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Asymptotic properties for the parameter estimation in Ornstein-Uhlenbeck process with discrete observations

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Abstract: In this paper, under discrete observations, we study Cramér-type moderate deviations (extended central limit theorem) for parameter estimation in Ornstein-Uhlenbeck process. Our results contain both stationary and explosive cases. For applications, we propose test statistics which can be used to construct rejection regions in the hypothesis testing for the drift coefficient, and the corresponding probability of type II error tends to zero exponentially. Simulation study shows that our test statistics have good finite-sample performances both in size and power. The main methods include the deviation inequalities for multiple Wiener-Itô integrals, as well as the asymptotic analysis techniques.

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1. Introduction and main results

1.1. Introduction

The Ornstein-Uhlenbeck (O-U) process is defined as

$$dX_t = \theta X_t dt + dW_t, \quad X_0 = 0, \quad t \geq 0 \tag{1.1}$$

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where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion and θ is an unknown parameter. It is a reduced Vasicek model ([8], [9], [45]) which is often used to describe term structure. This model can successfully characterize mean reversion property of short interest rate. The parameter θ indicates the reverting rate. In model (1.1), the average interest rate is 0. When $\theta < 0$, short rate X_t will always be pulled back to its average level if it is away from 0. However, the short rate X_t will deviate from average level much more drastically when $\theta > 0$. Therefore, $\theta < 0$ is called stationary case and $\theta > 0$ is called explosive case. The trivial case $\theta = 0$ indicates that the short rate X_t is completely random (in this case, the O-U process is Brownian motion).

It is of great importance to estimate parameter θ in financial market. Also interest rate is usually regularly reported, so we can consider the parameter estimation under discrete observations. Let $\{t_i \mid t_i = i\Delta, 0 \leq i \leq n, \Delta > 0\}$ be sampling times and $\{X_i \mid X_i = X_{t_i}, 0 \leq i \leq n\}$ be observed samples. Kasonga ([28]) proposed the following estimator

$$\hat{\theta}_{n,\Delta} = \begin{cases} \frac{1}{\Delta} \log\left(\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2}\right), & \text{if } \frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} > 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

He also studied the weak consistency property for $\hat{\theta}_{n,\Delta}$ in ([28]). In this article, we assume

$$(H): \Delta \to 0, \quad n\Delta \to \infty, \quad n \to \infty.$$

Thus, we are dealing with high frequency observations. Shimizu ([43]) obtained the asymptotic distribution of Kasonga's estimator $\hat{\theta}_{n,\Delta}$ under this high frequency observations. To be explicit, for explosive case, as $\Delta \to 0, n \to \infty$ (Theorem 2 in page 198 of [43]),

$$\frac{e^{\theta n\Delta}}{2\theta}\left(\hat{\theta}_{n,\Delta} - \theta\right) \xrightarrow{\mathcal{L}} \frac{\nu}{\eta}, \tag{1.3}$$

where ν and η are two independent standard normal variables; while for stationary case, as $\Delta \to 0, n \to \infty$ (Corollary 1 in page 199 of [43]),

$$\sqrt{\frac{n\Delta}{-2\theta}}(\hat{\theta}_{n,\Delta} - \theta) \xrightarrow{\mathcal{L}} \nu. \tag{1.4}$$

Moreover, there are also some research works on maximum likelihood estimator of θ in continuous observation case. For stationary case, one can refer to ([4], [6], [16]) for the large deviations, and ([17], [18], [20], [26]) for Cramér-type moderate deviations, and ([1], [11], [14], [15]) for the Berry-Esseen bounds. In explosive case, Bercu et al. ([5]), Bercu and Richou ([7]) obtained the large deviations for the maximum likelihood estimator of θ, while Jiang and Zhang ([27]) considered the Cramér-type moderate deviations. More study on the statistical inferences for Vasicek type processes can be found in Azmoodeh and Morlanes ([2]), Barboza and Viens ([3]), Dietz and Kutoyants ([10]), El Onsy et al. ([12]), Hu and Nualart ([22]), Hu et al. ([23], [24]), Nourdin and Tran ([37]), Tudor and Viens ([44]), Xiao et al. ([46], [47]), Zang and Zhang ([48], [49]) and the references therein.

In the discrete observation case, Shimizu ([42]) proved the asymptotic normality for the maximum likelihood estimator of θ in the stationary case. For more references, one can see Gobet [21], Kessler [30], Kutoyants [33], Prakasa Rao ([40]). For the explosive diffusion process, there are only literatures about consistency and asymptotic distribution of estimators, such as ([13], [25], [29], [31], [32] and the references therein). However, Cramér-type moderate deviations are still missing in both stationary case and explosive case.

In this paper, our motivation is to study the Cramér-type moderate deviations (so-called extended central limit theorem) for Kasonga's estimator $\hat{\theta}_{n,\Delta}$, and we handle both stationary case and explosive case. By Cramér-type moderate deviation, we mean there exists sequence α_n and β_n such that for every $\rho > 0$

$$\frac{1}{1-F(x)} P\left(\alpha_n\left(\hat{\theta}_{n,\Delta} - \theta\right) \geq x\right) \to 1 \qquad \forall 0 \leq x \leq \rho\beta_n,$$

where $F(x)$ is the limiting fluctuation distribution of $\hat{\theta}_{n,\Delta}$. Therefore, Cramér-type moderate deviation provides a nice comparison between tail probability of estimator and limiting fluctuation distribution. Especially one always need to take care of these issues in hypothesis testing. We benefit a lot from Gao et al. ([19]).

1.2. Main results

In the following, we will state our main results in explosive case and stationary cases as well.

1.2.1. Explosive case: $\theta > 0$

We denote $\mathbb{C} = \frac{\nu}{\eta}$, where ν and η are two independent standard normal variables. Then $\mathbb{C}$ follows standard Cauchy distribution. Let $F_{\mathbb{C}}(x)$ be its cumulative distribution function. Let $\beta_{n,\Delta}$ be positive numbers and satisfy

$$\beta_{n,\Delta} \to \infty, \quad \frac{n\Delta\beta_{n,\Delta}^8(\log\beta_{n,\Delta})^4}{e^{2\theta n\Delta}} \to 0. \tag{1.5}$$

Firstly, we show that the event $\left\{\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \leq 0\right\}$ is negligible in the sense of Cramér-type moderate deviation.

Theorem 1.1. *Under assumption* (H) *and (1.5), we have*

$$\lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} P\left(\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \leq 0\right) = 0.$$

Now, we state the Cramér-type moderate deviation for $\hat{\theta}_{n,\Delta}$.

Theorem 1.2. *Under assumption* (H) *and* (1.5), *for each* $\rho > 0$, *and for all* $0 \le x \le \rho\beta_{n,\Delta}$,

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{\Delta e^{\theta(n-1)\Delta}}{1 - e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta} - \theta\right) \ge x\right) = 1 \tag{1.6}$$

and

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{F_{\mathbb{C}}(-x)} P\left(\frac{\Delta e^{\theta(n-1)\Delta}}{1 - e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta} - \theta\right) \le -x\right) = 1. \tag{1.7}$$

Under assumption (H), one can easily show that

$$\lim_{\Delta\to 0, n\to\infty} \frac{\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}}{\frac{e^{\theta n\Delta}}{2\theta}} = 1.$$

Then we have the following corollary.

Corollary 1.1. *Under assumption* (H), *there exists* $\beta_{n,\Delta}$, *satisfying* (1.5) *and* $\lim_{\Delta\to 0, n\to\infty} \Delta\beta_{n,\Delta} = 0$, *then, for each* $\rho > 0$, *uniformly for* $0 \le x \le \rho\beta_{n,\Delta}$,

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{e^{\theta n\Delta}}{2\theta}\left(\hat{\theta}_{n,\Delta} - \theta\right) \ge x\right) = 1 \tag{1.8}$$

and

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{F_{\mathbb{C}}(-x)} P\left(\frac{e^{\theta n\Delta}}{2\theta}\left(\hat{\theta}_{n,\Delta} - \theta\right) \le -x\right) = 1. \tag{1.9}$$

However, the above results can not be directly applied in statistical inference, because the scaling factor of fluctuation limit depends on the unknown parameter θ. To circumvent this difficulty, one usually replace the unknown parameter by its estimation $\hat{\theta}_{n,\Delta}$. By consistency property of estimator $\hat{\theta}_{n,\Delta}$, one can also show a self-normalized version of fluctuation limiting theorem

$$\frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1 - e^{-2\hat{\theta}_{n,\Delta}\Delta}}\left(\hat{\theta}_{n,\Delta} - \theta\right) \xrightarrow{\mathcal{L}} \frac{\nu}{\eta}. \tag{1.10}$$

Then, the associated Cramér-type moderate deviations can be stated as follows.

Theorem 1.3. *Under assumption* (H), *there exists* $\beta_{n,\Delta}$ *satisfying* (1.5) *such that, for each* $\rho > 0$, *uniformly for* $0 \le x \le \rho\beta_{n,\Delta}$,

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1 - e^{-2\hat{\theta}_{n,\Delta}\Delta}}\left(\hat{\theta}_{n,\Delta} - \theta\right) \ge x\right) = 1 \tag{1.11}$$

and

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{F_{\mathbb{C}}(-x)} P\left(\frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1 - e^{-2\hat{\theta}_{n,\Delta}\Delta}}\left(\hat{\theta}_{n,\Delta} - \theta\right) \le -x\right) = 1. \tag{1.12}$$

Corollary 1.2. *Under assumption (H), let $b_{n,\Delta}$ be positive numbers, satisfying*

$$b_{n,\Delta} \to \infty, \quad \frac{b_{n,\Delta}}{n\Delta} \to 0. \tag{1.13}$$

Then the families

$$\left\{ \left| \frac{e^{\theta n\Delta}}{2\theta} \left(\hat{\theta}_{n,\Delta} - \theta \right) \right|^{\frac{1}{b_{n,\Delta}}}, n \geq 1, \Delta > 0 \right\}$$

and

$$\left\{ \left| \frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1 - e^{-2\hat{\theta}_{n,\Delta}\Delta}} \left(\hat{\theta}_{n,\Delta} - \theta \right) \right|^{\frac{1}{b_{n,\Delta}}}, n \geq 1, \Delta > 0 \right\}$$

satisfy the same large deviations with the speed $b_{n,\Delta}$ and rate function

$$I(x) = \begin{cases} \log x, & \text{if } x \geq 1; \\ 0, & \text{if } 0 < x < 1. \end{cases}$$

To be explicit, for any $x \geq 0$, we have

$$\lim_{\Delta \to 0, n \to \infty} \frac{1}{b_{n,\Delta}} \log P \left(\left| \frac{e^{\theta n\Delta}}{2\theta} \left(\hat{\theta}_{n,\Delta} - \theta \right) \right|^{\frac{1}{b_{n,\Delta}}} \geq x \right) = -I(x) \tag{1.14}$$

and

$$\lim_{\Delta \to 0, n \to \infty} \frac{1}{b_{n,\Delta}} \log P \left(\left| \frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1 - e^{-2\hat{\theta}_{n,\Delta}\Delta}} \left(\hat{\theta}_{n,\Delta} - \theta \right) \right|^{\frac{1}{b_{n,\Delta}}} \geq x \right) = -I(x). \tag{1.15}$$

1.2.2. Ergodic case: $\theta < 0$

In this subsection, we will strengthen the assumption (H) by

$$(H') : \Delta \to 0, \quad n\Delta \to \infty, \quad n\Delta^4 (\log n\Delta)^6 \to 0, \quad n \to \infty.$$

Under assumption (H'), let $\lambda_{n,\Delta}$ be positive numbers, satisfying

$$\lambda_{n,\Delta} \to \infty, \quad \frac{\lambda_{n,\Delta}}{(n\Delta)^{1/12}} \to 0. \tag{1.16}$$

Similar to the explosive case, we first show the event $\left\{ \frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \leq 0 \right\}$ is negligible in the sense of Cramér-type moderate deviation.

Theorem 1.4. *Under assumption (H') and (1.16), we have*

$$\lim_{\Delta \to 0, n \to \infty} \frac{1}{\lambda_{n,\Delta}^2} \log P \left(\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \leq 0 \right) = -\infty.$$

Theorem 1.5. *Then, for each $\rho > 0$, uniformly for $0 \le x \le \rho\lambda_{n,\Delta}$,*

$$\frac{1}{1-\Phi(x)} P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) \ge x\right) \to 1 \tag{1.17}$$

and

$$\frac{1}{\Phi(-x)} P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) \le -x\right) \to 1, \tag{1.18}$$

where $\Phi(x)$ is the standard normal distribution function.

Similarly, to apply the above result in statistical inference, we need to replace the unknown parameter θ by its estimation. Therefore, we consider a self-normalized version of $\hat{\theta}_{n,\Delta}$. To be explicit, from (1.4) and the fact that $\hat{\theta}_{n,\Delta}$ is a consistent estimator of θ, we can derive the following result immediately,

$$\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}(\hat{\theta}_{n,\Delta}-\theta) \xrightarrow{\mathcal{L}} \nu. \tag{1.19}$$

Then, the associated Cramér-type moderate deviation can be stated as follows.

Theorem 1.6. *Under assumption (H'), let $\lambda_{n,\Delta}$ satisfy (1.16). Then, for each $\rho > 0$, uniformly for $0 \le x \le \rho\lambda_{n,\Delta}$,*

$$\frac{1}{1-\Phi(x)} P\left(\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}(\hat{\theta}_{n,\Delta}-\theta) \ge x\right) \to 1 \tag{1.20}$$

and

$$\frac{1}{\Phi(-x)} P\left(\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}(\hat{\theta}_{n,\Delta}-\theta) \le -x\right) \to 1. \tag{1.21}$$

Corollary 1.3. *Under assumption (H'), let $\lambda_{n,\Delta}$ satisfy (1.16), the following families*

$$\left\{\frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right),\ n \ge 1, \Delta > 0\right\}$$

and

$$\left\{\frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta\right),\ n \ge 1, \Delta > 0\right\}$$

satisfy the same large deviation with speed $\lambda_{n,\Delta}^2$ and the rate function $\Lambda(x) = x^2/2$, $x \in \mathbb{R}$.

The remaining part of this paper is organized as follows. In Section 2, we apply our results to confidence interval and hypothesis testing problem. Simulation study shows that our test statistics have good finite-sample performances both in size and power. Then, the proofs of our main results will be presented in Section 3. Two key technical propositions will be proved in Section 4 by using deviation inequalities for multiple Wiener-Itô integrals and the asymptotic analysis techniques. Finally, we introduce the Cramér-type moderate deviations for multiple Wiener-Itô integrals in Appendix section. Throughout this paper, C, C_1, C_2 depending only on θ, denote positive constants whose values can differ at different places.

2. Statistical applications

2.1. Explosive case: $\theta > 0$

(I) Confidence interval. An application of (1.10) yields the following approximate $(1-\alpha)$ $(0<\alpha<1)$ confidence interval for θ:

$$\left(\hat{\theta}_{n,\Delta} - \frac{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}} t_{\alpha/2}(1), \hat{\theta}_{n,\Delta} + \frac{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}} t_{\alpha/2}(1)\right)$$

where $t_{\alpha/2}(1)$ is the upper $\alpha/2$-quantile of t-distribution with 1 degree of freedom. To assess the performance of above confidence interval, we need to use the Berry-Esseen bound for the estimator $\hat{\theta}_{n,\Delta}$. To the best of our knowledge, the Berry-Esseen bound for the explosive case remains open, which will be studied in the future work.

(II) Hypothesis testing. For the hypothesis testing problem

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta = \theta_1, \tag{2.1}$$

where $\theta_0 > 0, \theta_1 > 0$. We can propose the test statistic

$$\tilde{\Delta}_n = \frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}} \left(\hat{\theta}_{n,\Delta} - \theta_0\right). \tag{2.2}$$

Given $0<\alpha<1$, by Theorem 1.3, under H_0, for each $\rho > 0$,

$$\lim_{\Delta\to 0, n\to\infty} \sup_{x\in[0,\rho\beta_{n,\Delta}]} \left| \frac{P\left(|\tilde{\Delta}_n| \geq x\right)}{2-2F_{\mathbb{C}}(x)} - 1 \right| = 0.$$

Then, take the rejection region as $\left\{|\tilde{\Delta}_n| \geq t_{\alpha/2}(1)\right\}$ and the corresponding probability $\tilde{\gamma}_{n,\Delta}$ of type II error can be written formulated as

$$\tilde{\gamma}_{n,\Delta} = P\left(|\tilde{\Delta}_n| < t_{\alpha/2}(1)|H_1\right) := P_{H_1}\left(|\tilde{\Delta}_n| < t_{\alpha/2}(1)\right).$$

Applying Corollary 1.2, we can obtain the exponential decay rate for $\tilde{\gamma}_{n,\Delta}$.

Theorem 2.1. *Let $b_{n,\Delta}$ be defined by (1.13). The type II error $\tilde{\gamma}_{n,\Delta}$ tends to 0 with exponential decay rate $e^{-rb_{n,\Delta}}$ for all $r>0$, i.e. $\lim_{\Delta\to 0,n\to\infty} \frac{\log\tilde{\gamma}_{n,\Delta}}{b_{n,\Delta}} = -\infty$.*

Proof. We have

$$
\begin{aligned}
&\tilde{\gamma}_{n,\Delta} \\
&\leq P_{H_1}\left(\left|\frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta_1\right)\right|^{\frac{1}{b_{n,\Delta}}}\right. \\
&\left.\geq\left(\frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{\left|1-e^{-2\hat{\theta}_{n,\Delta}\Delta}\right|}|\theta_1-\theta_0|-t_{\alpha/2}(1)\right)^{\frac{1}{b_{n,\Delta}}}\right) \\
&\leq P\left(\left|\hat{\theta}_{n,\Delta}-\theta_1\right|\geq\theta_1/2\middle|H_1\right) \\
&+P_{H_1}\left(\left|\frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta_1\right)\right|^{\frac{1}{b_{n,\Delta}}}\right. \\
&\left.\geq\left(\frac{\Delta e^{\theta_1(n-1)\Delta/2}}{1-e^{-\theta_1\Delta}}|\theta_1-\theta_0|-t_{\alpha/2}(1)\right)^{\frac{1}{b_{n,\Delta}}}\right).
\end{aligned}
$$

Note that $\left(\frac{\Delta e^{\theta_1(n-1)\Delta/2}}{1-e^{-\theta_1\Delta}}|\theta_1-\theta_0|-t_{\alpha/2}(1)\right)^{\frac{1}{b_{n,\Delta}}} = O\left(e^{\frac{\theta_1(n-1)\Delta}{2b_{n,\Delta}}}\right)$. Applying Theorem 1.2, we complete the proof of this theorem. □

2.2. Ergodic case: $\theta < 0$

(I) Confidence interval. An application of (1.19) gives the following approximate $(1-\alpha)$ $(0<\alpha<1)$ confidence interval for θ:

$$
\left(\hat{\theta}_{n,\Delta}-\sqrt{\frac{-2\hat{\theta}_{n,\Delta}}{n\Delta}}z_{\alpha/2},\hat{\theta}_{n,\Delta}+\sqrt{\frac{-2\hat{\theta}_{n,\Delta}}{n\Delta}}z_{\alpha/2}\right)
$$

where $z_{\alpha/2}$ is the upper $\alpha/2$-quantile of standard normal distribution. Here, the quality of approximation for above confidence interval can be an assessed by the Berry-Esseen bound of estimator $\hat{\theta}_{n,\Delta}$ ([1], [11], [14], [15]).

(II) Hypothesis testing. For the hypothesis testing problem

$$
H_0:\theta=\theta_0 \qquad \text{versus} \qquad H_1:\theta=\theta_1, \tag{2.3}
$$

for any $\theta_0<0,\theta_1<0$. We introduce the test statistic

$$
\hat{\Delta}_n=\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta_0\right). \tag{2.4}
$$

Given $0<\alpha<1$, by Theorem 1.6, under H_0, for each $\rho>0$,

$$
\lim_{\Delta\to0,n\to\infty}\sup_{x\in[0,\rho\lambda_{n,\Delta}]}\left|\frac{P\left(|\hat{\Delta}_n|\geq x\right)}{2-2\Phi(x)}-1\right|=0.
$$

Then, take the rejection region as $\left\{|\hat{\Delta}_n| \geq z_{\alpha/2}\right\}$, and the corresponding probability $\hat{\gamma}_n$ of type II error can be written formulated as

$$\hat{\gamma}_n = P_{H_1}\left(|\hat{\Delta}_n| < z_{\alpha/2}\right).$$

Now, we can analyze its convergence rate explicitly.

Theorem 2.2. *Let $\lambda_{n,\Delta}$ be defined by (1.16). The type II error $\hat{\gamma}_n$ tends to 0 with exponential decay rate $e^{-r\lambda_{n,\Delta}^2}$ for all $r > 0$, i.e.* $\lim_{\Delta\to 0, n\to\infty} \frac{\log \hat{\gamma}_n}{\lambda_{n,\Delta}^2} = -\infty$.

Proof. It holds that

$$\begin{aligned}\hat{\gamma}_n &\leq P_{H_1}\left(\left|\frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta_1\right)\right| \geq \frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}|\theta_1-\theta_0| - \frac{z_{\alpha/2}}{\lambda_{n,\Delta}}\right)\\ &\leq P\left(\left|\hat{\theta}_{n,\Delta}-\theta_1\right| \geq \theta_1/2\middle|H_1\right)\\ &\quad + P_{H_1}\left(\left|\frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta_1\right)\right| \geq \frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-3\theta}}|\theta_1-\theta_0| - \frac{z_{\alpha/2}}{\lambda_{n,\Delta}}\right).\end{aligned}$$

By Corollary 1.3, we have $\lim_{\Delta\to 0, n\to\infty} \log \hat{\gamma}_n/\lambda_{n,\Delta}^2 = -\infty$. □

2.3. Simulation study

The sample path of (1.1) can be obtained easily by using the transition probability from X_{t_k} to $X_{t_{k+1}}$, that is $X_{t_{k+1}} = e^{\theta\Delta}X_{t_k} + \epsilon_{k+1}^{\Delta}(\theta)$, where $t_i = i\Delta$, $\epsilon_i^{\Delta}(s) := e^{t_i\theta}\int_{t_{i-1}}^{t_i} e^{-\theta s}dW_s \sim N\left(0, \frac{1}{2\theta}(e^{2\Delta\theta}-1)\right)$, $(i = 0, 1\cdots, n-1)$. Here, we always simulate $N = 1000$ times.

For the hypothesis testing (2.1) and (2.3), we conduct numerical simulations to evaluate the finite-sample performances of the following proposed tests

$$\text{Proposed I: reject } H_0: \theta = \theta_0, \text{ if } \left|\hat{\theta}_{n,\Delta}-\theta_0\right| \geq \frac{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}} t_{\alpha/2}(1), \theta_0, \theta_1 > 0$$

$$\text{Proposed II: reject } H_0: \theta = \theta_0, \text{ if } \left|\hat{\theta}_{n,\Delta}-\theta_0\right| \geq \sqrt{\frac{-2\hat{\theta}_{n,\Delta}}{n\Delta}} z_{\alpha/2}, \theta_0, \theta_1 < 0$$

The size and power for the test statistics are summarized as follows. Tables 1–3 are for the explosive case, while the remainders (Tables 4–6) are for the stationary case. We can learn that

(i) In the explosive case, from Tables 1–3, as $n\Delta$ or $|\theta_0 - \theta_1|$ becomes larger (but $n\Delta$ not larger than 5), the size of $\tilde{\Delta}_n$ is close to the significance level 0.050 and the power increases dramatically to 1. The reason is that the convergence speed of $\tilde{\Delta}_n$ is $e^{\theta_0(n-1)\Delta}$ under H_0 or $e^{\theta_1(n-1)\Delta}$ under H_1 (Theorem 1.2), which is quite fast. Moreover, by Theorem 2.1, $\tilde{\gamma}_{n,\Delta}$ tends to 0

TABLE 1

*(**Explosive case**) The Empirical C.I. means the empirical $1-\alpha$ confidence interval with confidence level $\alpha = 0.05$. Here, take $\Delta = 0.01$, significance level $\alpha = 0.05$ with various sample number n.*

$H_0: \theta_0 = 2$ versus $H_1: \theta_1 = 3$				
Samples	Estimated $\hat{\theta}_T$	Empirical C.I.	Power	Size
$n = 200$	1.739	$[0.397, 3.080]$	0.978	0.007
$n = 300$	1.968	$[1.834, 2.102]$	1.000	0.036
$n = 400$	1.992	$[1.975, 2.009]$	1.000	0.050
$n = 500$	1.999	$[1.997, 2.001]$	1.000	0.051

TABLE 2

*(**Explosive case**) Take $n = 100$, confidence and significance level $\alpha = 0.05$, with varies iteration step Δ.*

$H_0: \theta_0 = 2$ versus $H_1: \theta_1 = 3$				
Samples	Estimated $\hat{\theta}_T$	Empirical C.I.	Power	Size
$\Delta = 0.02$	1.710	$[0.338, 3.083]$	0.983	0.007
$\Delta = 0.03$	1.939	$[1.800, 2.077]$	1.000	0.044
$\Delta = 0.05$	1.999	$[1.998, 2.002]$	1.000	0.053

TABLE 3

*(**Explosive case**) Take $\Delta = 0.01$, $n = 350$, significance level $\alpha = 0.05$ with varies alternatives H_1.*

$H_0: \theta_0 = 2$ versus $H_1: \theta_1 = \theta$		
Samples	Power	Size
$\theta = 2.1$	0.985	0.055
$\theta = 2.5$	1.000	0.055
$\theta = 3.0$	1.000	0.055

TABLE 4

*(**Stationary case**) Take $\Delta = 0.01$, confidence and significance level $\alpha = 0.05$ with various sample number n.*

$H_0: \theta_0 = -2$ versus $H_1: \theta_1 = -3$				
Samples	Estimated $\hat{\theta}_T$	Empirical C.I.	Power	Size
$n = 5000$	-2.043	$[-2.603, -1.482]$	0.881	0.064
$n = 6000$	-2.040	$[-2.552, -1.529]$	0.939	0.061
$n = 7000$	-2.026	$[-2.497, -1.554]$	0.965	0.050

TABLE 5

*(**Stationary case**) Take $n = 2500$, confidence and significance level $\alpha = 0.05$, with varies iteration step Δ.*

$H_0: \theta_0 = -2$ versus $H_1: \theta_1 = -3$				
Samples	Estimated $\hat{\theta}_T$	Empirical C.I.	Power	Size
$\Delta = 0.01$	-2.081	$[-2.881, -1.282]$	0.578	0.049
$\Delta = 0.03$	-2.027	$[-2.482, -1.571]$	0.971	0.052
$\Delta = 0.05$	-2.016	$[-2.369, -1.664]$	0.996	0.055

TABLE 6

*(**Stationary case**) Take $\Delta = 0.01$, $n = 6500$, significance level $\alpha = 0.05$ with varies alternatives H_1.*

$H_0: \theta_0 = -2$ versus $H_1: \theta_1 = \theta$		
Samples	Power	Size
$\theta = -2.1$	0.071	0.058
$\theta = -2.5$	0.445	0.058
$\theta = -3.0$	0.948	0.058
$\theta = -4.0$	1.000	0.058

with exponential decay rate $e^{-rb_{n,\Delta}}$ for all $r > 0$, where $b_{n,\Delta}$ satisfies that

$$b_{n,\Delta} \to \infty, \quad \frac{b_{n,\Delta}}{n\Delta} \to 0.$$

(ii) In the stationary case, from Tables 4–6, as $n\Delta$ or $|\theta_0 - \theta_1|$ becomes larger, the size of $\hat{\Delta}_n$ is close to the significance level 0.050, and the power increases gradually to 1 which is slower than the explosive case. The above statement asserts our theory (Theorem 1.5, Theorem 2.2).

3. Proofs of the main results

For the Ornstein-Uhlenbeck process (1.1), the following formulas hold immediately

$$X_i = e^{\theta t_i} \int_0^{t_i} e^{-\theta t} dW_t \tag{3.1}$$

and

$$X_i = e^{\theta \Delta} X_{i-1} + \epsilon_i^{\Delta}(\theta), \tag{3.2}$$

where $\epsilon_i^{\Delta}(\theta) := e^{t_i\theta} \int_{t_{i-1}}^{t_i} e^{-\theta s} dW_s \sim N\left(0, \frac{1}{2\theta}(e^{2\Delta\theta} - 1)\right)$, $i = 1, 2, \cdots, n$.

3.1. Explosive case: $\theta > 0$

In this subsection, we will prove Theorem 1.1, Theorem 1.2, Theorem 1.3, Corollary 1.1 and Corollary 1.2. To begin with, it is crucial to note that on the event $\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} > 0$

$$\frac{\Delta e^{\theta(n-1)\Delta}}{1 - e^{-2\theta\Delta}} \left(\hat{\theta}_{n,\Delta} - \theta\right) = \frac{e^{\theta(n-1)\Delta}}{1 - e^{-2\theta\Delta}} \log\left(1 + \Delta e^{-\theta n\Delta} \frac{e^{-\theta(n+1)\Delta} U_n}{\Delta e^{-2\theta n\Delta} V_n}\right), \tag{3.3}$$

where

$$U_n = \sum_{i=1}^n \epsilon_i^{\Delta}(\theta) X_{i-1}, \quad V_n = \sum_{i=1}^n X_{i-1}^2. \tag{3.4}$$

Propositions 3.1 plays an important role in our following analysis, and its proof will be given in Section 4.

Proposition 3.1. *Under assumption (H), let $\beta_{n,\Delta}$ satisfy (1.5). Then, for each $\rho > 0$, uniformly for $0 \le x \le \rho\beta_{n,\Delta}$,*

$$\frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{e^{-\theta(n+2)\Delta} U_n}{(1 - e^{-2\theta\Delta}) e^{-2\theta n\Delta} V_n} \ge x\right) \to 1 \tag{3.5}$$

and

$$\frac{1}{F_{\mathbb{C}}(-x)} P\left(\frac{e^{-\theta(n+2)\Delta} U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta} V_n} \le -x\right) \to 1. \tag{3.6}$$

3.1.1. Proof of Theorem 1.1

By using (1.2) and (3.3), we have

$$\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} = e^{\theta\Delta}\left(1 + \Delta e^{-\theta n\Delta} \frac{e^{-\theta(n+1)\Delta} U_n}{\Delta e^{-2\theta n\Delta} V_n}\right).$$

Then, it follows that

$$\begin{aligned} P\left(\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \le 0\right) &= P\left(e^{\theta\Delta}\left(1 + \Delta e^{-\theta n\Delta} \frac{e^{-\theta(n+1)\Delta} U_n}{\Delta e^{-2\theta n\Delta} V_n}\right) \le 0\right) \\ &\le P\left(\left|\Delta e^{-\theta n\Delta} \frac{e^{-\theta(n+1)\Delta} U_n}{\Delta e^{-2\theta n\Delta} V_n}\right| \ge 1\right) \\ &\le P\left(\left|\frac{e^{-\theta(n+2)\Delta} U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta} V_n}\right| \ge \frac{e^{\theta(n-1)\Delta}}{(1-e^{-2\theta\Delta})}\right). \end{aligned}$$

Together with (1.5) and (3.8), we can get

$$\lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} P\left(\frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \le 0\right),$$

which completes the proof of this theorem.

3.1.2. Equivalence in Cramér-type moderate deviations

As an application of Theorem 1.1 and Proposition 3.1, we can get the following key result.

Lemma 3.1. *For $\beta_{n,\Delta}$ defined by (1.5), we have*

$$\begin{aligned} \lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} P\Bigg(&\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta} - \theta\right) - \frac{e^{-\theta(n+2)\Delta} U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta} V_n}\right| \\ &\ge \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\Bigg) = 0. \end{aligned} \tag{3.7}$$

Proof. Using (3.3) and the fact $|\log(1+x) - x| \le x^2$, $x > -\frac{1}{2}$, we can obtain for any $\epsilon > 0$

$$P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta} - \theta\right) - \frac{e^{-\theta(n+2)\Delta} U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta} V_n}\right| \ge \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)$$

$$\leq P\left(\frac{\sum_{i=1}^n X_iX_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \leq 0\right) + P\left(\Delta e^{-\theta(n-1)\Delta}\frac{e^{-\theta(n+1)\Delta}U_n}{\Delta e^{-2\theta n\Delta}V_n} \leq -\frac{1}{2}\right)$$
$$+ P\left(\frac{e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\Delta e^{-\theta n\Delta}\frac{e^{-\theta(n+1)\Delta}U_n}{\Delta e^{-2\theta n\Delta}V_n}\right)^2 \geq \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)$$
$$\leq P\left(\frac{\sum_{i=1}^n X_iX_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \leq 0\right) + P\left(\left|\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\right| \geq \frac{e^{\theta n\Delta}}{2(1-e^{-2\theta\Delta})}\right)$$
$$+ P\left(\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\right)^2 \geq \frac{e^{\theta(n-2)\Delta}}{(1-e^{-2\theta\Delta})\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)$$
$$\leq P\left(\frac{\sum_{i=1}^n X_iX_{i-1}}{\sum_{i=1}^n X_{i-1}^2} \leq 0\right) + 2P\left(\left|\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})\Delta e^{-2\theta n\Delta}V_n}\right| \geq \frac{\beta_{n,\Delta}}{\epsilon}\right).$$

Applying Theorem 1.1, Proposition 3.1 and the fact that

$$\lim_{x\to+\infty} x\left(1-F_{\mathbb{C}}(x)\right) = \frac{1}{\pi}, \tag{3.8}$$

we can complete the proof of (3.7). □

3.1.3. Proof of Theorem 1.2

By (3.8), for any $\epsilon > 0$, there exists some positive constant M such that

$$\sup_{x\geq M}\left|\pi x\left(1-F_{\mathbb{C}}(x)\right)-1\right| \leq \epsilon. \tag{3.9}$$

Now, for each $\rho > 0$ and $2M \leq x \leq \rho\beta_{n,\Delta}$, we have

$$\begin{aligned}
&P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n} \geq x + \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)\\
&\quad - P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) - \frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\right| \geq \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)\\
&\leq P\left(\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) \geq x\right)\\
&\leq P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n} \geq x - \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)\\
&\quad + P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) - \frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\right| \geq \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right).
\end{aligned} \tag{3.10}$$

Firstly, by using Lemma 3.1 and (4.24), we have

$$\lim_{\Delta\to 0, n\to\infty} \sup_{M\le x\le \rho\beta_{n,\Delta}} \frac{P(|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}(\hat{\theta}_{n,\Delta}-\theta)-\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}|\ge \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}})}{1-F_{\mathbb{C}}(x)}$$
$$= 0. \tag{3.11}$$

Secondly, from then mean value theorem and (4.24), it follows that,

$$\sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{1-F_{\mathbb{C}}\left(x\pm \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)} - 1\right| \le \frac{2\pi\rho}{(1+2M^2)(1-\epsilon)\log\beta_{n,\Delta}}. \tag{3.12}$$

Then, we get

$$\begin{aligned}
&\sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\ge x\pm \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)} - 1\right| \\
&\le \sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{1-F_{\mathbb{C}}\left(x\pm \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)} - 1\right| \\
&\quad + \sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\ge x\pm \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}\left(x\pm \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)} - 1\right| \\
&\qquad \cdot \left| \frac{1-F_{\mathbb{C}}\left(x\pm \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)}\right|.
\end{aligned}$$

Together with Proposition 3.1 and (3.12), we can obtain that

$$\lim_{\Delta\to 0, n\to\infty} \sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\ge x\pm \frac{2}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)} - 1\right| = 0. \tag{3.13}$$

Therefore, by (3.10)–(3.13), it holds that

$$\lim_{\Delta\to 0, n\to\infty} \sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{1}{1-F_{\mathbb{C}}(x)} P\left(\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\ge x\right) - 1\right| = 0. \tag{3.14}$$

Moreover, since the function $1-F_{\mathbb{C}}(x)$ is uniformly continuous, the standard discretized approximation argument shows that

$$\lim_{\Delta\to 0, n\to\infty} \sup_{x\in\mathbb{R}} \left| P\left(\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\ge x\right) - (1-F_{\mathbb{C}}(x))\right| = 0.$$

Therefore, it follows that

$$\lim_{\Delta\to 0, n\to\infty} \sup_{0\le x\le 2M} \left| \frac{1}{1-F_{\mathbb{C}}(x)} P\left(\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}} \left(\hat{\theta}_{n,\Delta}-\theta\right) \ge x \right) - 1 \right| = 0. \tag{3.15}$$

Finally, (1.6) can be obtained immediately by (3.14) and (3.15). Using the same procedure, we can prove (1.7), and the details are omitted here.

3.1.4. *Proof of Corollary 1.1*

In the sequel, we will show (1.8), while the proof of (1.9) is similar and the details are omitted here. In fact, we have

$$\begin{aligned} &\left| \frac{1}{1-F_{\mathbb{C}}(x)} P\left(\frac{e^{\theta n\Delta}}{2\theta} \left(\hat{\theta}_{n,\Delta}-\theta\right) \ge x \right) - 1 \right| \\ &\le \left| \frac{1-F_{\mathbb{C}}(\Upsilon_n x)}{1-F_{\mathbb{C}}(x)} \right| \cdot \left| \frac{1}{1-F_{\mathbb{C}}(\Upsilon_n x)} P\left(\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}} \left(\hat{\theta}_{n,\Delta}-\theta\right) \ge \Upsilon_n x \right) - 1 \right| \\ &\quad + \left| \frac{1-F_{\mathbb{C}}(\Upsilon_n x)}{1-F_{\mathbb{C}}(x)} - 1 \right|, \end{aligned}$$

where $\Upsilon_n = \frac{2\theta\Delta}{e^{\theta\Delta}-e^{-\theta\Delta}} = 1 + O(\Delta^2)$.

By Theorem 1.2, we only need to show

$$\lim_{\Delta\to 0, n\to\infty} \sup_{0\le x\le \rho\beta_{n,\Delta}} \left| \frac{1-F_{\mathbb{C}}(\Upsilon_n x)}{1-F_{\mathbb{C}}(x)} - 1 \right| = 0. \tag{3.16}$$

For positive constant M satisfying (4.24), we have

$$\sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{1-F_{\mathbb{C}}(\Upsilon_n x)}{1-F_{\mathbb{C}}(x)} - 1 \right| \le \frac{\rho^2\beta_{n,\Delta}^2 |1-\Upsilon_n|}{1+2M^2} \le \frac{C\beta_{n,\Delta}^2 \Delta^2}{1+2M^2},$$

which implies that

$$\lim_{\Delta\to 0, n\to\infty} \sup_{2M\le x\le \rho\beta_{n,\Delta}} \left| \frac{1-F_{\mathbb{C}}(\Upsilon_n x)}{1-F_{\mathbb{C}}(x)} - 1 \right| = 0.$$

Finally, it is easy to get that

$$\lim_{\Delta\to 0, n\to\infty} \sup_{0\le x\le 2M} \left| \frac{1-F_{\mathbb{C}}(\Upsilon_n x)}{1-F_{\mathbb{C}}(x)} - 1 \right| = 0.$$

Consequently, we can complete the proof of this corollary.

3.1.5. Proof of Theorem 1.3

To begin with, we can get the following formula

$$\begin{aligned}
&\frac{\Delta e^{\hat{\theta}_{n,\Delta}(n-1)\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\\
&=\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)+\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\frac{e^{-2\hat{\theta}_{n,\Delta}\Delta}-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}\\
&\quad+\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\left(e^{(\hat{\theta}_{n,\Delta}-\theta)(n-1)\Delta}-1\right)\frac{1-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}.
\end{aligned}$$

By Theorem 1.2 and the procedures in its proof, to get (1.11), it is sufficient to show

$$\begin{aligned}
&\lim_{\Delta\to 0,n\to\infty}\beta_{n,\Delta}P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|\left|\frac{1-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}-1\right|\geq\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)\\
&=0
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
\lim_{\Delta\to 0,n\to\infty}\beta_{n,\Delta}P\Bigg(&\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|\left|\left(e^{(\hat{\theta}_{n,\Delta}-\theta)(n-1)\Delta}-1\right)\frac{1-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}\right|\\
&\geq\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\Bigg)=0.
\end{aligned}\tag{3.18}$$

In fact, for any $L>0$, we have

$$\begin{aligned}
&P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|\left|\frac{1-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}-1\right|\geq\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)\\
&\leq P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|\geq L\beta_{n,\Delta}\right)\\
&\quad+P\left(\left|\frac{1-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}-1\right|\geq\frac{1}{L\beta_{n,\Delta}^2\log\beta_{n,\Delta}}\right).
\end{aligned}$$

By using Theorem 1.2 and (3.8), we have

$$\lim_{\Delta\to 0,n\to\infty}\beta_{n,\Delta}P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|\geq L\beta_{n,\Delta}\right)\leq\frac{1}{\pi L},$$

which implies

$$\lim_{L\to\infty}\lim_{\Delta\to 0,n\to\infty}\beta_{n,\Delta}P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|\geq L\beta_{n,\Delta}\right)=0.\tag{3.19}$$

Moreover,

$$P\left(\left|\frac{1-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}-1\right|\geq\frac{1}{L\beta_{n,\Delta}^2\log\beta_{n,\Delta}}\right)$$

$$\leq P\left(\left|\hat{\theta}_{n,\Delta}-\theta\right|\geq \theta/2\right)$$
$$+P\left(\frac{2e^{-2\theta\Delta}}{1-e^{-\theta\Delta}}\left(\Delta\left|\hat{\theta}_{n,\Delta}-\theta\right|+\Delta^{2}\left|\hat{\theta}_{n,\Delta}-\theta\right|^{2}\right)\geq \frac{1}{L\beta_{n,\Delta}^{2}\log\beta_{n,\Delta}}\right).$$

From Theorem 1.2 and (3.8), we have

$$\lim_{\Delta\to 0,n\to\infty}\beta_{n,\Delta}P\left(\left|\frac{1-e^{-2\theta\Delta}}{1-e^{-2\hat{\theta}_{n,\Delta}\Delta}}-1\right|\geq\frac{1}{L\beta_{n,\Delta}^{2}\log\beta_{n,\Delta}}\right)=0. \tag{3.20}$$

Therefore, together with (3.19) and (3.20), we can prove (3.17).

Finally, (3.18) can be proved similarly, and the details are omitted here.

3.1.6. Proof of Corollary 1.2

Notice that for any $x\geq 0$,

$$P\left(\left|\frac{e^{\theta n\Delta}}{2\theta}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|^{\frac{1}{b_{n,\Delta}}}\geq x\right)=P\left(\left|\frac{\Delta e^{\theta(n-1)\Delta}}{1-e^{-2\theta\Delta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right|\geq|\Upsilon_n|x^{b_{n,\Delta}}\right),$$

where $\Upsilon_n=\frac{2\theta\Delta}{e^{\theta\Delta}-e^{-\theta\Delta}}=1+O(\Delta^2)$.

In the case of $0\leq x<1$, we have $x^{b_{n,\Delta}}\to 0$ as $n\to\infty$. Then, Theorem 1.2 implies that

$$\frac{P\left(\left|\frac{e^{\theta n\Delta}}{2\theta}(\hat{\theta}_{n,\Delta}-\theta)\right|^{\frac{1}{b_{n,\Delta}}}\geq x\right)}{1-F_{\mathbb{C}}(|\Upsilon_n|x^{b_{n,\Delta}})}\to 1,\quad n\to\infty.$$

Hence, we can obtain that

$$\begin{aligned}&\lim_{\Delta\to 0,n\to\infty}\frac{1}{b_{n,\Delta}}\log P\left(\left|\frac{e^{\theta n\Delta}}{2\theta}(\hat{\theta}_{n,\Delta}-\theta)\right|^{\frac{1}{b_{n,\Delta}}}\geq x\right)\\&=\lim_{\Delta\to 0,n\to\infty}\frac{1}{b_{n,\Delta}}\log\left(1-F_{\mathbb{C}}(|\Upsilon_n|x^{b_{n,\Delta}})\right)=0.\end{aligned} \tag{3.21}$$

In the case of $x=1$, we have $x^{b_{n,\Delta}}=1$. By Theorem 1.2, it follows that

$$\frac{P\left(\left|\frac{e^{\theta n\Delta}}{2\theta}(\hat{\theta}_{n,\Delta}-\theta)\right|^{\frac{1}{b_{n,\Delta}}}\geq x\right)}{1-F_{\mathbb{C}}(|\Upsilon_n|)}\to 1,\quad n\to\infty.$$

Therefore,

$$\begin{aligned}&\lim_{\Delta\to 0,n\to\infty}\frac{1}{b_{n,\Delta}}\log P\left(\left|\frac{e^{\theta n\Delta}}{2\theta}(\hat{\theta}_{n,\Delta}-\theta)\right|^{\frac{1}{b_{n,\Delta}}}\geq x\right)\\&=\lim_{\Delta\to 0,n\to\infty}\frac{1}{b_{n,\Delta}}\log\left(1-F_{\mathbb{C}}(|\Upsilon_n|)\right)=0.\end{aligned} \tag{3.22}$$

In the case of $x > 1$, if $b_{n,\Delta}$ satisfies condition (1.13), it holds that

$$x^{b_{n,\Delta}} \to \infty, \quad \frac{n\Delta x^{b_{n,\Delta}} b_{n,\Delta}^4}{e^{2\theta n\Delta}} \to 0.$$

By Theorem 1.2, we have

$$\frac{P\left(\left|\frac{e^{\theta n\Delta}}{2\theta}(\hat{\theta}_{n,\Delta} - \theta)\right|^{\frac{1}{b_{n,\Delta}}} \geq x\right)}{1 - F_{\mathbb{C}}(|\Upsilon_n| x^{b_{n,\Delta}})} \to 1, \quad n \to \infty.$$

According to (3.8),

$$\begin{aligned} &\lim_{\Delta\to 0, n\to\infty} \frac{1}{b_{n,\Delta}} \log P\left(\left|\frac{e^{\theta n\Delta}}{2\theta}(\hat{\theta}_{n,\Delta} - \theta)\right|^{\frac{1}{b_{n,\Delta}}} \geq x\right) \\ &= \lim_{\Delta\to 0, n\to\infty} \frac{1}{b_{n,\Delta}} \log\left(1 - F_{\mathbb{C}}(|\Upsilon_n| x^{b_{n,\Delta}})\right) = -\log x. \end{aligned} \tag{3.23}$$

Then, the proof of (1.14) is completed by (3.21)–(3.23). Finally, by using Theorem 1.3 and the procedure in the proof of (1.14), we can show (1.15), and the details are omitted here.

3.2. Ergodic case: $\theta < 0$

In this subsection, we will prove Theorem 1.4, Theorem 1.5, Theorem 1.6 and Corollary 1.3. The below formula plays an important in our following analysis.

$$\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta} - \theta) = \sqrt{-\frac{n\Delta}{2\theta}} \cdot \frac{1}{\Delta} \log\left(1 + \frac{\Delta}{\sqrt{n\Delta}} \frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n}\right) \tag{3.24}$$

where U_n, V_n are defined by (3.4).

The following results state that $\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta} - \theta)$ is exponential equivalent to $\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{\sqrt{-2\theta} n^{-1} V_n}$. The proof of moderate deviation for $\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{\sqrt{-2\theta} n^{-1} V_n}$ (Proposition 3.2) will be postponed to Section 4.

Proposition 3.2. *Under assumption (H'), let $\lambda_{n,\Delta}$ satisfy (1.16). Then, for each $\rho > 0$, uniformly for $0 \leq x \leq \rho\lambda_{n,\Delta}$,*

$$\frac{1}{1 - \Phi(x)} P\left(\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{\sqrt{-2\theta} n^{-1} V_n} \geq x\right) \to 1 \tag{3.25}$$

and

$$\frac{1}{1 - \Phi(x)} P\left(\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{\sqrt{-2\theta} n^{-1} V_n} \leq -x\right) \to 1. \tag{3.26}$$

3.2.1. Proof of Theorem 1.4

Applying (1.2) and (3.24), we can obtain

$$\frac{\sum_{i=1}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_{i-1}^2} = e^{\theta\Delta}\left(1 + \frac{\Delta}{\sqrt{n\Delta}} \frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n}\right).$$

Consequently, it holds that

$$\begin{aligned} P\left(\frac{\sum_{i=1}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_{i-1}^2} \le 0\right) &= P\left(e^{\theta\Delta}\left(1 + \frac{\Delta}{\sqrt{n\Delta}} \frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n}\right) \le 0\right) \\ &\le P\left(\left|\frac{\Delta}{\sqrt{n\Delta}} \frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n}\right| \ge 1\right) \\ &\le P\left(\left|\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n}\right| \ge \sqrt{n/\Delta}\right). \end{aligned}$$

From (1.16) and Proposition 3.2, we can completes the proof of this theorem.

3.2.2. Exponential equivalence in Cramér-type moderate deviations

Now, by Theorem 1.4 and Proposition 3.2, we can have the following exponential equivalence.

Lemma 3.2. *Under assumption (H'), let $\lambda_{n,\Delta}$ satisfy (1.16), we have*

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{\lambda_{n,\Delta}^2} \log P\left(\left|\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta} - \theta) - \frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{\sqrt{-2\theta} n^{-1} V_n}\right| \ge \frac{1}{\lambda_{n,\Delta} \log \lambda_{n,\Delta}}\right) = -\infty.$$

Proof. Using (3.24) and the fact $|\log(1+x) - x| \le x^2$, $x > -\frac{1}{2}$, we can obtain for any $L > 0$

$$\begin{aligned} &P\left(\left|\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta} - \theta) - \frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{\sqrt{-2\theta} n^{-1} V_n}\right| \ge \frac{1}{\lambda_{n,\Delta} \log \lambda_{n,\Delta}}\right) \\ &\le P\left(\frac{\sum_{i=1}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_{i-1}^2} \le 0\right) + P\left(\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n} \le -\frac{1}{2}\sqrt{n/\Delta}\right) \\ &\quad + P\left(\left|\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n}\right| \ge \left(\frac{\sqrt{n/\Delta}}{\lambda_{n,\Delta} \log \lambda_{n,\Delta}}\right)^{1/2}\right) \\ &\le P\left(\frac{\sum_{i=1}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_{i-1}^2} \le 0\right) + 2P\left(\left|\frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{n^{-1} V_n}\right| \ge L\lambda_{n,\Delta}\right). \end{aligned}$$

From Theorem 1.4 and Proposition 3.2, it follows that

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{\lambda_{n,\Delta}^2} \log P\left(\left|\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta} - \theta) - \frac{(n\Delta)^{-1/2} e^{-\theta\Delta} U_n}{\sqrt{-2\theta} n^{-1} V_n}\right| \ge \frac{1}{\lambda_{n,\Delta} \log \lambda_{n,\Delta}}\right) \le -L^2/2,$$

which completes the proof of this lemma by letting $L \to \infty$. □

3.2.3. Proof of Theorem 1.5

Here, we only prove (1.20), since (1.21) can be obtained in the same way. Firstly, we can write that

$$\begin{aligned}
&P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\geq x+\frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\right)\\
&\quad-P\left(\left|\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta}-\theta)-\frac{(n\Delta)^{-1/2}e^{-\theta\Delta}U_n}{\sqrt{-2\theta}n^{-1}V_n}\right|\geq\frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\right)\\
&\leq P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\geq x\right)\\
&\leq P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\geq x-\frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\right)\\
&\quad+P\left(\left|\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta}-\theta)-\frac{(n\Delta)^{-1/2}e^{-\theta\Delta}U_n}{\sqrt{-2\theta}n^{-1}V_n}\right|\geq\frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\right).
\end{aligned}$$

Note that for all $x\geq 0$,

$$\frac{1}{2+\sqrt{2\pi}x}e^{-x^2/2}\leq 1-\Phi(x)\leq\frac{1}{2}e^{-x^2/2}. \tag{3.27}$$

Then, using Lemma 3.2, we have

$$\begin{aligned}
\lim_{\Delta\to 0,n\to\infty}\sup_{0\leq x\leq\rho\lambda_{n,\Delta}}\frac{1}{1-\Phi(x)}P\Bigg(&\left|\sqrt{-\frac{n\Delta}{2\theta}}(\hat{\theta}_{n,\Delta}-\theta)-\frac{(n\Delta)^{-1/2}e^{-\theta\Delta}U_n}{\sqrt{-2\theta}n^{-1}V_n}\right|\\
&\geq\frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\Bigg)=0.
\end{aligned} \tag{3.28}$$

Moreover, letting $\Xi_n=\frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}$,

$$\begin{aligned}
&\left|\frac{1}{1-\Phi(x)}P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\geq x\pm\Xi_n\right)\right|\\
&\leq\left|\frac{1-\Phi(x\pm\Xi_n)}{1-\Phi(x)}\right|\cdot\left|\frac{1}{1-\Phi(x\pm\Xi_n)}P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\geq x\pm\Xi_n\right)\right|\\
&\quad+\left|\frac{1-\Phi(x\pm\Xi_n)}{1-\Phi(x)}-1\right|.
\end{aligned}$$

By (3.27) and mean value theorem, we have

$$\sup_{0\leq x\leq\rho\lambda_{n,\Delta}}\left|\frac{1-\Phi(x\pm\Xi_n)}{1-\Phi(x)}-1\right|\leq\frac{C}{\log\lambda_{n,\Delta}}.$$

Together with Proposition 3.2,

$$\lim_{\Delta\to 0, n\to\infty} \sup_{0\le x\le \rho\lambda_{n,\Delta}} \left| \frac{1}{1-\Phi(x)} P\left(\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) \ge x \pm \Xi_n\right)\right| = 0. \quad (3.29)$$

Therefore, we can complete the proof of (1.20) by (3.28) and (3.29).

3.2.4. *Proof of Theorem 1.6*

By the method as in the proof of Theorem 1.5, we only need to show

$$\begin{aligned} \lim_{\Delta\to 0, n\to\infty} \frac{1}{\lambda_{n,\Delta}^2} \log P\Bigg(\left| \sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta\right) - \sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right| \\ \ge \frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\Bigg) = -\infty. \end{aligned} \quad (3.30)$$

In fact, we have

$$\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta\right) - \sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) = \frac{2(\hat{\theta}_{n,\Delta}-\theta)^2}{\sqrt{4\hat{\theta}_{n,\Delta}\theta}\left(\sqrt{-2\hat{\theta}_{n,\Delta}}+\sqrt{-2\theta}\right)}.$$

Hence, it holds that

$$\begin{aligned} &P\left(\left| \sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta\right) - \sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right)\right| \ge \frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\right) \\ &\le P\left(\left|\hat{\theta}_{n,\Delta}-\theta\right| \ge |\theta|/2\right) + P\left(\frac{2\sqrt{n\Delta}(\hat{\theta}_{n,\Delta}-\theta)^2}{2\sqrt{-\theta^3}+\sqrt{2\theta^3}} \ge \frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}\right). \end{aligned}$$

Then, we can complete the proof by Theorem 1.5.

3.2.5. *Proof of Corollary 1.3*

For any $x>0$, take $y=\lambda_{n,\Delta}x$, and it holds by (3.27)

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{\lambda_{n,\Delta}^2} \log\left(1-\Phi(y)\right) = -\frac{x^2}{2}.$$

Then, by virtue of Theorem 1.5,

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{\lambda_{n,\Delta}^2} \log P\left(\frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-2\theta}}\left(\hat{\theta}_{n,\Delta}-\theta\right) \ge x\right) = -\frac{x^2}{2}.$$

Similarly, we can also have by Theorem 1.6

$$\lim_{\Delta\to 0, n\to\infty} \frac{1}{\lambda_{n,\Delta}^2} \log P\left(\frac{1}{\lambda_{n,\Delta}}\sqrt{\frac{n\Delta}{-2\hat{\theta}_{n,\Delta}}}\left(\hat{\theta}_{n,\Delta}-\theta\right) \ge x\right) = -\frac{x^2}{2}.$$

4. Proofs of two technical propositions

4.1. Proof of Proposition 3.1

Recall that

$$U_n = \sum_{i=1}^{n} \epsilon_i^{\Delta}(\theta) X_{i-1}, \quad V_n = \sum_{i=1}^{n} X_{i-1}^2.$$

The above two terms can be rewritten as

$$e^{-\theta(n+2)\Delta} U_n = e^{-3\theta\Delta} \widetilde{U}_n \widetilde{X}_n + r_{1n}, \; (1 - e^{-2\theta\Delta}) e^{-2\theta n\Delta} V_n = e^{-2\theta\Delta} \widetilde{X}_{n-1}^2 - r_{2n}, \tag{4.1}$$

where

$$\widetilde{U}_n = \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_i^{\Delta}(\theta), \quad \widetilde{X}_i := e^{-\theta i\Delta} X_i, \quad i = 0, \cdots, n, \tag{4.2}$$

$$r_{1n} = e^{-3\theta\Delta} \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_i^{\Delta}(\theta) \left(\widetilde{X}_{i-1} - \widetilde{X}_n \right) \tag{4.3}$$

and

$$r_{2n} = e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-2\theta(n-i+1)\Delta} \left(\widetilde{X}_{i-1}^2 - \widetilde{X}_{i-2}^2 \right). \tag{4.4}$$

Then, we have the following key decomposition for $\frac{e^{-\theta(n+2)\Delta} U_n}{(1-e^{-2\theta\Delta}) e^{-2\theta n\Delta} V_n}$.

Lemma 4.1. *Define*

$$R_{1n} = \frac{e^{-\theta\Delta} \widetilde{U}_n}{\widetilde{X}_{n-1}} \cdot \frac{e^{-\theta n\Delta} \epsilon_n^{\Delta}(\theta)}{\widetilde{X}_{n-1}}, \quad R_{2n} = \frac{r_{1n}}{e^{-2\theta\Delta} \widetilde{X}_{n-1}^2 - r_{2n}}, \tag{4.5}$$

and

$$R_{3n} = \frac{r_{2n}}{e^{-2\theta\Delta} \widetilde{X}_{n-1}^2 - r_{2n}}.$$

We have

$$\frac{e^{-\theta(n+2)\Delta} U_n}{(1 - e^{-2\theta\Delta}) e^{-2\theta n\Delta} V_n} = \frac{e^{-n\Delta} \widetilde{U}_n}{\widetilde{X}_{n-1}} + R_{1n} + R_{2n} + \widetilde{R}_{3n}, \tag{4.6}$$

where

$$\widetilde{R}_{3n} = \left(\frac{e^{-n\Delta} \widetilde{U}_n}{\widetilde{X}_{n-1}} + R_{1n} \right) R_{3n}. \tag{4.7}$$

Proof. Using (4.1), we have

$$\frac{e^{-\theta(n+2)\Delta} U_n}{(1 - e^{-2\theta\Delta}) e^{-2\theta n\Delta} V_n} - \frac{e^{-\theta\Delta} \widetilde{U}_n \widetilde{X}_n}{\widetilde{X}_{n-1}^2} = R_{2n} + \frac{e^{-\theta\Delta} \widetilde{U}_n \widetilde{X}_n}{\widetilde{X}_{n-1}^2} R_{3n}.$$

Since $\widetilde{X}_n = \widetilde{X}_{n-1} + e^{-\theta n\Delta}\epsilon_n^{\Delta}(\theta)$, it holds that

$$e^{-\theta\Delta}\left(\frac{\widetilde{U}_n\widetilde{X}_n}{\widetilde{X}_{n-1}^2} - \frac{\widetilde{U}_n}{\widetilde{X}_{n-1}}\right) = R_{1n}.$$

Then, (4.6) can be proved immediately. □

Lemma 4.2. *Under assumption* (H)*, let* $\beta_{n,\Delta}$ *satisfy (1.5). Then, for each* $\rho > 0$*, uniformly for* $0 \le x \le \rho\beta_{n,\Delta}$*,*

$$\frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \ge x\right) \to 1 \tag{4.8}$$

and

$$\frac{1}{F_{\mathbb{C}}(-x)} P\left(\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \le -x\right) \to 1. \tag{4.9}$$

Proof. Note that $\left(e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}\right)$ is a two dimensional normal variable, satisfying

$$E\widetilde{X}_{n-1} = 0, \quad Var(\widetilde{X}_{n-1}) = \frac{1}{2\theta}\left(1 - e^{-2\theta(n-1)\Delta}\right) := \sigma^2_{\widetilde{X}_{n-1}},$$
$$Ee^{-\theta\Delta}\widetilde{U}_n = 0, \quad Var(e^{-\theta\Delta}\widetilde{U}_n) = \frac{1}{2\theta}\left(1 - e^{-2\theta n\Delta}\right) := \sigma^2_{e^{-\theta\Delta}\widetilde{U}_n},$$

and

$$Cov\left(e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}\right) = \frac{1}{2\theta}(n-1)\left(e^{2\theta\Delta} - 1\right)e^{-\theta(n+1)\Delta}.$$

Letting

$$\rho_{e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}} = \sigma^{-1}_{\widetilde{X}_{n-1}}\sigma^{-1}_{e^{-\theta\Delta}\widetilde{U}_n} Cov\left(e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}\right),$$

we can get that

$$e^{-\theta\Delta}\widetilde{U}_n = \rho_{e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}}\sigma_{e^{-\theta\Delta}\widetilde{U}_n}\sigma^{-1}_{\widetilde{X}_{n-1}}\widetilde{X}_{n-1} + \sqrt{1 - \rho^2_{e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}}}\sigma_{e^{-\theta\Delta}\widetilde{U}_n}\xi,$$

where ξ is a standard normal variable which is independent of $\widetilde{X}_{n-1}$. Therefore, it holds that

$$\begin{aligned}
&\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \\
&= \sqrt{1 - \rho^2_{e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}}}\sigma_{e^{-\theta\Delta}\widetilde{U}_n}\sigma^{-1}_{\widetilde{X}_{n-1}}\frac{\xi}{\sigma^{-1}_{\widetilde{X}_{n-1}}\widetilde{X}_{n-1}} + \rho_{e^{-\theta\Delta}\widetilde{U}_n, \widetilde{X}_{n-1}}\sigma_{e^{-\theta\Delta}\widetilde{U}_n}\sigma^{-1}_{\widetilde{X}_{n-1}} \\
&= A_n\frac{\xi}{\sigma^{-1}_{\widetilde{X}_{n-1}}\widetilde{X}_{n-1}} + B_n.
\end{aligned}$$

Moreover, we also have

$$A_n = 1 + O\left((n\Delta)^2 e^{-2\theta n\Delta}\right), \quad B_n = O\left(n\Delta e^{-\theta n\Delta}\right). \tag{4.10}$$

Since $\frac{\xi}{\sigma^{-1}_{\widetilde{X}_{n-1}}\widetilde{X}_{n-1}} \sim \mathbb{C}$, we have $P\left(\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \geq x\right) = 1 - F_{\mathbb{C}}\left(A_n^{-1}(x - B_n)\right)$.

For any $\epsilon > 0$, let M be positive constant such that (4.24) holds. Then

$$\begin{aligned}
&\sup_{2M \leq x \leq \rho\beta_{n,\Delta}} \left| \frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \geq x\right) - 1 \right| \\
&= \sup_{2M \leq x \leq \rho\beta_{n,\Delta}} \left| \frac{F_{\mathbb{C}}(x) - F_{\mathbb{C}}\left(A_n^{-1}(x - B_n)\right)}{1 - F_{\mathbb{C}}(x)} \right| \\
&\leq \frac{x^2}{(1-\epsilon)(1+2M^2)} O\left(n\Delta e^{-\theta n\Delta}\right).
\end{aligned}$$

Therefore

$$\lim_{\Delta\to 0, n\to\infty} \sup_{2M \leq x \leq \rho\beta_{n,\Delta}} \left| \frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \geq x\right) - 1 \right| = 0. \tag{4.11}$$

Furthermore, it is easy to obtain that

$$\lim_{\Delta\to 0, n\to\infty} \sup_{0 \leq x \leq 2M} \left| \frac{1}{1 - F_{\mathbb{C}}(x)} P\left(\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \geq x\right) - 1 \right| = 0.$$

Together with (4.11), we can prove (4.8). Moreover, (4.9) can be shown similarly, and the details are omitted here. □

In the remainder of this subsection, we will show that R_{1n}, R_{2n} and $\widetilde{R}_{3n}$ defined in (4.5) and (4.7), are negligible in the sense of Cramér-type moderate deviations. To do this, we need the following deviation inequality for the multiple Wiener-Itô integrals ([34], Theorem 2 or [35], Theorem 4.1).

Lemma 4.3. *For any symmetric function $f \in L^2\left([0,T]^n\right)$ and $x > 0$, we have*

$$P\Big(|I_n(f)| > x\Big) \leq C \exp\left\{-\frac{1}{2}\left(\frac{x}{\sqrt{n!}\|f\|_{L^2([0,T]^n)}}\right)^{\frac{2}{n}}\right\},$$

where $I_n(f)$ is the n-th Wiener-Itô integral of f with respect to the Wiener process ([36], [38]), and the constant $C > 0$ depends only on the multiplicity n of the integral.

Lemma 4.4. *Let R_{1n} be defined in (4.5). For $\beta_{n,\Delta}$ defined by (1.5), we have*

$$\lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} P\left(|R_{1n}| \geq \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right) = 0. \tag{4.12}$$

Proof. For $R_{1n} = \frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \cdot \frac{e^{-\theta n\Delta}\epsilon_n^\Delta(\theta)}{\widetilde{X}_{n-1}}$, we can wirte that

$$\begin{aligned} &P\left(|R_{1n}| \geq \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right) \\ &\leq P\left(\left|\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}}\right| \geq \frac{\beta_{n,\Delta}}{\epsilon}\right) + P\left(\left|\widetilde{X}_{n-1}\right| \leq \frac{\epsilon}{\beta_{n,\Delta}}\right) \\ &\quad + P\left(\left|e^{-\theta n\Delta}\epsilon_n^\Delta(\theta)\right| \geq \frac{\epsilon^2}{\beta_{n,\Delta}^3\log\beta_{n,\Delta}}\right). \end{aligned}$$

Using (3.8) and Lemma 4.2, we have

$$\lim_{\epsilon\to 0}\lim_{\Delta\to 0, n\to\infty}\beta_{n,\Delta}P\left(\left|\frac{e^{-\theta\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}}\right| \geq \frac{\beta_{n,\Delta}}{\epsilon}\right) = 0. \tag{4.13}$$

Note that $\widetilde{X}_{n-1} \sim N\Big(0, \frac{1}{2\theta}\left(1 - e^{-2\theta(n-1)\Delta}\right)\Big)$. Then

$$P\left(\left|\widetilde{X}_{n-1}\right| \leq \frac{\epsilon}{\beta_{n,\Delta}}\right) \leq \frac{C\epsilon}{\beta_{n,\Delta}},$$

which implies that

$$\lim_{\epsilon\to 0}\lim_{\Delta\to 0, n\to\infty}\beta_{n,\Delta}P\left(\left|\widetilde{X}_{n-1}\right| \leq \frac{\epsilon}{\beta_{n,\Delta}}\right) = 0. \tag{4.14}$$

From the fact $e^{-\theta n\Delta}\epsilon_n^\Delta(\theta) \sim N\left(0, \frac{1}{2\theta}e^{-2\theta n\Delta}(e^{2\theta\Delta} - 1)\right)$, it follows that

$$P\left(\left|e^{-\theta n\Delta}\epsilon_n^\Delta(\theta)\right| \geq \frac{\epsilon^2}{\beta_{n,\Delta}^3\log\beta_{n,\Delta}}\right) \leq 2\exp\left\{-\frac{C\epsilon^4 e^{2\theta n\Delta}}{\Delta\beta_{n,\Delta}^6(\log\beta_{n,\Delta})^2}\right\},$$

which implies immediately

$$\lim_{\Delta\to 0, n\to\infty}\beta_{n,\Delta}P\left(\left|e^{-\theta n\Delta}\epsilon_n^\Delta(\theta)\right| \geq \frac{\epsilon^2}{\beta_{n,\Delta}^3\log\beta_{n,\Delta}}\right) = 0. \tag{4.15}$$

Combing (4.13)–(4.15), we can obtain that

$$\lim_{\Delta\to 0, n\to\infty}\beta_{n,\Delta}P\left(|R_{1n}| \geq \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right) = 0. \qquad \square$$

Lemma 4.5. *Let R_{2n} and $\widetilde{R}_{3n}$ be defined in (4.7). For $\beta_{n,\Delta}$ defined by (1.5), we have*

$$\lim_{\Delta\to 0, n\to\infty}\beta_{n,\Delta}P\left(|R_{2n}| \vee |\widetilde{R}_{3n}| \geq \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right) = 0. \tag{4.16}$$

Proof. To begin with, we establish the deviation inequalities for r_{1n} and r_{2n} defined in (4.3) and (4.4). In fact, by (4.3), r_{1n} can be rewritten as

$$r_{1n} = e^{-3\theta\Delta} \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_i^{\Delta}(\theta) \left(\widetilde{X}_{i-1} - \widetilde{X}_i \right) + e^{-3\theta\Delta} \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_i^{\Delta}(\theta) \left(\widetilde{X}_i - \widetilde{X}_n \right) := r_{11n} + r_{12n}.$$

Straightforward calculation gives that

$$Er_{11n} = O\left(n\Delta e^{-\theta n\Delta}\right), \quad Var(r_{11n}) = O\left(n\Delta^2 e^{-2\theta n\Delta}\right) \tag{4.17}$$

and

$$Er_{12n} = 0, \quad Var(r_{12n}) = O\left(n\Delta e^{-2\theta n\Delta}\right). \tag{4.18}$$

Note that $r_{1n} - Er_{11n}$ is a second order Wiener-Itô integral. Using Lemma 4.3, we obtain for any $r > 0$

$$P\left(\left|r_{1n} - Er_{11n}\right| \geq r\right) \leq C_1 \exp\left\{-C_2 r (n\Delta)^{-1/2} e^{\theta n\Delta}\right\}. \tag{4.19}$$

Moreover, we also

$$Er_{2n} = O\left(n\Delta e^{-2\theta n\Delta}\right), \quad Var(r_{2n}) = O\left(e^{-2\theta n\Delta}\right) \tag{4.20}$$

Since $r_{2n} - Er_{2n}$ is a second order Wiener-Itô integral, using Lemma 4.3 again, we obtain for any $r > 0$

$$P\left(\left|r_{2n} - Er_{2n}\right| \geq r\right) \leq C_1 \exp\left\{-C_2 r e^{\theta n\Delta}\right\}. \tag{4.21}$$

Secondly, for $R_{2n} = \frac{r_{1n}}{e^{-2\theta\Delta}\widetilde{X}_{n-1}^2 - r_{2n}}$ and $R_{3n} = \frac{r_{2n}}{e^{-2\theta\Delta}\widetilde{X}_{n-1}^2 - r_{2n}}$, we have for any $\epsilon > 0$

$$P\left(|R_{2n}| \geq \frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right) \leq P\left(\left|e^{-2\theta\Delta}\widetilde{X}_{n-1}^2\right| \leq \frac{2\epsilon}{\beta_{n,\Delta}^2}\right) + P\left(|r_{2n}| \geq \frac{\epsilon}{\beta_{n,\Delta}^2}\right) + P\left(|r_{1n}| \geq \frac{\epsilon}{\beta_{n,\Delta}^3 \log\beta_{n,\Delta}}\right)$$

and

$$P\left(|R_{3n}| \geq \frac{\epsilon}{\beta_{n,\Delta}^2\log\beta_{n,\Delta}}\right) \leq P\left(\left|e^{-2\theta\Delta}\widetilde{X}_{n-1}^2\right| \leq \frac{2\delta}{\beta_{n,\Delta}^2}\right) + P\left(|r_{2n}| \geq \frac{\delta}{\beta_{n,\Delta}^2}\right) + P\left(|r_{2n}| \geq \frac{\epsilon\delta}{\beta_{n,\Delta}^4 \log\beta_{n,\Delta}}\right).$$

From (4.14), it follows that

$$\lim_{\delta\to 0} \lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} P\left(\left|e^{-2\theta\Delta}\widetilde{X}_{n-1}^2\right| \leq \frac{2\delta}{\beta_{n,\Delta}^2}\right) = 0.$$

Moreover, by (4.17)–(4.21) and the condition (1.5) for $\beta_{n,\Delta}$,

$$\lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} \max\left\{ P\left(|r_{1n}| \geq \frac{\epsilon}{\beta_{n,\Delta}^3 \log \beta_{n,\Delta}}\right), P\left(|r_{2n}| \geq \frac{\epsilon\delta}{\beta_{n,\Delta}^4 \log \beta_{n,\Delta}}\right)\right\} = 0.$$

Therefore, we have

$$\lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} \max\left\{ P\left(|R_{2n}| \geq \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right), P\left(|R_{3n}| \geq \frac{\epsilon}{\beta_{n,\Delta}^2 \log \beta_{n,\Delta}}\right)\right\} = 0. \tag{4.22}$$

Finally, we are ready to deal with the term $\widetilde{R}_{3n}$. In fact, for any $\epsilon > 0$

$$\begin{aligned} &P\left(\left|\widetilde{R}_{3n}\right| \geq \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right) \\ &\leq P\left(\left|\frac{e^{-n\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}}\right| \geq \frac{\beta_{n,\Delta}}{\epsilon}\right) + P\left(\left|R_{1n}\right| \geq \frac{\beta_{n,\Delta}}{\epsilon}\right) + P\left(\left|R_{3n}\right| \geq \frac{2\epsilon}{\beta_{n,\Delta}^2 \log \beta_{n,\Delta}}\right). \end{aligned}$$

Applying (4.12), (4.13) and (4.22), we have

$$\lim_{\Delta\to 0, n\to\infty} \beta_{n,\Delta} P\left(|\widetilde{R}_{3n}| \geq \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right) = 0. \tag{4.23}$$

Together with (4.22) and (4.23), we can complete the proof of this lemma. □

Proof of Proposition 3.1. By (3.8), for any $\epsilon > 0$, there exists some positive constant M such that

$$\sup_{x\geq M} |\pi x (1 - F_{\mathbb{C}}(x)) - 1| \leq \epsilon. \tag{4.24}$$

Now, for each $\rho > 0$ and $2M \leq x \leq \rho\beta_{n,\Delta}$, we have by (4.6)

$$\begin{aligned} &P\left(\frac{e^{-n\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \geq x + \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right) - P\left(\left|R_{1n} + R_{2n} + \widetilde{R}_{3n}\right| \geq \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right) \\ \leq &P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n} \geq x\right) \\ \leq &P\left(\frac{e^{-n\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}} \geq x - \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right) + P\left(\left|R_{1n} + R_{2n} + \widetilde{R}_{3n}\right| \geq \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right). \end{aligned} \tag{4.25}$$

Firstly, by using Lemma 4.4, Lemma 4.5, and (4.24), we have

$$\lim_{\Delta\to 0, n\to\infty} \sup_{M\leq x\leq \rho\beta_{n,\Delta}} \frac{P\left(\left|R_{1n} + R_{2n} + \widetilde{R}_{3n}\right| \geq \frac{1}{\beta_{n,\Delta} \log \beta_{n,\Delta}}\right)}{1 - F_{\mathbb{C}}(x)} = 0. \tag{4.26}$$

Secondly, we can write that

$$\sup_{2M\le x\le\rho\beta_{n,\Delta}}\left|\frac{P\left(\frac{e^{-n\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}}\ge x\pm\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)}-1\right|$$

$$\le\sup_{2M\le x\le\rho\beta_{n,\Delta}}\left|\frac{1-F_{\mathbb{C}}\left(x\pm\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)}-1\right|$$

$$+\sup_{2M\le x\le\rho\beta_{n,\Delta}}\left|\frac{1-F_{\mathbb{C}}\left(x\pm\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)}\right|\left|\frac{P\left(\frac{e^{-n\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}}\ge x\pm\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}\left(x\pm\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}-1\right|.$$

Together with Lemma 4.2 and (3.12), we can obtain that

$$\lim_{\Delta\to0,n\to\infty}\sup_{2M\le x\le\rho\beta_{n,\Delta}}\left|\frac{P\left(\frac{e^{-n\Delta}\widetilde{U}_n}{\widetilde{X}_{n-1}}\ge x\pm\frac{1}{\beta_{n,\Delta}\log\beta_{n,\Delta}}\right)}{1-F_{\mathbb{C}}(x)}-1\right|=0. \tag{4.27}$$

Therefore, by (4.25)–(4.27), it holds that

$$\lim_{\Delta\to0,n\to\infty}\sup_{2M\le x\le\rho\beta_{n,\Delta}}\left|\frac{1}{1-F_{\mathbb{C}}(x)}P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\ge x\right)-1\right|=0. \tag{4.28}$$

Moreover, since the function $1-F_{\mathbb{C}}(x)$ is uniformly continuous, the standard discretized approximation argument shows that

$$\lim_{\Delta\to0,n\to\infty}\sup_{x\in\mathbb{R}}\left|P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\ge x\right)-(1-F_{\mathbb{C}}(x))\right|=0.$$

Therefore, it follows that

$$\lim_{\Delta\to0,n\to\infty}\sup_{0\le x\le 2M}\left|\frac{1}{1-F_{\mathbb{C}}(x)}P\left(\frac{e^{-\theta(n+2)\Delta}U_n}{(1-e^{-2\theta\Delta})e^{-2\theta n\Delta}V_n}\ge x\right)-1\right|=0. \tag{4.29}$$

Finally, (3.5) can be obtained immediately by (4.28) and (4.29). Using the same procedure, we can prove (3.6), and the details are omitted here. □

4.2. Proof of Proposition 3.2

In this part, we will prove the Cramér-type moderate deviation for $\frac{(n\Delta)^{-1/2}e^{-\theta\Delta}U_n}{\sqrt{-2\theta n^{-1}V_n}}$, where U_n, V_n are defined by (3.4). Firstly, we will represent U_n by second order Wiener-Itô integral, while its forth moment can be estimated explicitly. Then, we can get the Cramér-type moderate deviation for U_n by using the results in Schulte and Thäle ([41]). Secondly, for the term V_n, we will show that it is exponential equivalent to its asymptotic expectation in the sense of moderate deviation.

Lemma 4.6. *For $s, t \in [0, n\Delta]$, define*

$$\varphi_{n,\Delta}(s,t) = \sum_{i=1}^{n} e^{(2i-1)\Delta\theta} e^{-\theta(t+s)} \Big(\mathbb{I}_{[(i-1)\Delta, i\Delta]}(t) \cdot \mathbb{I}_{[0,(i-1)\Delta]}(s) + \mathbb{I}_{[(i-1)\Delta, i\Delta]}(s) \cdot \mathbb{I}_{[0,(i-1)\Delta]}(t) \Big),$$

where $\mathbb{I}_A(\cdot)$ is the indicative function of A. Then

$$U_n = \frac{1}{2} I_2(\varphi_{n,\Delta}) \tag{4.30}$$

and

$$E\left(\|DI_2(\varphi(s,t))\|_{\mathbb{H}}^2 - E\,\|DI_2(\varphi(s,t)\|_{\mathbb{H}}^2 \right)^2 = O\left(n\Delta\right), \tag{4.31}$$

where $I_2(\cdot)$ is the second order Wiener-Itô integral with respect to Brownian motion, D is the Malliavin derivative operator and $\mathbb{H}$ is a Hilbert space associated with Brownian motion (see Nualart ([38])).

Proof. By straightforward calculation, we have

$$\begin{aligned} U_n &= \sum_{i=1}^{n} e^{(2i-1)\theta\Delta} \int_0^{n\Delta} e^{-\theta s} \mathbb{I}_{[(i-1)\Delta, i\Delta]}(s) dW_s \int_0^{n\Delta} e^{-\theta s} \mathbb{I}_{[0,(i-1)\Delta]}(s) dW_s \\ &= \sum_{i=1}^{n} e^{(2i-1)\theta\Delta} \Big(\int_0^{n\Delta} \int_0^{s} e^{-\theta t} \mathbb{I}_{[(i-1)\Delta, i\Delta]}(t) \cdot e^{-\theta s} \mathbb{I}_{[0,(i-1)\Delta]}(s) dW_t dW_s \\ &\quad + \int_0^{n\Delta} \int_0^{s} e^{-\theta t} \mathbb{I}_{[0,(i-1)\Delta]}(t) \cdot e^{-\theta s} \mathbb{I}_{[(i-1)\Delta, i\Delta]}(s) dW_t dW_s \Big) \\ &= \frac{1}{2} I_2(\varphi_{n,\Delta}). \end{aligned}$$

Using (4.30), we have for $s \in [0, n\Delta]$

$$\begin{aligned} D_s\left(I_2(\varphi_{n,\Delta})\right) &= 2 \sum_{i=1}^{n} e^{(2i-1)\theta\Delta} \mathbb{I}_{[0,(i-1)\Delta]}(s) \int_{(i-1)\Delta}^{i\Delta} e^{-\theta(t+s)} dW_t \\ &\quad + 2 \sum_{i=1}^{n} e^{(2i-1)\theta\Delta} \mathbb{I}_{[(i-1)\Delta, i\Delta]}(s) \int_0^{(i-1)\Delta} e^{-\theta(t+s)} dW_t \\ &= 2Y_s + 2\tilde{Y}_s, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{1}{4} \|DI_2(\varphi(s,t)\|_{\mathbb{H}}^2 &= \int_0^{n\Delta} Y_t^2 dt + \int_0^{n\Delta} \tilde{Y}_t^2 dt + 2 \int_0^{n\Delta} Y_t \tilde{Y}_t dt \\ &= A_{n,\Delta}^{(1)} + A_{n,\Delta}^{(2)} + A_{n,\Delta}^{(3)}. \end{aligned}$$

Firstly, for $A_{n,\Delta}^{(1)}$, we have

$$
\begin{aligned}
Y_t^2 = & \sum_{i=1}^n e^{2(i-1)\Delta\theta} e^{-2\theta t} \mathbb{I}_{[0,(i-1)\Delta]}(t) \left(\epsilon_i^\Delta(\theta)\right)^2 \\
& + 2\sum_{i=1}^n \sum_{j=1}^{i-1} e^{(i+j-2)\Delta\theta} e^{-2\theta t} \mathbb{I}_{[0,(j-1)\Delta]}(t) \epsilon_i^\Delta(\theta) \epsilon_j^\Delta(\theta),
\end{aligned}
$$

where $\epsilon_i^\Delta(\theta) = e^{i\Delta\theta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dW_s$. Therefore,

$$
\begin{aligned}
A_{n,\Delta}^{(1)} = & -\frac{1}{2\theta} \sum_{i=1}^n \left(1 - e^{2(i-1)\Delta\theta}\right) \left(\epsilon_i^\Delta(\theta)\right)^2 \\
& - \frac{1}{\theta} \sum_{i=1}^n \sum_{j=1}^{i-1} \left(e^{(i-j)\Delta\theta} - e^{(i+j-2)\Delta\theta}\right) \epsilon_i^\Delta(\theta) \epsilon_j^\Delta(\theta).
\end{aligned}
$$

Then

$$
EA_{n,\Delta}^{(1)} = n\frac{1-e^{2\theta\Delta}}{4\theta^2} - \frac{1-e^{2\theta n\Delta}}{4\theta^2}, \quad E(A_{n,\Delta}^{(1)})^2 = \frac{(n\Delta)^2}{4\theta^2} - \frac{n\Delta}{4\theta^3} + o(n\Delta),
$$

which implies that

$$
E\left(A_{n,\Delta}^{(1)} - E(A_{n,\Delta}^{(1)})\right)^2 = -\frac{n\Delta}{2\theta^3} + o(n\Delta). \tag{4.32}
$$

Secondly, for $A_{n,\Delta}^{(2)}$, it holds that

$$
\tilde{Y}_t^2 = \sum_{i=1}^n e^{2(2i-1)\Delta\theta} e^{-2\theta t} \mathbb{I}_{[(i-1)\Delta, i\Delta]}(t) \left(\int_0^{(i-1)\Delta} e^{-\theta s} dW_s\right)^2
$$

and

$$
A_{n,\Delta}^{(2)} = -\frac{1}{2\theta} \sum_{i=1}^n \left(e^{2(i-1)\Delta\theta} - e^{2i\Delta\theta}\right) \left(\int_0^{(i-1)\Delta} e^{-\theta s} dW_s\right)^2.
$$

Then, we can obtain that

$$
EA_{n,\Delta}^{(2)} = n\frac{1-e^{2\theta\Delta}}{4\theta^2} - \frac{1-e^{2\theta n\Delta}}{4\theta^2}, \quad E(A_{n,\Delta}^{(2)})^2 = \frac{n^2\Delta^2}{4\theta^2} - \frac{n\Delta}{4\theta^3} + o(n\Delta),
$$

which implies that

$$
E\left(A_{n,\Delta}^{(2)} - E(A_{n,\Delta}^{(2)})\right)^2 = -\frac{n\Delta}{2\theta^3} + o(n\Delta). \tag{4.33}
$$

Finally, for the $A^{(3)}_{n,\Delta}$, we can write that

$$Y_t\tilde{Y}_t = \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{2(i+j-1)\Delta\theta} e^{-2\theta t}\mathbb{I}_{[(j-1)\Delta, j\Delta]}(t)\int_0^{(j-1)\Delta} e^{-\theta s}dW_s \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dW_s,$$

and then

$$A^{(3)}_{n,\Delta} = -\frac{1}{\theta}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\left(e^{2(i-1)\Delta\theta} - e^{(2i-1)\Delta\theta}\right)\int_0^{(j-1)\Delta} e^{-\theta s}dW_s \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dW_s.$$

Since

$$EA^{(3)}_{n,\Delta} = 0, \quad E(A^{(3)}_{n,\Delta})^2 = o(n\Delta).$$

We can also obtain

$$E\left(A^{(3)}_{n,\Delta} - E(A^{(3)}_{n,\Delta})\right)^2 = o(n\Delta). \tag{4.34}$$

Together with (4.32)–(4.34), we can complete the proof of this lemma. □

Lemma 4.7. *Let $\lambda_{n,\Delta}$ defined by (1.16). For each $\rho > 0$, uniformly for $0 \le x \le \rho\lambda_{n,\Delta}$,*

$$\frac{1}{1-\Phi(x)}P\left(\pm\sqrt{-\frac{2\theta}{n\Delta}}U_n \ge x\right) \to 1.$$

Proof. Denote by $F_{n,\Delta} = \sqrt{-\frac{2\theta}{n\Delta}}U_n$. By (4.30), we have $F_{n,\Delta} = \sqrt{-\frac{\theta}{2n\Delta}}I_2(\varphi_{n,\Delta})$. Then, we can calculate variance of U_n as

$$\begin{aligned} Var(U_n) &= \frac{1}{4\theta^2}\left(e^{2\theta n\Delta} - 1 + n(1-e^{-2\theta\Delta})\right) \\ &= -\frac{1}{2\theta}n\Delta - \frac{1}{4\theta^2} - \frac{1}{2}n\Delta^2 + o(n\Delta^2). \end{aligned} \tag{4.35}$$

Using Lemma 2 in ([39]), we can obtain

$$E\left(\|DF_{n,\Delta}\|^2_{\mathbb{H}} - E\|DF_{n,\Delta}\|^2_{\mathbb{H}}\right)^2 = 4\theta^2\left\|\frac{\varphi_{n,\Delta}}{\sqrt{n\Delta}}\otimes_1\frac{\varphi_{n,\Delta}}{\sqrt{n\Delta}}\right\|^2_{\mathbb{H}^{\otimes 2}}.$$

From (4.31) in Lemma 4.6, it follows

$$\left\|\frac{\varphi_{n,\Delta}}{\sqrt{n\Delta}}\otimes_1\frac{\varphi_{n,\Delta}}{\sqrt{n\Delta}}\right\|^2_{\mathbb{H}^{\otimes 2}} = O\left((n\Delta)^{-1}\right).$$

According to Theorem 5 in ([41]) (see also Theorem A.1 in the Appendix A), we have for each $\rho > 0$, uniformly for $0 \le x \le \rho\lambda_{n,\Delta}$,

$$\frac{1}{1-\Phi(x)}P\left(\pm\sqrt{-\frac{2\theta}{n\Delta}}U_n \ge x\sqrt{-\frac{2\theta}{n\Delta}Var(U_n)}\right) \to 1. \tag{4.36}$$

Moreover, under the condition that $\lim_{\Delta\to 0, n\to\infty} n\Delta^4(\log n\Delta)^6 = 0$, we have

$$\sup_{0\le x\le \rho\lambda_{n,\Delta}} \left| x\Big(\sqrt{-\frac{2\theta}{n\Delta}Var(U_n)} - 1\Big)\right| \le \frac{1}{\lambda_{n,\Delta}\log\lambda_{n,\Delta}}.$$

By the procedure used in the proof of Theorem 1.5, we have

$$\lim_{\Delta\to 0, n\to\infty} \sup_{0\le x\le \rho\lambda_{n,\Delta}} \left| \frac{1-\Phi\Big(x\sqrt{-\frac{2\theta}{n\Delta}Var(U_n)}\Big)}{1-\Phi(x)} - 1\right| = 0,$$

which completes the proof of this lemma. □

Lemma 4.8. *For V_n defined by (3.4), we have*

$$EV_n = -\frac{n}{2\theta} + \frac{1-e^{2\theta n\Delta}}{2\theta(1-e^{2\theta\Delta})} \tag{4.37}$$

and for any $x > 0$

$$P\left(\frac{1}{n}|V_n - EV_n| \ge x\right) \le C_1 e^{-C_2\sqrt{n\Delta}x}. \tag{4.38}$$

Proof. Since

$$V_n = \sum_{i=1}^n X_{i-1}^2 = \sum_{i=1}^n e^{2(i-1)\Delta\theta}\left(\int_0^{(i-1)\Delta} e^{-\theta t}dW_t\right)^2,$$

we can obtain immediately that

$$EV_n = -\frac{n}{2\theta} + \frac{1-e^{2\theta n\Delta}}{2\theta(1-e^{2\theta\Delta})}, \quad Var(V_n) = -\frac{n}{2\theta^3\Delta} + o\big(n/\Delta\big).$$

Note that $V_n - EV_n$ can be represented as a second order Wiener-Itô integral. Then, using Lemma 4.3, we obtain (4.38). □

Proof of Proposition 3.2. For each $\rho > 0$ and $0 \le x \le \rho\lambda_{n,\Delta}$, we obtain

$$\begin{aligned}
&\frac{1}{1-\Phi(x)}P\left(\frac{(n\Delta)^{-1/2}e^{-\theta\Delta}U_n}{\sqrt{-2\theta n^{-1}V_n}} \ge x\right)\\
&\le \frac{1}{1-\Phi(x)}P\left(\left|n^{-1}V_n + \frac{1}{2\theta}\right| \ge \frac{\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\\
&\quad + \frac{P\left(\pm\sqrt{-\frac{2\theta}{n\Delta}}U_n \ge xe^{\theta\Delta}\left(1+\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}{1-\Phi\left(xe^{\theta\Delta}\left(1+\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}
\end{aligned}$$

$$\cdot \frac{1-\Phi\left(xe^{\theta\Delta}\left(1+\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}{1-\Phi(x)}.$$

Similarly,

$$\begin{aligned}
&\frac{1}{1-\Phi(x)}P\left(\frac{(n\Delta)^{-1/2}e^{-\theta\Delta}U_n}{\sqrt{-2\theta}n^{-1}V_n}\geq x\right)\\
&\geq -\frac{1}{1-\Phi(x)}P\left(\left|n^{-1}V_n+\frac{1}{2\theta}\right|\geq \frac{\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\\
&\quad+\frac{P\left(\pm\sqrt{-\frac{2\theta}{n\Delta}}U_n\geq xe^{\theta\Delta}\left(1-\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}{1-\Phi\left(xe^{\theta\Delta}\left(1-\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}\\
&\quad\cdot\frac{1-\Phi\left(xe^{\theta\Delta}\left(1-\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}{1-\Phi(x)}.
\end{aligned}$$

From Lemma 4.7, it follows immediately that

$$\lim_{\Delta\to 0,n\to\infty}\sup_{0\leq x\leq \rho\lambda_{n,\Delta}}\left|\frac{P\left(\pm\sqrt{-\frac{2\theta}{n\Delta}}U_n\geq xe^{\theta\Delta}\left(1\pm\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}{1-\Phi\left(xe^{\theta\Delta}\left(1\pm\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}-1\right|=0. \tag{4.39}$$

By Lemma 4.8, we have

$$\lim_{\Delta\to 0,n\to\infty}\frac{1}{\lambda_{n,\Delta}^2}\log P\left(\left|n^{-1}V_n+\frac{1}{2\theta}\right|\geq\frac{\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)=-\infty,$$

which implies by (3.27)

$$\lim_{\Delta\to 0,n\to\infty}\sup_{0\leq x\leq \rho\lambda_{n,\Delta}}\frac{1}{1-\Phi(x)}P\left(\left|n^{-1}V_n+\frac{1}{2\theta}\right|\geq\frac{\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)=0. \tag{4.40}$$

Moreover, by mean value theorem and (3.27), we have such that

$$\begin{aligned}
&\sup_{0\leq x\leq \rho\lambda_{n,\Delta}}\left|\frac{1-\Phi\left(xe^{\theta\Delta}\left(1-\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}{1-\Phi(x)}-1\right|\\
&\leq C_1\left(\Delta\lambda_{n,\Delta}^2\vee\frac{\lambda_{n,\Delta}^4\log n\Delta}{\sqrt{n\Delta}}\right)\exp\left\{-C_2\left(\Delta\lambda_{n,\Delta}^2\vee\frac{\lambda_{n,\Delta}^4\log n\Delta}{\sqrt{n\Delta}}\right)\right\}.
\end{aligned}$$

Since $\lim_{\Delta\to 0,n\to\infty}\Delta\lambda_{n,\Delta}^2\leq\lim_{\Delta\to 0,n\to\infty}\left(n\Delta^7\right)^{1/6}$, we can get that

$$\lim_{\Delta\to 0,n\to\infty}\sup_{0\leq x\leq \rho\lambda_{n,\Delta}}\left|\frac{1-\Phi\left(xe^{\theta\Delta}\left(1-\frac{2\theta\lambda_{n,\Delta}^2\log n\Delta}{\sqrt{n\Delta}}\right)\right)}{1-\Phi(x)}-1\right|=0. \tag{4.41}$$

Finally, together with (4.39)–(4.41), we can completes the proof of this proposition. □

Appendix A

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $< \cdot,\cdot >_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$. We denote for integer $q \geq 1$ by $\mathbb{H}^{\otimes q}$ the qth tensor power and by $\mathbb{H}^{\odot q}$ the qth symmetric tensor power of $\mathbb{H}$. Moreover, let $\{e_k, 1 \leq k \leq \dim \mathbb{H}\}$ be a complete orthonormal system in $\mathbb{H}$. For integers $p, q \geq 1$, $f \in \mathbb{H}^{\odot p}, g \in \mathbb{H}^{\odot q}$ and $1 \leq r \leq \min\{p, q\}$, we denote by $f \otimes_r g$ the rth contraction of f and g defined as

$$f \otimes_r g = \sum_{i_1,\dots,i_r=1}^{\dim \mathbb{H}} < f, e_{i_1} \otimes \dots \otimes e_{i_r} >_{\mathbb{H}^{\otimes r}} < g, e_{i_1} \otimes \dots \otimes e_{i_r} >_{\mathbb{H}^{\otimes r}} .$$

Now, let $\{h_n, n \geq 1\}$ be a sequence of elements in $\mathbb{H}^{\odot q}$ for some fixed integer $q \geq 2$. We shall state Theorem 1 and Theorem 5 in Schulte and Thäle ([41]) together as follows.

Theorem A.1. *Let $F_n := I_q(h_n)$ satisfying*

$$\mathrm{Var}(F_n) = q!\|h_n\|^2_{\mathbb{H}^{\otimes q}} = q!$$

and

$$\lim_{n\to\infty} K_n = \lim_{\Delta\to 0, n\to\infty} \max_{r=1,\cdots,q-1} \| h_n \otimes_r h_n \|_{\mathbb{H}^{\otimes 2(q-r)}} = 0$$

Define $\triangle_n = (q^{3q/2} K_n^{\alpha(q)})^{-1}$, with with

$$\alpha(q) := \frac{q+2}{3q+2} \quad (q \text{ even}), \quad \alpha(q) := \frac{q^2 - q - 1}{q(3q-5)} \quad (q \text{ odd}).$$

Then, there exist constants $c_0, c_1, c_2 > 0$ depending only on q such tat for $\triangle_n \geq c_0$ and $0 \leq x \leq c_1 \triangle_n^{1/(q-1)}$, such that

$$\left| \log \frac{P\left(\pm F_n/\sqrt{q!} \geq x\right)}{1 - \Phi(x)} \right| \leq c_2 \frac{\left(1 + x/\sqrt{q!}\right)^3}{\triangle_n^{1/(q-1)}}.$$

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