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Fluctuation theory for spectrally positive additive Lévy fields

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Abstract

A spectrally positive additive Lévy field is a multidimensional field obtained as the sum $\mathbf{X}_{\rm t} = {\rm X}^{(1)}_{t_1} + {\rm X}^{(2)}_{t_2} + \cdots + {\rm X}^{(d)}_{t_d}$, ${\rm t} = (t_1, \dots, t_d) \in \mathbb{R}^d_+$, where ${\rm X}^{(j)} = {}^t(X^{1,j}, \dots, X^{d,j})$, $j = 1, \dots, d$, are d independent $\mathbb{R}^d$-valued Lévy processes issued from $\mathbf{0} = {}^t(0, 0, \dots, 0)$, such that $X^{i,j}$ is non decreasing for $i \neq j$ and $X^{j,j}$ is spectrally positive. It can also be expressed as $\mathbf{X}_{\rm t} = \mathbb{X}_{\rm t} \cdot \mathbf{1}$, where $\mathbf{1} = {}^t(1, 1, \dots, 1)$ and $\mathbb{X}_{\rm t} = (X^{i,j}_{t_j})_{1 \le i,j \le d}$. The main interest of spaLf's lies in the Lamperti representation of multitype continuous state branching processes. In this work, we study the law of the first passage times $\mathbf{T}_{\rm r}$ of such fields at levels $-{\rm r}$, where ${\rm r} \in \mathbb{R}^d_+$. We prove that the field $\{(\mathbf{T}_{\rm r}, \mathbb{X}_{\mathbf{T}_{\rm r}}), {\rm r} \in \mathbb{R}^d_+\}$ has stationary and independent increments and we describe its law in terms of this of the spaLf $\mathbf{X}$. In particular, the Laplace exponent of $(\mathbf{T}_{\rm r}, \mathbb{X}_{\mathbf{T}_{\rm r}})$ solves a functional equation leaded by the Laplace exponent of $\mathbf{X}$. This equation extends in higher dimension a classical fluctuation identity satisfied by the Laplace exponents of the ladder processes. Then we give an expression of the distribution of $\{(\mathbf{T}_{\rm r}, \mathbb{X}_{\mathbf{T}_{\rm r}}), {\rm r} \in \mathbb{R}^d_+\}$ in terms of the distribution of $\{\mathbb{X}_{\rm t}, {\rm t} \in \mathbb{R}^d_+\}$ by the means of a Kemperman-type formula, well-known for spectrally positive Lévy processes.

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1 Introduction

A spectrally positive, additive Lévy field (spaLf) is defined by

$$\mathbf{X}_{\rm t} := \left(\sum_{j=1}^{d} X^{i,j}_{t_j},\, i = 1, \dots, d\right) = {\rm X}^{(1)}_{t_1} + \cdots + {\rm X}^{(d)}_{t_d}\,, \quad {\rm t} = (t_1, \dots, t_d) \in [0, \infty)^d\,,$$

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where $\mathrm{X}^{(j)} = {}^t(X^{1,j}, \dots, X^{d,j})$, $j = 1, \dots, d$, are d independent $\mathbb{R}^d$-valued Lévy processes such that $X^{i,j}$ are non decreasing for $i \neq j$ and $X^{j,j}$ is spectrally positive (here ${}^t\mathrm{u}$ means the transpose of the vector $\mathrm{u} \in \mathbb{R}^d$). SpaLf's can be considered as (non-trivial) extensions in higher dimension of spectrally positive Lévy processes and the purpose of this article is to develop fluctuation theory for such random fields. We refer to Chapter VII of [3] for a complete account on fluctuation theory for spectrally one sided Lévy processes, see also [9] and [13] (Chapter VII of [3] deals with the case of spectrally negative Lévy processes but the results can easily be transferred to the spectrally positive case). The particular pathwise features of spaLf's allow us to define their first passage times $\mathbf{T}_\mathrm{r} = (T_\mathrm{r}^{(1)}, \dots, T_\mathrm{r}^{(d)})$ at multivariate levels $-\mathrm{r} \in (-\infty, 0]^d$ as the smallest of the indices $\mathrm{t} = (t_1, \dots, t_d)$ satisfying $\mathbf{X}_\mathrm{t} = -\mathrm{r}$ in the usual partial order of $\mathbb{R}^d$. The distribution of the variables $(\mathbf{T}_\mathrm{r}, \mathbb{X}_{\mathbf{T}_\mathrm{r}})$, $\mathrm{r} \in [0, \infty)^d$ can then be related to the distribution of the field $\{\mathbb{X}_\mathrm{t}, \mathrm{t} \in [0, \infty)^d\}$, where $\mathbb{X}_\mathrm{t} = (X_{t_j}^{i,j})_{1 \le i,j \le d}$. In doing so we obtain some fluctuation-type identities in the general framework of multivariate stochastic fields. These results provide an intrinsic motivation for the present study that can be considered in the line of several works on additive Lévy processes from Khoshnevisan and Xiao, see for instance [11].

The original motivation comes from an extension of the Lukasiewicz-Harris coding of Bienaymé-Galton-Watson trees through downward skip free random walks. In [7], the authors proved that multitype Bienaymé-Galton-Watson trees can be coded by multivariate random fields

$$\left(\sum_{j=1}^{d} S_{n_j}^{i,j},\, i = 1, \dots, d \right), \quad n_j = 0, 1 \dots, \quad j = 1, \dots, d,$$

where ${}^t(S^{1,j}, \dots, S^{d,j})$, $j = 1, \dots, d$ are d independent $\mathbb{Z}^d$-valued random walks such that $S^{i,j}$ are non decreasing for $i \neq j$ and $S^{j,j}$ is downward skip free. These random fields are the discrete time counterparts of spaLf's which suggests the possibility of coding continuous multitype branching trees in an analogous way. It seems quite complicated to achieve such a result as the notion of continuous multitype tree is not clearly defined for general mechanisms. However, reducing the analysis to processes rather than trees, one may still consider the Lamperti representation which provides a pathwise relationship between branching processes and their mechanism. This representation can be extended to continuous time multitype branching processes by using spaLf's. It was done in [6] for the discrete valued case and in [5] and [10] for the continuous one. More specifically, let $\mathrm{Z} = (Z^{(1)}, \dots, Z^{(d)})$ be a continuous time multitype branching process issued from $\mathrm{r} \in [0, \infty)^d$. Then Z can be represented as the unique pathwise solution of the following equation,

$$(Z_t^{(1)}, \dots, Z_t^{(d)}) = \mathrm{r} + \left(\sum_{j=1}^{d} X^{1,j}_{\int_0^t Z_s^{(j)}\, ds}, \dots, \sum_{j=1}^{d} X^{d,j}_{\int_0^t Z_s^{(j)}\, ds} \right), \quad t \ge 0\,,$$

where $\mathrm{X}^{(j)}$, $j = 1, \dots, d$, are Lévy processes as described above. Now recall that $\mathbf{0}$ is an absorbing state for Z. Then it follows from the above equation that the path of Z up to its first passage time at $\mathbf{0}$ is entirely determined by the path of the spaLf

$$\{\mathbf{X}_\mathrm{t}, \mathrm{t} \in [0, \infty)^d\} = \left\{ \left(\sum_{j=1}^{d} X_{t_j}^{i,j} \right)_{1 \le i,j \le d}, \mathrm{t} \in [0, \infty)^d \right\}$$

up to its first passage time $\mathbf{T}_\mathrm{r}$ at level $-\mathrm{r}$. This fact which is plain in the case $d = 1$ will be proved in the general case in the upcoming paper [8], where extinction of continuous time multitype branching processes is characterized through path properties of spaLf's.

The next section consists in an important preliminary lemma for deterministic paths whose aim is to prove the existence of first passage times of spaLf's and to derive their first basic properties. Then in Section 3 we will turn our attention to the law of these first passage times. In particular we will prove that in analogy with the one dimensional case, their Laplace exponent is the inverse of the Laplace exponent of the spaLf. The situation for $d \geq 2$ differs significantly from the one dimensional case as we first need to give necessary and sufficient conditions for the multivariate hitting times $\mathbf{T}_{\mathrm{r}}$ to be finite on each coordinate, with positive probability, for all $\mathrm{r} \in [0,\infty)^d$. (When $d=1$, this is equivalent to saying that the spectrally positive Lévy process is not a subordinator.) Another fundamental difference concerns the matrix valued field $\mathbb{X}_{\mathbf{T}_{\mathrm{r}}}$ which is simply equal to $-\mathrm{r}$ on the set $\mathbf{T}_{\mathrm{r}} < \infty$, when $d=1$. In Section 4 we will focus on the law of the field $(\mathbf{T}_{\mathrm{r}}, \mathbb{X}_{\mathbf{T}_{\mathrm{r}}})$ and prove that its Laplace exponent solves a functional equation leaded by the Laplace exponent of the spaLf $\mathbf{X}$. This equation, see (4.1) in Theorem 4.1 below, can be compared to the classical Wiener-Hopf factorization involving the ladder processes of spectrally positive Lévy processes. Then in Theorem 4.3 the distribution of $(\mathbf{T}_{\mathrm{r}}, \mathbb{X}_{\mathbf{T}_{\mathrm{r}}})$ will be fully characterized in terms of the distribution of the original stochastic field $\mathbb{X}$, through an extension of Kemperman's formula, see Corollary VII.3 in [3]. More specifically, our result states that the measure

$$\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathrm{dt},\, X^{i,j}_{t_j} \in \mathrm{d}x_{i,j},\, 1 \leq i,j \leq d)\,\mathrm{dr}$$

is the image of the measure

$$\frac{\det(-(x_{i,j})_{i,j\in[d]})}{t_1 t_2 \dots t_d} \prod_{j=1}^{d} \mathbb{P}(X^{i,j}_{t_j} \in \mathrm{d}x_{i,j},\, i=1,\dots,d)\mathrm{d}t_1 \dots \mathrm{d}t_d,$$

through the mapping $(\mathrm{t}, (x_{i,j})_{i,j\in(d]}) \mapsto (\mathrm{t}, (x_{i,j})_{i,j\in[d]}, -(x_{i,j})_{i,j\in[d]} \cdot \mathbf{1})$, where we set $\mathbf{1} = (1,1,\dots,1)$. In order to prove it, we will use a similar identity recently obtained in [7] and [6] in the discrete time and space settings together with a discrete approximation.

2 A preliminary lemma in the deterministic setting

We use the notation $\mathbb{R}_+ = [0,\infty)$, $\overline{\mathbb{R}}_+ = [0,\infty]$ and $[d] = \{1,\dots,d\}$, where $d \geq 1$ is an integer. The zero vector of $\mathbb{R}^d$ will be denoted by $\mathbf{0}$. For $\mathrm{s} = (s_1,\dots,s_d)$ and $\mathrm{t} = (t_1,\dots,t_d) \in \overline{\mathbb{R}}^d_+$, we write $\mathrm{s} \leq \mathrm{t}$ if $s_i \leq t_i$ for all $i \in [d]$ and we write $\mathrm{s} < \mathrm{t}$ if $\mathrm{s} \leq \mathrm{t}$ and there exists $i \in [d]$ such that $s_i < t_i$.

Recall that a real valued function $x : \mathbb{R}_+ \to \mathbb{R}$ is said to be càdlàg, if it is right continuous on $\mathbb{R}_+$ and has left limits on $(0,\infty)$. Such a function is said to be downward skip free if for all $s \geq 0$, $x(s) - x(s-) \geq 0$, where we set $x(0-) = x(0)$. We also say that x has no negative jumps. We will use the notation x_t or $x(t)$ indifferently.

Definition 2.1. *We call $\mathcal{E}_d$, the set of matrix valued functions $\mathrm{x} = \{(x^{i,j}_{t_j})_{i,j\in[d]},\, \mathrm{t} \in \mathbb{R}^d_+\}$ such that for all i,j, $x^{i,j}$ is a càdlàg function and*

- (i) *$x^{i,j}_0 = 0$, for all $i,j \in [d]$,*
- (ii) *for all $i \in [d]$, $x^{i,i}$ is downward skip free,*
- (iii) *for all $i,j \in [d]$ such that $i \neq j$, $x^{i,j}$ is non decreasing.*

For $\mathrm{s} \in \overline{\mathbb{R}}^d_+$, we denote by $[d]_{\mathrm{s}}$ the set of indices of finite coordinates of s, that is $[d]_{\mathrm{s}} = \{i \in [d] : s_i < \infty\}$. For $i \neq j$, we set $x^{i,j}(\infty) = x^{i,j}(\infty-) = \lim_{s\to\infty} x^{i,j}(s)$.

Definition 2.2. *Let* $\mathrm{x} \in \mathcal{E}_d$ *and* $\mathrm{r} = (r_1, \dots, r_d) \in \mathbb{R}_+^d$. *Then* $\mathrm{s} \in \overline{\mathbb{R}}_+^d$ *is called a solution of the system* (r, x) *if it satisfies*

$$(\mathrm{r}, \mathrm{x}) \qquad r_i + \sum_{j=1}^{d} x^{i,j}(s_j-) = 0\,, \quad i \in [d]_{\mathrm{s}}\,. \tag{2.1}$$

(*In particular,* $\mathrm{s} = (\infty, \infty, \dots, \infty)$ *is always a solution of the system* (r, x) *since* $[d]_{\mathrm{s}} = \emptyset$.)

We emphasize that according to our definition, some of the coordinates of the smallest solution of the system (r, x) may be infinite. Note also that in (2.1) it is implicit that $\sum_{j \in [d] \setminus [d]_{\mathrm{s}}} x^{i,j}(s_j-) < \infty$, for all $i \in [d]_{\mathrm{s}}$, although by definition $s_j = \infty$, for $j \in [d] \setminus [d]_{\mathrm{s}}$. The next lemma is a continuous time and space counterpart of Lemma 1 in [6]. The proof of the present result follows a similar scheme, however we need to perform it here as it requires more care. It is done in the Appendix at the end of this paper.

Lemma 2.3. *Let* $\mathrm{x} \in \mathcal{E}_d$ *and* $\mathrm{r} = (r_1, \dots, r_d) \in \mathbb{R}_+^d$.

1. *There exists a solution* $\mathrm{s} = (s_1, \dots, s_d) \in \overline{\mathbb{R}}_+^d$ *of the system* (r, x) *such that any other solution* t *of* (r, x) *satisfies* $\mathrm{t} \geq \mathrm{s}$. *The solution* s *will be called* the smallest solution *of the system* (r, x).
2. *Let* s *and* s' *be the smallest solutions of the systems* (r, x) *and* $(\mathrm{r}', \mathrm{x})$*, respectively. If* $\mathrm{r}' \leq \mathrm{r}$*, then* $\mathrm{s}' \leq \mathrm{s}$. *Moreover if* $(\mathrm{r}_n)_{n \geq 0}$ *is non decreasing with* $\lim_{n \to \infty} \mathrm{r}_n = \mathrm{r}$ *then the sequence* $(\mathrm{s}_n)_{n \geq 0}$ *of smallest solutions of* $(\mathrm{r}_n, \mathrm{x})$ *satisfies* $\lim_{n \to \infty} \mathrm{s}_n = \mathrm{s}$.
3. *Let* s *be the smallest solution of* (r, x). *If* u *is such that for all* $i \in [d]_{\mathrm{u}}$, $\sum_{j=1}^{d} x^{i,j}(u_j-) \leq -r_i$*, then* $\mathrm{u} \geq \mathrm{s}$. *As a consequence, for all* $\mathrm{u} \in \mathbb{R}_+^d$ *such that* $\mathrm{u} < \mathrm{s}$*, there is* $i \in [d]$ *such that* $\sum_{j=1}^{d} x^{i,j}(u_j-) > -r_i$.
4. *The smallest solution* s *of* (r, x) *satisfies* $s_i = \inf \left\{ t : x_{t-}^{i,i} = \inf_{0 \leq u \leq s_i} x_u^{i,i} \right\}$*, for all* $i \in [d]_{\mathrm{s}}$.

3 Fluctuation theory for additive Lévy fields

Vectors of $\mathbb{R}^d$ will be denoted by $\mathrm{x} = (x_1, \dots, x_d)$ and $\mathrm{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ will be the i-th unit vector of $\mathbb{R}_+^d$. We recall the notation ${}^t\mathrm{x}$ for the transpose of any vector $\mathrm{x} \in \mathbb{R}^d$ and the notations $\mathbf{1} = {}^t(1, 1, \dots, 1)$, $\mathbf{0} = {}^t(0, 0, \dots, 0)$. We will set $\langle \mathrm{x}, \mathrm{y} \rangle$, $\mathrm{x}, \mathrm{y} \in \mathbb{R}^d$ for the usual scalar product on $\mathbb{R}^d$ and $|\mathrm{x}|$ for the Euclidian norm of x. A matrix $M = (m_{i,j})_{i,j \in [d]} \in M_d(\mathbb{R} \cup \{\infty\})$ is said to be irreducible if for all $i, j \in [d]$, there is a sequence $i = i_1, i_2, \dots, i_n = j$, for some $n \geq 1$, such that $m_{i_k, i_{k+1}} \neq 0$, for all $k = 1, \dots, n-1$. For two matrices A and B of $M_d(\mathbb{R})$, with columns $\mathrm{a}^{(1)}, \dots, \mathrm{a}^{(d)}$ and $\mathrm{b}^{(1)}, \dots, \mathrm{b}^{(d)}$, respectively, we define the following special product,

$$\langle\!\langle A, B \rangle\!\rangle = \sum_{j \in [d]} \langle \mathrm{a}^{(j)}, \mathrm{b}^{(j)} \rangle.$$

A matrix $A = (a_{i,j})_{i,j \in [d]}$ is called essentially nonnegative (or a Metzler matrix) if $a_{i,j}$ is nonnegative whenever $i \neq j$. For instance, for any element $\mathrm{x} = \{(x_{t_j}^{i,j})_{i,j \in [d]},\, \mathrm{t} \in \mathbb{R}_+^d\}$ of the set $\mathcal{E}_d$ introduced at the previous section, the matrix $\mathrm{x}_{\mathrm{t}} = (x_{t_j}^{i,j})_{i,j \in [d]}$ is essentially nonnegative for all $\mathrm{t} = (t_1, \dots, t_d) \geq \mathbf{0}$.

3.1 SpaLf's and their first hitting times

In this work, we shall consider d independent Lévy processes $\mathrm{X}^{(1)},\dots,\mathrm{X}^{(d)}$ on $\mathbb{R}_+^d$, such that with the notation $\mathrm{X}^{(j)}={}^t(X^{1,j},\dots,X^{d,j})$, for all $j\in[d]$, the process $X^{j,j}$ is a real spectrally positive Lévy process, that is, it has no negative jumps, and for all $i\neq j$, the Lévy process $X^{i,j}$ is a subordinator. We emphasize that the processes $X^{1,j},\dots,X^{d,j}$ are not necessarily independent. Moreover, we do not exclude the possibility for a process $X^{i,j}$ to be identically equal to 0 and note that for each $i\in[d]$, $X^{i,i}$ can be a subordinator. It is known, see Chap. VII, in [3], that the Lévy process $\mathrm{X}^{(j)}$ admits all negative exponential moments. We denote by φ_j its Laplace exponent, that is

$$\mathbb{E}[e^{-\langle\lambda,\mathrm{X}_t^{(j)}\rangle}]=e^{t\varphi_j(\lambda)}\,,\quad t\geq 0\,,\quad \lambda=(\lambda_1,\dots,\lambda_d)\in\mathbb{R}_+^d\,.$$

Then from Lévy Khintchine formula and the above assumptions on $\mathrm{X}^{(j)}$, φ_j has the following form,

$$\varphi_j(\lambda)=-\sum_{i=1}^d a_{i,j}\lambda_i+\frac{1}{2}q_j\lambda_j^2-\int_{\mathbb{R}_+^d}(1-e^{-\langle\lambda,\mathrm{x}\rangle}-\langle\lambda,\mathrm{x}\rangle 1_{\{|\mathrm{x}|<1\}})\,\pi_j(d\mathrm{x})\,,\quad \lambda\in\mathbb{R}_+^d,\qquad (3.1)$$

where $(a_{i,j})_{i,j\in[d]}$ is an essentially nonnegative matrix, $q_j\geq 0$ and π_j is a measure on $\mathbb{R}_+^d$ such that $\pi_j(\{\mathbf{0}\})=0$ and

$$\int_{\mathbb{R}_+^d}\left[(1\wedge|\mathrm{x}|^2)+\sum_{i\neq j}(1\wedge x_i)\right]\pi_j(d\mathrm{x})<\infty\,.$$

Note that for all $j\in[d]$, φ_j is log-convex, i.e. the function $\log\varphi_j$ is convex on $(0,\infty)^d$. In particular, φ_j is a convex function. Moreover, for all $i\neq j$ and $\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1},\dots,\lambda_d$, the function $\lambda_i\mapsto\varphi_j(\lambda)$ is non increasing.

Let us now define the multivariate stochastic field

$$\mathbf{X}_{\mathrm{t}}:=\mathrm{X}_{t_1}^{(1)}+\dots+\mathrm{X}_{t_d}^{(d)}=\left(\sum_{j=1}^d X_{t_j}^{i,j}\right)_{i\in[d]}\,,\quad\text{for}\quad \mathrm{t}=(t_1,\dots,t_d)\in\mathbb{R}_+^d\,.$$

Then $\mathbf{X}:=\{\mathbf{X}_{\mathrm{t}},\,\mathrm{t}\in\mathbb{R}_+^d\}$ is a particular case of additive Lévy field in the sense of [11]. Its law is characterized by the Laplace exponent $\varphi:=(\varphi_1,\dots,\varphi_d)$, that is

$$\mathbb{E}[e^{-\langle\lambda,\mathbf{X}_{\mathrm{t}}\rangle}]=e^{\langle\mathrm{t},\varphi(\lambda)\rangle}\,,\quad \mathrm{t},\lambda\in\mathbb{R}_+^d\,.$$

Such an additive Lévy field will be called a spectrally positive additive Lévy field (spaLf). This terminology is justified by the results of this section which extend fluctuation theory for spectrally positive Lévy processes. Let us also introduce the field of essentially nonnegative matrices

$$\{\mathbb{X}_{\mathrm{t}},\mathrm{t}\in\mathbb{R}_+^d\}=\{(X_{t_j}^{i,j})_{i,j\in[d]},\mathrm{t}\in\mathbb{R}_+^d\}.$$

Note that the spaLf $\mathbf{X}$ can be defined as $\mathbf{X}_{\mathrm{t}}=\mathbb{X}_{\mathrm{t}}\cdot\mathbf{1}$, where $\mathbf{1}={}^t(1,1,\dots,1)$. Moreover, we emphasize that the spaLf $\mathbf{X}$ carries on the same information as the field of essentially nonnegative matrices $\{\mathbb{X}_{\mathrm{t}},\mathrm{t}\in\mathbb{R}_+^d\}$. For this reason, the terminology 'spaLf' will refer indifferently to $\mathbf{X}$ or to $\mathbb{X}$.

Example. Let us give an example of a 2-dimensional spaLf. Assume that, for $j\in[2]$, the $X^{j,j}$'s are independent Brownian motions $B^{(j)}$ with drifts $a_j\in\mathbb{R}$, that is $X_t^{j,j}=B_t^{(j)}+a_jt$

and that for $i \neq j$, $X^{i,j}$ is a pure drift, that is $X_t^{i,j} = a_{ij}t$, $a_{ij} \geq 0$. Then the spaLf is written as follows,

$$\mathbf{X}_{\mathrm{t}} = {}^t(B_{t_1}^{(1)} + a_1 t_1 + a_{12}t_2, a_{21}t_1 + B_{t_2}^{(2)} + a_2 t_2), \quad \mathrm{t} = (t_1, t_2) \in \mathbb{R}_+^2,$$

the Laplace exponents φ_j are explicitly given by

$$\varphi_j(\lambda) = -\lambda_i a_{ij} - \lambda_j a_j + \frac{1}{2} q_j \lambda_j^2, \quad i \neq j, \lambda \in \mathbb{R}_+^2,$$

and the associated field of essentially nonnegative matrices is

$$\mathbb{X}_{\mathrm{t}} = \begin{pmatrix} B_{t_1}^{(1)} + a_1 t_1 & a_{12}t_2 \\ a_{21}t_1 & B_{t_2}^{(2)} + a_2 t_2 \end{pmatrix}.$$

Now let us define the first hitting times of negative levels of the spaLf $\mathbf{X}$. Let $\mathrm{r} = (r_1, \dots, r_d) \in \mathbb{R}_+^d$, since $\mathbb{X} \in \mathcal{E}_d$ a.s., according to Lemma 2.3 there is almost surely a smallest solution to the system

$$(\mathrm{r}, \mathbb{X}) \qquad \sum_{j=1}^d X_{s_j-}^{i,j} = -r_i,\ i \in [d]_{\mathrm{s}}\,. \tag{3.2}$$

We will denote by $\mathbf{T}_{\mathrm{r}} = (T_{\mathrm{r}}^{(1)}, \dots, T_{\mathrm{r}}^{(d)})$ this solution and use the notation

$$\mathbf{T}_{\mathrm{r}} = \inf\{\mathrm{t} : \mathbf{X}_{\mathrm{t}-} = -\mathrm{r}\}, \text{ with } \mathbf{X}_{\mathrm{t}-} = \left(\sum_{j=1}^d X_{t_j-}^{i,j}\right)_{i \in [d]}. \tag{3.3}$$

Then $\mathbf{T}_{\mathrm{r}}$ will be referred to as the (multivariate) first hitting time of level $-\mathrm{r}$ by the spaLf $\{\mathbf{X}_{\mathrm{t}}, \mathrm{t} \in \mathbb{R}_+^d\}$. Note that according to Lemma 2.3, some of the coordinates of $\mathbf{T}_{\mathrm{r}}$ can be infinite.

Proposition 3.1. *Let $\mathbf{X}$ be a spaLf and for $\mathrm{r} \in \mathbb{R}_+^d$, let $\mathbf{T}_{\mathrm{r}}$ be its first hitting time of level $-\mathrm{r}$ as defined above. Then,*

1. *for all $j \in [d]$ and $\mathrm{r} \in \mathbb{R}_+^d$, $\mathrm{X}^{(j)}_{T_{\mathrm{r}}^{(j)}-} = \mathrm{X}^{(j)}_{T_{\mathrm{r}}^{(j)}}$ a.s. on $\{T_{\mathrm{r}}^{(j)} < \infty\}$. In particular, for all $i \in [d]$,*
$$\sum_{j=1}^d X_{T_{\mathrm{r}}^{(j)}-}^{i,j} = \sum_{j=1}^d X_{T_{\mathrm{r}}^{(j)}}^{i,j} = -r_i \quad \text{a.s. on the set } \{T_{\mathrm{r}}^{(i)} < \infty\}\,. \tag{3.4}$$
2. *For all $\mathrm{r}' \in \mathbb{R}_+^d$ such that $\mathbb{P}(\mathbf{T}_{\mathrm{r}'} \in \mathbb{R}_+^d) > 0$, conditionally on $\{\mathbf{T}_{\mathrm{r}'} \in \mathbb{R}_+^d\}$, the field $\{\mathbf{T}_{\mathrm{r}+\mathrm{r}'} - \mathbf{T}_{\mathrm{r}'}, \mathrm{r} \in \mathbb{R}_+^d\}$ has the same law as the field $\{\mathbf{T}_{\mathrm{r}}, \mathrm{r} \in \mathbb{R}_+^d\}$ and it is independent of the field $\{\mathbf{T}_{\mathrm{r}}, \mathrm{r} \leq \mathrm{r}'\}$. In particular, for all $\mathrm{r}, \mathrm{r}' \in \mathbb{R}_+^d$,*
$$\mathbf{T}_{\mathrm{r}+\mathrm{r}'} \overset{(law)}{=} \mathbf{T}_{\mathrm{r}} + \tilde{\mathbf{T}}_{\mathrm{r}'}, \tag{3.5}$$
where $\tilde{\mathbf{T}}_{\mathrm{r}'}$ is an independent copy of $\mathbf{T}_{\mathrm{r}'}$.
3. *If $\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathbb{R}_+^d) > 0$ for some $\mathrm{r} \in (0,\infty)^d$, then $\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathbb{R}_+^d) > 0$ for all $\mathrm{r} \in \mathbb{R}_+^d$. Under this condition, there is a mapping $\phi = (\phi_1, \dots, \phi_d) : \mathbb{R}_+^d \to \mathbb{R}_+^d$ such that*
$$\mathbb{E}[e^{-\langle \lambda, \mathbf{T}_{\mathrm{r}} \rangle}] = e^{-\langle \mathrm{r}, \phi(\lambda) \rangle}, \quad \lambda \in \mathbb{R}_+^d, \ \mathrm{r} \in \mathbb{R}_+^d\,. \tag{3.6}$$
Moreover, $\phi(\lambda) > \mathbf{0}$ if $\lambda \in (0,\infty)^d$, the mapping ϕ is differentiable and each ϕ_i is a concave function.

Proof. The first assertion is a consequence of quasi-left continuity for Lévy processes. Indeed, let us denote by $(\mathcal{F}_t^{(j)})_{t\ge0}$ the natural filtration generated by $\mathrm{X}^{(j)}$ and set $\mathcal{F}_\infty^{(j)} = \sigma\left(\bigcup_{t\ge0}\mathcal{F}_t^{(j)}\right)$. Then for all $t_j\ge0$, the set

$$\{T_{\mathrm{r}}^{(j)}\le t_j\} = \bigcup_{\substack{\mathrm{u}\in(\mathbb{Q}\cup\{\infty\})^d\\ u_j=t_j}}\left\{\exists\,\mathrm{s}\le\mathrm{u}: r_i+\sum_{k=1}^d X^{i,k}_{s_k-}=0,\ \ i\in[d]_{\mathrm{s}}\right\}$$

belongs to the sigma-field $\mathcal{G}_{t_j}^{(j)} := \sigma\left(\mathcal{F}_{t_j}^{(j)}\cup\left(\bigcup_{i\ne j}\mathcal{F}_\infty^{(i)}\right)\right)$, so that $T_{\mathrm{r}}^{(j)}$ is a stopping time of the filtration $(\mathcal{G}_t^{(j)})_{t\ge0}$. Moreover, since the processes $\mathrm{X}^{(i)}$, $i\in[d]$ are independent, $\mathrm{X}^{(j)}$ is a Lévy process in the latter filtration. Now let us consider the sequence $(\mathbf{T}_{\mathrm{r}_n})_{n\ge1}$, where $\mathrm{r}_n=\mathrm{r}-\mathrm{e}_j/n$. Then from part 2. of Lemma 2.3, $T_{\mathrm{r}_n}^{(j)}$ is an increasing sequence of $(\mathcal{G}_t^{(j)})$-stopping times and this sequence satisfies $\lim_{n\to\infty}T_{\mathrm{r}_n}^{(j)}=T_{\mathrm{r}}^{(j)}$. Therefore from the quasi-left continuity of $\mathrm{X}^{(j)}$, see Proposition I.7 in [3], $\mathrm{X}^{(j)}_{T_{\mathrm{r}}^{(j)}-}=\mathrm{X}^{(j)}_{T_{\mathrm{r}}^{(j)}}$ a.s. on $\{T_{\mathrm{r}}^{(j)}<\infty\}$. It clearly implies (3.4).

In order to prove 2. it suffices to see that conditionally on $\{\mathbf{T}_{\mathrm{r}'}\in\mathbb{R}_+^d\}$, the stochastic field $\{\tilde{\mathbf{X}}_{\mathrm{t}},\mathrm{t}\in\mathbb{R}_+^{\mathrm{d}}\}=\{\mathbf{X}_{\mathbf{T}_{\mathrm{r}'}+\mathrm{t}}+\mathrm{r}',\ \mathrm{t}\in\mathbb{R}_+^{\mathrm{d}}\}$ is independent of $\{\mathbf{X}_{\mathrm{t}},\ \mathrm{t}\le\mathbf{T}_{\mathrm{r}'}\}$ and has the same law as $\{\mathbf{X}_{\mathrm{t}},\ \mathrm{t}\in\mathbb{R}_+^d\}$. We conclude by noticing that $\tilde{\mathbf{T}}_{\mathrm{r}}=\inf\{\mathrm{t}:\tilde{\mathbf{X}}_{\mathrm{t}}=-\mathrm{r}\}=\mathbf{T}_{\mathrm{r}+\mathrm{r}'}-\mathbf{T}_{\mathrm{r}'}$.

Assertion 3. follows from Lemma 2.3 and (3.5). Indeed, if there exists $\mathrm{r}\in(0,\infty)^d$ such that $\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in\mathbb{R}_+^d)>0$ then from Lemma 2.3, for all $\bar{\mathrm{r}}\le\mathrm{r}$, $\mathbf{T}_{\bar{\mathrm{r}}}\le\mathbf{T}_{\mathrm{r}}$ a.s. and in particular, $\mathbb{P}(\mathbf{T}_{\bar{\mathrm{r}}}\in\mathbb{R}_+^d)>0$. On the other hand, for all $\mathrm{r}'\in(0,\infty)^d$, identity (3.5) implies that $\mathbf{T}_{\mathrm{r}'}\overset{(law)}{=}\mathbf{T}^{(1)}_{\mathrm{r}^{(1)}}+\ldots+\mathbf{T}^{(p)}_{\mathrm{r}^{(p)}}$ where $p\ge1$, the $\mathrm{r}^{(i)}$'s are such that $\mathrm{r}^{(i)}\le\mathrm{r}$ for all $i\in[p]$, $\mathrm{r}^{(1)}+\ldots+\mathrm{r}^{(p)}=\mathrm{r}'$, and the $\mathbf{T}^{(i)}$'s are independent copies of $\mathbf{T}$. As a consequence, we obtain $\mathbb{P}(\mathbf{T}_{\mathrm{r}'}\in\mathbb{R}_+^d)=\prod_{i=1}^p\mathbb{P}(\mathbf{T}^{(i)}_{\mathrm{r}^{(i)}}\in\mathbb{R}_+^d)>0$. Now let us prove the second part of this assertion. Let $\mathrm{r}\in(0,\infty)^d$ be such that $\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in\mathbb{R}_+^d)>0$ and let $\lambda\in\mathbb{R}_+^d$, then by (3.5), for all $\mathrm{r}'\in(0,\infty)^d$,

$$\begin{aligned}0<f(\lambda,\mathrm{r}+\mathrm{r}') &= \mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}+\mathrm{r}'}\rangle}]\\ &= \mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}}\rangle}]\mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}'}\rangle}]=f(\lambda,\mathrm{r})f(\lambda,\mathrm{r}')\,.\end{aligned}$$

Since f is right continuous in r, this equation implies that $f(\lambda,\mathrm{r})=\mathrm{e}^{-\langle\mathrm{r},\phi(\lambda)\rangle}$, for some $\phi(\lambda)\in\mathbb{R}^d$. Furthermore take $\mathrm{r}=r\mathrm{e}_i$, for some $r>0$ and $i\in[d]$, so that $\mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}}\rangle}]=e^{-r\phi_i(\lambda)}$. Then from right continuity, $\mathbf{T}_{\mathrm{r}}>\mathbf{0}$ almost surely, so that $f(\lambda,\mathrm{r})<1$, for all $\lambda\in(0,\infty)^d$ and thus $\phi_i(\lambda)\in(0,\infty)$. On the other hand it is plain from (3.6), that the ϕ_j's are concave functions for all $j\in[d]$ and that ϕ is differentiable. $\square$

Note that in (3.4), if for some $j\ne i$, $T_{\mathrm{r}}^{(j)}=\infty$ with positive probability on the set $\{T_{\mathrm{r}}^{(i)}<\infty\}$, then $X^{i,j}\equiv0$, a.s. This is due to the fact that $X^{i,j}$ are subordinators for $i\ne j$, therefore either $X^{i,j}\equiv0$ a.s. or $X^{i,j}_\infty=\infty$ a.s.

Let us emphasize the following direct consequence of Proposition 3.1,

$$\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in\mathbb{R}_+^d)=e^{-\langle\mathrm{r},\phi(\mathbf{0})\rangle}\,, \tag{3.7}$$

so that in particular $\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in\mathbb{R}_+^d)=1$, for all $\mathrm{r}\in(0,\infty)^d$ if and only if $\phi(\mathbf{0})=\mathbf{0}$. Note also that Proposition 3.1 does not allow us a full description of the law of the d-dimensional stochastic field $\{\mathbf{T}_{\mathrm{r}},\mathrm{r}\in\mathbb{R}_+^d\}$. This is the case only when $d=1$. In particular for $d\ge2$, if r and r' are not ordered, then we do not know the joint law of $(\mathbf{T}_\mathrm{r}, \mathbf{T}_{\mathrm{r}'})$. Moreover, looking at part 2. of Proposition 3.1, one is tempted to think that, when $d \geq 2$, the field $\{\mathbf{T}_\mathrm{r},\, \mathrm{r} \in \mathbb{R}_+^d\}$ is a spaLf, but it is actually not the case. Indeed from the construction of this field, the processes $\{\mathbf{T}_{r\mathrm{e}_i},\, r \geq 0\}$, $i \in [d]$ are clearly not independent. However, it is easy to derive from Proposition 3.1, that each of these processes is a multivariate subordinator whose Laplace exponent is ϕ_i. The following result, proved in [4] for $d = 1$, provides an expression of its Lévy measure. Since it is a consequence of further results (e.g. Theorem 4.3), it will be proved at the end of this paper.

Proposition 3.2. *Assume that* $\mathbb{P}(\mathbf{T}_\mathrm{r} \in \mathbb{R}_+^d) > 0$ *for all* $\mathrm{r} \in (0,\infty)^d$. *Then for all* $i \in [d]$, *the process* $\{\mathbf{T}_{r\mathrm{e}_i},\, r \geq 0\}$ *is a multivariate subordinator whose Laplace exponent is* ϕ_i *given in* (3.6). *Assume moreover for all* $j \in [d]$ *and* $t_j > 0$, *the* j*-th column* $\mathrm{X}_{t_j}^{(j)}$ *of the matrix* $\mathbb{X}_\mathrm{t}$ *admits a density which is continuous on* $F_1 \times F_2 \times \cdots \times F_d$, *where* $F_i = \mathbb{R}_+$, *for* $i \neq j$ *and* $F_j = \mathbb{R}$. *Define the matrix* $\widehat{\mathbb{X}}_\mathrm{t} = (\widehat{X}_{t_j}^{i,j})_{i,j\in[d]}$ *by* $\widehat{X}_{t_i}^{i,i} = \sum_{j=1}^{d} X_{t_j}^{i,j}$ *and* $\widehat{X}_{t_j}^{i,j} = X_{t_j}^{i,j}$, $i \neq j$, *and let* $p_\mathrm{t} : M_d(\mathbb{R}) \to \mathbb{R}$ *be the density of* $\widehat{\mathbb{X}}_\mathrm{t}$. *Then the Lévy measure of the multivariate subordinator* $\{\mathbf{T}_{r\mathrm{e}_i},\, r \geq 0\}$ *is given by*

$$\nu_i(\mathrm{dt}) = \int_{\mathbb{R}_+^{d(d-1)}} \frac{\det(-\overline{\mathrm{x}}^{i,i})}{t_1 \dots t_d} p_\mathrm{t}(\mathrm{x}^0) \prod_{k\neq j} \mathrm{d}x_{k,j}\mathrm{dt}, \quad \text{if } d > 1 \text{ and } \nu(\mathrm{d}t) = \frac{p_t(0)}{t}\mathrm{d}t, \quad \text{if } d = 1.$$

Here $\overline{\mathrm{x}}^{i,i}$ *is the matrix* $\overline{\mathrm{x}} = (\overline{x}_{i,j})_{i,j\in[d]}$ *given by* $\overline{x}_{i,i} = -\sum_{j\neq i} x_{i,j}$ *and* $\overline{x}_{i,j} = x_{i,j}$ *for* $i \neq j$ *in which row and column of index* i *have been removed and* $\mathrm{x}^0 = (x_{i,j}^0)_{i,j\in[d]}$, *where* $x_{i,j}^0 = x_{i,j}$, *for* $i \neq j$ *and* $x_{i,i}^0 = 0$.

3.2 Inverting the Laplace transform of spaLf’s

We will now define a d-dimensional Lévy process whose law is obtained from the law of $\mathrm{X}^{(j)}$ through the Esscher transform associated to the martingale

$$(e^{-\langle \mu^{(j)}, \mathrm{X}_t^{(j)}\rangle - t\varphi_j(\mu^{(j)})})_{t\geq 0}\,,$$

for any $\mu^{(j)} \in \mathbb{R}_+^d$. Recall that $(\mathcal{F}_t^{(j)})_{t\geq 0}$ denotes the natural filtration generated by $\mathrm{X}^{(j)}$. Then for $t \geq 0$ and $A \in \mathcal{F}_t^{(j)}$, the law of this new Lévy process is defined by

$$\mathbb{P}^{\mu^{(j)}}(A) = \mathbb{E}[\mathbb{1}_A e^{-\langle \mu^{(j)}, \mathrm{X}_t^{(j)}\rangle - t\varphi_j(\mu^{(j)})}]\,.$$

Let us now consider d independent Lévy processes $\mathrm{X}^{\mu^{(j)},(j)}$, $j \in [d]$ with respective laws $\mathbb{P}^{\mu^{(j)}}$. The Laplace exponent of $\mathrm{X}^{\mu^{(j)},(j)}$ is given by

$$\varphi_j^{\mu^{(j)}}(\lambda) = \varphi_j(\lambda + \mu^{(j)}) - \varphi_j(\mu^{(j)})\,, \quad \lambda \in \mathbb{R}_+^d\,.$$

Moreover, a new spaLf is obtained by setting

$$\mathbf{X}_\mathrm{t}^\mu := \mathrm{X}_{t_1}^{\mu^{(1)},(1)} + \cdots + \mathrm{X}_{t_d}^{\mu^{(d)},(d)}\,, \;\; \mathrm{t} = (t_1, \dots, t_d) \in \mathbb{R}_+^d, \tag{3.8}$$

where $\mu = (\mu^{(1)}, \dots, \mu^{(d)}) \in M_d(\mathbb{R}_+)$ is the matrix whose columns are equal to $\mu^{(j)}$, $j \in [d]$. Let us set $\mathcal{F}_\mathrm{t} = \sigma\{\mathbf{X}_\mathrm{s},\, \mathrm{s} \leq \mathrm{t}\}$ for all $\mathrm{t} \in \mathbb{R}_+^d$, then $\mathcal{F}_\mathrm{t} = \sigma(\mathcal{F}_{t_1}^{(1)} \cup \mathcal{F}_{t_2}^{(2)} \cdots \cup \mathcal{F}_{t_d}^{(d)})$ and the law of the spaLf $\mathbf{X}^\mu$ is given by,

$$\mathbb{P}^\mu(A) = \mathbb{E}[\mathbb{1}_A e^{-\langle\!\langle \mu, \mathbb{X}_\mathrm{t}\rangle\!\rangle - \langle \mathrm{t}, \bar{\varphi}(\mu)\rangle}], \quad \mathrm{t} \in \mathbb{R}_+^d, \quad A \in \mathcal{F}_\mathrm{t}, \tag{3.9}$$

where we have set $\bar{\varphi}(\mu) = (\varphi_1(\mu^{(1)}), \dots, \varphi_d(\mu^{(d)}))$ and where we recall that $\langle\!\langle \mu, \mathbb{X}_{\mathrm{t}} \rangle\!\rangle = \sum_{j\in[d]} \langle \mu^{(j)}, \mathrm{X}^{(j)}_{t_j} \rangle$. We will refer to (3.9) as the Esscher transform of the additive field $\mathbf{X}$. The Laplace exponent of $\mathbf{X}^\mu$ is then

$$\varphi^\mu(\lambda) := (\varphi_1^{\mu^{(1)}}(\lambda), \dots, \varphi_d^{\mu^{(d)}}(\lambda)), \quad \lambda \in \mathbb{R}_+^d .$$

Let us denote by $J_\varphi(\lambda)$, $\lambda \in (0, +\infty)^d$, the transpose of the negative of the Jacobian matrix of φ, that is

$$J_\varphi(\lambda)_{i,j} := -\frac{\partial}{\partial \lambda_i} \varphi_j(\lambda), \quad i, j \in [d] . \tag{3.10}$$

Recall that since all processes $X^{i,j}$, $i, j \in [d]$, are spectrally positive Lévy processes, their expectation is always defined and $\mathbb{E}[X_1^{i,j}] \in (-\infty, \infty]$. Moreover φ is differentiable on $(0, \infty)^d$ and the partial derivatives of φ at $\mathbf{0}$ satisfy $\mathbb{E}[X_1^{i,j}] = -\lim_{\lambda\to\mathbf{0}} \frac{\partial}{\partial\lambda_i}\varphi_j(\lambda)$. We will set $\frac{\partial}{\partial\lambda_i}\varphi_j(\mathbf{0}) := \lim_{\lambda\to\mathbf{0}} \frac{\partial}{\partial\lambda_i}\varphi_j(\lambda)$, and

$$J_\varphi(\mathbf{0})_{i,j} := -\frac{\partial}{\partial \lambda_i} \varphi_j(\mathbf{0}) = \mathbb{E}[X_1^{i,j}] , \quad i, j \in [d] . \tag{3.11}$$

Then let us consider the following hypothesis:

(H) The set $D := \{\lambda \in \mathbb{R}_+^d : \varphi_j(\lambda) > 0,\ j \in [d]\}$ is non empty.

This hypothesis implies in particular that none of the processes $X^{j,j}$, $j \in [d]$ is a subordinator but it is actually stronger as we will see later on. Moreover since all $X^{i,j}$, $i \neq j$ are subordinators, it is clear that actually $D \subset (0, \infty)^d$.

Theorem 3.3. *Let* $\mathrm{r} = (r_1, \dots, r_d) \in \mathbb{R}_+^d$ *and let* $\mathbf{T}_\mathrm{r} = (T_\mathrm{r}^{(1)}, \dots, T_\mathrm{r}^{(d)}) \in \overline{\mathbb{R}}_+^d$ *be the first hitting time of level* $-\mathrm{r}$ *by the spaLf* $\mathbf{X}$*, then*

1. $\mathbf{T}_\mathrm{r} \in \mathbb{R}_+^d$ *holds with positive probability for some (and hence for all)* $\mathrm{r} \in \mathbb{R}_+^d$ *if and only if* (H) *holds.*
2. *Suppose that* (H) *holds, then* $\phi(\lambda) \in D$*, for all* $\lambda \in (0, \infty)^d$*. Moreover, the mapping* $\phi : (0, \infty)^d \to D$ *is a diffeomorphism whose inverse corresponds to the mapping* $\varphi : D \to (0, \infty)^d$*, that is*
$$\varphi(\phi(\lambda)) = \lambda , \quad \lambda \in (0, \infty)^d.$$

Proof. Assume that (H) holds, let $\mu \in D$ and let us consider the spaLf $\mathbf{X}^\mu$ whose law is defined in (3.9). In the present case, μ also denotes the matrix whose each column is equal to μ. Then as already observed $\mu \in (0, \infty)^d$, so that all the random variables $X_1^{\mu,i,j}$ are integrable and the mean matrix of $\mathbf{X}^\mu$ is given by

$$\mathbb{E}[X_1^{\mu,i,j}] = -\frac{\partial}{\partial \lambda_i} \varphi_j(\mu) , \quad i, j \in [d] .$$

It is actually the transpose of the negative of the Jacobian matrix of φ denoted by $J_\varphi(\mu)$ and defined in (3.10). Note that $J_\varphi(\mu)$ is an essentially nonnegative matrix so that from Lemma A.2 in [2], there is a real eigenvalue ρ^μ such that $\mathrm{Re}(\rho) < \rho^\mu$ for all the other eigenvalues ρ. Moreover, since φ_j is a differentiable convex function and $\varphi_j(\mathbf{0}) = 0$, one has

$$\sum_{i=1}^d \frac{\partial}{\partial \lambda_i} \varphi_j(\mu) \mu_i \geq \varphi_j(\mu) > 0 ,$$

so that from Theorem 3 of [1], $J_\varphi(\mu)^T$, and therefore $J_\varphi(\mu)$, is a stable matrix in the sense of [1]. In particular, $\rho^\mu < 0$.

Let us first assume that $J_\varphi(\mu)$ is irreducible. Then from Lemma A.3 in [2], we can choose an eigenvector $\mathrm{v}^\mu=(v_1^\mu,\dots,v_d^\mu)$ associated to ρ^μ such that $v_i^\mu>0$, for all $i\in[d]$. From the law of large numbers of Lévy processes, we obtain

$$\lim_{t\to+\infty} t^{-1}\mathbf{X}^\mu_{t\mathrm{v}^\mu}=\rho^\mu \mathrm{v}^\mu \quad a.s.$$

Therefore, from part 3. of Lemma 2.3, $\{\mathbf{X}^\mu_{\mathrm{t}},\,\mathrm{t}\in\mathbb{R}^d_+\}$ reaches each level $\alpha\mathrm{v}^\mu$, with $\alpha<0$, almost surely. Then from the definition (3.9) of the law of $\mathbf{X}^\mu$, the field $\{\mathbf{X}_{\mathrm{t}},\,\mathrm{t}\in\mathbb{R}^d_+\}$ reaches each level $\alpha\mathrm{v}^\mu$, $\alpha<0$, with positive probability and since $v_i^\mu>0$, $i\in[d]$, from part 2. of Lemma 2.3, it reaches each level $-\mathrm{r}\in\mathbb{R}^d_-$ with positive probability.

Now let us assume that $J_\varphi(\mu)$ is not irreducible that is there exists a permutation matrix P_σ and three matrices A_1,A_2 and B such that A_1 is of size $1\le p\le d-1$ and

$$P_\sigma^{-1}J_{\varphi(\mu)}P_\sigma=\begin{pmatrix}A_1 & 0\\ B & A_2\end{pmatrix}.$$

In particular, for all $(i,j)\in I\times J$ where $I=\{\sigma(1),...,\sigma(p)\}$ and $J=\{\sigma(p+1),...,\sigma(d)\}$,

$$\mathbb{E}[X_1^{\mu,i,j}]=0 \ \text{ that is } \ X_1^{\mu,i,j}=0 \ \text{ a.s.}$$

Therefore we can write for all $\mathrm{r}\in\mathbb{R}^d_+$,

$$\begin{aligned}\mathbb{P}(\mathbf{T}^\mu_{\mathrm{r}}\in\mathbb{R}^d_+) &= \mathbb{P}\left(\exists \mathrm{t}\in\mathbb{R}^d_+:\forall i\in[d],\sum_{j=1}^d X^{\mu,i,j}(t_j)=-r_i\right)\\ &= \mathbb{P}\Bigg(\exists \mathrm{t}\in\mathbb{R}^d_+:\forall i\in I,\sum_{j\in I} X^{\mu,i,j}(t_j)=-r_i\\ &\qquad \text{and } \forall i\in J,\sum_{j\in J}X^{\mu,i,j}(t_j)=-\left(r_i+\sum_{j\in I}X^{\mu,i,j}(t_j)\right)\Bigg).\end{aligned}$$

Let $\mathbf{T}^{\mu,I}_{\mathrm{r}}$ be the smallest solution of the system $(\mathrm{r}_I,\mathbb{X}^{\mu,I})$, where we set $\mathrm{r}_I=(r_i)_{i\in I}$ and $\mathbb{X}^{\mu,I}=(X^{\mu,i,j})_{i,j\in I}$. Then conditioning on the event $\{\mathbf{T}^{\mu,I}_{\mathrm{r}}\in\mathbb{R}^p_+\}$, we obtain

$$\mathbb{P}(\mathbf{T}^\mu_{\mathrm{r}}\in\mathbb{R}^d_+)=\mathbb{P}(\mathbf{T}^{\mu,J}_{\mathrm{r}'}\in\mathbb{R}^{d-p}_+|\mathbf{T}^{\mu,I}_{\mathrm{r}}\in\mathbb{R}^p_+)\mathbb{P}(\mathbf{T}^{\mu,I}_{\mathrm{r}}\in\mathbb{R}^p_+)\,,$$

where we have set $\mathrm{r}'=\left(r_i+\sum_{j\in I}X^{\mu,i,j}(T^{\mu,I,j}_{\mathrm{r}})\right)_{i\in J}$. Then $\mathbf{T}^{\mu,J}_{\mathrm{r}'}$ is the smallest solution of the system $(\mathrm{r}',\mathbb{X}^{\mu,J})$ with $\mathbb{X}^{\mu,J}=(X^{\mu,i,j})_{i,j\in J}$. Thus if A_1 and A_2 are irreducible, then we derive from the previous case that $\mathbb{P}(\mathbf{T}^{\mu,I}_{\mathrm{r}}\in\mathbb{R}^p_+)=1$ and $\mathbb{P}(\mathbf{T}^{\mu,J}_{\mathrm{r}'}\in\mathbb{R}^{d-p}_+|\mathbf{T}^{\mu,I}_{\mathrm{r}}\in\mathbb{R}^p_+)=1$. In other words, we have $\mathbb{P}(\mathbf{T}^\mu_{\mathrm{r}}\in\mathbb{R}^d_+)=1$ and then $\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in\mathbb{R}^d_+)>0$. On the other hand, if A_1 and/or A_2 are not irreducible, then we can repeat this argument.

Conversely, let us assume that $\mathbf{T}_{\mathrm{r}}\in\mathbb{R}^d_+$ holds with positive probability for all $\mathrm{r}\in\mathbb{R}^d_+$. Recall from part 3 of Proposition 3.1 the definition of the function ϕ. Let us show that for all $\lambda\in(0,\infty)^d$, $\varphi(\phi(\lambda))=\lambda$, which implies in particular that $\phi(\lambda)\in D$. It follows from the independence and stationarity of the increments of the spaLf $\{\mathbf{X}_{\mathrm{t}},\,\mathrm{t}\in\mathbb{R}^d_+\}$ that for all $\mathrm{r},\mathrm{t},\lambda\in\mathbb{R}^d_+$,

$$\begin{aligned}\mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}}\rangle}\mathbb{1}_{\{\mathrm{t}<\mathbf{T}_{\mathrm{r}}\}}] &= \int_{C_{\mathrm{r}}}\mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}}\rangle}\mathbb{1}_{\{\mathrm{t}<\mathbf{T}_{\mathrm{r}}\}}\,|\,\mathbf{X}_{\mathrm{t}}=\mathrm{x}]\mathbb{P}(\mathbf{X}_{\mathrm{t}}\in d\mathrm{x})\\ &= \int_{C_{\mathrm{r}}}e^{-\langle\lambda,\mathrm{t}\rangle}\mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}+\mathrm{x}}\rangle}]\mathbb{P}(\mathbf{X}_{\mathrm{t}}\in d\mathrm{x})\\ &= e^{-\langle\lambda,\mathrm{t}\rangle}e^{-\langle\mathrm{r},\phi(\lambda)\rangle}\left[e^{\langle\varphi(\phi(\lambda)),\mathrm{t}\rangle}-\int_{-\infty}^{-r_1}\cdots\int_{-\infty}^{-r_d}e^{-\langle\mathrm{x},\phi(\lambda)\rangle}\mathbb{P}(\mathbf{X}_{\mathrm{t}}\in d\mathrm{x})\right],\end{aligned}$$

where $C_{\rm r}$ is the union of all the sets $E_1 \times \cdots \times E_d$ with at least one $i \in [d]$ such that $E_i =]-r_i, +\infty[$ and for the others $j \in [d]$, $E_j = \mathbb{R}$. Then we derive the identity

$$1-e^{\langle {\rm r},\phi(\lambda)\rangle}\mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{\rm r}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{\rm r}\}^c}] = e^{-\langle \lambda,{\rm t}\rangle}\left[e^{\langle \varphi(\phi(\lambda)),{\rm t}\rangle} - \int_{-\infty}^{-r_1}\cdots\int_{-\infty}^{-r_d} e^{-\langle {\rm x},\phi(\lambda)\rangle}\mathbb{P}(\mathbf{X}_{\rm t}\in d{\rm x})\right]. \tag{3.12}$$

Let ${\rm r}', {\rm r}'' \in (0,\infty)^d$ be such that ${\rm r}' + {\rm r}'' = {\rm r}$, then from Proposition 3.1, $\mathbf{T}_{\rm r}$ can be decomposed as $\mathbf{T}_{\rm r} = \mathbf{T}_{{\rm r}'} + \tilde{\mathbf{T}}_{{\rm r}''}$, where $\tilde{\mathbf{T}}_{{\rm r}''}$ is an independent copy of $\mathbf{T}_{{\rm r}''}$. Moreover $\{{\rm t} < \mathbf{T}_{\rm r}\}^c \subset \{{\rm t} < \mathbf{T}_{{\rm r}'}\}^c \cap \{{\rm t} < \tilde{\mathbf{T}}_{{\rm r}''}\}^c$, so that

$$\mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{\rm r}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{\rm r}\}^c}] \le \mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{{\rm r}'}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{{\rm r}'}\}^c}]\mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{{\rm r}''}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{{\rm r}''}\}^c}].$$

If the coordinates of ${\rm r}$ are integers, then applying this identity recursively, we obtain,

$$\mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{\rm r}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{\rm r}\}^c}] \le \prod_{j=1}^{d}\mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{e_j}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{e_j}\}^c}]^{r_j}\,. \tag{3.13}$$

Then we can find ${\rm t}$ whose coordinates are sufficiently small so that for all j,

$$\mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{e_j}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{e_j}\}^c}] < \mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{e_j}\rangle}] = e^{-\phi_j(\lambda)}.$$

Therefore $\lim_{{\rm r}\to\infty} e^{\langle {\rm r},\phi(\lambda)\rangle}\prod_{j=1}^{d}\mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{e_j}\rangle}\mathbb{1}_{\{{\rm t}<\mathbf{T}_{e_j}\}^c}]^{r_j} = 0$ and from (3.13) we derive that the left member of (3.12) tends to 1, while the right member tends to $e^{-\langle \lambda,{\rm t}\rangle}e^{\langle \varphi(\phi(\lambda)),{\rm t}\rangle}$, which shows that $\varphi(\phi(\lambda)) = \lambda$. This is true in particular for all $\lambda \in (0,\infty)^d$ and hence D is not empty. This achieves the proof of both assertions 1. and 2. $\square$

From part 1. of Theorem 3.3, assuming (H) for a spaLf $\mathbf{X}$ ensures that $\mathbf{X}$ hits all negative levels in a finite time with positive probability. When $d = 1$, this is simply assuming that the spectrally positive Lévy process we consider is not a subordinator.

Example. Let us go back to our 2-dimensional example. Assume that $q_j > 0$, $j \in [2]$, where q_j is defined in (3.1). After some calculations, we obtain the following explicit form of the set D defined in hypothesis (H),

$$D = \left\{\lambda \in \mathbb{R}_+^2 : \lambda_1 > \left(\frac{a_1 + \sqrt{\Delta_1(\lambda_2)}}{q_1} \vee 0\right) \text{ and } \lambda_2 > \left(\frac{a_2 + \sqrt{\Delta_2(\lambda_1)}}{q_2} \vee 0\right)\right\},$$

where $\Delta_j(\lambda_i) = a_j^2 + 2a_{ij}q_j\lambda_i$ for all $j \in [2]$ and $i \ne j$. Note that this set is not empty and so assumption (H) holds. In particular, thanks to Theorem 3.3, the spaLf $\mathbf{X}$ reaches all the level $-{\rm r} \in \mathbb{R}_-^2$ with positive probability and according to the second part of this theorem, we know that the mapping φ admits an inverse ϕ on the set D. This inverse $\phi = (\phi_1, \phi_2)$ is given by

$$\phi_j(\lambda) = \frac{1}{q_j}\sqrt{2q_j\lambda_j + a_j^2 + 2a_{ij}q_j\phi_i(\lambda)} + \frac{a_j}{q_j}, \quad j \in [2], i \ne j, \lambda \in \mathbb{R}_+^2\,.$$

Moreover ϕ is the Laplace exponent of the field of first hitting times of negative levels by $\mathbf{X}$ defined for all ${\rm r} = (r_1, r_2) \in \mathbb{R}_+^2$ by

$$\mathbf{T}_{\rm r} = \inf\left\{{\rm t} \ge \mathbf{0} : \left\{\begin{array}{rcl} B_{t_1}^{(1)} + a_1t_1 + a_{12}t_2 & = & -r_1 \\ a_{21}t_1 + B_{t_2}^{(2)} + a_2t_2 & = & -r_2 \end{array}\right.\right\}.$$

3.3 Asymptotic behaviour of spaLf's

In order to carry on with the general study of the fluctuation of the spaLf $\mathbf{X}$, we shall now give a characterization of the condition $\phi(\mathbf{0}) = \mathbf{0}$ in terms of the Jacobian matrix $J_\varphi(\mathbf{0})$. As a first remark, note that if for some $j \in [d]$, $J_\varphi(\mathbf{0})_{j,j} > 0$, then $\lim_{t\to+\infty} X_t^{j,j} = +\infty$ a.s. and hence the field $\{\mathbf{X}_{\mathrm{t}},\, \mathrm{t} \in \mathbb{R}_+^d\}$ cannot reach all the levels $-\mathrm{r} \in \mathbb{R}_-^d$ with probability one. Therefore, by Proposition 3.1, $\phi(\mathbf{0}) > \mathbf{0}$ whenever there is j such that $J_\varphi(\mathbf{0})_{j,j} > 0$.

Recall that whenever the essentially nonnegative matrices $J_\varphi(\lambda)$, defined in (3.10) and (3.11) for $\lambda \in [0,\infty)^d$ have finite entries and are irreducible, according to the Perron-Frobenius theory, there are real eigenvalues ρ^λ with multiplicity equal to 1 and such that the real part of any other eigenvalue is less than ρ^λ, see Appendix A of [2]. We set $\rho^0 = \rho$.

Theorem 3.4. *Assume that (H) holds and that $J_\varphi(\mathbf{0})$ is irreducible, then*

1. *the values $\mathbf{0}$ and $\phi(\mathbf{0})$ are the only roots of the equation $\varphi(\lambda) = \mathbf{0}$, $\lambda \in \mathbb{R}_+^d$. Furthermore, either $\phi(\mathbf{0})$ is equal to $\mathbf{0}$ or it belongs to $(0,\infty)^d$.*
2. *If $\mathbb{E}[X_1^{i,j}] = \infty$, for some $i,j \in [d]$, then $\phi(\mathbf{0}) > \mathbf{0}$. Assume that $\mathbb{E}[X_1^{i,j}] < \infty$, for all $i,j \in [d]$, then $\phi(\mathbf{0}) = \mathbf{0}$ if and only if $\rho \le 0$.*

Proof. Let us assume that $J_\varphi(\mathbf{0})$ is irreducible. Since $\varphi : D \to (0,\infty)^d$ is the inverse of $\phi : (0,\infty)^d \to D$, $\phi(\mathbf{0})$ is the only solution of the equation $\varphi(\lambda) = \mathbf{0}$ on $\overline{D}$. Indeed, let $\mu \in \overline{D}$ such that $\varphi(\mu) = \mathbf{0}$ and $\mu_n \in D$ such that $\lim_{n\to+\infty} \mu_n = \mu$. Then by continuity, $\lim_{n\to+\infty} \varphi(\mu_n) = \mathbf{0}$ and $\phi(\mathbf{0}) = \lim_{n\to+\infty} \phi(\varphi(\mu_n)) = \lim_{n\to+\infty} \mu_n$, so that $\mu = \phi(\mathbf{0})$.

Now let $\mu \in \mathbb{R}_+^d \setminus \{\mathbf{0}, \phi(\mathbf{0})\}$ be a solution of the equation $\varphi(\lambda) = \mathbf{0}$ and $\mathrm{u} = \dfrac{\mu}{||\mu||}$. Then we consider, for all $j \in [d]$, the function $f_j : a \in \mathbb{R} \mapsto \varphi_j(\mu + a\mathrm{u})$. Let us first note that since φ_j is convex, so is f_j. Furthermore, for all $j \in [d]$, we have $f_j(0) = \varphi_j(\mu) = 0 = \varphi_j(\mathbf{0}) = f_j(-||\mu||)$. On the one hand, if there exists $j \in [d]$ such that $\mu_j = 0$, then for all $a \in \mathbb{R}$, $\mu_j + au_j = 0$ that is $f_j(a) = \varphi_j(\mu + a\mathrm{u}) \le 0$. Since 0 and $-||\mu|| < 0$ are zeros of the real convex function f_j, it implies that f_j is constant equal to 0. In other words, for all $t \ge 0$,

$$\mathbb{E}\left[e^{-\sum\limits_{i\neq j}(\mu_i + au_i)X_t^{i,j}}\right] = e^{t\varphi_j(\mu + a\mathrm{u})} = 1$$

and then for all $i \in [d]$, $X^{i,j} \equiv 0$ a.s. that is $J_\varphi(\mathbf{0})$ is reducible. Since we assumed $J_\varphi(\mathbf{0})$ irreducible, we necessarily have $\mu_j > 0$, $j \in [d]$ and then, by convexity, f_j is negative on $(-||\mu||, 0)$ and positive on $(0,+\infty)$. In other words, for all integers $j \in [d]$ and for all $\epsilon > 0$, $\varphi_j(\mu + \epsilon\mathrm{u}) > 0$ that is $\mu \in \overline{D}$ which is a contradiction. As a consequence, when $J_\varphi(\mathbf{0})$ is irreducible, there is at most two solutions of the equation $\varphi(\lambda) = \mathbf{0}$, $\lambda \in \mathbb{R}_+^d$ which are $\mathbf{0}$ and $\phi(\mathbf{0}) \in \overline{D}$. Furthermore, when $J_\varphi(\mathbf{0})$ is irreducible, we have seen that $\phi(\mathbf{0}) = \mathbf{0}$ or $\phi(\mathbf{0}) \in (0,\infty)^d$.

Let us now prove assertion 2. Suppose that $\mathbb{E}[X_1^{i,j}] = \infty$, for some $i,j \in [d]$. Then for all $\lambda \in (0,\infty)^d$ small enough, $\varphi_j(\lambda) < 0$. Indeed, let $\lambda \in (0,\infty)^d$. Since the spectrally positive Lévy process $\langle \lambda, X_t^{(j)} \rangle$ drifts to ∞, for all $\alpha \in (0,\infty)$ small enough, its characteristic exponent evaluated at α is negative, that is $\varphi_j(\alpha \cdot \lambda) < 0$. But if $\phi(\mathbf{0}) = \mathbf{0}$, since $\mathbf{0} \in \overline{D}$, there is $\lambda \in (0,\infty)^d$ small enough such that $\varphi_j(\lambda) > 0$. Therefore, $\phi(\mathbf{0}) > \mathbf{0}$.

Suppose now that $\mathbb{E}[X_1^{i,j}] < \infty$, for all $i,j \in [d]$ and that $\rho < 0$. Let $\mathrm{u} = (u_1, \dots, u_d)$ be the unique right eigenvector corresponding to ρ such that $u_i > 0$ for all $i \in [d]$, and $u_1 + \dots + u_d = 1$, see Lemma A.2 in [2]. Then from the law of large numbers,

$$\lim_{t\to+\infty} t^{-1}\mathbf{X}_{t\mathrm{u}} = \rho\mathrm{u}\,, \quad a.s.$$

Therefore, $\{\mathbf{X}_{\mathrm{t}},\, \mathrm{t}\in\mathbb{R}_+^d\}$ reaches a.s. all the levels $\alpha\mathrm{u}$, $\alpha<0$ and from Proposition 3.1 it reaches all the levels $-\mathrm{r}\in\mathbb{R}_-^d$ a.s. We conclude from (3.7) that $\phi(\mathbf{0})=\mathbf{0}$.

Assume that $\rho=0$. Let $\mathrm{u}=(u_1,\dots,u_d)$ be a right eigenvector corresponding to ρ, then from the law of large numbers,

$$\lim_{t\to+\infty} t^{-1}\mathbf{X}_{t\mathrm{u}}=\mathbf{0}\,,\quad a.s.$$

Therefore, for all $i\in[d]$, the process $Y^i=(Y^i_t)_{t\ge 0}$, defined for all $t\ge 0$, by $Y^i_t=\sum_{j=1}^d X^{i,j}_{tu_j}$ is a real Lévy process such that

$$\lim_{t\to+\infty} t^{-1}Y^i_t=0\,,\quad a.s.$$

that is, for all $i\in[d]$, Y^i oscillates. On the other hand, if $\phi(\mathbf{0})>\mathbf{0}$, then, by convexity of the φ_j's, there exists $\lambda\in\mathbb{R}_+^d$ such that $\varphi_j(\lambda)<0$, for all $j\in[d]$. Consequently, for all direction $\mathrm{v}\in\mathbb{R}_+^d$, we have

$$\mathbb{E}[e^{-\langle\lambda,\mathbf{X}_{t\mathrm{v}}\rangle}]=e^{\langle t\mathrm{v},\varphi(\lambda)\rangle}\underset{t\to+\infty}{\to}0\,.$$

It implies that for all direction $\mathrm{v}\in\mathbb{R}_+^d$, the Lévy process $\langle\lambda,\mathbf{X}_{t\mathrm{v}}\rangle$ tends to ∞ in probability (and hence almost surely), as $t\to\infty$. In particular, for $\mathrm{v}=\mathrm{u}$, there exists $i\in[d]$ such that Y^i_t tends to ∞ almost surely, as $t\to\infty$, which is a contradiction. In conclusion, $\phi(\mathbf{0})=\mathbf{0}$.

Conversely, assume that $\phi(\mathbf{0})=\mathbf{0}$ then $\mathbf{0}\in\overline{D}$ and by convexity, there exists $\mu\in(0,+\infty)^d$, small enough, such that $\varphi_i(\mu)>0$, for all $i\in[d]$. Recall from (3.7) and (3.9) the definition of the Esscher transform $\mathbf{X}^\mu$ of the spaLf $\mathbf{X}$, with $\mu^{(1)}=\cdots=\mu^{(d)}=\mu$. We have seen in the proof of Theorem 3.1 that the Perron-Frobenius eigenvalue of $J_\varphi(\mu)$ satisfies $\rho^\mu<0$. Since the φ_j's are $\mathcal{C}^\infty$-functions, for all $i,j\in[d]$, $\frac{\partial}{\partial\lambda_i}\varphi_j$ are continuous and hence $\lim_{\mu\to 0}J_\varphi(\mu)=J_\varphi(\mathbf{0})$. Furthermore, the eigenvalues of the matrix $J_\varphi(\mu)$ depend continuously on its entries because they are the roots of its characteristic polynomial whose coefficients are polynomial functions of the entries of the matrix. Then since $\rho^\mu=\max_{i\in[d]}\mathrm{Re}(\lambda_i^\mu)$ and $\rho=\max_{i\in[d]}\mathrm{Re}(\lambda_i)$ where λ_i^μ and λ_i are respectively the eigenvalues of $J_\varphi(\mu)$ and $J_\varphi(\mathbf{0})$, we have that $\lim_{\mu\to0}\rho^\mu=\rho\le 0$. □

Assuming (H), we will say that the additive Lévy field $(\mathbf{X}_{\mathrm{t}},\mathrm{t}\in\mathbb{R}_+^d)$ drifts to $-\infty$, oscillates or drifts to $+\infty$ according as $\rho<0$, $\rho=0$ or $\rho>0$.

Example. In our example, we already have the explicit form of φ, the set D and the inverse ϕ. Let us now find the solutions of the equation $\varphi(\lambda)=\mathbf{0}$, $\lambda\in\mathbb{R}_+^2$. Assume that $J_\varphi(\mathbf{0})$ is irreducible, that is $a_{ij}>0$ for all $i\neq j$. Then the solutions of the equation $\varphi(\lambda)=\mathbf{0}$, $\lambda\in\mathbb{R}_+^2$ are $\mathbf{0}=(0,0)$ and points of the form $\left(\frac{a_1+\sqrt{\Delta_1(\lambda_2)}}{q_1},\frac{a_2+\sqrt{\Delta_2(\lambda_1)}}{q_2}\right)$ where $\Delta_j(\lambda_i)=a_j^2+2a_{ij}q_j\lambda_i$, $j\in[2], i\neq j$. It is easy to check that there is only one solution of the second kind. It belongs to $(0,+\infty)^2$ or it is equal to $\mathbf{0}$. According to the expression of ϕ, $\phi(\mathbf{0})$ is this solution. We can show that $\phi(\mathbf{0})=\mathbf{0}$ if and only if $a_1<0$, $a_2<0$ and $a_1a_2\ge a_{1,2}a_{2,1}$. Furthermore, we can compute the Perron-Frobenius eigenvalue ρ of the Jacobian $J_\varphi(\mathbf{0})$. It has the form

$$\rho=\frac{a_1+a_2+\sqrt{(a_1-a_2)^2+4a_{1,2}a_{2,1}}}{2}\,.$$

Then it is easy to see that $\rho \leq 0$ if and only if $a_1 < 0$, $a_2 < 0$ and $a_1 a_2 \geq a_{1,2}a_{2,1}$. In conclusion, we find $\phi(\mathbf{0}) = \mathbf{0} \Leftrightarrow \rho \leq 0$.

Note that if $J_\varphi(\mathbf{0})$ is reducible then at least one of the $a_{i,j}$ is equal to zero, for $i, j \in [d]$. Then φ has at most four zeros. These are the values:

$$\mathbf{0}\ ,\ \left(\frac{2a_1}{q_1}, 0\right)\ ,\ \left(0, \frac{2a_2}{q_2}\right) \ \text{ and } \ \left(\frac{a_1 + \sqrt{\Delta_1(\frac{2a_2}{q_2})}}{q_1}, \frac{a_2 + \sqrt{\Delta_2(\frac{2a_1}{q_1})}}{q_2}\right) = \phi(\mathbf{0}),$$

whenever they belong to $\mathbb{R}_+^2$.

Remark 3.5. By carefully reading the proof of Theorem 3.4, it appears that we have proved a little more than what is in the statement.

Indeed, in part 1. we have proved that if there exists a solution to the equation $\varphi(\lambda) = \mathbf{0}$ in $(0, +\infty)^d$, then it is unique and equal to $\phi(\mathbf{0})$. This is when $J_\varphi(\mathbf{0})$ is irreducible but we can see from the proof that this is also true when $J_\varphi(\mathbf{0})$ is reducible. Let us also notice that in the reducible case, there may exist solutions $\lambda \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ with $\lambda_j = 0$ for some $j \in [d]$ as the above example shows.

Moreover it can be derived from arguments in the proof of part 2. that when $\phi(\mathbf{0}) > \mathbf{0}$, for each direction $\mathrm{v} \in \mathbb{R}_+^d$, almost surely, there is at least one coordinate of the field $\mathbf{X}$ which goes to $+\infty$.

4 On the distribution of the field $(\mathbf{T}_\mathrm{r}, \mathbb{X}_{\mathbf{T}_\mathrm{r}})$

Let us recall the definition of the matrix valued field $\mathbb{X} = \{\mathbb{X}_\mathrm{t}, \mathrm{t} \in \mathbb{R}_+^d\}$ given in the beginning of Section 3. As already noticed, this field carries on the same information as the spaLf $\mathbf{X}$. However, whereas the vector $\mathbf{X}_{\mathbf{T}_\mathrm{r}}$ is deterministic on the set $\{\mathbf{T}_\mathrm{r} \in \mathbb{R}_+^d\}$ (and is actually equal to $-\mathrm{r}$), the matrix $\mathbb{X}_{\mathbf{T}_\mathrm{r}}$ is random whenever $d \geq 2$. From another point of view, the fact that the field $\mathrm{r} \mapsto (\mathbf{T}_\mathrm{r}, \mathbb{X}_{\mathbf{T}_\mathrm{r}})$ has independent and stationary increments (see the next theorem) induces an analogy with fluctuation theory in dimension 1. More specifically, this bivariate field can be considered as the analogue of the scale process describing the fluctuations of any one dimensional Lévy process at its infimum. The aim of this section is to characterize the law of the field $\mathrm{r} \mapsto (\mathbf{T}_\mathrm{r}, \mathbb{X}_{\mathbf{T}_\mathrm{r}})$, first through its Laplace exponent and then from a Kemperman's type identity relating its law to that of the field $\mathbb{X}$.

4.1 Characterization through the Laplace transform

Recall that we denote by $\mu^{(j)}$ the j-th column of the matrix $\mu = (\mu_{i,j})_{i,j\in[d]}$. Then given a spaLf $\mathbf{X}$ we define the set

$$\mathcal{M}_\varphi = \{(\lambda, \mu) \in \mathbb{R}_+^d \times M_d(\mathbb{R}_+) : \lambda_j \geq \varphi_j(\mu^{(j)}),\ j \in [d]\}.$$

Theorem 4.1. *Assume that (H) holds. Let $\mathrm{r} = (r_1, \dots, r_d) \in \mathbb{R}_+^d$ and let $\mathbf{T}_\mathrm{r}$ be the first hitting time of level $-\mathrm{r}$ by the spaLf $\mathbf{X}$, then there exits a mapping $\Phi = (\Phi_1, \dots, \Phi_d) : \mathcal{M}_\varphi \to \mathbb{R}_+^d$ such that*

$$\mathbb{E}\left[e^{-\langle \lambda, \mathbf{T}_\mathrm{r}\rangle - \langle\!\langle \mu, \mathbb{X}_{\mathbf{T}_\mathrm{r}}\rangle\!\rangle} \mathbb{1}_{\{\mathbf{T}_\mathrm{r} \in \mathbb{R}_+^d\}}\right] = e^{-\langle \mathrm{r}, \Phi(\lambda,\mu)\rangle}, \quad (\lambda, \mu) \in \mathcal{M}_\varphi.$$

Moreover Φ satisfies the equations,

$$\varphi_j(\mu^{(j)} + \Phi(\lambda, \mu)) = \lambda_j, \quad j \in [d], \quad (\lambda, \mu) \in \mathcal{M}_\varphi, \tag{4.1}$$

and it is explicitly determined by

$$\Phi(\lambda, \mu) = \phi^\mu(\lambda_1 - \varphi_1(\mu^{(1)}), \dots, \lambda_d - \varphi_d(\mu^{(d)})) \tag{4.2}$$

where ϕ^μ is the inverse of the Laplace exponent $\varphi^\mu = (\varphi_1^{\mu^{(1)}}, \dots, \varphi_d^{\mu^{(d)}})$ of the Esscher transform $\mathbf{X}^\mu$ defined in (3.8).

Proof. Let us first note that the random field $\{\mathbf{M}_{\rm t},\, {\rm t} \in \mathbb{R}_+^d\} := \{e^{-\langle \bar\varphi(\mu), {\rm t}\rangle - \langle\!\langle \mu, \mathbb{X}_{\rm t}\rangle\!\rangle},\, {\rm t} \in \mathbb{R}_+^d\}$, where $\bar\varphi(\mu) = (\varphi_1(\mu^{(1)}), \dots, \varphi_d(\mu^{(d)}))$, is a multi-indexed martingale with respect to the filtration $\mathcal{F}_{\rm t} = \sigma\{\mathbf{X}_{\rm s},\, {\rm s} \le {\rm t}\} = \sigma(\mathcal{F}^{(1)}_{t_1} \cup \mathcal{F}^{(2)}_{t_2} \dots \cup \mathcal{F}^{(d)}_{t_d})$, ${\rm t} \in \mathbb{R}_+^d$ in the sense of [12]. Fix ${\rm r} = (r_1, \dots, r_d) \in \mathbb{R}_+^d$ and define the sequence of multivariate random times $\mathbf{T}_{n,{\rm r}} = (T^{(1)}_{n,{\rm r}}, \dots, T^{(d)}_{n,{\rm r}})$, $n \ge 1$ by

$$T^{(i)}_{n,{\rm r}} = \sum_{k\ge 0} 2^{-n}(k+1)\mathbb{1}_{\{2^{-n}k \le T^{(i)}_{\rm r} < 2^{-n}(k+1)\}} + \infty \cdot \mathbb{1}_{\{T^{(i)}_{\rm r} = \infty\}}.$$

Then $\mathbf{T}_{\rm r}$ and $\mathbf{T}_{n,{\rm r}}$, $n \ge 1$ are stopping times of the filtration $(\mathcal{F}_{\rm t})_{{\rm t}\in\mathbb{R}_+^d}$ in the sense of [12]. Moreover, for each $i \in [d]$, the sequence $(T^{(i)}_{n,{\rm r}})_{n\ge1}$ is non increasing and tends to $T^{(i)}_{\rm r}$ almost surely. Now for all ${\rm u} \in \mathbb{R}_+^d$, define $\mathbf{T}^{({\rm u})}_{n,{\rm r}}$ by

$$\mathbf{T}^{({\rm u})}_{n,{\rm r}} := \begin{cases} \mathbf{T}_{n,{\rm r}} & \text{on } \{\mathbf{T}_{n,{\rm r}} \le {\rm u}\} \\ {\rm u} & \text{on } \{\mathbf{T}_{n,{\rm r}} \le {\rm u}\}^c. \end{cases}$$

Then $\mathbf{T}^{({\rm u})}_{n,{\rm r}}$ is a stopping time (see for instance the proof of Lemma (2.3) in [12]). Moreover,

$$\mathbf{M}_{\mathbf{T}^{({\rm u})}_{n,{\rm r}}} = \sum_{{\rm v}\in {\rm D}_n, {\rm v}\le {\rm u}} \mathbf{M}_{\rm v}\mathbb{1}_{\{\mathbf{T}_{n,{\rm r}}={\rm v}\}} + \mathbf{M}_{\rm u}\mathbb{1}_{\{\mathbf{T}_{n,{\rm r}}\le {\rm u}\}^c} \le \sum_{{\rm v}\in {\rm D}_n, {\rm v}\le {\rm u}} \mathbf{M}_{\rm v} + \mathbf{M}_{\rm u},$$

where ${\rm D}_n := \{{\rm v} \in \mathbb{R}_+^d : {\rm v} = 2^{-n}k,\, k \ge 0\}$. Since the set $\{{\rm v} \in {\rm D}_n, {\rm v} \le {\rm u}\}$ is finite, $\mathbb{E}\left[\mathbf{M}_{\mathbf{T}^{({\rm u})}_{n,{\rm r}}}\right] < \infty$. Moreover $\mathbf{T}^{({\rm u})}_{n,{\rm r}}$ and $\mathbf{M}_{\rm u}$ clearly satisfy the conditions (2.4) and (2.5) of Lemma (2.3) in [12]. Therefore, in virtue of this lemma,

$$\mathbb{E}\left[\mathbf{M}_{\mathbf{T}^{({\rm u})}_{n,{\rm r}}}\right] = 1.$$

Then $\lim_{n\to\infty} \mathbf{T}^{({\rm u})}_{n,{\rm r}} = \mathbf{T}^{({\rm u})}_{\rm r}$ almost surely, where

$$\mathbf{T}^{({\rm u})}_{\rm r} := \begin{cases} \mathbf{T}_{\rm r} & \text{on } \{\mathbf{T}_{\rm r} \le {\rm u}\} \\ {\rm u} & \text{on } \{\mathbf{T}_{\rm r} \le {\rm u}\}^c, \end{cases}$$

so that by Fatou's Lemma and the right continuity of $\{\mathbf{M}_{\rm t},\, {\rm t} \in \mathbb{R}_+^d\}$, we obtain as n tends to ∞, $\mathbb{E}\left[\mathbf{M}_{\mathbf{T}^{({\rm u})}_{\rm r}}\right] \le 1$. Then by applying Fatou's Lemma again, we obtain as each coordinate of ${\rm u}$ tends to ∞ that $\mathbb{E}\left[\mathbf{M}_{\mathbf{T}_{\rm r}}\mathbb{1}_{\{\mathbf{T}_{\rm r}\in\mathbb{R}_+^d\}}\right] \le 1$. It implies that for all $(\lambda, \mu) \in \mathcal{M}_\varphi$, $\mathbb{E}\left[e^{-\langle \lambda, \mathbf{T}_{\rm r}\rangle - \langle\!\langle \mu, \mathbb{X}_{\mathbf{T}_{\rm r}}\rangle\!\rangle}\mathbb{1}_{\{\mathbf{T}_{\rm r}\in\mathbb{R}_+^d\}}\right] \le 1$.

Then we prove in the same way as for (3.5) in Proposition 3.1, that for all ${\rm r}, {\rm r}' \in \mathbb{R}_+^d$,

$$(\mathbf{T}_{{\rm r}+{\rm r}'}, \mathbb{X}_{\mathbf{T}_{{\rm r}+{\rm r}'}})\mathbb{1}_{\{\mathbf{T}_{{\rm r}+{\rm r}'}\in\mathbb{R}_+^d\}} \stackrel{(law)}{=} (\mathbf{T}_{\rm r} + \mathbf{T}'_{{\rm r}'}, \mathbb{X}_{\mathbf{T}_{\rm r}} + \mathbb{X}'_{\mathbf{T}'_{{\rm r}'}})\mathbb{1}_{\{\mathbf{T}_{\rm r}+\mathbf{T}'_{{\rm r}'}\in\mathbb{R}_+^d\}}, \tag{4.3}$$

where $\mathbb{X}'$ is an independent copy of $\mathbb{X}$ and $\mathbf{T}'$ is its first hitting time process. Recall that under assumption (H), $\mathbb{P}(\mathbf{T}_{\rm r} \in \mathbb{R}_+^d) > 0$ for all ${\rm r} \in \mathbb{R}_+^d$. The existence of the mapping Φ follows by using (4.3), in the same way as for the existence of the mapping ϕ in 3. of Proposition 3.1. (Note that in particular $\Phi(\lambda, 0) = \phi(\lambda)$, $\lambda \in \mathbb{R}_+^d$.)

Then it is readily seen that

$$(\mathbf{T}_{\rm r}, \mathbb{X}_{\mathbf{T}_{\rm r}}) = ({\rm r}, \mathbb{X}_{\rm r}) + (\tilde{\mathbf{T}}_{{\rm r}+\mathbf{X}_{\rm r}}, \tilde{\mathbb{X}}_{\tilde{\mathbf{T}}_{{\rm r}+\mathbf{X}_{\rm r}}}) \text{ a.s. on } \{\mathbf{T}_{\rm r} \in \mathbb{R}_+^d\}, \tag{4.4}$$

where $\tilde{\mathbb{X}}_{\rm t} = \mathbb{X}_{\rm r+t} - \mathbb{X}_{\rm r}$ and $\tilde{\mathbf{T}}_{\rm k} = \inf\{{\rm t} \ge \mathbf{0} : \tilde{\mathbf{X}}_{\rm t} = -{\rm k}\}$. Since $\mathbf{X}$ is a spaLf, for all ${\rm t} \in \mathbb{R}_+^d$, $\tilde{\mathbf{X}}_{\rm t}$ has the same law as $\mathbf{X}_{\rm t}$ and is independent of $\{\mathbf{X}_{\rm s} : {\rm s} \le {\rm r}\}$. Thus conditionally on $\{\mathbf{T}_{\rm r} \in \mathbb{R}_+^d\}$, $\tilde{\mathbf{T}}_{{\rm r}+\mathbf{X}_{\rm r}}$ and $\tilde{\mathbb{X}}_{\tilde{\mathbf{T}}_{{\rm r}+\mathbf{X}_{\rm r}}}$ are independent of $\mathbb{X}_{\rm r}$. Let $(\lambda, \mu) \in \mathcal{M}_\varphi$, then using (4.4), we obtain

$$e^{-\langle r,\Phi(\lambda,\mu)\rangle} = e^{-\langle \lambda,{\rm r}\rangle} \int_{M_d(\mathbb{R})} \mathbb{E}[e^{-\langle \lambda,\mathbf{T}_{{\rm r}+\bar{\rm x}}\rangle} e^{-\langle\!\langle \mu,\mathbb{X}_{\mathbf{T}_{{\rm r}+\bar{\rm x}}}\rangle\!\rangle} \mathbb{1}_{\{\mathbf{T}_{{\rm r}+\bar{\rm x}} \in \mathbb{R}_+^d\}}] e^{-\langle\!\langle \mu,{\rm x}\rangle\!\rangle} \mathbb{P}(\mathbb{X}_{\rm r} \in {\rm dx}),$$

where ${\rm x} = (x^{(1)}, \dots, x^{(d)})$ and $\bar{\rm x} = \sum_{j\in[d]} {\rm x}^{(j)} = \left(\sum_{j\in[d]} x^{1,j}, \dots, \sum_{j\in[d]} x^{d,j} \right)$. This equality can also be written as

$$\begin{aligned} e^{-\langle r,\Phi(\lambda,\mu)\rangle} &= e^{-\langle \lambda,{\rm r}\rangle} \int_{M_d(\mathbb{R})} e^{-\langle {\rm r}+\bar{\rm x},\Phi(\lambda,\mu)\rangle} e^{-\langle\!\langle \mu,{\rm x}\rangle\!\rangle} \mathbb{P}(\mathbb{X}_{\rm r} \in {\rm dx}) \\ &= e^{-\langle \lambda,{\rm r}\rangle} e^{-\langle r,\Phi(\lambda,\mu)\rangle} \mathbb{E}[e^{-\langle\!\langle \mu+\hat{\Phi}(\lambda,\mu),\mathbb{X}_{\rm r}\rangle\!\rangle}], \end{aligned}$$

where $\hat{\Phi}(\lambda, \mu)$ is the matrix whose all columns are equal to $\Phi(\lambda, \mu)$. Thanks to the independence of the ${\rm X}^{(j)}$'s, the latter equality is reduced to

$$e^{\langle \lambda,{\rm r}\rangle} = \prod_{j\in[d]} \mathbb{E}[e^{-\langle \mu^{(j)}+\Phi(\lambda,\mu),{\rm X}^{(j)}_{r_j}\rangle}] .$$

As a consequence, the Laplace exponent Φ of $(\mathbf{T}_{\rm r}, \mathbb{X}_{\mathbf{T}_{\rm r}})$ satisfy (4.1).

Now recall the definition of the Esscher transform ${\rm X}^{\mu^{(j)},(j)}$ of each ${\rm X}^{(j)}$ given after Proposition 3.1, with Laplace exponent

$$\varphi_j^{\mu^{(j)}}(\lambda) = \varphi_j(\lambda + \mu^{(j)}) - \varphi_j(\mu^{(j)}), \quad \lambda \in \mathbb{R}_+^d, \quad j \in [d] .$$

From these Esscher transforms we defined, see (3.8), the spaLf $\mathbf{X}^\mu$ by

$$\mathbf{X}_{\rm t}^\mu = \sum_{j\in[d]} {\rm X}_{t_j}^{\mu^{(j)},(j)}, \quad {\rm t} \in \mathbb{R}_+^d.$$

Let $D_\mu := \{\lambda \in \mathbb{R}_+^d : \varphi_j^{\mu^{(j)}}(\lambda) > 0, j \in [d]\}$. Then under assumption (H), from part 1. of Theorem 3.3 and from the absolute continuity relationship (3.9) between $\mathbf{X}$ and $\mathbf{X}^\mu$, the set D_μ is not empty. Moreover, thanks to Theorem 3.3, the Laplace exponent $\varphi^\mu = (\varphi_1^{\mu^{(1)}}, \dots, \varphi_d^{\mu^{(d)}})$ of $\mathbf{X}^\mu$ is a diffeomorphism from D_μ, whose inverse $\phi^\mu : (0, \infty)^d \to D_\mu$ is the Laplace exponent of the field $\{\mathbf{T}_{\rm r}^\mu, {\rm r} \in \mathbb{R}_+^d\}$, where $\mathbf{T}_{\rm r}^\mu := \inf\{{\rm t} \ge 0 : \mathbf{X}_{\rm t}^\mu = -{\rm r}\}$.

On the other hand, from (4.1), Φ satisfies

$$\varphi_j^{\mu^{(j)}}(\Phi(\lambda, \mu)) = \lambda_j - \varphi_j(\mu^{(j)}), \quad j \in [d], \quad (\lambda, \mu) \in \mathcal{M}_\varphi .$$

Thus the Laplace exponent Φ of the couple $(\mathbf{T}_{\rm r}, \mathbb{X}_{\mathbf{T}_{\rm r}})$ exists and is given for all $(\lambda, \mu) \in \mathcal{M}_\varphi$ such that $\lambda_j > \varphi_j(\mu^{(j)})$, $j \in [d]$ by

$$\Phi(\lambda, \mu) = \phi^\mu(\lambda_1 - \varphi_1(\mu^{(1)}), \dots, \lambda_d - \varphi_d(\mu^{(d)})). \tag{4.5}$$

Finally this relation is extended to the whole set $\mathcal{M}_\varphi$ by continuity. □

Remark 4.2. We emphasize that Theorem 4.1 provides an extension of the case $d = 1$. More specifically, (4.1) can be compared to relation (2), p. 191 in [3].

Example. An explicit form of Φ can be derived from our example. Let $\mu^{(1)} = {}^t(\mu_{1,1}, \mu_{2,1}) \in \mathbb{R}_+^2$, $\mu^{(2)} = {}^t(\mu_{1,2}, \mu_{2,2}) \in \mathbb{R}_+^2$ and $\mu = (\mu^{(1)}, \mu^{(2)})$. Then the Esscher transform $\mathbf{X}^\mu$ has Laplace exponent $\varphi^\mu = (\varphi_1^{\mu^{(1)}}, \varphi_2^{\mu^{(2)}})$ where for all $j \in [2]$ and $\lambda \in \mathbb{R}_+^2$,

$$\varphi_j^{\mu^{(j)}}(\lambda) = \varphi_j(\lambda) + q_j\mu_{i,j}\lambda_j = -a_{ij}\lambda_i - (a_j - q_j\mu_{j,j})\lambda_j + \frac{1}{2}q_j\lambda_j^2 .$$

Assume $q_j > 0, j \in [2]$. Hence after some calculations, we obtain

$$D_{\varphi^\mu} = \Bigg\{\lambda \in \mathbb{R}_+^2 : \lambda_1 > \left(\frac{a_1 - q_1\mu_{1,1} + \sqrt{\Delta_1^\mu(\lambda_2)}}{q_1} \vee 0\right) \text{ and } \lambda_2 > \left(\frac{a_2 - q_2\mu_{2,2} + \sqrt{\Delta_2^\mu(\lambda_1)}}{q_2} \vee 0\right)\Bigg\},$$

where $\Delta_j^\mu(\lambda_i) = (a_j - q_j\mu_{j,j})^2 + 2a_{ij}q_j\lambda_i$ for all $j \in [2]$ and $i \neq j$. Note that $D_\varphi \subset D_{\varphi^\mu}$. In particular, if (H) is satisfied then (H^μ) is satisfied too and both sets D_φ and D_{φ^μ} are non empty. Under this assumption, thanks to Theorem 3.3, the spaLf $\mathbf{X}^\mu$ reaches all the level $-\mathrm{r} \in \mathbb{R}_-^2$ with positive probability and according to the second part of this theorem, we know that the mapping φ^μ admits an inverse ϕ^μ on the set D^μ. This inverse $\phi^\mu = (\phi_1^\mu, \phi_2^\mu)$ is given by

$$\phi_j^\mu(\lambda) = \frac{1}{q_j}\sqrt{2q_j\lambda_j + (a_j - q_j\mu_{j,j})^2 + 2a_{ij}q_j\phi_i^\mu(\lambda)} + \frac{a_j}{q_j} - \mu_{j,j}, \quad j \in [2], i \neq j, \lambda \in \mathbb{R}_+^2 .$$

Then according to Theorem (4.1), the Laplace exponent $\Phi = (\Phi_1, \Phi_2)$ of the field $(\mathbf{T}_\mathrm{r}, \mathbb{X}_{\mathbf{T}_\mathrm{r}})$ is given for all $(\lambda, \mu) \in \mathcal{M}_\varphi$ and $j \in [2]$ by

$$\begin{aligned}\Phi_j(\lambda, \mu) &= \phi_j^\mu(\lambda_1 - \varphi_1(\mu^{(1)}), \lambda_2 - \varphi_2(\mu^{(2)})) \\ &= \frac{1}{q_j}\sqrt{2q_j(\lambda_j - \varphi_j(\mu^{(j)})) + (a_j - q_j\mu_{j,j})^2 + 2a_{ij}q_j\phi_i^\mu(\lambda_1 - \varphi_1(\mu^{(1)}), \lambda_2 - \varphi_2(\mu^{(2)}))} \\ &\quad + \frac{a_j}{q_j} - \mu_{j,j}.\end{aligned}$$

4.2 An explicit form of the distribution

Let us define the set

$$\widehat{M}_d(\mathbb{R}) = \{\mathrm{x} \in M_d(\mathbb{R}) : \mathrm{x} \text{ is essentially nonnegative and } \mathrm{x} \cdot \mathbf{1} \leq 0\}$$

endowed with some matrix norm, $\|\cdot\|$ and equipped with its Borel σ-field. From Theorem 4.1, the measure $\mathbb{P}(\mathbf{T}_\mathrm{r} \in \mathrm{dt},\, \mathbb{X}_\mathrm{t} \in \mathrm{dx})\mathrm{dr}$ on $\mathbb{R}_+^d \times \mathbb{R}_+^d \times \widehat{M}_d(\mathbb{R})$ has Laplace transform

$$\begin{aligned}\int_{\mathbb{R}_+^d \times \mathbb{R}_+^d \times \widehat{M}_d(\mathbb{R})} & e^{-\langle\alpha,\mathrm{r}\rangle - \langle\lambda,\mathrm{t}\rangle - \langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\, \mathbb{P}(\mathbf{T}_\mathrm{r} \in \mathrm{dt},\, \mathbb{X}_\mathrm{t} \in \mathrm{dx})\mathrm{dr} \\ &= [(\alpha_1 + \Phi_1(\lambda,\mu))(\alpha_2 + \Phi_2(\lambda,\mu)) \dots (\alpha_d + \Phi_d(\lambda,\mu))]^{-1}. \qquad (4.6)\end{aligned}$$

The following result shows that this measure can be expressed only in terms of the law of the spaLf.

Theorem 4.3. *Assume that (H) is satisfied. Then for all $\alpha \in \mathbb{R}_+^d$ and $(\lambda, \mu) \in \mathcal{M}_\varphi$,*

$$\begin{aligned}\int_{\mathbb{R}_+^d \times \mathbb{R}_+^d \times \widehat{M}_d(\mathbb{R})} & e^{-\langle\alpha,\mathrm{r}\rangle - \langle\lambda,\mathrm{t}\rangle - \langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\, \mathbb{P}(\mathbf{T}_\mathrm{r} \in \mathrm{dt},\, \mathbb{X}_\mathrm{t} \in \mathrm{dx})\mathrm{dr} \\ &= \int_{\mathbb{R}_+^d \times \widehat{M}_d(\mathbb{R})} e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle - \langle\lambda,\mathrm{t}\rangle - \langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\, \frac{det(-\mathrm{x})}{t_1 t_2 \dots t_d}\mathbb{P}(\mathbb{X}_\mathrm{t} \in \mathrm{dx})\,\mathrm{dt}\,. \qquad (4.7)\end{aligned}$$

In other words, the measure

$$\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathrm{dt},\, \mathbb{X}_{\mathrm{t}} \in \mathrm{dx})\mathrm{dr}, \quad \mathrm{t} \in \mathbb{R}_+^d,\ \mathrm{x} \in \widehat{M}_d(\mathbb{R}),\ \mathrm{r} \in \mathbb{R}_+^d,$$

is the image of the measure

$$\frac{\mathit{det}(-\mathrm{x})}{t_1 t_2 \dots t_d} \mathbb{P}(\mathbb{X}_{\mathrm{t}} \in \mathrm{dx})\,\mathrm{dt}, \quad \mathrm{t} \in \mathbb{R}_+^d,\ \mathrm{x} \in \widehat{M}_d(\mathbb{R}),$$

through the mapping $(t, \mathrm{x}) \mapsto (t, \mathrm{x}, -\mathrm{x} \cdot \mathbf{1})$.

When $d = 1$, the above identity can be read as

$$\mathbb{P}(T_x \in \mathrm{d}t)\mathrm{d}x = \frac{-x}{t} \mathbb{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t, \quad (t, x) \in (0, \infty) \times (-\infty, 0), \tag{4.8}$$

and is known as Kemperman's identity for spectrally positive Lévy processes. It can be found in [3], see Proposition VII.2.

We shall prove Theorem 4.3 through discrete approximation. As a first step, we need to recall the discrete time and space counterpart of spaLf's. Those are matrix valued fields of the form $\{\mathbb{S}_{\mathrm{n}},\, \mathrm{n} \in \mathbb{Z}_+^d\} = \{(S^{i,j}_{n_j})_{i,j \in [d]},\, \mathrm{n} \in \mathbb{Z}_+^d\}$, where the columns $\mathrm{S}^{(j)} = {}^t(S^{1,j}, \dots, S^{d,j})$, $j \in [d]$ are independent random walks. Moreover, all coordinates $S^{i,j}$ start from 0 and take their values in $k^{-1}\mathbb{Z}$, where $k \geq 1$ is some integer which will be fixed until mentioned otherwise. For $i \neq j$ they are non decreasing and for $i = j$ they are downward skip free, that is $S^{i,i}_n - S^{i,i}_{n-1} \geq -k^{-1}$, for all $n \geq 1$. This setting is introduced in [7] (for $k = 1$ and up to transposition of the matrix $\mathbb{S}$). Equivalently to the continuous case, we define the field $\mathbf{S} := \mathbb{S} \cdot \mathbf{1}$ and its first hitting time process

$$\mathbf{T}^{\mathbf{S}}_{\mathrm{r}} := \inf\{\mathrm{n} : \mathbf{S}_{\mathrm{n}} = -\mathrm{r}\}, \quad \mathrm{r} \in k^{-1}\mathbb{Z}_+^d,$$

see Lemma 2.2 in [7]. The field $\mathbb{S}$ (or equivalently $\mathbf{S}$) will be called a downward skip free random field (dsfrf for short). An essential result for the proof of Theorem 4.3, is the following extension of the ballot theorem

$$\mathbb{P}(\mathbf{T}^{\mathbf{S}}_{\mathrm{r}} = \mathrm{n}, \mathbf{S}_{\mathrm{n}} = \mathrm{x}) = \frac{k^d \mathrm{det}(-\mathrm{x})}{n_1 \dots n_d} \mathbb{P}(\mathbf{S}_{\mathrm{n}} = \mathrm{x}), \tag{4.9}$$

for all $\mathrm{n} \in \mathbb{N}^d$ and all essentially nonnegative matrix x of $M_d(k^{-1}\mathbb{Z})$ such that $\mathrm{x} \cdot \mathbf{1} = -\mathrm{r}$. (Here we have used the notation $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$.) Identity (4.9) is proved for $k = 1$ in [7], see Theorem 3.4 therein. Its extension to any $k \geq 1$ is straightforward.

The next step is to consider lattice valued spaLf's. Let us first define these processes. Let $\mathrm{X}^{(j)} = {}^t(X^{1,j}, \dots, X^{d,j})$, $j \in [d]$ be a family of d independent d-dimensional Lévy processes such that for $i \neq j$, $X^{i,j}$ is non-decreasing $k^{-1}\mathbb{Z}$-valued Lévy process and for each $j \in [d]$, $X^{j,j}$ is a $k^{-1}\mathbb{Z}$-valued Lévy process such that for all $t > 0$, $X^{j,j}_t - X^{j,j}_{t-} \geq -k^{-1}$. Then there exists a dsfrf $\mathbb{S}$ as defined above and d independent Poisson processes $N^{(j)}$, $j \in [d]$ also independent of $\mathbb{S}$ such that

$$X^{i,j}_t = S^{i,j}_{N^{(j)}_t}, \quad i, j \in [d], \quad t \geq 0. \tag{4.10}$$

The random fields $\{\mathbb{X}_{\mathrm{t}},\, \mathrm{t} \in \mathbb{R}_+^d\} = \{(X^{i,j}_{t_j})_{i,j \in [d]},\, \mathrm{t} \in \mathbb{R}_+^d\}$ and $\mathbf{X} = \mathbb{X} \cdot \mathbf{1}$ will be referred to as lattice valued spaLf's. Let $(e^{(j)}_n)_{n \geq 0}$, $j \in [d]$ be the sequences of exponentially distributed random variables satisfying

$$N^{(j)}_t = \sum_{n \geq 0} \mathbb{1}_{\{e^{(j)}_1 + \dots + e^{(j)}_n \leq t\}}.$$

The first hitting time process of $\mathbf{X}$ can be defined in the same way as for spaLf's in Lemma 2.3 and Proposition 3.1. It is denoted by

$$\mathbf{T}_{\mathrm{r}} = \inf\{\mathrm{t} : \mathbf{X}_{\mathrm{t}} = -\mathrm{r}\}, \quad \mathrm{r} \in k^{-1}\mathbb{Z}_+^d.$$

We can easily check that the latter is related to the first hitting time process of $\mathbf{S}$ through the identity,

$$T_{\mathrm{r}}^{(j)} = \sum_{l=1}^{T_{\mathrm{r}}^{(j),\mathbf{S}}} e_l^{(j)}, \quad j \in [d]. \tag{4.11}$$

The following proposition is a direct consequence of (4.9). Although it can also be found in [6] for $k = 1$, we give a more direct proof here.

Proposition 4.4. *Let* $\{\mathbb{X}_{\mathrm{t}},\, \mathrm{t} \in \mathbb{R}_+^d\} = \{(X_{t_j}^{i,j})_{i,j\in[d]},\, \mathrm{t} \in \mathbb{R}_+^d\}$ *be a lattice valued spaLf. Then for fixed* $\mathrm{r} \in k^{-1}\mathbb{Z}_+^d$*, the joint law of* $(\mathbf{T}_{\mathrm{r}}, \mathbb{X}_{\mathbf{T}_{\mathrm{r}}})$ *is given by*

$$\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathrm{dt},\, \mathbb{X}_{\mathrm{t}} = \mathrm{x}) = \frac{k^d \det(-\mathrm{x})}{t_1 t_2 \dots t_d} \mathbb{P}(\mathbb{X}_{\mathrm{t}} = \mathrm{x}) \mathrm{d}t_1 \mathrm{d}t_2 \dots \mathrm{d}t_d\,,$$

for all essentially nonnegative matrices x *of* $M_d(k^{-1}\mathbb{Z})$ *such that* $\mathrm{x} \cdot \mathbf{1} = -\mathrm{r}$.

Proof. Let r and $\mathrm{x} = (x_{i,j})_{i,j\in[d]}$ be as in the statement. Then the straightforward identity $\mathbb{S}_{\mathbf{T}_{\mathrm{r}}^{\mathbf{S}}} = \mathbb{X}_{\mathbf{T}_{\mathrm{r}}}$ together with expressions (4.10) and (4.11) allow us to write,

$$\begin{aligned}
\mathbb{P}(\mathbf{T}_r \in \mathrm{dt}, \mathbb{X}_{\mathrm{t}} = \mathrm{x}) &= \mathbb{P}\left(\sum_{l=1}^{T_{\mathrm{r}}^{(j),\mathbf{S}}} e_l^{(j)} \in \mathrm{d}t_j,\, j \in [d],\, \mathbb{S}_{\mathbf{T}_{\mathrm{r}}^{\mathbf{S}}} = \mathrm{x}\right) \\
&= \sum_{\mathrm{n}\in\mathbb{N}^d} \prod_{j\in[d]} \mathbb{P}\left(\sum_{l=1}^{n_j} e_l^{(j)} \in \mathrm{d}t_j\right) \mathbb{P}(\mathbf{T}_{\mathrm{r}}^{\mathbf{S}} = \mathrm{n}, \mathbb{S}_{\mathrm{n}} = \mathrm{x}) \\
&= \sum_{\mathrm{n}\in\mathbb{N}^d} \frac{\lambda_1^{n_1} t_1^{n_1} \dots \lambda_d^{n_d} t_d^{n_d}}{n_1! \dots n_d!} e^{-\langle\lambda,\mathrm{t}\rangle} \frac{k^{-d}\det(-\mathrm{x})}{t_1 \dots t_d} \mathbb{P}(\mathbb{S}_{\mathrm{n}} = \mathrm{x})\mathrm{dt} \\
&= \frac{k^{-d}\det(-\mathrm{x})}{t_1 \dots t_d} \sum_{\mathrm{n}\in\mathbb{N}^d} \prod_{j\in[d]} \mathbb{P}(N_{t_j}^{(j)} = n_j) \mathbb{P}(\mathbb{S}_{\mathrm{n}} = \mathrm{x})\mathrm{dt} \\
&= \frac{k^{-d}\det(-\mathrm{x})}{t_1 \dots t_d} \mathbb{P}(\mathbb{X}_{\mathrm{t}} = \mathrm{x})\mathrm{dt},
\end{aligned}$$

which proves our result. □

From now on, we will add k as a superscript to all objects referring to the discrete valued spaLf defined above. For instance, the latter will be denoted by $\mathbb{X}^{(k)} = (X^{i,j,k})_{i,j\in[d]}$ or $\mathbf{X}^{(k)}$, where $\mathrm{X}^{(j),k} = {}^t(X^{1,j,k}, \dots, X^{d,j,k})$. It is pretty clear that lattice valued spaLf's satisfy analogous properties to those of spaLf's introduced in Section 3. In particular, the discrete time field $\mathrm{r} \mapsto (\mathbf{T}_{\mathrm{r}}^{(k)}, \mathbb{X}_{\mathbf{T}_{\mathrm{r}}^{(k)}}^{(k)})$, $\mathrm{r} \in k^{-1}\mathbb{Z}_+^d$ has independent and stationary increments and can be treated in a very similar way as its continuous space counterpart involved in Theorem 4.1. That is why we will content ourselves with stating the next theorem as well as some preliminary results without giving any proof.

Recall the definition of the Laplace exponent $\varphi_j^{(k)}$ of $\mathrm{X}^{(j),k}$, that is

$$\mathbb{E}[e^{-\langle\lambda,\mathrm{X}_t^{(j),k}\rangle}] = e^{t\varphi_j^{(k)}(\lambda)}\,, \quad t \geq 0\,, \quad \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}_+^d\,.$$

Then as in Theorem 3.3, we can prove that the hypothesis

$$(H^{(k)}) \qquad D^{(k)} := \{\lambda \in \mathbb{R}_+^d : \varphi_j^{(k)}(\lambda) > 0,\, j \in [d]\} \text{ is non empty}$$

is equivalent to the fact that $\mathbf{T}_{\mathrm{r}}^{(k)} \in \mathbb{R}_+^d$ holds with positive probability, for all $\mathrm{r} \in k^{-1}\mathbb{Z}_+^d$. As in Theorem 3.3, the proof of this equivalence is based on the Esscher transform $\mathbf{X}^{(k),\mu}$, for $\mu \in M_d(\mathbb{R}_+)$ whose Laplace exponent is given by

$$\varphi_j^{(k),\mu^{(j)}}(\lambda) = \varphi_j^{(k)}(\lambda + \mu^{(j)}) - \varphi_j^{(k)}(\mu^{(j)}), \quad \lambda \in \mathbb{R}_+^d. \tag{4.12}$$

Let us define the set

$$\mathcal{M}_\varphi^{(k)} := \{(\lambda, \mu) \in \mathbb{R}_+^d \times M_d(\mathbb{R}_+) : \lambda_j \geq \varphi_j^{(k)}(\mu^{(j)}),\, j \in [d]\}.$$

The following theorem is the analog of Theorem 4.1 for lattice valued spaLf's.

Theorem 4.5. *Assume that* $(H^{(k)})$ *holds. Let* $\mathrm{r} = (r_1, \dots, r_d) \in k^{-1}\mathbb{Z}_+^d$ *and let* $\mathbf{T}_{\mathrm{r}}^{(k)}$ *be the first hitting time of level* $-\mathrm{r}$ *by the spaLf* $\mathbf{X}^{(k)}$*, then there exits a mapping* $\Phi^{(k)} : \mathcal{M}_\varphi^{(k)} \to \mathbb{R}_+^d$ *such that*

$$\mathbb{E}\left[e^{-\langle \lambda, \mathbf{T}_{\mathrm{r}}^{(k)}\rangle - \langle\!\langle \mu, \mathbb{X}^{(k)}_{\mathbf{T}_{\mathrm{r}}^{(k)}}\rangle\!\rangle} \mathbb{1}_{\{\mathbf{T}_{\mathrm{r}}^{(k)} \in \mathbb{R}_+^d\}}\right] = e^{-\langle \mathrm{r}, \Phi^{(k)}(\lambda,\mu)\rangle}, \quad (\lambda, \mu) \in \mathcal{M}_\varphi^{(k)}.$$

Moreover $\Phi^{(k)}$ *satisfies the equations,*

$$\varphi_j^{(k)}(\mu^{(j)} + \Phi^{(k)}(\lambda, \mu)) = \lambda_j, \quad j \in [d], \quad (\lambda, \mu) \in \mathcal{M}_\varphi^{(k)}, \tag{4.13}$$

and it is explicitly determined by

$$\Phi^{(k)}(\lambda, \mu) = \phi^{(k),\mu}(\lambda_1 - \varphi_1^{(k)}(\mu^{(1)}), \dots, \lambda_d - \varphi_d^{(k)}(\mu^{(d)})), \tag{4.14}$$

where $\phi^{(k),\mu}$ *is the inverse of the Laplace exponent* $\varphi^{(k),\mu}$ *of the Esscher transform* $\mathbf{X}^{(k),\mu}$ *recalled in* (4.12).

In order to end the proof of Theorem 4.3, we need to prove that any spaLf is the weak limit of a sequence of lattice valued spaLf's. The index k is now a variable that will be taken to infinity.

Lemma 4.6. *Let* Y *be a* d*-dimensional Lévy process whose all coordinates are spectrally positive. Then there exists a sequence of* $(k^{-1}\mathbb{Z})^d$*-valued Lévy processes* $\mathrm{Y}^{(k)}$ *which converges weakly in the* J_1 *Skohorod's topology toward* Y*. Moreover, the sequence* $(\mathrm{Y}^{(k)})$ *can be chosen so that for each* k*, all coordinates of* $\mathrm{Y}^{(k)}$ *take their values in the set* $\{-k^{-1}, 0, k^{-1}, 2k^{-1}, 3k^{-1}, \dots\}$.

The proof of this lemma is transferred to the Appendix. We have now gathered all necessary ingredients for the proof of Theorem 4.3.

Proof of Theorem 4.3. Let $(\mathbb{X}^{(k)})_{k\geq 1}$ be a sequence of lattice valued spaLf's such that each sequence of columns $(\mathrm{X}^{(j),k})_{k\geq 1}$, where $\mathrm{X}^{(j),k} = {}^t(X^{1,j,k}, \dots, X^{d,j,k})$, converges weakly to $\mathrm{X}^{(j)}$. The existence of such a sequence is ensured by Lemma 4.6. This convergence means in particular that

$$\lim_{k\to\infty} \varphi_j^{(k)}(\lambda) = \varphi_j(\lambda)\,, \quad \lambda \geq \mathbf{0}, \quad j \in [d]\,. \tag{4.15}$$

Since (H) is satisfied, by continuity of the functions φ_j and from (4.15), there is k_0 such that for all $k \geq k_0$, $(H^{(k)})$ is satisfied. Then let $k \geq k_0$ and let $\widehat{M}_{d,\mathrm{r}}(k^{-1}\mathbb{Z})$ be the set of essentially nonnegative matrices x of $M_d(k^{-1}\mathbb{Z})$ such that $\mathrm{x} \cdot \mathbf{1} = -\mathrm{r}$. We derive from Theorem 4.5 that for all $\alpha \in \mathbb{R}_+^d$ and $(\lambda, \mu) \in \mathcal{M}_\varphi^{(k)}$,

$$\sum_{\mathrm{r} \in k^{-1}\mathbb{Z}_+^d} k^{-d} e^{-\langle \alpha, \mathrm{r}\rangle} \int_{\mathbb{R}_+^d} \sum_{\mathrm{x} \in \widehat{M}_{d,\mathrm{r}}(k^{-1}\mathbb{Z})} e^{-\langle \lambda, \mathrm{t}\rangle - \langle\!\langle \mu, \mathrm{x}\rangle\!\rangle}\, \mathbb{P}(\mathbf{T}_{\mathrm{r}}^{(k)} \in \mathrm{dt},\, \mathbb{X}_{\mathrm{t}}^{(k)} = \mathrm{x})$$
$$= [k(1 - e^{-k^{-1}(\alpha_1 + \Phi_1^{(k)}(\lambda,\mu))}) \times \cdots \times k(1 - e^{-k^{-1}(\alpha_d + \Phi_d^{(k)}(\lambda,\mu))})]^{-1}. \tag{4.16}$$

Now take $(\lambda,\mu)\in\mathcal{M}_\varphi$ such that $\lambda_j>\varphi_j(\mu^{(j)})$ for all $j\in[d]$. Then by continuity of φ_j, $j\in[d]$, there is k_0' such that for all $k\ge k_0'$, $(\lambda,\mu)\in\mathcal{M}_\varphi^{(k)}$. Clearly $(\varphi_j^{(k),\mu^{(j)}})_{k\ge1}$ defined in (4.12) converges pointwise to $\varphi_j^{\mu^{(j)}}$, for all $j\in[d]$. Hence, the sequence of inverses $(\phi^{(k),\mu})_{k\ge1}$ also converges pointwise to ϕ^μ. Therefore, from (4.2), (4.14) and by continuity, $(\Phi^{(k)}(\lambda,\mu))_{k\ge1}$ converges to $\Phi(\lambda,\mu)$.

Now let us extend the definition of $\mathbf{T}^{(k)}_{\mathrm{r}}$ to all $\mathrm{r}\in\mathbb{R}_+^d$ by setting $\mathbf{T}^{(k)}_{\mathrm{r}}:=\mathbf{T}^{(k)}_{\mathrm{r}_k}$, where $\mathrm{r}_k=k^{-1}(\lfloor kr_1\rfloor,\dots,\lfloor kr_d\rfloor)$ and where $\lfloor x\rfloor$ denotes the lower integer part of x. Then by taking k to infinity in (4.16), we obtain from (4.6) that for all $\alpha\in\mathbb{R}_+^d$ and $(\lambda,\mu)\in\mathcal{M}_\varphi$ such that $\lambda_j>\varphi_j(\mu^{(j)})$, for all $j\in[d]$,

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{\mathrm{r}\in k^{-1}\mathbb{Z}_+^d}k^{-d}e^{-\langle\alpha,\mathrm{r}\rangle}\int_{\mathbb{R}_+^d}\sum_{\mathrm{x}\in\widehat{M}_{d,\mathrm{r}}(k^{-1}\mathbb{Z})}e^{-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\,\mathbb{P}(\mathbf{T}^{(k)}_{\mathrm{r}}\in\mathrm{dt},\,\mathbb{X}^{(k)}_{\mathrm{t}}=\mathrm{x})\\
&=\lim_{k\to\infty}\int_{\mathbb{R}_+^d\times\mathbb{R}_+^d\times\widehat{M}_d(\mathbb{R})}e^{-\langle\alpha,\mathrm{r}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\,\mathbb{P}(\mathbf{T}^{(k)}_{\mathrm{r}}\in\mathrm{dt},\,\mathbb{X}^{(k)}_{\mathrm{t}}\in\mathrm{dx})\mathrm{dr}\\
&=[(\alpha_1+\Phi_1(\lambda,\mu)))(\alpha_2+\Phi_2(\lambda,\mu)))\dots(\alpha_d+\Phi_d(\lambda,\mu))]^{-1}\\
&=\int_{\mathbb{R}_+^d\times\mathbb{R}_+^d\times\widehat{M}_d(\mathbb{R})}e^{-\langle\alpha,\mathrm{r}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\,\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in\mathrm{dt},\,\mathbb{X}_{\mathrm{t}}\in\mathrm{dx})\mathrm{dr}.
\end{aligned}\tag{4.17}$$

On the other hand, let $\widehat{M}_d(k^{-1}\mathbb{Z})$ be the set of essentially nonnegative matrices x of $M_d(k^{-1}\mathbb{Z})$ such that $\mathrm{x}\cdot\mathbf{1}\le0$. Then as a direct consequence of Proposition 4.4, we obtain that for all $\alpha\in\mathbb{R}_+^d$ and $(\lambda,\mu)\in\mathcal{M}_\varphi^{(k)}$,

$$\begin{aligned}
&\sum_{\mathrm{r}\in k^{-1}\mathbb{Z}_+^d}k^{-d}e^{-\langle\alpha,\mathrm{r}\rangle}\int_{\mathbb{R}_+^d}\sum_{\mathrm{x}\in\widehat{M}_{d,\mathrm{r}}(k^{-1}\mathbb{Z})}e^{-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\,\mathbb{P}(\mathbf{T}^{(k)}_{\mathrm{r}}\in\mathrm{dt},\,\mathbb{X}^{(k)}_{\mathrm{t}}=\mathrm{x})\\
&\quad=\int_{\mathbb{R}_+^d}\sum_{\mathrm{r}\in k^{-1}\mathbb{Z}_+^d,\,\mathrm{x}\in\widehat{M}_{d,\mathrm{r}}(k^{-1}\mathbb{Z})}e^{-\langle\alpha,\mathrm{r}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}=\mathrm{x})\,\mathrm{dt}\\
&\quad=\int_{\mathbb{R}_+^d}\sum_{\mathrm{x}\in\widehat{M}_d(k^{-1}\mathbb{Z})}e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}=\mathrm{x})\,\mathrm{dt}\\
&\quad=\int_{\mathbb{R}_+^d}\int_{\widehat{M}_d(\mathbb{R})}e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}\in\mathrm{dx})\,\mathrm{dt}\,.
\end{aligned}$$

Then it follows from the above calculation and from (4.17) that for all $\alpha\in\mathbb{R}_+^d$ and $(\lambda,\mu)\in\mathcal{M}_\varphi$ such that $\lambda_j>\varphi_j(\mu^{(j)})$, $j\in[d]$,

$$\begin{aligned}
&\int_{\mathbb{R}_+^d\times\mathbb{R}_+^d\times\widehat{M}_d(\mathbb{R})}e^{-\langle\alpha,\mathrm{r}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\,\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in\mathrm{dt},\,\mathbb{X}_{\mathrm{t}}\in\mathrm{dx})\mathrm{dr}\\
&\qquad=\lim_{k\to\infty}\int_{\mathbb{R}_+^d}\int_{\widehat{M}_d(\mathbb{R})}e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}\in\mathrm{dx})\,\mathrm{dt}\,.
\end{aligned}\tag{4.18}$$

Now, we derive from the weak convergence of $\mathbb{X}^{(k)}_{\mathrm{t}}$ toward $\mathbb{X}_{\mathrm{t}}$ for each t that

$$\lim_{k\to\infty}\int_{\widehat{M}_d(\mathbb{R})}e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\det(-\mathrm{x})\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}\in\mathrm{dx})=\int_{\widehat{M}_d(\mathbb{R})}e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\det(-\mathrm{x})\mathbb{P}(\mathbb{X}_{\mathrm{t}}\in\mathrm{dx}),$$

so that for all $\varepsilon>0$,

$$\begin{aligned}
&\lim_{k\to\infty}\int_{\{\mathrm{t}\ge\varepsilon\cdot\mathbf{1}\}}\int_{\widehat{M}_d(\mathbb{R})}e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}\in\mathrm{dx})\,\mathrm{dt}\\
&\qquad=\int_{\{\mathrm{t}\ge\varepsilon\cdot\mathbf{1}\}}\int_{\widehat{M}_d(\mathbb{R})}e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}_{\mathrm{t}}\in\mathrm{dx})\,\mathrm{dt}.
\end{aligned}$$

Then from Proposition 4.4,

$$\begin{aligned}
&\int_{\{\mathrm{t}\geq\varepsilon\cdot\mathbf{1}\}^c}\int_{\widehat{M}_d(\mathbb{R})} e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}\in d\mathrm{x})\,d\mathrm{t}\\
&\qquad=\int_{\{\mathrm{t}\geq\varepsilon\cdot\mathbf{1}\}^c\times\mathbb{R}^d_+\times\widehat{M}_d(\mathbb{R})} e^{-\langle\alpha,\mathrm{r}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\,\mathbb{P}(\mathbf{T}^{(k)}_{\mathrm{r}}\in d\mathrm{t},\ \mathbb{X}^{(k)}_{\mathrm{t}}=\mathrm{x})\,d\mathrm{r}\\
&\qquad=\int_{\mathbb{R}^d_+} e^{-\langle\alpha,\mathrm{r}\rangle}\mathbb{E}\left[e^{-\langle\lambda,\mathbf{T}^{(k)}_{\mathrm{r}}\rangle-\langle\!\langle\mu,\mathbb{X}^{(k)}_{\mathbf{T}^{(k)}_{\mathrm{r}}}\rangle\!\rangle}\mathbb{1}_{\{\mathbf{T}^{(k)}_{\mathrm{r}}\geq\varepsilon\cdot\mathbf{1}\}^c}\right]d\mathrm{r},
\end{aligned}$$

which entails from a trivial extension of (4.18) that,

$$\begin{aligned}
&\lim_{k\to\infty}\int_{\{\mathrm{t}\geq\varepsilon\cdot\mathbf{1}\}^c}\int_{\widehat{M}_d(\mathbb{R})} e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle-\langle\!\langle\mu,\mathrm{x}\rangle\!\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}^{(k)}_{\mathrm{t}}\in d\mathrm{x})\,d\mathrm{t}\\
&\qquad=\int_{\mathbb{R}^d_+} e^{-\langle\alpha,\mathrm{r}\rangle}\mathbb{E}[e^{-\langle\lambda,\mathbf{T}_{\mathrm{r}}\rangle-\langle\!\langle\mu,\mathbb{X}_{\mathbf{T}_{\mathrm{r}}}\rangle\!\rangle}\mathbb{1}_{\{\mathbf{T}_{\mathrm{r}}\geq\varepsilon\cdot\mathbf{1}\}^c}]\,d\mathrm{r}.
\end{aligned}\tag{4.19}$$

But from part 2. of Theorem 3.3, for all $i,j\in[d]$, $\lim_{s\to\infty}\phi_j(s\mathrm{e}_i)=\infty$, which implies that for all $\mathrm{r}>\mathbf{0}$ and all $i\in[d]$, $\mathbb{P}(T^{(i)}_{\mathrm{r}}>0)>0$. In particular,

$$\lim_{\varepsilon\to0}\mathbb{P}(\{\mathbf{T}_{\mathrm{r}}\geq\varepsilon\cdot\mathbf{1}\}^c)\leq\lim_{\varepsilon\to0}\sum_{i=1}^d\mathbb{P}(T^{(i)}_{\mathrm{r}}<\varepsilon)=0,$$

therefore by dominated convergence, expression (4.19) can be made arbitrarily small as ε tends to 0.

Then we have proved that the identity (4.7) is valid for all $\alpha\in\mathbb{R}^d_+$ and $(\lambda,\mu)\in\mathcal{M}_\varphi$ such that $\lambda_j>\varphi_j(\mu^{(j)})$, $j\in[d]$. Now let any $(\lambda,\mu)\in\mathcal{M}_\varphi$ and assume that $\lambda_i=\varphi_i(\mu^{(i)})$ for some $i\in[d]$. Then identity (4.7) is valid if we replace λ_i by $\lambda'_i=\lambda_i+\varepsilon_i$, for $\varepsilon_i>0$ and we obtain it for (λ,μ) by letting ε_i going to 0 and applying monotone convergence. ☐

Proof of Proposition 3.2. Assume first that $d>1$. Then taking $\mu=0$ in Theorem 4.3 gives

$$\begin{aligned}
\int_{\mathbb{R}^d_+\times\mathbb{R}^d_+} e^{-\langle\alpha,\mathrm{r}\rangle-\langle\lambda,\mathrm{t}\rangle}\,\mathbb{P}(\mathbf{T}_{\mathrm{r}}\in d\mathrm{t})d\mathrm{r} &= \int_{\mathbb{R}^d_+\times\widehat{M}_d(\mathbb{R})} e^{\langle\alpha,\mathrm{x}\cdot\mathbf{1}\rangle-\langle\lambda,\mathrm{t}\rangle}\frac{\det(-\mathrm{x})}{t_1t_2\dots t_d}\mathbb{P}(\mathbb{X}_{\mathrm{t}}\in d\mathrm{x})\,d\mathrm{t}\\
&= \int_{\mathbb{R}^d_+} e^{-\langle\lambda,\mathrm{t}\rangle}\mathbb{E}\left[e^{\langle\alpha,\mathbb{X}_{\mathrm{t}}\cdot\mathbf{1}\rangle}\frac{\det(-\mathbb{X}_{\mathrm{t}})}{t_1t_2\dots t_d}\mathbb{1}_{\{\mathbb{X}_{\mathrm{t}}\in\widehat{M}_d(\mathbb{R})\}}\right]d\mathrm{t}\,.
\end{aligned}\tag{4.20}$$

Note that from our assumptions the density $p_{\mathrm{t}}:M_d(\mathbb{R})\to\mathbb{R}$ of $\widehat{\mathbb{X}}_{\mathrm{t}}$ is continuous on the set of matrices whose columns belong to $F_1\times F_2\times\cdots\times F_d$. Let $\overline{M}_d(\mathbb{R})$ be the set of essentially nonnegative matrices whose elements of the diagonal are non-positive. Then

$$\mathbb{E}\left[e^{\langle\alpha,\mathbb{X}_{\mathrm{t}}\cdot\mathbf{1}\rangle}\frac{\det(-\mathbb{X}_{\mathrm{t}})}{t_1t_2\dots t_d}\mathbb{1}_{\{\mathbb{X}_{\mathrm{t}}\in\widehat{M}_d(\mathbb{R})\}}\right]=\int_{\overline{M}_d(\mathbb{R})} e^{\sum_{i=1}^d\alpha_ix_{i,i}}\frac{\det(-(\overline{\mathrm{x}}+D(\mathrm{x})))}{t_1\dots t_d}p_{\mathrm{t}}(\mathrm{x})d\mathrm{x},$$

where $D(\mathrm{x})=(d_{i,j})_{i,j\in[d]}$ is defined by $d_{i,i}=x_{i,i}$ and $d_{i,j}=0$ for $i\neq j$, and $\overline{x}=(\overline{x}_{i,j})_{i,j\in[d]}$

such that $\overline{x}_{i,i} = -\sum_{j\neq i} x_{i,j}$ and $\overline{x}_{i,j} = x_{i,j}$ for $i \neq j$. Let I_d be the identity matrix. Then

$$\begin{aligned}\int_{\overline{M}_d(\mathbb{R})} e^{\sum_{i=1}^d \alpha_i x_{i,i}} \frac{\det(-(\overline{\mathrm{x}} + D(\mathrm{x}))}{t_1 \dots t_d} p_{\mathrm{t}}(\mathrm{x})\mathrm{dx} \\ = \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^{d(d-1)}} e^{-\langle \alpha, \mathrm{r}\rangle} \frac{\det(-(\overline{\mathrm{x}} + \mathrm{r}I_d))}{t_1 \dots t_d} p_{\mathrm{t}}(\mathrm{x}^{\mathrm{r}}) \prod_{k\neq j} \mathrm{d}x_{k,j}\mathrm{dr},\end{aligned} \tag{4.21}$$

where x^{r} is the matrix x in which the variable $x_{i,i}$ has been replaced by r_i, for all $i \in [d]$. Then we derive from (4.20) and (4.21) that for fixed $\mathrm{r} \in \mathbb{R}_+^d$,

$$\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathrm{dt}) = \int_{\mathbb{R}_+^{d(d-1)}} \frac{\det(-(\overline{\mathrm{x}} + \mathrm{r}I_d))}{t_1 \dots t_d} p_{\mathrm{t}}(\mathrm{x}^{\mathrm{r}}) \prod_{k\neq j} \mathrm{d}x_{k,j}\mathrm{dt}\,. \tag{4.22}$$

Let $i \in [d]$ and $\mathrm{r} = r\mathrm{e}_i$, then

$$\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathrm{dt}) = \int_{\mathbb{R}_+^{d(d-1)}} \frac{r \det(-\overline{\mathrm{x}}^{i,i})}{t_1 \dots t_d} p_{\mathrm{t}}(\mathrm{x}^{\mathrm{r}}) \prod_{k\neq j} \mathrm{d}x_{k,j}\mathrm{dt},$$

where $\overline{\mathrm{x}}^{i,i}$ is the matrix obtained from $\overline{\mathrm{x}}$ by deleting the row and the column i. From Exercise 1. in Chapter I of [3], the Lévy measure of the subordinator $(\mathbf{T}_{r\mathrm{e}_i})_{r\geq 0}$ is the vague limit of $\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathrm{dt})/r$ as r tends to 0, on sets of the form $\{|\mathrm{t}| > a\}$, $a > 0$. Hence the expression of the statement follows from continuity property of p_{t}.

The expression for $d = 1$ is obtained in the same way by using the simpler form (4.8) of $\mathbb{P}(\mathbf{T}_{\mathrm{r}} \in \mathrm{dt})$ in this case. $\square$

A Appendix

Proof of Lemma 2.3. This proof is based on the observation that for each $i \in [d]$, as a function of t, the term $\sum_{j=1}^d x^{i,j}(t_j)$ has no negative jumps. Moreover, when t_i is fixed, it is non decreasing.

Let us set $v_i^{(1)} = r_i$ and for $n \geq 1$,

$$s_i^{(n)} = \inf\{t : x_{t-}^{i,i} = -v_i^{(n)}\} \;\text{ and }\; v_i^{(n+1)} = r_i + \sum_{j\neq i} x^{i,j}(s_j^{(n)}-)\,,$$

where $\inf \emptyset = \infty$. Set also $\mathrm{s}^{(0)} = \mathbf{0}$ and note that $[d]_{\mathrm{s}^{(0)}} = [d]$. Then since for $i \neq j$, the $x^{i,j}$'s are positive and non decreasing, we have

$$\mathrm{s}^{(n)} \leq \mathrm{s}^{(n+1)} \;\text{ and }\; [d]_{\mathrm{s}^{(n+1)}} \subseteq [d]_{\mathrm{s}^{(n)}}\,, \quad n \geq 0\,.$$

Let us set $\mathrm{s}^{(\infty)} = \lim_{n\to\infty} \mathrm{s}^{(n)}$. Then $\mathrm{s}^{(\infty)}$ is the smallest solution of the system (r, x) in the sense which is defined in part 1. of Lemma 2.3. Indeed, let $i \in [d]_{\mathrm{s}^{(\infty)}}$. By definition and since $x^{i,i}$ has no negative jumps, for all $n \geq 1$, $x^{i,i}(s_i^{(n)}-) = -v_i^{(n)}$. Moreover, since the processes $t \mapsto x^{i,j}(t-)$ are left continuous, $\lim_{n\to\infty} x^{i,i}(s_i^{(n)}-) = x^{i,i}(s_i^{(\infty)}-)$ and $\lim_{n\to\infty} v_i^{(n)} = r_i + \sum_{j\neq i} x^{i,j}(s_j^{(\infty)}-)$. Hence (2.1) is satisfied for $\mathrm{s}^{(\infty)}$, that is $r_i + \sum_{j=1}^d x^{i,j}(s_j^{(\infty)}-) = 0$, for all $i \in [d]_{\mathrm{s}^{(\infty)}}$. Now let $\mathrm{t} \in \overline{\mathbb{R}}_+^d$ satisfy (2.1), that is

$$r_i + \sum_{j\neq i} x^{i,j}(t_j-) + x^{i,i}(t_i-) = 0\,,\; i \in [d]_{\mathrm{t}}\,. \tag{A.1}$$

We can prove by induction that $\mathrm{t} \geq \mathrm{s}^{(n)}$, for all $n \geq 1$. Firstly for (A.1) to be satisfied, we should have $t_i \geq \inf\{s : x^{i,i}(s-) = -r_i\}$, for all $i \in [d]_{\mathrm{t}}$, hence $\mathrm{t} \geq \mathrm{s}^{(1)}$. Now assume that $\mathrm{t} \geq \mathrm{s}^{(n)}$. Then $[d]_{\mathrm{t}} \subseteq [d]_{\mathrm{s}^{(n)}}$ and from (A.1), for each $i \in [d]_{\mathrm{t}}$,

$$x^{i,i}(t_i-) = -\left(r_i + \sum_{j \neq i} x^{i,j}(t_j-)\right) \leq -\left(r_i + \sum_{j \neq i} x^{i,j}(s_j^{(n)}-)\right).$$

Therefore $t_i \geq \inf\left\{s : x^{i,i}(s-) = -\left(r_i + \sum_{j \neq i} x^{i,j}(s_j^{(n)}-)\right)\right\}$, so that $\mathrm{t} \geq \mathrm{s}^{(n+1)}$ and the first assertion is proved.

If $\mathrm{r}' \leq \mathrm{r}$, then one can easily prove by induction that, with obvious notation, ${\mathrm{s}'}^{(n)} \leq \mathrm{s}^{(n)}$ for all $n \geq 1$ and the first part of assertion 2. follows. For the second part, set $\mathrm{s}' := \lim_{n\to\infty} \mathrm{s}_n$. Then first part of assertion 2. yields $\mathrm{s}' \leq \mathrm{s}$. Moreover, from the left continuity of the functions $t \mapsto x^{i,j}_{t-}$, $r_i + \sum_{j=1}^{d} x^{i,j}(s'_j-) = 0$, $i \in [d]_{\mathrm{s}'}$ hence s' is a solution of (r, x) and thus $\mathrm{s}' = \mathrm{s}$.

Let $\mathrm{u} \in \mathbb{R}_+^d$, such that $\sum_{j=1}^{d} x^{i,j}(u_j-) \leq -r_i$, for all $i \in [d]_{\mathrm{u}}$ and set $r'_i = -\sum_{j=1}^{d} x^{i,j}(u_j-)$. Since $\mathrm{r}' \geq \mathrm{r}$, it follows from 2. that the smallest solution s' of the system $(\mathrm{r}', \mathrm{x})$ is such that $\mathrm{s}' \geq \mathrm{s}$. But since u is also a solution of $(\mathrm{r}', \mathrm{x})$, 1. implies $\mathrm{u} \geq \mathrm{s}'$ and the first assertion of 3. follows. The second assertion of 3. is a consequence of the first one. Indeed, $\mathrm{u} < \mathrm{s}$ implies that $\mathrm{u} \geq \mathrm{s}$ is not satisfied.

Assertion 4. follows from the above construction of $\mathrm{s} = \mathrm{s}^{(\infty)}$. Indeed, if there exists $i \in [d]_{\mathrm{s}}$ and $t_i < s_i$ such that $x^{i,i}(t_i-) \leq x^{i,i}(s_i-)$ then

$$\sum_{j \neq i} x^{i,j}(s_j-) + x^{i,i}(t_i-) \leq \sum_{j \in [d]} x^{i,j}(s_j-) = -r_i \tag{A.2}$$

and for all $k \in [d]_{\mathrm{s}} \setminus \{i\}$,

$$\sum_{j \neq i} x^{k,j}(s_j-) + x^{k,i}(t_i-) \leq \sum_{j \in [d]} x^{k,j}(s_j-) = -r_k\,. \tag{A.3}$$

Then set for all $k \in [d]_s$, $r'_k = -\left(\sum_{j \neq i} x^{k,j}(s_j-) + x^{k,i}(t_i-)\right)$ and for all $k \in [d] \setminus [d]_s$, $r'_k = r_k$. Let s' be the smallest solution of the system $(\mathrm{r}', \mathrm{x})$. From part 2. of the present lemma, since $\mathrm{r}' \geq \mathrm{r}$, $\mathrm{s}' \geq \mathrm{s}$. On the other hand, from (A.2), (A.3) and part 3. of the present lemma, $\mathrm{s} > (s_1, \dots, s_{i-1}, t_i, s_{i+1}, \dots, s_d) \geq \mathrm{s}'$ which is a contradiction. □

Proof of Lemma 4.6. Let us first assume that Y has bounded variation. Then the characteristic exponent ψ of Y can be written as

$$\psi(\lambda) = -i\langle \mathrm{a}, \lambda\rangle + \int_{(0,\infty)^d} (1 - e^{i\langle \lambda, \mathrm{x}\rangle})\, \pi(d\mathrm{x}), \quad \lambda \in \mathbb{R}^d,$$

where $\mathrm{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$ and the Lévy measure π satisfies $\int_{(0,\infty)^d} (1 \wedge |\mathrm{x}|)\, \pi(d\mathrm{x}) < \infty$.

Let $\pi^{(k)}$ be the restriction of π to the set $[k^{-1}, \infty)^d$ i.e. $\pi^{(k)}(d\mathrm{x}) = \mathbb{1}_{[k^{-1},\infty)^d}\pi(d\mathrm{x})$. For $x \in \mathbb{R}$, set $\mathrm{sign}(x) = \mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x<0\}}$. Then we consider the following sequence of $(k^{-1}\mathbb{Z})^d$-valued Lévy processes

$$\mathrm{Y}_t^{(k)} = k^{-1}\tilde{\mathrm{N}}_t^{(k)} + \sum_{n=0}^{N_t^{(k)}} \mathrm{Z}_n^{(k)},$$

where $\tilde{\mathrm{N}}^{(k)} = (\mathrm{sign}(a_1)\tilde{N}^{1,k},\dots,\mathrm{sign}(a_d)\tilde{N}^{d,k})$ and $\tilde{N}^{1,k},\dots,\tilde{N}^{d,k}$ are independent Poisson processes with respective intensities $k|a_j|$, $(N_t^{(k)})_{t\ge0}$ is a Poisson process with intensity $\pi([k^{-1},\infty)^d)$ and for each $k\ge1$, $(\mathrm{Z}_n^{(k)})_{n\ge0}$ is a sequence of i.i.d random variables such that $\mathrm{Z}_n^{(k)} \overset{(law)}{=} k^{-1}\lfloor k\mathrm{Z}_k\rfloor$ and Z_k has law $(\pi([k^{-1},\infty)^d))^{-1}\pi^{(k)}(d\mathrm{x})$. (Here $\lfloor \mathrm{x}\rfloor = (\lfloor x_1\rfloor,\dots,\lfloor x_d\rfloor)$ and we recall that $\lfloor x_i\rfloor$ denotes the lower integer part of $x_i\in\mathbb{R}$.) Moreover, the sequences $\{(\tilde{\mathrm{N}}_t^{(k)})_{t\ge0},k\ge1\}$, $\{(N_t^{(k)})_{t\ge0},k\ge1\}$ and $\{(\mathrm{Z}_n^{(k)})_{n\ge0},k\ge1\}$ are independent. Then we can check that $\mathrm{Y}^{(k)}$ has characteristic exponent

$$\psi_k(\lambda) = \sum_{j=1}^d k|a_j|\left(1-e^{i\frac{\lambda_j\,\mathrm{sign}(a_j)}{k}}\right) + \int_{(0,\infty)^d}(1-e^{i\langle\lambda,\mathrm{x}\rangle})\,\pi^{(k)}(d\mathrm{x}),\quad \lambda\in\mathbb{R}_+^d,$$

whose limit, as k tends to ∞, is $\psi(\lambda)$, for all $\lambda\in\mathbb{R}^d$. It proves that the sequence of random variables $(\mathrm{Y}_1^{(k)})_{k\ge1}$ converges weakly towards Y_1.

Then recall that from Theorem 2.7 in [14], which can be extended in higher dimension, see Section 5 in the same paper, the weak convergence of the sequence of random variables $(\mathrm{Y}_1^{(k)})_{k\ge1}$ toward Y_1 implies the weak convergence of the sequence of processes $\{(\mathrm{Y}_t^{(k)})_{t\ge0},\ k\ge1\}$ towards $(\mathrm{Y}_t)_{t\ge0}$ in the J_1 Skohorod's topology. Hence our result is proved in the case where Y has bounded variation.

Let us now assume that Y is any Lévy process as described in the statement and set $\Delta_s = \mathrm{Y}_s - \mathrm{Y}_{s-}$. Then it is well known that the sequence of processes

$$\mathrm{Z}_t^{(n)} := \sum_{s\le t}\mathbb{1}_{\{|\Delta_s|>n^{-1}\}}\Delta_s - t\int_{(0,\infty)^d} x\mathbb{1}_{\{|\mathrm{x}|>n^{-1}\}}\pi(d\mathrm{x}),\quad t\ge0,$$

converges weakly toward Y as n tends to ∞, see the proof of Theorem 1 of Chapter I in [3] and the above argument on weak convergence in the J_1 Skohorod's topology. Since, for each n, $\mathrm{Z}^{(n)}$ is a Lévy process with bounded variation whose all coordinates have no negative jumps, in application of what has just been proved, there is a sequence of $(k^{-1}\mathbb{Z})^d$-valued Lévy processes $\mathrm{Z}^{(n,k)}$, $k\ge1$ which converges weakly in the J_1 Skohorod's topology toward $\mathrm{Z}^{(n)}$. Moreover, for each k, all the coordinates of process $\mathrm{Z}^{(n,k)}$ take their values in the set $\{-k^{-1},0,k^{-1},2k^{-1},3k^{-1},\dots\}$. Then it suffices to set $\mathrm{Y}^{(k)} := \mathrm{Z}^{(k,k)}$ in order to obtain the desired sequence. $\square$

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