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# A unified approach to Stein's method for stable distributions*

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**Abstract:** In this article, we first review the connection between Lévy processes and infinitely divisible random variables, and discuss the classification of infinitely divisible distributions. Next, we establish a Stein identity for an infinitely divisible random variable via the Lévy-Khintchine representation of the characteristic function. In particular, we establish and unify the Stein identities for an $\alpha$-stable random variable available in the existing literature. Next, we derive the solutions of the Stein equations and its regularity estimates. Further, we derive error bounds for $\alpha$-stable approximations, and also obtain rates of convergence results in Wasserstein-$\delta$, $\delta < \alpha$ for $\alpha \in (0, 1)$ and Wasserstein-type distances for $\alpha \in (1, 2)$. Finally, we compare these results with existing literature.

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## Contents



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## 1. Introduction

Over the last five decades, Stein's method has been an important tool for studying approximation problems. Charles Stein [34] first introduced this method for normal approximation in 1972. Thereafter, Chen [13] developed this method for Poisson approximation in 1975. Several extensions of this method for various well-known probability distributions are studied in the literature. For a crisp overview of Stein's method related to classical distributions, we refer the reader to [26, 27] and references therein. The method is also extended to some families of distributions, such as the Pearson [32], variance-gamma [16] and discrete Gibbs measure families [15, 25]. We refer the reader to the web page of Yvik Swan [sites.google.com/site/steinsmethod/home] for the exhaustive historical development of Stein's method.

Let $X$ be a random variable of interest with distribution $F_X$, denoted by $X \sim F_X$. Then, the setup of Stein's method is given in three parts. In the first part, one identifies a suitable operator $\mathcal{A}$ (called Stein operator) such that $\mathbb{E}(\mathcal{A}f(X)) = 0$ for all $f \in \mathcal{F}$, where $\mathcal{F}$ is a suitable class of functions. In recent years, several approaches are developed to identify a suitable Stein operator. See, for example, the density approach [35], the generator approach [7], the probability generating function approach [37]. In the second part, one chooses a Stein equation as

$$\mathcal{A}f(x) = h(x) - \mathbb{E}h(X) \tag{1.1}$$

for $h \in \mathcal{H}$ (a class of test functions) and derives the solution of Stein equation. In the last part, one derives regularity estimates for the solution of (1.1) and "Stein factors". Further, if $Y \sim G_Y$ is another random variable of interest then the problem of $F_X$-approximation to $G_Y$ reduces to bounding the quantity $|\mathbb{E}h(Y) - \mathbb{E}h(X)| = |\mathbb{E}\mathcal{A}f(Y)|$ using Stein factors.

Recently, Arras and Houdré developed Stein's method for infinitely divisible distributions (IDD) with finite first moment [3], and for multivariate self-decomposa- ble distributions (with finite first moment (see [4]), and without finite first moment (see [5])). Note that $\alpha$-stable distributions is an important subclass of IDD and self-decomposable distributions. In this direction, Xu [39] developed Stein's method for symmetric $\alpha$-stable distributions with $\alpha \in (1, 2)$. Chen et al. [11] and Jin et al. [21] extended Xu's idea [39] and developed Stein's method for asymmetric $\alpha$-stable distributions with $\alpha \in (1, 2)$. Later, Chen et al. [12] developed Stein's method for multivariate $\alpha$-stable distributions with $\alpha \in (1,2)$. More recently, Chen et al. [10] developed Stein's method for $\alpha$-stable distributions with $\alpha \in (0,1]$. A detailed overview of these articles is given in Section 2.

It is clear from the existing literature that the techniques for developing Stein's method for $\alpha$-stable distributions depend on range of the tail parameter $\alpha$, with the cases $\alpha \in (0,1]$ and $\alpha \in (1,2)$ necessitating different approaches. This observation raises the following question.

> *(Question) For $\alpha \in (0,2)$, can one unify the Stein's method for $\alpha$-stable distributions?*

In this article, we establish a Stein identity for infinitely divisible random variables. In particular, we establish a unified Stein identity for $\alpha$-stable random variables with $\alpha \in (0,2)$. We solve our Stein equation in a unified way via the semigroup approach. Using the fine regularity estimates for the solution of $\alpha$-stable Stein equation, we derive error bounds for $\alpha$-stable approximations. Finally, we apply these results to obtain convergence rates in Wasserstein-$\delta$, $\delta < \alpha$ and Wasserstein-type distances for $\alpha \in (0,1)$ and $\alpha \in (1,2)$ respectively. We also compare our rates with the existing literature.

The organization of the article is as follows. In Section 2, we discuss some preliminaries and known results. In Section 3, we state our results concerning Stein identities for an infinitely divisible random variable, and in particular for an $\alpha$-stable random variable. Using regularity estimates for the solution of the $\alpha$-stable Stein equation, we compute bounds in appropriate probability metrics for $\alpha \in (0,1]$ and $\alpha \in (1,2)$ respectively. In Section 4, we discuss two applications of our results for $\alpha$-stable approximations and obtain the convergence rates. In Section 5, we provide the proofs of the results presented in Section 3.

## 2. Preliminaries and known results

In this section, we review the relationship between IDD and Lévy processes. We also establish the classification of $\alpha$-stable distributions based on Lévy processes. Further, we discuss the results on convergence rates for $\alpha$-stable approximations.

### *2.1. Infinite divisibility and Lévy processes*

Let us first define the concept of infinite divisibility.

**Definition 2.1.** [24, p.3] The distribution of a random variable $X$ is said to be infinitely divisible, if, for every $n \in \mathbb{N}$,

$$X \stackrel{d}{=} X_{n,1} + \ldots + X_{n,n},$$

where $\stackrel{d}{=}$ denotes the equality in distribution and $X_{n,1}, \ldots, X_{n,n}$ are independent and identically distributed (i.e., $X_{n,j} = X_n$, $j = 1, 2, \ldots n$). $X_n$ is called $n$-th factor of $X$.

In other words, a distribution function $F_X$ is infinitely divisible if, for each $n \in \mathbb{N}$, $F_X$ is the $n$-fold convolution of $F_{n,X_n}$ with itself (i.e., $F_X = F^{*n}_{n,X_n}$), where $F_{n,X_n}$ is the $n$-th factor of $F_X$. This can also be summarized using a notion of characteristic exponent as follows: Define $\eta(z) := \log \phi_X(z) = \log \mathbb{E}(e^{izX})$, $z \in \mathbb{R}$ to be the characteristic exponent of a random variable $X$. Then, the distribution of random variable $X$ ($F_X$) is infinitely divisible, if, for each $n \in \mathbb{N}$, there exist a characteristic exponent $\eta_n(\cdot)$, such that $\eta(z) = n\eta_n(z)$, $z \in \mathbb{R}$.

Next, we use this property of characteristic exponent for some familiar distributions and show that these are in fact infinitely divisible.

**Example 2.2** (Normal distribution [38])**.** *Let $X \sim \mathcal{N}(\beta, \sigma^2)$, where $\beta \in \mathbb{R}$, and $\sigma > 0$. Then the characteristic exponent of $X$ is given by*

$$\eta(z) = i\beta z - \frac{\sigma^2 z^2}{2} = n\left(i(\beta/n)z - \frac{(\sigma^2/n)z^2}{2}\right) = n\eta_n(z).$$

*Observe now that, for every $n \in \mathbb{N}$, $\eta_n(z)$ is the characteristic exponent of the random variable $X_n \sim \mathcal{N}(\beta/n, \sigma^2/n)$. Hence the distribution of $X$ is infinitely divisible.*

**Example 2.3** (Poisson Distribution [14])**.** *Let $N \sim Poisson(\lambda)$, where $\lambda > 0$. Then the characteristic exponent of $N$ is given by*

$$\eta(z) = \lambda(e^{iz} - 1) = n((\lambda/n)(e^{iz} - 1) = n\eta_n(z).$$

*Note that, for each $n \in \mathbb{N}$, $N_n \sim Poisson(\lambda/n)$. Hence the distribution of $N$ is infinitely divisible.*

Next, we recall the stochastic processes associated with these examples, namely, Brownian motion and Poisson process, and explore their connection with infinite divisibility.

**Definition 2.4** (Brownian Motion [38])**.** A real-valued stochastic process $\{X_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion, if

(i) $X_0 = 0$ a.s.
(ii) For any fixed $\omega \in \Omega$, $t \mapsto X_t$ is continuous a.s.
(iii) For $0 \leq s \leq t$, $X_t - X_s \overset{d}{=} X_{t-s}$.
(iv) For any partition of the interval $[0, t]$, $0 = t_0 < t_1 < \cdots < t_n = t$, the increments $X_{t_1} - X_0, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
(v) For $t > 0$, $X_t \sim \mathcal{N}(0, t)$.

From (v), it is clear that the process is generated from a standard normal random variable $X \sim \mathcal{N}(0, 1)$. Also, from (ii), we see that sample paths are continuous and not monotone.

**Definition 2.5** (Poisson Process [14])**.** A non-negative integer-valued stochastic process $\{N_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Poisson process, if

(i) $N_0 = 0$ a.s.

(ii) For any fixed $\omega \in \Omega$, $t \mapsto N_t$ is right continuous with left limits.
(iii) For $0 \leq s \leq t$, $N_t - N_s \stackrel{d}{=} N_{t-s}$.
(iv) For any partition of the interval $[0, t]$, $0 = t_0 < t_1 < \cdots < t_n = t$, the increments $N_{t_1} - N_0, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent.
(v) For $t > 0$, $N_t \sim Poisson(\lambda t)$.

From (v), it is clear that the process is generated from $N \sim Poisson(\lambda)$. Also, from (ii), we can see that the sample paths are right continuous and non-decreasing.

Observe now that these two processes may appear to be quite different from each other, but the distributions that generate these processes are infinitely divisible. Let us look at the processes closely. We can see that these two processes have common properties, such as right-continuous sample paths, stationary and independent increments (from (iii) and (iv)), and generated from infinitely divisible random variables (from (v)). These common properties lead us to introduce a general class of processes known as Lévy processes.

**Definition 2.6** (Lévy Process [24])**.** A real-valued stochastic process $\{X_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process, if

(i) $X_0 = 0$ a.s.
(ii) For any fixed $\omega \in \Omega$, $t \mapsto X_t$ is right continuous with left limits.
(iii) For $0 \leq s \leq t$, $X_t - X_s \stackrel{d}{=} X_{t-s}$.
(iv) For any partition of the interval $[0, t]$, $0 = t_0 < t_1 < \cdots < t_n = t$, the increments $X_{t_1} - X_0, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
(v) For $\varepsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

In short, a stochastic process can be characterized as Lévy process if its sample paths satisfy (ii) and have stationary and independent increments (from (iii) and (iv), respectively).

Next, we focus on the relation between infinite divisibility and Lévy processes. From the definition of Lévy process, it is clear that the distribution of $X_t$ is infinitely divisible. To see this, observe that

$$X_t = (X_t - X_{(n-1)h}) + (X_{(n-1)h} - X_{(n-2)h}) + \cdots + (X_h - X_0), \tag{2.1}$$

where $h = t/n$ and $n \in \mathbb{N}$, and these increments are independent and identically distributed with $X_0 = 0$. Hence, from Definition 2.1, the distribution of $X_t$ is infinitely divisible. We can also use the characteristic exponent to show that $X_t$ has IDD, for any $t > 0$. To see this, let $\eta_t(z) = \log \mathbb{E}(e^{izX_t})$, $z \in \mathbb{R}$. Assume first that $t = m \in \mathbb{N}$ then, from (2.1), with $h = m/n$, $\eta_m(z) = n\eta_{m/n}(z)$. Similarly, for $t \in \mathbb{Q}_+$, the set of positive rational numbers, say $t = m/n$, $\eta_t(z) = \eta_{m/n}(z) = (m/n)\eta_1(z)$ follows by choosing $h = n/m$ and (2.1). Now, for $t$ in positive irrationals, construct a decreasing sequence $\{t_n\}$ of positive rational numbers such that $t_n \to t$ as $n \to \infty$, then $\eta_t(z) = \lim_{n\to\infty} \eta_{t_n}(z) = \lim_{n\to\infty} t_n\eta_1(z) = t\eta_1(z)$. The last but one equality follows from continuity of sample paths (see, (ii) in the definition of Lévy process and dominated convergence theorem). Hence, using the characteristic exponent, we have proved that $\eta_t(z) = t\eta_1(z)$, $z \in \mathbb{R}$ and

$t > 0$. This shows that, for any $t > 0$, $X_t$ has IDD with characteristic exponent $\eta_t(\cdot)$ and can be generated using the distribution of $X_1$ with characteristic exponent $\eta_1(\cdot)$. The above discussion can now be summarized in the following theorem.

**Theorem 2.7.** *[14, p.81] Let $\{X_t\}_{t\geq 0}$ be a real-valued Lévy process. Then there exists an IDD $F$ such that $X_1 \sim F$.*

In literature, the expression of the characteristic exponent for Lévy processes has a specific representation known as Lévy-Khintchine representation. The following theorem provides this representation of the characteristic exponent.

**Theorem 2.8.** *[24, p.5] Let $\{X_t\}_{t\geq 0}$ be a real-valued Lévy process. Then there exists a triplet $(\beta, \sigma^2, \nu)$, where $\beta \in \mathbb{R}$, $\sigma \geq 0$ and $\nu$ is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge z^2)\nu(dz) < \infty$, such that*

$$\mathbb{E}(e^{izX_t}) = e^{t\eta(z)}, \quad \textit{for} \quad z \in \mathbb{R},$$

*with* $\eta(z) = i\beta z - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz\mathbf{1}_{\{|u|\leq 1\}}\right)\nu(du)$.

Note that $\eta(\cdot)$ is the characteristic exponent of $F$ and the measure $\nu(\cdot)$ is called a Lévy measure (need not be a probability measure).

This brings us to the important question. Given an IDD $F$, can we construct a Lévy process $\{X_t\}_{t\geq 0}$ such that $X_1 \sim F$? The following theorem provides the answer to this question, which ensures the existence of the triplet $(\beta, \sigma^2, \nu)$ associated with $F$. Hence, it also assures the existence of Lévy process.

**Theorem 2.9.** *[24, p.3] A distribution $F$ with characteristic exponent $\eta(\cdot)$ is infinitely divisible if and only if there exists a triplet $(\beta, \sigma^2, \nu)$, where $\beta \in \mathbb{R}$, $\sigma \geq 0$ and $\nu$, the Lévy measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge z^2)\nu(dz) < \infty$, with*

$$\eta(z) = i\beta z - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz\mathbf{1}_{\{|u|\leq 1\}})\nu(du). \tag{2.2}$$

The proofs of these theorems are quite lengthy and involved, we refer the interested readers to Sato [29] for more detailed discussion.

We have now established the fact that, for any IDD $F$ with triplet $(\beta, \sigma^2, \nu)$, there exist a unique Lévy process $\{X_t\}_{t\geq 0}$. Let us understand the concept through the following examples.

Ex.1. Let $X \sim \mathcal{N}(0, 1)$. Then $\eta(z) = -z^2/2$. On comparison with (2.2), we get the triplet $(\beta, \sigma^2, \nu) = (0, 1, 0)$ and the associated Lévy process is a Brownian motion as defined in Definition 2.4.

Ex.2. Let $N \sim Poisson(\lambda)$, $\lambda > 0$. Then $\eta(z) = \lambda(e^{iz} - 1)$. On comparison with (2.2), we get the triplet $(\beta, \sigma^2, \nu) = (\lambda, 0, \lambda\delta_1)$, where $\delta_1$ is the Dirac measure at $\{1\}$, and the associated Lévy process is a Poisson process as defined in Definition 2.5.

Ex.3. Let $X \sim Gamma(\lambda, \gamma)$, where $\lambda > 0, \gamma > 0$. Then $\eta(z) = -\gamma \log(1 - iz/\lambda)$. On comparison with (2.2), we get the triplet $(\beta, \sigma^2, \nu) = (\gamma(1 - e^{\lambda}), 0, \nu_0)$, where $\nu_0(du) = (\gamma e^{-\lambda u}/u)du$. For more details on computation of the triplet, we refer the readers to [24]. The associated Lévy process is known as a gamma process.

Ex.4. Let $X \sim Cauchy(x_0, c)$, $x_0 \in \mathbb{R}, c > 0$. Then $\eta(z) = ix_0 z - c|z|$. On comparison with (2.2), we get the triplet $(\beta, \sigma^2, \nu) = (x_0 - 2c\Gamma/\pi, 0, \nu_1)$ where $\Gamma = \int_0^\infty \left( \frac{\sin u}{u^2} - \frac{\mathbf{1}_{\{u:|u|\leq 1\}}(u)}{u} \right) du$, and $\nu_1(du) = (c/(\pi u^2))du$. The associated Lévy process is known as a 1-stable process.

In the examples discussed above, we see that the IDD and associated Lévy process are uniquely characterized by the triplet $(\beta, \sigma^2, \nu)$. Also, the behavior of $\nu$ is different in each of the examples. For Poisson distribution, $\nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(du) = \lambda < \infty$, for normal and gamma distribution, $\nu(\mathbb{R}) = 0$ and $\nu(\mathbb{R}) = \infty$, respectively, and for Cauchy distribution, $\int_{\{u:|u|\leq 1\}} u\nu(du) = \infty$. Also, observe that the behavior $\sigma$ is important for normal distribution. These two components of the triplet need to be further classified. Sato [29, Definition 11.9] has classified Lévy process $\{X_t\}_{t\geq 0}$ (infinitely divisible distribution $(X_1)$) into three different classes based on the triplet $(\beta, \sigma^2, \nu)$, as follows.

**(Type A)** $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$. (e.g. Poisson process).

**(Type B)** $\sigma = 0, \nu(\mathbb{R}) = \infty$, $\int_{\{|u|\leq 1\}} u\nu(du) < \infty$. (e.g. gamma process)

**(Type C)** $\int_{\{|u|\leq 1\}} u\nu(du) = \infty$ or $\sigma > 0$. (e.g. 1-stable process or Brownian motion)

The examples studied here are by no means exhaustive. The class of IDD is very rich and includes various well-known distribution like Student's t-distribution, Pareto distribution, $F$-distribution among many others.

Next, we focus on an important subclass of IDD, namely, non-Gaussian stable distributions. This class is characterized by the triplet $(\beta, \sigma^2, \nu) = (\beta, 0, \nu_\alpha)$, with $\beta \in \mathbb{R}$ and the Lévy measure $\nu_\alpha$ is given by

$$\nu_\alpha(du) = \left( m_1 \frac{1}{u^{1+\alpha}} \mathbf{1}_{(0,\infty)}(u) + m_2 \frac{1}{|u|^{1+\alpha}} \mathbf{1}_{(-\infty,0)}(u) \right) du, \qquad (2.3)$$

where $\alpha \in (0,2)$, $m_1, m_2 \in [0, \infty]$ and $m_1 + m_2 > 0$. (see [2, p.32]). Next, we give the characteristic exponent representation for non-Gaussian stable distributions.

**Definition 2.10.** [18, p.168] A real-valued random variable $X$ is said to have non-Gaussian stable (also called $\alpha$-stable) distribution, if there exists a triplet $(\beta, 0, \nu_\alpha)$, such that for all $z \in \mathbb{R}$, the characteristic exponent is given by

$$\eta_\alpha(z) = \log \phi_\alpha(z) = iz\beta + \int_{\mathbb{R}} \left( e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u) \right) \nu_\alpha(du), \qquad (2.4)$$

where $\beta \in \mathbb{R}$, $\alpha \in (0,2)$, and $\nu_\alpha$ is the Lévy measure defined in (2.3), and is denoted by $X \sim \mathcal{S}(\alpha, \beta, m_1, m_2)$.

Note here that $\beta \in \mathbb{R}$ is the location parameter and $\alpha \in (0,2)$ is stability parameter, useful in determining the decay of the tail of distribution of $X$.

Observe next that, based on the classification of IDD summarized earlier in this section, we can classify the $\alpha$-stable distributions as follows:

**(Type B)** $\alpha \in (0,1)$ (as $\nu_\alpha(\mathbb{R}) = \infty$, but $\int_{\{|u|\leq 1\}} u\nu_\alpha(du) < \infty$).
**(Type C)** $\alpha \in [1,2)$ (as $\int_{\{|u|\leq 1\}} u\nu_\alpha(du) = \infty$).

Observe now that, for $\alpha$-stable distributions of Type B ($\alpha \in (0,1)$), as $\int_{\{|u|\leq 1\}} u\nu_\alpha(du) < \infty$, the characteristic exponent given in (2.4) can be rewritten as

$$\eta_\alpha(z) = iz\beta_1 + \int_{\mathbb{R}} \left(e^{izu} - 1\right) \nu_\alpha(du), \tag{2.5}$$

where $\beta_1 = \beta - \int_{\{|u|\leq 1\}} u\nu_\alpha(du)$.

Also, for $\alpha$-stable distributions of Type C ($\alpha \in (1,2), \alpha \neq 1$), as $\int_{\{|u|\leq 1\}} u\nu_\alpha(du) = \infty$, but $\int_{\{|u|>1\}} u\nu_\alpha(du) < \infty$, the characteristic exponent given in (2.4) can be rewritten as

$$\eta_\alpha(z) = iz\beta_2 + \int_{\mathbb{R}} \left(e^{izu} - 1 - izu\right) \nu_\alpha(du), \tag{2.6}$$

where $\beta_2 = \beta + \int_{\{|u|>1\}} u\nu_\alpha(du)$.

Next, we show the connection between our representation and the various other representations of characteristic exponents available for $\alpha$-stable distributions ($\alpha$-stable random variables). Based on the well-known four parameter representation of $\alpha$-stable distributions, we observe that the parameters $\alpha \in (0,2)$, $\gamma_\alpha \in \mathbb{R}$, $d_\alpha \geq 0$ and $\theta \in [-1,1]$ denote the stability, shift, scale and skewness parameters, respectively (see, [28]). Note here that, on careful adjustments of the integrals in (2.4) with respect to $\nu_\alpha$, one can obtain a well-known form of characteristic exponent (characteristic function) of $\alpha$-stable random variables of both types from the Lévy-Khintchine representation (2.4) (see, [28]). The explicit forms are given below:

**(Type B)** Let $\theta = \frac{m_1 - m_2}{m_1 + m_2}$, $\gamma_\alpha = \beta - \frac{(m_1 - m_2)}{(1-\alpha)}$, and

$$d_\alpha = (m_1 + m_2) \cos\frac{\pi}{2}\alpha \int_0^\infty \left(1 - e^{-u}\right) \frac{du}{u^{1+\alpha}}.$$

Then

$$\eta_\alpha(z) = iz\gamma_\alpha - d_\alpha |z|^\alpha \left(1 - i\theta \frac{z}{|z|} \tan\frac{\pi}{2}\alpha\right).$$

**(Type C)** Here we classify further into two cases $\alpha = 1$ and $\alpha \in (1,2)$

**($\alpha = 1$).** Let $\theta = \frac{m_1 - m_2}{m_1 + m_2}$, $\gamma_1 = \beta + (m_1 + m_2) \int_0^\infty \left(\frac{\sin u}{u^2} - \frac{\mathbf{1}_{\{|u|\leq 1\}}(u)}{u}\right) du$, and $d_1 = (m_1 + m_2)\frac{\pi}{2}$. Then

$$\eta_1(z) = iz\gamma_1 - d_1 |z| (1 + i\theta \frac{z}{|z|} \frac{2}{\pi} \log|z|).$$

**($\alpha \in (1,2)$).** Let $\theta = \frac{m_1 - m_2}{m_1 + m_2}$, $\gamma_\alpha = \beta - \frac{(m_1 - m_2)}{(1-\alpha)}$, and

$$d_\alpha = (m_1 + m_2) \cos\frac{\pi}{2}\alpha \int_0^\infty \left(1 - e^{-u}\right) \frac{du}{u^{1+\alpha}}.$$

Then

$$\eta_\alpha(z) = iz\gamma_\alpha - d_\alpha|z|^\alpha\left(1 - i\theta\frac{z}{|z|}\tan\frac{\pi}{2}\alpha\right).$$

The derivation of these forms of characteristic exponents is given in the Appendix A. Observe also that, for $X \sim \mathcal{S}(\alpha, 0, m, m)$, characteristic exponent of a symmetric $\alpha$-stable random variable is given by

$$\eta_\alpha^s(z) = \int_{\mathbb{R}}(e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u))\nu_\alpha(du) = -d_\alpha|z|^\alpha, \quad z \in \mathbb{R},$$

where $d_\alpha > 0$ is the scale parameter given by

$$d_\alpha = \begin{cases} 2m\cos(\frac{\pi}{2}\alpha)\displaystyle\int_0^\infty(1-e^{-y})\frac{dy}{y^{1+\alpha}}, & \alpha \in (0,1), \\ 2m\cos(\frac{\pi}{2}\alpha)\displaystyle\int_0^\infty(1-y-e^{-y})\frac{dy}{y^{\alpha}}, & \alpha \in (1,2), \\ \pi m, \quad \alpha = 1. \end{cases}$$

If we choose the scale parameter $d_\alpha = 1$, then the characteristic exponent $\eta_\alpha^s$ simplifies to

$$\eta_\alpha^s(z) = -|z|^\alpha, \quad z \in \mathbb{R}.$$

Next, we discuss differentiability of the characteristic exponent of $\alpha$-stable random variables. Indeed, this discussion is related to the derivation of a Stein identity for $\alpha$-stable random variables. Note that, $|z|^\alpha$ is differentiable for all $z \in \mathbb{R}$, whenever $\alpha > 1$. This fact ensures that the characteristic exponent of an $\alpha$-stable random variable is differentiable for $\alpha \in (1,2)$. Note also that, for $\alpha \in (0,1]$, $|z|^\alpha$ is not differentiable at $z = 0$. Hence, the characteristic exponent of an $\alpha$-stable random variable is not differentiable at $z = 0$ for $\alpha \in (0,1]$. To fix this problem, we consider a tempered $\alpha$-stable random variable. For a tempered $\alpha$-stable random variable $Y_{\alpha,\gamma}$ its characteristic exponent is given by

$$\eta_{\alpha,\gamma}(z) = iz\beta + \int_{\mathbb{R}}(e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u))\nu_{\alpha,\gamma}(du), \quad z \in \mathbb{R}, \tag{2.7}$$

where $\alpha \in (0,1]$, tempering parameter $\gamma \in (0,\infty)$, and $\nu_{\alpha,\gamma}$ is the Lévy measure defined as

$$\nu_{\alpha,\gamma}(du) := \left(m_1\frac{e^{-\gamma u}}{u^{1+\alpha}}\mathbf{1}_{(0,\infty)}(u) + m_2\frac{e^{-\gamma|u|}}{|u|^{1+\alpha}}\mathbf{1}_{(-\infty,0)}(u)\right)du.$$

Cont and Tankov [14, Section 4.5] also show that characteristic exponent of $Y_{\alpha,\gamma}$ is differentiable for all $z \in \mathbb{R}$. Finally, we prove the limiting result for $\alpha$-stable random variables.

**Proposition 2.11.** *Let $\alpha \in (0,1]$ and suppose that $Y_{\alpha,\gamma}$ has a tempered $\alpha$-stable distribution. Then $Y_{\alpha,\gamma} \xrightarrow{d} Y_\alpha$, an $\alpha$-stable random variable as $\gamma \downarrow 0$.*

*Proof.* By [31, Section 3.3], we write the characteristic exponent of tempered $\alpha$-stable random variable as

$$\eta_{\alpha,\gamma}(z) = iz\beta + \int_{\mathbb{R}} (e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u))q_\gamma(u)\nu_\alpha(du), \quad z \in \mathbb{R},$$

where $\nu_\alpha$ is the Lévy measure defined in (2.3), and $q_\gamma$ is a tempering function given by

$$q_\gamma(u) = e^{-\gamma u}\mathbf{1}_{(0,\infty)}(u) + e^{-\gamma|u|}\mathbf{1}_{(-\infty,0)}(u).$$

Note that, $\eta_{\alpha,\gamma}$ is continuous on $\mathbb{R}$. Moreover, $\eta_{\alpha,\gamma}(0) = 0$.
Now, using dominated convergence theorem, we have

$$\begin{aligned}
|\eta_{\alpha,\gamma}(z) - \eta_\alpha(z)| &= \left| \int_{\mathbb{R}} (e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u))(q_\gamma(u) - 1)\nu_\alpha(du) \right| \\
&\leq \int_{\mathbb{R}} |(e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u))(q_\gamma(u) - 1)|\nu_\alpha(du) \\
&\to 0 \text{ as } \gamma \downarrow 0.
\end{aligned}$$

The desired conclusion follows. □

### *2.2. Probability metrics*

Here, we review some probability metrics used in this article.

(M1) **Wasserstein-$\delta$ distance**.
Let $\mathcal{H}_\delta = \left\{ h : \mathbb{R} \to (\mathbb{R}, d_\delta) \middle| |h^{(k)}(v) - h^{(k)}(x)| \leq d_\delta(v,x), k = 0, 1 \right\}$, where $d_\delta(v,x) := |v - x| \wedge |v - x|^\delta$, $h^{(1)}$ is the first derivative of $h$, with $h^{(0)} = h$ and the range of $h^{(k)}$ is endowed with metric $d_\delta$. Then, for any two random variables $V$ and $X$, the metric is given by

$$d_{W_\delta}(V,X) := \sup_{h \in \mathcal{H}_\delta} |\mathbb{E}[h(V)] - \mathbb{E}[h(X)]|, \quad \delta < \alpha \leq 1.$$

This metric is useful in studying $\alpha$-stable approximations of Type B and Type C, Case 1 ($\alpha \in (0,1]$). Chen et al. [10] use $d^*_{W_\delta}$ distance with $\delta \in (0,\alpha)$, for $\alpha$-stable approximations (Type B, and Type C, Case 1 ($\alpha \in (0,1]$)). The metric is given by

$$d^*_{W_\delta}(V,X) := \sup_{h \in \mathcal{H}^0_\delta} |\mathbb{E}[h(V)] - \mathbb{E}[h(X)]|, \quad \delta < \alpha \leq 1,$$

where $\mathcal{H}_\delta^0 = \left\{ h : \mathbb{R} \to (\mathbb{R}, d_\delta) \middle| |h^{(k)}(v) - h^{(k)}(x)| \leq d_\delta(v, x), k = 0 \right\}$. Note that, $\mathcal{H}_\delta \subseteq \mathcal{H}_\delta^0$. Hence, it can be shown that $d_{W_\delta}(V, X) \leq d^*_{W_\delta}(V, X)$.

(M2) **Wasserstein-type distance ([3])**. Let

$$\mathcal{H}_r = \left\{ h : \mathbb{R} \to \mathbb{R} \middle| h \text{ is } r \text{ times continuously differentiable and, } \|h^{(k)}\| \leq 1 \right\},$$

where $h^{(k)}$, $k = 0, 1, \dots, r$, is the $k$-th derivative of $h$, with $h^{(0)} = h$ and $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$. Then, for any two random variables $V$ and $X$, the metric is given by

$$d_{W_r}(V, X) := \sup_{h \in \mathcal{H}_r} |\mathbb{E}[h(V)] - \mathbb{E}[h(X)]|.$$

This metric is useful in studying $\alpha$-stable approximations of Type C, Case 2 ($\alpha \in (1, 2)$).

(M3) **Mallows $r$-distance ([22])**. For any $r > 0$, the Mallows $r$-distance is given by

$$d_r(V, X) := \left( \inf_{(V,X)} \mathbb{E}|V - X|^r \right)^{\frac{1}{r}},$$

where the infimum is taken over pairs $(V, X)$ whose marginal distribution functions are $F_V$ and $F_X$ respectively.

(M4) **Wasserstein $r$-distance ([6])**. For any $r \geq 1$, the Wasserstein $r$-distance is given by

$$W_r(V, X) := \left( \inf_{(V,X)} \mathbb{E}|V - X|^r \right)^{\frac{1}{r}},$$

where the infimum is taken over pairs $(V, X)$ whose marginal distribution functions are $F_V$ and $F_X$ respectively. By a duality argument, the Wasserstein-1 distance between two random variables can be defined as

$$W_1(V, X) = \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}h(V) - \mathbb{E}h(X)|,$$

where $\mathrm{Lip}(1) = \left\{ h : \mathbb{R} \to \mathbb{R} \middle| |h(v) - h(x)| \leq |v - x| \right\}$.

Finally, let us discuss the connections of these metrics. From [10, Subsection 1.2], we note that $d_{W_\delta}(V, X) \leq d^*_{W_\delta}(V, X) \leq W_1(V, X) \leq W_p(V, X)$, $p \geq 1$. We also note that the Mallows $r$-distance is the Wasserstein $r$-distance for $r \geq 1$ (see, [3, Section 2.3] for more details). Moreover, the metrics defined in (M2) and (M4) have the following order relationship.

$$d_{W_r}(V, X) \leq d_{W_1}(V, X) \leq W_1(V, X) \leq W_p(V, X), \quad r, p \geq 1.$$

### 2.3. Literature review

Here, we review the known results on convergence rates for $\alpha$-stable approximations. The generalized CLT states that the sum of i.i.d. random variables when scaled and centered appropriately, converges to an $\alpha$-stable distribution. To be more precise, assume that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences in $(0,\infty)$ and $\mathbb{R}$ respectively and let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d random variables. Then $S_n := a_n \sum_{i=1}^{n} Y_i - b_n$, $n \in \mathbb{N}$ converges weakly to an $\alpha$-stable distribution with stability index $\alpha \in (0,2]$, see [18].

The problem of convergence rates for $\alpha$-stable approximations is studied by many authors, see [8, 9, 20, 22, 23] for more details. In [22], the authors consider the generalized CLT, and derive the convergence rate for $\alpha$-stable approximation in the Mallows $r$-distance $d_r$, for $r > 0$, using the following framework.

Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with distribution function $F$ such that $F(y) = \frac{c_1+b(y)}{|y|^{\alpha}}$ for $y < 0$ and $1 - F(y) = \frac{c_2+b(y)}{|y|^{\alpha}}$ for $y > 0$, where $c_1, c_2 > 0$ and $b(y) = O(|y|^{-d})$, $d > 0$. In [22, Theorem 1.2], it is shown that the partial sum $S_n = n^{-\frac{1}{\alpha}} \sum_{i=1}^{n} Y_i$ converges weakly to an $\alpha$-stable distribution with a rate $n^{\frac{1}{r}-\frac{1}{\alpha}}$ in the Mallows $r$-distance $d_r$, where $r \in (\alpha, 2]$.

Let us now review the results in the literature related to $\alpha$-stable distribution approximation and Stein's method. Recently, Xu [39] develops Stein's method for symmetric $\alpha$-stable distributions with $\alpha \in (1,2)$. The author derives the convergence rate for $\alpha$-stable approximation in the Wasserstein-1 distance $W_1$, using the following framework.

Let $S_n = \sum_{i=1}^{n} Z_i$ be a sum of $n$ centered i.i.d random variables. By [39, Theorem 1.4], a Stein operator for symmetric $\alpha$-stable random variable is given by

$$\mathcal{A}f(x) = \Delta^{\frac{\alpha}{2}} f(x) - \frac{1}{\alpha} x f'(x),$$

where $\Delta^{\frac{\alpha}{2}}$ is the fractional Laplacian and $f \in \mathcal{F}$ (a class of functions $f$ with first and second derivatives bounded by a constant depending on $\alpha$ and that $\Delta^{\frac{\alpha}{2}} f$ is $\gamma$-Hölder continuous for any $0 < \gamma < 1$).

It is shown that

$$\mathbb{E}\left(S_n f'(S_n)\right) = \sum_{i=1}^{n} \int_{-N}^{N} \mathbb{E}\left(K_i(t,N) f''(S_n(i)+t)\right) dt + R, \tag{2.8}$$

where $N > 0$ is an arbitrary number, $S_n(i) = S_n - Z_i$,

$$K_i(t,N) = \mathbb{E}\left(Z_i \mathbf{1}_{\{0\leq t\leq Z_i\leq N\}} - Z_i \mathbf{1}_{\{-N\leq Z_i\leq t\leq 0\}}\right),$$

and $R$ is a remainder.

It is also shown that

$$\Delta^{\frac{\alpha}{2}} f(S_n) = \int_{-N}^{N} \mathcal{K}_{\alpha}(t,N) f''(S_n+t) dt + R', \tag{2.9}$$

where $\mathcal{K}_\alpha(t,N) = \frac{m_\alpha}{\alpha(\alpha-1)}(|t|^{1-\alpha} - N^{1-\alpha})$ with $m_\alpha = \left(\int_{\mathbb{R}} \frac{1-\cos y}{|y|^{1+\alpha}} dy\right)^{-1}$.

Using (2.8) and (2.9), it can be seen that

$$\mathbb{E}\left(\mathcal{A}f(S_n)\right) = \sum_{i=1}^{n} \int_{-N}^{N} \mathbb{E}\left(\frac{\mathcal{K}_\alpha(t,N)}{n} - \frac{K_i(t,N)}{\alpha}\right) f''(S_n(i)+t)dt + R'', \tag{2.10}$$

where $R''$ is an another remainder. Hence,

$$|\mathbb{E}\mathcal{A}f(S_n)| \leq \left(\sum_{i=1}^{n} \int_{-N}^{N} \mathbb{E}\left|\frac{\mathcal{K}_\alpha(t,N)}{n} - \frac{K_i(t,N)}{\alpha}\right| dt\right) \|f''\| + \|R''\|,$$

where $\|f''\| = \sup_{x\in\mathbb{R}} |f''(x)|$. Therefore, to obtain a rate of convergence, it is sufficient to bound $\|f''\|$ and the remainder $\|R''\|$. In [39, Example 1], the author derives a convergence rate $n^{-\frac{2-\alpha}{\alpha}}$ for $\alpha$-stable approximations in the Wasserstein-1 distance. Note that, Johnson and Samworth [22] show a convergence rate $n^{(\frac{1}{r}-\frac{1}{\alpha})}$ for $\alpha$-stable approximations in the Mallows $r$-distance for some $r \in (\alpha, 2]$. Hence, at $r = 2$, they show that the convergence rate is at most $n^{\frac{1}{2}-\frac{1}{\alpha}}$. Xu [39] proves that $S_n$ converges to an symmetric $\alpha$-stable distribution with the rate $n^{-\frac{(2-\alpha)}{\alpha}}$ in the Wasserstein-1 distance. Xu also mention that the convergence rate $n^{\frac{1}{2}-\frac{1}{\alpha}}$ is not accessible by his Stein's method setup.

In Section 4, we show that $S_n$ converges to an $\alpha$-stable distribution with a rate $n^{-\frac{2-\alpha}{\alpha}}$ for $\alpha \in (1,2)$ in the $d_{W_2}$ distance, which is faster rate than the rate obtained in [22], whenever $r \in (\alpha, 2)$. We also show that the rate $n^{\frac{1}{2}-\frac{1}{\alpha}}$ is accessible in the $d_{W_1}$ distance using our Stein's method setup.

Next, Arras and Houdré [3] develop Stein's method for IDD with finite first moment. In [3, Theorem 3.1], the authors obtain a Stein characterization for IDD using covariance representation given in [19, Proposition 2]. In [3, Section 6], the authors derive a bound for approximation of self-decomposable distribution in the Wasserstein-type distance $d_{W_2}$ which, in turn, helps to obtain bounds for $\alpha$-stable approximations with $\alpha \in (1,2)$. However, there is no discussion on convergence rates.

Further, Jin et al. [21] and Chen et al. [11] extend Xu's idea [39], and develop Stein's method for asymmetric $\alpha$-stable distributions with $\alpha \in (1,2)$. In [21], the authors obtain a kernel discrepancy type bound as (2.10), and derive the convergence rate $n^{-\frac{2-\alpha}{\alpha}}$ for asymmetric $\alpha$-stable approximations in the Wasserstein-1 distance. In [11], the authors use the leave-one-out approach developed by Stein [34], and derive the convergence rate $n^{-\frac{2-\alpha}{\alpha}}$ for $\alpha$-stable approximations in the Wasserstein-1 distance. Later, Chen et al. [12] extend the leave-one-out approach for multivariate case, and develop Stein's method for multivariate $\alpha$-stable distributions with $\alpha \in (1,2)$.

More recently, Chen et al. [10] develop Stein's method for $\alpha$-stable distributions with $\alpha \in (0,1]$. Due to lack of finite first moment, the strategy in deriving rate of convergence for $\alpha$-stable approximation differs significantly from the case

$\alpha \in (1,2)$, obtained in [3, 21, 39]. The authors derive the convergence rate for $\alpha$-stable approximation in the $d^*_{W_\delta}$ distance, using the following framework.

Let $S_n = \frac{n^{-1/\alpha}}{\sigma_\alpha}\sum_{i=1}^n Y_i$, $\sigma_\alpha > 0$ be a partial sum of i.i.d. random variables in the domain of normal attraction of an $\alpha$-stable distribution. We discuss the domain of normal attraction in Definition 3.15 in more detail. For any $\alpha$-stable random variable $X$, Chen et al. [10] show that

$$d^*_{W_\delta}(S_n, X) \leq \sup_{f \in \mathcal{F}_{\alpha,\theta}} \left| \mathbb{E}\left(\mathcal{L}^{\alpha,\theta} f(S_n)\right) - \frac{1}{\alpha}\mathbb{E}\left(S_n f'(S_n)\right)\right|, \tag{2.11}$$

where $\theta \in [-1,1]$ is a skewness parameter, $\mathcal{L}^{\alpha,\theta}$ is a generator of an $\alpha$-stable Lévy process, and $\mathcal{F}_{\alpha,\theta}$ is a class of smooth functions. In [10, Section 5, Example 1], the authors derive the convergence rate $n^{-1}$ for $\alpha$-stable approximations in the $d^*_{W_\delta}$ distance for $\alpha \in (0,1)$. Note that, there is a limitation of the rate $n^{-1}$. This rate is not flexible as it does not depend on $\alpha$. In Section 4, we show that our rate is $n^{-(\frac{1}{\alpha}-1)}$ with $\alpha \in (0,1)$, which is flexible. In comparison with the rate derived in [10], we see that our rate is faster ($\alpha \in (0, 0.5)$), same ($\alpha = 0.5$) and slower ($\alpha \in (0.5, 1)$).

## 3. Main results

In this section, we discuss the three important components of Stein's method for IDD, as mentioned in the Introduction. First, we obtain a Stein identity for infinitely divisible random variables. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space defined by

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : \lim_{|x|\to\infty} |x^m \frac{d^n}{dx^n} f(x)| = 0, \text{ for all } m, n \in \mathbb{N} \right\},$$

where $C^\infty(\mathbb{R})$ is the class of infinitely differentiable functions on $\mathbb{R}$. It is important to note that the Fourier transform on $\mathcal{S}(\mathbb{R})$ is automorphism. This enables us to identify the elements of dual space $\mathcal{S}^*(\mathbb{R})$ with $\mathcal{S}(\mathbb{R})$. In particular, if $f \in \mathcal{S}(\mathbb{R})$, and $\widehat{f}(u) = \int_{\mathbb{R}} e^{-iux} f(x)dx,\ u \in \mathbb{R}$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$. Similarly, if $\widehat{f} \in \mathcal{S}(\mathbb{R})$, and $f(x) = \int_{\mathbb{R}} e^{iux}\widehat{f}(u)du,\ x \in \mathbb{R}$, then $f \in \mathcal{S}(\mathbb{R})$, see [33].

Now, we state our first result on Stein identity for infinitely divisible random variables.

**Theorem 3.1.** *Let $X \sim IDD(\beta, \sigma^2, \nu)$ with characteristic exponent given in (2.2) which we assume to be differentiable. Then,*

$$\mathbb{E}\left((X-\beta)g(X) - \sigma^2 g'(X) - \int_{\mathbb{R}} u(g(X+u) - g(X)\mathbf{1}_{\{|u|\leq 1\}}(u))\nu(du)\right) = 0,\ \ g \in \mathcal{S}(\mathbb{R}). \tag{3.1}$$

**Remark 3.2.** (i) Note that existence of finite first moment implies differentiable characteristic function. But, the converse is not always true, see [30, p.75]. To understand this fact, let us consider a symmetric-Pareto random variable $X$ with density given by

$$f_X(x) = \begin{cases} \frac{c}{x^2}, & |x| \geq a \\ 0, & |x| < a, \end{cases}$$

where $a > 0$, and $c$ is a normalizing constant. The integral $\int_{|x|\geq a} \frac{1}{x} dx$ is divergent. Hence, the first moment of the symmetric-Pareto random variable is not finite. But, the characteristic function of symmetric-Pareto random variable is differentiable. We refer the reader to Appendix A for the detailed discussion of this fact. In [36], the author also shows that the Pareto distribution is infinitely divisible and hence the Lévy-Khintchine representation of the characteristic function for the symmetric-Pareto random variable can be derived. Consequently, a Stein identity for this random variable follows from Theorem 3.1.

(ii) Observe that the differentiability of the characteristic function plays a crucial role in deriving the Stein identity. Indeed, for a non-differentiable characteristic function, our approach for infinitely divisible random variables does not follow easily. For example, the characteristic function of a Cauchy random variable is not differentiable (see, Section 2, Ex.4.). We need to modify our approach to handle this problem (see, Theorem 3.3).

(iii) Arras and Houdré [3, Theorem 3.1] provide a Stein identity for infinitely divisible random variables using covariance representation. They assume finite first moment and the function space as bounded Lipschitz. It is important to note that the assumption of the finite first moment is an artefact of the technique of covariance representation. Our proof of Theorem 3.1 in Section 5 is without any assumption on the first moment, and we consider the Schwartz space $\mathcal{S}(\mathbb{R})$ as a suitable function space. Observe that, for $f \in \mathcal{S}(\mathbb{R})$, $f$ is bounded Lipschitz and the Stein identity given in [3, Theorem 3.1] follows from the proof of Theorem 3.1 in Section 5.

(iv) Observe also that several random variables such as Poisson, negative binomial, normal, Laplace, and gamma can be viewed as infinitely divisible by choosing appropriate triplet $(\beta, \sigma^2, \nu)$. Also, these random variables have differentiable characteristic function. Consequently, Stein identities for these random variables can be easily derived using Theorem 3.1. In particular, $\alpha$-stable random variables are also infinitely divisible, but the derivation of Stein identity is not straightforward (see, Chen et al. [10, 11]). As noted in Section 2, the characteristic function of $\alpha$-stable variable is differentiable for $\alpha \in (1, 2)$, but the differentiability of the characteristic function fails at $\alpha \in (0, 1]$. Therefore, we modify our approach in deriving the Stein identity for $\alpha$-stable random variables with $\alpha \in (0, 1]$ (see Section 5).

Next, we establish a Stein identity for $\alpha$-stable random variables.

**Theorem 3.3.** *Let $X \sim \mathcal{S}(\alpha, \beta, m_1, m_2)$ with characteristic exponent given* *in* (2.4)*. Then,*

$$\mathbb{E}\left((X-\beta)g(X)-\int_{\mathbb{R}}(g(X+u)-g(X)\mathbf{1}_{\{|u|\leq 1\}}(u))u\nu_\alpha(du)\right)=0,\quad g\in\mathcal{S}(\mathbb{R}).\tag{3.2}$$

Note here that $\sigma = 0$, as $X$ has a non-Gaussian stable distribution. The following corollary immediately follows for symmetric $\alpha$-stable random variables by setting $m_1 = m_2 = m$, $\beta = 0$ and adjusting $\nu_\alpha$.

**Corollary 3.4.** *Let $X \sim \mathcal{S}(\alpha, 0, m, m)$, for $\alpha \in (0, 2)$. Then a Stein identity for $X$ is given by*

$$\mathbb{E}\left(Xg(X)-m\int_0^\infty\frac{g(X+u)-g(X-u)}{u^\alpha}du\right)=0,\quad g\in\mathcal{S}(\mathbb{R}).\tag{3.3}$$

In the following remark, we discuss and review the Stein identities available in the literature and in $(ii)$, we compare them with our Stein identities ((3.2) and (3.3)).

**Remark 3.5.** (i) For $\alpha \in (0, 1)$, the following literature is available.

- Chen et al. [10, Proposition 2.4] provide a Stein identity for $\alpha$-stable random variables with $\alpha \in (0, 1)$, using Barbour's generator approach [7]. We note that the authors choose the scale and the location parameters to be 1 and 0 respectively.
- Arras and Houdré [5, Theorem 3.1] also provide a Stein identity for $\alpha$-stable random variables with $\alpha \in (0, 1)$, using a truncation technique.

For $\alpha = 1$, the following literature is available.

- Chen et al. [10, Proposition 2.4] provide a Stein identity for a 1-stable random variable, using Barbour's generator approach [7]. We note that the authors choose the scale and the location parameters to be 1 and 0 respectively. We also note that the authors set the skewness parameter to zero.
- Arras and Houdré [5, Theorem 3.2] also provide a Stein identity for a 1-stable random variable, using a truncation technique.

For $\alpha \in (1, 2)$, the following literature is available.

- Chen et al. [11, Theorem 1.2] provide a Stein identity for $\alpha$-stable random variables with $\alpha \in (1, 2)$, using Barbour's generator approach [7]. We note that the authors choose the scale and the location parameters to be 1 and 0 respectively.

For the symmetric case, the following literature is available.

- Xu [39, Theorem 4.1] provides a Stein identity for symmetric $\alpha$-stable random variables with $\alpha \in (0, 2)$, using invariant measure property of Lévy-type operators [1].

- Arras and Houdré [3, Examples 3.3, (viii)] also provide a Stein identity for symmetric $\alpha$-stable random variables with $\alpha \in (1, 2)$, using covariance representation of functions of infinitely divisible random variables [19].

We see from the existing literature that the techniques for deriving Stein identity for $\alpha$-stable random variables depend on ranges of $\alpha$ ($\alpha \in (0, 1]$ and $\alpha \in (1, 2)$) and are different.

(ii) Our Stein identity given in Theorem 3.3 is derived using the Lévy-Khintchine representation of the characteristic exponent given in (2.4) without any assumption on the scale, location and skewness parameter. To the best of our knowledge, the Stein identity given in Theorem 3.3 provides a unified perspective to all Stein identities available in the literature. Observe that, under the assumptions of Chen et al. [10, 11], their Proposition 2.4 and Theorem 1.2 can be retrieved from Theorem 3.3. Using Proposition A.4, we see that Stein identities given in [5, Theorem 3.1 and Theorem 3.2] exactly match with our Stein identities. For the symmetric case, our Stein identity given in Corollary 3.4 is comparable ($g$ replaced with $g'$) to the Stein identities given in [3, Example 3.3, (viii) and Remark 5.3, (iv)] and [39, Theorem 4.1].

As noted in Section 1, the linchpin of Stein's method is the Stein operator $\mathcal{A}$, and the properties of $\mathcal{A}$ play a crucial role in the success of this method. In this context, we adopt the following definition of Stein operator from [17].

**Definition 3.6.** [17, p.1] For a given target random variable $X \sim F_X$ (where $X \sim F_X$ means that the random variable $X$ has distribution $F_X$), a suitable operator $\mathcal{A}_X$ is said to be a Stein operator, if $\mathcal{A}_X$ acts on a class of functions $\mathcal{F}$ such that $\mathbb{E}(\mathcal{A}_X g(X)) = 0$ for all $g \in \mathcal{F}$.

**Remark 3.7.** It is now clear from Theorem 3.1 that, for an infinitely divisible random variable $X$, $\mathcal{A}_X g(x) := (-x+\beta)g(x) + \sigma^2 g'(x) + \int_{\mathbb{R}} u(g(x+u) - g(x)\mathbf{1}_{\{|u|\le 1\}}(u))\nu(du)$ is an operator acting on $\mathcal{S}(\mathbb{R})$ such that $\mathbb{E}\left(\mathcal{A}_X g(X)\right) = 0$ for all $g \in \mathcal{S}(\mathbb{R})$. Then, by the above definition, $\mathcal{A}_X$ is a Stein operator for an infinitely divisible random variable $X$. Also, for any $g \in \mathcal{S}(\mathbb{R})$, $\mathcal{A}_X^\alpha g(x) := (-x+\beta)g(x) + \int_{\mathbb{R}} u(g(x+u) - g(x)\mathbf{1}_{\{|u|\le 1\}}(u))\nu_\alpha(du)$ is a Stein operator for an $\alpha$-stable random variable. Observe also that $\mathcal{A}_X^\alpha$ is an integral operator, where domain of the operator is $\mathcal{F} = \overline{\mathcal{S}(\mathbb{R})}$, the closure of $\mathcal{S}(\mathbb{R})$ (see, [39] and references therein [33] for more details).

The existing literature on Stein's method for $\alpha$-stable distributions (see, [3, 10, 11, 12, 21, 39]) suggests a variety of techniques for deriving a Stein operator depending on the stability parameter $\alpha \in (0, 1]$ or $(1, 2)$. As mentioned in Section 1, the purpose of this article is to unify Stein's method for $\alpha$-stable distributions. To achieve this, let us use the Stein operator $\mathcal{A}_X^\alpha$ to set up Stein equation. For any $h \in \mathcal{H}_X$ (a class of smooth functions), Stein equation of an $\alpha$-stable random variable $X$ is

$$\mathcal{A}_X^\alpha g(x) = h(x) - \mathbb{E}(h(X)). \tag{3.4}$$

To solve (3.4), we use well-known semigroup approach (see, [7]), and this can be motivated as follows. Recall first that, for $X \sim \mathcal{S}(\alpha, 0, m, m)$ with $d_\alpha = 1$, characteristic function simplifies to

$$\phi^s_\alpha(z) = \exp\left(-|z|^\alpha\right), \quad z \in \mathbb{R}.$$

Also, observe that, for any $z \in \mathbb{R}$, $\phi^s_\alpha(z) = \phi^s_\alpha(e^{-t}z)\phi^s_\alpha((1-e^{-t})z), \quad t \geq 0$, where $\phi^s_\alpha(e^{-t}z)$ and $\phi^s_\alpha((1-e^{-t})z)$ denote the characteristic functions of $e^{-t}X$ and $(1-e^{-t})X$ respectively. Note that $e^{-t}X$ and $(1-e^{-t})X$ are symmetric $\alpha$-stable random variables. Let us now generalize this idea for non-symmetric case. One can define a characteristic function, for all $z \in \mathbb{R}$, and $t \geq 0$, by

$$\phi_t(z) := \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)} = \int_{\mathbb{R}} e^{izu} F_{X_{(t)}}(du), \tag{3.5}$$

where $F_{X_{(t)}}$ is the distribution function of $X_{(t)}$ and $\phi_\alpha$ is the characteristic function of $\alpha$-stable random variables given in (2.4). The property given in (3.5) is also known as self-decomposability (see, [29]).

Henceforth throughout the article, let $\mathcal{F} = \overline{\mathcal{S}(\mathbb{R})}$, the closure of $\mathcal{S}(\mathbb{R})$. Following Barbour's approach [7] and using (3.5), we choose a family of operators $(P^\alpha_t)_{t\geq 0}$, for all $x \in \mathbb{R}$, as

$$P^\alpha_t(g)(x) := \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(z) e^{izxe^{-t}} \phi_t(z)dz, \quad g \in \mathcal{F}. \tag{3.6}$$

Using (3.5), we get

$$\begin{aligned} P^\alpha_t(g)(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{g}(z) e^{izxe^{-t}} e^{izu} F_{X_{(t)}}(du)dz \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{g}(z) e^{iz(u+xe^{-t})} F_{X_{(t)}}(du)dz \\ &= \int_{\mathbb{R}} g(u+xe^{-t}) F_{X_{(t)}}(du), \end{aligned} \tag{3.7}$$

where the last step follows by applying inverse Fourier transform.

**Proposition 3.8.** *The family of operators $(P^\alpha_t)_{t\geq 0}$ given in (3.6) is a $\mathbb{C}_0$-semigroup on $\mathcal{F}$.*

*Proof.* For each $g \in \mathcal{F}$, it is easy to show that $P^\alpha_0 g(x) = g(x)$ and $\lim_{t\to\infty} P^\alpha_t(g)(x) = \mathbb{E}g(X)$. Now, for any $s, t \geq 0$, we have

$$\phi_{t+s}(z) = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-(t+s)}z)} = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-s}z)}\frac{\phi_\alpha(e^{-s}z)}{\phi_\alpha(e^{-(t+s)}z)} = \phi_s(z)\phi_t(e^{-s}z) \tag{3.8}$$

We need to show that $P^\alpha_{t+s}(g)(x) = P^\alpha_t(P^\alpha_s g)(x)$ for all $g \in \mathcal{F}$.
Using (3.8), we have

$$LHS = P^\alpha_{t+s}(g)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(z) e^{izxe^{-(t+s)}} \phi_{t+s}(z)dz$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(z) e^{izxe^{-(t+s)}} \phi_s(z)\phi_t(e^{-s}z)dz. \tag{3.9}$$

$$\begin{aligned}
\text{RHS} &= P_t^{\alpha}(P_s^{\alpha}(g))(x) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{P_s^{\alpha}(g)}(z) e^{izxe^{-t}} \phi_t(z)dz \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-ivz} P_s^{\alpha}(g)(v)dv \right) e^{izxe^{-t}} \phi_t(z)dz \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-ivz} \left( \int_{\mathbb{R}} \widehat{g}(w) e^{iwe^{-s}v} \phi_s(w)dw \right) dv \right) e^{izxe^{-t}} \phi_t(z)dz \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \widehat{g}(w)\phi_s(w) \int_{\mathbb{R}} e^{izxe^{-t}} \phi_t(z) \left( \int_{\mathbb{R}} e^{iv(e^{-s}w-z)}dv \right) dzdw \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \widehat{g}(w)\phi_s(w) \int_{\mathbb{R}} e^{izxe^{-t}} \phi_t(z) 2\pi\delta(e^{-s}w - z)dzdw \\
&\qquad\qquad\qquad\qquad \text{(where } \delta \text{ is the Dirac-}\delta \text{ measure )} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(w)\phi_s(w) e^{ie^{-s}wxe^{-t}} \phi_t(e^{-s}w)dw \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(z) e^{izxe^{-(t+s)}} \phi_s(z)\phi_t(e^{-s}z)dz \\
&= P_{t+s}^{\alpha}(g)(x) = \text{LHS} \quad \text{(from (3.9))},
\end{aligned}$$

and the desired conclusion follows. □

Next, we find the generator of the semigroup defined in (3.6).

**Lemma 3.9.** *Let $(P_t^{\alpha})_{t\geq 0}$ be a $\mathbb{C}_0$-semigroup as defined in (3.6). Then, its generator $\mathcal{T}_{\alpha}$ is given by*

$$\mathcal{T}_{\alpha}g(x) = (-x+\beta)g'(x) + \int_{\mathbb{R}} \left( g'(x+u) - g'(x)\mathbf{1}_{\{|u|\leq 1\}}(u) \right) u\nu_{\alpha}(du), \quad g \in \mathcal{S}(\mathbb{R}),$$

*where $\alpha \in (0,1) \cup (1,2)$.*

*Proof.* The proof of this lemma is split into two parts.

(**i**) $\alpha \in (\mathbf{0}, \mathbf{1})$: For all $g \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\mathcal{T}_{\alpha}g(x) &= \lim_{t\to 0^+} \frac{1}{t} \left( P_t^{\alpha}(g)(x) - g(x) \right) \\
&= \frac{1}{2\pi} \lim_{t\to 0^+} \int_{\mathbb{R}} \widehat{g}(z) e^{izx} \frac{1}{t} \left( e^{izx(e^{-t}-1)} \phi_t(z) - 1 \right) dz \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(z) e^{izx} \left( -x + \beta - \int_{\{|u|\leq 1\}} u\nu_{\alpha}(du) + \int_{\mathbb{R}} e^{izu} u\nu(du) \right) (iz)dz \\
&\qquad\qquad\qquad\qquad \text{(using Prop. A.2 )}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(z)e^{izx}\left(-x+\beta_1+\int_{\mathbb{R}} e^{izu}u\nu(du)\right)(iz)dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(\text{where } \beta_1=\beta-\int_{\{|u|\le 1\}} u\nu_\alpha(du)\right) \\
&= (-x+\beta_1)g'(x)+\int_{\mathbb{R}} g'(x+u)u\nu_\alpha(du) \\
&= (-x+\beta)g'(x)+\int_{\mathbb{R}} (g'(x+u)-g'(x)\mathbf{1}_{\{|u|\le 1\}}(u))u\nu_\alpha(du),
\end{aligned}
$$

where the last equality follows by splitting $\beta_1$ (see, (2.5)).

(**ii**) $\alpha \in (\mathbf{1},\mathbf{2})$: For all $g\in\mathcal{S}(\mathbb{R})$,

$$
\begin{aligned}
\mathcal{T}_\alpha g(x) &= \lim_{t\to 0^+}\frac{1}{t}\left(P_t^\alpha(g)(x)-g(x)\right) \\
&= \frac{1}{2\pi}\lim_{t\to 0^+}\int_{\mathbb{R}} \widehat{g}(z)e^{izx}\frac{1}{t}\left(e^{izx(e^{-t}-1)}\phi_t(z)-1\right)dz \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(z)e^{izx}\left(-x+\beta+\int_{\{|u|>1\}} u\nu_\alpha(du)+\int_{\mathbb{R}}(e^{izu}-1)u\nu(du)\right)(iz)dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{using Prop. A.3}) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(z)e^{izx}\left(-x+\beta_2+\int_{\mathbb{R}}(e^{izu}-1)u\nu(du)\right)(iz)dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(\text{where } \beta_2=\beta+\int_{\{|u|>1\}} u\nu_\alpha(du)\right) \\
&= (-x+\beta_2)g'(x)+\int_{\mathbb{R}} (g'(x+u)-g'(x))u\nu_\alpha(du) \\
&= (-x+\beta)g'(x)+\int_{\mathbb{R}} (g'(x+u)-g'(x)\mathbf{1}_{\{|u|\le 1\}}(u))u\nu_\alpha(du),
\end{aligned}
$$

where the last equality follows by splitting $\beta_2$ (see, (2.6)).

This completes the proof. □

Next we handle the case $\alpha = 1$, using tempered 1-stable random variable $Y_{1,\gamma}$. Recall that the characteristic exponent of $Y_{1,\gamma}$ is given in (2.7). Let $\phi_{1,\gamma}(z) := e^{\eta_{1,\gamma}(z)}$, $z\in\mathbb{R}$ be the characteristic function of $Y_{1,\gamma}$. Then, for all $z\in\mathbb{R}$, we define

$$
\phi_{1,t,\gamma}(z) := \frac{\phi_{1,\gamma}(z)}{\phi_{1,\gamma}(e^{-t}z)}, \quad t\ge 0. \tag{3.10}
$$

Using [29, Corollary 15.11], it can be easily shown that $\phi_{1,t,\gamma}$ is a well-defined characteristic function.

Now, using (3.10), define a family of operators $(P_t^{1,\gamma})_{t\ge 0}$, for all $x\in\mathbb{R}$, by

$$
P_t^{1,\gamma}(f)(x) := \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(z)e^{izxe^{-t}}\phi_{1,t,\gamma}(z)dz = \int_{\mathbb{R}} g(u+xe^{-t})F_{Y_{(\gamma,t)}}(du), \quad g\in\mathcal{F}. \tag{3.11}
$$

Following similar steps as Proposition 3.8, one can show that $(P_t^{1,\gamma})_{t\geq 0}$, is a $\mathbb{C}_0$-semigroup on $\mathcal{F}$.

Next, we obtain a generator for the semigroup defined in (3.11).

**Lemma 3.10.** *Let $(P_t^{1,\gamma})_{t\geq 0}$ be a $\mathbb{C}_0$-semigroup as defined in (3.11). Then, its generator $\mathcal{T}_{1,\gamma}$ is given by*

$$\mathcal{T}_{1,\gamma}g(x)=(-x+\beta)g'(x)+\int_{\mathbb{R}}\left(g'(x+u)-g'(x)\mathbf{1}_{\{|u|\leq 1\}}(u)\right)u\nu_{1,\gamma}(du),\;\; g\in\mathcal{S}(\mathbb{R}).$$

*Proof.* For all $g \in \mathcal{S}(\mathbb{R})$, we get

$$\mathcal{T}_{1,\gamma}(g)(x) = \lim_{t\to 0^+}\frac{1}{t}\left(P_t^{1,\gamma}(g)(x) - g(x)\right).$$

$$\begin{aligned}
\mathcal{T}_{1,\gamma}(g)(x) &= \frac{1}{2\pi}\lim_{t\to 0^+}\int_{\mathbb{R}}\widehat{g}(z)e^{izx}\frac{1}{t}\left(e^{izx(e^{-t}-1)}\frac{\phi_{1,\gamma}(z)}{\phi_{1,\gamma}(e^{-t}z)}-1\right)dz\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g}(z)e^{izx}\left(-x+\beta+\int_{\mathbb{R}}(e^{i\xi u}-1_{\{|u|\leq 1\}})u\nu_{1,\gamma}(du)\right)(iz)dz\\
&= (-x+\beta)g'(x)+\int_{\mathbb{R}}(g'(x+u)-g'(x)\mathbf{1}_{\{|u|\leq 1\}})u\nu_{1,\gamma}(du),
\end{aligned}$$

where the last but one equality follows by doing computations similar to Proposition A.3.

This completes the proof. □

Now, observe that, $\lim_{\gamma\to 0^+} P_t^{1,\gamma}g(x) = P_t^1 g(x)$, $g \in \mathcal{F}$, as defined in (3.6). Hence $\lim_{\gamma\to 0^+}\mathcal{T}_{1,\gamma} = \mathcal{T}_1$, where $\mathcal{T}_1$ is given by

$$\mathcal{T}_1 g(x) = (-x+\beta)g'(x)+\int_{\mathbb{R}}(g'(x+u)-g'(x)\mathbf{1}_{\{|u|\leq 1\}})u\nu_1(du),\quad g\in\mathcal{S}(\mathbb{R}).$$

**Remark 3.11.** Observe that the Stein equation given in (3.4) has an integral form and the Stein equations studied in literature have integro-differential form (see for example [3, Lemma 5.8]). Indeed, on careful adjustments of the integrals with respect to $\nu_\alpha$ for $\alpha \in (0,2)$, we see that the operator $\mathcal{T}_\alpha$ is also a Stein operator for an $\alpha$-stable random variable, see [10, 11, 21, 39]. In these articles, the authors use $\mathcal{T}_\alpha$ to set up their Stein equations. However, we consider $\mathcal{A}_X^\alpha$ to set up our Stein equation. Hence, the properties for the solution of this equation are different, which help us to obtain optimal convergence rates.

Next, we provide the solution of the $\alpha$-stable Stein equation (3.4).

**Theorem 3.12.** *(i) For $\alpha \in (0,1]$, let $X \sim \mathcal{S}(\alpha,\beta,m_1,m_2)$ and $h \in \mathcal{H}_\delta$, $\delta \in (0,\alpha)$. Then, the Stein equation for $X$ is given by*

$$\mathcal{A}_X^\alpha g(x) = h(x) - \mathbb{E}h(X), \tag{3.12}$$

*and*

$$g_h^\alpha(x) = -\int_0^\infty e^{-t}\int_{\mathbb{R}} h'(u+xe^{-t})F_{X_{(t)}}(du)dt \tag{3.13}$$

*solves the $\alpha$-stable Stein equation (3.12).*

(ii) *For $\alpha \in (1,2)$, let $X \sim \mathcal{S}(\alpha,\beta,m_1,m_2)$ and $h \in \mathcal{H}_2$. Then, the Stein equation for $X$ is given by*

$$\mathcal{A}_X^\alpha g(x) = h(x) - \mathbb{E}h(X), \tag{3.14}$$

*and*

$$g_h^\alpha(x) = -\int_0^\infty e^{-t}\int_{\mathbb{R}} h'(u+xe^{-t})F_{X_{(t)}}(du)dt \tag{3.15}$$

*solves the $\alpha$-stable Stein equation (3.14).*

In the following remark, we review the techniques used by several authors to solve Stein equations under various constraints and justify our claim of unification.

**Remark 3.13.** (i) Chen et al. [10, 11] derive the solution of a Stein equation for $\alpha$-stable random variables with $\alpha \in (0,1]$ and $\alpha \in (1,2)$ respectively using Barbour's generator approach [7], and the transition density function of $\alpha$-stable processes. Xu [39] also uses Barbour's generator approach to solve the Stein equation for a symmetric $\alpha$-stable random variable with $\alpha \in (1,2)$. Arras and Houdré [3] provide the semigroup approach to solve the Stein equation for an infinitely divisible random variable with finite first moment. Recently, Arras and Houdré [5, Remark 4.3] show that semigroup approach for deriving the solution of the Stein equation is also applicable for multivariate $\alpha$-stable random vectors with $\alpha \in (0,1)$, and they also mention that the semigroup approach is also applicable for $\alpha = 1$, but requires different estimates.

(ii) Note that, for both parts of Theorem 3.12, we use only the semigroup approach to solve the Stein equation for an $\alpha$-stable random variable with $\alpha \in (0,2)$, and this unifies the method of solving the Stein equation for $\alpha$-stable random variables.

### *3.1. Properties for the solution of the Stein equation*

Let us now study regularity estimates for the solution of our Stein equation. Recall that the Lévy measure $\nu_\alpha$ for $\alpha$-stable distributions is given by $\nu_\alpha(du) = \left(m_1\frac{1}{|u|^{1+\alpha}}\mathbf{1}_{(0,\infty)}(u) + m_2\frac{1}{|u|^{1+\alpha}}\mathbf{1}_{(-\infty,0)}(u)\right)du$, where $m_1, m_2 \in [0,\infty)$, $m_1 + m_2 > 0$ and $\alpha \in (0,2)$. In the following theorem, we establish estimates of $g_h^\alpha$, which play a crucial role in the $\alpha$-stable approximation problem.

**Theorem 3.14.** (*i*) *Let $\alpha \in (0,1)$. For $h \in \mathcal{H}_\delta$, $0 < \delta < \alpha$, let $g_h^\alpha$ be defined in (3.13). Then, for any $x, y \in \mathbb{R}$*

$$\|g_h^\alpha\| \le \|h'\|, \tag{3.16}$$

$$|g_h^\alpha(x) - g_h^\alpha(y)| \leq \frac{1}{\delta+1}|x-y|^\delta. \tag{3.17}$$

*Define* $A_0 g_h^\alpha(x) := \int_{\mathbb{R}} u g_h^\alpha(x+u)\nu_\alpha(du)$. *Then*

$$\|A_0 g_h^\alpha\| \leq C_{\alpha,\delta,m_1,m_2} := \frac{\alpha(m_1+m_2)}{\delta(\alpha-\delta)} + \frac{\alpha(m_1-m_2)}{1-\alpha}. \tag{3.18}$$

*(ii) Let* $\alpha = 1$. *For* $h \in \mathcal{H}_\delta$, $0 < \delta < 1$, *let* $g_h^1$ *be defined in* (3.13). *Then, for any* $x, y \in \mathbb{R}$

$$\left\|g_h^1\right\| \leq \|h'\|, \tag{3.19}$$

$$\left|g_h^1(x) - g_h^1(y)\right| \leq \frac{1}{\delta+1}|x-y|^\delta. \tag{3.20}$$

*Define* $A_1 g_h^1(x) := \int_{\mathbb{R}} u(g_h^1(x+u) - u g_h^1 \mathbf{1}_{\{|u|\leq 1\}})\nu_1(du)$. *Then*

$$\left\|A_1 g_h^1\right\| \leq C_{1,\delta,m_1,m_2} := \frac{2(m_1+m_2)}{\delta(1-\delta^2)}. \tag{3.21}$$

*(iii) Let* $\alpha \in (1,2)$. *For* $h \in \mathcal{H}_2$, *let* $g_h^\alpha$ *be defined in* (3.15). *Then,* $g_h^\alpha$ *is differentiable on* $\mathbb{R}$,

$$\|g_h^\alpha\| \leq \|h'\| \quad \textit{and} \quad \|(g_h^\alpha)'\| \leq \frac{1}{2}\|h''\|. \tag{3.22}$$

*Let* $m_1 = m_2 = m$. *Define* $A_2 g_h^\alpha(x) := \int_{\mathbb{R}} (g_h^\alpha(x+u) - g_h^\alpha(x)) u\nu_\alpha(du)$. *For any* $x, y \in \mathbb{R}$

$$|A_2 g_h^\alpha(x) - A_2 g_h^\alpha(y)| \leq C_{\alpha,m}\|h''\||x-y|^{2-\alpha}, \tag{3.23}$$

*where* $C_{\alpha,m} := 2m\left(\frac{1}{2-\alpha} + \frac{1}{\alpha-1}\right)$ *is a positive constant.*

Next, we provide Wasserstein-$\delta$ distance error bounds for $\alpha$-stable approximations with $\alpha \in (0,1]$. Before stating our result, let us first define the domain of normal attraction of an $\alpha$-stable distribution.

**Definition 3.15.** [10, p.6] A real-valued random variable $Y$ is said to be in the domain of normal attraction of an $\alpha$-stable distribution with $\alpha \in (0,1]$ if its CDF, $F_Y$ satisfies

$$1 - F_Y(y) = \frac{A + e(y)}{|y|^\alpha}(1+\theta) \text{ and } F_Y(-y) = \frac{A + e(-y)}{|y|^\alpha}(1-\theta), \tag{3.24}$$

where $y > 1$, $\alpha \in (0,1]$, $\theta = \frac{m_1 - m_2}{m_1 + m_2} \in [-1,1]$, $A(>0)$ a constant and $e : \mathbb{R} \to \mathbb{R}$ is a bounded differentiable function vanishing at $\pm\infty$. Since $e$ is bounded, we denote $K := \sup_{y\in\mathbb{R}} |e(y)|$.

We denote $Y \in D_\alpha$, if $Y$ is in the domain of normal attraction of an $\alpha$-stable distribution, and for a positive constant $L$, the function $e$ defined in (3.24) is $C^2$ with the domain $\{|y| > L\}$, and it satisfies $ye'(y) \to 0$ and $y^2 e''(y) \to 0$ as $|y| \to \infty$.

**Theorem 3.16.** *Let $Y_1, Y_2, \dots, Y_n$ be a sequence of i.i.d random variables such that $Y_i \in D_\alpha$ and $X \sim \mathcal{S}(\alpha, \beta, m_1, m_2)$ with $\alpha \in (0,1]$. Define $S_n := n^{-1/\alpha}(Y_1 + Y_2 + \cdots + Y_n)$. Then,*

***(a)*** *for $\alpha \in (0,1)$*

$$\begin{aligned} d_{W_\delta}(S_n, X) &\leq C^{A,K}_{\alpha,\delta,m_1,m_2} n^{-1} + C_{1,\delta,L} n^{1-\frac{(1+\delta)}{\alpha}} \\ &\quad + C_{2,\delta} n^{1-\frac{(1+\delta)}{\alpha}} \sup_{L<|y|<n^{\frac{1}{\alpha}}} (\alpha|e(y)| + |ye'(y)|) \int_{L<|y|<n^{\frac{1}{\alpha}}} |y|^{\delta-\alpha} dy \\ &\quad + C_{\alpha,\delta,m_1,m_2} n^{-\frac{(1-\alpha)}{\alpha}} + n^{-\frac{(1-\alpha)}{\alpha}} \int_{|y|<n^{\frac{1}{\alpha}}} |y| dF_Y(y) + R_{\alpha,n}, \end{aligned} \tag{3.25}$$

*where $C^{A,K}_{\alpha,\delta,m_1,m_2}, C_{1,\delta,L}, C_{2,\delta}$, $C_{\alpha,\delta,m_1,m_2}$ are positive constants, and*

$$R_{\alpha,n} = \beta_1 + 2 \sup_{|y|>n^{\frac{1}{\alpha}}} (\alpha|e(y)| + |ye'(y)|) \int_{|y|>1} \eta_{\alpha,\beta,\delta,m_1,m_2}(y)|y|^{-1-\alpha} dy.$$

***(b)*** *For $\alpha = 1$*

$$\begin{aligned} d_{W_\delta}(S_n, X) &\leq C^{A,K}_{1,\delta,m_1,m_2} n^{-1} + \frac{1}{\delta+1} n^{-\delta} \left(L^2 + m_1 + m_2\right) \\ &\quad + \frac{n^{-\delta}}{1+\delta} \int_{L<|u|<\frac{1}{a}} \frac{|e(u) - ue'(u)|}{|u|^{1-\delta}} du \\ &\quad + n^{-1} \int_{|u|>1} \frac{|e(nu) - nue'(nu)|}{|u|} du + R_{1,n}, \end{aligned} \tag{3.26}$$

*where $C^{A,K}_{1,\delta,m_1,m_2}, C_{1,\delta,m_1,m_2}$ are positive constants, and*

$$R_{1,n} = \beta + 2K + 2C_{1,\delta,m_1,m_2} \int_0^n dF_{|Y|}(y) + \left| \int_0^n \frac{e(y) - e(-y)}{y} dy \right|.$$

**Remark 3.17.** Note that, in view of Theorem 3.16, we consider only real-valued random variables $Y_i \in D_\alpha$. Indeed, integer-valued random variables in general do not belong to $D_\alpha$, see [10]. The problem for developing an approach that allow to handle integer-valued sums is still open. Recently, Chen et al. [10] also provide bounds in the $d^*_{W_\delta}$ distance for $\alpha$-stable approximations of a partial sum of a sequence of i.i.d random variables, belong to $D_\alpha$. Our bounds given in (3.25) and (3.26) are similar to the bounds given in [10, Theorem 4]. Note also that, our bounds include non-zero location parameter on $\alpha$-stable approximations with $\alpha \in (0,1]$. Chen et al. [10], Chen and Xu [9] give error bounds for $\alpha$-stable approximations with $\alpha \in (0,1]$ by choosing the location parameter to be 0.

Now, we present Wasserstein-type distance error bound for $\alpha$-stable approximations of a partial sum of a sequence of i.i.d random variables with $\alpha \in (1,2)$. To derive this bound, we apply kernel decomposition method introduced by Xu [39] for symmetric $\alpha$-stable approximations. Later, Arras and Houdré [3] generalized it for IDD with finite first moment. Before stating our result, define

$$K_{\nu_\alpha}(t,N) = \mathbf{1}_{[0,N]}(t)\int_t^N u\nu_\alpha(du) + \mathbf{1}_{[-N,0]}(t)\int_{-N}^t (-u)\nu_\alpha(du), \text{ and}$$
$$K_i(t,N) = \mathbb{E}\left(Z_i \mathbf{1}_{\{0\le t\le Z_i\le N\}} - Z_i\mathbf{1}_{\{-N\le Z_i\le t\le 0\}}\right),$$

where $N > 0$ is an arbitrary number. Our theorem is as follows.

**Theorem 3.18.** *Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of i.i.d random variables with $\mathbb{E}Y_i = 0$ and $\mathbb{E}|Y_i| < \infty$. Let $X \sim \mathcal{S}(\alpha, 0, m, m)$ with $\alpha \in (1,2)$. Define $Z_i = n^{-\frac{1}{\alpha}}Y_i$ and $S_n = Z_1 + Z_2 + \ldots + Z_n$. Then,*

$$d_{W_2}(S_n, X) \le \frac{1}{2}\sum_{i=1}^n \int_{-N}^N \left|\frac{K_{\nu_\alpha}(t,N)}{n} - K_i(t,N)\right| dt + R_{n,N}, \quad N > 0, \tag{3.27}$$

*where* $R_{n,N} = 2\left(\int_{|u|>N}|u|\nu_\alpha(du) + \sum_{i=1}^n \mathbb{E}\left[|Z_i|\mathbf{1}_{\{|Z_i|>N\}}\right]\right) + \frac{C_{\alpha,m}}{n}\sum_{i=1}^n \mathbb{E}|Z_i|^{2-\alpha}$.

**Remark 3.19.** In the existing literature, several authors use this kernel discrepancy type bound for $\alpha$-stable approximations with $\alpha \in (1,2)$, see [3, 21, 39]. In these articles, we note that the derivation of this bound heavily depends on the upper bound of the second derivative for the solution of $\alpha$-stable Stein equation. However, our kernel discrepancy bound given in (3.27) mainly depends on upper bound of the first derivative for the solution of $\alpha$-stable Stein equation, as our Stein equation given in (3.4) is an integral equation, and our bound given in (3.27) is comparable to the bounds given in [3, 21, 39].

## 4. Applications

In this section, we discuss the convergence rates for $\alpha$-stable approximations using two examples and we compare them with existing literature.

**Example 4.1** (Pareto distribution with $\alpha \in (0,1)$ [10, 23])**.** *Assume that $Y_1, Y_2, \ldots, Y_n$ be i.i.d random variables having a Pareto distribution with $\alpha \in (0,1)$, i.e.,*

$$P(Y_1 > y) = \frac{1}{2|y|^\alpha},\ y \ge 1, \quad P(Y_1 \le y) = \frac{1}{2|y|^\alpha},\ y \le -1.$$

*From the Definition 3.15, we observe that $Y_i \in D_\alpha$ with $L = 1$, and (3.24) holds for $\theta = 0$, $A = \frac{1}{2}$, $e(y) = \frac{|y|^\alpha - 1}{2}\mathbf{1}_{(-1,1)}(y)$ and $0$ for $|y| > 1$, and $K = \frac{1}{2}$. Let $X \sim \mathcal{S}(\alpha, \beta, m_1, m_2)$ with $\alpha \in (0,1)$, and $S_n = n^{-1/\alpha}(Y_1 + Y_2 + \cdots + Y_n)$. Then, by Theorem 3.16, Case 1, one can verify that*

$$d_{W_\delta}(S_n, X) \le Cn^{-(\frac{1}{\alpha}-1)},$$

*where $C$ is some positive constant. Moreover,* $d_{W_\delta}(S_n, X) = O(n^{-(\frac{1}{\alpha}-1)})$.

In the following remark, we recall the convergence rates available in the literature and in *(iii)*, we compare them with our convergence rate.

**Remark 4.2.** (i) The reference [23] shows a convergence rate $d_{\mathrm{Kol}}(S_n, X) \leq C_\alpha n^{-1}$, for $\alpha \in (0, 1]$, where an exact value of $C_\alpha$ was not given.

(ii) Chen et al. [10] prove that the rate $n^{-1}$ is valid for the $d^*_{W_\delta}$ distance, whenever $\alpha \in (0, 1)$.

(iii) For $\alpha \in (0, 1)$, our rate is $d_{W_\delta}(S_n, X) \leq Cn^{-(\frac{1}{\alpha}-1)}$, which is flexible with respect to $\alpha$. In comparison to the rates derived in [10, 23], we see that our rate is faster ($\alpha \in (0, 0.5)$), same ($\alpha = 0.5$) and slower ($\alpha \in (0.5, 1)$).

**Example 4.3** (Pareto distribution with $\alpha \in (1, 2)$ [22, 39])**.** *Assume that $Y_1, Y_2, \ldots, Y_n$ be i.i.d random variables having a Pareto distribution with $\alpha \in (1, 2)$, i.e.,*

$$P(Y_1 > y) = \frac{1}{2|y|^\alpha},\ y \geq 1, \quad P(Y_1 \leq y) = \frac{1}{2|y|^\alpha},\ y \leq -1.$$

*Assume also that $\beta = 0$ and $m_1 = m_2 = m$. Let $X \sim \mathcal{S}(\alpha, 0, m, m)$ with $\alpha \in (1, 2)$, and $Z_i = n^{-\frac{1}{\alpha}} Y_i$ and $S_n = Z_1 + Z_2 + \ldots + Z_n$. Now, using Theorem 3.18, we show $d_{W_2}(S_n, X) = O(n^{-(\frac{2-\alpha}{\alpha})})$.*

*Let us first compute the terms in Remainder $R_{N,n}$. Observe that the first term is zero, and the second term of $R_{N,n}$ is given by*

$$\begin{aligned}
2\sum_{i=1}^{n} \mathbb{E}\left(|Z_i|\mathbf{1}_{|Z_i|>N}\right) &= 2n^{-\frac{1}{\alpha}} \left(\int_{n^{\frac{1}{\alpha}}N}^{\infty} xp(x)dx + \int_{-\infty}^{n^{\frac{1}{\alpha}}N} xp(x)dx\right) \\
&= \frac{4}{\alpha-1} N^{1-\alpha} = D_0 N^{1-\alpha}.
\end{aligned}$$

*The last term is given by*

$$\begin{aligned}
\frac{C_{\alpha,m}}{n}\sum_{i=1}^{n} \mathbb{E}|Z_i|^{2-\alpha} &= \frac{C_{\alpha,m}}{n}\sum_{i=1}^{n}\int_{|x|>1} (n^{-\frac{1}{\alpha}}|x|)^{2-\alpha} p(x)dx \\
&= \frac{C_{\alpha,m}}{\alpha-1} n^{-\frac{2-\alpha}{\alpha}} = D_1 n^{-\frac{2-\alpha}{\alpha}}.
\end{aligned}$$

*Hence,*

$$R_{N,n} = D_0 N^{1-\alpha} + D_1 n^{-\frac{2-\alpha}{\alpha}}.$$

*For any $N > 0$, we have*

$$\begin{aligned}
K_{\nu_\alpha}(t, N) &= \mathbf{1}_{[0,N]}(t)\int_t^N u\nu_\alpha(du) + \mathbf{1}_{[-N,0]}(t)\int_{-N}^{t} (-u)\nu_\alpha(du) \\
&= \frac{m}{1-\alpha}\left(N^{1-\alpha} - t^{1-\alpha}\right) + \frac{m}{1-\alpha}\left(N^{1-\alpha} - (-t)^{1-\alpha}\right)
\end{aligned}$$

$$= \frac{m}{\alpha-1}\left(|t|^{1-\alpha} - N^{1-\alpha}\right).$$

*Using symmetry of Pareto distribution, we have*

$$K_i(t,N) = \frac{\alpha}{2n(\alpha-1)}\left((|t| \wedge n^{-\frac{1}{\alpha}})^{1-\alpha} - N^{1-\alpha}\right).$$

*Then, from Theorem 3.18, we obtain*

$$d_{W_2}(S_n, X) \leq D_0 N^{1-\alpha} + D_1 n^{-(\frac{2-\alpha}{\alpha})} + D_2 n^{-(\frac{2-\alpha}{\alpha})},$$

*where $D_0, D_1$ and $D_2$ are positive constants. Since $N$ is arbitrary, let $N \to \infty$. Hence, $d_{W_2}(S_n, X) = O(n^{-(\frac{2-\alpha}{\alpha})})$. By [3, Lemma A.4], we have*

$$d_{W_1}(S_n, X) \leq D_3 \sqrt{n^{-(\frac{2-\alpha}{\alpha})}} = D_3 n^{\frac{1}{2}-\frac{1}{\alpha}}, \tag{4.1}$$

*where $D_3$ is an another positive constant. Moreover, $d_{W_1}(S_n, X) = O(n^{\frac{1}{2}-\frac{1}{\alpha}})$.*

In the following remark, we recall the convergence rates available in the literature and in $(iii)$, we compare them with our convergence rates.

**Remark 4.4.** (i) Johnson and Samworth [22] show that $S_n = n^{-\frac{1}{\alpha}} \sum_{i=1}^n Y_i$ converges to an $\alpha$-stable distribution with a rate $n^{(\frac{1}{r}-\frac{1}{\alpha})}$ in the Mallows $r$-distance for some $r \in (\alpha, 2]$. Hence, at $r = 2$, they show that the convergence rate is at most $n^{\frac{1}{2}-\frac{1}{\alpha}}$.

(ii) Xu [39] proves that $S_n$ converges to an symmetric $\alpha$-stable distribution with a rate $n^{-\frac{(2-\alpha)}{\alpha}}$ in the Wasserstein-1 distance. From [39, Example 1], it is clear that the convergence rate $n^{\frac{1}{2}-\frac{1}{\alpha}}$ is not accessible.

(iii) Note that, we obtain a rate $n^{-\frac{2-\alpha}{\alpha}}$ with $\alpha \in (1, 2)$ in the $d_{W_2}$ distance, which is faster rate than the rate obtained in [22], whenever $r \in (\alpha, 2)$. Observe also that, at $r = 2$, the rate obtained in [22, Theorem 1.2] becomes $n^{\frac{1}{2}-\frac{1}{\alpha}}$. From (4.1), it immediately follows that the rate $n^{\frac{1}{2}-\frac{1}{\alpha}}$ is accessible in the $d_{W_1}$ distance using our estimates.

## 5. Proofs

### *5.1. Proof of Theorem 3.1*

Recall first that for $X \sim IDD(\beta, \sigma^2, \nu)$, the characteristic exponent $\eta$ is given by

$$\eta(z) = \log \phi_X(z) = iz\beta - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u))\nu(du), \quad z \in \mathbb{R}. \tag{5.1}$$

Differentiating (5.1) with respect to $z$, we have

$$\phi_X'(z) = \left(i\beta - \sigma^2 z + i\int_{\mathbb{R}} u(e^{izu} - \mathbf{1}_{\{|u|\leq 1\}}(u))\nu(du)\right)\phi_X(z). \tag{5.2}$$

Recall from Section 2, $F_X$ is the distribution function (cumulative distribution function) of $X$ and if $\phi'_X$ exists on $\mathbb{R}$, then,

$$\phi_X(z) = \int_{\mathbb{R}} e^{izx} F_X(dx) \quad \text{and} \quad \phi'_X(z) = i\int_{\mathbb{R}} x e^{izx} F_X(dx), \quad z \in \mathbb{R}. \tag{5.3}$$

Using (5.3) in (5.2) and rearranging the integrals, we have

$$\begin{aligned}
0 &= i\int_{\mathbb{R}} x e^{izx} F_X(dx) \\
&- \left( i \left( \beta + \int_{\mathbb{R}} u(e^{izu} - \mathbf{1}_{\{|u|\le 1\}}(u))\nu(du) \right) \phi_X(z) - \sigma^2 z \phi_X(z) \right) \\
&= i \left( \int_{\mathbb{R}} (x-\beta) e^{izx} F_X(dx) - \left( \int_{\mathbb{R}} u e^{izu} \nu(du) \right) \phi_X(z) \right. \\
&+ \left. \left( \int_{\mathbb{R}} u \mathbf{1}_{\{|u|\le 1\}}(u) \nu(du) \right) \phi_X(z) - i\sigma^2 z \phi_X(z) \right) \\
&= \int_{\mathbb{R}} (x-\beta) e^{izx} F_X(dx) - \left( \int_{\mathbb{R}} u e^{izu} \nu(du) \right) \phi_X(z) \\
&+ \left( \int_{\mathbb{R}} u \mathbf{1}_{\{|u|\le 1\}}(u) \nu(du) \right) \phi_X(z) - iz\sigma^2 \phi_X(z).
\end{aligned} \tag{5.4}$$

The second integral of (5.4) can be written as

$$\begin{aligned}
\left( \int_{\mathbb{R}} u e^{izu} \nu(du) \right) \phi_X(z) &= \int_{\mathbb{R}} \int_{\mathbb{R}} u e^{izu} e^{izx} F_X(dx) \nu(du) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} u e^{iz(u+x)} \nu(du) F_X(dx) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} u e^{izy} \nu(du) F_X(d(y-u)) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} u e^{izx} \nu(du) F_X(d(x-u)) \\
&= \int_{\mathbb{R}} e^{izx} \int_{\mathbb{R}} u F_X(d(x-u)) \nu(du).
\end{aligned} \tag{5.5}$$

Substituting (5.5) in (5.4), we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}} (x-\beta) e^{izx} F_X(dx) - \int_{\mathbb{R}} e^{izx} \int_{\mathbb{R}} u F_X(d(x-u)) \nu(du) \\
&+ \left( \int_{\mathbb{R}} u \mathbf{1}_{\{|u|\le 1\}}(u) \nu(du) \right) \phi_X(z) - iz\sigma^2 \phi_X(z) \\
&= \int_{\mathbb{R}} e^{izx} \left( (x-\beta) F_X(dx) - \int_{\mathbb{R}} u F_X(d(x-u)) \nu(du) \right. \\
&+ \left. \left( \int_{\mathbb{R}} u \mathbf{1}_{\{|u|\le 1\}}(u) \nu(du) \right) F_X(dx) - iz\sigma^2 F_X(dx) \right).
\end{aligned} \tag{5.6}$$

On applying Fourier transform to (5.6), multiplying with $g \in \mathcal{S}(\mathbb{R})$, and integrating over $\mathbb{R}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}} g(x)\left((x-\beta)+\int_{\mathbb{R}} u\mathbf{1}_{\{|u|\leq 1\}}(u)\nu(du)\right)F_X(dx)\\
&-\int_{\mathbb{R}} ug(x)F_X(d(x-u))\nu(du)-\sigma^2\int_{\mathbb{R}} g'(x)F_X(dx)=0. \qquad (5.7)
\end{aligned}$$

The third integral of (5.7) can be seen as

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} ug(x)F_X(d(x-u))\nu(du) &= \int_{\mathbb{R}}\int_{\mathbb{R}} ug(y+u)F_X(dy)\nu(du)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} ug(x+u)F_X(dx)\nu(du)\\
&= \mathbb{E}\left(\int_{\mathbb{R}} ug(X+u)\nu(du)\right). \qquad (5.8)
\end{aligned}$$

Substituting (5.8) in (5.7) and rearranging the integrals, we have

$$\mathbb{E}\left((X-\beta)g(X)-\int_{\mathbb{R}} u(g(X+u)-g(X)\mathbf{1}_{\{|u|\leq 1\}}(u))\nu(du)-\sigma^2 g'(X)\right)=0.$$

This proves the theorem.

### 5.2. Proof of Theorem 3.3

Recall first that, for $X \sim \mathcal{S}(\alpha,\beta,m_1,m_2)$, characteristic exponent $\eta_\alpha$ is given by

$$\eta_\alpha(z)=\log\phi_\alpha(z)=iz\beta+\int_{\mathbb{R}}(e^{izu}-1-izu\mathbf{1}_{\{|u|\leq 1\}}(u))\nu_\alpha(du),\quad z\in\mathbb{R}.$$

Following similar steps to the proof of Theorem 3.1, we get the result for $\alpha \in (1,2)$.

For $\alpha \in (0,1]$, as the characteristic exponent $\eta_\alpha$ is not differentiable on $\mathbb{R}$. Let us consider tempered $\alpha$-stable random variable $Y_{\alpha,\gamma}$ with characteristic exponent (see, Section 2) given by

$$\eta_{\alpha,\gamma}(z)=iz\beta+\int_{\mathbb{R}}(e^{izu}-1-izu\mathbf{1}_{\{|u|\leq 1\}}(u))\nu_{\alpha,\gamma}(du),\quad z\in\mathbb{R},$$

where $\alpha\in(0,1]$, $\gamma\in(0,\infty)$, and $\nu_{\alpha,\gamma}$ is the Lévy measure defined as

$$\nu_{\alpha,\gamma}(du):=\left(m_1\frac{e^{-\gamma u}}{u^{1+\alpha}}\mathbf{1}_{(0,\infty)}(u)+m_2\frac{e^{-\gamma|u|}}{|u|^{21+\alpha}}\mathbf{1}_{(-\infty,0)}(u)\right)du.$$

Observe that $Y_{\alpha,\gamma}$ is infinitely divisible and its characteristic exponent $\eta_{\alpha,\gamma}$ is differentiable on $\mathbb{R}$. Also, by Proposition 2.11 $\eta_{\alpha,\gamma}\to\eta_\alpha$, whenever $\gamma\downarrow 0$.

Now, applying Theorem 3.1, we get the Stein identity for $Y_{\alpha,\gamma}$ as follows.

$$\mathbb{E}\left[(Y_{\alpha,\gamma}-\beta)\,g(Y_{\alpha,\gamma})-\int_{\mathbb{R}}\left(g(Y_{\alpha,\gamma}+u)-g(Y_{\alpha,\gamma})\mathbf{1}_{\{|u|\le 1\}}(u)\right)u\nu_{\alpha,\gamma}(du)\right]=0, \tag{5.9}$$

where $g\in\mathcal{S}(\mathbb{R})$. Now, taking limit as $\gamma\downarrow 0$, (5.9) reduces to

$$\mathbb{E}\left[(X-\beta)\,g(X)-\int_{\mathbb{R}}\left(g(X+u)-g(X)\mathbf{1}_{\{|u|\le 1\}}(u)\right)u\nu_{\alpha}(du)\right]=0,\ \ g\in\mathcal{S}(\mathbb{R}).$$

This proves the theorem.

### *5.3. Proof of Theorem 3.12*

For the proof of this theorem, we use the connection between the operators $\mathcal{A}_X^{\alpha}$ and $\mathcal{T}_{\alpha}$.

**Proof of** (**i**). The proof of this part is split into two parts.

**(a)** $\alpha\in(0,1)$**:** We have

$$\begin{aligned}
\mathcal{A}_X^{\alpha}g_h^{\alpha}(x)&=(-x+\beta)g_h^{\alpha}(x)+\int_{\mathbb{R}}\left(g_h^{\alpha}(x+u)-g_h^{\alpha}(x)\mathbf{1}_{\{|u|\le 1\}}(u)\right)u\nu_{\alpha}(du)\\
&=\mathcal{T}_{\alpha}(\tilde{g}_h^{\alpha})(x),\quad \left(\text{where } \tilde{g}_h^{\alpha}(x)=-\int_0^{\infty}\left(P_t^{\alpha}(h)(x)-\mathbb{E}h(X)\right)dt, h\in\mathcal{H}_{\delta}\right)\\
&=-\int_0^{\infty}\mathcal{T}_{\alpha}P_t^{\alpha}(h)(x)dt\\
&=-\int_0^{\infty}\frac{d}{dt}P_t^{\alpha}(h)(x)dt\\
&=P_0h(x)-P_{\infty}h(x)\\
&=h(x)-\mathbb{E}h(X)\quad(\text{by Proposition 3.8}).
\end{aligned}$$

Hence, for $\alpha\in(0,1)$, $g_h^{\alpha}$ is the solution of (3.12). Now, it remains to show that, $g_h^{\alpha}$ is well-defined. Let us first consider $\tilde{g}_h^{\alpha}:\mathbb{R}\to\mathbb{R}$ be defined as

$$\tilde{g}_h^{\alpha}(x)=-\int_0^{\infty}\left(P_t^{\alpha}(h)(x)-\mathbb{E}h(X)\right)dt,\quad h\in\mathcal{H}_{\delta},\delta\in(0,\alpha),$$

where $P_t^{\alpha}$ is the semigroup as defined in (3.6). We show that, for any $h\in\mathcal{H}_{\delta}$, $0<\delta<\alpha$, $\tilde{g}_h^{\alpha}$ is well-defined.

Using (3.7), we have

$$\begin{aligned}
|P_t^{\alpha}(h)(x)-\mathbb{E}h(X)|&=\left|\int_{\mathbb{R}}h(r+e^{-t}x)F_{X_{(t)}}(dr)-\int_{\mathbb{R}}h(r)F_X(dr)\right|\\
&=\left|\int_{\mathbb{R}}(h(r+e^{-t}x)-h(r))F_{X_{(t)}}(dr)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}} h(r) F_{X_{(t)}}(dr) - \int_{\mathbb{R}} h(r) F_X(dr) \Big| \\
&\leq \min\left\{e^{-t}|x|, (e^{-t}|x|)^{\delta}\right\} + \left| \int_{\mathbb{R}} \widehat{h}(z) \left(\phi_t(z) - \phi_\alpha(z)\right) dz \right| \\
&\leq \min\left\{e^{-t}|x|, (e^{-t}|x|)^{\delta}\right\} + \int_{\mathbb{R}} |\widehat{h}(z)| |\phi_t(z) - \phi_\alpha(z)| dz.
\end{aligned} \tag{5.10}
$$

Now, let us calculate an upper bound between the difference of two characteristic functions $\phi_t$ and $\phi_\alpha$. For all $t > 0$ and $z \in \mathbb{R}$,

$$
|\phi_t(z) - \phi_\alpha(z)| = \left| \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)} - \phi_\alpha(z) \right| \leq \left| \phi_\alpha(e^{-t}z) - 1 \right| = |e^{\omega_t(z)} - 1|,
$$

where $\omega_t(z) = iz\beta_1 e^{-t} + e^{-t\alpha} \int_{\mathbb{R}} (e^{izu} - 1)\nu_\alpha(du)$, and $\beta_1 = \beta - \int_{\{|u| \leq 1\}} u\nu_\alpha(du)$. Now, from [29, Lemma 7.9], the function $z \to e^{s\omega_t(z)}$ is a characteristic function for all $s \in (0, \infty)$, thus, for all $z \in \mathbb{R}$ and $t > 0$

$$
\begin{aligned}
&|\phi_t(z) - \phi_\alpha(z)| \\
&\quad \leq \left| \int_0^1 \frac{d}{ds} (\exp(s\omega_t(z))) ds \right| \\
&\quad \leq |\omega_t(z)| \\
&\quad \leq |z||\beta_1| e^{-t} + e^{-t\alpha} \left| \int_{\mathbb{R}} (e^{izu} - 1)\nu_\alpha(du) \right| \\
&\quad = |z||\beta_1| e^{-t} \\
&\qquad + e^{-t\alpha} \left| \int_{\{|u|>1\}} (e^{izu} - 1)\nu_\alpha(du) + \int_{\{|u| \leq 1\}} (e^{izu} - 1)\nu_\alpha(du) \right| \\
&\quad \leq |z||\beta_1| e^{-t} + 2e^{-t\alpha} \left| \int_{\{|u|>1\}} \nu_\alpha(du) \right| \\
&\qquad + e^{-t\alpha} \left( \left| \int_{\{|u| \leq 1\}} (\cos(zu) - 1)\nu_\alpha(du) \right| + \left| \int_{\{|u| \leq 1\}} \sin(zu)\nu_\alpha(du) \right| \right) \\
&\quad = |z||\beta_1| e^{-t} + 2\frac{m_1 + m_2}{\alpha} e^{-t\alpha} + |z|^\alpha \left(M_1 + M_2\right) e^{-t\alpha},
\end{aligned} \tag{5.11}
$$

where $M_1 = \left| \int_{\{|u| \leq z\}} (\cos v - 1)\nu_\alpha(dv) \right|$ and $M_2 = \left| \int_{\{|u| \leq z\}} \sin v\nu_\alpha(dv) \right|$.

Using (5.11) in (5.10), one can easily show that $\int_0^\infty |P_t^\alpha(h)(x) - \mathbb{E}h(X)| dt < \infty$. Hence, $\tilde{g}_h^\alpha$ is well-defined.

By dominated convergence theorem, we see that $\tilde{g}_h^\alpha$ is differentiable and

$$
(\tilde{g}_h^\alpha)'(x) = - \lim_{\zeta \to \infty} \frac{d}{dx} \int_0^\zeta (P_t^\alpha(h)(x) - \mathbb{E}h(X)) dt
$$

$$= -\lim_{\zeta\to\infty}\int_0^{\zeta}\frac{d}{dx}\left(\int_{\mathbb{R}} h(xe^{-t}+u)F_{X_{(t)}}(du)\right)dt$$
$$= -\int_0^{\infty} e^{-t}\int_{\mathbb{R}} h'(u+xe^{-t})F_{X_{(t)}}(du)dt = g_h^{\alpha}(x),$$

the desired conclusion follows.

**(b)** $\alpha = 1$**:** To solve (3.12) for $\alpha = 1$, consider a Stein equation (see, (5.9)) for tempered 1-stable random variable $Y_{1,\gamma}$ is given by

$$(-x+\beta)\,g(x) + \int_{\mathbb{R}}\left(g(x+u) - g(x)\mathbf{1}_{\{|u|\le 1\}}(u)\right)u\nu_{1,\gamma}(du) = h(x) - \mathbb{E}h(Y_{1,\gamma}), \tag{5.12}$$

where $h \in \mathcal{H}_{\delta}$. Following similar steps to proof as the Case 1 of (i), and using (3.11), we see that the function $g_h^{1,\gamma}(x) = -\int_0^{\infty} e^{-t}\int_{\mathbb{R}} h'(u+xe^{-t})F_{Y_{(\gamma,t)}}(du)dt$ solves (5.12) i.e.,

$$\begin{aligned}(-x+\beta)\,g_h^{1,\gamma}(x) &+ \int_{\mathbb{R}}\left(g_h^{1,\gamma}(x+u) - g_h^{1,\gamma}(x)\mathbf{1}_{\{|u|\le 1\}}(u)\right)u\nu_{1,\gamma}(du)\\ &= h(x) - \mathbb{E}h(Y_{\gamma}).\end{aligned} \tag{5.13}$$

Observe that,

$$\begin{aligned}\lim_{\gamma\to 0^+} g_h^{1,\gamma}(x) &= -\lim_{\gamma\to 0^+}\int_0^{\infty} e^{-t}\int_{\mathbb{R}} h'(u+xe^{-t})F_{Y_{(\gamma,t)}}(du)dt\\ &= -\int_0^{\infty} e^{-t}\int_{\mathbb{R}} h'(u+xe^{-t})F_{X_{(t)}}(du)dt\\ &= g_h^1(x).\end{aligned}$$

Also,

$$\lim_{\gamma\to 0^+}\mathbb{E}h(Y_{\gamma}) = \mathbb{E}h(X), \quad (\text{since } Y_{\gamma} \xrightarrow{d} X).$$

Hence taking limit as $\gamma \to 0^+$ on (5.13), we get

$$(-x+\beta)\,g_h^1(x) + \int_{\mathbb{R}}\left(g_h^1(x+u) - g_h^1(x)\mathbf{1}_{\{|u|\le 1\}}(u)\right)u\nu_1(du) = h(x) - \mathbb{E}h(X).$$

Hence, for $\alpha = 1$, $g_h^1$ is the solution of (3.12). Note here that, on careful adjustments of the integrals and suitably adjustments of parameters as previous case, we see that the function $\tilde{g}_h^1(x) = -\int_0^{\infty}\left(P_t^1(h)(x) - \mathbb{E}h(X)\right)dt$, $h \in \mathcal{H}_{\delta},\ \delta \in (0,1)$ is well-defined and $(\tilde{g}_h^1)'(x) = g_h^1(x)$ for all $x \in \mathbb{R}$.

**Proof of** (**ii**). Following similar steps to proof of Case 1 of **(i)**, it immediately shows that $g_h^{\alpha}$ (where $\alpha \in (1,2)$ and $h \in \mathcal{H}_2$) is the solution of (3.14). So, it remains to show that $g_h^{\alpha}$ is well-defined. Let us consider a function $\tilde{g}_h^{\alpha} : \mathbb{R} \to \mathbb{R}$ defined as

$$\tilde{g}_h^{\alpha}(x) = -\int_0^{\infty}\left(P_t^{\alpha}(h)(x) - \mathbb{E}h(X)\right)dt, \quad h \in \mathcal{H}_2,$$

where $P_t^\alpha$ is the semigroup as defined in (3.6). Now, we show that for any $h \in \mathcal{H}_2$ and $\alpha \in (1,2)$, $\tilde{g}_h^\alpha$ is well-defined and $(\tilde{g}_h^\alpha)'(x) = g_h^\alpha(x)$ for all $x \in \mathbb{R}$.

Using (3.7), we have

$$\begin{aligned}
|P_t^\alpha(h)(x) - \mathbb{E}h(X)| &= \left| \int_{\mathbb{R}} h(r + e^{-t}x) F_{X_{(t)}}(dr) - \int_{\mathbb{R}} h(r) F_X(dr) \right| \\
&= \left| \int_{\mathbb{R}} (h(r + e^{-t}x) - h(r)) F_{X_{(t)}}(dr) \right. \\
&\quad \left. + \int_{\mathbb{R}} h(r) F_{X_{(t)}}(dr) - \int_{\mathbb{R}} h(r) F_X(dr) \right| \\
&\le e^{-t}|x||h'| + \left| \int_{\mathbb{R}} \widehat{h}(z) \left(\phi_t(z) - \phi_\alpha(z)\right) dz \right| \\
&\le e^{-t}|x||h'| + \int_{\mathbb{R}} |\widehat{h}(z)||\phi_t(z) - \phi_\alpha(z)| dz \qquad (5.14)
\end{aligned}$$

Now, let us calculate an upper bound between the difference of two characteristic functions $\phi_t$ and $\phi_\alpha$. For all $t > 0$ and $z \in \mathbb{R}$,

$$|\phi_t(z) - \phi_\alpha(z)| = \left| \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)} - \phi_1(z) \right| \le \left|\phi_\alpha(e^{-t}z) - 1\right| = |e^{\omega_t(z)} - 1|,$$

where $\omega_t(z) = e^{-t\alpha}(iz\tilde{\beta} + \int_{\mathbb{R}}(e^{izu} - 1 - izu\mathbf{1}_{\{|u|\le 1\}})\nu_\alpha(du))$, $\tilde{\beta} = \beta e^{(\alpha-1)t} + \int_{\mathbb{R}}(u\mathbf{1}_{\{|u|\le 1\}} - u\mathbf{1}_{\{|u|\le e^{-t}\}})\nu_\alpha(du)$. Note that the function $z \to e^{s\omega_t(z)}$ is a characteristic function for all $s \in (0,\infty)$. Indeed, $e^{s\omega_t(z)}$ is a characteristic function of an $\alpha$-stable random variable with different parameters.

Thus, for all $z \in \mathbb{R}$ and $t > 0$,

$$\begin{aligned}
|\phi_t(z) - \phi_\alpha(z)| &\le \left| \int_0^1 \frac{d}{ds}(\exp(s\omega_t(z))) ds \right| \\
&\le |\omega_t(z)| \\
&\le e^{-t\alpha} \left| iz\tilde{\beta} + \int_{\mathbb{R}} (e^{izu} - 1 - izu\mathbf{1}_{\{|u|\le 1\}})\nu_\alpha(du) \right| \\
&\le C_\alpha e^{-t\alpha}(1 + |z|^2), \quad C_\alpha > 0, \qquad (5.15)
\end{aligned}$$

where the last inequality is followed by [2, p.30, Ex. 1.2.16]. Using (5.15) in (5.14), one can easily show that $\int_0^\infty |P_t^\alpha(h)(x) - \mathbb{E}h(X)| dt < \infty$. Hence, $\tilde{g}_h^\alpha(x)$ is well-defined. The rest of this part follows from similar computations as Case 1 of **(i)**.

### *5.4. Proof of Theorem 3.14*

Recall the definition of $(P_t^\alpha)_{t\ge 0}$,

$$P_t^\alpha(g)(x) = \int_{\mathbb{R}} g(r + e^{-t}x) F_{X_{(t)}}(dr), \quad g \in \mathcal{F},$$

where $\alpha \in (0,2)$ and $F_{X_{(t)}}$ is the distribution function of $X_{(t)}$ (see, (3.5)).

*Proof of (i).* Suppose $\alpha \in (0,1)$ and $h \in \mathcal{H}_\delta$, $\delta \in (0,\alpha)$.

Let

$$g_h^\alpha(x) = -\int_0^\infty e^{-t} \int_{\mathbb{R}} h'(xe^{-t}+u) F_{X_{(t)}}(du)dt.$$

It is clear that

$$\begin{aligned}\|g_h^\alpha\| &\le \left|\int_0^\infty e^{-t}dt|| \int_{\mathbb{R}} \|h'\| F_{X_{(t)}}(du)\right| \\ &= \|h'\|,\end{aligned}$$

the desired conclusion follows.

Now observe that, for any $x, y \in \mathbb{R}$ and $h \in \mathcal{H}_\delta$,

$$\begin{aligned}|g_h^\alpha(x) - g_h^\alpha(y)| &\le \int_0^\infty e^{-t} \int_{\mathbb{R}} \left|h'(xe^{-t}+z) - h'(ye^{-t}+z)\right| F_{X_{(t)}}(dz)dt \\ &\le \int_0^\infty e^{-t} \int_{\mathbb{R}} |x-y|^\delta e^{-t\delta} F_{X_{(t)}}(dz)dt \\ &= |x-y|^\delta \int_0^\infty e^{-t(1+\delta)}dt\end{aligned}$$

$$|g_h^\alpha(x) - g_h^\alpha(y)| \le \frac{1}{1+\delta} |x-y|^\delta,$$

the desired conclusion follows.

For $\alpha \in (0,1)$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} u g_h^\alpha(x+u)\nu_\alpha(du)\right| &= \alpha \left|\int_{\mathbb{R}}\int_0^\infty (P_t^\alpha h(x+u) - P_t^\alpha h(x))dt\nu_\alpha(du)\right| \\ &\qquad\qquad \text{(using Proposition A.4)} \\ &\le \alpha \left|\int_{|u|>1}\int_0^\infty (P_t^\alpha h(x+u) - P_t^\alpha h(x))dt\nu_\alpha(du)\right| \\ &+ \alpha \left|\int_{|u|\le 1}\int_0^\infty (P_t^\alpha h(x+u) - P_t^\alpha h(x))dt\nu_\alpha(du)\right| \\ &:= \text{I+II}.\end{aligned}$$

Now observe that

$$\begin{aligned}\text{I} &= \alpha \left|\int_{|u|>1}\int_0^\infty (P_t^\alpha h(x+u) - P_t^\alpha h(x))dt\nu_\alpha(du)\right| \\ &= \alpha \left|\int_{|u|>1}\int_0^\infty \int_{\mathbb{R}} \left(h((x+u)e^{-t}+y) - h(xe^{-t}+y)\right) F_{X_{(t)}}(dy)dt\nu_\alpha(du)\right|\end{aligned}$$

$$\leq \alpha \int_{|u|>1} |u|^{\delta} \int_0^{\infty} e^{-t\delta} dt \nu_\alpha(du)$$
$$= \frac{\alpha(m_1+m_2)}{\delta(\alpha-\delta)}. \tag{5.16}$$

Consider,

$$\begin{aligned}
\mathrm{II} =& \alpha \left| \int_{|u|\leq 1} \int_0^{\infty} (P_t^{\alpha} h(x+u) - P_t^{\alpha} h(x)) dt \nu_\alpha(du) \right| \\
&= \alpha \left| \int_{|u|\leq 1} \int_0^{\infty} \int_{\mathbb{R}} \big( h((x+u)e^{-t}+y) - h(xe^{-t}+y) \big) F_{X_{(t)}}(dy) dt \nu_\alpha(du) \right| \\
&\leq \alpha \int_{|u|\leq 1} |u| \int_0^{\infty} e^{-t} dt \nu_\alpha(du) \\
&\leq \frac{\alpha(m_1-m_2)}{1-\alpha}.
\end{aligned} \tag{5.17}$$

Hence, by (5.16) and (5.17), we get

$$|A_0 g_h^{\alpha}(x)| \leq \frac{\alpha(m_1+m_2)}{\delta(\alpha-\delta)} + \frac{\alpha(m_1-m_2)}{1-\alpha} := C_{\alpha,\delta,m_1,m_2},$$

the desired conclusion follows. □

*Proof of (ii).* The proofs of first two properties are similar to previous case. To prove the third property, we split $g_h^1$ in terms of the semigroup $P_t^1$ defined in (3.6). We write

$$\begin{aligned}
A_1 g_h^1 :=& \int_{\mathbb{R}} u \left( g_h^1(x+u) - g_h^1(x) \mathbf{1}_{\{|u|\leq 1\}} \right) \nu_1(du) \\
&= \int_{\{|u|\leq 1\}} u \left( g_h^1(x+u) - g_h^1(x) \right) \nu_1(du) + \int_{\{|u|>1\}} u g_h^1(x+u) \nu_1(du) \\
&= \int_{\{|u|\leq 1\}} u \int_0^{\infty} \left( e^{-t} \left( \int_{\mathbb{R}} h'((x+u)e^{-t}+y) - h'(xe^{-t}+y) \right) F_{X_{(t)}}(dy) \right) dt \nu_1(du) \\
&+ \int_{\{|u|>1\}} \int_0^{\infty} (P_t^1 h(x+u) - P_t^1 h(x)) dt \nu_1(du) \text{ (using Fubini's theorem)} \\
&:= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By similar computation as Case 1 of **(i)**, it is easy to show that

$$\begin{aligned}
\mathrm{I} =& \int_{\{|u|\leq 1\}} u \int_0^{\infty} \left( e^{-t} \left( \int_{\mathbb{R}} h'((x+u)e^{-t}+y) - h'(xe^{-t}+y) \right) F_{X_{(t)}}(dy) \right) dt \nu_1(du) \\
&\leq \frac{1}{1+\delta} \int_{\{|u|\leq 1\}} |u|^{1+\delta} \nu_1(du) = \frac{m_1+m_2}{\delta(1+\delta)}.
\end{aligned} \tag{5.18}$$

Consider,

$$\begin{aligned}
\mathrm{II} &= \int_{\{|u|>1\}} \int_0^\infty (P_t^1 h(x+u) - P_t^1 h(x)) dt \nu_1(du) \\
&\le \frac{1}{\delta} \int_{\{|u|>1\}} |u|^\delta \nu_1(du) = \frac{m_1+m_2}{\delta(1-\delta)} \qquad (5.19)
\end{aligned}$$

Hence, by (5.18) and (5.19), we get

$$\|A_1 g_h^1\| \le \frac{2(m_1+m_2)}{\delta(1-\delta^2)} := C_{1,\delta,m_1,m_2},$$

the desired conclusion follows. □

*Proof of (iii).* Suppose $\alpha \in (1,2)$ and $h \in \mathcal{H}_2$.

Let

$$g_h^\alpha(x) = -\int_0^\infty e^{-t} \int_{\mathbb{R}} h'(xe^{-t}+u) F_{X_{(t)}}(du) dt.$$

Then,

$$(g_h^\alpha)'(x) = -\int_0^\infty e^{-2t} \int_{\mathbb{R}} h'(xe^{-t}+u) F_{X_{(t)}}(du) dt.$$

It is also easy to show that

$$\|(g_h^\alpha)\| \le \|h'\|, \text{ and } \|(g_h^\alpha)'\| \le \frac{1}{2}\|h''\|.$$

Let $m_1 = m_2 = m$. Let $A_2 g_h^\alpha(x) = \int_{\mathbb{R}} (g_h^\alpha(x+u) - g_h^\alpha(x)) u \nu_\alpha(du)$. Then, for any $x, y \in \mathbb{R}$

$$\begin{aligned}
|A_2 g_h^\alpha(x) - A_2 g_h^\alpha(y)| &\le \int_{\mathbb{R}} |(g_h^\alpha(x+u) - g_h^\alpha(y+u)) - (g_h^\alpha(x) - g_h^\alpha(y))| \, |u| \nu_\alpha(du) \\
&= m \left( \int_{|u|>|x-y|} + \int_{-|x-y|}^{|x+y|} \right) |(g_h^\alpha(x+u) - g_h^\alpha(y+u)) \\
&\quad -(g_h^\alpha(x) - g_h^\alpha(y))| \frac{du}{|u|^\alpha} \\
&=: \mathrm{I}+\mathrm{II}
\end{aligned}$$

Now observe that

$$\begin{aligned}
\mathrm{I} &= m \int_{|u|>|x-y|} |(g_h^\alpha(x+u) - g_h^\alpha(y+u)) - (g_h^\alpha(x) - g_h^\alpha(y))| \frac{du}{|u|^\alpha} \\
&\le m \int_{|u|>|x-y|} (|g_h^\alpha(x+u) - g_h^\alpha(y+u)| + |(g_h^\alpha(x) - g_h^\alpha(y))|) \frac{du}{|u|^\alpha}
\end{aligned}$$

$$\leq 4m\|(g_h^\alpha)'\||x-y|\int_{|x-y|}^{\infty} u^{-\alpha}du$$
$$\leq 2m\|h''\|\frac{|x-y|^{2-\alpha}}{\alpha-1}. \tag{5.20}$$

Consider,

$$\begin{aligned}
\mathrm{II} &= m\int_{-|x-y|}^{|x+y|} |(g_h^\alpha(x+u)-g_h^\alpha(y+u))-(g_h^\alpha(x)-g_h^\alpha(y))|\,\frac{du}{|u|^\alpha} \\
&\leq m\int_{-|x-y|}^{|x-y|} (|g_h^\alpha(x+u)-g_h^\alpha(x)|+|(g_h^\alpha(y+u)-g_h^\alpha(y))|)\,\frac{du}{|u|^\alpha} \\
&\leq 4m\|(g_h^\alpha)'\|\int_0^{|x-y|} u^{1-\alpha}du \\
&\leq 2m\|h''\|\frac{|x-y|^{2-\alpha}}{2-\alpha}. \qquad (5.21)
\end{aligned}$$

Hence, by (5.20) and (5.21), we get

$$\|A_2 g_h^\alpha(x)-A_2 g_h^\alpha(y)\| \leq C_{\alpha,m}\|h''\||x-y|^{2-\alpha},$$
$$\text{where } C_{\alpha,m} = 2m\left(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\right).$$

□

## *5.5. Proof of Theorem 3.16*

Recall that $(Y_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that $Y_1 \in D_\alpha$ (see, Definition 3.15). Denote

$$S_n = n^{-\frac{1}{\alpha}}(Y_1+Y_2+\ldots+Y_n), \text{ and}$$
$$S_{n,i} = S_n - n^{-\frac{1}{\alpha}}Y_i.$$

Note that, $S_{n,i}$ and $Y_i$ are independent. To prove this theorem, we first derive some lemmas. With the help of these lemmas, we obtain Wasserstein-$\delta$ distance error bounds for $\alpha$-stable approximations with $\alpha \in (0,1]$.

### *5.5.1. Proof of (a)*

To prove this part of Theorem 3.16, we use the following lemmas. Recall the Lévy measure $\nu_\alpha$ for $\alpha$-stable distributions is given by

$$\nu_\alpha(du) = \left(m_1\frac{1}{|u|^{1+\alpha}}\mathbf{1}_{(0,\infty)}(u) + m_2\frac{1}{|u|^{1+\alpha}}\mathbf{1}_{(-\infty,0)}(u)\right)du,$$

where $m_1, m_2 \in [0,\infty)$, $m_1+m_2>0$ and $\alpha \in (0,2)$.

**Lemma 5.1.** *Let $\alpha \in (0,1)$. Let $g_h^\alpha$ be a function defined in (3.13). Then, for any $a > 0$,*

$$\int_{\mathbb{R}} u g_h^\alpha(x+u)\nu_\alpha(du) = a^{1-\alpha}\int_{\mathbb{R}} u g_h^\alpha(x+au)\nu_\alpha(du)$$

*Proof.* We write

$$\begin{aligned}\int_{\mathbb{R}} u g_h^\alpha(x+u)\nu_\alpha(du) &= \int_{\mathbb{R}} u g_h^\alpha(x+u)\frac{(m_1 1_{(0,\infty)}(u) + m_2 1_{(-\infty,0)}(u))}{|u|^{\alpha+1}}du \\ &= a^{1-\alpha}\int_{\mathbb{R}} u g_h^\alpha(x+au)\frac{(m_1 1_{(0,\infty)}(u) + m_2 1_{(-\infty,0)}(u))}{|u|^{\alpha+1}}du \\ &= a^{1-\alpha}\int_{\mathbb{R}} u g_h^\alpha(x+au)\nu_\alpha(du),\end{aligned}$$

the desired conclusion follows. □

**Lemma 5.2.** *Let $\alpha \in (0,1)$. Let $Y \in D_\alpha$ and $g_h^\alpha$ be a function defined in (3.13). Then, for $0 < a < 1$ and $z \in \mathbb{R}$,*

$$\mathbb{E}\left(\left|\int_{\mathbb{R}} u\left(g_h^\alpha(z+aY+u) - g_h^\alpha(z+u)\right)\nu_\alpha(du)\right|\right) \le C^{A,K}_{\alpha,\delta,m_1,m_2}a^\alpha$$

*Proof.* We write

$$\mathbb{E}\left(\left|\int_{\mathbb{R}} u\left(g_h^\alpha(z+aY+u) - g_h^\alpha(z+u)\right)\nu_\alpha(du)\right|\right) := \text{I}+\text{II},$$

where

$$\begin{aligned}\text{I}: &= \mathbb{E}\left(\left|\int_{\mathbb{R}} u\left(g_h^\alpha(z+aY+u) - g_h^\alpha(z+u)\right)\nu_\alpha(du)\right|\mathbf{1}_{|Y|>a^{-1}}\right),\\ \text{II}: &= \mathbb{E}\left(\left|\int_{\mathbb{R}} u\left(g_h^\alpha(z+aY+u) - g_h^\alpha(z+u)\right)\nu_\alpha(du)\right|\mathbf{1}_{|Y|\le a^{-1}}\right).\end{aligned}$$

For $\alpha \in (0,1)$, one can write by (3.18) and (3.24),

$$\begin{aligned}\text{I} &\le 2C_{\alpha,\delta,m_1,m_2}P(|Y| \ge a^{-1}) \\ &\le 4C_{\alpha,\delta,m_1,m_2}\left(A + \sup_{|y|\ge a^{-1}}|e(y)|\right)a^\alpha \\ &\le 4C_{\alpha,\delta,m_1,m_2}(A+K)a^\alpha. \end{aligned} \tag{5.22}$$

It is also easy to show that

$$\text{II} \le C_\alpha a^\alpha. \tag{5.23}$$

Hence, by (5.22) and (5.23), we have

$$\begin{aligned}\mathbb{E}\left(\left|\int_{\mathbb{R}} u\left(g_h^\alpha(z+aY+u) - g_h^\alpha(z+u)\right)\nu_\alpha(du)\right|\right) &\le \left(4C_{\alpha,\delta,m_1,m_2} + C_\alpha\right)a^\alpha \\ &\le C^{A,K}_{\alpha,\delta,m_1,m_2}a^\alpha,\end{aligned}$$

the desired conclusion follows. □

Recall the definition of $D_\alpha$ in Definition 3.15. We see that the function $e$ satisfies certain conditions with the domain $\{|y| > L\}$. These conditions play an important role for proving the following lemma.

**Lemma 5.3.** *Let $\alpha \in (0,1)$. Let $Y \in D_\alpha$ and $X$ be a random variable with finite $\delta$-th moment, which is independent of $Y$. For any $0 < a < \frac{1}{L}$ and $g_h^\alpha$ defined in (3.13), define*

$$\mathcal{J}_1 := \left|\mathbb{E}\left(Y g_h^\alpha(X+aY)\right) - \mathbb{E}\left(Y \mathbf{1}_{(-1,1)}(aY)\right)\mathbb{E}\left(g_h^\alpha(X)\right)\right|,$$

*Then,*

$$\begin{aligned}\mathcal{J}_1 \le{}& C_{1,\delta,L}a^\delta + C_{2,\delta}a^\delta \sup_{L<|y|<\frac{1}{a}} \left(\alpha|e(y)| + |ye'(y)|\right) \int_{L<|y|<\frac{1}{a}} |y|^{\delta-\alpha} dy \\ &+ 2a^{\alpha-1} \sup_{|y|>a^{-1}} \left(\alpha|e(y)| + |ye'(y)|\right) \int_{|y|>1} \eta_{\alpha,\beta,\delta,m_1,m_2}(y)|y|^{-1-\alpha} dy,\end{aligned}$$

*where $C_{1,\delta,L} = \frac{2}{1+\delta}L^{1+\delta}$ and $C_{2,\delta} = \frac{2}{1+\delta}$.*

*Proof.* We have by (3.24),

$$\mathcal{J}_1 = \left|\mathbb{E}\left(\int_{\mathbb{R}} (y g_h^\alpha(X+ay) - y\mathbf{1}_{(-1,1)}(ay) g_h^\alpha(X)) dF_Y(y)\right)\right|.$$

Since $e$ is in $C^2$, for any $|y| > L$,

$$dF_Y(y) = \frac{A\alpha + \alpha e(y) - ye'(y)}{|y|^{1+\alpha}} \kappa_\theta(y) dy,$$

where $\kappa_\theta(y) = (1+\theta)\mathbf{1}_{(0,\infty)}(y) + (1-\theta)\mathbf{1}_{(-\infty,0)}(y)$.

Thus, we have

$$\begin{aligned}\mathcal{J}_1 \le{}& \mathbb{E}\int_{|y|<L} |y| \left|g_h^\alpha(X+ay) - g_h^\alpha(X)\right| dF_Y(y) \\ &+ \mathbb{E}\int_{L<|y|<\frac{1}{a}} |y| \left|g_h^\alpha(X+ay) - g_h^\alpha(X)\right| dF_Y(y) \\ &+ \mathbb{E}\int_{|y|>\frac{1}{a}} |y g_h^\alpha(X+ay)| dF_Y(y) \\ :={}& \text{I+II+III}.\end{aligned}$$

It is easy to verify by (3.17),

$$\begin{aligned}\text{I} &\le \frac{1}{1+\delta}a^\delta \int_{|y|<L} |y|^{1+\delta} dF_Y(y) \\ &\le \frac{2}{1+\delta}L^{1+\delta}a^\delta,\end{aligned}$$

and

$$\begin{aligned}
\text{II} &\leq \frac{2a^{\delta}}{1+\delta}\int_{L<|y|<\frac{1}{a}} \frac{|\alpha e(y) - ye'(y)|}{|y|^{\alpha-\delta}}dy \\
&\leq \frac{2a^{\delta}}{1+\delta}\int_{L<|y|<\frac{1}{a}} |y|^{\delta-\alpha}\left(\alpha|e(y)| + |ye'(y)|\right)dy \\
&\leq \frac{2a^{\delta}}{1+\delta}\sup_{L<|y|<\frac{1}{a}}\left(\alpha|e(y)| + |ye'(y)|\right)\int_{L<|y|<\frac{1}{a}} |y|^{\delta-\alpha}dy.
\end{aligned}$$

For the third term, we have,

$$\begin{aligned}
\text{III} &\leq 2\int_{|y|>\frac{1}{a}} \frac{|\alpha e(y) - ye'(y)||yg_h^{\alpha}(X+ay)|}{|y|^{\alpha+1}}dy \\
&\leq 2\sup_{|y|>a^{-1}}\left(\alpha|e(y)| + |ye'(y)|\right)\int_{|y|>\frac{1}{a}} \frac{|yg_h^{\alpha}(X+ay)|}{|y|^{\alpha+1}}dy \\
&\leq 2a^{\alpha-1}\sup_{|y|>a^{-1}}\left(\alpha|e(y)| + |ye'(y)|\right)\int_{|y|>1} \eta_{\alpha,\beta,\delta,m_1,m_2}(y)|y|^{-1-\alpha}dy,
\end{aligned}$$

where the last inequality follows by Lemma 5.1 and Proposition A.5. Combining the estimates obtained in I, II and III, the desired conclusion follows. □

*Proof of (a).* With the help of above lemmas, we now derive bound in the $d_{W_\delta}$ distance for $\alpha$-stable approximation with $\alpha \in (0,1)$.

By (3.4), we have

$$\begin{aligned}
|\mathbb{E}[h(S_n) - h(X)]| &= \left|\mathbb{E}\left[-S_n g_h^{\alpha}(S_n) + \beta_1 g_h^{\alpha}(S_n) + \int_{\mathbb{R}} g_h^{\alpha}(S_n+u)u\nu_{\alpha}(du)\right]\right| \\
&\leq \text{I+II+III},
\end{aligned}$$

where,

$$\begin{aligned}
\text{I} &:= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left|\int_{\mathbb{R}} u\left(g_h^{\alpha}(S_{n,i} + n^{-\frac{1}{\alpha}}Y_i + u) - g_h^{\alpha}(S_{n,i}+u)\right)\nu_{\alpha}(du)\right| \\
\text{II} &:= n^{-\frac{1}{\alpha}}\sum_{i=1}^{n}\mathbb{E}\left|-Y_i g_h^{\alpha}(S_n) + Y_i\mathbf{1}_{(-1,1)}(|n^{-\frac{1}{\alpha}}Y_i|)\mathbb{E}g_h^{\alpha}(S_{n,i})\right| \\
\text{III} &:= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left|\int_{\mathbb{R}} ug_h^{\alpha}(S_{n,i}+u)\nu_{\alpha}(du) + \beta_1 g_h^{\alpha}(S_n)\right. \\
&\qquad \left. - n^{1-\frac{1}{\alpha}}Y_i\mathbf{1}_{(-1,1)}(|n^{-\frac{1}{\alpha}}Y_i|)\mathbb{E}g_h^{\alpha}(S_{n,i})\right|.
\end{aligned}$$

For $\alpha \in (0,1)$, we have by Lemma 5.2 with $a = n^{-\frac{1}{\alpha}}$,

$$\text{I} \leq \frac{C^{A,K}_{\alpha,\delta,m_1,m_2}}{n}.$$

By Lemma 5.3 with $a = n^{-\frac{1}{\alpha}}$, we have

$$\mathrm{II} \le C_{1,\delta,L} n^{1-\frac{(1+\delta)}{\alpha}} + C_{2,\delta} n^{1-\frac{(1+\delta)}{\alpha}} \sup_{L<|y|<n^{\frac{1}{\alpha}}} (\alpha|e(y)| + |ye'(y)|) \int_{L<|y|<n^{\frac{1}{\alpha}}} |y|^{\delta-\alpha} dy$$
$$+ 2 \sup_{|y|>n^{\frac{1}{\alpha}}} (\alpha|e(y)| + |ye'(y)|) \int_{|y|>1} \eta_{\alpha,\beta,\delta,m_1,m_2}(y)|y|^{-1-\alpha} dy.$$

Using Lemma 5.1 with $a = n^{-\frac{1}{\alpha}}$, we have

$$\mathrm{III} \le C_{\alpha,\delta,m_1,m_2} n^{-\frac{(1-\alpha)}{\alpha}} + \beta_1 + n^{-\frac{(1-\alpha)}{\alpha}} \int_{|y|<n^{\frac{1}{\alpha}}} |y| dF_Y(y).$$

Combining the estimates obtained in I, II and III, the desired conclusion follows.

### 5.5.2. *Proof of (b)*

The following two lemmas play an important role for deriving bound in the $d_{W_\delta}$ distance for 1-stable approximation. Recall that $A_1 g_h^1(x) := \int_{\mathbb{R}} u(g_h^1(x+u) - u g_h^1 \mathbf{1}_{\{|u|\le 1\}})\nu_1(du)$, where $g_h^1$ is defined in (3.13) and $\nu_1$ is the Lévy measure (see (2.3)).

**Lemma 5.4.** *Let $\alpha = 1$. Let $Y \in D_\alpha$ and $g_h^1$ be defined in (3.13). Then, for any $0 < a < 1$ and $z \in \mathbb{R}$,*

$$\mathbb{E}\left(\left|A_1 g_h^1(z) - A_1 g_h^1(z+aY)\right|\right) \le C^{A,K}_{1,\delta,m_1,m_2} a + 2C_{1,\delta,m_1,m_2} \int_0^{\frac{1}{a}} dF_{|Y|}(y),$$

*where $C^{A,K}_{1,\delta,m_1,m_2}$ and $C_{1,\delta,m_1,m_2}$ are constants.*

□

*Proof.* We write

$$\mathbb{E}\left(\left|A_1 g_h^1(z) - A_1 g_h^1(z+aY)\right|\right) := \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} := \mathbb{E}\left(\left|A_1 g_h^1(z) - A_1 g_h^1(z+aY)\right| \mathbf{1}_{|y|>\frac{1}{a}}\right),$$
$$\mathrm{II} := \mathbb{E}\left(\left|A_1 g_h^1(z) - A_1 g_h^1(z+aY)\right| \mathbf{1}_{|y|\le\frac{1}{a}}\right)$$

When $\alpha = 1$, one can write by (3.21) and (3.24),

$$\mathrm{I} \le 2C_{1,\delta,m_1,m_2} P(|Y| \ge a^{-1})$$

$$\leq 4C_{1,\delta,m_1,m_2}\left(A+\sup_{|y|\geq a^{-1}}|e(y)|\right)a$$
$$\leq 4C_{1,\delta,m_1,m_2}(A+K)a,$$

and

$$\begin{aligned}\mathrm{II} &\leq 2C_{1,\delta,m_1,m_2}P(|Y|<\frac{1}{a})\\ &= C_{1,\delta,m_1,m_2}\int_0^{\frac{1}{a}} dF_{|Y|}(y),\end{aligned}$$

Combining the estimates obtained in I and II, the desired conclusion follows. ☐

**Lemma 5.5.** *Let $\alpha = 1$. Let $Y \in D_\alpha$ and $X$ be a random variable with $\delta$-th finite moment such that $X$ and $Y$ are independent. For any $0 < a < \frac{1}{L}$, define*

$$\mathcal{J}_2 := \left|\mathbb{E}\left(Yg_h^1(X+aY)\right) - \mathbb{E}\left(Y\mathbf{1}_{(-1,1)}(aY)\right)\mathbb{E}\left(g_h^1(X)\right) - \mathbb{E}\left(A_0 g_h^1(X)\right)\right|.$$

*Then,*

$$\begin{aligned}\mathcal{J}_2 \leq \frac{1}{\delta+1}a^\delta\left(L^2+m_1+m_2\right) + \frac{a^\delta}{1+\delta}\int_{L<|u|<\frac{1}{a}}\frac{|e(u)-ue'(u)|}{|u|^{1-\delta}}du\\ + a\int_{|u|>1}\frac{|e(u/a)-u/ae'(u/a)|}{|u|}du.\end{aligned}$$

*Proof.* We have

$$\begin{aligned}\mathbb{E}A_1 g_h^1(X) &= \mathbb{E}\int_{\mathbb{R}} u\left(g_h^1(X+u) - g_h^1(X)\mathbf{1}_{\{|u|\leq 1\}}(u)\right)\nu_1(du)\\ &= \mathbb{E}\int_{\mathbb{R}} u\left(g_h^1(X+aY) - g_h^1(X)\mathbf{1}_{\{|aY|\leq 1\}}(aY)\right)\nu_1(du)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left(Yg_h^1(X+aY)\right) - \mathbb{E}\left(Y\mathbf{1}_{(-1,1)}(aY)\right)\mathbb{E}\left(g_h^1(X)\right)\\ =&\mathbb{E}\left(\int_{\mathbb{R}}\left(ug_h^1(X+au) - u\mathbf{1}_{(-1,1)}(au)g_h^1(X)\right)dF_Y(u)\right).\end{aligned}$$

Since $e$ is $C^2$, for any $|y| > L$

$$dF_Y(y) = \frac{A\alpha + e(y) - ye'(y)}{|y|^2}\kappa_\theta(y)dy,$$

where $\kappa_\theta(y) = (1+\theta)\mathbf{1}_{(0,\infty)}(y) + (1-\theta)\mathbf{1}_{(-\infty,0)}(y)$.

Thus we have,

$$\mathcal{J}_2 \leq \mathbb{E}\int_{|u|<L}\left|ug_h^1(X+au) - ug_h^1(X)\right|(dF_y(u)+\nu_1(du))$$

$$+\mathbb{E}\int_{L<|u|<\frac{1}{a}}\left|ug_h^1(X+au)-ug_h^1(X)\right|\frac{|\alpha e(u)-ue'(u)|}{|u|^2}du$$
$$+\mathbb{E}\int_{|u|>\frac{1}{a}}\left|ug_h^1(X+au)\right|\frac{|\alpha e(u)-ue'(u)|}{|u|^2}du$$
$$:=\text{I+II+III}$$

Moreover, by (3.20), it is easy to verify

$$\begin{aligned}\text{I}&\le\frac{1}{\delta+1}a^\delta\int_{|u|<L}u^{1+\delta}(dF_Y(u)+\nu_1(du))\\&\le\frac{1}{\delta+1}a^\delta\left(L^2+m_1+m_2\right).\end{aligned}$$

Using (3.20), we also have

$$\text{II}\le\frac{a^\delta}{1+\delta}\int_{L<|u|<\frac{1}{a}}\frac{|e(u)-ue'(u)|}{|u|^{1-\delta}}du.$$

For the third term, using (3.19), it can be immediately shown that

$$\text{III}\le a\int_{|u|>1}\frac{|e(u/a)-u/ae'(u/a)|}{|u|}du.$$

Combining the estimates obtained in I, II and III, the desired conclusion follows. □

**Proof of (b)** With the help of above lemmas, we now find bound in the $d_{W_\delta}$ distance for 1-stable approximation. By (3.4), we have

$$\begin{aligned}|\mathbb{E}[h(S_n)-h(X)]|&=\Big|\mathbb{E}[(-S_n+\beta)g_h^1(S_n)\\&\quad+\int_{\mathbb{R}}(g_h^1(S_n+u)-g_h^1(S_n)\mathbf{1}_{\{|u|\le1\}}(u))u\nu_1(du)]\Big|\\&\le\text{I+II+III},\end{aligned}$$

where

$$\text{I}:=\frac{1}{n}\sum_{i=1}^n\left|\mathbb{E}A_1g_h^1(S_{n,i})-\mathbb{E}A_1g_h^1(S_n)\right|$$
$$\text{II}:=\frac{1}{n}\sum_{i=1}^n\Big|\mathbb{E}\left(Y_ig_h^1(S_{n,i}+\frac{1}{n}Y_i)\right)-\mathbb{E}\left(Y_i\mathbf{1}_{(-1,1)}(|\frac{1}{n}Y_i|)g_h^1(S_{n,i})\right)$$
$$-\mathbb{E}\left(A_1g_h^1(S_{n,i})\right)\Big|$$
$$\text{III}:=\frac{1}{n}\sum_{i=1}^n\left|\mathbb{E}\left(Y_i\mathbf{1}_{(-1,1)}(|\frac{1}{n}Y_i|)g_h^1(S_{n,i})\right)-\beta\mathbb{E}g_h^1(S_n)\right|$$

For $\alpha = 1$, we have by Lemma 5.4 with $a = \frac{1}{n}$,

$$\mathrm{I} \leq C^{A,K}_{1,\delta,m_1,m_2}\frac{1}{n} + 2C_{1,\delta,m_1,m_2}\int_0^n dF_{|Y|}(y)$$

By Lemma 5.5 with $a = \frac{1}{n}$, we have

$$\mathrm{II} \leq \frac{1}{\delta+1}n^{-\delta}\left(L^2 + m_1 + m_2\right) + \frac{n^{-\delta}}{1+\delta}\int_{L<|u|<\frac{1}{a}} \frac{|e(u) - ue'(u)|}{|u|^{1-\delta}}du$$
$$+ \frac{1}{n}\int_{|u|>1}\frac{|e(nu) - nue'(nu)|}{|u|}du.$$

Using (3.24) and (3.19), we have

$$\mathrm{III} \leq \left|\int_0^n \frac{e(y) - e(-y)}{y}dy\right| + 2K + \beta.$$

Combining the estimates obtained in I, II and III, the desired conclusion follows.

### 5.6. *Proof of Theorem 3.18*

Recall that $(Y_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with $\mathbb{E}Y_i = 0$ and $\mathbb{E}|Y_i| < \infty$ for $1 \leq i \leq n$. Let $Z_i = n^{-\frac{1}{\alpha}}Y_i$ and define,

$$S_n = Z_1 + Z_2 + \ldots + Z_n \text{ and}$$
$$S_n(i) = S_n - Z_i.$$

Note that $S_n(i)$ and $S_n$ are independent. To prove this theorem, we use the following lemmas.

**Lemma 5.6.** *Let $\nu_\alpha$ be a Lévy measure for $\alpha$-stable distributions with $\alpha \in (1,2)$. Let $g_h^\alpha$ be a function defined in (3.15). Then for any $N > 0$,*

$$\int_{\mathbb{R}}\left(g_h^\alpha(S_n+u) - g_h^\alpha(S_n)\right)u\nu_\alpha(du) = \int_{-N}^{N} K_{\nu_\alpha}(t,N)(g_h^\alpha)'(S_n+t)dt + R_N(S_n),$$

*where*

$$K_{\nu_\alpha}(t,N) = \mathbf{1}_{[0,N]}(t)\int_t^N u\nu_\alpha(du) + \mathbf{1}_{[-N,0]}(t)\int_{-N}^{t}(-u)\nu_\alpha(du), \quad \textit{and}$$
$$R_N(S_n) = \int_{|u|>N}\left(g^\alpha(S_n+u) - g^\alpha(S_n)\right)u\nu_\alpha(du).$$

The proof of this lemma follows by similar computations [3, Lemma 5.3].

**Lemma 5.7.** *Let $g_h^\alpha$ be a function defined in (3.15). Then for any $N > 0$, we have,*

$$\mathbb{E}\big[S_n g^\alpha(S_n)\big] = \sum_{i=1}^n \int_{-N}^N \mathbb{E}\big[K_i(t,N)(g_h^\alpha)'(S_n(i)+t)\big]dt + R_1,$$

*where*

$$K_i(t,N) = \mathbb{E}\big[Z_i 1_{\{0\le t\le Z_i\le N\}} - Z_i 1_{\{-N\le Z_i\le t\le 0\}}\big], \text{ and}$$
$$R_1 = \sum_{i=1}^n \mathbb{E}\big[\xi_i\{g_h^\alpha(S_n) - g^\alpha(S_n(i)))\}\big]1_{\{|\xi_i|\ge N\}}.$$

The proof of this lemma follows by similar computations [39, Lemma 4.5].

Next, we derive a result using the above two lemmas which is as follows.

**Lemma 5.8.** *Let $g_h^\alpha$ be a function defined in (3.15). Then,*

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}}(g_h^\alpha(S_n+u) - g_h^\alpha(S_n))u\nu_\alpha(du) - S_n g_h^\alpha(S_n)\right]\\
&= \sum_{i=1}^n \int_{-N}^N \mathbb{E}\big(\frac{K_{\nu_\alpha}(t,N)}{n} - K_i(t,N)\big)(g_h^\alpha)'(S_n(i)+t)dt\\
&+ \frac{1}{n}\sum_{i=1}^n \mathbb{E}(R_N(S_n(i))) + R_1 + R_2,
\end{aligned}$$

*where $R_N(x)$ and $R_1$ are defined in Lemmas 5.6 and 5.7 respectively,*

$$\begin{aligned}
R_2 = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\bigg[&\int_{\mathbb{R}}(g_h^\alpha(S_n+u) - g_h^\alpha(S_n))u\nu_\alpha(du)\\
&- \int_{\mathbb{R}}(g_h^\alpha(S_n(i)+u) - g_h^\alpha(S_n(i)))u\nu_\alpha(du)\bigg].
\end{aligned}$$

*Proof.* We have,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}}(g_h^\alpha(S_n+u) - g_h^\alpha(S_n))u\nu_\alpha(du) - S_n g_h^\alpha(S_n)\right]\\
&= \frac{1}{n}\sum_{i=1}^n \mathbb{E}\big[\int_{\mathbb{R}}(g_h^\alpha(S_n(i)+u) - g_h^\alpha(S_n(i)))u\nu_\alpha(du)\\
&- S_n g_h^\alpha(S_n)\big] + R_1 + R_2 + \frac{1}{n}\sum_{i=1}^n \mathbb{E}\big[R_N(S_n(i))\big]\\
&= \sum_{i=1}^n \int_{-N}^N \mathbb{E}\big(\frac{K_{\nu_\alpha}(t,N)}{n} - K_i(t,N)\big)(g_h^\alpha)'(S_n(i)+t)dt\\
&+ \frac{1}{n}\sum_{i=1}^n \mathbb{E}(R_N(S_n(i))) + R_1 + R_2,
\end{aligned}$$

the desired conclusion follows. □

**Proof of Theorem 3.18** With the help of above three lemmas, we now find bound in the $d_{W_2}$ distance for $\alpha$-stable approximation with $\alpha \in (1,2)$. By (3.4), we have

$$\mathbb{E}[h(S_n) - h(X)] = \mathbb{E}\left(-S_n g_h^\alpha(S_n) + \int_{\mathbb{R}} (g_h^\alpha(S_n+u) - g_h^\alpha(S_n)) u\nu_\alpha(du)\right) + \mathbb{E}\left(\beta + \int_{|u|>1} u\nu_\alpha(du)\right) g_h^\alpha(S_n).$$

To get a bound on $\mathbb{E}[h(S_n) - h(X)]$, it is sufficient to bound the right hand side of the above equality relation. By Lemma 5.8 and (3.22), we have

$$\begin{aligned}&\left|\sum_{i=1}^n \int_{-N}^N \mathbb{E}\big(\frac{K_{\nu_\alpha}(t,N)}{n} - K_i(t,N)\big)(g_h^\alpha)'(S_n(i)+t)dt\right| \\ &\leq \frac{1}{2}||h''|| \sum_{i=1}^n \int_{-N}^N \left|\frac{K_{\nu_\alpha}(t,N)}{n} - K_i(t,N)\right|.\end{aligned}$$

Note that,

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^n \mathbb{E}(R_N(S_n(i)))\right| &\leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}\int_{|u|>N} |f'(S_n(i)+u) - f'(S_n(i))|\, u\nu_\alpha(du) \\ &\leq 2||h'|| \int_{|u|>N} |u|\nu_\alpha(du), \quad \text{and}\end{aligned}$$

$$\begin{aligned}|R_1| &= \left|\sum_{i=1}^n \mathbb{E}\big[Z_i\{f'(S_n) - f'(S_n(i)))\}\big] 1_{\{|Z_i|\geq N\}}\right| \\ &\leq 2||h'|| \sum_{i=1}^n \mathbb{E}\big[|Z_i| 1_{\{|Z_i|>N\}}\big].\end{aligned}$$

Using (3.23), we have

$$\begin{aligned}|R_2| &\leq \frac{1}{n}\sum_{i=1}^n \Big|\mathbb{E}[\int_{\mathbb{R}} (g_h^\alpha(S_n+u) - g_h^\alpha(S_n)) u\nu_\alpha(du) \\ &\quad - \int_{\mathbb{R}} (g_h^\alpha(S_n(i)+u) - g_h^\alpha(S_n(i))) u\nu_\alpha(du)]\Big| \\ &\leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big|\int_{\mathbb{R}} [(g_h^\alpha(S_n+u) - g_h^\alpha(S_n(i)+u)) \\ &\quad -(g_h^\alpha(S_n) - g_h^\alpha(S_n(i)))] u\nu_\alpha(du)|\end{aligned}$$

$$\leq C_{\alpha,m}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}|Z_i|^{2-\alpha}.$$

Also, for $m_1 = m_2 = m$ and $\beta = 0$, we have

$$\begin{aligned}\left|\mathbb{E}\left(\beta+\int_{\{|u|>1\}}u\nu_\alpha(du)\right)g_h^\alpha(S_n)\right| &= \left|\left(\beta+\int_{\{|u|>1\}}u\nu_\alpha(du)\right)\mathbb{E}g_h^\alpha(S_n\right| \\ &= 0,\end{aligned}$$

where the last equality holds as the integral $\int_{\{|u|>1\}}u\nu_\alpha(du) = 0$. Combining all the estimates above, we get the inequality of the theorem, as desired.

## Appendix A: Appendix

In this section, we prove some technical results used in the previous sections.

**Proposition A.1.** *Let $X \sim \mathcal{S}(\alpha,\beta,m_1,m_2)$. Then, its characteristic exponent $\eta_\alpha$ given in (2.4) can be written in the following form.*

$$\eta_\alpha(z) = \begin{cases} iz\gamma_\alpha - d_\alpha|z|^\alpha\left(1 - i\theta\frac{z}{|z|}\tan\frac{\pi}{2}\alpha\right), & \alpha\in(0,2)\setminus\{1\}, \\ iz\gamma_1 - d_1|z|(1+i\theta\frac{z}{|z|}\frac{2}{\pi}\log|z|), & \alpha = 1,\end{cases}$$

*where $\alpha \in (0,2)$, $\gamma_\alpha \in \mathbb{R}$, $d_\alpha \geq 0$ and $\theta \in [-1,1]$.*

*Proof.* Recall that for $X \sim \mathcal{S}(\alpha,\beta,m_1,m_2)$, the characteristic exponent is given by

$$\eta_\alpha(z) = iz\beta + \int_{\mathbb{R}}(e^{izu} - 1 - izu\mathbf{1}_{\{|u|\leq 1\}}(u))\nu_\alpha(du), \quad z\in\mathbb{R},$$

where $\nu_\alpha$ is the Lévy measure given by

$$\nu_\alpha(du) = \left(m_1\frac{1}{u^{1+\alpha}}\mathbf{1}_{(0,\infty)}(u) + m_2\frac{1}{|u|^{1+\alpha}}\mathbf{1}_{(-\infty,0)}(u)\right)du.$$

Now, we have to consider three different cases to proceed to the derivations of these expressions.

**(i)** $\alpha \in (0,1)$

As noted in Section 2, for $\alpha \in (0,1)$, the integral $\int_{\{|u|\leq 1\}}u\nu_\alpha(du) < \infty$. Indeed $\int_{\{|u|\leq 1\}}u\nu_\alpha(du) = \frac{m_1-m_2}{1-\alpha}$ Denote $\beta_1 = \beta - \frac{m_1-m_2}{1-\alpha}$. So, one can write $\eta_\alpha$ as

$$\eta_\alpha(z) = iz\beta_1 + \int_{\mathbb{R}}(e^{izu}-1)\nu_\alpha(du), \quad z\in\mathbb{R} \tag{A.1}$$

Suppose $z > 0$, then from (A.1)

$$\eta_\alpha(z) = iz\beta_1 + \int_0^\infty(e^{izu}-1)\nu_\alpha(du) + \int_{-\infty}^0(e^{izu}-1)\nu_\alpha(du)$$

$$= iz\beta_1 + m_1 \int_0^\infty (e^{izu} - 1)\frac{du}{u^{1+\alpha}} + m_2 \int_{-\infty}^0 (e^{izu} - 1)\frac{du}{|u|^{1+\alpha}}$$

$$= iz\beta_1 + z^\alpha \left( m_1 \int_0^\infty (e^{iv} - 1)\frac{du}{v^{1+\alpha}} + m_2 \int_0^\infty (e^{-iv} - 1)\frac{dv}{v^{1+\alpha}} \right) \quad \text{(A.2)}$$

Applying Cauchy's Theorem of contour integration on (A.2), we have

$$\eta_\alpha(z) = iz\beta_1 + z^\alpha \left( m_1 e^{-i\frac{\pi}{2}\alpha} L(\alpha) + m_2 e^{i\frac{\pi}{2}\alpha} L(\alpha) \right),$$

where $L(\alpha) = \int_0^\infty (e^{-y} - 1)\frac{dy}{y^{1+\alpha}} < 0$, see [18, p.164].

Thus,

$$\eta_\alpha(z) = iz\beta_1 + z^\alpha L(\alpha) \left( (m_1 + m_2)\cos(\frac{\pi}{2}\alpha) + i(m_2 - m_1)\sin(\frac{\pi}{2}\alpha) \right)$$

$$= iz\beta_1 + z^\alpha L(\alpha)(m_1 + m_2)\cos(\frac{\pi}{2}\alpha) \left( 1 + i\frac{m_2 - m_1}{m_1 + m_2}\tan(\frac{\pi}{2}\alpha) \right)$$

For $z < 0$,

$$\eta_\alpha(z) = \overline{\eta_\alpha(-z)}$$

$$= iz\beta_1 + (-z)^\alpha L(\alpha)(m_1 + m_2)\cos(\frac{\pi}{2}\alpha) \left( 1 + i\frac{m_2 - m_1}{m_1 + m_2}\frac{z}{|z|}\tan(\frac{\pi}{2}\alpha) \right).$$

Therefore, for any $z \in \mathbb{R}$

$$\eta_\alpha(z) = iz\beta_1 + |z|^\alpha L(\alpha)(m_1 + m_2)\cos(\frac{\pi}{2}\alpha) \left( 1 + i\frac{m_2 - m_1}{m_1 + m_2}\frac{z}{|z|}\tan(\frac{\pi}{2}\alpha) \right)$$

$$= iz\gamma_\alpha - d_\alpha |z|^\alpha \left( 1 - i\theta\frac{z}{|z|}\tan(\frac{\pi}{2}\alpha) \right),$$

where $\gamma_\alpha = \beta_1 = \beta - \frac{m_1 - m_2}{1-\alpha}$, $d_\alpha = (m_1 + m_2)\cos(\frac{\pi}{2}\alpha)\int_0^\infty (1 - e^{-y})\frac{dy}{y^{1+\alpha}}$ and $\theta = \frac{m_1 - m_2}{m_1 + m_2}$.

**(ii)** $\alpha \in (1, 2)$

As noted in Section 2, for $\alpha \in (1, 2)$, the integral $\int_{\{|u|>1\}} u\nu_\alpha(du) < \infty$. Indeed $\int_{\{|u|>1\}} u\nu_\alpha(du) = \frac{m_1 - m_2}{1-\alpha}$ Denote $\beta_2 = \beta - \frac{m_1 - m_2}{1-\alpha}$. So, one can write $\eta_\alpha$ as

$$\eta_\alpha(z) = iz\beta_2 + \int_{\mathbb{R}} (e^{izu} - 1 - izu)\nu_\alpha(du), \quad z \in \mathbb{R} \quad \text{(A.3)}$$

Suppose $z > 0$, then from (A.3)

$$\eta_\alpha(z) = iz\beta_2 + \int_0^\infty (e^{izu} - 1 - izu)\nu_\alpha(du) + \int_{-\infty}^0 (e^{izu} - 1 - izu)\nu_\alpha(du)$$

$$= iz\beta_2 + m_1 \int_0^\infty (e^{izu} - 1 - izu)\frac{du}{u^{1+\alpha}} + m_2 \int_{-\infty}^0 (e^{izu} - 1 - izu)\frac{du}{|u|^{1+\alpha}}$$

$$= iz\beta_2 + z^\alpha \left( m_1 \int_0^\infty (e^{iv} - 1 - iv)\frac{du}{v^{1+\alpha}} + m_2 \int_0^\infty (e^{-iv} - 1 + iv)\frac{dv}{v^{1+\alpha}} \right) \tag{A.4}$$

Applying Cauchy's Theorem of contour integration on (A.4), we have

$$\eta_\alpha(z) = iz\beta_2 + z^\alpha \left( m_1 e^{-i\frac{\pi}{2}\alpha} M(\alpha) + m_2 e^{i\frac{\pi}{2}\alpha} M(\alpha) \right),$$

where $M(\alpha) = \int_0^\infty (e^{-y} - 1 + y)\frac{dy}{y^{1+\alpha}} > 0$, see [18, p.164].

Thus, for any $z > 0$

$$\begin{aligned}\eta_\alpha(z) &= iz\beta_2 + z^\alpha M(\alpha) \left( (m_1 + m_2)\cos(\frac{\pi}{2}\alpha) + i(m_2 - m_1)\sin(\frac{\pi}{2}\alpha) \right) \\ &= iz\beta_2 + z^\alpha M(\alpha)(m_1 + m_2)\cos(\frac{\pi}{2}\alpha) \left( 1 + i\frac{m_2 - m_1}{m_1 + m_2}\tan(\frac{\pi}{2}\alpha) \right)\end{aligned}$$

For $z < 0$,

$$\begin{aligned}\eta_\alpha(z) &= \overline{\eta_\alpha(-z)} \\ &= iz\beta_2 + (-z)^\alpha M(\alpha)(m_1 + m_2)\cos(\frac{\pi}{2}\alpha) \left( 1 + i\frac{m_2 - m_1}{m_1 + m_2}\frac{z}{|z|}\tan(\frac{\pi}{2}\alpha) \right).\end{aligned}$$

Therefore, for any $z \in \mathbb{R}$

$$\begin{aligned}\eta_\alpha(z) &= iz\beta_2 + |z|^\alpha M(\alpha)(m_1 + m_2)\cos(\frac{\pi}{2}\alpha) \left( 1 + i\frac{m_2 - m_1}{m_1 + m_2}\frac{z}{|z|}\tan(\frac{\pi}{2}\alpha) \right) \\ &= iz\gamma_\alpha - d_\alpha |z|^\alpha \left( 1 - i\theta\frac{z}{|z|}\tan(\frac{\pi}{2}\alpha) \right),\end{aligned}$$

where $\gamma_\alpha = \beta_1 = \beta - \frac{m_1 - m_2}{1-\alpha}$, $d_\alpha = (m_1 + m_2)\cos(\frac{\pi}{2}\alpha)\int_0^\infty (1 - e^{-y} - y)\frac{dy}{y^{1+\alpha}}$ and $\theta = \frac{m_1 - m_2}{m_1 + m_2}$.

**(iii)** $\alpha = 1$

For $z \in \mathbb{R}$, it is easy to show that

$$\int_0^\infty \frac{\cos zu - 1}{u^2} du = -\frac{\pi}{2} z$$

Now, suppose $z > 0$, then

$$\begin{aligned}\eta_1(z) = iz\beta &+ \int_0^\infty (e^{izu} - 1 - izu\mathbf{1}_{\{|u| \le 1\}})\nu_1(du) \\ &+ \int_{-\infty}^0 (e^{izu} - 1 - izu\mathbf{1}_{\{|u| \le 1\}})\nu_1(du)\end{aligned} \tag{A.5}$$

Let us consider second integral of (A.5). Then, we have

$$\int_0^\infty (e^{izu} - 1 - izu\mathbf{1}_{\{|u| \le 1\}})\nu_1(du)$$

$$
\begin{aligned}
&= m_1 \left( \int_0^\infty \frac{\cos zu - 1}{u^2} du + i \int_0^\infty (\sin zu - zu\mathbf{1}_{\{|u|\le 1\}}) \frac{du}{u^2} \right) \\
&= m_1 \left( -\frac{\pi}{2} z + i \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \left( \frac{\sin zu}{u^2} - z \frac{u\mathbf{1}_{\{|u|\le 1\}}}{u^2} \right) du \right) \qquad \text{(A.6)}
\end{aligned}
$$

Using the transformation $zu = v$ and changing suitably the limit of integration on (A.6), we have

$$
\begin{aligned}
&\int_0^\infty (e^{izu} - 1 - izu\mathbf{1}_{\{|u|\le 1\}}) \nu_1(du) \\
&= m_1 \left( -\frac{\pi}{2} z + \lim_{\epsilon \to 0^+} \left( -z \int_\epsilon^{\epsilon z} \frac{\sin v}{v^2} dv + z \int_\epsilon^\infty \left( \frac{\sin v}{v^2} - \frac{\mathbf{1}_{\{|v|\le 1\}}}{v} \right) dv \right) \right) \\
&= m_1 \left( -\frac{\pi}{2} z - iz \log z + iz \int_0^\infty \left( \frac{\sin v}{v^2} - \frac{\mathbf{1}_{\{|v|\le 1\}}}{v} \right) dv \right) \qquad \text{(A.7)}
\end{aligned}
$$

The last equality of (A.7) follows, since $\lim_{\epsilon\to 0^+} \int_\epsilon^{\epsilon z} \frac{\sin v}{v^2} dv = \lim_{\epsilon\to 0^+} \int_\epsilon^{\epsilon z} \frac{1}{v} dv = \log z$. If we set $\Gamma = \int_0^\infty \left( \frac{\sin v}{v^2} - \frac{\mathbf{1}_{\{|v|\le 1\}}}{v} \right) dv$, then (A.7) simplifies to

$$
\int_0^\infty (e^{izu} - 1 - izu\mathbf{1}_{\{|u|\le 1\}}) \nu_1(du) = m_1 \left( -\frac{\pi}{2} z - iz \log z + iz\Gamma \right)
$$

Similarly, the last integral of (A.5) leads to

$$
\int_{-\infty}^0 (e^{izu} - 1 - izu\mathbf{1}_{\{|u|\le 1\}}) \nu_1(du) = m_2 \left( -\frac{\pi}{2} z + iz \log(-z) - iz\Gamma \right)
$$

Thus, for any $z > 0$

$$
\begin{aligned}
\eta_1(z) &= iz\beta - (m_1 + m_2) \frac{\pi}{2} z + i(m_2 - m_1) z \log z + iz(m_1 - m_2)\Gamma \\
&= iz(\beta + (m_1 - m_2)\Gamma) - (m_1 + m_2) \frac{\pi}{2} z \left( 1 - i \frac{(m_2 - m_1)}{m_1 + m_2} \frac{2}{\pi} \log z \right)
\end{aligned}
$$

For any $z < 0$,

$$
\begin{aligned}
\eta_1(z) &= \overline{\eta_1(-z)} \\
&= iz(\beta + (m_1 - m_2)\Gamma) - (m_1 + m_2) \frac{\pi}{2} (-z) \left( 1 - i \frac{(m_2 - m_1)}{m_1 + m_2} \frac{z}{|z|} \frac{2}{\pi} \log(-z) \right).
\end{aligned}
$$

Therefore, for any $z \in \mathbb{R}$

$$
\eta_1(z) = iz(\beta + (m_1 - m_2)\Gamma) - (m_1 + m_2) \frac{\pi}{2} |z| \left( 1 - i \frac{(m_2 - m_1)}{m_1 + m_2} \frac{z}{|z|} \frac{2}{\pi} \log |z| \right)
$$

$$= iz\gamma_1 - d_1|z|\left(1 + i\theta\frac{z}{|z|}\frac{2}{\pi}\log|z|\right),$$

where $\gamma_1 = \beta + (m_1 - m_2)\Gamma$, $d_1 = (m_1 + m_2)\frac{\pi}{2}$ and $\theta = \frac{(m_1 - m_2)}{m_1 + m_2}$.

This completes the proof. $\square$

**Proposition A.2.** *Let $x, z \in \mathbb{R}$ and $\alpha \in (0,1)$. Then, for all $t \geq 0$,*

$$\lim_{t\to 0^+}\frac{1}{t}\left(e^{izx(e^{-t}-1)}\phi_t(z) - 1\right) = \left(-x + \beta_1 + \int_{\mathbb{R}} ue^{izu}\nu_\alpha(du)\right)(iz), \tag{A.8}$$

*where $\beta_1 = \beta - \int_{\{|u|\leq 1\}} u\nu_\alpha(du)$, and $\nu_\alpha$ is the Lévy measure given in (2.3).*

*Proof.* Recall from Section 2, if $X$ be a $\alpha$-stable random variable with $\alpha \in (0,1)$ one can write

$$\phi_t(z) = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)} = \exp\left(iz\beta_1(1 - e^{-t}) + \int_{\mathbb{R}}(e^{izu} - e^{iue^{-t}z})\nu_\alpha(du)\right), \quad t \geq 0,$$

where $\beta_1 = \beta - \int_{\{|u|\leq 1\}} u\nu_\alpha(du)$ (see (2.5)).

Now, let us consider LHS of (A.8),

$$\begin{aligned}
&\lim_{t\to 0^+}\frac{1}{t}\left(e^{izx(e^{-t}-1)}\phi_t(z) - 1\right)\\
&= \lim_{t\to 0^+}\frac{1}{t}\left(\exp\left(izx(e^{-t} - 1) + iz\beta_1(1 - e^{-t}) + \int_{\mathbb{R}}(e^{izu} - e^{iue^{-t}z})\nu_\alpha(du)\right) - 1\right)\\
&= \lim_{t\to 0^+}\frac{1}{t}\left(\exp(A + iB) - 1\right),
\end{aligned}\tag{A.9}$$

where

$$\begin{aligned}
A &= \int_{\mathbb{R}}(\cos(zu) - \cos(zue^{-t}))\nu_\alpha(du) \text{ and}\\
B &= \left(zx(e^{-t} - 1) + z\beta_1(1 - e^{-t}) + \int_{\mathbb{R}}(\sin(zu) - \sin(zue^{-t}))\nu_\alpha(du)\right).
\end{aligned}$$

Applying Euler's formula for complex exponential to (A.9), and rearranging the limits, we have

$$\lim_{t\to 0^+}\frac{1}{t}\left(e^{izx(e^{-t}-1)}\phi_t(z) - 1\right) = \lim_{t\to 0^+}\frac{e^A\cos(B) - 1}{t} + i\lim_{t\to 0^+}\frac{e^A\sin(B)}{t}. \tag{A.10}$$

It is easy to show that at $t = 0$, $e^A\cos(B) - 1 = 0$ and $e^A\sin(B) = 0$. Thus, on applying L'Hospital rule on (A.10), taking limit as $t$ tend to $0^+$, and using dominated convergence theorem, we have

$$\lim_{t\to 0^+}\frac{1}{t}\left(e^{izx(e^{-t}-1)}\phi_t(z) - 1\right)$$

$$= \left( \int_{\mathbb{R}} iu \sin(zu)\nu_\alpha(du) - x + \beta_1 + \int_{\mathbb{R}} u\cos(zu)\nu_\alpha(du) \right)(iz)$$
$$= \left( -x + \beta_1 + \int_{\mathbb{R}} u(\cos(zu) + i\sin(zu))\nu_\alpha(du) \right)(iz)$$
$$= \left( -x + \beta_1 + \int_{\mathbb{R}} ue^{izu}\nu_\alpha(du) \right)(iz),$$

the desired conclusion follows. □

**Proposition A.3.** *Let $x, z \in \mathbb{R}$ and $\alpha \in (1,2)$. Then, for all $t \geq 0$,*

$$\lim_{t\to 0^+} \frac{1}{t}\left( e^{izx(e^{-t}-1)}\phi_t(z) - 1 \right) = \left( -x + \beta_2 + \int_{\mathbb{R}} u(e^{izu} - 1)\nu_\alpha(du) \right)(iz),$$

*where $\beta_2 = \beta + \int_{\{|u|>1\}} u\nu_\alpha(du)$, and $\nu_\alpha$ is the Lévy measure given in (2.3).*

*Proof.* Recall from Section 2, if $X$ be a $\alpha$-stable random variable with $\alpha \in (1,2)$, one can write

$$\phi_t(z) = \frac{\phi_\alpha(z)}{\phi_\alpha(e^{-t}z)}$$
$$= \exp\left( iz\beta_2(1-e^{-t}) + \int_{\mathbb{R}} (e^{izu} - e^{iue^{-t}z} - iuz(1-e^{-t}))\nu_\alpha(du) \right), \quad t \geq 0, \tag{A.11}$$

where $\beta_2 = \beta + \int_{\{|u|>1\}} u\nu_\alpha(du)$.

Now, let us consider LHS of (A.11),

$$\lim_{t\to 0^+} \frac{1}{t}\left( e^{izx(e^{-t}-1)}\phi_t(z) - 1 \right)$$
$$= \lim_{t\to 0^+} \frac{1}{t}\left( \exp\left(C + iD\right) - 1 \right), \tag{A.12}$$

where

$$C = \int_{\mathbb{R}} (\cos(zu) - \cos(zue^{-t}))\nu_\alpha(du) \text{ and}$$
$$D = \left( zx(e^{-t}-1) + z\beta_2(1-e^{-t}) + \int_{\mathbb{R}} (\sin(zu) - \sin(zue^{-t}) - zu(1-e^{-t}))\nu_\alpha(du) \right).$$

Applying Euler's formula for complex exponential to (A.12), and rearranging the limits, we have

$$\lim_{t\to 0^+} \frac{1}{t}\left( e^{izx(e^{-t}-1)}\phi_t(z) - 1 \right) = \lim_{t\to 0^+} \frac{e^C\cos(D) - 1}{t} + i\lim_{t\to 0^+} \frac{e^C\sin(D)}{t}. \tag{A.13}$$

It is easy to show that at $t = 0$, $e^C\cos(D) - 1 = 0$ and $e^C\sin(D) = 0$. Thus, on applying L'Hospital rule on (A.13), taking limit as $t$ tend to $0^+$, and using dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{t\to 0^+}\frac{1}{t}\left(e^{izx(e^{-t}-1)}\phi_t(z)-1\right)\\
&=\left(\int_{\mathbb{R}} iu\sin(zu)\nu_\alpha(du)-x+\beta_2+\int_{\mathbb{R}}u(\cos(zu)-1)\nu_\alpha(du)\right)(iz)\\
&=\left(-x+\beta_2+\int_{\mathbb{R}}u(\cos(zu)+i\sin(zu)-1)\nu_\alpha(du)\right)(iz)\\
&=\left(-x+\beta_2+\int_{\mathbb{R}}u(e^{izu}-1)\nu_\alpha(du)\right)(iz),
\end{aligned}$$

the desired conclusion follows. □

**Proposition A.4.** *Let $\alpha\in(0,2)$. Then,*

$$\frac{1}{\alpha}\int_{\mathbb{R}} ug'(x+u)\nu_\alpha(du)=\int_{\mathbb{R}}\left(g(x+u)-g(x)\right)\nu_\alpha(du),\quad g\in\mathcal{S}(\mathbb{R}),$$

*where $\nu_\alpha$ is the Lévy measure given in (2.3).*

*Proof.* We use Fubini's theorem and change in the order of integration in the following proof.

For $\alpha\in(0,2)$, we have

$$\begin{aligned}
\int_{\mathbb{R}} ug'(x+u)\nu_\alpha(du)&=m_1\int_0^\infty\frac{ug'(x+u)}{u^{1+\alpha}}du+m_2\int_{-\infty}^0\frac{ug'(x+u)}{(-u)^{1+\alpha}}du\\
&=m_1\int_0^\infty\frac{g'(x+u)}{u^{\alpha}}du-m_2\int_{-\infty}^0\frac{g'(x+u)}{(-u)^{\alpha}}du\\
&=\alpha m_1\int_0^\infty g'(x+u)\int_u^\infty\frac{1}{z^{1+\alpha}}dzdu\\
&-\alpha m_2\int_{-\infty}^0 g'(x+u)\int_{-\infty}^u\frac{1}{(-z)^{1+\alpha}}dzdu\\
&=\alpha m_1\int_0^\infty\frac{1}{z^{1+\alpha}}\int_0^z g'(x+u)dudz\\
&-\alpha m_2\int_{-\infty}^0\frac{1}{(-z)^{1+\alpha}}\int_z^0 g'(x+u)dudz\\
&=\alpha\int_0^\infty\left(g(x+z)-g(x)\right)\frac{m_1}{z^{1+\alpha}}\\
&+\alpha\int_{-\infty}^0\left(g(x+z)-g(x)\right)\frac{m_2}{(-z)^{1+\alpha}}\\
&=\alpha\int_{\mathbb{R}}\left(g(x+u)-g(x)\right)\nu_\alpha(du),
\end{aligned}$$

the desired conclusion follows. □

**Proposition A.5.** *Let $\alpha \in (0,1)$ and $h \in \mathcal{H}_\delta$ with $\delta \in (0,\alpha)$. Then,*

$$|xg_h^\alpha(x)| \leq \eta_{\alpha,\beta,\delta,m_1,m_2}(x) := |\beta_1|\|h'\| + C_{\alpha,\delta,m_1,m_2} + |x| \wedge |x|^\delta + \mathbb{E}|X|^\delta.$$

*Proof.* For $\alpha \in (0,1)$, we have by (3.4),

$$|xg_h^\alpha(x)| = \left| -\beta_1 g_h^\alpha(x) - \int_{\mathbb{R}} u g_h^\alpha(x+u)\nu_\alpha(du) + (h(x)-h(0)) - (\mathbb{E}h(X) - \mathbb{E}h(0)) \right|$$

Thus, by (3.16) and (3.18), we have

$$|xg_h^\alpha(x)| \leq |\beta_1|\|h'\| + C_{\alpha,\delta,m_1,m_2} + |x| \wedge |x|^\delta + \mathbb{E}|X|^\delta := \eta_{\alpha,\beta,\delta,m_1,m_2}(x),$$

the desired conclusion follows. □

### *A.1. A continuous distribution without finite first moment and differentiable characteristic function*

In this appendix, we present that the characteristic function of symmetric-Pareto distribution is differentiable. For more examples on probability distributions having no finite first moment and yet a differentiable characteristic function, we refer the reader to [30].

Let $X$ be a symmetric-Pareto random variable with density

$$f_X(x) = \begin{cases} \frac{c}{x^2}, & |x| \geq a \\ 0, & |x| < a, \end{cases}$$

where $a > 0$, and $c > 0$ is a normalizing constant. Then its characteristic function is given by

$$\phi(t) = \int_{\mathbb{R}} e^{itx} \frac{c}{x^2} dx = 2c \int_a^\infty \frac{\cos tx}{x^2} dx$$

Observe that, $\phi(t)$ is even. We can write the difference $\frac{1-\phi(t)}{2c}$ for $t > 0$ as follows.

$$\frac{1-\phi(t)}{2c} = \int_a^\infty \frac{1-\cos tx}{x^2} dx = \int_a^{1/t} \frac{1-\cos tx}{x^2} dx + \int_{1/t}^\infty \frac{1-\cos tx}{x^2} dx$$

Observe also that $1-\phi(t)$ is a real-valued and non-negative function. For an arbitrary $u \in \mathbb{R}$, we have $0 \leq 1 - \cos u \leq \min\{2, u^2\}$. This fact implies that $1-\phi(t)$ is not greater than some constant multiplied by the function $h(t)$ where

$$h(t) = t^2 \int_a^{1/t} dx + 2 \int_{1/t}^{\infty} \frac{1}{x^2} dx.$$

However, since $h(t) = o(t)$ as $t \to 0$, we find that

$$\phi(t) = 1 + o(t) \text{ as } t \to 0.$$

Therefore the characteristic function $\phi(t)$ is differentiable at $t = 0$.

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