On the Manhattan pinball problem[*]

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Abstract

We consider the periodic Manhattan lattice with alternating orientations going north-south and east-west. Place obstructions on vertices independently with probability $0 < p < 1$. A particle is moving on the edges with unit speed following the orientation of the lattice and it will turn only when encountering an obstruction. The problem is that for which value of p is the trajectory of the particle closed almost surely. We prove this is true for $p > \frac{1}{2} - \varepsilon$ with some $\varepsilon > 0$.

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1 Introduction

We consider the Manhattan pinball problem which is a network model of a quantum localization problem (see [3] and also [12, Page 237],[5]). There are equivalent ways to state the problem formally. We follow [12, Page 238] and state it by using bond percolation. Consider the $\mathbb{Z}^2$ lattice embedded into the plane $\mathbb{R}^2$. Denote the tilted lattice by

$$\widetilde{\mathbb{Z}^2} = \left\{ \left(x + \frac{1}{2}, y + \frac{1}{2} \right) : x, y \text{ are integers and } x - y \text{ is even} \right\},$$

and for any $a, b \in \widetilde{\mathbb{Z}^2}$, a and b are connected by an edge if and only if $|a - b| = \sqrt{2}$. Here, $|\cdot|$ is the Euclidean distance on $\mathbb{R}^2$. See Figure 1 for an illustration. Given $0 < p < 1$, we consider the *Bernoulli bond percolation* on $\widetilde{\mathbb{Z}^2}$. We declare each edge of $\widetilde{\mathbb{Z}^2}$ to be closed with probability p and open with probability $1 - p$, independent of all other edges. We use ω to denote a configuration of open and closed edges of $\widetilde{\mathbb{Z}^2}$ and then place a two-sided plane mirror on each closed edge in ω. Suppose a ray of light starts from the origin of $\mathbb{Z}^2$ and initially moves towards one of the two directions following the Manhattan orientation. When the light reaches a closed edge of $\widetilde{\mathbb{Z}^2}$, it is deflected through a right angle by the mirror on the edge. When the light reaches an open edge of $\widetilde{\mathbb{Z}^2}$, it passes through without deflection and does not change direction. Let $L(\omega)$ denote the trajectory of the light and we view $L(\omega)$ as a piece-wise linear path in $\mathbb{R}^2$.

Definition 1.1. Given a nonnegative integer n, denote by

$$Q_n = \left\{ (x, y) \in \widetilde{\mathbb{Z}^2} : |x + y - 1| \le n \text{ and } |x - y| \le n \right\}$$

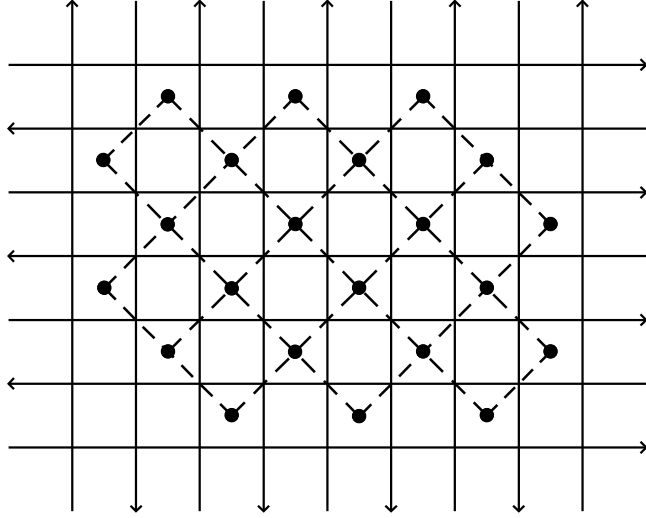[†]Department of Mathematics, University of Pennsylvania, Philadelphia, PA. E-mail: linjun@sas.upenn.edu

Figure 1: The black dots in the picture illustrate the lattice $\widetilde{\mathbb{Z}^2}$. The dashed lines are edges of $\widetilde{\mathbb{Z}^2}$. The arrows indicate the orientation of the Manhattan lattice $\mathbb{Z}^2$.

the tilted box centered at $(\frac{1}{2}, \frac{1}{2})$. We say a trajectory $L(\omega)$ is n-*bounded* if any point (x, y) on $L(\omega)$ satisfies $|x + y - 1| \le n$ and $|x - y| \le n$.

Our main result is the following

Theorem 1.2. *Let $\mathcal{E}_n$ denote the event that $L(\omega)$ is n-bounded. Then there exists $\varepsilon_0 > 0$ such that following holds. For each $p > \frac{1}{2} - \varepsilon_0$, there are $\alpha, c > 0$ such that for $n > \alpha$, we have*

$$\mathbb{P}_p[\mathcal{E}_n] \ge 1 - \exp(-cn). \tag{1.1}$$

Theorem 1.2 implies that almost surely, the trajectory of the light is bounded in $\mathbb{R}^2$ when $p > \frac{1}{2} - \varepsilon_0$. This result is previously known to hold for $p > \frac{1}{2}$ by using bond percolation, see e.g. [3], [12, Page 238]. It is suggested by physicists that the same happens for any $p \in (0, 1)$ (see [3]) and the average diameter of the trajectory is conjectured to scale as $\exp(cp^{-2})$ when p tends to 0. Interestingly, this assertion is in agreement with the prediction that all states in the two dimensional *random Schrödinger model* are localized (see [1]). For mathematical results on 2D Anderson localization with singular potential, see e.g. [6],[4],[7],[10]. We also refer the reader to [12] for a relationship between Manhattan pinball problem and the quantum localization problem.

Remark 1.3. The Manhattan pinball model is equivalent to the following. Given $p \in (0, 1)$, we designate each vertex of $\mathbb{Z}^2$ a mirror with probability p independently. If vertex u is designated a mirror, we position on u a two-sided mirror which coincides with the edge of $\widetilde{\mathbb{Z}^2}$ which crosses u (see Figure 1). Note that, there are two types of mirrors classified by their directions: *north-west (NW) mirror* and *north-east (NE) mirror*. If vertex $u = (x, y)$ is designated a mirror, we only position on u an NE mirror if $x - y$ is even and an NW mirror otherwise.

It is interesting to compare the Manhattan model with the classical Lorentz gas model on $\mathbb{Z}^2$ (see e.g. [8, Section 13.3], [9], [11]). In the classical Lorentz gas model, we first designate each vertex of $\mathbb{Z}^2$ a mirror with probability p independently. Then if a vertex u is designated a mirror, we position on u an NW mirror with probability $\frac{1}{2}$ and an NE mirror otherwise. In both models, after placing the mirrors, a ray of light traveling along
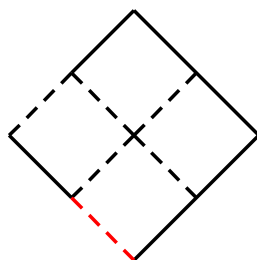
Figure 2: The figure illustrates the configuration G_0 where solid black edges are closed and dashed edges are open.
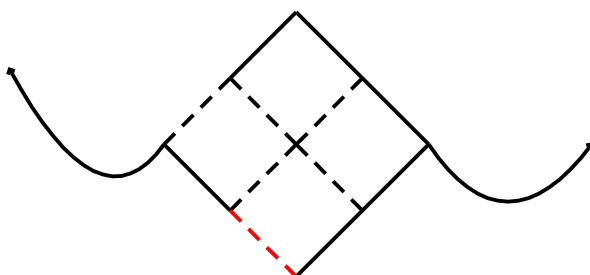

Figure 3: The picture contains a doubly-infinite closed path of $\widetilde{\mathbb{Z}^2}$ if and only if the enhancement is activated so that the red edge is turned to be closed. This means the enhancement is essential.

the edges of $\mathbb{Z}^2$ is reflected when it hits a mirror and keeps its direction unchanged otherwise.

2   Enhancement

The proof of Theorem 1.2 uses an *enhancement* procedure which we define now. Denote by G_0 the finite configuration illustrated in Figure 2 and its caption. Given any configuration ω, we enhance it by the following procedure. For any translated copy of G_0 which appears in ω, we close the open edge which coincides with the red edge (illustrated in Figure 2). We denote the configuration after this procedure by $\tilde{\omega}$.

The procedure above is a special case of the general enhancement (see e.g. [8, Section 3.3] and [2]). Intuitively, we have added closed edges to ω in a systematic way and we expect that the closed edges of resulting configuration $\tilde{\omega}$ are more "percolative" than closed edges of ω. This is an informal statement of [8, Theorem (3.16)]. For [8, Theorem (3.16)] to apply, the enhancement is required to be *essential* (a concept defined in [8, Page 64]) which is true for our case (see Figure 3 and its caption).

Remark 2.1. Note that our enhancement is not *monotonic* in the sense of [8, Section 3.3]. However, [8, Theorem (3.16)] does not require monotonicity.

Remark 2.2. There are other choices of the enhancement G_0 for our proof of Theorem 1.2. We will see in Section 3 that, our choice of G_0 makes the proof easier because of an important feature contained in Lemma 2.3 below.

Lemma 2.3. *Suppose ω is a configuration and $\overline{G_0}$, $\overline{\overline{G_0}}$ are two different translated copies of G_0 that appear in ω. Then any open edge in $\overline{G_0}$ does not coincide with the red edge in $\overline{\overline{G_0}}$.*
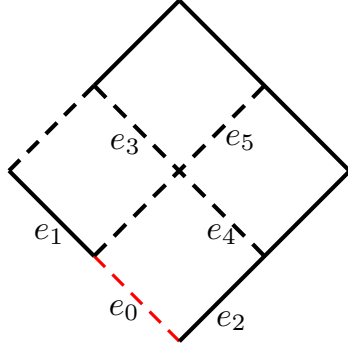
Figure 4: Illustration of the labelling of six edges e_i ($0 \leq i \leq 5$) in G_0 where e_0 is the red edge. e_1 and e_2 are closed while e_3, e_4, e_5 and e_0 are open.

Proof. For simplicity, we first label six edges e_i ($0 \leq i \leq 5$) in G_0 as illustrated in Figure 4 and its caption. For $0 \leq i \leq 5$, denote by $\overline{e_i}$ the translation of e_i in $\overline{G_0}$ and denote by $\overline{\overline{e_i}}$ the translation of e_i in $\overline{\overline{G_0}}$.

Assume the red edge in $\overline{\overline{G_0}}$ (i.e. $\overline{\overline{e_0}}$) coincides with an open edge in $\overline{G_0}$ and we will arrive at contradiction. Firstly, by the direction of edge $\overline{\overline{e_0}}$ and since $\overline{\overline{e_0}}$ is open, we must have $\overline{\overline{e_0}} = \overline{e_0}$ or $\overline{\overline{e_0}} = \overline{e_3}$ or $\overline{\overline{e_0}} = \overline{e_4}$. Since $\overline{G_0} \neq \overline{\overline{G_0}}$, we have $\overline{\overline{e_0}} \neq \overline{e_0}$. If $\overline{\overline{e_0}} = \overline{e_3}$, then we have $\overline{\overline{e_2}} = \overline{e_5}$ which contradicts with the fact that $\overline{\overline{e_2}}$ is closed and $\overline{e_5}$ is open. If $\overline{\overline{e_0}} = \overline{e_4}$, then we have $\overline{\overline{e_1}} = \overline{e_3}$ which contradicts with the fact that $\overline{\overline{e_1}}$ is closed and $\overline{e_3}$ is open. Thus we arrive at contradiction and our lemma follows. $\square$

Definition 2.4. For each positive integer n, let A_n denote the event that there exists a path of closed edges of $\widetilde{\mathbb{Z}^2}$ joining $(\frac{1}{2}, \frac{1}{2})$ to some vertex in ∂Q_n where $\partial Q_n = Q_n \setminus Q_{n-1}$.

The following proposition follows from the proof of [8, Theorem (3.16)] and the fact that the enhancement which we constructed is essential. For more details on adapting arguments in the proof of [8, Theorem (3.16)] to prove Proposition 2.5, see Appendix A.

Proposition 2.5 (Theorem (3.16) in [8])**.** There exists $\varepsilon_1 > 0$ such that, for each $p > \frac{1}{2} - \varepsilon_1$,

$$\mathbb{P}_p[\{\omega : \tilde{\omega} \in A_n\}] \geq \mathbb{P}_{\frac{1}{2}+\varepsilon_1}[A_n] \tag{2.1}$$

for large enough n.

Definition 2.6. For any positive integer n, let A_n' denote the event that there is a path of closed edges of $\widetilde{\mathbb{Z}^2}$ in the tilted rectangle

$$T_n = \{(x, y) \in \widetilde{\mathbb{Z}^2} : 1 \leq x + y - 1 \leq n, |x - y| \leq 2n\}$$

joining some vertex on northwest side $\{(x, y) \in T_n : x - y = -2n\}$ to some vertex on southeast side $\{(x, y) \in T_n : x - y = 2n\}$. See Figure 5 for an illustration.

In fact, by the same argument in Appendix A (see Remark A.1), we can substitute A_n in Proposition 2.5 by A_n' and we have

Proposition 2.7. There exists $\varepsilon_1 > 0$ such that, for each $p > \frac{1}{2} - \varepsilon_1$,

$$\mathbb{P}_p[\{\omega : \tilde{\omega} \in A_n'\}] \geq \mathbb{P}_{\frac{1}{2}+\varepsilon_1}[A_n'] \tag{2.2}$$

for large enough n.

The following well-known lemma states that in the supercritical phase (i.e. $p > \frac{1}{2}$), the crossing event A_n' happens with high probability.
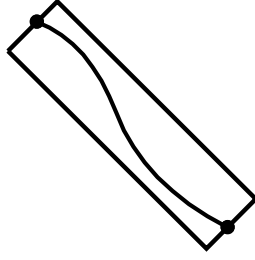
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Figure 5: The picture illustrates a closed path of $\widetilde{\mathbb{Z}^2}$ in T_n which joins some vertex on northwest side to southeast side.
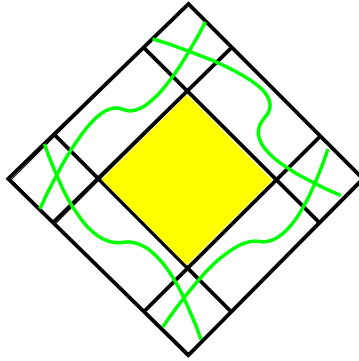

Figure 6: The four green curves illustrate four closed paths of $\widetilde{\mathbb{Z}^2}$ that cross tilted rectangles $T^{(i)}(i = 1, 2, 3, 4)$ in Q_{2n}. If such paths exist, then Q_n (the yellow region) lies in the interior of a closed circuit.

Lemma 2.8. *For any $p > \frac{1}{2}$, there is constant $c > 0$ such that*

$$\mathbb{P}_p[A'_n] \geq 1 - \exp(-cn)$$

for large enough n.

Proof. Follow the argument prior to [8, Lemma (11.22)]. □

Let A''_n denote the event that Q_n lies in the interior of a circuit of $\widetilde{\mathbb{Z}^2}$ which consists of closed edges in Q_{2n}. Together with Proposition 2.7 and Lemma 2.8, we have

Proposition 2.9. There exists $\varepsilon_1, c_1 > 0$ such that, for each $p > \frac{1}{2} - \varepsilon_1$,

$$\mathbb{P}_p[\{\omega : \tilde{\omega} \in A''_n\}] \geq 1 - \exp(-c_1 n) \tag{2.3}$$

for large enough n.

Proof. Consider the following four tilted rectangles contained in Q_{2n}:

1. $T^{(1)} = \{(x, y) \in \widetilde{\mathbb{Z}^2} : n + 1 \leq x + y - 1 \leq 2n, |x - y| \leq 2n\}$,
2. $T^{(2)} = \{(x, y) \in \widetilde{\mathbb{Z}^2} : -2n \leq x + y - 1 \leq -n - 1, |x - y| \leq 2n\}$,
3. $T^{(3)} = \{(x, y) \in \widetilde{\mathbb{Z}^2} : n + 1 \leq x - y \leq 2n, |x + y - 1| \leq 2n\}$,
4. $T^{(4)} = \{(x, y) \in \widetilde{\mathbb{Z}^2} : -2n \leq x - y \leq -n - 1, |x + y - 1| \leq 2n\}$.

See Figure 6 for an illustration. If these four tilted rectangles are crossed in the 'long direction' by closed paths of $\widetilde{\mathbb{Z}^2}$ as indicated in Figure 6, then Q_n lies in the interior of a closed circuit. By symmetry, each of these four crossing events has the same probability as A'_n. Thus our proposition follows from Proposition 2.7 and Lemma 2.8. □
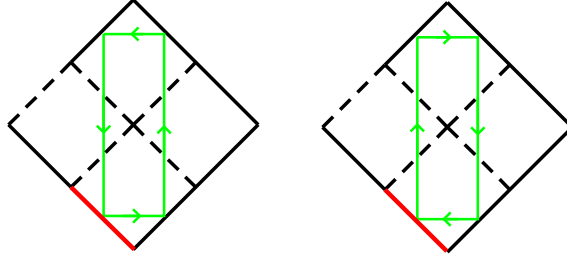
Figure 7: $e_* \in E$ is a (translated) red edge which is open under ω. After enhancement, e_* is turned to be closed. After closing e_*, the light comes from the north of e_* will travel along the green loop (see the first figure); and the light comes from the east of e_* will travel along the *inverse* of green loop (see the second figure).

3   Proof of Theorem 1.2

Definition 3.1. Denote by $\widetilde{\mathbb{E}^2}$ the set of edges of $\widetilde{\mathbb{Z}^2}$. Given an edge $e \in \widetilde{\mathbb{E}^2}$ connecting $x, y \in \widetilde{\mathbb{Z}^2}$, we also think of e as the segment in $\mathbb{R}^2$ whose end points are x, y. Given a configuration ω and $e \in \widetilde{\mathbb{E}^2}$, we say $L(\omega)$ *intersects* e (or equivalently, e intersects $L(\omega)$) if $L(\omega)$ (as a path) intersects e (as a segment). For any $e \in \widetilde{\mathbb{E}^2}$, we say e is *inside* Q_n if e connects two vertices in Q_n, otherwise we say it is *outside* Q_n.

Proof of Theorem 1.2. By Proposition 2.9, it suffices to prove $\{\omega : \tilde{\omega} \in A_n''\} \subset \mathcal{E}_{2n+10}$ for $n > 100$. Thus we fix an ω and suppose $\tilde{\omega} \in A_n''$. Our goal is to prove $\omega \in \mathcal{E}_{2n+10}$. Let configuration ω_0 be obtained by letting $\omega_0 = \omega$ inside Q_{100} and $\omega_0 = \tilde{\omega}$ outside Q_{100}. Recall that $L(\omega_0)$ is the trajectory of the light under the configuration ω_0. By definition of A_n'' and $n > 100$, the closed circuit of ω_0 in $Q_{2n} \setminus Q_{n-1}$ gives rise to an enclosure of mirrors surrounding the origin and thus traps the light. Hence, $L(\omega_0)$ is $2n$-bounded.

Note that, for each open edge e in ω, e is closed in the configuration ω_0 if and only if

1. e is outside Q_{100},

2. e is at the position of the (translated) red edge in a translated copy of G_0.

We denote by $E \subset \widetilde{\mathbb{E}^2}$ the set of edges which are open in ω but are closed in ω_0.

Claim 3.2. Suppose $e_* \in E$ and the light following the trajectory $L(\omega_0)$ approaches e_*. Then the light must approach e_* from the west or south.

Proof. Let $\overline{G_0}$ be the translated copy of G_0 which appears in ω such that e_* is at the position of the red edge in $\overline{G_0}$. By definition of E, e_* is open in ω but closed in ω_0. By Lemma 2.3, among all edges in $\overline{G_0}$, only e_* belongs to E. Thus, except e_*, other edges in $\overline{G_0}$ have the same open-closed status in ω and ω_0. Assume our claim is not true, then the light comes from the east or north of e_*. Then $L(\omega_0)$ must be the green loop or its inverse as in Figure 7 and its caption. Observe that, for any two points $a_1, a_2 \in \mathbb{R}^2$ on the green loop (or its inverse), we have $|a_1 - a_2| \leq \sqrt{10}$. Since $L(\omega_0)$ starts from the origin, any point (x, y) on $L(\omega_0)$ satisfies $|x + y - 1| \leq 2\sqrt{10} + 1 < 10$ and $|x - y| \leq 2\sqrt{10} < 10$. We deduce from Definition 1.1 that $L(\omega_0)$ is 10-bounded. Since e_* intersects $L(\omega_0)$, e_* is inside Q_{11}. This contradicts with the fact that edges in E are outside Q_{100}. $\qquad\square$

We now start from ω_0 and open edges in E one by one to reach the initial configuration ω and we keep track of the trajectory. Suppose $e_1, e_2, \cdots, e_k$ are all the edges in E which intersect $L(\omega_0)$. For $1 \leq i \leq k$, let ω_i be obtained from ω_0 by opening edges $e_1, e_2, \cdots, e_i$.
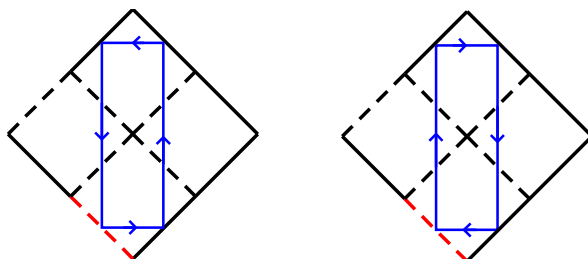
Figure 8: After opening the red edge of G_0, the light comes from the west of the red edge will travel an extra blue path (see the first figure); and the light comes from the south of the red edge will travel an extra path which is the inverse of blue path (see the second figure). After travelling the extra path, the light will continue along its original trajectory (i.e. the trajectory when the red edge was still closed).

Claim 3.3. Suppose $0 \leq i \leq k-1$. Then $L(\omega_{i+1})$ is obtained from $L(\omega_i)$ by attaching a translated copy of the blue path (the first figure in Figure 8) or its inverse (the second figure in Figure 8). Moreover, this newly attached path intersects e_{i+1} but does not intersect other edges in E.

Proof. We prove by induction on i. For $i = 0$, let $\overline{G_0}$ be the translated copy of G_0 which appears in ω and has translated red edge e_1. By Lemma 2.3, among edges of $\overline{G_0}$, only e_1 belongs to E and other edges do not belong to E. Thus the enhancement will only turn e_1 to be closed and will not affect other edges in $\overline{G_0}$. Suppose a light is traveling along $L(\omega_0)$. By Claim 3.2, the light will approach e_1 from the west or south. Now, under configuration ω_0, we open the edge e_1 and consider how the trajectory of light is affected. If the light comes from the west of e_1, it will travel an extra path which is a translated copy of the blue path (the first figure in Figure 8); if the light comes from the south of e_1, it will travel an extra translated copy of the inverse of blue path (the second figure in Figure 8). Thus $L(\omega_1)$ is obtained from $L(\omega_0)$ by attaching a translated copy of the blue path (or its inverse) which intersects e_1 but does not intersect other edges in E.

Assume our claim holds for $i < k-1$ and we prove it for $i+1$. By inductive hypothesis, $L(\omega_{i+1})$ is obtained from $L(\omega_0)$ by attaching $i+1$ translated copies of blue path (or its inverse). Thus the light traveling along $L(\omega_{i+1})$ will still approach e_{i+2}. By the same argument as in the proof of Claim 3.2, the light must approach e_{i+2} from the west or south. By the same argument as in the case $i = 0$, after opening e_{i+2}, the light will travel an extra path which is a translated copy of the blue path (or its inverse). Thus $L(\omega_{i+2})$ is obtained from $L(\omega_{i+1})$ by attaching a translated copy of the blue path (or its inverse) which intersects e_{i+2} but does not intersect other edges in E. By induction principle, our claim follows. $\qquad \square$

Claim 3.4. For each $1 \leq i \leq k$ and each $e \in E$, if e intersects $L(\omega_i)$, then $e \in \{e_1, e_2, \cdots, e_k\}$. In particular, we have $L(\omega_k) = L(\omega)$.

Proof. By Claim 3.3, $L(\omega_i)$ is obtained from $L(\omega_0)$ by attaching i translated copies of the blue path or its inverse. If $e \in E$ intersects $L(\omega_i)$, then either e intersects $L(\omega_0)$ or e intersects those i newly attached translated copies of blue path (or its inverse). If $e \in E$ intersects $L(\omega_0)$, then $e \in \{e_1, e_2, \cdots, e_k\}$ by definition of E. Otherwise, suppose e intersects a newly attached translated copy of the blue path (or its inverse). By Claim 3.3, this newly attached path intersects only one edge in E which is one of e_i's. Thus we have $e \in \{e_1, e_2, \cdots, e_k\}$.
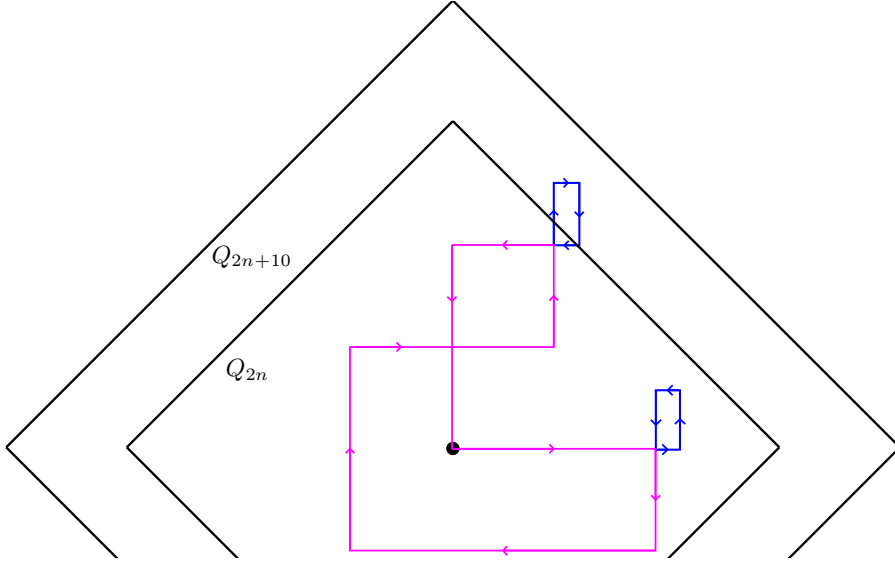
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Figure 9: $L(\omega)$ is obtained from $L(\omega_0)$ (the pink path) by attaching translated copies of the blue path (the first figure in Figure 8) or its inverse (the second figure in Figure 8) to $L(\omega_0)$.

Taking $i = k$, we deduce that for any edge e in E, if e intersects $L(\omega_k)$, then $e \in \{e_1, e_2, \cdots, e_k\}$. Thus edges in $E \setminus \{e_1, e_2, \cdots, e_k\}$ do not intersect $L(\omega_k)$ and opening these edges will not affect the original trajectory of light $L(\omega_k)$. Since ω is obtained from ω_k by opening edges in $E \setminus \{e_1, e_2, \cdots, e_k\}$, we have $L(\omega_k) = L(\omega)$. $\qquad\square$

Finally, by Claim 3.4 and Claim 3.3, $L(\omega) = L(\omega_k)$ is obtained from $L(\omega_0)$ by attaching k translated copies of the blue path (or its inverse) to $L(\omega_0)$ (see Figure 9). Here, we used the fact that each newly attached path intersects one of e_i's and $L(\omega_0)$ intersects each of e_i's. By Figure 8, observe that, for any two points $a_1, a_2 \in \mathbb{R}^2$ on the blue path (or its inverse), we have $|a_1 - a_2| \le \sqrt{10}$. Thus for any point $(x, y) \in \mathbb{R}^2$ on $L(\omega)$, there is a point $(x', y') \in \mathbb{R}^2$ on $L(\omega_0)$ such that $|(x, y) - (x', y')| \le \sqrt{10}$. Since $L(\omega_0)$ is $2n$-bounded, we have

$$|x + y - 1| \le |x' + y' - 1| + |x - x'| + |y - y'| \le |x' + y' - 1| + 2\sqrt{10} \le 2n + 2\sqrt{10} \quad (3.1)$$

and

$$|x - y| \le |x' - y'| + |x - x'| + |y - y'| \le |x' - y'| + 2\sqrt{10} \le 2n + 2\sqrt{10}. \quad (3.2)$$

Here, the last inequalities in (3.1) and (3.2) are by the definition of $2n$-bounded (see Definition 1.1). Since $2n + 2\sqrt{10} < 2n + 10$, we have $L(\omega)$ is $(2n + 10)$-bounded and our theorem follows. $\qquad\square$

A Essential enhancement

We give here a sketch of the proof of Proposition 2.5 and the same argument works for the proof of Proposition 2.7 (see Remark A.1). We follow the argument in [8, Section 3.3] and adapt notations in [8, Section 3.3] while keeping in mind that our enhancement is to add more closed edges rather than open edges.

Denote by Ω the edge configuration space. Each configuration $\omega \in \Omega$ can be viewed as a function $\omega : \widetilde{\mathbb{E}^2} \to \{0, 1\}$ such that $\omega(e) = 1$ if e is closed under ω and $\omega(e) = 0$ otherwise. For any $e \in \widetilde{\mathbb{E}^2}$ and $x \in \widetilde{\mathbb{Z}^2}$, we denote by $e + x$ the translate of the edge e by
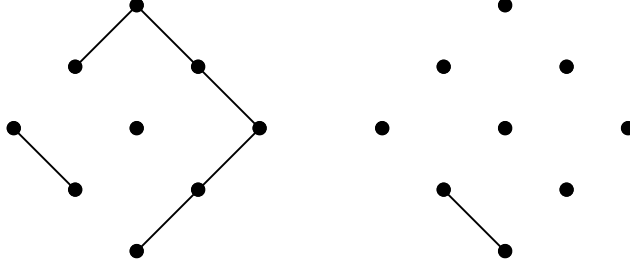
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Figure 10: The left hand side illustrates graph μ_* and the right hand side illustrates graph μ_{**}. In both graphs, the black dots illustrate vertices in $Q = Q_1$ and the segments illustrate the edges.

the vector $x \in \widetilde{\mathbb{Z}^2}$; similarly, for any subgraph G of $\widetilde{\mathbb{Z}^2}$, let $G + x$ denote the translate by x of G.

Let $\mathcal{G}$ be the set of all simple graphs on the vertex set $Q := Q_1$. For any configuration ω, let $\omega_Q \in \mathcal{G}$ be the graph on Q with edge set

$$\{e \in \widetilde{\mathbb{E}^2} : e \text{ is inside } Q \text{ and } \omega(e) = 1\}.$$

Let $\mu_{tri} \in \mathcal{G}$ be the graph on Q with empty edge set. Let $\mu_*, \mu_{**} \in \mathcal{G}$ be the graphs defined in Figure 10 and its caption. Let $F : \mathcal{G} \to \mathcal{G}$ be the *enhancement function* defined as follows:

$$F(\mu) = \begin{cases} \mu_{**} & \text{if } \mu = \mu_* \\ \mu_{tri} & \text{otherwise.} \end{cases} \tag{A.1}$$

For each $x \in \widetilde{\mathbb{Z}^2}$ and $\omega \in \Omega$, we write $F(x, \omega) = F((\tau_x \omega)_Q)$ where τ_x is the shift operator on Ω given by $\tau_x \omega(e) = \omega(e + x)$. Let $\Xi = \{0, 1\}^{\widetilde{\mathbb{Z}^2}}$. For each $\omega \in \Omega$ and $\xi \in \Xi$, denote the enhanced configuration by (more precisely, the graph of closed edges under the enhanced configuration)

$$G^{\mathbf{enh}}(\omega, \xi) = G(\omega) \cup \left\{ \bigcup_{x:\xi(x)=1} \{x + F(x, \omega)\} \right\} \tag{A.2}$$

where $G(\omega)$ is the graph of closed edges under ω. In writing the union of graphs, we mean the graph with vertex set $\widetilde{\mathbb{Z}^2}$ having the union of the appropriate edge sets; wherever this union contains two or more edges between the same pair of vertices, these edges coalesce into a single edge.

Thus we have associated with each pair $(\omega, \xi) \in \Omega \times \Xi$ an enhanced graph $G^{\mathbf{enh}}(\omega, \xi)$. We consider making an enhancement stochastically at each vertex $x \in \widetilde{\mathbb{Z}^2}$. Suppose $p, s \in [0, 1]$. For each $e \in \widetilde{\mathbb{E}^2}$ and each $x \in \widetilde{\mathbb{Z}^2}$, we declare $\omega(e) = 1$ with probability p and declare $\xi(x) = 1$ with probability s, independent of all other edges and vertices. This endows the space $\Omega \times \Xi$ with the product probability measure $\mathbb{P}_{p,s}$.

Proof of Proposition 2.5. Given two configurations $\omega, \omega' \in \Omega$, we write $\omega \leq \omega'$ if $\omega \leq \omega'$ as functions in $\{0, 1\}^{\widetilde{\mathbb{E}^2}}$. Since $A_n \subset \{\omega : \tilde{\omega} \in A_n\}$, we have $\mathbb{P}_p[\{\omega : \tilde{\omega} \in A_n\}] \geq \mathbb{P}_p[A_n]$ for each $p \in (0, 1)$. Thus for $p \geq \frac{1}{2} + \varepsilon_1$, we have

$$\mathbb{P}_p[\{\omega : \tilde{\omega} \in A_n\}] \geq \mathbb{P}_p[A_n] \geq \mathbb{P}_{\frac{1}{2}+\varepsilon_1}[A_n] \tag{A.3}$$

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where the second inequality is because A_n is an increasing event. i.e. For any two configurations $\omega \leq \omega'$ such that $\omega \in A_n$, we have $\omega' \in A_n$. Thus it suffices to prove our proposition under the condition $p \in (\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1)$.

Let $B_n \subset \Omega \times \Xi$ denote the event that $(\frac{1}{2}, \frac{1}{2})$ is connected to ∂Q_n in the graph $G^{\mathbf{enh}}(\omega, \xi)$. Define $\xi_0, \xi_* \in \Xi$ by letting $\xi_0(x) = 0$ and $\xi_*(x) = 1$ for each $x \in \widetilde{\mathbb{Z}^2}$. Then the edges of $G^{\mathbf{enh}}(\omega, \xi_0)$ are exactly the closed edges of ω and the edges of $G^{\mathbf{enh}}(\omega, \xi_*)$ are closed edges of $\tilde{\omega}$. Thus $\omega \in A_n$ is equivalent to $(\omega, \xi_0) \in B_n$ and $\tilde{\omega} \in A_n$ is equivalent to $(\omega, \xi_*) \in B_n$. This implies

$$\mathbb{P}_p[A_n] = \mathbb{P}_{p,0}[B_n] \tag{A.4}$$

and

$$\mathbb{P}_p[\{\omega : \tilde{\omega} \in A_n\}] = \mathbb{P}_{p,1}[B_n]. \tag{A.5}$$

Hence, it suffices to prove that, there is $\varepsilon_1 > 0$ such that

$$\mathbb{P}_{p,1}[B_n] \geq \mathbb{P}_{\frac{1}{2}+\varepsilon_1,0}[B_n] \tag{A.6}$$

for each $p \in (\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1)$ and large enough n. Denote $\theta_n(p, s) = \mathbb{P}_{p,s}(B_n)$ for each $p, s \in [0, 1]$. By adapting the same argument in the proof of [8, Theorem (3.16)] (from Page 67 to Page 71 in [8]), we deduce [8, equation (3.25)]. Namely, there exists a continuous function $\nu : (0, 1) \times (0, 1) \to (0, \infty)$ such that

$$\frac{\partial \theta_n}{\partial p}(p, s) \leq \nu(p, s) \frac{\partial \theta_n}{\partial s}(p, s) \tag{A.7}$$

for each $(p, s) \in (0, 1) \times (0, 1)$ and each $n > n_0$ where n_0 is a constant independent of p, s. Now we pick $\eta \in (0, \frac{1}{10})$ and choose $K > 1$ such that $\nu(p, s) \leq K$ on $[\eta, 1 - \eta]^2$. Let $\varepsilon_1 > 0$ be small enough such that $\frac{1}{2} + \varepsilon_1 K < 1 - \eta$. Denote function $\phi(p) = \frac{1}{2} - K(p - \frac{1}{2})$. Suppose $n > n_0$. Then for $p \in [\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1]$, we have $(p, \phi(p)) \in [\eta, 1 - \eta]^2$ and $\nu(p, \phi(p)) \leq K$. Since $\theta_n(p, s)$ is a polynomial of p, s, we have for $p \in (\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1)$,

$$\begin{aligned}
\frac{d}{dp} \theta_n(p, \phi(p)) &= \frac{\partial \theta_n}{\partial p}(p, \phi(p)) - K \frac{\partial \theta_n}{\partial s}(p, \phi(p)) \\
&\leq (\nu(p, \phi(p)) - K) \frac{\partial \theta_n}{\partial s}(p, \phi(p)) \\
&\leq 0.
\end{aligned} \tag{A.8}$$

Here, we also used the fact that $\frac{\partial \theta_n}{\partial s}(p, s) \geq 0$ for $p, s \in (0, 1)$ (which is a direct consequence of [8, equation (3.17)]). By (A.8), $\theta_n(p, \phi(p))$ is non-increasing in p for $p \in [\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1]$. Thus we have

$$\theta_n\left(\frac{1}{2} + \varepsilon_1, 0\right) \leq \theta_n\left(\frac{1}{2} + \varepsilon_1, \phi\left(\frac{1}{2} + \varepsilon_1\right)\right) \leq \theta_n(p, \phi(p)) \leq \theta_n(p, 1) \tag{A.9}$$

for $p \in (\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1)$. Here, the first and third inequality is because $\theta_n(p, s)$ is non-decreasing in s (since $\frac{\partial \theta_n}{\partial s}(p, s) \geq 0$ for $p, s \in (0, 1)$). Hence, (A.6) holds and our proposition follows. $\qquad \square$

Remark A.1. In order to adapt the above argument to prove Proposition 2.7, substitute A_n by A'_n and substitute B_n by B'_n where B'_n denotes the following event: under the graph $G^{\mathbf{enh}}(\omega, \xi)$, there is a path inside T_n that joins the northwest side of T_n to the southeast side of T_n. In order to achieve (A.7), adapt the argument from Page 67 to Page 71 in [8] to the event $A = B'_n$.

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