Electronic Journal of Probability

Electron. J. Probab. **25** (2020), article no. 151, 1–42.
ISSN: 1083-6489 https://doi.org/10.1214/20-EJP556

Large deviations for stochastic porous media equations

Rangrang Zhang*

Abstract

In this paper, we establish the Freidlin-Wentzell type large deviation principle for porous medium-type equations perturbed by small multiplicative noise. The porous medium operator $\Delta(|u|^{m-1}u)$ is allowed. Our proof is based on weak convergence approach.

Keywords: large deviations; porous media equations; weak convergence approach; kinetic solution.
MSC2020 subject classifications: Primary 60F10, Secondary 60H15; 35R60.
Submitted to EJP on September 23, 2019, final version accepted on November 18, 2020.

1 Introduction

In this paper, we are interested in the asymptotic behaviour of porous media equations with small multiplicative noise. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]}, (\{\beta_k(t)\}_{t\in[0,T]})_{k\in\mathbb{N}})$ be a stochastic basis and fix any $T > 0$. Without loss of generality, here the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to be complete and $\{\beta_k(t)\}_{t\in[0,T]}, k \in \mathbb{N}$, are one-dimensional real-valued i.i.d. $\{\mathcal{F}_t\}_{t\in[0,T]}$-Wiener processes. We use $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$. Fix any $N \in \mathbb{N}$, let $\mathbb{T}^N \subset \mathbb{R}^N$ denote the N-dimensional torus (suppose the periodic length is 1). We are concerned with the following porous media equations with stochastic forcing

$$\begin{cases} du(t,x) = \Delta(|u(t,x)|^{m-1}u(t,x))dt + \Phi(u(t,x))dW(t) & \text{in } \mathbb{T}^N \times (0,T], \\ u(\cdot,0) = u_0(\cdot) \in L^{m+1}(\mathbb{T}^N) & \text{on } \mathbb{T}^N, \end{cases} \tag{1.1}$$

for $m \in (1,\infty)$. Here $u : (\omega,x,t) \in \Omega \times \mathbb{T}^N \times [0,T] \mapsto u(\omega,x,t) := u(x,t) \in \mathbb{R}$ is a random field, that is, the equation is periodic in the space variable $x \in \mathbb{T}^N$, the coefficient $\Phi : \mathbb{R} \to \mathbb{R}$ is measurable and fulfills certain conditions specified later, and W is a cylindrical Wiener process defined on a given (separable) Hilbert space U with the form $W(t) = \sum_{k\geq 1} \beta_k(t)e_k, t \in [0,T]$, where $(e_k)_{k\geq 1}$ is a complete orthonormal base in the

*School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, China.
E-mail: rrzhang@amss.ac.cn

Hilbert space U. Clearly, equation (1.1) can be viewed as a special case of a class of SPDE of the type

$$\begin{cases} du(t,x) = \Delta A(u(t,x))dt + \Phi(u(t,x))dW(t) \quad \text{in } \mathbb{T}^N \times (0,T], \\ u(\cdot,0) = u_0(\cdot) \in L^{m+1}(\mathbb{T}^N) \quad \text{on } \mathbb{T}^N, \end{cases} \tag{1.2}$$

where $A : \mathbb{R} \to \mathbb{R}$ is a differential function. Concrete conditions on A will be presented in subsection 2.1.

The porous media equations have been intensively investigated because of the interests stemmed from physics and mathematics (see, e.g., [4, 10, 11] and the references therein). Having a stochastic forcing term is very natural and important for various modeling problems arising in a wide variety of fields, e.g., physics, engineering, biology and so on. Up to now, the Cauchy problem for (1.1) has been studied by a lot of researchers utilizing different approaches. For example, a monotone operator approach is employed in the space H^{-1} by some pioneer works, e.g., [1, 26, 27, 28, 29]. When applied to the Nemytskii type diffusion coefficients, the monotone condition could be verified if Φ are affine linear functions of u, otherwise, the map $u \to \Phi(u)$ are not known to be Lipschitz continuous in H^{-1}, even if Φ is smooth. In order to relax the assumptions on Φ, alternative approaches based on L^1-techniques have been proposed. In the deterministic setting, this has been realized via the theory of accretive operators going back to Crandall-Ligget [9], entropy solutions studied by Otto [25], Kružkov [21] and kinetic solution studied by Lions et al. [22] and Chen-Perthame [8]. In the stochastic setting, an entropy solution was first introduced by Kim in [20] when studying the conservation laws driven by additive noise, wherein the author proposed a method of compensated compactness to prove the existence of a stochastic weak entropy solution via vanishing viscosity approximation. Moreover, a Kruzkov-type method was applied there to prove the uniqueness. Later, Vallet and Wittbold [30] extended the results of Kim to the multi-dimensional Dirichlet problem with additive noise. Concerning the conservation laws driven by multiplicative noise, for Cauchy problem over the whole spatial space, Feng and Nualart [17] introduced a notion of strong entropy solutions in order to prove the uniqueness of the entropy solution. By using a kinetic formulation, Debussche and Vovelle [12] solved the Cauchy problem for stochastic conservation laws in any dimensional torus. In view of the equivalence between kinetic solution and entropy solution, the authors of [12] also obtained the existence and uniqueness of entropy solution.

The literature concerning the entropy and kinetic solutions to degenerate parabolic equations (1.2) driven by stochastic forcing is quite extensive, let us mention some relevant works. For instance, Bauzet et al. [3] proved the existence and uniqueness of entropy solutions to (1.2) under the assumptions that A is assumed to be globally Lipschitz continuous and when Φ is Lipschitz, a behavior $A(u) = |u|^{m-1}u$ near the origin is allowed only for $m > 2$. Moreover, by using a kinetic formulation, Gess and Hofmanová [19] showed the global well-posedness of (1.2), where the boundedness of A' is released, Φ is assumed to be Lipschitz and $\sqrt{A'(u)}$ is γ-Hölder continuous with $\gamma > \frac{1}{2}$ which forces $m > 2$. Recently, based on a notion of entropy solutions, Dareiotis et al. [11] established the well-posedness of (1.2) in the full range $m \in (1, \infty)$ under mild assumptions on the Nemytskii type diffusion coefficient Φ, where the authors proved an L^1-contraction estimates as well as a generalized L^1-stability estimates. There are also a lot of interest on the stochastic fast diffusion equations, that is, $m \in (0, 1]$ (see [2, 28]). To learn invariant measures for the stochastic porous media equations, we can refer the readers to [4, 10].

From statistical mechanics point of view, exploring asymptotic behaviors for vanishing the noise force is important and interesting for studying stochastic porous media, in which establishing large deviations is a core step for finer analysis as well as gaining deeper insight for the described physical evolutions. There are several works on large deviation principle (LDP) for the stochastic porous media equations, we mention some of them. Röckner et al. [29] established the LDP for a class of generalized stochastic porous media equations for both small noise and short time in the space $C([0,T];H^{-1})$ by utilizing the monotonicity of the porous medium operator in H^{-1}. Later, Liu [23] established LDP for the distributions of stochastic evolution equations with general monotone drift and small multiplicative noise. As an application, the author of [23] proved the LDP holds for stochastic porous media equations in the space $C([0,T];H^{-1})$. The purpose of this paper is to prove that the kinetic solution to the stochastic porous medium-type equations (1.2) satisfies Freidlin-Wentzell type LDP in the space $L^1([0,T];L^1(\mathbb{T}^N))$, which is a more delicate result compared with [29] and [23]. On the other hand, Dong et al. [15] established the LDP for quasilinear parabolic SPDE, where the authors handled the hard term $\mathrm{div}(B(u)\nabla u)$ with B being uniformly positive definite, bounded and Lipschitz. For our model, the term $\Delta A(u)$ can be written as $\mathrm{div}(a^2(u)\nabla u)$ with $a^2(r) = A'(r)$, hence, it has similar structure as the term $\mathrm{div}(B(u)\nabla u)$. However, the function $a^2(r)$ is neither bounded nor Lipschitz, thus, our case is much more complex and difficult than [15].

To study the Freidlin-Wentzell's LDP for SPDE, we employ an important tool called the weak convergence approach, which is developed by Dupuis and Ellis in [16]. The key idea of this approach is to establish certain variational representation formula about the Laplace transform of bounded continuous functionals, which leads to the equivalence between the LDP and the Laplace principle. In particular, Boué, Dupuis in [5] and Budhiraja, Dupuis in [6] have proved an elegant variational representation formula for Brownian functionals. Recently, Matoussi et al. [24] proposed a sufficient condition to verify the large deviation criteria of Budhiraja et al. [7] for functionals of Brownian motions, which turns out to be more suitable to deal with SPDEs arising from fluid mechanics (e.g., see [14]). Therefore, in this paper, we adopt this new sufficient condition.

To our knowledge, the present paper is the first work towards establishing the LDP directly for the kinetic solution to the stochastic porous medium-type equations (1.2). The starting point for our research was the paper of Dareiotis et al. [11], where the global well-posedness of entropy solution to (1.2) was established. According to the equivalence between entropy solution and kinetic solution (see Proposition 2.6 in the below), we firstly deduce the existence and uniqueness of kinetic solution to (1.2). Due to the fact that the kinetic solutions are living in a rather irregular space comparing to various type solutions for parabolic SPDEs, it is indeed a challenge to establish LDP for the stochastic porous media equations with general noise force. In order to prove the LDP holds for the kinetic solution in the space $L^1([0,T];L^1(\mathbb{T}^N))$, our proof strategy mainly consists of the following procedures. As an important part of the proof, we need to obtain the global well-posedness of the associated skeleton equations. To show the uniqueness, we establish a general result concerning the stability of the strong solution map on the coefficients by utilizing the doubling of variables method. For showing the existence result, we adopt a similar approach as [11]. To complete the proof of the large deviation principle, we also need to study the weak convergence of the small noise perturbations of the problem (1.2) in the random directions of the Cameron-Martin space of the driving Brownian motions. To verify the convergence of the randomly perturbed equation to the corresponding unperturbed equation in $L^1([0,T];L^1(\mathbb{T}^N))$, an auxiliary approximating process is introduced and the doubling of variables method is employed.

The rest of the paper is organized as follows. The mathematical formulation of stochastic porous media equations is presented in Section 2. In Section 3, we introduce the weak convergence method and state our main result. Section 4 is devoted to the study of the associated skeleton equations. The large deviation principle is proved in Section 5.

2 Preliminaries

Let us firstly introduce some notations which will be used later. C_b represents the space of bounded, continuous functions and C_b^1 stands for the space of bounded, continuously differentiable functions having bounded first order derivative. Let $\|\cdot\|_{L^p(\mathbb{T}^N)}$ denote the norm of Lebesgue space $L^p(\mathbb{T}^N)$ for $p\in(0,\infty]$. In particular, set $H=L^2(\mathbb{T}^N)$ with the corresponding norm $\|\cdot\|_H$. For all $a\geq 0$, let $H^a(\mathbb{T}^N)=W^{a,2}(\mathbb{T}^N)$ be the usual Sobolev space of order a with the norm

$$\|u\|^2_{H^a(\mathbb{T}^N)}=\sum_{|\alpha|=|(\alpha_1,\dots,\alpha_N)|=\alpha_1+\dots+\alpha_N\leq a}\int_{\mathbb{T}^N}|D^\alpha u(x)|^2dx.$$

$H^{-a}(\mathbb{T}^N)$ stands for the topological dual of $H^a(\mathbb{T}^N)$, whose norm is denoted by $\|\cdot\|_{H^{-a}(\mathbb{T}^N)}$. Moreover, we use the brackets $\langle\cdot,\cdot\rangle$ to denote the duality between $C_c^\infty(\mathbb{T}^N\times\mathbb{R})$ and the space of distributions over $\mathbb{T}^N\times\mathbb{R}$. With a slight abuse of the notation $\langle\cdot,\cdot\rangle$, we set

$$\langle F,G\rangle:=\int_{\mathbb{T}^N}\int_{\mathbb{R}}F(x,\xi)G(x,\xi)dxd\xi,\quad F\in L^p(\mathbb{T}^N\times\mathbb{R}),G\in L^q(\mathbb{T}^N\times\mathbb{R}),$$

for $1\leq p<\infty$ and $q:=\frac{p}{p-1}$, the conjugate exponent of p. In particular, when $p=1$, we set $q=\infty$ by convention. For a measure m on $\mathbb{T}^N\times[0,T]\times\mathbb{R}$, the shorthand $m(\phi)$ is defined by

$$m(\phi):=\langle m,\phi\rangle=\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}\phi(x,t,\xi)dm(x,t,\xi),\quad \phi\in C_b(\mathbb{T}^N\times[0,T]\times\mathbb{R}).$$

For $T>0$, let $\mathcal{B}([0,T])$ be the Borel σ-algebra on $[0,T]$ and denote by $\mathcal{P}_T\subset\mathcal{B}([0,T])\otimes\mathcal{F}$ the predictable σ-algebra. Let E be a separable space, a process $(u(t))$ with values in E is said to be predictable if there exists a sequence of E-valued, $\mathcal{P}_T$-measurable simple functions which converges to u at every point (t,ω) in a set of full measure in $[0,T]\times\Omega$. Moreover, we denote by $L^p_{\mathcal{P}}(\Omega\times[0,T];E)$ the set of functions $v\in L^p(\Omega\times[0,T];E)$ which are equal $\mathcal{L}\times\mathbb{P}$-almost everywhere to a predictable function u, where $\mathcal{L}$ is the Lebesgue measure on $[0,T]$.

2.1 Hypothesis

Set

$$a(r)=\sqrt{A'(r)},\quad \Psi(r)=\int_0^r a(s)ds. \tag{2.1}$$

Following [11], we impose conditions on the nonlinearity A via assumptions on Ψ, with some constants $m>1$, $K\geq 1$, which are fixed throughout the whole paper. Precisely, we assume

Hypothesis H The initial value u_0 satisfies $\|u_0\|^{m+1}_{L^{m+1}(\mathbb{T}^N)}<\infty$. The function $A:\mathbb{R}\to\mathbb{R}$ is differentiable, strictly increasing and odd. The function a is differentiable away from the origin and satisfies the bounds

$$|a(0)|\leq K,\quad |a'(r)|\leq K|r|^{\frac{m-3}{2}},\quad \text{if r}>0, \tag{2.2}$$

as well as

$$Ka(r)\geq I_{|r|\geq 1},\quad K|\Psi(r)-\Psi(s)|\geq\begin{cases}|r-s|, & \text{if } |\text{r}|\vee|\text{s}|\geq 1,\\ |r-s|^{\frac{m+1}{2}}, & \text{if } |\text{r}|\vee|\text{s}|<1.\end{cases} \tag{2.3}$$

For each $u\in\mathbb{R}$, the map $\Phi(u):U\to H$ is defined by $\Phi(u)e_k=g^k(\cdot,u)$, where each $g^k(\cdot,u)$ is a regular function on $\mathbb{T}^N$. Denote by $g=(g^1,g^2,\dots)$. More precisely, we assume that $g : \mathbb{T}^N \times \mathbb{R} \to l_2$ satisfies the bounds,

$$G(x,u) = |g(x,u)|_{l^2} := \Big(\sum_{k\geq 1} |g^k(x,u)|^2\Big)^{\frac{1}{2}} \leq K(1+|u|), \tag{2.4}$$

$$|g(x,u) - g(y,v)|_{l^2} := \Big(\sum_{k\geq 1} |g^k(x,u) - g^k(y,v)|^2\Big)^{\frac{1}{2}} \leq K(|x-y| + |u-v|), \tag{2.5}$$

for $x, y \in \mathbb{T}^N, u, v \in \mathbb{R}$. For $g = (g^1, g^2, \ldots), \tilde{g} = (\tilde{g}^1, \tilde{g}^2, \ldots)$ as above, we set

$$d(g,\tilde{g}) := \sup_{u\in\mathbb{R}, x\in\mathbb{T}^N} \frac{\sum_{k\geq 1} |g^k(x,u) - \tilde{g}^k(x,u)|^2}{(1+|u|)^{m+1}} \tag{2.6}$$

Remark 2.1. The condition (2.5) on g is a special case of Assumption 2.2 in [11] with parameters $\kappa = \frac{1}{2}$ and $\bar{\kappa} = 1$.

Based on the above notations, equation (1.2) can be rewritten as

$$\begin{cases} du(t,x) = \Delta A(u(t,x))dt + \sum_{k\geq 1} g^k(x, u(t,x))d\beta_k(t) \quad \text{in } \mathbb{T}^N \times (0,T], \\ u(\cdot, 0) = u_0(\cdot) \in L^{m+1}(\mathbb{T}^N) \quad \text{on } \mathbb{T}^N. \end{cases} \tag{2.7}$$

We denote by $\mathcal{E}(A, g, u_0)$ the Cauchy problem (2.7).

2.2 Entropy solution

Set

$$\Psi_l(r) := \int_0^r l(s)a(s)ds, \quad \forall\, l \in C_b(\mathbb{R}). \tag{2.8}$$

Clearly, by (2.1), it gives that $\Psi = \Psi_1$.

We recall the following entropy solution of (2.7) introduced by [11].

Definition 2.1 (Entropy solution)**.** An entropy solution of $\mathcal{E}(A, g, u_0)$ is a predictable stochastic process $u : \Omega \times [0,T] \to L^{m+1}(\mathbb{T}^N)$ such that

(i) $u \in L^{m+1}(\Omega \times [0,T]; L^{m+1}(\mathbb{T}^N))$,

(ii) for all $l \in C_b(\mathbb{R})$, we have $\Psi_l(u) \in L^2(\Omega \times [0,T]; H^1(\mathbb{T}^N))$ and

$$\partial_i \Psi_l(u) = l(u)\partial_i \Psi(u),$$

(iii) for all convex function $\eta \in C^2(\mathbb{R})$ with η'' compactly supported and all non-negative function $\phi \in C_c^2(\mathbb{T}^N \times [0,T))$, we have

$$\begin{aligned} -\int_0^T \int_{\mathbb{T}^N} \eta(u)\partial_t \phi dxdt \leq \int_{\mathbb{T}^N} \eta(u_0)\phi(0)dx + \int_0^T \int_{\mathbb{T}^N} q_\eta(u)\Delta\phi dxdt \\ + \int_0^T \int_{\mathbb{T}^N} \Big(\frac{1}{2}\phi\eta''(u)G^2(x,u) - \phi\eta''(u)|\nabla\Psi(u)|^2\Big)dxdt \\ + \sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} \phi\eta'(u)g^k(x,u)dxd\beta_k(t), \quad a.s., \end{aligned} \tag{2.9}$$

where q_η is any function satisfying $q_\eta'(\xi) = \eta'(\xi)a^2(\xi)$.

Remark 2.2. (η, q_η) is called entropy-entropy flux pair. If $\eta(r) = \pm r$, it follows from (2.9) that any entropy solution is a weak solution of (2.7) (weak in both space and time). The entropy solution u is a strong solution in the probabilistic sense.

Referring to Theorem 2.1 in [11], the following global well-posedness of $\mathcal{E}(A, g, u_0)$ is proved.

Theorem 2.3. Let (A, g, u_0) satisfy Hypothesis H, then there exists a unique entropy solution to (2.7) with initial condition u_0. Moreover, if $\tilde{u}$ is the unique entropy solution to (2.7) with initial value $\tilde{u}_0$, then

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \mathbb{E}\|u(t)-\tilde{u}(t)\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|u_0-\tilde{u}_0\|_{L^1(\mathbb{T}^N)}. \tag{2.10}$$

2.3 Kinetic solution

In this subsection, we pay attention to the definition of kinetic solution to (2.7). Keeping in mind that we are working on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]}, (\beta_k(t))_{k\in\mathbb{N}})$.

Definition 2.2 (Kinetic measure)**.** A map m from Ω to the set of non-negative, finite measures over $\mathbb{T}^N \times [0,T] \times \mathbb{R}$ is said to be a kinetic measure, if

1. m is measurable, that is, for each $\phi \in C_b(\mathbb{T}^N \times [0,T) \times \mathbb{R})$, $\langle m, \phi\rangle : \Omega \to \mathbb{R}$ is measurable,

2. m vanishes for large ξ, i.e.,

$$\lim_{R\to+\infty} \mathbb{E}[m(\mathbb{T}^N \times [0,T) \times B_R^c)] = 0, \tag{2.11}$$

where $B_R^c := \{\xi \in \mathbb{R}, |\xi| \geq R\}$,

3. for every $\phi \in C_b(\mathbb{T}^N \times \mathbb{R})$, the process

$$(\omega, t) \in \Omega \times [0,T) \mapsto \int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} \phi(x,\xi)dm(x,s,\xi) \in \mathbb{R}$$

is predictable.

Let $\mathcal{M}_0^+(\mathbb{T}^N \times [0,T) \times \mathbb{R})$ stand for the space of all non-negative bounded measures m satisfying (2.11).

Definition 2.3 (Kinetic solution)**.** Let $u_0 \in L^{m+1}(\mathbb{T}^N)$. A measurable function $u : \mathbb{T}^N \times [0,T] \times \Omega \to \mathbb{R}$ is called a kinetic solution to (2.7) with initial datum u_0, if

1. $u \in L^{m+1}_{\mathcal{P}}(\Omega \times [0,T]; L^{m+1}(\mathbb{T}^N))$,

2. for all $l \in C_b(\mathbb{R})$, we have $\Psi_l(u) \in L^2(\Omega \times [0,T]; H^1(\mathbb{T}^N))$ and

$$\partial_i \Psi_l(u) = l(u)\partial_i \Psi(u),$$

3. there exists a kinetic measure m such that $f := I_{u>\xi}$ satisfies that for any $\varphi \in C_c^2(\mathbb{T}^N \times [0,T) \times \mathbb{R})$,

$$\begin{aligned}
&\int_0^T \langle f(t), \partial_t\varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f(t), a^2(\xi)\Delta\varphi(t)\rangle dt \\
&= -\sum_{k\geq 1}\int_0^T \int_{\mathbb{T}^N} g^k(x,u(x,t))\varphi(x,t,u(x,t))dxd\beta_k(t) \\
&\quad -\frac{1}{2}\int_0^T\int_{\mathbb{T}^N} \partial_\xi\varphi(x,t,u(x,t))G^2(x,u(t,x))dxdt + m(\partial_\xi\varphi) + n(\partial_\xi\varphi),\ a.s.,
\end{aligned} \tag{2.12}$$

where $u(t) = u(\cdot,t,\cdot)$, $G^2 = \sum_{k=1}^\infty |g^k|^2$, $a(\cdot)$ is defined by (2.1) and $n : \Omega \to \mathcal{M}_0^+(\mathbb{T}^N \times [0,T) \times \mathbb{R})$ is defined as follows: for any $\phi \in C_b(\mathbb{T}^N \times [0,T) \times \mathbb{R})$,

$$n(\phi) = \int_0^T \int_{\mathbb{T}^N}\int_{\mathbb{R}} \phi(x,t,\xi)|\nabla\Psi(u)|^2 d\delta_{u(t,x)=\xi}dxdt. \tag{2.13}$$

Let (X,λ) be a finite measure space. For some measurable function $u: X\to\mathbb{R}$, define $f: X\times\mathbb{R}\to[0,1]$ by $f(z,\xi)=I_{u(z)>\xi}$ a.e. we use $\bar{f}:=1-f$ to denote its conjugate function. Define $\Lambda_f(z,\xi):=f(z,\xi)-I_{0>\xi}$, which can be viewed as a correction to f. Note that Λ_f is integrable on $X\times\mathbb{R}$ if u is.

It is shown in [12] that for each kinetic solution u to (2.7), almost surely, the function $f=I_{u(x,t)>\xi}$ admits possibly different left and right weak limits at any point $t\in[0,T]$. More precisely, the following results are obtained.

Proposition 2.4 (Left and right weak limits)**.** Let u be the kinetic solution to (2.7) with initial datum u_0. Set $f=I_{u>\xi}$ and $f_0=I_{u_0>\xi}$. Then f admits, almost surely, left and right limits respectively at every point $t\in[0,T]$. More precisely, for any $t\in[0,T]$, there exist kinetic functions $f^{t\pm}$ on $\Omega\times\mathbb{T}^N\times\mathbb{R}$ such that $\mathbb{P}$-a.s.

$$\langle f(t-r),\varphi\rangle\to\langle f^{t-},\varphi\rangle$$

and

$$\langle f(t+r),\varphi\rangle\to\langle f^{t+},\varphi\rangle$$

as $r\to0$ for all $\varphi\in C_c^2(\mathbb{T}^N\times\mathbb{R})$. Moreover, almost surely,

$$\langle f^{t+}-f^{t-},\varphi\rangle=-\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}\partial_\xi\varphi(x,\xi)I_{\{t\}}(s)dm(x,s,\xi).$$

In particular, almost surely, the set of $t\in[0,T]$ fulfilling that $f^{t+}\neq f^{t-}$ is countable.

For $f=I_{u>\xi}$, define $f^{\pm}$ by $f^{\pm}(t)=f^{t\pm}$, $t\in[0,T]$. Since we are dealing with the filtration associated to Brownian motion, both $f^{\pm}$ are clearly predictable as well. Also $f=f^+=f^-$ almost everywhere in time and we can take any of them in an integral with respect to the Lebesgue measure or in a stochastic integral. However, if the integral is with respect to a measure–typically a kinetic measure in this article, the integral is not well defined for f and may differ if one chooses either f^+ or f^-. Due to [12], the weak form (2.12) satisfied by $f=I_{u>\xi}$ can be strengthened to be weak only respect to x and ξ. This property plays an important role in establishing a comparison principle which allows to prove uniqueness. Concretely,

Lemma 2.5. *The function $f=I_{u>\xi}$ satisfying (2.12) fulfills that for any $t\in[0,T]$ and $\varphi\in C_c^2(\mathbb{T}^N\times\mathbb{R})$,*

$$\begin{aligned}
&-\langle f^+(t),\varphi\rangle+\langle f_0,\varphi\rangle+\int_0^t\langle f(s),a^2(\xi)\Delta\varphi\rangle ds\\
=&-\sum_{k\geq1}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}g^k(x,\xi)\varphi(x,\xi)d\nu_{x,s}(\xi)dxd\beta_k(s)\\
&-\frac{1}{2}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\partial_\xi\varphi(x,\xi)G^2(x,\xi)d\nu_{x,s}(\xi)dxds\\
&+\langle m,\partial_\xi\varphi\rangle([0,t])+\langle n,\partial_\xi\varphi\rangle([0,t]),\quad a.s.,
\end{aligned}\tag{2.14}$$

where $\nu:=-\partial_\xi f=\delta_{u=\xi}$ and $\langle q,\partial_\xi\varphi\rangle([0,t])=\int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}}\partial_\xi\varphi(x,\xi)dm(x,s,\xi)$ for $q=m,n$.

As stated in the introduction, the starting point of this paper is the equivalence between entropy solution and kinetic solution. Now, we give a brief proof.

Proposition 2.6. Let $u_0\in L^{m+1}(\mathbb{T}^N)$. The kinetic solution to (2.7) in the sense of Definition 2.3 is equivalent to the entropy solution to (2.7) in the sense of Definition 2.1.

Proof. Let us first prove that a kinetic solution is an entropy solution. To achieve it, we choose test functions $\varphi(x,t,\xi)=\phi(x,t)\eta'(\xi)$, where the non-negative function

$\phi(x,t) \in C_c^2(\mathbb{T}^N \times [0,T))$ and the convex function $\eta \in C^2(\mathbb{R})$ with $\eta'' \geq 0$ compactly supported. Assume that $u(x,t)$ is a kinetic solution to (2.7), then from (2.12), we deduce that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} I_{u(x,t)>\xi} \eta'(\xi) \partial_t \phi(x,t) d\xi dx dt + \int_{\mathbb{T}^N} \int_{\mathbb{R}} I_{u_0>\xi} \eta'(\xi) \phi(0) d\xi dx \\
= & -\int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} I_{u(x,t)>\xi} a^2(\xi) \eta'(\xi) \Delta \phi(t) d\xi dx dt \\
& + \int_0^T \int_{\mathbb{T}^N} \phi(x,t) \eta''(u(x,t)) |\nabla \Psi(u)|^2 dx dt \\
& - \sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} g^k(x,u(x,t)) \phi(x,t) \eta'(u(x,t)) dx d\beta_k(t) \\
& - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \phi(x,t) \eta''(u(x,t)) G^2(x,u(t,x)) dx dt + m(\phi(x,t)\eta''(u(x,t))). \qquad (2.15)
\end{aligned}$$

In view of $\phi \in C_c^2(\mathbb{T}^N \times [0,T))$, we deduce that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} I_{u(x,t)>\xi} \eta'(\xi) \partial_t \phi(x,t) d\xi dx dt + \int_{\mathbb{T}^N} \int_{\mathbb{R}} I_{u_0>\xi} \eta'(\xi) \phi(0) d\xi dx \\
= & \int_0^T \int_{\mathbb{T}^N} (\eta(u(x,t)) - \eta(-\infty)) \partial_t \phi(x,t) dx dt + \int_{\mathbb{T}^N} (\eta(u_0) - \eta(-\infty)) \phi(0) dx \\
= & \int_0^T \int_{\mathbb{T}^N} \eta(u(x,t)) \partial_t \phi(x,t) dx dt + \int_{\mathbb{T}^N} \eta(u_0) \phi(0) dx. \qquad (2.16)
\end{aligned}$$

Taking into account that $q_\eta'(\xi) = a^2(\xi)\eta'(\xi)$ and $\phi \in C_c^2(\mathbb{T}^N \times [0,T))$, we arrive at

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} I_{u(x,t)>\xi} a^2(\xi) \eta'(\xi) \Delta \phi(x,t) d\xi dx dt \\
= & \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} I_{u(x,t)>\xi} q_\eta'(\xi) \Delta \phi(x,t) d\xi dx dt \\
= & \int_0^T \int_{\mathbb{T}^N} (q_\eta(u(x,t)) - q_\eta(-\infty)) \Delta \phi(x,t) dx dt \\
= & \int_0^T \int_{\mathbb{T}^N} q_\eta(u(x,t)) \Delta \phi(x,t) dx dt. \qquad (2.17)
\end{aligned}$$

Based on (2.16)-(2.17) and by $m(\phi(x,t)\eta''(u(x,t))) \geq 0$, it follows that u satisfies (2.9). Moreover, (ii) in Definition 2.1 is implied by condition 2 in Definition 2.3.

Conversely, we suppose that $u(x,t)$ is an entropy solution to (2.7) satisfying (2.9). For any non-negative function $\phi(x,t) \in C_c^2(\mathbb{T}^N \times [0,T))$ and any convex function $\eta \in C_c^3(\mathbb{R})$ with $\eta'' \geq 0$, we define a linear form m as follows:

$$\begin{aligned}
m(\phi \otimes \eta'') := & \int_0^T \int_{\mathbb{T}^N} \eta(u) \partial_t \phi dx dt + \int_{\mathbb{T}^N} \eta(u_0) \phi(0) dx \\
& + \int_0^T \int_{\mathbb{T}^N} q_\eta(u) \Delta \phi dx dt \\
& + \int_0^T \int_{\mathbb{T}^N} \Big(\frac{1}{2} \phi \eta''(u) G^2(x,u) - \phi \eta''(u) |\nabla \Psi(u)|^2 \Big) dx dt \\
& + \sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} \phi \eta'(u) g^k(x,u) dx d\beta_k(t). \qquad (2.18)
\end{aligned}$$

Since u is an entropy solution, we know that m is non-negative for any $\phi \geq 0$. By density, m can be extended to a linear, non-negative functional on $C_c(\mathbb{T}^N \times [0,T) \times \mathbb{R})$, by Riesz representative theorem, there exists a non-negative Radon measure l such that $l(\varphi) = m(\varphi)$ for $\varphi \in C_c(\mathbb{T}^N \times [0,T) \times \mathbb{R})$. For the convenience, we denote $l = m$. Moreover, for any $R > 0$, define $\theta_R(s) = I_{s \geq R}, \Theta_R(\xi) = \int_0^\xi \int_0^r \theta_R(s) ds dr$, and denote by $\{\tilde{\eta}_n\}_{n \geq 1}$ a sequence of smooth convex functions approximating Θ_R. Fix some $\iota > 0$ such that $T - 2\iota > 0$. Now, we choose a sequence $\tilde{\phi}_n \in C_c^\infty([0,T))$ satisfying $\|\tilde{\phi}_n\|_{L^\infty([0,T])} \vee \|\tilde{\phi}_n'\|_{L^1([0,T])} \leq 1$ and

$$\lim_{n \to \infty} \|\tilde{\phi}_n - \alpha_\iota\|_{H_0^1([0,T))} = 0, \tag{2.19}$$

where $\alpha_\iota : [0,T] \to \mathbb{R}$ satisfies that $\alpha_\iota(0) = 1$ and $\alpha_\iota' = -\iota^{-1} I_{T-2\iota, T-\iota}$. Clearly, for $\tau \in [0,T]$,

$$\alpha_\iota(\tau) = \begin{cases} 1, & 0 \leq \tau \leq T - 2\iota, \\ 1 - \iota^{-1}(\tau - (T - 2\iota)), & T - 2\iota < \tau \leq T - \iota, \\ 0, & T - \iota < \tau \leq T. \end{cases} \tag{2.20}$$

Taking $\phi = \tilde{\phi}_n$ and $\eta = \tilde{\eta}_n$ as test functions in (2.18), letting $n \to \infty$ and $\iota \to 0$, and by using the Lebesgue dominated convergence theorem, we deduce that $\lim_{R \to \infty} \mathbb{E}m(\mathbb{T}^N \times [0,T) \times [R, \infty)) = 0$. Similarly, we get $\lim_{R \to \infty} \mathbb{E}m(\mathbb{T}^N \times [0,T) \times (-\infty, -R]) = 0$. Hence, m is a finite non-negative measure satisfying (2.11). The condition 1 and 3 in Definition 2.2 are readily deduced from (2.18). Thus, m is a kinetic measure.

Taking $\eta(\xi) = \int_{-\infty}^\xi \varsigma$ in (2.18), by using (2.16)-(2.17), we conclude that (2.12) holds for $\varphi(x,t,\xi) = \phi(x,t)\varsigma(\xi)$. Since the test functions $\varphi(x,t,\xi) = \phi(x,t)\varsigma(\xi)$ form a dense subset of $C_c^2(\mathbb{T}^N \times [0,T) \times \mathbb{R})$, we get (2.12) holds for any $\varphi \in C_c^2(\mathbb{T}^N \times [0,T) \times \mathbb{R})$. □

On the basis of Proposition 2.6, we deduce from Theorem 2.3 that

Theorem 2.7 (Existence, Uniqueness)**.** Let $u_0 \in L^{m+1}(\mathbb{T}^N)$. Assume Hypothesis H holds, then there exists a unique kinetic solution u to (2.7) with initial datum u_0.

3 Freidlin-Wentzell large deviations and statement of the main result

We start with a brief account of notions of large deviations. Let $\{X^\varepsilon\}_{\varepsilon > 0}$ be a family of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in some Polish space $\mathcal{E}$.

Definition 3.1 (Rate function)**.** A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if the level set $\{x \in \mathcal{E} : I(x) \leq M\}$ is compact for each $M < \infty$.

Definition 3.2 (Large deviation principle)**.** The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle with rate function I if for each Borel subset A of $\mathcal{E}$

$$-\inf_{x \in A^o} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in A) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in A) \leq -\inf_{x \in \bar{A}} I(x),$$

where A^o and $\bar{A}$ denote the interior and closure of A in $\mathcal{E}$, respectively.

Suppose $W(t)$ is a cylindrical Wiener process on a Hilbert space U defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ (that is, the paths of W take values in $C([0,T]; \mathcal{U})$, where $\mathcal{U}$ is another Hilbert space such that the embedding $U \subset \mathcal{U}$ is Hilbert-

Schmidt). Now we define

$$\mathcal{A} := \{\phi : \phi \text{ is a } U\text{-valued } \{\mathcal{F}_t\}\text{-predictable process such that } \int_0^T |\phi(s)|_U^2 ds < \infty \ \mathbb{P}\text{-}a.s.\};$$
$$S_M := \{h \in L^2([0,T];U) : \int_0^T |h(s)|_U^2 ds \leq M\};$$
$$\mathcal{A}_M := \{\phi \in \mathcal{A} : \phi(\omega) \in S_M,\ \mathbb{P}\text{-}a.s.\}.$$

Here and in the sequel of this paper, we will always refer to the weak topology on the set S_M.

Suppose for each $\varepsilon > 0, \mathcal{G}^\varepsilon : C([0,T];\mathcal{U}) \to \mathcal{E}$ is a measurable map and let $X^\varepsilon := \mathcal{G}^\varepsilon(W)$. Now, we list below sufficient conditions for the large deviation principle of the sequence X^ε as $\varepsilon \to 0$.

Condition A There exists a measurable map $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to \mathcal{E}$ such that the following conditions hold

(a) For every $M < \infty$, let $\{h^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$. If h_ε converges to h as S_M-valued random elements in distribution, then $\mathcal{G}^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h^\varepsilon(s)ds)$ converges in distribution to $\mathcal{G}^0(\int_0^\cdot h(s)ds)$.

(b) For every $M < \infty$, the set $K_M = \{\mathcal{G}^0(\int_0^\cdot h(s)ds) : h \in S_M\}$ is a compact subset of $\mathcal{E}$.

The following result is due to Budhiraja et al. in [6].

Theorem 3.1. If $\{\mathcal{G}^\varepsilon\}$ satisfies condition A, then X^ε satisfies the large deviation principle on $\mathcal{E}$ with the following good rate function I defined by

$$I(f) = \inf_{\{h \in L^2([0,T];U) : f = \mathcal{G}^0(\int_0^\cdot h(s)ds)\}} \Big\{\frac{1}{2}\int_0^T |h(s)|_U^2 ds\Big\}, \quad \forall f \in \mathcal{E}. \tag{3.1}$$

By convention, $I(f) = \infty$, if $\Big\{h \in L^2([0,T];U) : f = \mathcal{G}^0(\int_0^\cdot h(s)ds)\Big\} = \emptyset$.

Recently, Matoussi, Sabagh and Zhang [24] proposed a new sufficient condition (Condition B below) to verify the assumptions in Condition A (hence the large deviation principle). It turns out this new sufficient condition is suitable for establishing the large deviation principle for (2.7).

Condition B There exists a measurable map $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to \mathcal{E}$ such that the following two items hold

(i) For every $M < +\infty$, and for any family $\{h^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_M$ and any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big(\rho(Y^\varepsilon, Z^\varepsilon) > \delta\Big) = 0,$$

where $Y^\varepsilon := \mathcal{G}^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h^\varepsilon(s)ds\right)$, $Z^\varepsilon := \mathcal{G}^0\left(\int_0^\cdot h^\varepsilon(s)ds\right)$, and $\rho(\cdot,\cdot)$ stands for the metric in the space $\mathcal{E}$.

(ii) For every $M < +\infty$ and any family $\{h^\varepsilon; \varepsilon > 0\} \subset S_M$ that converges to some element h as $\varepsilon \to 0$, $\mathcal{G}^0\left(\int_0^\cdot h^\varepsilon(s)ds\right)$ converges to $\mathcal{G}^0\left(\int_0^\cdot h(s)ds\right)$ in the space $\mathcal{E}$.

3.1 Statement of the main result

In this paper, we are concerned with the following SPDE driven by small multiplicative noise

$$\begin{cases} du^\varepsilon(x,t) = \Delta(A(u^\varepsilon))dt + \sqrt{\varepsilon}\sum_{k\geq 1} g^k(x,u^\varepsilon(t,x))d\beta_k(t), \\ u^\varepsilon(0) = u_0, \end{cases} \tag{3.2}$$

where $u_0 \in L^{m+1}(\mathbb{T}^N)$. Under Hypothesis H, by Theorem 2.7, (3.2) admits a unique kinetic solution $u^\varepsilon \in L^1([0,T];L^1(\mathbb{T}^N))$ a.s. Therefore, there exists a Borel-measurable map

$$\mathcal{G}^\varepsilon : C([0,T];\mathcal{U}) \to L^1([0,T];L^1(\mathbb{T}^N))$$

such that $u^\varepsilon(\cdot) = \mathcal{G}^\varepsilon(W(\cdot))$.

Let $h \in L^2([0,T];U)$ with $h(t) = \sum_{k\geq 1} h_k(t)e_k$, we consider the following skeleton equation

$$\begin{cases} du^h = \Delta(A(u^h))dt + \sum_{k\geq 1} g^k(x,u^h(t,x))h_k(t)dt, \\ u^h(0) = u_0. \end{cases} \tag{3.3}$$

The solution u^h, whose existence and uniqueness will be proved in the following section, defines a measurable map $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to L^1([0,T];L^1(\mathbb{T}^N))$ such that $\mathcal{G}^0\Big(\int_0^\cdot h(s)ds\Big) := u^h(\cdot)$.

We are ready to proceed with the statement of our main result.

Theorem 3.2. Assume Hypothesis H holds, then u^ε satisfies the large deviation principle on $L^1([0,T];L^1(\mathbb{T}^N))$ with the good rate function I given by (3.1).

4 Skeleton equation

4.1 Existence and uniqueness of solutions to skeleton equation

In this subsection, we fix $h \in S_M$, and assume $h(t) = \sum\limits_{k\geq 1} h_k(t)e_k$, where $\{e_k\}_{k\geq 1}$ is an orthonormal basis of U. Now, we introduce two kinds of definitions of solutions to the skeleton equation (3.3).

Definition 4.1 (Entropy solution)**.** An entropy solution of (3.3) is a measurable function $u^h : [0,T] \to L^{m+1}(\mathbb{T}^N)$ such that

(i) $u^h \in L^{m+1}([0,T];L^{m+1}(\mathbb{T}^N))$,

(ii) for all $l \in C_b(\mathbb{R})$, we have $\Psi_l(u^h) \in L^2([0,T];H^1(\mathbb{T}^N))$ and

$$\partial_i\Psi_l(u^h) = l(u^h)\partial_i\Psi(u^h),$$

(iii) for all convex function $\eta \in C^2(\mathbb{R})$ with η'' compactly supported and all non-negative function $\phi \in C_c^2(\mathbb{T}^N \times [0,T))$, we have

$$\begin{aligned} &-\int_0^T\int_{\mathbb{T}^N}\eta(u^h)\partial_t\phi dxdt \\ &\leq \int_{\mathbb{T}^N}\eta(u_0)\phi(0)dx + \int_0^T\int_{\mathbb{T}^N}q_\eta(u)\Delta\phi dxdt - \int_0^T\int_{\mathbb{T}^N}\phi\eta''(u^h)|\nabla\Psi(u^h)|^2dxdt \\ &\quad + \sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N}g^k(x,u^h(x,t))\phi\eta'(u^h)h_k(t)dxdt, \end{aligned} \tag{4.1}$$

where q_η is any function satisfying $q'_\eta(\xi) = \eta'(\xi)a^2(\xi)$.

Definition 4.2 (Kinetic solution)**.** Let $u_0 \in L^{m+1}(\mathbb{T}^N)$. A measurable function $u^h : \mathbb{T}^N \times [0,T] \to \mathbb{R}$ is said to be a kinetic solution to (3.3), if

(1) there exists $C_m > 0$ such that

$$\int_0^T \|u^h(t)\|^{m+1}_{L^{m+1}(\mathbb{T}^N)} dt \leq C_m, \tag{4.2}$$

(2) for all $l \in C_b(\mathbb{R})$, we have $\Psi_l(u^h) \in L^2([0,T]; H^1(\mathbb{T}^N))$ and

$$\partial_i \Psi_l(u^h) = l(u^h)\partial_i \Psi(u^h). \tag{4.3}$$

(3) there exists a measure $m_h \in \mathcal{M}_0^+(\mathbb{T}^N \times [0,T) \times \mathbb{R})$ such that $f_h := I_{u^h > \xi}$ satisfies that for all $\varphi \in C_c^2(\mathbb{T}^N \times [0,T] \times \mathbb{R})$,

$$\begin{aligned} &\int_0^T \langle f_h(t), \partial_t \varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f_h(t), a^2(\xi)\Delta\varphi(t)\rangle dt \\ = &-\sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} g^k(x, u^h(x,t))\varphi(x,t,u^h(x,t)) h_k(t) dx dt \\ &+ m_h(\partial_\xi \varphi) + n_h(\partial_\xi \varphi), \end{aligned} \tag{4.4}$$

where $f_0(x,\xi) = I_{u_0 > \xi}$ and $n_h \in \mathcal{M}_0^+(\mathbb{T}^N \times [0,T) \times \mathbb{R})$ satisfies that for any $\phi \in C_b(\mathbb{T}^N \times [0,T) \times \mathbb{R})$,

$$n_h(\phi) = \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \phi(x,t,\xi)|\nabla \Psi(u^h)|^2 d\delta_{u^h(t,x)=\xi} dx dt. \tag{4.5}$$

Similarly to Lemma 2.5, (4.4) can also be strengthened to be weak only respect to x and ξ, that is, for all $t \in [0,T]$ and $\varphi \in C_c^2(\mathbb{T}^N \times \mathbb{R})$

$$\begin{aligned} &-\langle f_h^+(t), \varphi\rangle + \langle f_0, \varphi\rangle + \int_0^t \langle f_h(s), a^2(\xi)\Delta\varphi\rangle ds \\ = &-\sum_{k\geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g^k(x,\xi)\varphi(x,\xi) h_k(s) d\nu^h_{x,s}(\xi) dx ds \\ &+ \langle m_h, \partial_\xi \varphi\rangle([0,t]) + \langle n_h, \partial_\xi \varphi\rangle([0,t]), \end{aligned} \tag{4.6}$$

where $\nu^h_{x,s}(\xi) = -\partial_\xi I_{u^h > \xi} = \delta_{u^h = \xi}$.

In the rest part, we shall prove the uniqueness of kinetic solutions to (3.3) firstly, then, based on the uniqueness, we show the existence part.

Consider

$$\begin{cases} d\tilde{u}^h(t,x) = \Delta(\tilde{A}(\tilde{u}^h))dt + \sum_{k\geq 1} \tilde{g}^k(x, \tilde{u}^h(t,x)) h_k(t) dt, \\ \tilde{u}^h(0) = \tilde{u}_0, \end{cases} \tag{4.7}$$

whose kinetic solution is denoted by $\tilde{u}^h$. Moreover, define

$$\tilde{a}(r) = \sqrt{\tilde{A}'(r)}, \quad \tilde{\Psi}(r) = \int_0^r \tilde{a}(s) ds.$$

In the following, we devote to proving a comparison theorem associated to kinetic solutions u^h and $\tilde{u}^h$. It states that

Proposition 4.1. Assume that (A, g, u_0) and $(\tilde{A}, \tilde{g}, \tilde{u}_0)$ satisfy Hypothesis H, and let u^h, $\tilde{u}^h$ be kinetic solutions of (3.3) and (4.7), respectively. Set $f_1 = I_{u^h>\xi}$, $f_2 = I_{\tilde{u}^h>\xi}$, and the corresponding measures are denoted by m_1, n_1 and m_2, n_2. Then, for any $0 < t < T$, and non-negative test functions $\rho \in C^\infty(\mathbb{T}^N), \psi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)\Big(f_1^\pm(x,t,\xi)\bar{f}_2^\pm(y,t,\zeta) + \bar{f}_1^\pm(x,t,\xi)f_2^\pm(y,t,\zeta)\Big)d\xi d\zeta dxdy \\
\le &\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)\Big(f_{1,0}(x,\xi)\bar{f}_{2,0}(y,\zeta) + \bar{f}_{1,0}(x,\xi)f_{2,0}(y,\zeta)\Big)d\xi d\zeta dxdy \\
&+ 2K_1 + K_2 + K_3 + 2K_4, \qquad (4.8)
\end{aligned}$$

where

$$\begin{aligned}
K_1 =& -\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)d\nu_{x,s}^1(\xi)dxdn_2(y,\zeta,s) \\
&-\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)d\nu_{y,s}^2(\zeta)dydn_1(x,\xi,s), \\
K_2 =& \int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\bar{f}_2(a^2(\xi)+\tilde{a}^2(\zeta))\Delta_x\rho(x-y)\psi(\xi-\zeta)d\xi d\zeta dxdyds, \\
K_3 =& \int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_1 f_2(a^2(\xi)+\tilde{a}^2(\zeta))\Delta_x\rho(x-y)\psi(\xi-\zeta)d\xi d\zeta dxdyds, \\
K_4 =& \sum_{k\ge1}\int_0^t \int_{(\mathbb{T}^N)^2}\rho(x-y)\int_{\mathbb{R}^2} \chi_1(\xi,\zeta)(g^k(x,\xi)-\tilde{g}^k(y,\zeta))h_k(s) \\
&\qquad d\nu_{x,s}^1\otimes d\nu_{y,s}^2(\xi,\zeta)dxdyds,
\end{aligned}$$

with

$$\begin{aligned}
\chi_1(\xi,\zeta) = \int_{-\infty}^{\xi}\psi(\xi'-\zeta)d\xi' = \int_{-\infty}^{\xi-\zeta}\psi(y)dy,\ f_{1,0}(x,\xi) = I_{u_0(x)>\xi}, \\
f_{2,0}(x,\xi) = I_{\tilde{u}_0(x)>\xi}, \quad \nu_{x,s}^1(\xi) = \delta_{u^h=\xi}, \quad \nu_{y,s}^2(\zeta) = \delta_{\tilde{u}^h=\zeta}.
\end{aligned}$$

Proof. Let $\varphi_1 \in C_c^\infty(\mathbb{T}_x^N\times\mathbb{R}_\xi)$ and $\varphi_2 \in C_c^\infty(\mathbb{T}_y^N\times\mathbb{R}_\zeta)$. By (4.6), we have

$$\begin{aligned}
\langle f_1^+(t),\varphi_1\rangle = &\langle f_{1,0},\varphi_1\rangle + \int_0^t \langle f_1(s), a^2(\xi)\Delta_x\varphi_1(s)\rangle ds \\
&+ \sum_{k\ge1}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} g^k(x,\xi)\varphi_1(x,\xi)h_k(s)d\nu_{x,s}^1(\xi)dxds \\
&- \langle m_1, \partial_\xi\varphi_1\rangle([0,t]) - \langle n_1, \partial_\xi\varphi_1\rangle([0,t]).
\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}
\langle \bar{f}_2^+(t),\varphi_2\rangle = &\langle \bar{f}_{2,0},\varphi_2\rangle + \int_0^t \langle \bar{f}_2(s), \tilde{a}^2(\zeta)\Delta_y\varphi_2(s)\rangle ds \\
&- \sum_{k\ge1}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} \tilde{g}^k(y,\zeta)\varphi_2(y,\zeta)h_k(s)d\nu_{y,s}^2(\zeta)dyds \\
&+ \langle m_2, \partial_\zeta\varphi_2\rangle([0,t]) + \langle n_2, \partial_\zeta\varphi_2\rangle([0,t]).
\end{aligned}$$

Denote the duality distribution over $\mathbb{T}_x^N\times\mathbb{R}_\xi\times\mathbb{T}_y^N\times\mathbb{R}_\zeta$ by $\langle\langle\cdot,\cdot\rangle\rangle$. Setting $\alpha(x,\xi,y,\zeta) =$

$\varphi_1(x,\xi)\varphi_2(y,\zeta)$ and using the integration by parts formula, we have

$$\begin{aligned}
\langle\langle f_1^+(t)\bar{f}_2^+(t),\alpha\rangle\rangle &= \langle\langle f_{1,0}\bar{f}_{2,0},\alpha\rangle\rangle + \int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\bar{f}_2(a^2(\xi)\Delta_x+\tilde{a}^2(\zeta)\Delta_y)\alpha d\xi d\zeta dxdyds \\
&\quad -\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+(s,x,\xi)\alpha\tilde{g}^k(y,\zeta)h_k(s)d\xi d\nu_{y,s}^2(\zeta)dxdyds \\
&\quad +\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_2^+(s,y,\zeta)\alpha g^k(x,\xi)h_k(s)d\zeta d\nu_{x,s}^1(\xi)dxdyds \\
&\quad +\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+(s,x,\xi)\partial_\zeta\alpha d\xi dxdm_2(y,\zeta,s) \\
&\quad -\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_2^+(s,y,\zeta)\partial_\xi\alpha d\zeta dydm_1(x,\xi,s) \\
&\quad +\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+(s,x,\xi)\partial_\zeta\alpha d\xi dxdn_2(y,\zeta,s) \\
&\quad -\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_2^+(s,y,\zeta)\partial_\xi\alpha d\zeta dydn_1(x,\xi,s) \\
&=: \langle\langle f_{1,0}\bar{f}_{2,0},\alpha\rangle\rangle + I_1+I_2+I_3+I_4+I_5+I_6+I_7. \qquad (4.9)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\langle\langle \bar{f}_1^+(t)f_2^+(t),\alpha\rangle\rangle &= \langle\langle \bar{f}_{1,0}f_{2,0},\alpha\rangle\rangle + \int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_1 f_2(a^2(\xi)\Delta_x+\tilde{a}^2(\zeta)\Delta_y)\alpha d\xi d\zeta dxdyds \\
&\quad +\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_1^+(s,x,\xi)\alpha\tilde{g}^k(y,\zeta)h_k(s)d\xi d\nu_{y,s}^2(\zeta)dxdyds \\
&\quad -\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_2^+(s,y,\zeta)\alpha g^k(x,\xi)h_k(s)d\nu_{x,s}^1(\xi)d\zeta dxdyds \\
&\quad -\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_1^+(s,x,\xi)\partial_\zeta\alpha d\xi dxdm_2(y,\zeta,s) \\
&\quad +\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_2^+(s,y,\zeta)\partial_\xi\alpha d\zeta dydm_1(x,\xi,s) \\
&\quad -\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_1^+(s,x,\xi)\partial_\zeta\alpha d\xi dxdn_2(y,\zeta,s) \\
&\quad +\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_2^+(s,y,\zeta)\partial_\xi\alpha d\zeta dydn_1(x,\xi,s) \\
&=: \langle\langle \bar{f}_{1,0}f_{2,0},\alpha\rangle\rangle + \bar{I}_1+\bar{I}_2+\bar{I}_3+\bar{I}_4+\bar{I}_5+\bar{I}_6+\bar{I}_7. \qquad (4.10)
\end{aligned}$$

By a density argument, (4.9) and (4.10) remain true for any test function $\alpha\in C_c^\infty(\mathbb{T}_x^N\times\mathbb{R}_\xi\times\mathbb{T}_y^N\times\mathbb{R}_\zeta)$. The assumption that α is compactly supported can be relaxed thanks to (2.11), (4.2) and (2) in Definition 4.2. Using a truncation argument of α, it is easy to see that (4.9) and (4.10) remain true if $\alpha\in C_b^\infty(\mathbb{T}_x^N\times\mathbb{R}_\xi\times\mathbb{T}_y^N\times\mathbb{R}_\zeta)$ is compactly supported in a neighbourhood of the diagonal

$$\Big\{(x,\xi,x,\xi); x\in\mathbb{T}^N,\xi\in\mathbb{R}\Big\}.$$

Taking $\alpha=\rho(x-y)\psi(\xi-\zeta)$, then we have the following remarkable identities

$$(\nabla_x+\nabla_y)\alpha=0,\quad (\partial_\xi+\partial_\zeta)\alpha=0. \qquad (4.11)$$

With the aid of (4.11), we get

$$
\begin{aligned}
I_4 = & -\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^+(s,x,\xi)\partial_\xi \alpha d\xi dx dm_2(y,\zeta,s) \\
= & -\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha d\nu^1_{x,s}(\xi) dx dm_2(y,\zeta,s) \leq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
I_5 = & \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s,y,\zeta)\partial_\zeta \alpha dm_1(x,\xi,s) d\zeta dy \\
= & -\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha dm_1(x,\xi,s) d\nu^2_{y,s}(\zeta) dy \leq 0.
\end{aligned}
$$

Similarly, we have $\bar{I}_4 + \bar{I}_5 \leq 0$. By utilizing (4.11) again, it follows that

$$
\begin{aligned}
I_6 + I_7 = & -\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha d\nu^1_{x,s}(\xi) dx dn_2(y,\zeta,s) \\
& -\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha d\nu^2_{y,s}(\zeta) dy dn_1(x,\xi,s) \\
=: & \, K_1.
\end{aligned}
$$

By the same argument, we have $\bar{I}_6 + \bar{I}_7 = K_1$. Moreover, it is readily to deduce from (4.11) that

$$
\begin{aligned}
I_1 = & \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a^2(\xi) + \tilde{a}^2(\zeta)) \Delta_x \alpha d\xi d\zeta dx dy ds \\
=: & \, K_2,
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{I}_1 = & \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_1 f_2 (a^2(\xi) + \tilde{a}^2(\zeta)) \Delta_x \alpha d\xi d\zeta dx dy ds \\
=: & \, K_3.
\end{aligned}
$$

For some $\xi, \zeta \in \mathbb{R}$, set

$$
\chi_1(\xi,\zeta) = \int_{-\infty}^{\xi} \psi(\xi' - \zeta) d\xi',
$$

then, I_2 can be written as

$$
\begin{aligned}
I_2 = & -\sum_{k\geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^+(s,x,\xi)\rho(x-y)\partial_\xi \chi_1(\xi,\zeta) \tilde{g}^k(y,\zeta) h_k(s) d\xi d\nu^2_{y,s}(\zeta) dx dy ds \\
= & -\sum_{k\geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}} \rho(x-y) \tilde{g}^k(y,\zeta) h_k(s) \Big(\int_{\mathbb{R}} f_1^+(s,x,\xi) \partial_\xi \chi_1(\xi,\zeta) d\xi \Big) d\nu^2_{y,s}(\zeta) dx dy ds \\
= & -\sum_{k\geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x-y) \chi_1(\xi,\zeta) \tilde{g}^k(y,\zeta) h_k(s) d\nu^1_{x,s} \otimes d\nu^2_{y,s}(\xi,\zeta) dx dy ds. \qquad (4.12)
\end{aligned}
$$

Similarly, for $\xi, \zeta \in \mathbb{R}$, let

$$
\chi_2(\zeta,\xi) = \int_{\zeta}^{\infty} \psi(\xi - \zeta') d\zeta',
$$

it yields

$$
\begin{aligned}
I_3 =& -\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^+(s,y,\zeta)\rho(x-y)\partial_\zeta\chi_2(\zeta,\xi)g^k(x,\xi)h_k(s)d\nu^1_{x,s}(\xi)d\zeta dxdyds\\
=& -\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho(x-y)g^k(x,\xi)h_k(s)\Big(\int_{\mathbb{R}}\bar{f}_2^+(s,y,\zeta)\partial_\zeta\chi_2(\zeta,\xi)d\zeta\Big)d\nu^1_{x,s}(\xi)dxdyds\\
=&\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2(\zeta,\xi)\rho(x-y)g^k(x,\xi)h_k(s)d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds.
\end{aligned}\tag{4.13}
$$

Note that $\chi_1(\xi,\zeta)=\chi_2(\zeta,\xi)=\int_{-\infty}^{\xi-\zeta}\psi(y)dy$. Combining (4.12) and (4.13), it follows that

$$
\begin{aligned}
&I_2+I_3\\
=&\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\rho(x-y)\int_{\mathbb{R}^2}\chi_1(\xi,\zeta)(g^k(x,\xi)-\tilde{g}^k(y,\zeta))h_k(s)d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
=:&\, K_4.
\end{aligned}
$$

Similarly, we have $\bar{I}_2+\bar{I}_3=K_4$. Based on the above, the equation (4.8) is established for f_i^+. To obtain the result for f_i^-, we take $t_n\uparrow t$, write (4.8) for $f_i^+(t_n)$ and let $n\to\infty$. $\square$

Theorem 4.2. Assume that (A,g,u_0) and $(\tilde{A},\tilde{g},\tilde{u}_0)$ satisfy Hypothesis H and let $u:=u^h,\tilde{u}:=\tilde{u}^h$ be kinetic solutions to (3.3) and (4.7), respectively. We claim that

(1) if $A=\tilde{A}$ and $g=\tilde{g}$, then for a.e. $t\in[0,T]$,

$$
\|u(t)-\tilde{u}(t)\|_{L^1(\mathbb{T}^N)}\leq e^{K(T+M)}\|u_0-\tilde{u}_0\|_{L^1(\mathbb{T}^N)},\tag{4.14}
$$

(2) furthermore, for all $\gamma,\delta\in(0,1)$, $\lambda\in[0,1]$ and $\alpha\in(0,1\wedge\frac{m}{2})$, we have for a.e. $t\in[0,T]$,

$$
\begin{aligned}
&\|u(t)-\tilde{u}(t)\|_{L^1(\mathbb{T}^N)}\\
\leq&\mathcal{E}_t(\gamma,\delta)+e^{K(T+M)}\Big[\|u_0-\tilde{u}_0\|_{L^1(\mathbb{T}^N)}+\mathcal{E}_0(\gamma,\delta)\\
&+N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big(1+\|u\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|\tilde{u}\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\
&+N_0\gamma^{-2}\|I_{|u|\geq R_\lambda}(1+|u|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}+N_0\gamma^{-2}\|I_{|\tilde{u}|\geq R_\lambda}(1+|\tilde{u}|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}\\
&+K(\gamma+2\delta)(T+M)+d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)(T+M)\Big],
\end{aligned}
$$

where $\mathcal{E}_0(\gamma,\delta),\mathcal{E}_t(\gamma,\delta)\to 0$ as $\gamma,\delta\to 0$, the constant N_0 is independent of γ,δ,λ, and R_λ is defined by

$$
R_\lambda:=\sup\{R\in[0,\infty]:|a(r)-\tilde{a}(r)|\leq\lambda,\forall|r|<R\}.
$$

Proof of Theorem 4.2. Let ρ_γ,ψ_δ be approximations to the identity on $\mathbb{T}^N$ and $\mathbb{R}$, respectively. That is, let $\rho\in C^\infty(\mathbb{T}^N)$, $\psi\in C_c^\infty(\mathbb{R})$ be symmetric nonnegative functions such as $\int_{\mathbb{T}^N}\rho=1$, $\int_{\mathbb{R}}\psi=1$ and supp$\psi\subset(-1,1)$. We define

$$
\rho_\gamma(x)=\frac{1}{\gamma^N}\rho\Big(\frac{x}{\gamma}\Big),\quad \psi_\delta(\xi)=\frac{1}{\delta}\psi\Big(\frac{\xi}{\delta}\Big).
$$

Letting $\rho := \rho_\gamma(x-y)$ and $\psi := \psi_\delta(\xi-\zeta)$ in Proposition 4.1, we get from (4.8) that

$$\begin{aligned}
&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))d\xi d\zeta dxdy\\
\le&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_{1,0}(x,\xi)\bar{f}_{2,0}(y,\zeta)+\bar{f}_{1,0}(x,\xi)f_{2,0}(y,\zeta))d\xi d\zeta dxdy\\
&+2R_1+R_2+R_3+2R_4, \qquad (4.15)
\end{aligned}$$

where R_1, R_2, R_3, R_4 in (4.15) are the corresponding K_1, K_2, K_3, K_4 in the statement of Proposition 4.1 with ρ, ψ replaced by ρ_γ, ψ_δ, respectively. For simplicity, we still denote by $\chi_1(\xi,\zeta)$ with ρ, ψ replaced by ρ_γ, ψ_δ.

For any $t\in[0,T]$, define the error term

$$\begin{aligned}
&\mathcal{E}_t(\gamma,\delta)\\
:=&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta\\
&-\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))d\xi dx. \qquad (4.16)
\end{aligned}$$

By utilizing $\int_{\mathbb{R}}\psi_\delta(\xi-\zeta)d\zeta=1$ and $\int_0^\delta\psi_\delta(\zeta')d\zeta'=\int_{-\delta}^0\psi_\delta(\zeta')d\zeta'=\frac{1}{2}$, we get

$$\begin{aligned}
&\Big|\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\xi)d\xi dxdy\\
&\quad-\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta\Big|\\
=&\Big|\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}}f_1^{\pm}(x,t,\xi)\int_{\mathbb{R}}\psi_\delta(\xi-\zeta)(\bar{f}_2^{\pm}(y,t,\xi)-\bar{f}_2^{\pm}(y,t,\zeta))d\zeta d\xi dxdy\Big|\\
\le&\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}}f_1^{\pm}(x,t,\xi)\int_{\xi-\delta}^{\xi}\psi_\delta(\xi-\zeta)(\bar{f}_2^{\pm}(y,t,\xi)-\bar{f}_2^{\pm}(y,t,\zeta))d\zeta d\xi dxdy\\
&+\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)f_1^{\pm}(x,t,\xi)\int_{\xi}^{\xi+\delta}\psi_\delta(\xi-\zeta)(\bar{f}_2^{\pm}(y,t,\zeta)-\bar{f}_2^{\pm}(y,t,\xi))d\zeta d\xi dxdy\\
\le&\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_0^\delta\psi_\delta(\zeta')\int_{\mathbb{R}}f_1^{\pm}(x,t,\xi)(\bar{f}_2^{\pm}(y,t,\xi)-\bar{f}_2^{\pm}(y,t,\xi-\zeta'))d\xi d\zeta' dxdy\\
&+\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{-\delta}^0\psi_\delta(\zeta')\int_{\mathbb{R}}f_1^{\pm}(x,t,\xi)(\bar{f}_2^{\pm}(y,t,\xi-\zeta')-\bar{f}_2^{\pm}(y,t,\xi))d\xi d\zeta' dxdy\\
\le&\,\delta\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\Big(\int_0^\delta\psi_\delta(\zeta')d\zeta'\Big)\Big(\int_{\mathbb{R}}\nu^2_{y,t}(d\xi)\Big)dxdy\\
&+\delta\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\Big(\int_{-\delta}^0\psi_\delta(\zeta')d\zeta'\Big)\Big(\int_{\mathbb{R}}\nu^2_{y,t}(d\xi)\Big)dxdy\\
\le&\frac{1}{2}\delta+\frac{1}{2}\delta=\delta, \qquad (4.17)
\end{aligned}$$

where we have taken into account the facts that $\bar{f}_2^{\pm}(y,t,\xi)$ is increasing in ξ, and $f_1^{\pm}(x,t,\xi)\le 1$. Similarly, it yields

$$\begin{aligned}
&\Big|\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\xi)d\xi dxdy\\
&-\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta)\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta\Big|\le\delta. \qquad (4.18)
\end{aligned}$$

Moreover, when γ is small enough, it follows that

$$\begin{aligned}
&\Big|\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\xi)d\xi dydx-\int_{\mathbb{T}^N}\int_{\mathbb{R}}f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)d\xi dx\Big|\\
=&\Big|\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)f_1^{\pm}(x,t,\xi)(\bar{f}_2^{\pm}(y,t,\xi)-\bar{f}_2^{\pm}(x,t,\xi))d\xi dydx\Big|\\
\le&\sup_{|z|<\gamma}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f_1^{\pm}(x,t,\xi)|\bar{f}_2^{\pm}(x-z,t,\xi)-\bar{f}_2^{\pm}(x,t,\xi)|d\xi dx\\
\le&\sup_{|z|<\gamma}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|-f_2^{\pm}(x-z,t,\xi)+I_{0>\xi}-I_{0>\xi}+f_2^{\pm}(x,t,\xi)|d\xi dx\\
=&\sup_{|z|<\gamma}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\Lambda_{f_2^{\pm}}(x-z,t,\xi)-\Lambda_{f_2^{\pm}}(x,t,\xi)|d\xi dx.
\end{aligned}$$

In view of the integrability of Λ_{f_2}, we have

$$\begin{aligned}
&\lim_{\gamma\to0}\Big|\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\xi)d\xi dxdy-\int_{\mathbb{T}^N}\int_{\mathbb{R}}f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)d\xi dx\Big|\\
&=0. \qquad (4.19)
\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}
&\lim_{\gamma\to0}\Big|\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\xi)d\xi dxdy-\int_{\mathbb{T}^N}\int_{\mathbb{R}}\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi)d\xi dx\Big|\\
&=0. \qquad (4.20)
\end{aligned}$$

Based on (4.17)-(4.20), we have

$$\lim_{\gamma,\delta\to0}\mathcal{E}_t(\gamma,\delta)=0. \qquad (4.21)$$

In particular, when $t=0$, we get

$$\lim_{\gamma,\delta\to0}\mathcal{E}_0(\gamma,\delta)=0. \qquad (4.22)$$

From now on, we devote to making estimates of R_1, R_2, R_3 and R_4. Notice that

$$\begin{aligned}
R_1+R_2=&\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar{f}_2(a(\xi)-\tilde{a}(\zeta))^2\Delta_x\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)d\xi d\zeta dxdyds\\
&+2\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar{f}_2a(\xi)\tilde{a}(\zeta)\Delta_x\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)d\xi d\zeta dxdyds\\
&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)d\nu^1_{x,s}(\xi)dxdn_2(y,\zeta,s)\\
&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)d\nu^2_{y,s}(\zeta)dydn_1(x,\xi,s)\\
=:&\,J_1+J_2+J_3+J_4.
\end{aligned}$$

We will prove $J_2+J_3+J_4\le0$ by using similar method as [13]. From the definition of n_1 and n_2, we have

$$\begin{aligned}
J_3+J_4=&-\int_0^t\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\psi_\delta(u-\tilde{u})|\nabla_y\tilde{\Psi}(\tilde{u})|^2dxdyds\\
&-\int_0^t\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\psi_\delta(u-\tilde{u})|\nabla_x\Psi(u)|^2dxdyds.
\end{aligned}$$

Due to the chain formula (4.3), for any $l\in C_b(\mathbb{R})$, it yields

$$\partial_i\int_{\mathbb{R}} fl(\xi)a(\xi)d\xi=\partial_i\int_{\mathbb{R}}\Lambda_f l(\xi)a(\xi)d\xi=\partial_i\Psi_l(u)=l(u)\partial_i\Psi(u),$$

where $\Lambda_f=f-I_{0>\xi}$. Then, we deduce that

$$\begin{aligned}J_2&=2\sum_{i=1}^N\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(\partial_{x_i}f_1)a(\xi)\tilde a(\zeta)(\partial_{y_i}f_2)\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)d\xi d\zeta dxdyds\\&=2\sum_{i=1}^N\int_0^t\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\psi_\delta(u-\tilde u)\partial_{x_i}\Psi(u)\partial_{y_i}\tilde\Psi(\tilde u)dxdyds.\end{aligned}$$

Based on the above estimates, we get

$$\begin{aligned}J_2+J_3+J_4=&\ -\sum_{i=1}^N\int_0^t\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\psi_\delta(u-\tilde u)|\partial_{y_i}\tilde\Psi(\tilde u)-\partial_{x_i}\Psi(u)|^2dxdyds\\&\le 0.\end{aligned}$$

Thus, $R_1+R_2\le J_1$. By the same method as above, we have

$$R_1+R_3\le\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar f_1f_2(a(\xi)-\tilde a(\zeta))^2\Delta_x\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)d\xi d\zeta dxdyds.$$

Therefore,

$$\begin{aligned}&2R_1+R_2+R_3\\&\le\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(f_1\bar f_2+\bar f_1f_2)(a(\xi)-\tilde a(\zeta))^2\Delta_x\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)d\xi d\zeta dxdyds\\&\le 2\sum_{i=1}^N\int_0^t\int_{(\mathbb{T}^N)^2}|\partial^2_{x_iy_i}\rho_\gamma(x-y)|\int_\xi^\zeta\int_\xi^\zeta\psi_\delta(\xi'-\zeta')|a(\xi')-\tilde a(\zeta')|^2d\xi'd\zeta'\\&\qquad d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds.\end{aligned}$$

By similar arguments as in the proof of (4.18) in [11], we get for all $\alpha\in(0,1\wedge\frac{m}{2})$, there exists a positive constant N_0 independent of γ,δ,λ such that

$$\begin{aligned}&2R_1+R_2+R_3\\&\le N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big[1+\int_0^T\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(|\xi|^m+|\zeta|^m)d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\Big]\\&\quad+N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}I_{|\xi|\ge R_\lambda}(1+|\xi|)^m d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\&\quad+N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}I_{|\zeta|\ge R_\lambda}(1+|\zeta|)^m d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\&\le N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big(1+\|u\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|\tilde u\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\&\quad+N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2}I_{|u|\ge R_\lambda}(1+|u|)^m dxdyds\\&\quad+N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2}I_{|\tilde u|\ge R_\lambda}(1+|\tilde u|)^m dxdyds.\end{aligned}\tag{4.23}$$

For the remain term R_4, it can be bounded as

$$
\begin{aligned}
&R_4\\
&\leq \int_0^t \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y) \int_{\mathbb{R}^2} \chi_1(\xi,\zeta) \sum_{k\geq 1} |g^k(x,\xi)-\tilde{g}^k(y,\zeta)||h_k(s)| d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
&\leq \int_0^t \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y) \int_{\mathbb{R}^2} \chi_1(\xi,\zeta) \sum_{k\geq 1} |g^k(x,\xi)-g^k(y,\zeta)||h_k(s)| d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
&\quad + \int_0^t \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y) \int_{\mathbb{R}^2} \chi_1(\xi,\zeta) \sum_{k\geq 1} |g^k(y,\zeta)-\tilde{g}^k(y,\zeta)||h_k(s)|\\
&\qquad\qquad d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
&:= L_1+L_2.
\end{aligned}
\tag{4.24}
$$

By Hölder inequality and (2.5), we deduce that

$$
\begin{aligned}
L_1 \leq\ & \int_0^t \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y) \int_{\mathbb{R}^2} \chi_1(\xi,\zeta) \Big(\sum_{k\geq 1} |g^k(x,\xi)-g^k(y,\zeta)|^2\Big)^{\frac{1}{2}} \Big(\sum_{k\geq 1} |h_k(s)|^2\Big)^{\frac{1}{2}}\\
&\qquad d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
\leq\ & K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y)|x-y| \int_{\mathbb{R}^2} \chi_1(\xi,\zeta) d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
&+ K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y) \int_{\mathbb{R}^2} \chi_1(\xi,\zeta)|\xi-\zeta| d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
=:\ & L_{1,1}+L_{1,2}.
\end{aligned}
$$

By utilizing

$$
\begin{aligned}
\int_{\mathbb{R}^2} \chi_1(\xi,\zeta) d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta) \leq 1,\\
\int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y)|x-y| dxdy \leq \gamma,
\end{aligned}
$$

we get

$$
L_{1,1} \leq K\gamma \int_0^t |h(s)|_U ds. \tag{4.25}
$$

To dealing with the term $L_{1,2}$, we adopt the similar method as [14]. Taking into account $\nu^1=\delta_{u=\xi}$ and $\nu^2=\delta_{\tilde{u}=\zeta}$, it follows that

$$
\begin{aligned}
L_{1,2} \leq\ & K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\gamma(x-y)|\xi-\zeta| d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds\\
=\ & K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y)(u-\tilde{u})^+ dxdyds\\
&+ K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y)(u-\tilde{u})^- dxdyds\\
=\ & K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}} \rho_\gamma(x-y)(f_1^{\pm}(x,s,\xi)\bar{f}_2^{\pm}(y,s,\xi)+\bar{f}_1^{\pm}(x,s,\xi)f_2^{\pm}(y,s,\xi))\\
&\qquad d\xi dxdyds,
\end{aligned}
$$

where we have used the following identities

$$\int_{\mathbb{R}} I_{u>\xi}\overline{I_{\tilde{u}>\xi}}d\xi = (u-\tilde{u})^{+}, \quad \int_{\mathbb{R}} \overline{I_{u>\xi}}I_{\tilde{u}>\xi}d\xi = (u-\tilde{u})^{-}. \tag{4.26}$$

Then, we deduce from (4.17) and (4.18) that

$$\begin{aligned} L_{1,2} \leq\ & 2\delta K\int_0^t |h(s)|_U ds \\ & + K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} (f_1^{\pm}\bar{f}_2^{\pm} + \bar{f}_1^{\pm}f_2^{\pm})\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta ds. \end{aligned} \tag{4.27}$$

Hence, combining (4.25) and (4.27), we get

$$\begin{aligned} L_1 \leq\ & K(\gamma+2\delta)\int_0^t |h(s)|_U ds \\ & + K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} (f_1^{\pm}\bar{f}_2^{\pm} + \bar{f}_1^{\pm}f_2^{\pm})\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta ds. \end{aligned} \tag{4.28}$$

Utilizing Hölder inequality, we deduce that

$$\begin{aligned} L_2 \leq\ & \int_0^t \int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y)\int_{\mathbb{R}^2}\chi_1(\xi,\zeta)\Big(\sum_{k\geq 1}|g^k(y,\zeta)-\tilde{g}^k(y,\zeta)|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\geq 1}|h_k(s)|^2\Big)^{\frac{1}{2}} \\ & \qquad d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds \\ \leq\ & \int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}^2} d^{\frac{1}{2}}(g,\tilde{g})(1+|\zeta|)^{\frac{m+1}{2}}d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds \\ \leq\ & d^{\frac{1}{2}}(g,\tilde{g})\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}^2}(1+|\zeta|)^{\frac{m+1}{2}}d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds \\ \leq\ & d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)\int_0^t |h(s)|_U\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)dxdyds \\ \leq\ & d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)\int_0^t |h(s)|_U ds. \end{aligned} \tag{4.29}$$

Combining (4.24), (4.28) and (4.29), it yields

$$\begin{aligned} R_4 \leq\ & K(\gamma+2\delta)\int_0^t |h(s)|_U ds + d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)\int_0^t |h(s)|_U ds \\ & + K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} (f_1^{\pm}\bar{f}_2^{\pm} + \bar{f}_1^{\pm}f_2^{\pm})\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta ds. \end{aligned} \tag{4.30}$$

Collecting (4.15), (4.23) and (4.30), we conclude that

$$\begin{aligned} & \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm})(y,t,\zeta)d\xi d\zeta dxdy \\ \leq\ & \int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})d\xi dx + \mathcal{E}_0(\gamma,\delta) \\ & + N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big(1+\|u\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|\tilde{u}\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big) \\ & + N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2} I_{|u|\geq R_\lambda}(1+|u|)^m dxdyds \\ & + N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2} I_{|\tilde{u}|\geq R_\lambda}(1+|\tilde{u}|)^m dxdyds \end{aligned}$$

$$+ 2K(\gamma+2\delta)\int_0^t |h(s)|_U ds + 2d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)\int_0^t |h(s)|_U ds$$
$$+ 2K\int_0^t |h(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} (f_1^{\pm}\bar{f}_2^{\pm}+\bar{f}_1^{\pm}f_2^{\pm})\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta ds,$$

where $\mathcal{E}_0(\gamma,\delta)$ is defined by (4.16).

Applying Gronwall inequality and utilizing $\int_0^t |h(s)|_U ds \le (T+M)/2$, we obtain

$$\begin{aligned}
&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))d\xi d\zeta dxdy\\
&\le e^{K(T+M)}\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})d\xi dx+\mathcal{E}_0(\gamma,\delta)\\
&\quad + N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big(1+\|u\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|\tilde{u}\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\
&\quad + N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2} I_{|u|\ge R_\lambda}(1+|u|)^m dxdyds\\
&\quad + N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2} I_{|\tilde{u}|\ge R_\lambda}(1+|\tilde{u}|)^m dxdyds\\
&\quad + K(\gamma+2\delta)(T+M)+d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)(T+M)\Big].
\end{aligned}\tag{4.31}$$

Then, it follows that

$$\begin{aligned}
&\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))dxd\xi\\
&= \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))d\xi d\zeta dxdy\\
&\quad + \mathcal{E}_t(\gamma,\delta)\\
&\le \mathcal{E}_t(\gamma,\delta)+e^{K(T+M)}\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})d\xi dx+\mathcal{E}_0(\gamma,\delta)\\
&\quad + N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big(1+\|u\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|\tilde{u}\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\
&\quad + N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2} I_{|u|\ge R_\lambda}(1+|u|)^m dxdyds\\
&\quad + N_0\gamma^{-2}\int_0^T\int_{(\mathbb{T}^N)^2} I_{|\tilde{u}|\ge R_\lambda}(1+|\tilde{u}|)^m dxdyds\\
&\quad + K(\gamma+2\delta)(T+M)+d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)(T+M)\Big],
\end{aligned}\tag{4.32}$$

where $\mathcal{E}_0(\gamma,\delta),\mathcal{E}_t(\gamma,\delta)\to 0$, as $\gamma,\delta\to 0$.

When $A=\tilde{A}, g=\tilde{g}$, we can take $\lambda=0$ and $R_\lambda=+\infty$. Then, by (4.2), we deduce from (4.32) that

$$\begin{aligned}
&\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))dxd\xi\\
&\le \mathcal{E}_t(\gamma,\delta)+e^{K(T+M)}\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})d\xi dx+\mathcal{E}_0(\gamma,\delta)\\
&\quad + N_0C_m\gamma^{-2}\delta^{2\alpha}+K(\gamma+2\delta)(T+M)\Big].
\end{aligned}$$

Taking $\delta=\gamma^{\frac{3}{2\alpha}}$, we get

$$\begin{aligned}&\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))dxd\xi\\ &\leq \mathcal{E}_t(\gamma,\delta)+e^{K(T+M)}\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})d\xi dx+\mathcal{E}_0(\gamma,\delta)\\ &\quad+N_0C_m\gamma+K(\gamma+2\gamma^{\frac{3}{2\alpha}})(T+M)\Big].\end{aligned}\tag{4.33}$$

Letting $\gamma\to 0$, we deduce from (4.33) that

$$\begin{aligned}&\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))dxd\xi\\ &\leq e^{K(T+M)}\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})d\xi dx.\end{aligned}\tag{4.34}$$

Since $f_1=I_{u>\xi}, f_2=I_{\tilde{u}>\xi}, f_{1,0}=I_{u_0>\xi}, f_{2,0}=I_{\tilde{u}_0>\xi}$, with the aid of (4.26), we deduce from (4.34) that

$$\|u(t)-\tilde{u}(t)\|_{L^1(\mathbb{T}^N)}\leq e^{K(T+M)}\|u_0-\tilde{u}_0\|_{L^1(\mathbb{T}^N)}.$$

We complete the proof of (1).

Now, it remains to prove (2). Utilizing (4.26) again, we deduce from (4.32) that

$$\begin{aligned}&\|u(t)-\tilde{u}(t)\|_{L^1(\mathbb{T}^N)}\\ &\leq \mathcal{E}_t(\gamma,\delta)+e^{K(T+M)}\Big[\|u_0-\tilde{u}_0\|_{L^1(\mathbb{T}^N)}+\mathcal{E}_0(\gamma,\delta)\\ &\quad+N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big(1+\|u\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|\tilde{u}\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\ &\quad+N_0\gamma^{-2}\|I_{|u|\geq R_\lambda}(1+|u|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}+N_0\gamma^{-2}\|I_{|\tilde{u}|\geq R_\lambda}(1+|\tilde{u}|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}\\ &\quad+K(\gamma+2\delta)(T+M)+d^{\frac{1}{2}}(g,\tilde{g})\Big(1+\|\tilde{u}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)(T+M)\Big],\end{aligned}$$

which implies (2). □

Now, we are in a position to show the uniqueness.

Theorem 4.3. Assume (A,g,u_0) satisfy Hypothesis H, then the skeleton equation (3.3) has at most one kinetic solution.

Proof of Theorem 4.3. Taking $u_0=\tilde{u}_0$ in (4.14), we get

$$\|u(t)-\tilde{u}(t)\|_{L^1(\mathbb{T}^N)}=0,\quad a.e.\ t\in[0,T],$$

which implies the uniqueness of solutions to (3.3). □

Now, we devote to proving the existence of kinetic solutions to (3.3).

Theorem 4.4 (Existence)**.** Assume (A,g,u_0) satisfy Hypothesis H, then for any $T>0$, (3.3) has a kinetic solution u^h on the time interval $[0,T]$.

Proof of Theorem 4.4. By the similar method as Proposition 2.6, we get the equivalence between entropy solution and kinetic solution to (3.3), hence, it suffices to prove the existence of entropy solutions. For technical reasons, we follow the spirit of Dareiotis et al. in [11] to introduce the approximations of the coefficients of (3.3). Define

$$g_n:=\rho_{1/n}^{\otimes(N+1)}*g(\cdot,-n\vee(\cdot\wedge n)),\quad u_{0,n}:=\rho_{1/n}^{\otimes N}*(-n\vee(u_0(\cdot)\wedge n)).\tag{4.35}$$

It's easy to verify that if g and u_0 satisfy Hypothesis H with a constant $K \geq 1$, then the same holds for g_n and $u_{0,n}$ with constant $2K$. It is also clear that $g_n \in C^\infty(\mathbb{T}^N \times \mathbb{R})$ with first order derivatives bounded by $C(n, K)$. Also, $u_{0,n}$ is a bounded $C^k(\mathbb{T}^N)$-valued function for any $k \in \mathbb{N}$ and

$$d(g, g_n) \to 0, \quad \|u_0 - u_{0,n}\|^{m+1}_{L^{m+1}(\mathbb{T}^N)} \to 0, \text{ as } n \to \infty. \tag{4.36}$$

Now, we focus on the approximation of A. Taking a symmetric mollifier $\bar{\rho}_\theta$ supported on $[-\theta, \theta]$, for instance, $\bar{\rho}_\theta(r) := \int_{\mathbb{R}} \rho_\theta(r+s)\rho_\theta(s)ds$. Set $\theta_n := \sup\{\theta \in (0,1] : |a(r) - a(\zeta)| \leq 1/n, \forall |r| \leq 3n, |\zeta - r| \leq 3\theta\} > 0$. Then, define

$$A_n(r) = \int_0^r a_n^2(\zeta)d\zeta, \quad \Psi_n(r) = \int_0^r a_n(s)ds, \quad a_n(r) = \bar{\rho}_{\theta_n} * (2/n + a(3\theta_n \vee |r| \wedge 3n)). \tag{4.37}$$

Referring to Proposition 5.1 in [11], we know that for all n, $a_n \in C^\infty(\mathbb{R})$ satisfying $a_n(r) \geq 2/n$,

$$\sup_{|r| \leq n} |a(r) - a_n(r)| \leq 4/n, \tag{4.38}$$

$$|a_n(r)| \leq C(n, m, K), \quad |a_n'(r)| \leq 2K|r|^{\frac{m-3}{2}}, \ \forall r \in \mathbb{R}. \tag{4.39}$$

Then, by (4.37), it yields $A_n \in C^\infty(\mathbb{R})$ with first order derivatives bounded by $C(n, m, K)$. Moreover, if A satisfies Hypothesis H with a constant $K \geq 1$, then A_n satisfy Hypothesis H with constant $3K$.

Based on $(A_n, g_n, u_{0,n})$ defined above, for $h(t) = \sum_{k \geq 1} h_k(t)e_k$, let us consider the following approximation

$$\begin{cases} du_n^h(t,x) = \Delta A_n(u_n^h(t,x))dt + \sum_{k \geq 1} g_n^k(x, u_n^h(t,x))h_k(t)dt, \\ u_n^h = u_{0,n}. \end{cases} \tag{4.40}$$

Note that $\Delta A_n(u_n^h(t,x)) = \mathrm{div}(a_n^2(u_n^h(t,x))\nabla u_n^h(t,x))$ and $\frac{4}{n^2} \leq a_n^2(u_n^h(t,x)) \leq C(n, m, K)$. Referring to Section 4 in [15] with $B = 0, A = a_n^2$, we know that for each n, (4.40) admits a unique solution $u_n^h \in C([0,T]; H) \cap L^2([0,T]; H^1(\mathbb{T}^N))$. Moreover, using integration by parts formula, we get the following energy estimates:

$$\sup_{t \in [0,T]} \|u_n^h(t)\|_H^p + \int_0^T \|\nabla \Psi_n(u_n^h(s))\|_H^p ds \leq C(M, K, T, p)(1 + \|u_{0,n}\|_H),$$
$$\sup_{t \in [0,T]} \|u_n^h(t)\|^{m+1}_{L^{m+1}(\mathbb{T}^N)} \leq C(M, K, T, m)(1 + \|u_{0,n}\|_{L^{m+1}(\mathbb{T}^N)}),$$

for all $p \geq 2$.

Applying integration by parts formula for the function $u \to \int_{\mathbb{T}^N} \int_0^u A_n(r)drdx$ and using the above estimates, it yields

$$\int_0^T \|\nabla A_n(u_n^h(s))\|_H^2 ds \leq C(M, K, T, m)(1 + \|u_{0,n}\|_{L^{m+1}(\mathbb{T}^N)}).$$

Note that $u_{0,n}$ are bounded by n, which implies that the right hand side of the above inequalities are finite. Moreover, since $u_{0,n} \leq u_0$, we get for all $p \geq 2$,

$$\sup_{t \in [0,T]} \|u_n^h(t)\|_H^p + \int_0^T \|\nabla \Psi_n(u_n^h(s))\|_H^p ds \leq C(M, K, T, p)(1 + \|u_0\|_H), \tag{4.41}$$

$$\sup_{t \in [0,T]} \|u_n^h(t)\|^{m+1}_{L^{m+1}(\mathbb{T}^N)} + \int_0^T \|\nabla A_n(u_n^h(s))\|_H^2 ds \leq C(M, K, T, m)(1 + \|u_0\|_{L^{m+1}(\mathbb{T}^N)}). \tag{4.42}$$

Since $a_n \geq \frac{2}{n} > 0$, we have $|\nabla u_n^h| \leq N(n)|\nabla \Psi_n(u_n^h)|$, then by (4.41), it yields

$$\int_0^T \|\nabla u_n^h(t)\|_H^p dt \leq N(n)C(M,K,T,p)(1+\|u_0\|_H). \tag{4.43}$$

In the following, for the sake of convenience, denote $u_n = u_n^h$ and $u = u^h$.

We will show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1([0,T];L^1(\mathbb{T}^N))$. For any $\alpha \in (0, 1\wedge \frac{m}{2})$, and $n \leq n'$, we apply Theorem 4.2 to u_n and $u_{n'}$ with setting $\delta = \gamma^{\frac{3}{2\alpha}}$ and $\lambda = \frac{8}{n}$. Due to (4.38), we get $R_\lambda \geq n$. By using (4.42), for a.e. $t \in [0,T]$, we have

$$\begin{aligned}
&\|u_n(t) - u_{n'}(t)\|_{L^1(\mathbb{T}^N)} \\
&\leq \mathcal{E}_t(\gamma,\delta) + e^{K(T+M)}\Big[\|u_{0,n} - u_{0,n'}\|_{L^1(\mathbb{T}^N)} + \mathcal{E}_0(\gamma,\delta) \\
&\quad + N_0(\gamma^{-2}n^{-2}+\gamma)\Big(1+\|u_n\|^m_{L^m([0,T]\times\mathbb{T}^N)} + \|u_{n'}\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big) \\
&\quad + N_0\gamma^{-2}\|I_{|u_n|\geq n}(1+|u_n|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} + N_0\gamma^{-2}\|I_{|u_{n'}|\geq n}(1+|u_{n'}|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} \\
&\quad + K(\gamma + 2\gamma^{\frac{3}{2\alpha}})(T+M) + d^{\frac{1}{2}}(g_n,g_{n'})\Big(1+\|u_{n'}\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)(T+M)\Big] \\
&\leq \mathcal{E}_t(\gamma,\gamma^{\frac{3}{2\alpha}}) + e^{K(T+M)}\Big[\|u_{0,n}-u_0\|_{L^1(\mathbb{T}^N)} + \|u_0 - u_{0,n'}\|_{L^1(\mathbb{T}^N)} + \mathcal{E}_0(\gamma,\gamma^{\frac{3}{2\alpha}}) \\
&\quad + N_0(\gamma^{-2}n^{-2}+\gamma)C_m + N_0\gamma^{-2}\|I_{|u_n|\geq n}(1+|u_n|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} \\
&\quad + N_0\gamma^{-2}\|I_{|u_{n'}|\geq n}(1+|u_{n'}|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} + K(\gamma+2\gamma^{\frac{3}{2\alpha}})(T+M) \\
&\quad + d^{\frac{1}{2}}(g_n,g)C_m(T+M) + d^{\frac{1}{2}}(g,g_{n'})C_m(T+M)\Big] \\
&=: M(\gamma) + e^{K(T+M)}\Big[\|u_{0,n}-u_0\|_{L^1(\mathbb{T}^N)} + \|u_0-u_{0,n'}\|_{L^1(\mathbb{T}^N)} + N_0\gamma^{-2}n^{-2}C_m \\
&\quad + N_0\gamma^{-2}\|I_{|u_n|\geq n}(1+|u_n|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} + N_0\gamma^{-2}\|I_{|u_{n'}|\geq n}(1+|u_{n'}|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} \\
&\quad + d^{\frac{1}{2}}(g_n,g)C_m(T+M) + d^{\frac{1}{2}}(g,g_{n'})C_m(T+M)\Big],
\end{aligned}$$

where

$$M(\gamma) = \mathcal{E}_t(\gamma,\gamma^{\frac{3}{2\alpha}}) + e^{K(T+M)}\Big[\mathcal{E}_0(\gamma,\gamma^{\frac{3}{2\alpha}}) + N_0\gamma C_m + K(\gamma+2\gamma^{\frac{3}{2\alpha}})(T+M)\Big] \to 0$$

as $\gamma \to 0$ and the constant N_0 is independent of $\gamma, \delta, \lambda, n, n'$.

For any $\iota > 0$, let $\gamma > 0$ be a constant such that $M(\gamma) < \iota$. By (4.36), we can choose n_0 big enough such that for $n_0 \leq n \leq n'$,

$$\begin{aligned}
&\|u_{0,n}-u_0\|_{L^1(\mathbb{T}^N)} + \|u_0-u_{0,n'}\|_{L^1(\mathbb{T}^N)} + N_0\gamma^{-2}n^{-2}C_m + d^{\frac{1}{2}}(g_n,g)C_m(T+M) \\
&\quad + d^{\frac{1}{2}}(g,g_{n'})C_m(T+M) \leq 5\iota.
\end{aligned}$$

Due to the uniform integrability of $(1+|u_n|)^m$ (see (4.42)), for such n_0, we also have

$$N_0\gamma^{-2}\|I_{|u_n|\geq n}(1+|u_n|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} + N_0\gamma^{-2}\|I_{|u_{n'}|\geq n}(1+|u_{n'}|)\|^m_{L^m([0,T]\times\mathbb{T}^N)} \leq \iota.$$

Hence, for $n_0 \leq n \leq n'$, we get for a.e. $t \in [0,T]$,

$$\|u_n(t) - u_{n'}(t)\|_{L^1(\mathbb{T}^N)} \leq 7\iota,$$

which implies that $(u_n)_{n\in\mathbb{N}}$ converges in $L^1([0,T];L^1(\mathbb{T}^N))$. Moreover, by passing to a subsequence, we may assume that

$$\lim_{n\to\infty} u_n = u, \quad a.e.\ (t,x) \in [0,T]\times\mathbb{T}^N. \tag{4.44}$$

In addition, it follows from (4.42) that for any $q < m+1$,

$$(|u_n(t,x)|^q)_{n\geq 1} \text{ is uniformly integrable on } [0,T]\times\mathbb{T}^N. \tag{4.45}$$

Taking $q=2$ in (4.45), we get $u_n \to u$ strongly in $L^2([0,T];L^2(\mathbb{T}^N))$.

Next, we prove that u is an entropy solution of (3.3). Utilizing (4.42) and Fatou lemma, we deduce that (i) of Definition 4.1 holds.

Let $l \in C_b(\mathbb{R})$ and η be the same as in Definition 4.1. Recall the entropy function

$$\Psi_l(r) := \int_0^r l(s)a(s)ds, \quad \forall\, l\in C_b(\mathbb{R}), \quad q_\eta \text{ is any function satisfying } q'_\eta(\xi)=\eta'(\xi)a^2(\xi).$$

Analogously, we define $\Psi_{n,l}, q_{n,\eta}$ similar to the above with a replaced by a_n. For each n, we have $\Psi_{n,l}(u_n) \in L^2([0,T];H^1(\mathbb{T}^N))$ and $\partial_i\Psi_{n,l}(u_n) = l(u_n)\partial_i\Psi_n(u_n)$. Also, we have $\Psi_{n,l}(r) \leq \|l\|_{L^\infty(\mathbb{R})}3K|r|^{\frac{m+1}{2}}$ for all $r\in\mathbb{R}$, which combined with (4.41) and (4.42) yields

$$\sup_n \int_0^T \|\Psi_{n,l}(u_n)\|^2_{H^1(\mathbb{T}^N)}dt < \infty.$$

Hence, for a subsequence, we have $\Psi_{n,l}(u_n) \rightharpoonup v_l$, $\Psi_n(u_n) \rightharpoonup v$ weakly for some $v_l, v \in L^2([0,T];H^1(\mathbb{T}^N))$. With the aid of (4.38), (4.44) and (4.45), we deduce that $v_l = \Psi_l(u), v=\Psi(u)$. Moreover, by $\partial_i\Psi_n(u_n) \rightharpoonup \partial_i\Psi(u)$ weakly in $L^2([0,T];H)$, for any $\phi\in C^\infty(\mathbb{T}^N)$, it holds that

$$\begin{aligned}
\int_0^T\int_{\mathbb{T}^N}\partial_i\Psi_l(u)\phi dxdt &= \lim_{n\to\infty}\int_0^T\int_{\mathbb{T}^N}\partial_i\Psi_{l,n}(u_n)\phi dxdt\\
&= \lim_{n\to\infty}\int_0^T\int_{\mathbb{T}^N} l(u_n)\partial_i\Psi_n(u_n)\phi dxdt\\
&= \int_0^T\int_{\mathbb{T}^N} l(u)\partial_i\Psi(u)\phi dxdt,
\end{aligned}$$

where we have used $l(u_n)\to l(u)$ strongly in $L^2([0,T];H)$. Hence, (ii) of Definition 4.1 is obtained.

In the following, we will show (iii) in Definition 4.1. Let η and ϕ be as in (iii). By integration by parts formula, we get

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{T}^N}\eta(u_n)\partial_t\phi dxdt\\
\leq &\int_{\mathbb{T}^N}\eta(u_{0,n})\phi(0)dx + \int_0^T\int_{\mathbb{T}^N} q_{n,\eta}(u_n)\Delta\phi dxdt\\
&-\int_0^T\int_{\mathbb{T}^N}\phi\eta''(u_n)|\nabla\Psi_n(u_n)|^2dxdt + \sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N}\phi\eta'(u_n)g_n^k(u_n)h_k(t)dxdt.
\end{aligned}\tag{4.46}$$

On the basis of (4.36)-(4.39), (4.44) and (4.45), we have

$$\begin{aligned}
\lim_{n\to\infty}\int_0^T\int_{\mathbb{T}^N}\eta(u_n)\partial_t\phi dxdt &= \int_0^T\int_{\mathbb{T}^N}\eta(u)\partial_t\phi dxdt,\\
\lim_{n\to\infty}\int_{\mathbb{T}^N}\eta(u_{0,n})\phi(0)dx &= \int_{\mathbb{T}^N}\eta(u_0)\phi(0)dx,\\
\lim_{n\to\infty}\int_0^T\int_{\mathbb{T}^N} q_{n,\eta}(u_n)\Delta\phi dxdt &= \int_0^T\int_{\mathbb{T}^N} q_\eta(u)\Delta\phi dxdt,\\
\lim_{n\to\infty}\sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N}\phi\eta'(u_n)g_n^k(u_n)h_k(t)dxdt &= \sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N}\phi\eta'(u)g^k(u)h_k(t)dxdt.
\end{aligned}$$

Set $\tilde{l}(r) = \sqrt{\eta''(r)}$, then $\partial_i \Psi_{n,\tilde{l}}(u_n) = \sqrt{\eta''(u_n)}\partial_i \Psi_n(u_n)$. From the above, we have obtained (after passing to a subsequence) $\partial_i \Psi_{n,\tilde{l}}(u_n) \rightharpoonup \partial_i \Psi_{\tilde{l}}(u)$ in $L^2([0,T];H)$. In particular, we have $\partial_i \Psi_{n,\tilde{l}}(u_n) \rightharpoonup \partial_i \Psi_{\tilde{l}}(u)$ in $L^2([0,T]\times\mathbb{T}^N,\bar{\mu})$, where $d\bar{\mu} := dx \otimes dt$. This implies that

$$\int_0^T \int_{\mathbb{T}^N} \phi \eta''(u) |\nabla \Psi(u)|^2 dxdt \leq \liminf_{n\to\infty} \int_0^T \int_{\mathbb{T}^N} \phi \eta''(u_n) |\nabla \Psi_n(u_n)|^2 dxdt.$$

Hence, taking $\liminf$ on both sides of (4.46), and by choosing an appropriate subsequence, we obtain that u satisfies (iii) of Definition 4.1. $\square$

In view of Theorem 4.3 and Theorem 4.4, define $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to L^1([0,T];L^1(\mathbb{T}^N))$ by

$$\mathcal{G}^0(\check{h}) := \begin{cases} u^h, & \text{if } \check{h} = \int_0^{\cdot} h(s)ds, \text{ for some } h \in L^2([0,T];U), \\ 0, & \text{otherwise}, \end{cases}$$

where u^h is the solution of equation (3.3).

4.2 The continuity of the skeleton equation

In this part, we aim to prove the continuity of the map $\mathcal{G}^0$. Namely, let u^{h^ε} be the kinetic solution of (3.3) with h replaced by h^ε, we will show that u^{h^ε} converges to the kinetic solution u^h of (3.3) in $L^1([0,T];L^1(\mathbb{T}^N))$, when $h^\varepsilon \to h$ weakly in $L^2([0,T];U)$. To achieve it, we need the auxiliary approximating process $(A_n, g_n, u_{0,n})$ defined by (4.35)-(4.39).

For any family $\{h^\varepsilon; \varepsilon > 0\} \subset S_M$ with $h^\varepsilon(t) = \sum_{k\geq 1} h^\varepsilon_k(t) e_k$, consider the following approximation equation

$$\begin{cases} du_n^{h^\varepsilon}(t,x) = \Delta A_n(u_n^{h^\varepsilon}(t,x))dt + \sum_{k\geq 1} g_n^k(x, u_n^{h^\varepsilon}(t,x)) h_k^\varepsilon(t) dt, \\ \qquad u_n^{h^\varepsilon} = u_{0,n}. \end{cases} \tag{4.47}$$

Referring to Section 4 in [15], for each n, the Cauchy problem $\mathcal{E}(A_n, g_n, u_{0,n})$ has a unique solution $u_n^{h^\varepsilon} \in C([0,T];H) \cap L^2([0,T];H^1(\mathbb{T}^N))$ satisfying the following energy estimates uniformly in n:

$$\sup_\varepsilon \left\{ \sup_{t\in[0,T]} \|u_n^{h^\varepsilon}(t)\|_H^p + \int_0^T \|\nabla \Psi_n(u_n^{h^\varepsilon}(s))\|_H^p ds \right\} \leq C(M,K,T,p,\|u_0\|_H), \tag{4.48}$$

$$\sup_\varepsilon \left\{ \sup_{t\in[0,T]} \|u_n^{h^\varepsilon}(t)\|^{m+1}_{L^{m+1}(\mathbb{T}^N)} + \int_0^T \|\nabla A_n(u_n^{h^\varepsilon}(s))\|_H^2 ds \right\} \leq C(M,K,T,m,\|u_0\|_{L^{m+1}(\mathbb{T}^N)}), \tag{4.49}$$

for all $p \geq 2$, where we have used $u_{0,n} \leq u_0$. Since $a_n \geq \frac{2}{n} > 0$, we have $|\nabla u_n^{h^\varepsilon}| \leq N(n)|\nabla \Psi_n(u_n^{h^\varepsilon})|$, then by (4.48), it yields

$$\sup_{\varepsilon>0} \int_0^T \|\nabla u_n^{h^\varepsilon}(t)\|_H^p dt \leq N(n) C(M,K,T,p,\|u_0\|_H). \tag{4.50}$$

With the approximation process (4.47) in hand, for any $\varepsilon > 0$ and $n \geq 1$, we have

$$\begin{aligned} &\|u^{h^\varepsilon} - u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} \\ &\leq \|u_n^{h^\varepsilon} - u^{h^\varepsilon}\|_{L^1([0,T];L^1(\mathbb{T}^N))} + \|u_n^{h^\varepsilon} - u_n^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} + \|u_n^h - u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))}. \end{aligned}$$

In order to establish the continuity of the skeleton equation, several steps are involved.

Firstly, we show the uniform convergence of the sequence $\{u_n^h; n \geq 1\}$ to u^h over $h \in S_M$.

Proposition 4.5. Assume (A, g, u_0) satisfies Hypothesis H, then

$$\lim_{n\to\infty}\sup_{h\in S_M}\|u_n^h-u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))}=0.$$

Proof. As discussed above, under assumption, the triple $(A_n, g_n, u_{0,n})$ satisfies Hypothesis H. Applying (2) of Theorem 4.2 with u_n^h and u^h, for all $\gamma,\delta\in(0,1)$, $\lambda\in[0,1]$, $\alpha\in(0,1\wedge\frac{m}{2})$ and a.e. $t\in[0,T]$, we have

$$\begin{aligned}
&\|u_n^h(t)-u^h(t)\|_{L^1(\mathbb{T}^N)}\\
&\leq \mathcal{E}_t(\gamma,\delta)+e^{2K\int_0^t|h(s)|_U ds}\Big[\|u_0-u_{0,n}\|_{L^1(\mathbb{T}^N)}+\mathcal{E}_0(\gamma,\delta)\\
&\quad+N_0\gamma^{-2}(\lambda^2+\delta^{2\alpha})\Big(1+\|u_n^h\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|u^h\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\
&\quad+N_0\gamma^{-2}\|I_{|u_n^h|\geq R_\lambda}(1+|u_n^h|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}+N_0\gamma^{-2}\|I_{|u^h|\geq R_\lambda}(1+|u^h|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}\\
&\quad+K(\gamma+2\delta)\int_0^t|h(s)|_U ds+d^{\frac{1}{2}}(g,g_n)\Big(1+\|u^h\|^{\frac{m+1}{2}}_{L^{\frac{m+1}{2}}([0,T]\times\mathbb{T}^N)}\Big)\int_0^t|h(s)|_U ds\Big],
\end{aligned}$$

where $\mathcal{E}_0(\gamma,\delta),\mathcal{E}_t(\gamma,\delta)\to 0$ as $\gamma,\delta\to 0$, the constant N_0 is independent of $\gamma,\delta,\lambda,n,h$ and R_λ is defined by

$$R_\lambda=\sup\{R\in[0,\infty]:|a(r)-a_n(r)|\leq\lambda,\forall|r|<R\}.$$

For $n\geq 4$, we can choose $\lambda=\frac{4}{n}$, by (4.38), we deduce that $R_\lambda\geq n$. Furthermore, let $\gamma=n^{-\frac{1}{2}}$ and $\delta=\gamma^{\frac{3}{2\alpha}}$, by (4.49) and (4.2), it follows that

$$\begin{aligned}
&\sup_{h\in S_M}\int_0^T\|u_n^h(t)-u^h(t)\|_{L^1(\mathbb{T}^N)}dt\\
&\leq\int_0^T\mathcal{E}_t(\gamma,\delta)dt+e^{K(T+M)}T\Big[\|u_0-u_{0,n}\|_{L^1(\mathbb{T}^N)}+\mathcal{E}_0(\gamma,\delta)\\
&\quad+N_0\Big(\frac{16}{n}+\frac{1}{\sqrt{n}}\Big)C(M,K,T,m,\|u_0\|_{L^m(\mathbb{T}^N)})\\
&\quad+N_0\sup_{h\in S_M}n\|I_{|u_n^h|\geq n}(1+|u_n^h|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}\\
&\quad+N_0\sup_{h\in S_M}n\|I_{|u^h|\geq n}(1+|u^h|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}\\
&\quad+K(n^{-\frac{1}{2}}+2n^{-\frac{3}{4\alpha}})(T+M)+d^{\frac{1}{2}}(g,g_n)C_m(T+M)\Big].
\end{aligned}\tag{4.51}$$

By the Lebesgue convergence theorem, we have $\int_0^T\mathcal{E}_t(\gamma,\delta)dt\to 0$, as $\gamma,\delta\to 0$.

Note that

$$\begin{aligned}
n\|I_{|u_n^h|\geq n}(1+|u_n^h|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}&\leq\int_0^T\int_{\mathbb{T}^N}I_{|u_n^h|\geq n}|u_n^h(x,t)|(1+|u_n^h(x,t)|)^m dxdt\\
&\leq C_m\int_0^T\int_{\mathbb{T}^N}I_{|u_n^h|\geq n}(1+|u_n^h(x,t)|^{m+1})dxdt,
\end{aligned}$$

hence, by (4.49), we get $\sup_{h\in S_M}n\|I_{|u_n^h|\geq n}(1+|u_n^h|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}\to 0$, as $n\to\infty$. Similarly, with the help of (4.2), we obtain $\sup_{h\in S_M}n\|I_{|u^h|\geq n}(1+|u^h|)\|^m_{L^m([0,T]\times\mathbb{T}^N)}\to 0$, as $n\to\infty$. Letting $n\to\infty$ in (4.51), we get the desired result. □

In the following, we show the compactness of $\{u_n^{h^\varepsilon};\varepsilon>0\}$. For simplicity, set $u_n^\varepsilon:=u_n^{h^\varepsilon}$.

As in [18], we introduce the following space. Let Y be a separable Banach space with the norm $\|\cdot\|_Y$. Given $p>1, \beta\in(0,1)$, let $W^{\beta,p}([0,T];Y)$ be the Sobolev space of all functions $u\in L^p([0,T];Y)$ such that

$$\int_0^T\int_0^T \frac{\|u(t)-u(s)\|_Y^p}{|t-s|^{1+\beta p}}dtds<\infty,$$

which is endowed with the norm

$$\|u\|^p_{W^{\beta,p}([0,T];Y)}=\int_0^T \|u(t)\|_Y^p dt+\int_0^T\int_0^T \frac{\|u(t)-u(s)\|_Y^p}{|t-s|^{1+\beta p}}dtds.$$

The following result can be found in [18].

Lemma 4.6. *Let $B_0\subset B\subset B_1$ be three Banach spaces. Assume that both B_0 and B_1 are reflexive, and B_0 is compactly embedded in B. Let $p\in(1,\infty)$ and $\beta\in(0,1)$ be given. Let Λ be the space*

$$\Lambda:=L^p([0,T];B_0)\cap W^{\beta,p}([0,T];B_1)$$

endowed with the natural norm. Then the embedding of Λ in $L^p([0,T];B)$ is compact.

To obtain the compactness, we also need the following lemma.

Lemma 4.7. *Assume $B_2\subset B_3$ are two Banach spaces with compact embedding, and the real numbers $\beta\in(0,1)$, $p>1$ satisfy $\beta p>1$, then the space $C([0,T];B_2)\cap W^{\beta,p}([0,T];B_3)$ is compactly embedded into $C([0,T];B_3)$.*

Proof. Clearly, the space $W^{\beta,p}([0,T];B_3)$ is continuously embedded into $C^\iota([0,T];B_3)$ for all $\iota\in(0,\beta-1/p)$. Thus, if a set $\mathcal{R}$ is bounded in $W^{\beta,p}([0,T];B_3)\cap C([0,T];B_2)$, it is bounded in $C^\iota([0,T];B_3)$. It follows that the functions in $\mathcal{R}$ are uniformly equicontinuous in $C([0,T];B_3)$. Since $B_2\subset B_3$ is compactly embedded, for each $s\in[0,T]$ the set $\{f(s):f\in\mathcal{R}\}$ is bounded in B_2 and thus relatively compact in B_3. We can apply Ascoli-Arzelà theorem to conclude that $\mathcal{R}$ is relatively compact in $C([0,T];B_3)$. □

With the help of Lemma 4.6, we obtain the following result.

Proposition 4.8. For any $n\geq 1$, $\{u_n^\varepsilon;\varepsilon>0\}$ is compact in $L^2([0,T];H)$.

Proof. From (4.47), u_n^ε can be written as

$$\begin{aligned}u_n^\varepsilon(t)&=u_{0,n}+\int_0^t\Delta A_n(u_n^\varepsilon)ds+\int_0^t\sum_{k\geq1}g_n^k(x,u_n^\varepsilon(t,x))h_k^\varepsilon(s)ds\\&=:I_1^\varepsilon+I_2^\varepsilon(t)+I_3^\varepsilon(t).\end{aligned}$$

Clearly, $\|I_1^\varepsilon\|_H\leq C_1$. Next,

$$\begin{aligned}\|\Delta A_n(u_n^\varepsilon)\|_{H^{-1}(\mathbb{T}^N)}&=\sup_{\|v\|_{H^1(\mathbb{T}^N)}\leq1}|\langle v,\Delta A_n(u_n^\varepsilon)\rangle|\\&=\sup_{\|v\|_{H^1(\mathbb{T}^N)}\leq1}|\langle\nabla v,\nabla A_n(u_n^\varepsilon)\rangle|\\&\leq C\|\nabla A_n(u_n^\varepsilon)\|_H\end{aligned}$$

which yields

$$\begin{aligned}\|I_2^\varepsilon(t)-I_2^\varepsilon(s)\|^2_{H^{-1}(\mathbb{T}^N)}&=\|\int_s^t\Delta A_n(u_n^\varepsilon)(l)dl\|^2_{H^{-1}(\mathbb{T}^N)}\\&\leq C(t-s)\int_s^t\|\Delta A_n(u_n^\varepsilon)(l)\|^2_{H^{-1}(\mathbb{T}^N)}dl\\&\leq C(t-s)\int_s^t\|\nabla A_n(u_n^\varepsilon)(l)\|^2_H dl.\end{aligned}$$

Hence, by (4.49), we have for $\beta \in (0, \frac{1}{2})$,

$$\begin{aligned}
&\sup_{\varepsilon} \|I_2^\varepsilon\|^2_{W^{\beta,2}([0,T];H^{-1}(\mathbb{T}^N))} \\
&\leq \int_0^T \|I_2^\varepsilon(t)\|^2_{H^{-1}(\mathbb{T}^N)} dt + \int_0^T \int_0^T \frac{\|I_2^\varepsilon(t) - I_2^\varepsilon(s)\|^2_{H^{-1}(\mathbb{T}^N)}}{|t-s|^{1+2\beta}} dsdt \\
&\leq C_2(\beta).
\end{aligned}$$

Moreover, by Hölder inequality and (2.4), it follows that

$$\begin{aligned}
\|\sum_{k\geq 1} g_n^k(x, u_n^\varepsilon(l,x)) h_k^\varepsilon(l)\|_H^2 &\leq \int_{\mathbb{T}^N} \sum_{k\geq 1} |g_n^k(x, u_n^\varepsilon(l,x))|^2 \sum_{k\geq 1} |h_k^\varepsilon(l)|^2 dx \\
&\leq K^2(1 + \|u_n^\varepsilon(l)\|_H^2)|h^\varepsilon(l)|_U^2,
\end{aligned}$$

then, by Hölder inequality, we get

$$\begin{aligned}
\|I_3^\varepsilon(t) - I_3^\varepsilon(s)\|_H^2 &= \|\int_s^t \sum_{k\geq 1} g_n^k(x, u_n^\varepsilon(l,x)) h_k^\varepsilon(l) dl\|_H^2 \\
&\leq (t-s) \int_s^t \|\sum_{k\geq 1} g_n^k(x, u_n^\varepsilon(l,x)) h_k^\varepsilon(l)\|_H^2 dl \\
&\leq K^2(t-s)(1 + \sup_{t\in[0,T]} \|u_n^\varepsilon(t)\|_H^2) \int_s^t |h^\varepsilon(l)|_U^2 dl \\
&\leq K^2 M(t-s)(1 + \sup_{t\in[0,T]} \|u_n^\varepsilon(t)\|_H^2).
\end{aligned}$$

Thus, we deduce from (4.48) that for $\beta \in (0, \frac{1}{2})$,

$$\begin{aligned}
&\sup_{\varepsilon} \|I_3^\varepsilon\|^2_{W^{\beta,2}([0,T];H)} \\
&\leq \int_0^T \|I_3^\varepsilon(t)\|_H^2 dt + \int_0^T \int_0^T \frac{\|I_3^\varepsilon(t) - I_3^\varepsilon(s)\|_H^2}{|t-s|^{1+2\beta}} dsdt \\
&\leq C_3(\beta).
\end{aligned}$$

Collecting all the above estimates, we conclude that for $\beta \in (0, \frac{1}{2})$,

$$\sup_{\varepsilon} \|u_n^\varepsilon\|^2_{W^{\beta,2}([0,T];H^{-1}(\mathbb{T}^N))} \leq C(\beta).$$

Taking into account that $u_n^\varepsilon \in C([0,T];H) \cap L^2([0,T];H^1(\mathbb{T}^N))$ and applying Lemma 4.6 with $B_0 = H^1(\mathbb{T}^N)$, $B_1 = H^{-1}(\mathbb{T}^N)$ and $B = H$, we conclude the desired result. ☐

Arguing similarly to the above, on the basis of the estimates (4.48) and (4.49), we have $\{u_n^\varepsilon; \varepsilon > 0\}$ are bounded in $W^{\beta,p}([0,T];H^{-1}(\mathbb{T}^N)) \cap C([0,T];H)$ for any $p \geq 2$. Choosing $\beta p > 1$ and applying Lemma 4.7 with $B_2 = H, B_3 = H^{-1}(\mathbb{T}^N)$, we get

Proposition 4.9. For any $n \geq 1$, $\{u_n^{h_\varepsilon}; \varepsilon > 0\}$ is compact in $L^2([0,T];H) \cap C([0,T];H^{-1}(\mathbb{T}^N))$.

Now, we are ready to prove the continuity of $\mathcal{G}^0$.

Theorem 4.10. Assume $h^\varepsilon \to h$ weakly in $L^2([0,T];U)$. Then u^{h^ε} converges to u^h in $L^1([0,T];L^1(\mathbb{T}^N))$, where u^{h^ε} is the kinetic solution of (3.3) with h replaced by h^ε.

Proof of Theorem 4.10. Fix any $n \geq 1$. For the solution $u_n^{h^\varepsilon}$ of (4.47), we shall firstly prove that when $h^\varepsilon \to h$ weakly in $L^2([0,T];U)$, we have

$$\lim_{\varepsilon\to 0} \|u_n^{h^\varepsilon} - u_n^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} = 0, \tag{4.52}$$

where u_n^h is the solution of (4.47) with h^ε replaced by h. This will be achieved if we show that for any sequence $\varepsilon_m \to 0$, one can find a subsequence $\varepsilon_{m_k} \to 0$ such that

$$\lim_{k\to\infty} \|u_n^{h^{\varepsilon_{m_k}}} - u_n^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} = 0. \tag{4.53}$$

From (4.48), (4.50) and Proposition 4.9, we know that for the sequence $\varepsilon_m \to 0$, there exists a subsequence $\{m_k,\ k \geq 1\}$ and an element $u^* \in L^\infty([0,T];H) \cap L^2([0,T];H) \cap L^2([0,T];H^1(\mathbb{T}^N)) \cap C([0,T];H^{-1}(\mathbb{T}^N))$ such that

$$u_n^{h^{\varepsilon_{m_k}}} \to u^* \text{ strongly in } L^2([0,T];H) \cap C([0,T];H^{-1}(\mathbb{T}^N)), \tag{4.54}$$
$$u_n^{h^{\varepsilon_{m_k}}} \to u^* \text{ weakly in } L^2([0,T];H^1(\mathbb{T}^N)). \tag{4.55}$$

Clearly, we also have $h^{\varepsilon_{m_k}} \to h$ weakly in $L^2([0,T];U)$. Thus, we only need to prove $u^* = u_n^h$.

From (4.47), we know that for a test function $\phi \in C^\infty(\mathbb{T}^N)$, it holds that

$$\langle u_n^{h^{\varepsilon_{m_k}}}(t), \phi\rangle - \langle u_{0,n}, \phi\rangle = \int_0^t \langle A_n(u_n^{h^{\varepsilon_{m_k}}}), \Delta\phi\rangle ds + \int_0^t \langle \sum_{k\geq 1} g_n^k(x, u_n^{h^{\varepsilon_{m_k}}}(s,x)) h_k^{\varepsilon_{m_k}}, \phi\rangle ds. \tag{4.56}$$

Due to (4.54), we get

$$|\langle u_n^{h^{\varepsilon_{m_k}}}(t) - u^*(t), \phi\rangle| \to 0, \quad \text{as } k \to \infty.$$

By the Lipschitz property of A_n and using (4.54) again, we deduce that

$$\begin{aligned}
&\int_0^t \langle A_n(u_n^{h^{\varepsilon_{m_k}}}) - A_n(u^*), \Delta\phi\rangle ds \\
&\leq C(n,m,K)\|\Delta\phi\|_{L^\infty(\mathbb{T}^N)} T^{\frac{1}{2}} \Big(\int_0^t \|u_n^{h^{\varepsilon_{m_k}}} - u^*\|_H^2 ds\Big)^{\frac{1}{2}} \to 0.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^t \langle \sum_{k\geq 1} \Big(g_n^k(x, u_n^{h^{\varepsilon_{m_k}}}) h_k^{\varepsilon_{m_k}} - g_n^k(x,u^*)h_k\Big), \phi\rangle ds \\
&= \int_0^t \langle \sum_{k\geq 1} (g_n^k(x, u_n^{h^{\varepsilon_{m_k}}}) - g_n^k(x,u^*)) h_k^{\varepsilon_{m_k}}, \phi\rangle ds + \int_0^t \langle \sum_{k\geq 1} g_n^k(x,u^*)(h_k^{\varepsilon_{m_k}} - h_k), \phi\rangle ds.
\end{aligned}$$

With the aid of (4.54), we get

$$\begin{aligned}
&\int_0^t \langle \sum_{k\geq 1} (g_n^k(x, u_n^{h^{\varepsilon_{m_k}}}) - g_n^k(x,u^*)) h_k^{\varepsilon_{m_k}}, \phi\rangle ds \\
&\leq \|\phi\|_{L^\infty(\mathbb{T}^N)} \int_0^t \int_{\mathbb{T}^N} \left(\sum_{k\geq 1} |g_n^k(x, u_n^{h^{\varepsilon_{m_k}}}) - g_n^k(x,u^*)|^2\right)^{\frac{1}{2}} \left(\sum_{k\geq 1} |h_k^{\varepsilon_{m_k}}|^2\right)^{\frac{1}{2}} dxds \\
&\leq CK \left(\int_0^t \|u_n^{h^{\varepsilon_{m_k}}} - u^*\|_H^2 ds\right)^{\frac{1}{2}} \left(\int_0^t |h^{\varepsilon_{m_k}}(s)|_U ds\right)^{\frac{1}{2}} \\
&\leq CKM^{\frac{1}{2}} \left(\int_0^t \|u_n^{h^{\varepsilon_{m_k}}} - u^*\|_H^2 ds\right)^{\frac{1}{2}} \to 0.
\end{aligned}$$

Since $h^{\varepsilon_{m_k}} \to h$ weakly in $L^2([0,T];U)$, we get

$$\int_0^t \langle \sum_{k\geq 1} g_n^k(x,u^*)(h_k^{\varepsilon_{m_k}} - h_k), \phi\rangle ds \to 0.$$

Thus, letting $k \to \infty$ in (4.56), we obtain

$$\langle u^*(t), \phi\rangle - \langle u_{0,n}, \phi\rangle = \int_0^t \langle A_n(u^*), \Delta\phi\rangle dt + \int_0^t \langle g_n(u^*)h, \phi\rangle dt,$$

hence, u^* is the solution to (4.47) with h^ε replaced by h. By the uniqueness of (4.47), we conclude that $u^* = u_n^h$. Thus, (4.53) is proved, which implies (4.52).

Note that for any $\varepsilon > 0$ and $n \geq 1$, we have

$$\begin{aligned} &\|u^{h^\varepsilon} - u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} \\ &\leq \|u_n^{h^\varepsilon} - u^{h^\varepsilon}\|_{L^1([0,T];L^1(\mathbb{T}^N))} + \|u_n^{h^\varepsilon} - u_n^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} + \|u_n^h - u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))}. \end{aligned} \tag{4.57}$$

For any $\iota > 0$, by Proposition 4.5, there exists N_0 such that for all $\varepsilon > 0$,

$$\|u_{N_0}^{h^\varepsilon} - u^{h^\varepsilon}\|_{L^1([0,T];L^1(\mathbb{T}^N))} \leq \frac{\iota}{3} \text{ and } \|u_{N_0}^h - u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} \leq \frac{\iota}{3}.$$

Letting $n = N_0$ in (4.57), it follows that

$$\|u^{h^\varepsilon} - u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} \leq \frac{2\iota}{3} + \|u_{N_0}^{h^\varepsilon} - u_{N_0}^h\|_{L^1([0,T];L^1(\mathbb{T}^N))}.$$

Using (4.52), we know that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, it holds that

$$\|u_{N_0}^{h^\varepsilon} - u_{N_0}^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} \leq \frac{\iota}{3}.$$

Thus, we conclude that

$$\lim_{\varepsilon\to 0} \|u^{h^\varepsilon} - u^h\|_{L^1([0,T];L^1(\mathbb{T}^N))} \leq \iota.$$

Due to the arbitrary of ι, we obtain the desired result. □

5 Large deviations

For any family $\{h^\varepsilon; 0 < \varepsilon \leq 1\} \subset \mathcal{A}_M$ with $h^\varepsilon(t) = \sum_{k\geq 1} h_k^\varepsilon(t)e_k$, we consider the following equation

$$\begin{cases} d\bar{u}^\varepsilon = \Delta A(\bar{u}^\varepsilon)dt + \sum_{k\geq 1} g^k(x,\bar{u}^\varepsilon)h_k^\varepsilon(t)dt + \sqrt{\varepsilon}\sum_{k\geq 1} g^k(x,\bar{u}^\varepsilon)d\beta_k(t), \\ \bar{u}^\varepsilon(0) = u_0. \end{cases} \tag{5.1}$$

Combining Theorem 2.7, Theorem 4.3 and Theorem 4.4, we know that (5.1) admits a unique kinetic solution $\bar{u}^\varepsilon$ with initial data $u_0 \in L^{m+1}(\mathbb{T}^N)$ satisfying

$$\sup_{0<\varepsilon\leq 1} \mathbb{E}\left(\int_0^T \int_{\mathbb{T}^N} |\bar{u}^\varepsilon|^{m+1} dxdt\right) \leq C_m. \tag{5.2}$$

Moreover, for all $l \in C_b(\mathbb{R})$, we have $\Psi_l(\bar{u}^\varepsilon) \in L^2(\Omega\times[0,T];H^1(\mathbb{T}^N))$ and

$$\partial_i \Psi_l(\bar{u}^\varepsilon) = l(\bar{u}^\varepsilon)\partial_i\Psi(\bar{u}^\varepsilon).$$

Also, there exists a kinetic measure $\bar{m}^\varepsilon \in \mathcal{M}_0^+(\mathbb{T}^N \times [0,T) \times \mathbb{R})$ such that $\bar{f}^\varepsilon := I_{\bar{u}^\varepsilon > \xi}$ fulfills that for all $\varphi \in C_c^2(\mathbb{T}^N \times [0,T) \times \mathbb{R})$,

$$\begin{aligned}
&\int_0^T \langle \bar{f}^\varepsilon(t), \partial_t \varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle \bar{f}^\varepsilon(t), a^2(\xi)\Delta\varphi(t)\rangle dt \\
= &-\sqrt{\varepsilon}\sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}} g^k(x,\xi)\varphi(x,t,\xi)d\nu^\varepsilon_{x,t}(\xi)dxd\beta_k(t) \\
&- \frac{\varepsilon}{2}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}} \partial_\xi\varphi(x,t,\xi)G^2(x,\xi)d\nu^\varepsilon_{x,t}(\xi)dxdt \\
&- \sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}} \varphi(x,t,\xi)g^k(x,\xi)h^\varepsilon_k(t)d\nu^\varepsilon_{x,t}(\xi)dxdt + \bar{m}^\varepsilon(\partial_\xi\varphi) + \bar{n}^\varepsilon(\partial_\xi\varphi), \quad a.s., \qquad (5.3)
\end{aligned}$$

where $\nu^\varepsilon_{x,t}(\xi) = -\partial_\xi f = \delta_{\bar{u}^\varepsilon(x,t)=\xi}$ and for any $\phi \in C_b(\mathbb{T}^N \times [0,T) \times \mathbb{R})$,

$$\bar{n}^\varepsilon(\phi) = \int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}} \phi(x,t,\xi)|\nabla\Psi(\bar{u}^\varepsilon)|^2 d\delta_{\bar{u}^\varepsilon(x,t)=\xi}dxdt.$$

According to the definition of $\mathcal{G}^\varepsilon$, it is clear that $\mathcal{G}^\varepsilon\Big(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h^\varepsilon(s)ds\Big) = \bar{u}^\varepsilon(\cdot)$.

Due to Theorem 3.1 (the sufficient condition B) and Theorem 4.10, in order to establish the main result, we only need to prove the following result.

Theorem 5.1. For every $M < \infty$, let $\{h^\varepsilon : 0 < \varepsilon \leq 1\} \subset \mathcal{A}_M$. Then

$$\Big\|\mathcal{G}^\varepsilon\Big(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h^\varepsilon(s)ds\Big) - \mathcal{G}^0\Big(\int_0^\cdot h^\varepsilon(s)ds\Big)\Big\|_{L^1([0,T];L^1(\mathbb{T}^N))} \to 0 \quad \text{in probability},$$

as $\varepsilon \to 0$.

Proof of Theorem 5.1. Recall that $\bar{u}^\varepsilon = \mathcal{G}^\varepsilon\Big(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h^\varepsilon(s)ds\Big)$ is the kinetic solution to (5.1) with corresponding measures denoted by m_1^ε and n_1^ε. Let $v^\varepsilon := \mathcal{G}^0\Big(\int_0^\cdot h^\varepsilon(s)ds\Big)$ be the kinetic solution to the skeleton equation (3.3) with h replaced by h^ε and the corresponding measures are denoted by m_2^ε and n_2^ε. Also, we have

$$\sup_{0<\varepsilon\leq 1}\mathbb{E}\int_0^T \|v^\varepsilon(t)\|^{m+1}_{L^{m+1}(\mathbb{T}^N)}dt < +\infty. \qquad (5.4)$$

Set $f_1(x,t,\xi) := I_{\bar{u}^\varepsilon(x,t)>\xi}$ and $f_2(y,t,\zeta) := I_{v^\varepsilon(y,t)>\zeta}$. By the same procedure as Lemma 2.5, we deduce from (5.3) that for all $\varphi_1(x,\xi) \in C_c^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi)$,

$$\begin{aligned}
\langle f_1^\pm(t), \varphi_1\rangle =& \langle f_{1,0}, \varphi_1\rangle\delta_0([0,t]) + \int_0^t \langle f_1(s), a^2(\xi)\Delta_x\varphi_1(x,\xi)\rangle ds \\
&+ \sqrt{\varepsilon}\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} g^k(x,\xi)\varphi_1(x,\xi)d\nu^{1,\varepsilon}_{x,s}(\xi)dxd\beta_k(s) \\
&+ \frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} \partial_\xi\varphi_1(x,\xi)G^2(x,\xi)d\nu^{1,\varepsilon}_{x,s}(\xi)dxds \\
&+ \sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} \varphi_1(x,\xi)g^k(x,\xi)h^\varepsilon_k(s)d\nu^{1,\varepsilon}_{x,s}(\xi)dxds \\
&- \langle m_1^\varepsilon, \partial_\xi\varphi_1\rangle([0,t]) - \langle n_1^\varepsilon, \partial_\xi\varphi_1\rangle([0,t]), \quad a.s.,
\end{aligned}$$

where $f_{1,0} = I_{u_0>\xi}$ and $\nu^{1;\varepsilon}_{x,s}(\xi) = -\partial_\xi f_1^\pm(s,x,\xi) = \partial_\xi\bar{f}_1^\pm(s,x,\xi) = \delta_{\bar{u}^{\varepsilon,\pm}(x,s)=\xi}$.

Similarly, in view of (4.6), for all $\varphi_2(y,\zeta)\in C_c^\infty(\mathbb{T}_y^N\times\mathbb{R}_\zeta)$, we have

$$\begin{aligned}\langle \bar{f}_2^{\pm}(t),\varphi_2\rangle=&\langle \bar{f}_{2,0},\varphi_2\rangle\delta_0([0,t])+\int_0^t\langle \bar{f}_2(s),a^2(\zeta)\Delta_y\varphi_2(y,\zeta)\rangle ds\\&-\sum_{k\geq1}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}g^k(y,\zeta)\varphi_2(y,\zeta)h_k^\varepsilon(s)d\nu_{y,s}^{2,\varepsilon}(\zeta)dyds\\&+\langle m_2^\varepsilon,\partial_\zeta\varphi_2\rangle([0,t])+\langle n_2^\varepsilon,\partial_\zeta\varphi_2\rangle([0,t]),\end{aligned}$$

where $f_{2,0}=I_{u_0>\zeta}$ and $\nu_{y,s}^{2;\varepsilon}(\zeta)=\partial_\zeta\bar{f}_2^{\pm}(s,y,\zeta)=-\partial_\zeta f_2^{\pm}(s,y,\zeta)=\delta_{v^{\varepsilon,\pm}(y,s)=\zeta}$.

Denote the duality distribution over $\mathbb{T}_x^N\times\mathbb{R}_\xi\times\mathbb{T}_y^N\times\mathbb{R}_\zeta$ by $\langle\langle\cdot,\cdot\rangle\rangle$. Setting $\alpha(x,\xi,y,\zeta)=\varphi_1(x,\xi)\varphi_2(y,\zeta)$. Using Itô formula for continuous semimartingales, we obtain that

$$\langle f_1^+(t),\varphi_1\rangle\langle \bar{f}_2^+(t),\varphi_2\rangle=\langle\langle f_1^+(t)\bar{f}_2^+(t),\alpha\rangle\rangle$$

satisfies

$$\begin{aligned}\langle\langle f_1^{\pm}(t)\bar{f}_2^{\pm}(t),\alpha\rangle\rangle=&\langle\langle f_{1,0}\bar{f}_{2,0},\alpha\rangle\rangle+\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar{f}_2(a^2(\xi)\Delta_x+a^2(\zeta)\Delta_y)\alpha d\xi d\zeta dxdyds\\&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^{\pm}(s,y,\zeta)\partial_\xi\alpha dn_1^\varepsilon(x,\xi,s)d\zeta dy\\&+\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1^{\pm}(s,x,\xi)\partial_\zeta\alpha dn_2^\varepsilon(y,\zeta,s)d\xi dx\\&+\frac{\varepsilon}{2}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\partial_\xi\alpha\bar{f}_2^{\pm}(s,y,\zeta)G^2(x,\xi)d\nu_{x,s}^{1;\varepsilon}(\xi)d\zeta dxdyds\\&+\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^{\pm}(s,y,\zeta)\alpha g^k(x,\xi)h_k^\varepsilon(s)d\zeta d\nu_{x,s}^{1;\varepsilon}(\xi)dxdyds\\&-\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1^{\pm}(s,x,\xi)\alpha g^k(y,\zeta)h_k^\varepsilon(s)d\xi d\nu_{y,s}^{2;\varepsilon}(\zeta)dxdyds\\&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^{\pm}(s,y,\zeta)\partial_\xi\alpha dm_1^\varepsilon(x,\xi,s)d\zeta dy\\&+\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1^{\pm}(s,x,\xi)\partial_\zeta\alpha dm_2^\varepsilon(y,\zeta,s)d\xi dx\\&+\sqrt{\varepsilon}\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^{\pm}(s,y,\zeta)g^k(x,\xi)\alpha d\zeta d\nu_{x,s}^{1;\varepsilon}(\xi)dxdyd\beta_k(s)\\=:&\langle\langle f_{1,0}\bar{f}_{2,0},\alpha\rangle\rangle+J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8+J_9,\quad a.s.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\langle\langle \bar{f}_1^{\pm}(t)f_2^{\pm}(t),\alpha\rangle\rangle=&\langle\langle \bar{f}_{1,0}f_{2,0},\alpha\rangle\rangle+\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_1f_2(a^2(\xi)\Delta_x+a^2(\zeta)\Delta_y)\alpha d\xi d\zeta dxdyds\\&+\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_2^{\pm}(s,y,\zeta)\partial_\xi\alpha dn_1^\varepsilon(x,\xi,s)d\zeta dy\\&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_1^{\pm}(s,x,\xi)\partial_\zeta\alpha dn_2^\varepsilon(y,\zeta,s)d\xi dx\\&-\frac{\varepsilon}{2}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\partial_\xi\alpha f_2^{\pm}(s,y,\zeta)G^2(x,\xi)d\nu_{x,s}^{1;\varepsilon}(\xi)d\zeta dxdyds\\&-\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_2^{\pm}(s,y,\zeta)\alpha g^k(x,\xi)h_k^\varepsilon(s)d\zeta d\nu_{x,s}^{1;\varepsilon}(\xi)dxdyds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_1^{\pm}(s,x,\xi)\alpha g^k(y,\zeta)h_k^{\varepsilon}(s)d\xi d\nu_{y,s}^{2,\varepsilon}(\zeta)dxdyds\\
&+\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_2^{\pm}(s,y,\zeta)\partial_\xi\alpha dm_1^{\varepsilon}(x,\xi,s)d\zeta dy\\
&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_1^{\pm}(s,x,\xi)\partial_\zeta\alpha dm_2^{\varepsilon}(y,\zeta,s)d\xi dx\\
&-\sqrt{\varepsilon}\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_2^{\pm}(s,y,\zeta)g^k(x,\xi)\alpha d\zeta d\nu_{x,s}^{1,\varepsilon}(\xi)dxdyd\beta_k(s)\\
=:&\ \langle\langle\bar{f}_{1,0}f_{2,0},\alpha\rangle\rangle+\bar{J}_1+\bar{J}_2+\bar{J}_3+\bar{J}_4+\bar{J}_5+\bar{J}_6+\bar{J}_7+\bar{J}_8+\bar{J}_9,\quad a.s.
\end{aligned}
$$

By the same technology as Proposition 4.1, we can take $\alpha(x,y,\xi,\zeta)=\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)$, where ρ_γ and ψ_δ are approximations to the identity on $\mathbb{T}^N$ and $\mathbb{R}$, respectively. Then, we have

$$
\begin{aligned}
&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))d\xi d\zeta dxdy\\
\leq&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_{1,0}(x,\xi)\bar{f}_{2,0}(y,\zeta)+\bar{f}_{1,0}(x,\xi)f_{2,0}(y,\zeta))d\xi d\zeta dxdy\\
&+\sum_{i=1}^{9}(\tilde{J}_i+\tilde{\bar{J}}_i),\quad a.s.,
\end{aligned}
\tag{5.5}
$$

where $\tilde{J}_i,\tilde{\bar{J}}_i$ in (5.5) are the corresponding $J_i,\bar{J}_i$ with $\alpha(x,y,\xi,\zeta)=\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)$, for $i=1,\ldots,9$.

In view of (4.11), it yields

$$
\begin{aligned}
\tilde{J}_7=&\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^{\pm}(s,y,\zeta)\partial_\zeta\alpha dm_1^{\varepsilon}(x,\xi,s)d\zeta dy\\
=&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha dm_1^{\varepsilon}(x,\xi,s)d\nu_{y,s}^{2,\varepsilon,\pm}(\zeta)dy\leq 0,\quad a.s.,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{J}_8=&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1^{\pm}(s,x,\xi)\partial_\xi\alpha dm_2^{\varepsilon}(y,\zeta,s)d\xi dx\\
=&-\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha dm_2^{\varepsilon}(y,\zeta,s)d\nu_{x,s}^{1,\varepsilon,\pm}(\xi)dx\leq 0,\quad a.s.
\end{aligned}
$$

By the same method as above, we deduce that $\tilde{\bar{J}}_7+\tilde{\bar{J}}_8\leq 0$, a.s.

Moreover, by using the same approach as Proposition 4.1 and Theorem 4.2, applying (4.23) with $\lambda=0$ and $R_\lambda=+\infty$, we have for any $\alpha\in(0,1\wedge\frac{m}{2})$, there exists a constant N_0 independent of $\gamma,\delta,\varepsilon$ such that

$$
\sum_{i=1}^{3}(\tilde{J}_i+\tilde{\bar{J}}_i)\leq N_0\gamma^{-2}\delta^{2\alpha}(1+\|\bar{u}^{\varepsilon}\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|v^{\varepsilon}\|^m_{L^m([0,T]\times\mathbb{T}^N)}),\quad a.s.
$$

By using (2.4), we have

$$
\begin{aligned}
\tilde{\bar{J}}_4&=\tilde{J}_4\\
&=\frac{\varepsilon}{2}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha G^2(x,\xi)d\nu_{x,s}^{1,\varepsilon}\otimes d\nu_{y,s}^{2,\varepsilon}(\xi,\zeta)dxdyds\\
&\leq\frac{\varepsilon}{2}K^2\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha(1+|\xi|^2)d\nu_{x,s}^{1,\varepsilon}\otimes d\nu_{y,s}^{2,\varepsilon}(\xi,\zeta)dxdyds
\end{aligned}
$$

$$\leq \frac{\varepsilon}{2}K^2\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds$$
$$+\frac{\varepsilon}{2}K^2\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha|\xi|^2 d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds.$$

Clearly, it holds that

$$\begin{aligned}&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdy\\ &\leq \|\psi_\delta\|_{L^\infty(\mathbb{T}^N)}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdy\\ &\leq \|\psi_\delta\|_{L^\infty(\mathbb{T}^N)}\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)dxdy\\ &\leq \delta^{-1}.\end{aligned}\tag{5.6}$$

Moreover, it follows that

$$\begin{aligned}&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha|\xi|^2 d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdy\\ \leq\ &\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}^2}\psi_\delta(\xi-\zeta)|\xi|^2 d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdy\\ \leq\ &\|\psi_\delta\|_{L^\infty(\mathbb{T}^N)}\|\rho_\gamma\|_{L^\infty(\mathbb{T}^N)}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}|\xi|^2 d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdy\\ \leq\ &C\delta^{-1}\gamma^{-N}\|\bar{u}^\varepsilon(s)\|^2_{L^2(\mathbb{T}^N)}.\end{aligned}\tag{5.7}$$

Hence, combining (5.6) and (5.7), we deduce that

$$\tilde{\tilde{J}}_4=\tilde{J}_4\leq\frac{\varepsilon}{2}K^2T\delta^{-1}+\frac{\varepsilon}{2}CK^2\delta^{-1}\gamma^{-N}\|\bar{u}^\varepsilon\|^2_{L^2([0,T];L^2(\mathbb{T}^N))},\quad a.s.$$

Recall

$$\chi_2(\zeta,\xi)=\int_\zeta^\infty\psi(\xi-\zeta')d\zeta',\tag{5.8}$$

using the similar arguments as in the proof of Proposition 4.1, we have

$$\begin{aligned}\tilde{\tilde{J}}_5+\tilde{\tilde{J}}_6&=\tilde{J}_5+\tilde{J}_6\\ &=\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2(\zeta,\xi)\rho_\gamma(x-y)\Big(g^k(x,\xi)-g^k(y,\zeta)\Big)h^\varepsilon_k(s)\\ &\qquad d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds\\ &\leq\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2(\zeta,\xi)\rho_\gamma(x-y)|g^k(x,\xi)-g^k(y,\zeta)||h^\varepsilon_k(s)|\\ &\qquad d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds\\ &\leq\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2(\zeta,\xi)\rho_\gamma(x-y)\Big(\sum_{k\geq1}|g^k(x,\xi)-g^k(y,\zeta)|^2\Big)^{\frac12}\Big(\sum_{k\geq1}|h^\varepsilon_k(s)|^2\Big)^{\frac12}\\ &\qquad d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds\\ &\leq K\int_0^t|h^\varepsilon(s)|_U\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2(\zeta,\xi)\rho_\gamma(x-y)|x-y|d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds\\ &\quad+K\int_0^t|h^\varepsilon(s)|_U\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}^2}\chi_2(\zeta,\xi)|\xi-\zeta|d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds\\ &=:\tilde{J}_{5,1}+\tilde{J}_{6,1}.\end{aligned}$$

By

$$\int_{(\mathbb{T}^N)^2} \rho_\gamma(x-y)|x-y|dxdy \le \gamma,$$
$$\int_{(\mathbb{T}^N)^2} \chi_2(\zeta,\xi)d\nu^{1,\varepsilon}_{x,s} \otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta) \le 1, \quad a.s.,$$

it follows that

$$\tilde{J}_{5,1} \le K\gamma(T+M), \quad a.s.$$

Using the same method as the proof of (4.27) in Theorem 4.2, we have

$$\begin{aligned}\tilde{J}_{6,1} &\le 2K\delta(T+M)\\ &\quad + K\int_0^t |h^\varepsilon(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}\bar{f}_2^{\pm} + \bar{f}_1^{\pm} f_2^{\pm})d\xi d\zeta dxdyds, a.s.\end{aligned}$$

Based on the above estimates, it yields

$$\begin{aligned}\tilde{J}_5 + \tilde{J}_6 &= \tilde{J}_5 + \tilde{J}_6\\ &\le K(\gamma+2\delta)(T+M) + K\int_0^t |h^\varepsilon(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho_\gamma(x-y)\psi_\delta(\xi-\zeta)\\ &\qquad \times (f_1^{\pm}\bar{f}_2^{\pm} + \bar{f}_1^{\pm} f_2^{\pm})d\xi d\zeta dxdyds,\ a.s.\end{aligned}$$

Combining all the previous estimates, it follows that

$$\begin{aligned}&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta) + \bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))d\xi d\zeta dxdy\\ \le\ & \int_{\mathbb{T}^N}\int_{\mathbb{R}} (f_{1,0}(x,\xi)\bar{f}_{2,0}(x,\xi) + \bar{f}_{1,0}(x,\xi)f_{2,0}(x,\xi))dxd\xi + \mathcal{E}_0(\gamma,\delta)\\ &+ N_0\gamma^{-2}\delta^{2\alpha}(1 + \|\bar{u}^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)} + \|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}) + \varepsilon K^2 T\delta^{-1}\\ &+ \varepsilon C K^2\delta^{-1}\gamma^{-N}\|\bar{u}^\varepsilon\|^2_{L^2([0,T];L^2(\mathbb{T}^N))} + 2K(2\delta+\gamma)(T+M) + |\tilde{J}_9|(t) + |\tilde{\bar{J}}_9|(t)\\ &+ 2K\int_0^t |h^\varepsilon(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}\bar{f}_2^{\pm} + \bar{f}_1^{\pm} f_2^{\pm})d\xi d\zeta dxdyds,\ a.s.\end{aligned}$$

Applying Gronwall inequality, we get

$$\begin{aligned}&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta) + \bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))d\xi d\zeta dxdy\\ \le\ & e^{K(T+M)}\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}} (f_{1,0}\bar{f}_{2,0} + \bar{f}_{1,0}f_{2,0})dxd\xi + \mathcal{E}_0(\gamma,\delta)\Big]\\ &+ e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}(1 + \|\bar{u}^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)} + \|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}) + \varepsilon K^2 T\delta^{-1}\\ &+ \varepsilon C K^2\delta^{-1}\gamma^{-N}\|\bar{u}^\varepsilon\|^2_{L^2([0,T];L^2(\mathbb{T}^N))} + 2K(2\delta+\gamma)(T+M) + |\tilde{J}_9|(t) + |\tilde{\bar{J}}_9|(t)\Big],\ a.s.\end{aligned}$$

Thus, collecting all the above estimates, we deduce that

$$\begin{aligned}
&\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))dxd\xi\\
=&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))\\
&\qquad\times\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta+\mathcal{E}_t(\gamma,\delta)\\
\leq& e^{K(T+M)}\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})dxd\xi+\mathcal{E}_0(\gamma,\delta)\Big]\\
&+e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}(1+\|\bar{u}^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)})+\varepsilon K^2T\delta^{-1}\\
&+\varepsilon CK^2\delta^{-1}\gamma^{-N}\|\bar{u}^\varepsilon\|^2_{L^2([0,T];L^2(\mathbb{T}^N))}+2K(2\delta+\gamma)(T+M)+|\tilde{J}_9|(t)+|\tilde{\tilde{J}}_9|(t)\Big]\\
&+\mathcal{E}_t(\gamma,\delta)\\
=:& e^{K(T+M)}\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{1,0}\bar{f}_{2,0}+\bar{f}_{1,0}f_{2,0})dxd\xi+e^{K(T+M)}(|\tilde{J}_9|(t)+|\tilde{\tilde{J}}_9|(t))\\
&+r(\varepsilon,\gamma,\delta,t),\ a.s.,
\end{aligned}\tag{5.9}$$

where the remainder is given by

$$\begin{aligned}
&r(\varepsilon,\gamma,\delta,t)\\
=&e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}(1+\|\bar{u}^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)})+\varepsilon K^2T\delta^{-1}\\
&+\varepsilon CK^2\delta^{-1}\gamma^{-N}\|\bar{u}^\varepsilon\|^2_{L^2([0,T];L^2(\mathbb{T}^N))}+2K(2\delta+\gamma)(T+M)+\mathcal{E}_0(\gamma,\delta)\Big]\\
&+\mathcal{E}_t(\gamma,\delta).
\end{aligned}\tag{5.10}$$

Applying the Burkholder-Davis-Gundy inequality, utilizing (5.8) and (2.4), it yields

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|\tilde{J}_9|(t)\\
\leq&\sqrt{\varepsilon}\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^{\pm}(s,y,\zeta)g^k(x,\xi)\alpha d\zeta d\nu^{1,\varepsilon}_{x,s}(\xi)dxdyd\beta_k(s)\Big|\\
=&\sqrt{\varepsilon}\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^{\pm}(s,y,\zeta)\partial_\zeta\chi_2(\zeta,\xi)\rho_\gamma(x-y)\\
&\qquad\times g^k(x,\xi)d\zeta d\nu^{1,\varepsilon}_{x,s}(\xi)dxdyd\beta_k(s)\Big|\\
=&\sqrt{\varepsilon}\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{k\geq1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2(\zeta,\xi)\rho_\gamma(x-y)\\
&\qquad\times g^k(x,\xi)d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyd\beta_k(s)\Big|\\
\leq&\sqrt{\varepsilon}\mathbb{E}\Big[\int_0^T\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2^2(\zeta,\xi)\rho_\gamma^2(x-y)\Big(\sum_{k\geq1}g_k^2(x,\xi)\Big)d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds\Big]^{\frac{1}{2}}\\
\leq&\sqrt{\varepsilon}K\mathbb{E}\Big[\int_0^T\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\chi_2^2(\zeta,\xi)\rho_\gamma^2(x-y)(1+|\xi|^2)d\nu^{1,\varepsilon}_{x,s}\otimes d\nu^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds\Big]^{\frac{1}{2}}\\
\leq&\sqrt{\varepsilon}K\gamma^{-N}\Big[\mathbb{E}\int_0^T\int_{\mathbb{T}^N}(1+|\bar{u}^\varepsilon(x,s)|^2)dxds\Big]^{\frac{1}{2}}\\
\leq&\sqrt{\varepsilon}K\gamma^{-N}(T+C_1)^{\frac{1}{2}},
\end{aligned}$$

where we have used (5.2). By the same method as above, we deduce that

$$\mathbb{E}\sup_{t\in[0,T]}|\tilde{\bar{J}}_9|(t)\leq\sqrt{\varepsilon}K\gamma^{-N}(T+C_1)^{\frac{1}{2}}.$$

In the following, we aim to make estimate of error term $\sup_{t\in[0,T]}\mathcal{E}_t(\gamma,\delta)$. For any $t\in[0,T]$,

$$\begin{aligned}
\mathcal{E}_t(\gamma,\delta):=&\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))d\xi dx\\
&-\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))\rho_\gamma(x-y)\\
&\qquad\times\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta\\
=&\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(x,t,\xi))d\xi dx\\
&-\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\xi))d\xi dxdy\Big]\\
&+\Big[\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\xi)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\xi))d\xi dxdy\\
&-\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}(f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_1^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))\rho_\gamma(x-y)\\
&\qquad\times\psi_\delta(\xi-\zeta)dxdyd\xi d\zeta\Big]\\
=:&H_1+H_2.
\end{aligned}$$

Applying the same method as (4.17) and (4.18), it follows that

$$|H_2|\leq 2\delta,\quad a.s. \tag{5.11}$$

Moreover, it is readily to derive

$$\begin{aligned}
|H_1|\leq&\Big|\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}}I_{\bar{u}^{\varepsilon,\pm}(x,t)>\xi}(I_{v^{\varepsilon,\pm}(x,t)\leq\xi}-I_{v^{\varepsilon,\pm}(y,t)\leq\xi})d\xi dxdy\Big|\\
&+\Big|\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}}I_{\bar{u}^{\varepsilon,\pm}(x,t)\leq\xi}(I_{v^{\varepsilon,\pm}(x,t)>\xi}-I_{v^{\varepsilon,\pm}(y,t)>\xi})d\xi dxdy\Big|\\
\leq&2\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)|v^{\varepsilon,\pm}(x,t)-v^{\varepsilon,\pm}(y,t)|dxdy,\quad a.s.
\end{aligned}$$

Due to (5.11) and applying (4.31) with $\lambda=0,R_\lambda=+\infty$, we have

$$\begin{aligned}
&\int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)|v^{\varepsilon,\pm}(x,t)-v^{\varepsilon,\pm}(y,t)|dxdy\\
=&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\rho_\gamma(x-y)(f_2^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\xi)+\bar{f}_2^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\xi))d\xi dxdy\\
\leq&\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho_\gamma(x-y)\psi_\delta(\xi-\zeta)(f_2^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)+\bar{f}_2^{\pm}(x,t,\xi)f_2^{\pm}(y,t,\zeta))d\xi d\zeta dxdy\\
&+2\delta\\
\leq&e^{K(T+M)}\Big[\int_{\mathbb{T}^N}\int_{\mathbb{R}}(f_{2,0}\bar{f}_{2,0}+\bar{f}_{2,0}f_{2,0})d\xi dx+\mathcal{E}_0(\gamma,\delta)\Big]\\
&+2e^{K(T+M)}[N_0\gamma^{-2}\delta^{2\alpha}\Big(1+2\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)+2K(\gamma+2\delta)(T+M)]+2\delta\\
=&e^{K(T+M)}\mathcal{E}_0(\gamma,\delta)\\
&+2e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}\Big(1+2\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)+2K(\gamma+2\delta)(T+M)\Big]+2\delta,
\end{aligned}$$

where $\mathcal{E}_0(\gamma,\delta)$ is defined by (4.16) and N_0 is independent of $\gamma,\delta,\varepsilon$. Then,

$$|H_1| \leq 4\delta + 2e^{K(T+M)}\mathcal{E}_0(\gamma,\delta) + 4e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}\Big(1+2\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big) + 2K(\gamma+2\delta)(T+M)\Big],\ a.s.$$

Combining all the above estimates, we conclude that

$$\begin{aligned}&\sup_{t\in[0,T]}\mathcal{E}_t(\gamma,\delta)\\ &\leq 6\delta + 2e^{K(T+M)}\mathcal{E}_0(\gamma,\delta) + 4e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}\Big(1+2\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\ &\qquad + 2K(\gamma+2\delta)(T+M)\Big],\ a.s.\end{aligned}$$

Hence, we deduce from (5.10) that

$$\begin{aligned}&\sup_{t\in[0,T]} r(\varepsilon,\gamma,\delta,t)\\ &\leq e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}\Big(1+\|\bar{u}^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)\\ &\quad + \varepsilon K^2T\delta^{-1} + \varepsilon CK^2\delta^{-1}\gamma^{-N}\|\bar{u}^\varepsilon\|^2_{L^2([0,T];L^2(\mathbb{T}^N))} + 2K(2\delta+\gamma)(T+M)\Big]\\ &\quad + 6\delta + 2e^{K(T+M)}\mathcal{E}_0(\gamma,\delta)\\ &\quad + 4e^{K(T+M)}\Big[N_0\gamma^{-2}\delta^{2\alpha}\Big(1+2\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big) + 2K(\gamma+2\delta)(T+M)\Big],\quad a.s.\end{aligned}$$

Letting

$$\delta = \varepsilon^{\frac{3}{4}},\quad \gamma = \varepsilon^{\frac{1}{4N+1}},$$

then, we deduce that

$$\mathbb{E}\sup_{t\in[0,T]}|\tilde{J}_9|(t) \leq K(T+C_1)^{\frac12}\varepsilon^{\frac{2N+1}{8N+2}} \to 0,\quad \text{as }\ \varepsilon\to 0,$$

and

$$\mathbb{E}\sup_{t\in[0,T]}|\tilde{\tilde{J}}_9|(t) \to 0,\quad \text{as } \varepsilon\to 0.$$

Taking $\alpha\in(\frac{1}{3N}, 1\wedge\frac{m}{2})$, it yields $b_0 := \frac{3\alpha}{2}-\frac{2}{4N+1} > 0$. Utilizing (5.2) and (5.4), it follows that

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]} r(\varepsilon,\gamma,\delta,t)\\ &\leq e^{K(T+M)}\Big[N_0\varepsilon^{b_0}\Big(1+\sup_{0<\varepsilon\leq 1}\mathbb{E}\|\bar{u}^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}+\sup_{0<\varepsilon\leq 1}\mathbb{E}\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big)+\varepsilon^{\frac14}K^2T\\ &\quad + \varepsilon^{\frac14-\frac{N}{4N+1}}CK^2\sup_{0<\varepsilon\leq 1}\mathbb{E}\|\bar{u}^\varepsilon\|^2_{L^2([0,T]\times\mathbb{T}^N)} + 2K(2\varepsilon^{\frac34}+\varepsilon^{\frac{1}{4N+1}})(T+M)\Big]\\ &\quad + 6\varepsilon^{\frac34} + 2e^{K(T+M)}\mathcal{E}_0(\gamma,\delta)\\ &\quad + 4e^{K(T+M)}\Big[N_0\varepsilon^{b_0}\Big(1+2\sup_{0\leq\varepsilon\leq 1}\mathbb{E}\|v^\varepsilon\|^m_{L^m([0,T]\times\mathbb{T}^N)}\Big) + 2K(2\varepsilon^{\frac34}+\varepsilon^{\frac{1}{4N+1}})(T+M)\Big]\\ &\to 0,\quad \text{as}\quad \varepsilon\to 0.\end{aligned}$$

Applying identities in (4.26) with $f_1 = I_{\bar{u}^\varepsilon>\xi}$, $f_2 = I_{v^\varepsilon>\xi}$, $f_{1,0} = I_{u_0>\xi}$ and $f_{2,0} = I_{u_0>\xi}$, we deduce from (5.9) that

$$\|\bar{u}^\varepsilon(t)-v^\varepsilon(t)\|_{L^1(\mathbb{T}^N)} \leq e^{K(T+M)}(|\tilde{J}_9|(t)+|\tilde{\tilde{J}}_9|(t)) + r(\varepsilon,\gamma,\delta,t),\quad a.s.$$

Hence, we get

$$\begin{aligned}
&\mathbb{E}\|\bar{u}^\varepsilon - v^\varepsilon\|_{L^1([0,T];L^1(\mathbb{T}^N))} \\
&\leq T\mathbb{E} \operatorname*{ess\,sup}_{0\leq t\leq T} \|\bar{u}^\varepsilon(t) - v^\varepsilon(t)\|_{L^1(\mathbb{T}^N)} \\
&\leq Te^{K(T+M)}\mathbb{E}\Big(\sup_{t\in[0,T]}|\tilde{J}_9|(t) + \sup_{t\in[0,T]}|\tilde{\tilde{J}}_9|(t)\Big) + T\mathbb{E}\sup_{t\in[0,T]} r(\varepsilon,\gamma,\delta,t) \\
&\to 0,
\end{aligned}$$

which implies $\|\bar{u}^\varepsilon - v^\varepsilon\|_{L^1([0,T];L^1(\mathbb{T}^N))} \to 0$ in probability as $\varepsilon \to 0$. □

References

[1] Barbu, V., Da Prato, G. and Röckner, M.: Existence of strong solutions for stochastic porous media equation under general monotonicity conditions. *Ann. Probab.* **37** (2009), 428–452. MR-2510012

[2] Barbu, V. and Röckner, M.: On a random scaled porous media equation. *J. Differential Equations* **251** (2011), 2494–2514. MR-2825337

[3] Bauzet, C., Vallet, G. and Wittbold, P.: A degenerate parabolic-hyperbolic Cauchy problem with a stochastic force. *J. Hyperbolic Differ. Equ.* **12** (2015), 501–533. MR-3401975

[4] Bogachev, V.I., Da Prato, G. and Röckner, M.: Invariant measures of generalized stochastic equations of porous media. *Dokl. Akad. Nauk* **396** (2004), 7–11. MR-2115900

[5] Boué, M. and Dupuis, P.: A variational representation for certain functionals of Brownian motion. *Ann. Probab.* **26** (1998), 1641–1659. MR-1675051

[6] Budhiraja, A. and Dupuis, P.: A variational representation for positive functionals of infinite dimensional Brownian motion. *Probab. Math. Statist.* **20** (2000), Acta Univ. Wratislav. No. 2246, 39–61. MR-1785237

[7] Budhiraja, A., Dupuis, P. and Maroulas, V.: Large deviations for infinite dimensional stochastic dynamical systems. *Ann. Probab.* **36** (2008), 1390–1420. MR-2435853

[8] Chen, G. and Perthame, B.: Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003), 645–668. MR-1981403

[9] Crandall, M. G. and Liggett, T. M.: Generation of semi-groups of nonlinear transformations on general Banach spaces. *Amer. J. Math.* **93** (1971), 265–298. MR-287357

[10] Da Prato, G. and Röckner, M.: Invariant measures for a stochastic porous medium equation. Stochastic analysis and related topics in Kyoto, 13-29, Adv. Stud. Pure Math., 41, Math. Soc. Japan, Tokyo, 2004. MR-2083701

[11] Dareiotis, K., Gerencsér, M. and Gess, B.: Entropy solutions for stochastic porous media equations. *J. Differential Equations* **266** (2019), 3732–3763. MR-3912697

[12] Debussche, A. and Vovelle, J.: Scalar conservation laws with stochastic forcing. *J. Funct. Anal.* **259** (2010), 1014–1042. MR-2652180

[13] Debussche, A., Hofmanová, M. and Vovelle, J.: Degenerate parabolic stochastic partial differential equations: quasilinear case. *Ann. Probab.* **44** (2016), 1916–1955. MR-3502597

[14] Dong, Z., Wu, J.-L., Zhang, R. and Zhang, T.: Large deviation principles for first-order scalar conservation laws with stochastic forcing. *Ann. Appl. Probab.* **30** (2020), 324–367. MR-4068313

[15] Dong, Z., Zhang, R. and Zhang, T.: Large deviations for quasilinear parabolic stochastic partial differential equations. *Potential Anal.* **53** (2020), 183–202. MR-4117985

[16] Dupuis, P. and Ellis, R.S.: A weak convergence approach to the theory of large deviations. *Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons, Inc.*, New York, 1997, A Wiley-Interscience Publication. MR-1431744

[17] Feng, J. and Nualart, D.: Stochastic scalar conservation laws. *J. Funct. Anal.* **255** (2008), 313–373. MR-2419964

[18] Flandoli, F. and Gatarek, D.: Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields* **102** (1995), 367–391. MR-1339739

[19] Gess, B. and Hofmanová, M.: Well-posedness and regularity for quasilinear degenerate parabolic-hyperbolic SPDE. *Ann. Probab.* **46** (2018), 2495–2544. MR-3846832

[20] Kim, J.U.: On a stochastic scalar conservation law. *Indiana Univ. Math. J.* **52** (2003), 227–256. MR-1970028

[21] Kružkov, S. N.: First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)* **81 (123)** (1970), 228–255. MR-0267257

[22] Lions, P.L., Perthame, B. and Tadmor, E.: A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.* **7** (1994), 169–191. MR-1201239

[23] Liu, W.: Large deviations for stochastic evolution equations with small multiplicative noise. *Appl. Math. Optim.* **61** (2010), 27–56. MR-2575313

[24] Matoussi, A., Sabbagh, W. and Zhang, T.: Large deviation principles of obstacle problems for quasilinear stochastic pdes, arXiv:math.PR/1712.02169

[25] Otto, F.: L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations. *J. Differential Equations* **131** (1996), 20–38. MR-1415045

[26] Pardoux, E.: Équations aux dérivées partielles stochastiques de type monotone. Séminaire sur les Équations aux Dérivées Partielles (1974-1975), III, Exp. No. 2, 10 pp. *Collège de France*, Paris, 1975. MR-0651582

[27] Prévôt, C. and Röckner, M.: A concise course on stochastic partial differential equations. Lecture Notes in Mathematics, vol. 1905, *Springer*, Berlin, 2007. MR-2329435

[28] Ren, J., Röckner, M. and Wang, F.: Stochastic generalized porous media and fast diffusion equations. *J. Differential Equations* **238** (2007), 118–152. MR-2334594

[29] Röckner, M. Wang, F. and Wu, L.: Large deviations for stochastic generalized porous media equations. *Stochastic Process. Appl.* **116** (2006), 1677–1689. MR-2307054

[30] Vallet, G. and Wittbold, P.: On a stochastic first-order hyperbolic equation in a bounded domain. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **12** (2009), 613–651. MR-2590159

Acknowledgments. We are grateful to the anonymous referees for their constructive comments and valuable suggestions. This work is partly supported by National Natural Science Foundation of China (No. 11801032,11971227), Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (No. 2008DP173182) and Beijing Institute of Technology Research Fund Program for Young Scholars.