_{B_1^2}dxdx'\int_{\mathbb{R}^2}d\xi e^{-i(x-x')\cdot\xi}\ \frac{\sin((t_1-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\frac{\sin((t_2-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\|\xi\|^{\beta-2},\end{aligned}$$

where in the last equality we made the change of variables $\xi\to\xi R^{-1}$.

The Fourier transform of $x\in\mathbb{R}^2\longmapsto\mathbf{1}_{\{\|x\|\le 1\}}$ is $\xi\in\mathbb{R}^2\longmapsto 2\pi\|\xi\|^{-1}J_1(\|\xi\|)$ (see, for instance, Lemma 2.1 in [20]), where J_1 is the Bessel function of first kind with order 1 introduced in (1.6). Then, we can rewrite $\Psi_R(t_1,t_2;s)$ as

$$c_\beta\int_{\mathbb{R}^2}\Big[2\pi\|\xi\|^{-1}J_1(\|\xi\|)\Big]^2\left(\frac{\sin((t_1-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\frac{\sin((t_2-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\right)\|\xi\|^{\beta-2}d\xi.$$

Since $\sin((t-s)\|\xi\|R^{-1})/(\|\xi\|R^{-1})$ is uniformly bounded over $s\in(0,t]$ and converges to $t-s$ as $R\to\infty$, then the statement (i) holds true and

$$\Psi_R(t_1,t_2;s)\xrightarrow{R\to\infty}4\pi^2c_\beta\kappa_\beta(t_1-s)(t_2-s).$$

by the dominated convergence theorem with the dominance condition $\kappa_\beta<\infty$. ☐

Remark 2.3. By inverting the Fourier transform, we have

$$(2\pi)^2c_\beta\kappa_\beta = c_\beta\int_{\mathbb{R}^2}(2\pi)^2J_1(\|\xi\|)^2\|\xi\|^{-2}\|\xi\|^{\beta-2}d\xi = \int_{B_1^2}\|y-z\|^{-\beta}dydz.$$

2.2 Basic stochastic analysis

Let $\mathfrak{H}$ be defined (see (1.8) and (1.9)) as the completion of $C_c^\infty(\mathbb{R}_+\times\mathbb{R}^2)$ under the inner product

$$\langle f,g\rangle_{\mathfrak{H}} = \int_{\mathbb{R}_+\times\mathbb{R}^4}f(s,y)g(s,z)\|y-z\|^{-\beta}dydzds \text{ for } f,g\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^2).$$

Consider an isonormal Gaussian process associated to the Hilbert space $\mathfrak{H}$, denoted by $W=\big\{W(\phi):\phi\in\mathfrak{H}\big\}$. That is, W is a centered Gaussian family of random variables such that $\mathbb{E}\big[W(\phi)W(\psi)\big]=\langle\phi,\psi\rangle_{\mathfrak{H}}$ for any $\phi,\psi\in\mathfrak{H}$. As the noise is white in time, a martingale structure naturally appears. First we define $\mathscr{F}_t$ to be the σ-algebra generated by $\mathbb{P}$-null sets and $\big\{W(\phi):\phi\in C^\infty(\mathbb{R}_+\times\mathbb{R}^2)$ has compact support contained in $[0,t]\times\mathbb{R}^2\big\}$, so we have a filtration $\mathbb{F} = \{\mathscr{F}_t : t \in \mathbb{R}_+\}$. If $\{\Phi(s,y) : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\}$ is an $\mathbb{F}$-adapted random field such that $\mathbb{E}\big[\|\Phi\|_{\mathfrak{H}}^2\big] < +\infty$, then

$$M_t = \int_{[0,t]\times\mathbb{R}^2} \Phi(s,y) W(ds,dy),$$

interpreted as the Dalang-Walsh integral ([5, 23]), is a square-integrable $\mathbb{F}$-martingale with quadratic variation given by

$$\langle M\rangle_t = \int_{[0,t]\times\mathbb{R}^4} \Phi(s,y)\Phi(s,z)\|y-z\|^{-\beta} dydzds = \big\|\Phi(\bullet,*)\mathbf{1}_{\{\bullet\le t\}}\big\|_{\mathfrak{H}}^2.$$

Let us record a suitable version of Burkholder-Davis-Gundy inequality (BDG for short); see *e.g.* [12, Theorem B.1].

Lemma 2.4 (BDG)**.** *If* $\{\Phi(s,y) : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\}$ *is an adapted random field with respect to* $\mathbb{F}$ *such that* $\|\Phi\|_{\mathfrak{H}} \in L^p(\Omega)$ *for some* $p \ge 2$*, then*

$$\left\| \int_{[0,t]\times\mathbb{R}^2} \Phi(s,y) W(ds,dy) \right\|_p^2 \le 4p \left\| \int_{[0,t]\times\mathbb{R}^4} \Phi(s,z)\Phi(s,y)\|y-z\|^{-\beta} dydzds \right\|_{p/2}. \tag{2.3}$$

We refer interested readers to the book [12] for a nice introduction to Dalang-Walsh's theory. For our purpose, we will often apply BDG as follows. If Φ is $\mathbb{F}$-adapted and $\|G_{t-\bullet}(x-*)\Phi(\bullet,*)\|_{\mathfrak{H}} \in L^p(\Omega)$ for some $p \ge 2$, then BDG implies

$$\begin{aligned}&\left\| \int_{[0,t]\times\mathbb{R}^2} G_{t-s}(x-y)\Phi(s,y) W(ds,dy) \right\|_p^2 \\ &\quad \le 4p \left\| \int_{[0,t]\times\mathbb{R}^4} G_{t-s}(x-z)G_{t-s}(x-y)\Phi(s,y)\Phi(s,z)\|y-z\|^{-\beta} dsdzdy \right\|_{p/2},\end{aligned} \tag{2.4}$$

by viewing $\int_{[0,t]\times\mathbb{R}^2} G_{t-s}(x-y)\Phi(s,y)W(ds,dy)$ as the martingale

$$\left\{ \int_{[0,r]\times\mathbb{R}^2} G_{t-s}(x-y)\Phi(s,y)W(ds,dy) : r\in[0,t] \right\} \text{ evaluated at at time } t.$$

Now let us recall some basic facts on the Malliavin calculus associated with W. For any unexplained notation and result, we refer to the book [16]. We denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let $\mathcal{S}$ be the space of simple functionals of the form $F = f(W(h_1),\dots,W(h_n))$ for $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$, $1 \le i \le n$. Then, the Malliavin derivative DF is the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),\dots,W(h_n))h_i\,.$$

The derivative operator D is closable from $L^p(\Omega)$ into $L^p(\Omega;\mathfrak{H})$ for any $p\ge 1$ and we define $\mathbb{D}^{1,p}$ to be the completion of $\mathcal{S}$ under the norm $\|F\|_{1,p} = \big(\mathbb{E}\big[|F|^p\big] + \mathbb{E}\big[\|DF\|_{\mathfrak{H}}^p\big]\big)^{1/p}$.

The *chain rule* for D asserts that if $F_1, F_2 \in \mathbb{D}^{1,2}$ and $h_1, h_2 : \mathbb{R}\to\mathbb{R}$ are Lipschitz, then $h_1(F_1)h_2(F_2) \in \mathbb{D}^{1,1}$ and $h_i(F_i) \in \mathbb{D}^{1,2}$ with

$$D\big(h_1(F_1)h_2(F_2)\big) = h_2(F_2)Y_1DF_1 + h_1(F_1)Y_2DF_2, \tag{2.5}$$

where Y_i is some $\sigma\{F_i\}$-measurable random variable bounded by the Lipschitz constant of h_i for $i=1,2$; when the h_i are differentiable, we have $Y_i = h_i'(F_i)$, $i=1,2$ (see, for instance, [16, Proposition 1.2.4]).

We denote by δ the adjoint of D given by the duality formula

$$\mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF\rangle_{\mathfrak{H}}] \tag{2.6}$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}\,\delta \subset L^2(\Omega;\mathfrak{H})$, the domain of δ. The operator δ is also called the Skorohod integral and in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod (see *e.g.* [8, 18]). In our context, the Dalang-Walsh integral coincides with the Skorohod integral: Any adapted random field Φ that satisfies $\mathbb{E}\big[\|\Phi\|_{\mathfrak{H}}^2\big] < \infty$ belongs to the domain of δ and

$$\delta(\Phi) = \int_0^\infty \int_{\mathbb{R}^2} \Phi(s,y) W(ds,dy).$$

The proof of this result is analogous to the case of integrals with respect to the Brownian motion (see [16, Proposition 1.3.11]), by just replacing real processes by $\mathfrak{H}_0$-valued processes, where $\mathfrak{H}_0$ is defined in (1.18). As a consequence, the equation (1.3) can be written as

$$u(t,x) = 1 + \delta\big(G_{t-\bullet}(x-*)\sigma(u(\bullet,*))\big).$$

The operators D and δ satisfy the commutation relation

$$[D,\delta]V := (D\delta - \delta D)(V) = V. \tag{2.7}$$

By Fubini's theorem and the duality formula (2.6), we can interchange the Skorohod integral and Lebesgue integral: Suppose $f_x \in \mathrm{Dom}\delta$ is adapted for each x in some finite measure space (E,μ) such that $\int_E f_x\mu(dx)$ also belongs to $\mathrm{Dom}\delta$ and $\mathbb{E}\int_E \|f_x\|_{\mathfrak{H}}^2\mu(dx) < \infty$, then

$$\delta\left(\int_E f_x\mu(dx)\right) = \int_E \delta(f_x)\mu(dx) \text{ almost surely.} \tag{2.8}$$

Indeed, for any $F \in \mathcal{S}$,

$$\begin{aligned}\mathbb{E}\left[F\delta\left(\int_E f_x\mu(dx)\right)\right] &= \mathbb{E}\big\langle DF, \int_E f_x\mu(dx)\big\rangle_{\mathfrak{H}} = \int_E \mathbb{E}\big\langle DF, f_x\big\rangle_{\mathfrak{H}}\mu(dx)\\ &= \int_E \mathbb{E}\big[F\delta(f_x)\big]\mu(dx) = \mathbb{E}\left[F\int_E \delta(f_x)\mu(dx)\right],\end{aligned}$$

which gives us (2.8). In particular, the equalities in (1.12) are valid.

With the help of the derivative operator, we can represent $F \in \mathbb{D}^{1,2}$ as a stochastic integral. This is the content of the following two-parameter Clark-Ocone formula, see *e.g.* [3, Proposition 6.3] for a proof.

Lemma 2.5 (Clark-Ocone formula)**.** *Given $F \in \mathbb{D}^{1,2}$, we have almost surely*

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathbb{E}\big[D_{s,y}F|\mathscr{F}_s\big]W(ds,dy).$$

We end this section with the following useful fact: If $\{\Phi_s : s \in \mathbb{R}_+\}$ is a jointly measurable and integrable process satisfying $\int_{\mathbb{R}_+}\big(\mathrm{Var}(\Phi_s)\big)^{1/2}ds < \infty$, then

$$\sqrt{\mathrm{Var}\left(\int_{\mathbb{R}_+}\Phi_s ds\right)} \le \int_{\mathbb{R}_+}\sqrt{\mathrm{Var}(\Phi_s)}ds. \tag{2.9}$$

3 Gaussian fluctuation of the spatial averages

We follow the three steps described in our introduction.

3.1 Limiting covariance structure

Proposition 3.1. *Suppose* $t_1, t_2 \in (0,\infty)$. *We have, with* $\xi(s) = \mathbb{E}\big[\sigma(u(s,0))\big]$,

$$\frac{\mathbb{E}\big[F_R(t_1)F_R(t_2)\big]}{R^{4-\beta}} \xrightarrow{R\to\infty} 4\pi^2 c_\beta \kappa_\beta \int_0^{t_1\wedge t_2} (t_1-s)(t_2-s)\xi^2(s)ds \tag{3.1}$$

with $\kappa_\beta = \int_{\mathbb{R}^2} d\xi \|\xi\|^{\beta-4} J_1(\|\xi\|)^2 \in (0,\infty)$. *In particular, for any* $t>0$,

$$\mathrm{Var}\big(F_R(t)\big)R^{\beta-4} \xrightarrow{R\to\infty} 4\pi^2 c_\beta \kappa_\beta \int_0^t (t-s)^2 \xi^2(s)ds.$$

Proof. Recall that $F_R(t) = \int_0^t \int_{\mathbb{R}^2} \varphi_{t,R}(s,y)\sigma(u(s,y))W(ds,dy)$. Then, by Itô's isometry,

$$\mathbb{E}\big[F_R(t_1)F_R(t_2)\big] = \int_0^{t_1\wedge t_2}\int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\|y-z\|^{-\beta}\mathbb{E}\big[\sigma(u(s,y))\sigma(u(s,z))\big]dydzds.$$

We claim that

$$\begin{aligned} R^{\beta-4}\int_0^{t_1\wedge t_2}\int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big]dydzds \\ \longrightarrow 0 \quad \text{as } R\to\infty. \end{aligned} \tag{3.2}$$

Assuming (3.2), we can deduce from Lemma 2.2, the stationarity of the process $\{u(t,x): x\in\mathbb{R}^2\}$ and dominated convergence that

$$\lim_{R\to\infty}\frac{\mathbb{E}\big[F_R(t_1)F_R(t_2)\big]}{R^{4-\beta}} = \lim_{R\to\infty}\int_0^{t_1\wedge t_2} \xi^2(s)\Psi_R(t_1,t_2;s)ds = \text{RHS of (3.1)},$$

where $\xi(s) = \mathbb{E}[\sigma(u(s,0))]$ is uniformly bounded over $s\in[0,t_1\wedge t_2]$.

We need to prove (3.2) now and it is enough to show for any $s\in(0,t_1\wedge t_2]$

$$\lim_{\|y-z\|\to\infty}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big] = 0. \tag{3.3}$$

Indeed, if (3.3) holds for any given $s\in(0,t_1\wedge t_2]$, then for arbitrarily small $\varepsilon>0$, there is some $K=K(\varepsilon,s)$ such that $\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big]<\varepsilon$, for $\|y-z\|\geq K$. By Lemma 2.2, we deduce

$$\begin{aligned} &R^{\beta-4}\int_{\|y-z\|\geq K}\varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,,z))\big]dydz \\ &\leq \varepsilon\Psi_R(t_1,t_2;s)\lesssim\varepsilon, \end{aligned}$$

while using the uniform L^2-boundedness of $u(t,x)$, we get

$$\begin{aligned} &R^{\beta-4}\int_{\|y-z\|<K}\varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big]dydz \\ &\lesssim R^{\beta-4}\int_{\|y-z\|<K}\varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}dydz \\ &= R^{\beta-4}\int_{\|\xi\|<K}d\xi\|\xi\|^{-\beta}\left(\int_{\mathbb{R}^2}\varphi_{t,R}(s,z+\xi)\varphi_{t,R}(s,z)dz\right)\lesssim R^{\beta-2}\int_{\|\xi\|<K}d\xi\|\xi\|^{-\beta} \text{ by (2.2)} \\ &\lesssim R^{\beta-2}\xrightarrow{R\to\infty}0. \end{aligned}$$

That is, we just proved for any $s \in (0, t_1 \wedge t_2]$,

$$R^{\beta-4} \int_{\mathbb{R}^4} \varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)), \sigma(u(s,z))\big]dydz \xrightarrow{R\to\infty} 0,$$

where the LHS is uniformly bounded in $R > 0$ and $s \in (0, t_1 \wedge t_2]$ in view of Lemma 2.2. Then the claim (3.2) follows from the dominated convergence.

It remains to verify (3.3). By Theorem 1.3, for any $0 < s < t$,

$$\|D_{s,y}u(t,x)\|_p \lesssim G_{t-s}(x-y).$$

By Lemma 2.5,

$$\sigma(u(s,y)) = \mathbb{E}\big[\sigma(u(s,y))\big] + \int_0^s \int_{\mathbb{R}^2} \mathbb{E}\Big[D_{r,\gamma}\big(\sigma(u(s,y))\big)|\mathscr{F}_r\Big]W(dr, d\gamma).$$

As a consequence,

$$\mathbb{E}\big[\sigma(u(s,y))\sigma(u(s,z))\big] = \xi^2(s) + T(s,y,z),$$

where

$$T(s,y,z) = \int_0^s \int_{\mathbb{R}^4} \mathbb{E}\Big(\mathbb{E}\big[D_{r,\gamma}(\sigma(u(s,y)))|\mathcal{F}_r\big]\mathbb{E}\big[D_{r,\gamma'}\big(\sigma(u(s,z))\big)|\mathcal{F}_r\big]\Big)\|\gamma-\gamma'\|^{-\beta}d\gamma d\gamma' dr.$$

By the chain-rule (2.5) for the derivative operator,

$$D_{r,\gamma}\big(\sigma(u(s,y))\big) = \Sigma_{s,y} D_{r,\gamma}u(s,y)$$

with $\Sigma_{s,y}$ an adapted random field uniformly bounded by L, where we recall that L is the Lipschitz constant of σ. This implies,

$$\begin{aligned}\Big|\mathbb{E}\Big(\mathbb{E}\big[D_{r,\gamma}(\sigma(u(s,y)))|\mathcal{F}_r\big]\mathbb{E}\big[D_{r,\gamma'}\big(\sigma(u(s,z))\big)|\mathcal{F}_r\big]\Big)\Big| &\lesssim \big\|D_{r,\gamma}u(s,y)\big\|_2\big\|D_{r,\gamma'}u(s,z)\big\|_2 \\ &\lesssim G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z).\end{aligned}$$

Thus,

$$|T(s,y,z)| \lesssim \int_0^s \int_{\mathbb{R}^4} G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z)\|\gamma-\gamma'\|^{-\beta}d\gamma d\gamma' dr.$$

Suppose $\|y-z\| > 2s$, then

$$G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z)\|\gamma-\gamma'\|^{-\beta} \leq G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z)\big(\|y-z\|-2s\big)^{-\beta}$$

from which we get

$$|T(s,y,z)| \lesssim \big(\|y-z\|-2s\big)^{-\beta}\int_0^s \int_{\mathbb{R}^4} G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z)d\gamma d\gamma' dr \xrightarrow{\|y-z\|\to\infty} 0.$$

This implies (3.3) and hence concludes our proof. □

3.2 Convergence of finite-dimensional distributions

As it was explained in the introduction, a basic ingredient for the convergence of finite-dimensional distributions is the following estimate

$$R^{2\beta-8}\mathrm{Var}\big(\langle DF_R(t_1), V_{t_2,R}\rangle_{\mathfrak{H}}\big) \lesssim R^{-\beta} \text{ for } R \geq t_1 + t_2, \tag{3.4}$$

where we recall that $V_{t,R}(s,y) = \varphi_{t,R}(s,y)\sigma(u(s,y))$ and $\varphi_{t,R}$ is defined in (1.13).

Note that the Malliavin-Stein bound (1.15) and the above bound (3.4) with $t_1 = t_2 = t$ lead to the quantitative CLT in (1.7). In fact, from (3.4) and (1.15), we have for any fixed $t > 0$ and $Z \sim \mathcal{N}(0,1)$,

$$d_{\rm TV}\big(F_R(t)/\sigma_R, Z\big) \leq \frac{2}{\sigma_R^2}\sqrt{\operatorname{Var}\big(\langle DF_R(t), V_{t,R}\rangle_{\mathfrak{H}}\big)} \lesssim \frac{1}{\sigma_R^2} R^{4-\frac{3\beta}{2}},\ R \geq 2t;$$

by Proposition 3.1, $\sigma_R^2 R^{\beta-4}$ converges to some explicit positive constant, see (3.1). So we can write, for all $R \geq R_t$

$$d_{\rm TV}\big(F_R(t)/\sigma_R, Z\big) \leq CR^{-\beta/2},$$

where R_t is some constant that does not depend on R. As the total variation distance is aways bounded by 1, we can write for $R \leq R_t$,

$$d_{\rm TV}\big(F_R(t)/\sigma_R, Z\big) \leq 1 \leq (R_t)^{\beta/2} R^{-\beta/2}, \forall R \leq R_t.$$

Therefore, the bound (1.7) follows.

Note that (3.4), together with Proposition 1.6, implies the convergence in law of the finite dimensional distributions. In fact, fix any integer $m \geq 1$ and choose m points $t_1, \dots, t_m \in (0\,,\infty)$, then consider the random vector $\Phi_R = \big(F_R(t_1), \dots, F_R(t_m)\big)$ and let $\mathbf{G} = (\mathcal{G}_1\,, \dots, \mathcal{G}_m)$ denote a centered Gaussian random vector with covariance matrix $(C_{i,j})_{1\leq i,j\leq m}$ given by

$$C_{i,j} := 4\pi^2 c_\beta \kappa_\beta \int_0^{t_i\wedge t_j} (t_i - s)(t_j - s)\xi^2(s)ds.$$

Recall from (1.12) that $F_R(t_i) = \delta(V_{t_i,R})$ for all $i = 1, \dots, m$. Then, by (1.11) we can write

$$\begin{aligned} &\big|\mathbb{E}(h(R^{\frac{\beta}{2}-2}\Phi_R)) - \mathbb{E}(h(\mathbf{G}))\big| \\ &\leq \frac{m}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^m \mathbb{E}\left(\big|C_{i,j} - R^{\beta-4}\langle DF_R(t_i)\,, V_{t_j,R}\rangle_{\mathfrak{H}}\big|^2\right)} \end{aligned} \tag{3.5}$$

for every $h \in C^2(\mathbb{R}^m)$ with bounded second partial derivatives. Thus, in view of (3.5), in order to show the convergence in law of $R^{\frac{\beta}{2}-2}\Phi_R$ to $\mathbf{G}$, it suffices to show that for any $i, j = 1, \dots, m$,

$$\lim_{R\to\infty} \mathbb{E}\left(\big|C_{i,j} - R^{\beta-4}\langle DF_R(t_i)\,, V_{t_j,R}\rangle_{\mathfrak{H}}\big|^2\right) = 0. \tag{3.6}$$

Notice that, by the duality relation (2.6) and the convergence (3.1), we have

$$\begin{aligned} R^{\beta-4}\mathbb{E}\Big(\big\langle DF_R(t_i)\,, V_{t_j,R}\big\rangle_{\mathfrak{H}}\Big) &= R^{\beta-4}\mathbb{E}\big[F_R(t_i)\delta(V_{t_j,R})\big] \\ &= R^{\beta-4}\mathbb{E}\big[F_R(t_i)F_R(t_j)\big] \xrightarrow{R\to\infty} C_{i,j}. \end{aligned} \tag{3.7}$$

Therefore, the convergence (3.6) follows immediately from (3.7) and (3.4). Hence the finite-dimensional distributions of $\{R^{\frac{\beta}{2}-2}F_R(t) : t \in \mathbb{R}_+\}$ converge to those of $\mathcal{G}$ as $R \to \infty$.

The rest of this subsection is then devoted to the proof of (3.4).

Proof of (3.4). Recall from (1.12) that

$$F_R(t) = \int_{B_R} (u(t,x) - 1)dx = \delta(V_{t,R}) \quad \text{with} \quad V_{t,R}(s,y) = \varphi_{t,R}(s,y)\sigma(u(s,y)).$$

The commutation relation (2.7) implies for $s \le t$,

$$D_{s,y}F_R(t) = D_{s,y}\delta(V_{t,R}) = V_{t,R}(s,y) + \delta(D_{s,y}V_{t,R}). \tag{3.8}$$

By the chain rule for the derivative operator (see (2.5))

$$D_{s,y}[V_{t,R}(r,z)] = \varphi_{t,R}(r,z)D[\sigma(u(r,z))] = \varphi_{t,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z), \tag{3.9}$$

where $\Sigma_{r,z}$ is an adapted random field bounded by the Lipschitz constant of σ. Substituting (3.9) into (3.8), yields, for $s \le t$,

$$D_{s,y}F_R(t) = \varphi_{t,R}(s,y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}^2} \varphi_{t,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z)W(dr,dz).$$

Then, for $t_1, t_2 \in (0,\infty)$, we can write $\langle DF_R(t_1), V_{t_2,R}\rangle_{\mathfrak{H}} = A_1 + A_2$, with

$$A_1 = \langle V_{t_1,R}, V_{t_2,R}\rangle_{\mathfrak{H}} = \int_0^{t_1\wedge t_2} \int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\sigma(u(s,y))\sigma(u(s,z))\|y-z\|^{-\beta}dydzds$$

and

$$\begin{aligned} A_2 = \int_0^{t_1\wedge t_2} \int_{\mathbb{R}^4} \left(\int_s^{t_1} \int_{\mathbb{R}^2} \varphi_{t_1,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z)W(dr,dz) \right) \\ \times \|y-y'\|^{-\beta} V_{t_2,R}(s,y')dsdydy'. \end{aligned}$$

(i) *Estimation of* $\mathrm{Var}(A_1)$. From (2.9), we deduce that $\mathrm{Var}(A_1)$ is bounded by

$$\left(\int_0^{t_2\wedge t_1} \left(\text{Var} \int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\sigma(u(s,y))\sigma(u(s,z))\|y-z\|^{-\beta}dydz \right)^{1/2} ds \right)^2. \tag{3.10}$$

Note that the variance term in (3.10) is equal to

$$\begin{aligned} \int_{\mathbb{R}^8} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\varphi_{t_1,R}(s,y')\varphi_{t_2,R}(s,z')\|y-z\|^{-\beta}\|y'-z'\|^{-\beta} \\ \times \text{Cov}\Big[\sigma(u(s,y))\sigma(u(s,z)), \sigma(u(s,y'))\sigma(u(s,z'))\Big] dydzdy'dz'. \end{aligned} \tag{3.11}$$

To estimate the covariance term, we apply the Clark-Ocone formula (see Lemma 2.5) to write

$$\begin{aligned} &\sigma(u(s,y))\sigma(u(s,z)) - \mathbb{E}[\sigma(u(s,y))\sigma(u(s,z))] \\ &\quad = \int_0^s \int_{\mathbb{R}^2} \mathbb{E}\Big\{ D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big) | \mathscr{F}_r \Big\} W(dr,d\gamma). \end{aligned}$$

Then we apply Itô's isometry to obtain

$$\begin{aligned} &\text{Cov}\Big[\sigma(u(s,y))\sigma(u(s,z)), \sigma(u(s,y'))\sigma(u(s,z'))\Big] \\ &= \int_0^s \int_{\mathbb{R}^4} \mathbb{E}\bigg[\mathbb{E}\big\{ D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big) | \mathscr{F}_r \big\} \mathbb{E}\big\{ D_{r,\gamma'}\big(\sigma(u(s,y'))\sigma(u(s,z'))\big) | \mathscr{F}_r \big\} \bigg] \\ &\quad \times \|\gamma-\gamma'\|^{-\beta} d\gamma d\gamma' dr, \end{aligned} \tag{3.12}$$

where, by the chain rule (2.5),

$$D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big) = \sigma(u(s,y))\Sigma_{s,z}D_{r,\gamma}u(s,z) + \sigma(u(s,z))\Sigma_{s,y}D_{r,\gamma}u(s,y).$$

Then by Cauchy-Schwarz inequality and Theorem 1.3, we can see that the covariance term (3.12) is bounded by

$$\int_0^s \int_{\mathbb{R}^4} \Big\| D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big)\Big\|_2 \Big\| D_{r,\gamma'}\big(\sigma(u(s,y'))\sigma(u(s,z'))\big)\Big\|_2 \|\gamma-\gamma'\|^{-\beta} d\gamma d\gamma' dr$$
$$\lesssim \int_0^s dr \int_{\mathbb{R}^4} d\gamma d\gamma' \|\gamma-\gamma'\|^{-\beta} \Big(\|D_{r,\gamma}u(s,z)\|_4 + \|D_{r,\gamma}u(s,y)\|_4 \Big)$$
$$\times \Big(\|D_{r,\gamma'}u(s,z')\|_4 + \|D_{r,\gamma'}u(s,y')\|_4 \Big)$$
$$\lesssim \int_0^s dr \int_{\mathbb{R}^4} d\gamma d\gamma' \|\gamma-\gamma'\|^{-\beta} \big(G_{s-r}(z-\gamma) + G_{s-r}(y-\gamma)\big)\big(G_{s-r}(z'-\gamma') + G_{s-r}(y'-\gamma')\big).$$

Now we can plug the last estimate into (3.11) for further computations:

$$\mathrm{Var}\left(\int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\sigma(u(s,y))\sigma(u(s,z))\|y-z\|^{-\beta} dydz \right)$$
$$\lesssim \int_0^s dr \int_{\mathbb{R}^{12}} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\varphi_{t_1,R}(s,y')\varphi_{t_2,R}(s,z')\|y-z\|^{-\beta}\|y'-z'\|^{-\beta}\|\gamma-\gamma'\|^{-\beta}$$
$$\times \big(G_{s-r}(z-\gamma) + G_{s-r}(y-\gamma)\big)\big(G_{s-r}(z'-\gamma') + G_{s-r}(y'-\gamma')\big) d\gamma d\gamma' dydzdy'dz'. \quad (3.13)$$

In order to obtain $\mathrm{Var}(A_1) \lesssim R^{8-3\beta}$, it is enough to show $\sup_{s\le t_1\wedge t_2} \mathcal{T}_s \lesssim R^{8-3\beta}$ with

$$\mathcal{T}_s := \int_0^s dr \int_{\mathbb{R}^{12}} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\varphi_{t_1,R}(s,y')\varphi_{t_2,R}(s,z')\|y-z\|^{-\beta}\|y'-z'\|^{-\beta}$$
$$\times \|\gamma-\gamma'\|^{-\beta} G_{s-r}(z-\gamma)G_{s-r}(z'-\gamma') d\gamma d\gamma' dydzdy'dz'$$

as other terms from (3.13) can be estimated *in the same way* with *the same bound*.

For $s \in (0, t_1\wedge t_2]$, we write, using (1.13),

$$\mathcal{T}_s = \int_0^s dr \int_{B_R^4} \int_{\mathbb{R}^{12}} G_{t_1-s}(x_1-y)G_{t_1-s}(x_1'-y')G_{t_2-s}(x_2-z)G_{t_2-s}(x_2'-z')G_{s-r}(z-\gamma)$$
$$\times G_{s-r}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta}\|y-z\|^{-\beta}\|y'-z'\|^{-\beta} d\gamma d\gamma' dydzdy'dz'dx_1dx_1'dx_2dx_2'.$$

Making the change of variables

$$(\gamma,\gamma',y,z,y',z',x_1,x_1',x_2,x_2') \to R(\gamma,\gamma',y,z,y',z',x_1,x_1',x_2,x_2')$$

and using $G_t(Rz) = R^{-1}G_{tR^{-1}}(z)$ for every $t, R > 0$ yields

$$R^{-14+3\beta}\mathcal{T}_s = \int_0^s dr \int_{B_1^4} \int_{\mathbb{R}^{12}} G_{\frac{t_1-s}{R}}(x_1-y)G_{\frac{t_1-s}{R}}(x_1'-y')G_{\frac{t_2-s}{R}}(x_2-z)G_{\frac{t_2-s}{R}}(x_2'-z')$$
$$\times G_{\frac{s-r}{R}}(z-\gamma)G_{\frac{s-r}{R}}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta}\|y-z\|^{-\beta}\|y'-z'\|^{-\beta} d\gamma d\gamma' dydzdy'dz'dx_1dx_1'dx_2dx_2'.$$

Using the fact (2.1), we can integrate out x_1, x_1', x_2, x_2' to bound $R^{-14+3\beta}\mathcal{T}_s$ by

$$R^{-10+3\beta}(t_1-s)^2(t_2-s)^2 \int_0^s dr \int_{\mathbb{R}^{12}} \mathbf{1}_{\{\|y\|\vee\|y'\|\vee\|z\|\vee\|z'\|\vee\|\gamma\|\vee\|\gamma'\|\le 1+(t_1+t_2)R^{-1}\}}$$
$$\times G_{\frac{s-r}{R}}(z-\gamma)G_{\frac{s-r}{R}}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta}\|y-z\|^{-\beta}\|y'-z'\|^{-\beta} d\gamma d\gamma' dydzdy'dz'. \quad (3.14)$$

Suppose $R \ge t_1+t_2$ and notice that

$$\sup_{z\in B_2} \int_{B_2} \|y-z\|^{-\beta} dy \le \int_{B_4} \|y\|^{-\beta} dy = \frac{2\pi}{2-\beta} 4^{2-\beta} < \infty.$$

Therefore, integrating out y, y' in (3.14), we obtain

$$\mathcal{T}_s \lesssim R^{10-3\beta} \int_0^s dr \int_{\mathbb{R}^8} \mathbf{1}_{\{\|z\|\vee\|z'\|\vee\|\gamma\|\vee\|\gamma'\|\leq 2\}} G_{\frac{s-r}{R}}(z-\gamma) \times G_{\frac{s-r}{R}}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta} d\gamma d\gamma' dz dz'.$$

We further integrate out z, z' and use (2.1) again to write

$$\sup_{s\leq t_1\wedge t_2} \mathcal{T}_s \lesssim R^{8-3\beta} \int_{\mathbb{R}^8} \mathbf{1}_{\{\|\gamma\|\vee\|\gamma'\|\leq 2\}} \|\gamma-\gamma'\|^{-\beta} d\gamma d\gamma' \lesssim R^{8-3\beta}.$$

So we obtain $\mathrm{Var}(A_1) \lesssim R^{8-3\beta}$ for $R \geq t_1 + t_2$, where the implicit constant does not depend on R.

Next we estimate the variance of A_2.

(ii) *Estimate of* $\mathrm{Var}(A_2)$. Using again (2.9), we write

$$\mathrm{Var}(A_2) \leq \left(\int_0^{t_1\wedge t_2} \left\{ \mathrm{Var} \int_{\mathbb{R}^4} \left(\int_s^{t_1} \int_{\mathbb{R}^2} \varphi_{t_1,R}(r,z)\Sigma_{r,z} D_{s,y}u(r,z) W(dr,dz) \right) \|y-y'\|^{-\beta} \times \varphi_{t_2,R}(s,y')\sigma(u(s,y'))dydy' \right\}^{1/2} ds \right)^2 =: \left(\int_0^{t_1\wedge t_2} \sqrt{\mathcal{U}_s} ds \right)^2.$$

As before, we will show $\sup_{s\leq t_2\wedge t_1} \mathcal{U}_s \lesssim R^{8-3\beta}$.

First note that

$$\int_s^{t_1} \int_{\mathbb{R}^2} \varphi_{t_1,R}(r,z)\Sigma_{r,z} D_{s,y}u(r,z) W(dr,dz) = \mathfrak{M}_{s,y}(t_1),$$

where $\{\mathfrak{M}_{s,y}(\tau) : \tau \in [s,t_1]\}$ is the square-integrable martingale given by

$$\mathfrak{M}_{s,y}(\tau) := \int_s^{\tau} \int_{\mathbb{R}^2} \varphi_{t_1,R}(r,z)\Sigma_{r,z} D_{s,y}u(r,z) W(dr,dz).$$

Then we deduce from the martingale property that

$$\mathbb{E}\big[\sigma(u(s,y'))\mathfrak{M}_{s,y}(t_1)\big] = \mathbb{E}\big[\sigma(u(s,y'))\mathbb{E}(\mathfrak{M}_{s,y}(t_1)|\mathscr{F}_s)\big] = 0,$$

that is, $\mathfrak{M}(t_1)$ and $\sigma\big(u(s,y')\big)$ are uncorrelated. Moreover, by Itô's formula,

$$\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,\widetilde{y}}(t_1) = \underbrace{\int_s^{t_1} \mathfrak{M}_{s,y}(\tau) d\mathfrak{M}_{s,\widetilde{y}}(\tau) + \int_s^{t_1} \mathfrak{M}_{s,\widetilde{y}}(\tau) d\mathfrak{M}_{s,y}(\tau)}_{\text{martingale}-\text{part}} + \langle \mathfrak{M}_{s,y}, \mathfrak{M}_{s,\widetilde{y}} \rangle_{t_1},$$

where the bracket $\langle \mathfrak{M}_{s,y}, \mathfrak{M}_{s,\widetilde{y}} \rangle_{t_1}$ between both martingales is equal to

$$\int_s^{t_1} \int_{\mathbb{R}^4} \varphi_{t_1,R}(r,z)\Sigma_{r,z}\big(D_{s,y}u(r,z)\big)\varphi_{t_1,R}(r,\widetilde{z})\Sigma_{r,\widetilde{z}}\big(D_{s,\widetilde{y}}u(r,\widetilde{z})\big)\|z-\widetilde{z}\|^{-\beta} dz d\widetilde{z} dr.$$

So, using the estimate (1.10), we obtain

$$\begin{aligned} &\mathbb{E}\Big[\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,\widetilde{y}}(t_1)\sigma(u(s,y'))\sigma(u(s,\widetilde{y}'))\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\big(\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,\widetilde{y}}(t_1)|\mathscr{F}_s\big)\sigma(u(s,y'))\sigma(u(s,\widetilde{y}'))\Big] \lesssim \big\|\langle \mathfrak{M}_{s,y}, \mathfrak{M}_{s,\widetilde{y}} \rangle_{t_1}\big\|_2 \\ &\lesssim \int_s^{t_1} \int_{\mathbb{R}^4} \varphi_{t_1,R}(r,z)\|D_{s,y}u(r,z)\|_4 \varphi_{t_1,R}(r,\widetilde{z})\|D_{s,\widetilde{y}}u(r,\widetilde{z})\|_4 \|z-\widetilde{z}\|^{-\beta} dz d\widetilde{z} dr \\ &\lesssim \int_s^{t_1} \int_{\mathbb{R}^4} \varphi_{t_1,R}(r,z) G_{r-s}(y-z)\varphi_{t_1,R}(r,\widetilde{z}) G_{r-s}(\widetilde{y}-\widetilde{z})\|z-\widetilde{z}\|^{-\beta} dz d\widetilde{z} dr. \end{aligned}$$

As a consequence, the variance-term $\mathcal{U}_s$ is indeed a second moment and

$$\begin{aligned}\mathcal{U}_s &= \int_{\mathbb{R}^8} dydy'd\widetilde{y}d\widetilde{y'}\|y-y'\|^{-\beta}\|\widetilde{y}-\widetilde{y'}\|^{-\beta}\varphi_{t_2,R}(s,y')\varphi_{t_2,R}(s,\widetilde{y'})\\ &\qquad\times\mathbb{E}\Big[\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,y'}(t_1)\sigma(u(s,y'))\sigma(u(s,\widetilde{y'}))\Big]\\ &\lesssim \int_s^{t_1} dr\int_{\mathbb{R}^{12}} dzd\widetilde{z}dydy'd\widetilde{y}d\widetilde{y'}\|y-y'\|^{-\beta}\|\widetilde{y}-\widetilde{y'}\|^{-\beta}\|z-\widetilde{z}\|^{-\beta}\\ &\qquad\times\varphi_{t_2,R}(s,y')\varphi_{t_2,R}(s,\widetilde{y'})\varphi_{t_1,R}(r,z)\varphi_{t_1,R}(r,\widetilde{z})G_{r-s}(y-z)G_{r-s}(\widetilde{y}-\widetilde{z}),\end{aligned}$$

which has the same kind of expression as $\mathcal{T}_s$. The same arguments that led to the uniform estimate of $\mathcal{T}_s$ yields

$$\sup_{s\le t_1\wedge t_2}\mathcal{U}_s\lesssim R^{8-3\beta},$$

for $R\ge t_1+t_2$, thus we obtain $\mathrm{Var}(A_2)\lesssim R^{8-3\beta}$ for $R\ge t_1+t_2$. Hence, for $R\ge t_1+t_2$,

$$R^{2\beta-8}\mathrm{Var}\big(\langle DF_R(t_1),V_{t_2,R}\rangle_{\mathfrak{H}}\big)\lesssim R^{2\beta-8}\big[\mathrm{Var}(A_2)+\mathrm{Var}(A_1)\big]\lesssim R^{-\beta}.$$

This completes the proof of (3.4). □

3.3 Tightness

Set $q=\frac{2}{4-\beta}\in(1/2,1)$. As explained in the introduction, by the Kolmogorov-Chentsov criterion for tightness, it is enough to prove the inequality (1.16): For any $T>0$, $p\ge 2$ and for any $0\le s<t\le T\le R$,

$$\big\|F_R(t)-F_R(s)\big\|_p\lesssim R^{1/q}\sqrt{t-s}, \tag{3.15}$$

where the implicit constant does not depend on t,s or R.

Proof of (3.15). Recall that $F_R(t)=\int_0^t\int_{\mathbb{R}^2}\varphi_{t,R}(s,y)\sigma(u(s,y))W(ds,dy)$. Then by BDG inequality (2.3) and (1.19) we have, with the convention that $\varphi_{s,R}(r,y)=0$ if $r>s$,

$$\begin{aligned}\big\|F_R(t)-F_R(s)\big\|_p^2 &\lesssim \bigg\|\int_{[0,t]\times\mathbb{R}^4}\big(\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big)\sigma(u(r,y))\big(\varphi_{t,R}(r,z)-\varphi_{s,R}(r,z)\big)\\ &\qquad\times\sigma(u(r,z))\|y-z\|^{-\beta}dydzdr\bigg\|_{p/2}\\ &\lesssim \bigg\|\int_0^t dr\left(\int_{\mathbb{R}^2}\Big|\big(\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big)\sigma(u(r,y))\Big|^{2q}dy\right)^{1/q}\bigg\|_{p/2}.\end{aligned}$$

Applying Minkowski's inequality yields

$$\begin{aligned}\big\|F_R(t)-F_R(s)\big\|_p^2 &\lesssim \int_0^t dr\left(\int_{\mathbb{R}^2}\big|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big|^{2q}\|\sigma(u(r,y))\|_p^{2q}dy\right)^{1/q}\\ &\lesssim \int_0^t dr\left(\int_{\mathbb{R}^2}\big|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big|^{2q}dy\right)^{1/q}.\end{aligned}\tag{3.16}$$

Note that

$$\begin{aligned}\big|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)\big| &= \mathbf{1}_{\{r\ge s\}}\int_{B_R}G_{t-r}(x-y)dx\\ &\quad+\mathbf{1}_{\{r<s\}}\int_{B_R}\mathbf{1}_{\{\|x-y\|<s-r\}}\big[G_{s-r}(x-y)-G_{t-r}(x-y)\big]dx\\ &\quad+\mathbf{1}_{\{r<s\}}\int_{B_R}\mathbf{1}_{\{\|x-y\|\ge s-r\}}G_{t-r}(x-y)dx\\ &=: S_1+S_2+S_3.\end{aligned}$$

The first summand S_1 is bounded by $\mathbf{1}_{\{r\geq s\}}(t-r)\mathbf{1}_{\{\|y\|\leq R+t\}} \leq (t-s)\mathbf{1}_{\{\|y\|\leq R+t\}}$, in view of Lemma 2.1-(2). For the second summand, we can write

$$\begin{aligned}
S_2 &\leq \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+s\}} \int_{B_R} \mathbf{1}_{\{\|x\|<s-r\}} \big[G_{s-r}(x) - G_{t-r}(x)\big] dx \\
&\leq \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+s\}} \int_{\{\|x\|<s-r\}} \left(\frac{1}{2\pi\sqrt{(s-r)^2 - \|x\|^2}} - \frac{1}{2\pi\sqrt{(t-r)^2-\|x\|^2}} \right) dx \\
&= \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+s\}} \sqrt{t-s}\Big(\sqrt{t+s-2r} - \sqrt{t-s}\Big) \quad \text{by explicit computation} \\
&\lesssim \sqrt{t-s}\,\mathbf{1}_{\{\|y\|\leq R+s\}};
\end{aligned}$$

In the same way, the third summand can be bounded as follows

$$S_3 \leq \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+t\}} \int_{\mathbb{R}^2} \mathbf{1}_{\{s-r\leq \|x\|<t-r\}} G_{t-r}(x)dx \lesssim \mathbf{1}_{\{\|y\|\leq R+t\}}\sqrt{t-s}.$$

Therefore, we can continue with (3.16) to write

$$\big\|F_R(t) - F_R(s)\big\|_p^2 \lesssim \int_0^t dr \left(\int_{\mathbb{R}^2} (t-s)^q \mathbf{1}_{\{\|y\|\leq R+t\}} dy \right)^{1/q} \lesssim (t-s)(R+t)^{2/q}.$$

This implies (3.15). □

4 Fundamental estimate on the Malliavin derivative

This section is devoted to the proof of Theorem 1.3. After a useful lemma, we study the convergence and moment estimates for the Picard approximation in Section 4.1. The main body of the proof of Theorem 1.3 is given in Section 4.2 and we leave proofs of two technical lemmas to Section 4.3. Recall that $\beta \in (0,2)$ is fixed throughout this paper.

Lemma 4.1. *Given any random field $\{\Phi(r,z) : (r,z) \in \mathbb{R}_+ \times \mathbb{R}^2\}$, we have for any $x \in \mathbb{R}^2$, $0 \leq s < t < \infty$ and $p \geq 2$,*

$$\begin{aligned}
&\left\| \int_s^t dr \int_{\mathbb{R}^4} dydz\, G_{t-r}(x-y)G_{t-r}(x-z)\Phi(r,z)\Phi(r,y)\|y-z\|^{-\beta} \right\|_{p/2} \\
&\leq K_\beta t^{\frac{(2-2q)^2}{2q}} \int_s^t dr \int_{\mathbb{R}^2} dz\, G_{t-r}^{2q}(x-z)\big\|\Phi(r,z)\big\|_p^2, \qquad (4.1)
\end{aligned}$$

where $q = \frac{2}{4-\beta} \in (1/2,1)$ and the constant K_β only depends on β.

Proof. By (1.19), there exists some constant C_β that only depends on β such that

$$\begin{aligned}
&\int_{\mathbb{R}^4} dydz\, G_{t-r}(x-y)G_{t-r}(x-z)\Phi(r,z)\Phi(r,y)\|y-z\|^{-\beta} \\
&\leq C_\beta \left(\int_{\mathbb{R}^2} dy\, G_{t-r}^{2q}(x-y)|\Phi(r,y)|^{2q} \right)^{1/q} \\
&\leq C_\beta \left(\frac{(2\pi)^{1-2q}}{2-2q}(t-r)^{2-2q} \right)^{\frac{1}{q}-1} \int_{\mathbb{R}^2} dy\, G_{t-r}^{2q}(x-y)|\Phi(r,y)|^2 \\
&\leq K_\beta t^{\frac{(2-2q)^2}{2q}} \int_{\mathbb{R}^2} dy\, G_{t-r}^{2q}(x-y)|\Phi(r,y)|^2,
\end{aligned}$$

where we have used the fact that $G_{t-r}^{2q}(y)dy$, with $2q<2$, is a finite measure on $\mathbb{R}^2$ with total mass $\frac{(2\pi)^{1-2q}}{2-2q}(t-r)^{2-2q}$ in view of (2.1) and we have put $K_\beta = C_\beta\big(\frac{(2\pi)^{1-2q}}{2-2q}\big)^{\frac{1}{q}-1}$. Therefore, a further application of Minkowski's inequality yields the bound in (4.1). □

4.1 Moment estimates for the Picard approximation

Recall the Picard iteration introduced in (1.17): $u_0(t,x)=1$ and

$$u_{n+1}(t,x)=1+\int_0^t\int_{\mathbb{R}^2}G_{t-s}(x-y)\sigma\big(u_n(s,y)\big)W(ds,dy)\text{ for }n\geq 0. \tag{4.2}$$

Using the estimates (2.4) and (4.1), we can write with $2q=\frac{4}{4-\beta}\in(1,2)$, $p\geq 2$ and $n\geq 1$,

$$\begin{aligned}\|u_n(t,x)\|_p^2&\leq 2+8p\\ &\times\left\|\int_{[0,t]\times\mathbb{R}^4}G_{t-s}(x-z)G_{t-s}(x-y)\sigma(u_n(s,y))\sigma(u_n(s,z))\|y-z\|^{-\beta}dsdzdy\right\|_{p/2}\\ &\leq 2+8pK_\beta t^{\frac{(2-2q)^2}{2q}}\int_0^t ds\int_{\mathbb{R}^2}G_{t-s}^{2q}(x-y)\|\sigma(u_{n-1}(s,y))\|_p^2dy.\end{aligned}$$

Then, using (2.1), we can write

$$\begin{aligned}\|u_n(t,x)\|_p^2&\leq 2+8pK_\beta t^{\frac{(2-2q)^2}{2q}}\int_0^t ds\int_{\mathbb{R}^2}G_{t-s}^{2q}(x-y)\Big(2\sigma(0)^2+2L^2\|u_{n-1}(s,y)\|_p^2\Big)dy\\ &\leq 2+\frac{16pK_\beta(2\pi)^{1-2q}}{(2-2q)(3-2q)}t^{\frac{(2-2q)^2}{2q}+3-2q}\sigma(0)^2\\ &\quad+16pK_\beta t^{\frac{(2-2q)^2}{2q}}L^2\int_0^t ds\int_{\mathbb{R}^2}G_{t-s}^{2q}(x-y)\|u_{n-1}(s,y)\|_p^2dy,\end{aligned}$$

where L is the Lipschitz constant of σ. This leads to

$$H_n(t)\leq c_1+c_2\int_0^t dsH_{n-1}(s), \tag{4.3}$$

where $H_n(t)=\sup_{x\in\mathbb{R}^2}\|u_n(t,x)\|_p^2$,

$$c_1:=2+\frac{pK_\beta^*\sigma(0)^2}{3-2q}t^{\frac{(2-2q)^2}{2q}+3-2q}\quad\text{and}\quad c_2:=pK_\beta^*L^2t^{\frac{(2-2q)^2}{2q}+2-2q},$$

where $K_\beta^*=\frac{16K_\beta(2\pi)^{1-2q}}{2-2q}=16C_\beta\big(\frac{(2\pi)^{1-2q}}{2-2q}\big)^{1/q}$ is a constant depending only on β. Therefore, by iterating the inequality (4.3) and taking into account that $H_0(t)=1$, yields

$$H_n(t)\leq c_1\exp(c_2t).$$

In what follows, we will denote by C_β^* a generic constant that only depends on β and may be different from line to line. In this way, we obtain

$$\|u_n(t,x)\|_p\leq\big(\sqrt{2}+\sqrt{p}C_\beta^*t^{\frac{3-\beta}{2}}|\sigma(0)|\big)\exp\big(pC_\beta^*t^{2-\beta}L^2\big).$$

As a consequence,

$$\|\sigma(u_n(t,x))\|_p\leq|\sigma(0)|+L\big(\sqrt{2}+\sqrt{p}C_\beta^*t^{\frac{3-\beta}{2}}|\sigma(0)|\big)\exp\big(pC_\beta^*t^{2-\beta}L^2\big)=:\kappa_{p,t}. \tag{4.4}$$

4.2 Proof of Theorem 1.3

The proof will be done in several steps.

Step 1. In this step, we will establish the following estimate (4.5) for the p-norm of the Malliavin derivative of the Picard iteration.

Proposition 4.2. *For any $n \geq 3$ and any $p \geq 2$*

$$\big\|D_{s,y}u_{n+1}(t,x)\big\|_p \leq C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y), \tag{4.5}$$

for almost all $(s,y) \in [0,t] \times \mathbb{R}^2$, where $\kappa_{p,t}$ is defined in (4.4) *and the constant $C_{\beta,p,t,L}$ is given by*

$$C_{\beta,p,t,L} := 1 + \sqrt{p}LC^*_\beta t^{\frac{1}{q}-\frac{1}{2}} + pC^*_\beta L^2 t^{\frac{2}{q}-1} + \sum_{k=3}^{\infty} \frac{(pC^*_\beta L^2)^{k/2}}{\sqrt{(k-2)!}} t^{k(\frac{1}{q}-\frac{1}{2})}, \tag{4.6}$$

with C^*_β a constant only depending on β.

One key ingredient for proving Proposition 4.2 is the following Lemma 4.3, which is a consequence of the technical Lemma 1.9. Both Lemma 1.9 and Lemma 4.3 will be proved in Section 4.3.

Lemma 4.3. *For $q \in (1/2,1)$, $\delta \in [1,1/q]$ and $s < t$, we have*

$$K_{s,t}(z) := \int_s^t dr\big[G^{2q}_{t-r} * G^{2q}_{r-s}(z)\big]^\delta \lesssim (t-s)^{1-\delta(2q-1)}G^{\delta(2q-1)}_{t-s}(z).$$

where the implicit constant only depends on q.

Proof of Proposition 4.2. Fix $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$ and $p \geq 2$. Let us first establish the following weaker estimate:

$$u_n(t,x) \in \mathbb{D}^{1,p} \text{ and } \|D_{s,y}u_n(t,x)\|_p \leq CG_{t-s}(x-y), \tag{4.7}$$

for almost all $(s,y) \in [0,t] \times \mathbb{R}^2$, where the constant C may depend on n. It follows from (4.2) that the claim (4.7) holds true for $n = 0,1$, because $D_{s,y}u_0(t,x) = 0$ and $D_{s,y}u_1(t,x) = \sigma(1)G_{t-s}(x-y)$. Now suppose the claim (4.7) holds true for $n \geq 1$, then taking the Malliavin derivative in both sides of equality (4.2) and using the commutation relationship (2.7) and the chain rule (2.5), we obtain

$$D_{s,y}u_{n+1}(t,x) = G_{t-s}(x-y)\sigma\big(u_n(s,y)\big) + \int_s^t\int_{\mathbb{R}^2} G_{t-r}(x-z)\Sigma^{(n)}_{r,z}D_{s,y}u_n(r,z)W(dr,dz),$$

where $\big\{\Sigma^{(n)}_{s,y} : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\big\}$ is an adapted random field that is uniformly bounded by L, for each n. We recall that the constant L is the Lipschitz constant of the function σ appearing in (1.1). It follows that

$$\begin{aligned}\big\|D_{s,y}u_{n+1}(t,x)\big\|_p^2 &\leq 2\kappa^2_{p,t}G^2_{t-s}(x-y) + 8p\bigg\|\int_s^t\int_{\mathbb{R}^4} G_{t-r}(x-z)G_{t-r}(x-z')\Sigma^{(n)}_{r,z} \\ &\qquad \times D_{s,y}u_n(r,z)\Sigma^{(n)}_{r,z'}D_{s,y}u_n(r,z')\|z-z'\|^{-\beta}dzdz'dr\bigg\|_{p/2} \quad \text{by BDG (2.4)} \\ &\leq 2\kappa^2_{p,t}G^2_{t-s}(x-y) + 8pL^2C^2_n\int_s^t \big\|G_{t-r}(x-\bullet)G_{r-s}(y-\bullet)\big\|^2_{\mathfrak{H}_0}dr,\end{aligned}$$

by applying Minkowski's inequality and using the induction hypothesis, where $\kappa_{p,t}$ is defined in (4.4) and $\mathfrak{H}_0$ has been introduced in (1.18). Note that Lemma 1.8 (see (1.19)) implies

$$\int_s^t \big\|G_{t-r}(x-\bullet)G_{r-s}(y-\bullet)\big\|^2_{\mathfrak{H}_0} \leq C_\beta\int_s^t dr\big(G^{2q}_{t-r} * G^{2q}_{r-s}\big)^{1/q}(x-y) \leq C^*_\beta t^{\frac{1}{q}-1}G^{2-\frac{1}{q}}_{t-s}(x-y),$$

where the last inequality follows from Lemma 4.3 with $\delta = 1/q$ and C^*_β is a constant that only depends on β. Finally, using

$$G^{2-\frac{1}{q}}_{t-s}(x-y) \le \big[2\pi(t-s)\big]^{\frac{1}{q}} G^2_{t-s}(x-y), \tag{4.8}$$

we get $\|D_{s,y}u_{n+1}(t,x)\|_p \le C_{n+1}G_{t-s}(x-y)$ with $C_{n+1} = \sqrt{2\kappa^2_{p,t} + pL^2C^2_nC^*_\beta t^{\frac{2}{q}-1}}$ and thus by routine computations, we can show $u_{n+1}(t,x) \in \mathbb{D}^{1,p}$; see also **Step 2**. This shows (4.7) for each n. Moreover, we point out that $D_{s,y}u_{n+1}(t,x) = 0$ if $s \ge t$.

To obtain the uniform estimate in (4.5), we proceed with the finite iterations

$$\begin{aligned}
&D_{s,y}u_{n+1}(t,x) = G_{t-s}(x-y)\sigma\big(u_n(s,y)\big) \\
&\quad + \int_s^t \int_{\mathbb{R}^2} G_{t-r_1}(x-z_1)\Sigma^{(n)}_{r_1,z_1} G_{r_1-s}(z_1-y)\sigma(u_{n-1}(s,y))W(dr_1,dz_1) \\
&\quad + \sum_{k=2}^n \int_s^t \cdots \int_s^{r_{k-1}} \int_{\mathbb{R}^{2k}} G_{r_k-s}(z_k-y)\sigma\big(u_{n-k}(s,y)\big) \\
&\qquad \times \prod_{j=1}^k G_{r_{j-1}-r_j}(z_{j-1}-z_j)\Sigma^{(n+1-j)}_{r_j,z_j} W(dr_j,dz_j) =: \sum_{k=0}^n T^{(n)}_k,
\end{aligned}$$

where $T^{(n)}_k$ denotes the kth item in the sum and $r_0 = t, z_0 = x$. For example, $T^{(n)}_0 = G_{t-s}(x-y)\sigma\big(u_n(s,y)\big)$ and

$$T^{(n)}_1 = \int_s^t \int_{\mathbb{R}^2} G_{t-r_1}(x-z_1)\Sigma^{(n)}_{r_1,z_1} G_{r_1-s}(z_1-y)\sigma(u_{n-1}(s,y))W(dr_1,dz_1).$$

We are going to estimate the p-norm of each of term $T^{(n)}_k$ for $k = 0, \dots, n$.

Case $k=0$: It is clear that

$$\|T^{(n)}_0\|_p \le \kappa_{p,t}G_{t-s}(x-y), \tag{4.9}$$

where $\kappa_{p,t}$ is the constant defined in (4.4).

Case $k=1$: Applying (2.4), Minkowski's inequality and (1.19), we can write

$$\begin{aligned}
\|T^{(n)}_1\|^2_p &\le 4p \bigg\| \int_s^t \int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)G_{t-r_1}(x-z_1')G_{r_1-s}(z_1-y)G_{r_1-s}(z_1'-y) \\
&\qquad \times \|z_1-z_1'\|^{-\beta}\Sigma^{(n)}_{r_1,z_1}\Sigma^{(n)}_{r_1',z_1'}\sigma^2(u_{n-1}(s,y))dz_1dz_1'dr_1 \bigg\|_{p/2} \\
&\le 4pL^2\kappa^2_{p,t}\int_s^t \big\|G_{t-r_1}(x-\bullet)G_{r_1-s}(y-\bullet)\big\|^2_{\mathfrak{H}_0}dr_1 \quad \text{with } \mathfrak{H}_0 \text{ introduced in (1.18)} \\
&\le 4pL^2\kappa^2_{p,t}C_\beta \int_s^t \left(\int_{\mathbb{R}^2} G^{2q}_{t-r_1}(x-z_1)G^{2q}_{r_1-s}(z_1-y)dz_1\right)^{1/q} dr_1,
\end{aligned}$$

with $q = 2/(4-\beta)$. Then, we can deduce immediately from Lemma 4.3 (with $\delta = 1/q$) that

$$\|T^{(n)}_1\|^2_p \le pL^2\kappa^2_{p,t}C^*_\beta t^{\frac{1}{q}-1}G^{2-\frac{1}{q}}_{t-s}(x-y), \tag{4.10}$$

for some generic constant C^*_β, which only depends on β. Taking (4.8) into account, we obtain

$$\|T^{(n)}_1\|_p \le \sqrt{p}L\kappa_{p,t}C^*_\beta t^{\frac{1}{q}-\frac{1}{2}}G_{t-s}(x-y). \tag{4.11}$$

Case $k=2$: We can write

$$T_2^{(n)} = \int_s^t \int_{\mathbb{R}^2} W(dr_1, dz_1) G_{t-r_1}(x-z_1) \Sigma_{r_1,z_1}^{(n)} N_{r_1,z_1}$$

with N_{r_1,z_1} defined to be

$$N_{r_1,z_1} = \int_s^{r_1} \int_{\mathbb{R}^2} G_{r_2-s}(z_2-y)\sigma\big(u_{n-2}(r_2,z_2)\big) G_{r_1-r_2}(z_1-z_2)\Sigma_{r_2,z_2}^{(n-1)} W(dr_2,dz_2),$$

which is clearly $\mathscr{F}_{r_1}$-measurable. Applying again (2.4), Minkowski's inequality and (1.19), we can bound $\|T_2^{(n)}\|_p^2$ by

$$\begin{aligned}
&4p\left\| \int_s^t \int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)G_{t-r_1}(x-z_1')\|z_1-z_1'\|^{-\beta}\Sigma_{r_1,z_1}^{(n)}\Sigma_{r_1',z_1'}^{(n)} N_{r_1,z_1}N_{r_1,z_1'}dz_1dz_1'dr_1 \right\|_{p/2} \\
&\le 4pL^2 \int_s^t \int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)G_{t-r_1}(x-z_1')G_{r_1-s}(z_1-y)G_{r_1-s}(z_1'-y) \\
&\qquad \times \|N_{r_1,z_1}\|_p \|N_{r_1,z_1'}\|_p \|z_1-z_1'\|^{-\beta} dz_1dz_1'dr_1 \\
&\le 4pL^2 C_\beta \int_s^t \left(\int_{\mathbb{R}^2} G_{t-r_1}^{2q}(x-z_1)\|N_{r_1,z_1}\|_p^{2q}dz_1 \right)^{1/q} dr_1. \qquad (4.12)
\end{aligned}$$

The same arguments used to obtain the bound (4.11) for $\|T_1^{(n)}\|_p$ yield

$$\|N_{r_1,z_1}\|_p \le \sqrt{p} L \kappa_{p,t} C_\beta^* t^{\frac{1}{q}-\frac{1}{2}} G_{r_1-s}(z_1-y). \qquad (4.13)$$

Substituting (4.13) into (4.12) and applying Lemma 4.3 with $\delta = 1/q$, we obtain

$$\begin{aligned}
\|T_2^{(n)}\|_p^2 &\le 4pL^2C_\beta(\sqrt{p}L\kappa_{p,t}C_\beta^* t^{\frac{1}{q}-\frac{1}{2}})^2 \int_s^t \left(\int_{\mathbb{R}^2} G_{t-r_1}^{2q}(x-z_1)G_{r_1-s}^{2q}(z_1-y)dz_1 \right)^{1/q} dr_1 \\
&\le p^2L^4\kappa_{p,t}^2 C_\beta^* t^{\frac{3}{q}-2} G_{t-s}^{2-\frac{1}{q}}(x-y),
\end{aligned}$$

which implies

$$\|T_2^{(n)}\|_p \le pL^2\kappa_{p,t}C_\beta^* t^{\frac{3}{2q}-1} G_{t-s}^{1-\frac{1}{2q}}(x-y).$$

In view of (4.8), we obtain

$$\|T_2^{(n)}\|_p \le pL^2\kappa_{p,t}C_\beta^* t^{\frac{2}{q}-1} G_{t-s}(x-y).$$

Case $3 \le k \le n$: The strategy to handle these cases will be slightly different. We need to get rid of the power $\frac{1}{q}$ in order to iterate the integrals in the time variables and obtain a summable series. We can write

$$T_k^{(n)} = \int_s^t \int_{\mathbb{R}^2} W(dr_1,dz_1)G_{t-r_1}(x-z_1)\Sigma_{r_1,z_1}^{(n)} \widehat{N}_{r_1,z_1}$$

with $\widehat{N}_{r_1,z_1}$ defined to be

$$\begin{aligned}
\widehat{N}_{r_1,z_1} = \int_{s<r_k<\cdots<r_2<r_1} \int_{\mathbb{R}^{2k-2}} & G_{r_k-s}(z_k-y)\sigma\big(u_{n-k}(s,y)\big) \\
&\times \prod_{j=2}^k G_{r_{j-1}-r_j}(z_{j-1}-z_j)\Sigma_{r_j,z_j}^{(n+1-j)} W(dr_j,dz_j),
\end{aligned}$$

which is $\mathscr{F}_{r_1}$-measurable. Then, we deduce from (2.4) and (4.1) that

$$
\begin{aligned}
\big\|T_k^{(n)}\big\|_p^2 &\leq 4p \bigg\| \int_s^t dr_1 \int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)\Sigma_{r_1,z_1}^{(n)} \widehat{N}_{r_1,z_1} G_{t-r_1}(x-z_1')\Sigma_{r_1,z_1'}^{(n)} \widehat{N}_{r_1,z_1'} \\
&\qquad\qquad \times \|z_1'-z_1\|^{-\beta} dz_1 dz_1' \bigg\|_{p/2} \\
&\leq 4pK_\beta L^2 t^{\frac{(2-2q)^2}{2q}} \int_s^t dr_1 \int_{\mathbb{R}^2} dz_1 G_{t-r_1}^{2q}(x-z_1) \|\widehat{N}_{r_1,z_1}\|_p^2.
\end{aligned}
$$

Now we can iterate the above process to obtain

$$
\begin{aligned}
\big\|T_k^{(n)}\big\|_p^2 &\leq \left(4pL^2K_\beta t^{\frac{(2-2q)^2}{2q}}\right)^{k-1} \int_s^t dr_1 \int_s^{r_1} dr_2 \cdots \int_s^{r_{k-2}} dr_{k-1} \int_{\mathbb{R}^{2k-2}} dz_1 \cdots dz_{k-1} \\
&\quad \times G_{t-r_1}^{2q}(x-z_1) G_{r_1-r_2}^{2q}(z_1-z_2) \cdots G_{r_{k-2}-r_{k-1}}^{2q}(z_{k-2}-z_{k-1}) \big\|\widetilde{N}_{r_{k-1},z_{k-1}}\big\|_p^2, \quad (4.14)
\end{aligned}
$$

where $\widetilde{N}_{r_{k-1},z_{k-1}}$ is defined to be

$$
\int_s^{r_{k-1}} \int_{\mathbb{R}^2} W(dr_k, dz_k) \sigma\big(u_{n-k}(s,y)\big) G_{r_{k-1}-r_k}(z_{k-1}-z_k) \Sigma_{r_k,z_k}^{(n+1-k)} G_{r_k-s}(z_k-y).
$$

Therefore, the same arguments for estimating $\|T_1^{(n)}\|_p^2$ (see (4.10)), lead to

$$
\big\|\widetilde{N}_{r_{k-1},z_{k-1}}\big\|_p^2 \leq p\kappa_{p,t}^2 L^2 C_\beta^* t^{\frac{1}{q}-1} G_{r_{k-1}-s}^{2-\frac{1}{q}}\big(z_{k-1}-y\big), \tag{4.15}
$$

with C_β^* being a generic constant that only depends on β. On the other hand, applying Lemma 4.3 with $\delta = 1$, we can write

$$
\begin{aligned}
&\int_{r_{k-1}}^{r_{k-3}} dr_{k-2} \int_{\mathbb{R}^2} dz_{k-2} G_{r_{k-3}-r_{k-2}}^{2q}(z_{k-3}-z_{k-2}) G_{r_{k-2}-r_{k-1}}^{2q}(z_{k-2}-z_{k-1}) \\
&\qquad \lesssim t^{2-2q} G_{r_{k-3}-r_{k-1}}^{2q-1}(z_{k-3}-z_{k-1}), \qquad (4.16)
\end{aligned}
$$

with the convention $z_0 = x$ and $r_0 = t$. Plugging the estimates (4.15) and (4.16) into (4.14), yields

$$
\begin{aligned}
\big\|T_k^{(n)}\big\|_p^2 &\leq \kappa_{p,t}^2 \left(pL^2 C_\beta^* t^{\frac{2(1-q)^2}{q}}\right)^k t^{5-4q-\frac{1}{q}} \\
&\quad \times \int_s^t dr_1 \int_s^{r_1} dr_2 \cdots \int_s^{r_{k-3}} dr_{k-1} \int_{\mathbb{R}^{2k-4}} dz_1 \cdots dz_{k-3} dz_{k-1} \\
&\quad \times G_{t-r_1}^{2q}(x-z_1) \cdots G_{r_{k-4}-r_{k-3}}^{2q}(z_{k-4}-z_{k-3}) \\
&\quad \times G_{r_{k-3}-r_{k-1}}^{2q-1}(z_{k-3}-z_{k-1}) G_{r_{k-1}-s}^{2-\frac{1}{q}}\big(z_{k-1}-y\big)
\end{aligned}
$$

By Cauchy-Schwartz inequality and (2.1),

$$
\begin{aligned}
&\int_{\mathbb{R}^2} G_{r_{k-3}-r_{k-1}}^{2q-1}(z_{k-3}-z_{k-1}) G_{r_{k-1}-s}^{2-\frac{1}{q}}\big(z_{k-1}-y\big) dz_{k-1} \\
&\qquad \leq \left[\int_{\mathbb{R}^2} G_{r_{k-3}-r_{k-1}}^{4q-2}(z) dz \int_{\mathbb{R}^2} G_{r_{k-1}-s}^{4-\frac{2}{q}}(z) dz\right]^{1/2} \leq C_\beta^* t^{1-2q+\frac{1}{q}}.
\end{aligned}
$$

In this way, we obtain

$$
\begin{aligned}
\big\|T_k^{(n)}\big\|_p^2 &\leq \kappa_{p,t}^2 (pL^2 C_\beta^* t^{\frac{2(1-q)^2}{q}})^k t^{6(1-q)} \mathbf{1}_{\{\|x-y\|<t-s\}} \qquad (4.17) \\
&\quad \times \int_s^t dr_1 \int_s^{r_1} dr_2 \cdots \int_s^{r_{k-3}} dr_{k-1} \int_{\mathbb{R}^{2k-6}} dz_1 \cdots dz_{k-3} \\
&\quad \times G_{t-r_1}^{2q}(x-z_1) \cdots G_{r_{k-4}-r_{k-3}}^{2q}(z_{k-4}-z_{k-3})
\end{aligned}
$$

The indicator function $\mathbf{1}_{\{\|x-y\|<t-s\}}$ appears in (4.17), because

$$\mathbf{1}_{\{\|z_{k-1}-y\|<r_{k-1}-s, \|z_{k-3}-z_{k-1}\|<r_{k-3}-r_{k-1}, \dots, \|x-z_1\|<t-r_1\}} \leq \mathbf{1}_{\{\|x-y\|<t-s\}}.$$

Now, we can perform the integration with respect to $dz_{k-3}, \dots, dz_1$ one by one to get

$$\begin{aligned}&\int_{\mathbb{R}^{2k-6}} dz_1 \cdots dz_{k-3} G^{2q}_{t-r_1}(x-z_1) G^{2q}_{r_1-r_2}(z_1-z_2) \cdots G^{2q}_{r_{k-4}-r_{k-3}}(z_{k-4}-z_{k-3}) \\ &\quad= \left(\frac{(2\pi)^{1-2q}}{2-2q}\right)^{k-3} \times \prod_{j=1}^{k-3} \big(r_{j-1}-r_j\big)^{2-2q} \leq \left(\frac{(2\pi)^{1-2q}}{2-2q} t^{2-2q}\right)^{k-3},\end{aligned}$$

in view of the equality (2.1). Together with the integration on the simplex $\{t > r_1 > \cdots > r_{k-3} > r_{k-1} > s\}$, we get

$$\big\|T^{(n)}_k\big\|_p^2 \leq \frac{(pC^*_\beta L^2)^k}{(k-2)!} \kappa^2_{p,t} t^{k(\frac{2}{q}-1)-2} \mathbf{1}_{\{\|x-y\|<t-s\}}.$$

Thus, taking into account that

$$\mathbf{1}_{\{\|x-y\|<t-s\}} \leq \big[2\pi(t-s)\big]^2 G^2_{t-s}(x-y),$$

we obtain for $k \in \{3, \dots, n\}$,

$$\big\|T^{(n)}_k\big\|_p \leq \kappa_{p,t} \frac{(pC^*_\beta L^2)^{k/2}}{\sqrt{(k-2)!}} t^{\,k(\frac{1}{q}-\frac{1}{2})} G_{t-s}(x-y). \tag{4.18}$$

Hence, we deduce from (4.9), (4.11) and (4.18) that for any $n \geq 3$,

$$\big\|D_{s,y} u_{n+1}(t,x)\big\|_p \leq \sum_{k=0}^n \big\|T^{(n)}_k\big\|_p \leq C_{\beta,p,t,L} \kappa_{p,t} G_{t-s}(x-y),$$

where the constant $C_{\beta,p,t,L}$ is defined in (4.6). This proves Proposition 4.2. $\square$

Step 2. We are going to show that $D_{s,y}u(t,x)$ is a real-valued random variable. As a consequence of (1.19), (4.5) and (2.1), we have for any $p \geq 2$ and with $q = 2/(4-\beta)$

$$\begin{aligned}&\mathbb{E}\Big[\|Du_{n+1}(t,x)\|^p_{\mathfrak{H}}\Big]^{2/p} = \bigg\|\int_{\mathbb{R}_+} ds \big\|D_{s,\bullet} u_{n+1}(t,x)\big\|^2_{\mathfrak{H}_0}\bigg\|_{p/2} \\ &\lesssim \bigg\|\int_{\mathbb{R}_+} ds \left(\int_{\mathbb{R}^2} |D_{s,y}u_{n+1}(t,x)|^{2q} dy\right)^{1/q}\bigg\|_{p/2} \\ &\lesssim \int_{\mathbb{R}_+} ds \left(\int_{\mathbb{R}^2} \big\|D_{s,y}u_{n+1}(t,x)\big\|^{2q}_p dy\right)^{1/q} \text{ by applying Minkowski twice} \\ &\lesssim \int_{\mathbb{R}_+} ds \left(\int_{\mathbb{R}^2} G^{2q}_{t-s}(x-y) dy\right)^{1/q} \lesssim \int_0^t (t-s)^{\frac{2-2q}{q}} ds \lesssim 1.\end{aligned}$$

One can first read from the above estimates that $\{Du_{n+1}(t,x), n \geq 1\}$ is uniformly bounded in $L^p\big(\Omega; \mathfrak{H}\big)$, which together with the L^p-convergence of $u_n(t,x)$ to $u(t,x)$ implies the convergence of $Du_{n+1}(t,x)$ to $Du(t,x)$ in the weak topology on $L^p\big(\Omega; \mathfrak{H}\big)$ up to a subsequence; this fact is well-known in the literature, see for instance [14]. One can deduce from the same arguments that $\{Du_{n+1}(t,x), n \geq 1\}$ is uniformly bounded in $L^p\big(\Omega; L^{2q}(\mathbb{R}_+ \times \mathbb{R}^2)\big)$:

$$\begin{aligned}\big\|Du_{n+1}(t,x)\big\|^p_{L^p(\Omega;L^{2q}(\mathbb{R}_+\times\mathbb{R}^2))} &= \left\|\int_{\mathbb{R}_+\times\mathbb{R}^2} |D_{s,y}u_{n+1}(t,x)|^{2q} dyds\right\|^{\frac{p}{2q}}_{\frac{p}{2q}} \\ &\leq \left(\int_{\mathbb{R}_+\times\mathbb{R}^2} \big\|D_{s,y}u_{n+1}(t,x)\big\|^{2q}_p dyds\right)^{\frac{p}{2q}} \lesssim \left(\int_{\mathbb{R}_+\times\mathbb{R}^2} G^{2q}_{t-s}(x-y)dyds\right)^{\frac{p}{2q}} \lesssim 1.\end{aligned}$$

So up to a subsequence, $Du_n(t,x)$ also converges to $Du(t,x)$ in the weak topology on $L^p\big(\Omega; L^{2q}(\mathbb{R}_+ \times \mathbb{R}^2)\big)$. In particular, we have $(2q < 2 \leq p < \infty)$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \left\|\int_{\mathbb{R}_+\times\mathbb{R}^2} |D_{s,y}u(t,x)|^{2q} dyds\right\|_{\frac{p}{2q}} < +\infty$$

and $D_{s,y}u(t,x)$ is a real function in (s,y).

Step 3. Let us prove the lower bound. By Lemma 2.5, we can write

$$u(t,x) - 1 = \int_0^t \int_{\mathbb{R}^2} \mathbb{E}\big[D_{s,y}u(t,x)|\mathscr{F}_s\big] W(ds,dy),$$

so that a comparison with (1.3) yields $\mathbb{E}\big[D_{s,y}u(t,x)|\mathscr{F}_s\big] = G_{t-s}(x-y)\sigma(u(s,y))$ almost everywhere in $\Omega \times \mathbb{R}_+ \times \mathbb{R}^2$. It follows that

$$\big\|\mathbb{E}[D_{s,y}u(t,x)|\mathscr{F}_s]\big\|_p = G_{t-s}(x-y)\big\|\sigma(u_{s,y})\big\|_p,$$

thus by conditional Jensen, we have

$$\big\|D_{s,y}u(t,x)\big\|_p \geq G_{t-s}(x-y)\big\|\sigma(u_{s,y})\big\|_p,$$

which is exactly the lower bound in (1.10).

Step 4. We are finally in a position to prove the upper bound in (1.10). Put $p^\star = p/(p-1)$, which is the conjugate exponent for p. Let us pick a nonnegative function $M \in C_c(\mathbb{R}_+ \times \mathbb{R}^2)$ and random variable $\mathcal{Z} \in L^{p^\star}(\Omega)$ with $\|\mathcal{Z}\|_{p^\star} \leq 1$. Since $Du_n(t,x)$ converges to $Du(t,x)$ in the weak topology on $L^p\big(\Omega; L^{2q}(\mathbb{R}_+ \times \mathbb{R}^2)\big)$ along some subsequence (say $Du_{n_k}(t,x)$), we have, in view of (4.5)

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}^2} M(s,y)\mathbb{E}\big[ZD_{s,y}u(t,x)\big]dsdy &= \lim_{k\to\infty}\int_{\mathbb{R}_+\times\mathbb{R}^2} M(s,y)\mathbb{E}\big[ZD_{s,y}u_{n_k}(t,x)\big]dsdy \\ &\leq C_{\beta,p,t,L}\kappa_{p,t}\int_{\mathbb{R}_+\times\mathbb{R}^2} M(s,y)G_{t-s}(x-y)dsdy.\end{aligned}$$

This implies that for almost all $(s,y) \in [0, t \times \mathbb{R}^2$,

$$\mathbb{E}\big[ZD_{s,y}u(t,x)\big] \leq C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y)$$

Taking the supremum over $\{\mathcal{Z} : \|\mathcal{Z}\|_{p^\star} \leq 1\}$ yields

$$\|D_{s,y}u(t,x)\|_p \leq C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y),$$

which finishes the proof.

4.3 Proof of technical lemmas

For convenience, let us recall Lemma 1.9 below.

Lemma 1.9. For $t > s$, with $\|z\| = \mathbf{w} > 0$ and $q \in (1/2, 1)$

$$G_t^{2q} * G_s^{2q}(z) \lesssim \mathbf{1}_{\{\mathbf{w}<s\}} \big[t^2 - (s-\mathbf{w})^2\big]^{1-2q} + \big[t^2 - (s+\mathbf{w})^2\big]^{1-2q} \mathbf{1}_{\{t>s+\mathbf{w}\}}$$
$$+ \mathbf{1}_{\{|s-\mathbf{w}|<t<s+\mathbf{w}\}} \big[(\mathbf{w}+s)^2 - t^2\big]^{-q+\frac{1}{2}} \big[t^2 - (s-\mathbf{w})^2\big]^{-q+\frac{1}{2}}, \tag{1.20}$$

where the implicit constant depends only on q.

Proof of Lemma 1.9. We are interested in estimating

$$\mathbf{I} = \int_{\mathbb{R}^2} \big(t^2 - \|x\|^2\big)_+^{-q} \big(s^2 - \|x-z\|^2\big)_+^{-q} dx,$$

where $(v)_+^{-q} = v^{-q}$ for $v > 0$ and $(v)_+^{-q} = 0$ for $v \le 0$. Because the convolution of two radial functions is radial, the quantity $\mathbf{I}$ depends only on s, t and $\|z\|$. Hence, we can assume additionally that $z = (\mathbf{w}, 0)$, where $\mathbf{w} > 0$. Note that the integral $\mathbf{I}$ vanishes if $t + s<\mathbf{w}$ and we can write, putting $x = (\xi, \eta)$,

$$\mathbf{I} = \int_{\mathbb{R}^2} \big(t^2 - \xi^2 - \eta^2\big)_+^{-q} \big(s^2 - (\xi-\mathbf{w})^2 - \eta^2\big)_+^{-q} d\xi d\eta.$$

Making the change of variables $(x, y) = \big(\xi^2 + \eta^2, (\mathbf{w}-\xi)^2 + \eta^2\big)$ yields

$$\mathbf{I} = \frac{1}{2} \int_D (t^2 - x)^{-q} (s^2 - y)^{-q} \big[(\sqrt{x} + \mathbf{w})^2 - y\big]^{-1/2} \big[y - (\sqrt{x} - \mathbf{w})^2\big]^{-1/2} dxdy, \tag{4.19}$$

where

$$D = \Big\{(x, y) \in \mathbb{R}^2 : 0 < x < t^2, 0 < y < s^2, \big(\sqrt{x} - \mathbf{w}\big)^2 < y < \big(\sqrt{x} + \mathbf{w}\big)^2\Big\}.$$

To derive the expression (4.19) for $\mathbf{I}$, we have used the fact that the Jacobian of the change of variables is

$$\left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right| = 4\mathbf{w}|\eta| = 2\big[(\sqrt{x} + \mathbf{w})^2 - y\big]^{1/2} \big[y - (\sqrt{x} - \mathbf{w})^2\big]^{1/2}.$$

Then, integrating first in the variable y yields

$$\mathbf{I} = \frac{1}{2} \int_0^{t^2} dx (t^2 - x)^{-q} \int_{D(x)} dy\ (s^2 - y)^{-q} \big[(\sqrt{x} + \mathbf{w})^2 - y\big]^{-1/2} \big[y - (\sqrt{x} - \mathbf{w})^2\big]^{-1/2}$$
$$=: \frac{1}{2} \int_0^{t^2} (t^2 - x)^{-q} \mathcal{S}_q(x) dx,$$

where

$$D(x) = \big\{y \in \mathbb{R} : (x, y) \in D\big\} = \Big\{y \in \mathbb{R} : y < s^2, \big(\sqrt{x} - \mathbf{w}\big)^2 < y < \big(\sqrt{x} + \mathbf{w}\big)^2\Big\}$$

and

$$\mathcal{S}_q(x) = \int_{D(x)} dy\ (s^2 - y)^{-q} \big[(\sqrt{x} + \mathbf{w})^2 - y\big]^{-1/2} \big[y - (\sqrt{x} - \mathbf{w})^2\big]^{-1/2}.$$

Let us first deal with $\mathcal{S}_q(x)$ for every $x \in (0, t^2)$. There are two possible cases, depending on the value of x:

(A) When $(\sqrt{x}-\mathbf{w})^2 < s^2 < (\sqrt{x}+\mathbf{w})^2$,

$$\begin{aligned}
\mathcal{S}_q(x) &= \int_{(\sqrt{x}-\mathbf{w})^2}^{s^2} (s^2-y)^{-q}\big[(\sqrt{x}+\mathbf{w})^2-y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2}dy \\
&\leq \mathrm{Beta}(1/2,1-q)\big[(\sqrt{x}+\mathbf{w})^2-s^2)\big]^{-1/2}\big[s^2-(\sqrt{x}-\mathbf{w})^2\big]^{-q+\frac{1}{2}} \\
&\lesssim \big[(\sqrt{x}+\mathbf{w})^2-s^2)\big]^{-1/2}\big[s^2-(\sqrt{x}-\mathbf{w})^2\big]^{-q+\frac{1}{2}}. \qquad (4.20)
\end{aligned}$$

Throughout this section, $\mathrm{Beta}(a,b)$ denotes the usual beta function:

$$\mathrm{Beta}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,\ \ a,b\in(0,\infty).$$

(B) When $(\sqrt{x}-\mathbf{w})^2 < (\sqrt{x}+\mathbf{w})^2 < s^2$,

$$\begin{aligned}
\mathcal{S}_q(x) &= \int_{(\sqrt{x}-\mathbf{w})^2}^{(\sqrt{x}+\mathbf{w})^2} (s^2-y)^{-q}\big[(\sqrt{x}+\mathbf{w})^2-y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2}dy \\
&\leq (s^2-(\sqrt{x}+\mathbf{w})^2)^{-q}\int_{(\sqrt{x}-\mathbf{w})^2}^{(\sqrt{x}+\mathbf{w})^2} \big[(\sqrt{x}+\mathbf{w})^2-y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2}dy \\
&= \mathrm{Beta}(1/2,1/2)\big[s^2-(\sqrt{x}+\mathbf{w})^2\big]^{-q} \lesssim \big[s^2-(\sqrt{x}+\mathbf{w})^2\big]^{-q}.
\end{aligned}$$

Note that three positive numbers a,b,c can form sides of a triangle if and only if *the sum of any two of them is strictly bigger than the third one*, which is equivalent to saying that $|a-b|<c<a+b$. It follows that

$$\begin{aligned}
(\sqrt{x}-\mathbf{w})^2 < s^2 < (\sqrt{x}+\mathbf{w})^2 &\Leftrightarrow \sqrt{x},\mathbf{w},s \text{ can be the sides of a triangle} \\
&\Leftrightarrow (s-\mathbf{w})^2 < x < (s+\mathbf{w})^2.
\end{aligned}$$

Furthermore, it is trivial that $(\sqrt{x}-\mathbf{w})^2 < (\sqrt{x}+\mathbf{w})^2 < s^2 \Leftrightarrow x < (s-\mathbf{w})^2$ and $s > \mathbf{w}$.

Now we decompose the integral $2\mathbf{I} = \int_0^{t^2}(t^2-x)^{-q}\mathcal{S}_q(x)dx$ into two parts corresponding to the cases **(A)** and **(B)**:

$$2\mathbf{I} = \mathbf{I_A} + \mathbf{I_B},$$

where

$$\mathbf{I_A} = \int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\mathcal{S}_q(x)dx \quad\text{and}\quad \mathbf{I_B} = \int_0^{(s-\mathbf{w})^2\wedge t^2}(t^2-x)^{-q}\mathcal{S}_q(x)dx.$$

Estimation of $\mathbf{I_A}$. We first write, using (4.20),

$$\begin{aligned}
\mathbf{I_A} &\lesssim \int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\big[(\sqrt{x}+\mathbf{w})^2-s^2)\big]^{-1/2}\big[s^2-(\sqrt{x}-\mathbf{w})^2\big]^{-q+\frac{1}{2}}dx \\
&= \int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\big[(\mathbf{w}+s)^2-x\big]^{-q+\frac{1}{2}}\big[x-(\mathbf{w}-s)^2\big]^{-q+\frac{1}{2}}\big[(\sqrt{x}+\mathbf{w})^2-s^2\big]^{q-1}dx.
\end{aligned}$$

Recall in this case $\sqrt{x}+\mathbf{w}>s$, which implies $(\sqrt{x}+\mathbf{w})^2-s^2 > x-(s-\mathbf{w})^2>0$. Therefore,

$$\mathbf{I_A} \lesssim \int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\big[(\mathbf{w}+s)^2-x\big]^{-q+\frac{1}{2}}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx.$$

Now we consider the following two sub-cases:

(A1) If $s+\mathbf{w}<t$, then for $(s-\mathbf{w})^2<x<(s+\mathbf{w})^2<t$, we have, with $\gamma=2-q^{-1}$,

$$(t^2-x)^{-q}\le\big[t^2-(s+\mathbf{w})^2\big]^{-q\gamma}\big[(s+\mathbf{w})^2-x\big]^{-q+q\gamma}$$
$$=\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}\big[(s+\mathbf{w})^2-x\big]^{q-1}.$$

Thus,

$$\mathbf{I_A}\lesssim\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}\int_{(s-\mathbf{w})^2}^{(s+\mathbf{w})^2}\big[(\mathbf{w}+s)^2-x\big]^{-1/2}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx$$
$$=\mathrm{Beta}(1/2,1/2)\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}.$$

(A2) If $(s-\mathbf{w})^2<t^2<(s+\mathbf{w})^2$ (*i.e.* $s,\mathbf{w},t$ form triangle sides), then

$$\mathbf{I_A}\lesssim\int_{(s-\mathbf{w})^2}^{t^2}(t^2-x)^{-q}\big[(\mathbf{w}+s)^2-x\big]^{-q+\frac{1}{2}}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx$$
$$\le\big[(\mathbf{w}+s)^2-t^2\big]^{-q+\frac{1}{2}}\int_{(s-\mathbf{w})^2}^{t^2}\big(t^2-x\big)^{-q}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx$$
$$\lesssim\big[(\mathbf{w}+s)^2-t^2\big]^{-q+\frac{1}{2}}\big[t^2-(s-\mathbf{w})^2\big]^{-q+\frac{1}{2}}$$

because $\int_a^b(b-x)^{-q}(x-a)^{-1/2}dx=\mathrm{Beta}(1/2,1-q)(b-a)^{-q+\frac{1}{2}}$ for any $0\le a<b<\infty$ and for any $q<1$.

Combining **(A1)** and **(A2)**, we have obtained

$$\mathbf{I_A}\lesssim\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}\mathbf{1}_{\{t>s+\mathbf{w}\}}$$
$$+\mathbf{1}_{\{|s-\mathbf{w}|<t<s+\mathbf{w}\}}\big[(\mathbf{w}+s)^2-t^2\big]^{\frac{1-2q}{2}}\big[t^2-(s-\mathbf{w})^2\big]^{\frac{1-2q}{2}}. \tag{4.21}$$

Estimation of $\mathbf{I_B}$. In this case, $\sqrt{x}<s-\mathbf{w}$ and $\mathbf{w}<s$, then

$$s^2-(\sqrt{x}+\mathbf{w})^2>(s-\mathbf{w})^2-x>0.$$

Therefore, $\mathcal{S}_q(x)\lesssim\big[(s-\mathbf{w})^2-x\big]^{-q}$ and the quantity $\mathbf{I_B}$ can be bounded as follows

$$\mathbf{I_B}=\int_0^{(s-\mathbf{w})^2}(t^2-x)^{-q}\mathcal{S}_q(x)dx\lesssim\int_0^{(s-\mathbf{w})^2}(t^2-x)^{-q}\big[(s-\mathbf{w})^2-x\big]^{-q}dx$$
$$\lesssim\big[t^2-(s-\mathbf{w})^2\big]^{1-2q}, \tag{4.22}$$

because for any $0<a<b<\infty$ and any $p,q\in(1/2,1)$

$$\int_0^a(b-x)^{-p}(a-x)^{-q}dx=\int_0^a(b-a+y)^{-p}y^{-q}dy=(b-a)^{1-p-q}\int_0^{\frac{a}{b-a}}y^{-q}(1+y)^{-p}dy$$
$$\le(b-a)^{1-p-q}\int_0^\infty y^{-q}(1+y)^{-p}dy\lesssim(b-a)^{1-p-q}.$$

Our proof is done by combining the estimates (4.21) and (4.22) to get (1.20). □

Now let us apply Lemma 1.9 to prove Lemma 4.3.

Proof of Lemma 4.3. Put $\mu = (t-r) \wedge (r-s)$ and $\nu = (t-r) \vee (r-s)$ and assume $\mu \neq \nu$. We apply Lemma 1.9 to write

$$\begin{aligned}
\left(G^{2q}_{t-r} * G^{2q}_{r-s}(z)\right)^{\delta} &\lesssim \Big(\mathbf{1}_{\{\mathbf{w}<\mu\}}\left[\nu^2-(\mu-\mathbf{w})^2\right]^{1-2q} + \left[\nu^2-(\mu+\mathbf{w})^2\right]^{1-2q}\mathbf{1}_{\{\nu>\mu+\mathbf{w}\}} \\
&\quad + \mathbf{1}_{\{|\mu-\mathbf{w}|<\nu<\mu+\mathbf{w}\}}\left[(\mathbf{w}+\mu)^2-\nu^2\right]^{-q+\frac12}\left[\nu^2-(\mu-\mathbf{w})^2\right]^{-q+\frac12}\Big)^{\delta} \\
&\lesssim \mathbf{1}_{\{\mathbf{w}<\mu\}}\left[\nu^2-(\mu-\mathbf{w})^2\right]^{\delta(1-2q)} + \left[\nu^2-(\mu+\mathbf{w})^2\right]^{\delta(1-2q)}\mathbf{1}_{\{\nu>\mu+\mathbf{w}\}} \\
&\quad + \mathbf{1}_{\{|\mu-\mathbf{w}|<\nu<\mu+\mathbf{w}\}}\left[(\mathbf{w}+\mu)^2-\nu^2\right]^{\delta(\frac12-q)}\left[\nu^2-(\mu-\mathbf{w})^2\right]^{\delta(\frac12-q)},
\end{aligned}$$

where $\mathbf{w} = \|z\| > 0$ and $0 > \delta(1-2q) \geq \frac{1}{q} - 2 > -1$. Define

$$\begin{aligned}
K^{(1)}_{s,t}(z) :&= \int_s^t dr \mathbf{1}_{\{\mathbf{w}<\mu\}}\left[\nu^2-(\mu-\mathbf{w})^2\right]^{\delta(1-2q)} \\
&= \int_s^t dr \mathbf{1}_{\{\mathbf{w}<\mu\}}\left[(\nu+\mu-\mathbf{w})(\nu-\mu+\mathbf{w})\right]^{\delta(1-2q)}
\end{aligned}$$

and note that $t-r > r-s$ if and only if $r < \frac{t+s}{2}$. Then, by exact computations and decomposing the integral in the intervals $[s,(t+s)/2]$ and $[(t+s)/2,t]$, yields

$$\begin{aligned}
K^{(1)}_{s,t}(z) &= \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}}\int_{s+\mathbf{w}}^{(t+s)/2}(t-s-\mathbf{w})^{\delta(1-2q)}(t+s+\mathbf{w}-2r)^{\delta(1-2q)}dr \\
&\quad + \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}}\int_{(t+s)/2}^{t-\mathbf{w}}(t-s-\mathbf{w})^{\delta(1-2q)}(2r+\mathbf{w}-t-s)^{\delta(1-2q)}dr \\
&= 2\times \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}}(t-s-\mathbf{w})^{\delta(1-2q)}\frac{1}{2(\delta(1-2q)+1)} \\
&\quad \times \left[(t-s-\mathbf{w})^{\delta(1-2q)+1} - \mathbf{w}^{\delta(1-2q)+1}\right] \\
&\leq \frac{(t-s)^{\delta(1-2q)+1}}{\delta(1-2q)+1}\mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}}(t-s-\mathbf{w})^{\delta(1-2q)} \\
&\lesssim (t-s)^{\delta(1-2q)+1}(t-s)^{\delta(1-2q)}\mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}} \\
&\lesssim (t-s)^{\delta(1-2q)+1}\left[(t-s)^2-\|z\|^2\right]^{\delta(\frac12-q)}\mathbf{1}_{\{\|z\|<t-s\}}. \qquad (4.23)
\end{aligned}$$

By the same arguments, we can get

$$\begin{aligned}
K^{(2)}_{s,t}(z) :&= \int_s^t dr\left[\nu^2-(\mu+\mathbf{w})^2\right]^{\delta(1-2q)}\mathbf{1}_{\{\nu>\mu+\mathbf{w}\}} \\
&= \int_s^t dr\left[(\nu+\mu+\mathbf{w})(\nu-\mu-\mathbf{w})\right]^{\delta(1-2q)}\mathbf{1}_{\{\nu>\mu+\mathbf{w}\}} \\
&= \mathbf{1}_{\{t-s>\mathbf{w}\}}(t-s+\mathbf{w})^{\delta(1-2q)}\int_s^{(t+s-\mathbf{w})/2}\left(t+s-2r-\mathbf{w}\right)^{\delta(1-2q)}dr \\
&\quad + \mathbf{1}_{\{t-s>\mathbf{w}\}}(t-s+\mathbf{w})^{\delta(1-2q)}\int_{(t+s+\mathbf{w})/2}^{t}\left(2r-s-t-\mathbf{w}\right)^{\delta(1-2q)}dr \\
&= \mathbf{1}_{\{t-s>\mathbf{w}\}}(t-s+\mathbf{w})^{\delta(1-2q)}\frac{1}{2(\delta(1-2q)+1)}(t-s-\mathbf{w})^{\delta(1-2q)+1}\times 2 \\
&\lesssim (t-s)^{\delta(1-2q)+1}\left[(t-s)^2-\|z\|^2\right]^{\delta(\frac12-q)}\mathbf{1}_{\{\|z\|<t-s\}}. \qquad (4.24)
\end{aligned}$$

Similarly, we first write

$$\begin{aligned}K^{(3)}_{s,t}(z) &:= \int_s^t dr \mathbf{1}_{\{|\mu-\mathbf{w}|<\nu<\mu+\mathbf{w}\}}\big[(\mathbf{w}+\mu)^2-\nu^2\big]^{\delta(\frac12-q)}\big[\nu^2-(\mu-\mathbf{w})^2\big]^{\delta(\frac12-q)}\\ &= \int_s^t dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}}\big[(\mu+\nu)^2-\mathbf{w}^2\big]^{\delta(\frac12-q)}(\mathbf{w}+\mu-\nu)^{\delta(\frac12-q)}(\mathbf{w}+\nu-\mu)^{\delta(\frac12-q)}\\ &= \big[(t-s)^2-\mathbf{w}^2\big]^{\delta(\frac12-q)}\int_s^t dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}}(\mathbf{w}+\mu-\nu)^{\delta(\frac12-q)}(\mathbf{w}+\nu-\mu)^{\delta(\frac12-q)}.\end{aligned}$$

Recall $t-r>r-s$ if and only if $r<\frac{t+s}{2}$. Then

$$\begin{aligned}&\int_s^{(t+s)/2} dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}}(\mathbf{w}+\mu-\nu)^{\delta(\frac12-q)}(\mathbf{w}+\nu-\mu)^{\delta(\frac12-q)}\\ &= \mathbf{1}_{\{\mathbf{w}<t-s\}}\int_{\frac{t+s-\mathbf{w}}{2}}^{\frac{t+s}{2}} dr\,(\mathbf{w}-t-s+2r)^{\delta(\frac12-q)}(\mathbf{w}+t+s-2r)^{\delta(\frac12-q)}\\ &= \mathbf{1}_{\{\mathbf{w}<t-s\}}2^{\delta(1-2q)}\int_a^b (r-a)^{-\delta(\frac12-q)}(c-r)^{\delta(\frac12-q)}dr,\end{aligned}$$

where $a=\dfrac{t+s-\mathbf{w}}{2}<b=\dfrac{t+s}{2}<c=\dfrac{t+s+\mathbf{w}}{2}$. It is easy to show that

$$\begin{aligned}\int_a^b (r-a)^{\delta(\frac12-q)}(c-r)^{\delta(\frac12-q)}dr &= (c-a)^{\delta(1-2q)+1}\int_0^{\frac{b-a}{c-a}} t^{\delta(\frac12-q)}(1-t)^{\delta(\frac12-q)}dt\\ &\le (c-a)^{\delta(1-2q)+1}\int_0^1 t^{\delta(\frac12-q)}(1-t)^{\delta(\frac12-q)}dt\\ &= \mathrm{Beta}(\delta(\frac12-q)+1,\delta(\frac12-q)+1))(c-a)^{\delta(1-2q)+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_s^{(t+s)/2} dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}}(\mathbf{w}+\mu-\nu)^{\delta(\frac12-q)}(\mathbf{w}+\nu-\mu)^{\delta(\frac12-q)}\\ &\quad\lesssim \mathbf{1}_{\{\mathbf{w}<t-s\}}\mathbf{w}^{\delta(1-2q)+1}\le (t-s)^{\delta(1-2q)+1}\mathbf{1}_{\{\|z\|<t-s\}}.\end{aligned}$$

In the same manner, we can get

$$\begin{aligned}&\int_{(t+s)/2}^t dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}}(\mathbf{w}+\mu-\nu)^{\delta(\frac12-q)}(\mathbf{w}+\nu-\mu)^{\delta(\frac12-q)}\\ &= \mathbf{1}_{\{\mathbf{w}<t-s\}}\int_{\frac{t+s}{2}}^{\frac{t+s+\mathbf{w}}{2}} dr\,(\mathbf{w}-t-s+2r)^{\delta(\frac12-q)}(\mathbf{w}+t+s-2r)^{\delta(\frac12-q)}\\ &= \mathbf{1}_{\{\mathbf{w}<t-s\}}2^{\delta(1-2q)}\int_b^c (c-r)^{\delta(\frac12-q)}(r-a)^{\delta(\frac12-q)}dr\\ &\le \mathbf{1}_{\{\mathbf{w}<t-s\}}2^{\delta(1-2q)}(c-a)^{\delta(1-2q)+1}\mathrm{Beta}(\delta(\frac12-q)+1,\delta(\frac12-q)+1)\\ &\lesssim \mathbf{1}_{\{\mathbf{w}<t-s\}}\mathbf{w}^{\delta(1-2q)+1}\le (t-s)^{\delta(1-2q)+1}\mathbf{1}_{\{\|z\|<t-s\}},\end{aligned}$$

where $a=\dfrac{t+s-\mathbf{w}}{2}<b=\dfrac{t+s}{2}<c=\dfrac{t+s+\mathbf{w}}{2}$. Thus, we obtain

$$K^{(3)}_{s,t}(z)\lesssim (t-s)^{\delta(1-2q)+1}\big[(t-s)^2-\|z\|^2\big]_+^{\delta(\frac12-q)}\mathbf{1}_{\{\|z\|<t-s\}}, \tag{4.25}$$

with $\delta(q-\frac12)\le 1-\frac{1}{2q}\in(0,\frac12)$. Combining the estimates (4.23), (4.24) and (4.25) allows us to finish the proof. □

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