Space-time coupled evolution equations and their stochastic solutions

John Herman[*]   Ifan Johnston[†]   Lorenzo Toniazzi[‡]

Abstract

We consider a class of space-time coupled evolution equations (CEEs), obtained by a subordination of the heat operator. Our CEEs reformulate and extend known governing equations of non-Markovian processes arising as scaling limits of continuous time random walks, with widespread applications. In particular we allow for initial conditions imposed on the past, general spatial operators on Euclidean domains and a forcing term. We prove existence, uniqueness and stochastic representation for solutions.

1 Introduction

We study the space-time coupled evolution equation (CEE)

$$\begin{cases} H^\nu u(t,x) = -f(t,x), & \text{in } (0,T] \times \Omega, \\ u(t,x) = \phi(t,x), & \text{in } (-\infty,0] \times \Omega, \end{cases} \tag{1.1}$$

where f and ϕ are given data and

$$H^\nu u(t,x) = \int_0^\infty \left(e^{r\mathcal{L}} u(t-r,x) - u(t,x) \right) \nu(r) \, dr, \quad t > 0, \tag{1.2}$$

so that $-H^\nu = (\partial_t - \mathcal{L})^\nu$ is the subordination of the heat operator $(\partial_t - \mathcal{L})$ by an infinite Lévy measure ν. Here the Markovian semigroup $\{e^{r\mathcal{L}}\}_{r \geq 0}$ acts on the space variable

---

[*]University of Warwick, United Kingdom. E-mail: j.a.Herman@warwick.ac.uk
[†]University of Warwick, United Kingdom. E-mail: i.johnston@warwick.ac.uk
[‡]University of Otago, New Zealand. E-mail: ltoniazzi@maths.otago.ac.nz

$\Omega \subset \mathbb{R}^d$, and we denote the associated stochastic process by $B = r \mapsto B_r$. As our main result, we prove the stochastic representation for the solution $u(t,x)$ to (1.1) to be

$$\mathbf{E}\left[\phi\left(t - S^\nu_{\tau_0(t)}, B^x_{S^\nu_{\tau_0(t)}}\right) \mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right] + \mathbf{E}\left[\int_0^{\tau_0(t) \wedge \tau_0(\tau_\Omega(x))} f\left(t - S^\nu_s, B^x_{S^\nu_s}\right) ds\right], \quad (1.3)$$

where S^ν is the Lévy subordinator with Lévy measure ν, S^ν is independent of B^x, with x denoting the starting point of B, $\tau_0(t) = \inf\{r > 0 : t - S^\nu_r < 0\}$ is the inverse of S^ν and $\tau_\Omega(x)$ is the life time of B^x, $x \in \Omega$. Let us first clarify the solution (1.3) by viewing the evolution equation (1.1) as a Dirichlet problem with space-time boundary conditions. Assume for simplicity $\tau_\Omega = \infty$ and observe that in (1.1) our operator (1.2) is subject to the *exterior/absorbing temporal* boundary condition $u = \phi$ on $(-\infty, 0] \times \Omega$. At the same time (1.2) is the generator of the space-time coupled Markov process

$$r \mapsto \left(t - S^\nu_r, B^x_{S^\nu_r}\right), \quad (t,x) \in \mathbb{R} \times \Omega. \quad (1.4)$$

Assuming $f = 0$, then we expect the solution kernel to be given by the absorption of the space-time process (1.4) on $(-\infty, 0] \times \Omega$ on its first attempt to exit $(0, T] \times \Omega$, which indeed happens at time $\tau_0(t)$. If the space-time forcing term f is non-zero, it is now natural to expect (1.3) to be the solution. (This absorption interpretation can be seen in the more standard case of the fractional Laplacian with exterior boundary condition [28, Theorem 1.3], or in a general setting in [29].) It is important to observe that

$$Y = t \mapsto Y_t = B_{S^\nu_{\tau_0(t)}}$$

is a non-Markovian process that is trapped when $t \mapsto \tau_0(t)$ is constant. Then there is a clear intuition for the initial condition on the "past" in problem (1.1), as the time parameters of ϕ are weighted according to the overshoot $S^\nu_{\tau_0(t)} - t$, which is the waiting/trapping time of Y_t (caused by the time change τ_0). As an example, let $\phi(t,x) = \mathbb{1}_{\{t < -1\}} \mathbb{1}_{\{x > 0\}}$ ($f = 0$ and $\Omega = \mathbb{R}$), then the solution is

$$u(t,x) = \mathbf{P}\left[Y^x_t \text{ will not move for at least 1 time-unit } \& \ Y^x_t \text{ is positive}\right],$$

and thus the modeller's choice of ϕ allows to gain control over the length of the traps in combination with the spatial position of the process. This implies that imposing initial conditions on the past in (1.1) results in a "finer" probabilistic model when compared to only imposing standard initial conditions at 0 (i.e. ϕ independent of time, so that in the example above $u(t,x) = \mathbf{P}[Y^x_t \text{ is positive}]$).

Select now time independent initial data $\phi(t) = \phi_0$, $f = 0$, $d = 1$ and let $r \mapsto B_r$ be a Lévy process with density $p_r(\cdot)$. Notating $\Phi(dy, dr) = p_r(y)\nu(r)\, dy\, dr$, we can now write $H^\nu u(t,x)$ as

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \left(\mathbb{1}_{\{t-r>0\}} u(t-r, x-y) - u(t,x)\right) \Phi(dy, dr) + \int_{\mathbb{R}} \phi_0(x-y)\Phi(dy, (t, \infty)),$$

and the CEE (1.1) is a particular case of [24, Theorem 4.1, eq. (4.1)]. In [24], problem (1.1) appears in Fourier-Laplace space as

$$p(\gamma, \xi) = \frac{1}{\gamma} \frac{\psi_\nu(\gamma + \psi_B(\xi)) - \psi_\nu(\psi_B(\xi))}{\psi_\nu(\gamma + \psi_B(\xi))}, \quad \gamma > 0, \ \xi \in \mathbb{R},$$

and it is shown that the Fourier-Laplace transform of the law of Y satisfies the above identity, where ψ_B is the Fourier symbol of B and ψ_ν the Laplace symbol of S^ν. The authors in [24] also show that Y arises as the scaling limit of overshoot continuous time

random walks (OCTRWs). The overshoot is reflected in the time change of B living above t, in the sense that $S^\nu_{\tau_0(t)} > t$ [6, Theorem III.4]. Recall again that Y is trapped when τ_0 is constant, like the fractional-kinetic process $t \mapsto B_{\tau_0(t)}$ [33, Chapter 2.4]. But the duration of a waiting time induced by τ_0 equals the length of the *last* discontinuity of $S^\nu_{\tau_0}$, mirrored in the coupling of space (B_{S^ν}) and time (τ_0). In particular, if the subordination is performed by an α-stable subordinator S^α, then Y scales like B, because $S^\alpha_{\tau_0(t)} = tS^\alpha_{\tau_0(1)}$. This implies that if B is a Brownian motion, the order of the mean square displacements (MSDs) of Y_t and B_t are both t, in contrast with the fractional kinetic process, whose MSD is of order t^α [33, Chapter 2.4]. The related literature known to us deals with variations of the CEE (1.1) in Fourier-Laplace space, mostly motivated by central limit theorems for coupled OCTRWs. See [30, 36] for multidimensional extensions of OCTRW limits, [31] for explicit densities in certain fractional cases, and [41, 25] for alternatives to the first derivative in time. Due to their peculiar properties OCTRWs are popular models appearing for instance in physics and finance [40, 42, 43, 19, 39, 26]. Worth mentioning that the OCTRW limit first appeared in [24] as the overshooting counterpart of CTRW limits studied in [5, 3, 32], which result in different CEEs. In this latter case, the counterpart of (1.1) expects the solution to be the subordination of B by $S^\nu_{\tau_0(t)-}$, for S^ν_{s-} the left continuous modification of S^ν_s. We could not treat this case, as our method relies on Dynkin formula, and we could not recover a suitable version for the left continuous process S^ν_{s-}. However, this case is treated in the general setting of space-time Feller semigroups in [4], as discussed below. Note that, although related, problem (1.1) is different from [35, problem (1.1)], as the latter does not impose initial conditions, and in turn it does not describe anomalous diffusion.

We present two main results. The first one is Theorem 3.5, where we prove wellposedness and stochastic representation for solutions. We call these *generalised solutions*, and they are (carefully chosen) pointwise limits of potentials of the space-time process (1.4) killed outside $(0, T] \times \Omega$. The second main result is Theorem 4.8, where we prove that (1.3) is a *weak solution* for (1.1) for weak data and $e^{\mathcal{L}}$ self-adjoint. We could not prove uniqueness of weak solutions, which appears to be a subtle problem already for the simpler (uncoupled) Marchaud-Caputo EE [1]. We mention that we assume existence of densities for the processes B and S^ν to work with weak data, but this assumption could be dropped by working with smoother data, as we discuss in Remark 3.6-(ii).

To the best of our knowledge, the novel contribution of this article is the following. A general probabilistic method to treat wellposedness and stochastic representation for the CEE (1.1) when it features: initial conditions on the past, general spatial operators on Euclidean domains and a forcing term. Moreover, our proof method tightly follows [17] and [38], which treat the rather different uncoupled EEs of Caputo/Marchaud-type. Therefore proposing a unified method for a large class of fractional/nonlocal EEs with initial conditions on the past. Besides the introduction of initial conditions on the past for CEEs, this work appears to be the first one that formulates and solves the governing equation of Y in differential form, without relying on Fourier-Laplace transform techniques. This was also part of the contribution of [4], which treats different CEEs, as mentioned above. We remark that [4] and our work share the idea of considering the generator of the respective space-time coupled process killed when the process in the time variable crosses a barrier ($t - S^\nu$ in (1.4) crossing 0 in this work). Then the potential of this space-time killed process is a bounded operator and so one can invert its generator (compare the proofs of [4, Theorem 4.1] and Lemma 3.3, respectively). Unfortunately, as mentioned above, the rest of our strategy does not appear to be compatible with the left continuous modifications of the processes in [4] and thus we do not know whether one can impose initial conditions on the past for the processes treated in [4].

The article is organised as follows; Section 2 introduces general notation, our assumptions and the main semigroup results used to treat the operator H^ν; Section 3 proves Theorem 3.5 and presents some concrete fundamental solutions to (1.1); Section 4 proves Theorem 4.8.

2 Notation and subordinated heat operators

We denote by $\mathbb{R}^d$, $\mathbb{N}$, $\mathbb{1}$, a.e., $a \vee b$ and $a \wedge b$ the d-dimensional Euclidean space, the positive integers, the indicator function, the statement almost everywhere with respect to Lebesgue measure, the maximum and the minimum between $a, b \in \mathbb{R}$, respectively. We denote by $\Gamma(\beta)$ the Gamma function for $\beta \in (-1, 0) \cup (0, \infty)$, and we recall the standard identity $\Gamma(\beta + 1) = \Gamma(\beta)\beta$. We denote by $\overline{E}$ the topological closure of a set E. If E is a locally compact space, then we write $C(E)$ for the real-valued continuous functions on E. We write $C_\infty(E)$ for the Banach space of functions in $C(E)$ *vanishing at infinity* with the supremum norm [10, page 1]. This means that for every $f \in C_\infty(E)$ and $\epsilon > 0$ there exists a compact set $K \subset E$ such that $\sup_{x \in E \setminus K} |f(x)| \leq \epsilon$, moreover we canonically extend f to $E \cup \{\partial\}$ by $f(\partial) = 0$ for ∂ a cemetery state and if E is not compact we set $E \cup \{\partial\}$ to be the one-point compactification of E, otherwise ∂ is an isolated point. We denote by $B(E)$ the Banach space of real-valued bounded Borel measurable functions on E with the supremum norm. We mostly work with the space-time Banach spaces

$$C_\infty((-\infty, T] \times \Omega), \quad C_\infty([0, T] \times \Omega) \quad \text{and} \quad C_\infty((0, T] \times \Omega),$$

for some $\Omega \subset \mathbb{R}^d$ and any $T > 0$, with the convention that we extend the functions in $C_\infty((0, T] \times \Omega)$ to zero on $\{0\} \times \Omega$ and for f in any of the above three spaces we write $f(0) = x \mapsto f(0, x)$. We define $C_\infty^1(-\infty, T] = \{f, f' \in C_\infty(-\infty, T]\}$ for any $T \geq 0$, and $C_c^1(0, T)$ to be the space of continuously differentiable functions in $C(\mathbb{R})$ with compact support in $(0, T)$. For two sets of real-valued functions F and G we define

$$F \cdot G := \{f \cdot g : f \in F, \, g \in G\}.$$

For a sequence of functions $\{f_n\}_{n \geq 1}$ and a function f, we write $f_n \to f$ *bpw (bpw a.e.)* if f_n converges to f pointwise (a.e.) as $n \to \infty$, and the supremum (essential supremum) norms of all f_n's are uniformly bounded in n. We denote by $L^1(\Omega)$, $L^2(\Omega)$ and $L^\infty(\Omega)$ the standard Banach spaces of Lebesgue integrable, square-integrable and essentially bounded real-valued functions on Ω, respectively. We denote by $\|\cdot\|_X$ the norm of a Banach space X.

The notation we use for an E-valued stochastic process started at $x \in E$ is $X^x = \{X_s^x\}_{s \geq 0} = s \mapsto X_s^x$. Note that the symbol t will often be used to denote the starting point of a stochastic process with state-space $E \subset \mathbb{R}$. By a *strongly continuous contraction semigroup* $e^{\mathcal{G}}$ we mean a collection of bounded linear operators $e^{s\mathcal{G}} : X \to X$, $s \geq 0$, where X is a Banach space, such that $e^{(s+r)\mathcal{G}} = e^{s\mathcal{G}} e^{r\mathcal{G}}$ for every $s, r > 0$, $e^{0\mathcal{G}}$ is the identity operator and $\lim_{s \downarrow 0} e^{s\mathcal{G}} f = f$ in X and $\sup_s \|e^{s\mathcal{G}} f\|_X \leq \|f\|_X$ for every $f \in X$. The generator of $e^{\mathcal{G}}$ is defined as the pair $(\mathcal{G}, \text{Dom}(\mathcal{G}))$, where $\text{Dom}(\mathcal{G}) := \{f \in X : \mathcal{G}f := \lim_{s \downarrow 0} s^{-1}(e^{s\mathcal{G}} f - f) \text{ exists in } X\}$. We say that a set $C \subset \text{Dom}(\mathcal{G})$ is a *core for* $(\mathcal{G}, \text{Dom}(\mathcal{G}))$ if the generator equals the closure of the restriction of $\mathcal{G}$ to C. Recall that $\text{Dom}(\mathcal{G})$ is dense in X. For a given $\lambda \geq 0$ we define the *resolvent of* $e^{\mathcal{G}}$ by $(\lambda - \mathcal{G})^{-1} := \int_0^\infty e^{-\lambda s} e^{s\mathcal{G}} \, ds$, and recall from [18, Theorem 1.1] that for $\lambda > 0$, $(\lambda - \mathcal{G})^{-1} : X \to \text{Dom}(\mathcal{G})$ is a bijection and it solves the abstract resolvent equation

$$\mathcal{G}(\lambda - \mathcal{G})^{-1} f = \lambda(\lambda - \mathcal{G})^{-1} f - f, \quad f \in X,$$

and if $(-\mathcal{G})^{-1} : X \to X$ is bounded, then the above statement holds for $\lambda = 0$ [18, Theorem 1.1']. Also, for any $f \in \text{Dom}(\mathcal{G})$ and $C = \|\mathcal{G}f\|_X \vee \|2f\|_X$ we have

$$\|e^{s\mathcal{G}} f - f\|_X \leq C(s \wedge 1) \quad \text{for all } s \geq 0. \tag{2.1}$$

By a *Feller semigroup* we mean a strongly continuous contraction semigroup $e^{\mathcal{G}}$ on any of the Banach spaces $C_\infty(E)$ defined above such that for each $s > 0$, $e^{s\mathcal{G}}f \geq 0$ if $f \geq 0$. Feller semigroups are in one-to-one correspondence with Feller processes, where a *Feller process* is a time-homogeneous sub-Markov process $\{X_s\}_{s\geq 0}$ such that $e^{s\mathcal{G}}f(x) := \mathbf{E}[f(X_s^x)]$, $f \in C_\infty(E)$ is a Feller semigroup [10, Chapter 1.2]. We recall that every Feller process admits a càdlàg (right continuous with left limits) modification which enjoys the strong Markov property [10, Theorem 1.19 and Theorem 1.20], and we always work with such modification. For further discussions on these terminologies and notation we refer to [10].

2.1 The spatial operator $\mathcal{L}$

Definition 2.1. *We define $(\mathcal{L}, Dom(\mathcal{L}))$ to be the generator of a Feller semigroup $\{e^{r\mathcal{L}}\}_{r\geq 0}$ on $C_\infty(\Omega)$, where the set $\Omega \subset \mathbb{R}^d$ is either open or the closure of an open set. We denote the associated Feller process by $B^x = s \mapsto B_s^x$, when started at $x \in \Omega$. As usual, the Feller process $s \mapsto B_s^x$ is defined to be in the cemetery if $s \geq \tau_\Omega(x)$, defining the life times $\tau_\Omega(x) = \inf\{s > 0 : B_s^x \notin \Omega\}$, $x \in \Omega$, so that $B_s^x = B_{s\wedge\tau_\Omega(x)}^x$.*

We will use the following assumption for the spatial semigroup $e^{\mathcal{L}}$.

(H1) The operator $e^{r\mathcal{L}}$ allows a density with respect to Lebesgue measure for each $r > 0$, i.e. there exists a function $y \mapsto p_r^\Omega(x, y) \in L^1(\Omega)$ for each $x \in \Omega$ such that $e^{r\mathcal{L}}f(x) = \int_\Omega f(y)p_r^\Omega(x, y)\, dy$ for every $f \in C_\infty(E)$.

In the examples below we say that a stochastic process $s \mapsto X_s$ with state-space E is *strong Feller* if $f(\cdot) \mapsto \mathbf{E}[f(X_s)]$ maps $B(E)$ to $C(E) \cap B(E)$ for each $s > 0$. If $E = \mathbb{R}^d$ we say that $\Omega \subset \mathbb{R}^d$ is a *regular set* if Ω is open and for each $z \in \partial\Omega$, $\mathbf{P}[\inf\{s > 0 : X_s^z \notin \Omega\} = 0] = 1$. (Here $\partial\Omega$ denotes the Euclidean boundary of Ω.)

Example 2.2. We mention some examples of Feller semigroups/processes that satisfy (H1), including several nonlocal and fractional derivatives on $\mathbb{R}^d$ and on bounded domains.

(i) Diffusion processes in $\Omega = \mathbb{R}^d$ with generator $\mathrm{div}(A(x)\nabla)$, where $A : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is a matrix valued function which is bounded, measurable, positive, symmetric and uniformly elliptic [37, Theorem II.3.1, p. 341]. In this case the density $(t, x, y) \mapsto p_t^\Omega(x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and the process is strong Feller (which follows by the Aronson estimate [37, formula (I.0.10)]).

(ii) All strong Feller Lévy processes ($\Omega = \mathbb{R}^d$). Indeed this is a characterisation [22, Lemma 2.1, p. 338]. See [27, Chapter 5.5] for a discussion. This class includes all stable Lévy processes.

(iii) Pure jump Lévy or Lévy-type processes ($\Omega = \mathbb{R}^d$) with generators characterised by:

(a) Lévy measures $\kappa(dy)$ for $d = 1$ such that $\int_{[-z,z]} y^2\, \kappa(dy) \geq z^{2-\alpha}$ for all small z and some $\alpha \in (0, 2)$ [34, Proposition 28.3];

(b) absolutely continuous Lévy measures $\kappa(dy) = \kappa(y)dy$, such that $\int_{\mathbb{R}^d\setminus\{0\}} \kappa(y)\, dy = \infty$ [34, Theorem 27.7];

(c) absolutely continuous Lévy-type measures $\kappa(x, dy) = \kappa(x, y)dy$ such that the respective symbols satisfy the Hölder continuity-type conditions in [27, Theorem 2.14] (see also [27, Theorem 3.3]).

(iv) Clearly any Feller process X taking values in $\mathbb{R}^d$ such that its density is continuous. If X is also strong Feller and $\Omega \subset \mathbb{R}^d$ is a regular set, then the process killed upon

the first exit from Ω is a Feller process on Ω [14, p. 68], and it has a continuous density (which can be proved by the strong Markov property as in [13, formula (4.1)]). This case includes the regional fractional Laplacian $(-\Delta)_\Omega^\beta$ [13].

(v) Any subordination of a Feller process by a Lévy subordinator which itself satisfies (H1), which is a straightforward consequence of [23, Theorem 4.3.5]. This case includes the spectral fractional Laplacian $(-\Delta_\Omega)^\beta$ [9, 8].

(vi) We mention the articles [12, 20] and references therein for related discussions about some jump-type generators with symmetric and non-symmetric kernels.

(vii) The 1-d reflected Brownian motion [7, Chapter 6.2], so that $\Omega = [0, \infty)$, and $\mathcal{L} = \partial_x^2$, endowed with the Neumann boundary condition on $(0, T] \times \{0\}$.

(viii) The restriction to $C_\infty(\overline{\Omega})$ of the $L^2(\Omega)$ semigroup generated by the divergence operator $\operatorname{div}(A(x)\nabla)$ with Neumann boundary conditions on a Lipschitz open bounded connected set $\Omega \subset \mathbb{R}^d$, for the same coefficients A as in Example 2.2-(i). This is a consequence of [21, Theorem 3.10, Section 2.1.2].

(ix) The reflected spectrally negative β-stable Lévy process on $\Omega = [0, \infty)$, for $\beta \in (1, 2)$ [2, Theorem 2.1, Corollary 2.4]. In this case

$$\mathcal{L}u(x) = \partial_x^\beta u(x) = \int_0^x u''(y) \frac{(x-y)^{1-\beta}}{\Gamma(2-\beta)}\, dy, \quad x > 0,$$

for u in the core given in [2, Theorem 2.1], which features $u'(0) = 0$. Note that ∂_x^β is the Caputo derivative of order $\beta \in (1, 2)$ [16].

For our notion of weak solution in Section 4 we will use a stronger assumption for the spatial semigroup. In this assumption below we could allow $\Omega = \mathbb{R}^d$, but we do not as it would affect the clarity of the exposition, as we would have to consider extra cases in several steps in Section 4.

(H1′) the set Ω is a bounded open subset of $\mathbb{R}^d$, and $e^\mathcal{L}$ is a Feller semigroup on $X = C_\infty(\Omega)$ or $X = C_\infty(\overline{\Omega})$ such that assumption (H1) holds and $e^\mathcal{L}$ is self-adjoint, in the sense that for each $r > 0$

$$\int_\Omega e^{r\mathcal{L}} v(x)\, w(x)\, dx = \int_\Omega v(x)\, e^{r\mathcal{L}} w(x)\, dx, \quad v, w \in X. \tag{2.2}$$

Example 2.3.

(i) Assumption (H1′) holds for several processes obtained by killing a Feller process on $\mathbb{R}^d$ upon exiting a regular bounded domain Ω. This is for example the case of the Dirichlet Laplacian $-\Delta_\Omega$, the regional fractional Laplacian $(-\Delta)_\Omega^\beta$ and the spectral fractional Laplacian $(-\Delta_\Omega)^\beta$, $\beta \in (0, 1)$. These killed semigroups are Feller, as explained in Example 2.2-(iv)-(v). Property (2.2) follows by the eigenfunction decomposition of the $L^2(\Omega)$ extension of the killed Feller semigroup [15, 13, 8], along with $C_\infty(\Omega) \subset L^2(\Omega)$. More generally, one can use the theory regular symmetric Dirichlet forms, for example combining [10, Proposition 3.15] with [11, Corollary 3.2.4-(ii)]. These examples correspond to 0 boundary conditions on $\partial\Omega$ or Ω^c.

(ii) Assumption (H1′) holds for the Feller semigroup of Example 2.2-(viii), as an immediate consequence of the semigroup being generated by a (symmetric) regular Dirichlet form [21, Theorem 3.10]. One can also consider an appropriate subordination of the Feller semigroup of Example 2.2-(viii), as mentioned in Example 2.2-(v). Then (H1) still holds along with property (2.2), which can be seen by the eigenfunction expansion to the subordinated semigroup. These examples correspond Neumann boundary conditions on $\partial\Omega$.

2.2 Subordinators and subordinated heat operators

In this section we define the simple space-time process $r \mapsto (t-r, B_r^x)$ with state-space $(-\infty, T] \times \Omega$ (and suitable killed/absorbed versions) and then we subordinate it to obtain the space-time process (1.4) (and the respective subordinated killed/absorbed versions). Basic semigroup theory then leads us to Theorem 2.9, which gives a description of these processes in terms of generators of space-time Feller semigroups. We will always assume the following.

(H0) Denote by $\nu : (0, \infty) \to [0, \infty)$ any continuous function such that

$$\int_0^\infty (r \wedge 1)\nu(r)\, dr < \infty \quad \text{and} \quad \int_0^\infty \nu(r)\, dr = \infty.$$

Definition 2.4. *We denote by $S^\nu = \{S_r^\nu\}_{r \geq 0}$ the Lévy subordinator for ν, characterised by the log-Laplace transforms $\log \mathbf{E}[e^{-kS_r^\nu}] = r \int_0^\infty (e^{-ks} - 1)\nu(s)\, ds$, for $r, k > 0$. We define the first exit/passage times*

$$\tau_0(t) := \inf\{r > 0 : S_r^\nu > t\}, \quad t > 0.$$

Remark 2.5.

(i) Recall that for each $r > 0$, the random variable S_r^ν allows a density p_r^ν (with respect to the Lebesgue measure) [34, Theorem 27.7]. Also, $\tau_0(t) = \inf\{r > 0 : S_r^\nu \geq t\}$ almost surely for every $t > 0$ as S^ν is increasing [6, Chapter III.2], and for every $t \in (0, T]$

$$\int_0^\infty \int_0^t p_s^\nu(t - z)\, dz\, ds = \mathbf{E}[\tau_0(t)] \leq \mathbf{E}[\tau_0(T)] < \infty,$$

see for example [6, page 74]. In particular $\sup_{t \in (0, T]} \mathbf{E}[\tau_0(t)] < \infty$.

(ii) To obtain the stable subordinator case select

$$\nu(r) := r^{-1-\alpha}/|\Gamma(-\alpha)|, \quad r > 0, \quad \alpha \in (0, 1),$$

then $S^\nu = S^\alpha$ is the α-stable subordinator, characterised by the Laplace transforms $\mathbf{E}[e^{-kS_r^\alpha}] = e^{-rk^\alpha}$, for $r, k > 0$. Denote its densities by p_r^α, $r > 0$, and recall that $\mathbf{E}[\tau_0(t)] = t^\alpha/\Gamma(\alpha + 1)$ [8, Example 5.8]. We refer to [8, Chapter 5.2.2] for other examples of subordination kernels ν.

We define three semigroups that correspond to three different space-time valued processes related to the heat operator $-\partial_t + \mathcal{L}$. Namely the "free" process $s \mapsto (t-s, B_s^x)$, the "absorbed at 0" process $s \mapsto ((t-s) \vee 0, B_s^x)$, and the "killed at 0" process $s \mapsto ((t-s), B_s^x)$ for $t > s$ and ∂ otherwise. It is straightforward to prove that such semigroups are Feller and we omit the proof.

Definition 2.6. *Define the operators $e^{s(-\partial_t)}u(t) := u(t - s)$ and $e^{s(-\partial_{t,0})}u(t) := u((t-s) \vee 0)$, $t \in \mathbb{R}$, $s \geq 0$, acting on the time variable. With the semigroup $e^{\mathcal{L}}$ acting on the Ω-variable, define the three Feller semigroups*

$$\begin{aligned}
e^{s\mathcal{H}} &:= e^{s(-\partial_t)}e^{s\mathcal{L}}, & \text{on } C_\infty((-\infty, T] \times \Omega), & \quad s \geq 0, \\
e^{s\mathcal{H}_0} &:= e^{s(-\partial_{t,0})}e^{s\mathcal{L}}, & \text{on } C_\infty([0, T] \times \Omega), & \quad s \geq 0, \\
e^{s\mathcal{H}_0,kill} &:= e^{s\mathcal{H}_0}, & \text{on } C_\infty((0, T] \times \Omega), & \quad s \geq 0,
\end{aligned}$$

with the respective generators denoted by

$$(\mathcal{H}, Dom(\mathcal{H})), \quad (\mathcal{H}_0, Dom(\mathcal{H}_0)), \quad \text{and} \quad (\mathcal{H}_0^{kill}, Dom(\mathcal{H}_0^{kill})).$$

We now define three semigroups that respectively correspond to subordinating the three semigroups in Definition 2.6 by an the independent Lévy subordinator S^ν.

Definition 2.7. *For appropriate functions u, we define for $r > 0$*

$$e^{r\mathcal{H}^\nu}u(t,x) = \int_0^\infty e^{s\mathcal{H}}u(t,x)\, p_r^\nu(s)\, ds, \qquad\qquad t \in \mathbb{R}, \quad (2.3)$$

$$e^{r\mathcal{H}_0^\nu}u(t,x) = \int_0^t e^{s\mathcal{H}}u(t,x)\, p_r^\nu(s)\, ds + \int_t^\infty e^{s\mathcal{L}}u(0,x)p_r^\nu(s)\, ds, \qquad t \in [0,T], \quad (2.4)$$

$$e^{r\mathcal{H}_0^\nu,\mathrm{kill}}u(t,x) = \int_0^t e^{s\mathcal{H}}u(t,x)\, p_r^\nu(s)\, ds, \qquad\qquad t \in (0,T], \quad (2.5)$$

and $e^{r\mathcal{H}^\nu}u(t,x) = e^{r\mathcal{H}_0^\nu}u(t,x) = e^{r\mathcal{H}_0^\nu,\mathrm{kill}}u(t,x) = u(t,x)$, for $r = 0$.

Remark 2.8.

(i) Note that $e^{r\mathcal{H}}u(t,x) = e^{r\mathcal{L}}u(t-r,x) = \mathbf{E}\left[u(t-r, B_r^x)\right]$. Also, for each $r > 0$,

$$e^{r\mathcal{H}_0^\nu}u(t,x) = e^{r\mathcal{H}_0^\nu,\mathrm{kill}}u(t,x) \quad \text{if} \quad u(0) = 0.$$

Moreover, for each $r > 0$, $e^{r\mathcal{H}_0^\nu,\mathrm{kill}}$ maps $B((0,T] \times \Omega)$ to itself.

(ii) If u is independent of time, then

$$e^{r\mathcal{H}_0^\nu}u(t,x) = \int_0^\infty e^{s\mathcal{L}}u(x)\, p_r^\nu(s)\, ds = \mathbf{E}\left[u\left(B_{S_r^\nu}^x\right)\right]$$

is independent of time.

The next theorem shows that the operators in Definition 2.7 define Feller semigroups, it gives a pointwise representation for the generators on "nice" cores, and finally it connects the domains of the generators of $e^{r\mathcal{H}_0^\nu}$ and $e^{r\mathcal{H}_0^\nu,\mathrm{kill}}$. These statements serve various purposes, but let us outline our main line of thinking. Our strategy is to reduce (1.1) to (3.1) with an appropriate forcing term, as suggested by the simple Lemma 4.6 (here we use the generators pointwise representation). Hence we solve problem (3.1) in the framework of abstract resolvent equations (Theorem 3.5). To do so, we use Theorem 2.9-(iv) to reduce problem (3.1) to the 0 initial condition version, easily solved by inverting $\mathcal{H}_0^{\nu,\mathrm{kill}}$ (Lemma 3.3). Moreover, Theorem 2.9 allows us to access Dynkin formula.

Theorem 2.9. *Assume (H0). Then, with the notation of Definitions 2.1, 2.6 and 2.7:*

(i) *The operators $e^{r\mathcal{H}^\nu}$, $r \geq 0$ form a Feller semigroup on $C_\infty((-\infty,T] \times \Omega)$. We denote the generator of the semigroup by $(\mathcal{H}^\nu, Dom(\mathcal{H}^\nu))$.*

   Moreover, $Dom(\mathcal{H})$ is a core for $(\mathcal{H}^\nu, Dom(\mathcal{H}^\nu))$, and for $g \in Dom(\mathcal{H})$

$$\mathcal{H}^\nu g(t,x) = H^\nu g(t,x) := \int_0^\infty \left(e^{r\mathcal{H}}g(t,x) - g(t,x)\right)\nu(r)\, dr. \quad (2.6)$$

(ii) *The operators $e^{r\mathcal{H}_0^\nu}$, $r \geq 0$ form a Feller semigroup on $C_\infty([0,T] \times \Omega)$. We denote the generator of the semigroup by $(\mathcal{H}_0^\nu, Dom(\mathcal{H}_0^\nu))$.*

   Moreover, $Dom(\mathcal{H}_0)$ is a core for $(\mathcal{H}_0^\nu, Dom(\mathcal{H}_0^\nu))$, and

$$\mathcal{H}_0^\nu g(t,x) = H_0^\nu g(t,x), \quad \text{for } g \in Dom(\mathcal{H}_0), \quad (2.7)$$

   where

$$H_0^\nu g(t,x) := \int_0^t \left(e^{r\mathcal{H}}g(t,x) - g(t,x)\right)\nu(r)\, dr + \int_t^\infty \left(e^{r\mathcal{L}}g(0,x) - g(t,x)\right)\nu(r)\, dr.$$

(iii) *The operators $e^{r\mathcal{H}_0^{\nu,kill}}$, $r \geq 0$ form a Feller semigroup on $C_\infty((0,T] \times \Omega)$. We denote the generator of the semigroup by $(\mathcal{H}_0^{\nu,kill}, Dom(\mathcal{H}_0^{\nu,kill}))$.*

Moreover, $Dom(\mathcal{H}_0^{kill})$ is a core for $(\mathcal{H}_0^{\nu,kill}, Dom(\mathcal{H}_0^{\nu,kill}))$, and

$$\mathcal{H}_0^{\nu,kill}g = H_0^\nu g, \quad \text{for } g \in Dom(\mathcal{H}_0^{kill}).$$

(iv) *In addition, it holds that $\mathcal{H}_0^\nu = \mathcal{H}_0^{\nu,kill}$ on $Dom(\mathcal{H}_0^{\nu,kill})$, and*

$$Dom(\mathcal{H}_0^{\nu,kill}) = Dom(\mathcal{H}_0^\nu) \cap \{g(0) = 0\}. \tag{2.8}$$

Proof. The statements (i), (ii) and (iii) are all consequences of [23, Theorem 4.3.5 and Proposition 4.3.7] along with preservation of positive functions and the contraction property, which are easily checked directly from the definitions (2.3), (2.4) and (2.5), respectively.

(iv) To prove (2.8), we note that the inclusion '$\subset$' is clear because, $Dom(\mathcal{H}_0^{\nu,kill}) \subset C_\infty((0,T] \times \Omega)$, and the two semigroups (2.4) and (2.5) agree on $C_\infty((0,T] \times \Omega)$ by Remark 2.8-(i). For the opposite inclusion '$\supset$', we show that

$$\text{if } g \in \text{Dom}(\mathcal{H}_0^\nu), \text{ then } g - g(0) \subset \text{Dom}(\mathcal{H}_0^{\nu,\text{kill}}).$$

Consider the resolvent representation for g for a given $\lambda > 0$ and $g_\lambda \in C_\infty([0,T] \times \Omega)$ given by

$$g(t,x) = \int_0^\infty e^{-r\lambda} e^{r\mathcal{H}_0^\nu} g_\lambda(t,x) \, dr,$$

and

$$g(0,x) = \int_0^\infty e^{-r\lambda} e^{r\mathcal{H}_0^\nu} g_\lambda(0,x) \, dr = \int_0^\infty e^{-r\lambda} e^{r\mathcal{H}_0^\nu} (g_\lambda(0))(t,x) \, dr,$$

where we use Remark 2.8-(ii). Then

$$g(t,x) - g(0,x) = \int_0^\infty e^{-r\lambda} e^{r\mathcal{H}_0^\nu} (g_\lambda - g_\lambda(0))(t,x) \, dr \in \text{Dom}(\mathcal{H}_0^{\nu,\text{kill}})$$

as $g_\lambda - g_\lambda(0) \in C_\infty((0,T] \times \Omega)$ and $e^{r\mathcal{H}_0^\nu} = e^{r\mathcal{H}_0^\nu,\text{kill}}$ on $C_\infty((0,T] \times \Omega)$, and (2.8) is proved. We can now conclude equating resolvent equations, as for any $g \in \text{Dom}(\mathcal{H}_0^{\nu,\text{kill}})$, for a positive $\lambda > 0$ and a respective $g_\lambda \in C_\infty((0,T] \times \Omega)$

$$\mathcal{H}_0^{\nu,\text{kill}}g = \lambda g - g_\lambda = \mathcal{H}_0^\nu g. \qquad \square$$

Remark 2.10.

(i) Let us stress that Theorem 2.9-(iv), although unsurprising, is a vital technical ingredient for this work. This is because it allows to obtain uniqueness of our notion of "solution in the domain of the generator" to (3.1) for non-zero initial data (see the proof of Lemma 3.3-(i)). This notion of solution is our building block for weak solutions to (1.1) in Section 4.

(ii) To see that $H^\nu u$ is well defined pointwise for $u \in \text{Dom}(\mathcal{H})$ simply use the general bound in (2.1) along with (H0).

(iii) Theorem 2.9 holds with the exact same proof if we replace S^ν with an arbitrary non-decreasing Lévy process and replace $e^{\mathcal{L}}$ with an arbitrary strongly continuous contraction semigroup. (Of course the statement will have to take into account the more general representation [23, Eq. (4.132)] for the subordinated generator.) This theorem can also be generalised to more general heat operators $(\partial_t^\mu - \mathcal{L})$ where $-\partial_t^\mu$ is the generator of a non-increasing Lévy-type process, but this is beyond the scope of this work.

The proof of Theorem 2.9-(i) guarantees that the next definition makes sense.

Definition 2.11. *We denote by $(\mathcal{L}^\nu, Dom(\mathcal{L}^\nu))$ the generator of the Feller semigroup*

$$e^{r\mathcal{L}^\nu}(\cdot) := \int_0^\infty e^{s\mathcal{L}}(\cdot)p_r^\nu(s)\,ds, \quad r > 0,$$

on $C_\infty(\Omega)$ induced by the Feller process $r \mapsto B_{S_r^\nu}$.

Remark 2.12. The life time of the Feller process $r \mapsto B_{S_r^\nu}$ is

$$\inf\{s > 0 : B_{S_s^\nu}^x \notin \Omega\} = \inf\{s > 0 : S_s^\nu \geq \tau_\Omega(x)\} = \tau_0(\tau_\Omega(x)),$$

for each $x \in \Omega$, where we used $B_s^x = B_{s \wedge \tau_\Omega(x)}^x$ and its independence with respect to S_s^ν.

We will later use the following simple lemma.

Lemma 2.13. *Suppose $\phi_0 \in Dom(\mathcal{L}^\nu)$ and constantly extend $\phi_0(x)$ to $[0, T]$ for each $x \in \Omega$. Then $\phi_0 \in Dom(\mathcal{H}_0^\nu) \subset C_\infty([0, T] \times \Omega)$ and*

$$\mathcal{H}_0^\nu \phi_0 = \mathcal{L}^\nu \phi_0.$$

Proof. This is straightforward, because

$$r^{-1}\left(e^{r\mathcal{H}_0^\nu}(\phi_0)(t, x) - \phi_0(t, x)\right) = r^{-1}\left(\int_0^\infty e^{s\mathcal{L}}\phi_0(x)p_r^\nu(s)\,ds - \phi_0(x)\right)$$
$$= r^{-1}\left(e^{r\mathcal{L}^\nu}\phi_0(x) - \phi_0(x)\right) \to \mathcal{L}^\nu \phi_0,$$

as $r \downarrow 0$, uniformly in both t and x. $\qquad\square$

3 Generalised solutions

We prove existence, uniqueness and stochastic representation for generalised solutions to the 'Caputo-type' problem

$$\begin{cases} \mathcal{H}_0^\nu u(t, x) = -g, & \text{in } (0, T] \times \Omega, \\ u(0, x) = \phi_0(x), & \text{in } \{0\} \times \Omega, \end{cases} \tag{3.1}$$

under assumptions (H0) and (H1). In particular, we will obtain the probabilistic representation

$$u(t, x) = \mathbf{E}\left[\phi_0\left(B_{S_{\tau_0(t)}^\nu}^x\right)\mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right] + \mathbf{E}\left[\int_0^{\tau_0(t) \wedge \tau_0(\tau_\Omega(x))} g\left(t - S_r^\nu, B_{S_r^\nu}^x\right)dr\right], \tag{3.2}$$

for the solution to (3.1).

Remark 3.1.

(i) Recalling Remark 2.12, observe that if $g(\partial) = 0$ for ∂ the cemetery state of $C_\infty([0, T] \times \Omega)$, then

$$\mathbf{E}\left[\int_0^{\tau_0(t) \wedge \tau_0(\tau_\Omega(x))} g\left(t - S_r^\nu, B_{S_r^\nu}^x\right)dr\right] = \mathbf{E}\left[\int_0^{\tau_0(t)} g\left(t - S_r^\nu, B_{S_r^\nu \wedge \tau_\Omega(x)}^x\right)dr\right]$$
$$= \int_0^\infty \mathbf{E}\left[\mathbb{1}_{\{t - S_r^\nu > 0\}}g\left(t - S_r^\nu, B_{S_r^\nu}^x\right)\right]dr.$$

Similarly, if $\phi_0(\partial) = 0$, for ∂ the cemetery state of $C_\infty(\Omega)$, then

$$\mathbf{E}\left[\phi_0\left(B_{S_{\tau_0(t)}^\nu}^x\right)\right] = \mathbf{E}\left[\phi_0\left(B_{S_{\tau_0(t)}^\nu \wedge \tau_\Omega(x)}^x\right)\right] = \mathbf{E}\left[\phi_0\left(B_{S_{\tau_0(t)}^\nu}^x\right)\mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right].$$

(ii) Problem (3.1) corresponds to problem (1.1) for time independent initial condition $\phi(t) = \phi_0$, in a similar way as Caputo and Marchaud evolution equations are related in [38]. By the appropriate choice of forcing term g, problem (3.1) formally rewrites as problem (1.1), which we make rigorous in Section 4.

We first assume some compatibility condition on the forcing term and the initial data in order to construct the following kind of strong solution by inverting $\mathcal{H}_0^\nu$.

Definition 3.2. *The function u is a solution in the domain of generator to* (3.1) *if*

$$\mathcal{H}_0^\nu u = -g, \ \ on \ (0, T] \times \Omega, \ \ u(0) = \phi_0, \ \ u \in Dom(\mathcal{H}_0^\nu). \tag{3.3}$$

Lemma 3.3. *Assume (H0), and let $g \in C_\infty([0, T] \times \Omega)$ and $\phi_0 \in Dom(\mathcal{L}^\nu)$ such that $g + \mathcal{L}^\nu \phi_0 \in C_\infty((0, T] \times \Omega)$.*

(i) *Then there exists a unique solution in the domain of the generator to* (3.1).

(ii) *Moreover, the solution in the domain of the generator allows the stochastic representation* (3.2).

Proof. i) We first claim that

$$(-\mathcal{H}_0^{\nu, \text{kill}})^{-1}(g + \mathcal{L}^\nu \phi_0) = \int_0^\infty \left(\int_0^t e^{s\mathcal{H}} (g + \mathcal{L}^\nu \phi_0) \, p_r^\nu(s) \, ds \right) dr,$$

is the unique solution to the abstract evolution equation

$$\mathcal{H}_0^{\nu, \text{kill}} u = -g - \mathcal{L}^\nu \phi_0, \ on \ (0, T] \times \Omega, \ \ u(0) = 0, \ \ u \in \text{Dom}(\mathcal{H}_0^{\nu, \text{kill}}). \tag{3.4}$$

Let $f \in C_\infty((0, T] \times \Omega)$. Then

$$\begin{aligned} (-\mathcal{H}_0^{\nu, \text{kill}})^{-1} f(t, x) &= \int_0^\infty e^{r\mathcal{H}_0^\nu} f(t, x) \, dr \\ &= \int_0^\infty \left(\int_0^t e^{s\mathcal{H}} f(t, x) \, p_r^\nu(s) \, ds \right) dr \\ &\leq \|f\|_{C_\infty((0,T] \times \Omega)} \int_0^\infty \int_0^t p_r^\nu(s) \, ds \, dr \\ &\leq \|f\|_{C_\infty((0,T] \times \Omega)} \mathbf{E}[\tau_0(T)]. \end{aligned}$$

Moreover, using $e^{r\mathcal{H}_0^\nu} f \in C_\infty((0, T] \times \Omega)$, Dominated Convergence Theorem (DCT) proves that $(-\mathcal{H}_0^{\nu, \text{kill}})^{-1}$ maps $C_\infty((0, T] \times \Omega)$ into itself. Then [18, Theorem 1.1'] proves the claim. We recall that by Theorem 2.9-(iv)

$$\mathcal{H}_0^\nu \tilde{u} = \mathcal{H}_0^{\nu, \text{kill}} \tilde{u}, \ \ \text{if} \ \tilde{u} \in \text{Dom}(\mathcal{H}_0^{\nu, \text{kill}}) = \text{Dom}(\mathcal{H}_0^\nu) \cap \{f(0) = 0\}.$$

To conclude, it is now enough to show that $\tilde{u} = u - \phi_0$ is a solution to (3.4) if and only if u is a solution to (3.3). For the 'only if' direction, define

$$u := \tilde{u} + \phi_0.$$

Then $u \in \text{Dom}(\mathcal{H}_0^\nu)$ as $\tilde{u} \in \text{Dom}(\mathcal{H}_0^\nu)$ by Theorem 2.9-(iv), $\phi_0 \in \text{Dom}(\mathcal{H}_0^\nu)$ by Lemma 2.13 and u solves

$$\mathcal{H}_0^\nu(\tilde{u} + \phi_0) = \mathcal{H}_0^\nu \tilde{u} + \mathcal{L}^\nu \phi_0 = -g,$$

along with $u(0) = \phi_0$. The 'if' direction is similar and omitted.

ii) Fix $(t, x) \in (0, T] \times \Omega$. First compute

$$\int_0^\infty e^{r \mathcal{H}_0^\nu, \text{kill}} (\mathcal{L}^\nu \phi_0)(t, x) \, dr = \mathbf{E} \left[\int_0^\infty \int_0^\infty \mathcal{L}^\nu \phi_0(B_s^x) \mathbb{1}_{\{t-s>0\}} p_r^\nu(s) \, ds \, dr \right]$$
$$= \mathbf{E} \left[\int_0^\infty \mathcal{L}^\nu \phi_0(B_{S_r^\nu}^x) \mathbb{1}_{\{t-S_r^\nu>0\}} \, dr \right]$$
$$= \mathbf{E} \left[\int_0^{\tau_0(t)} \mathcal{L}^\nu \phi_0(B_{S_r^\nu}^x) \, dr \right],$$

where we use $\{t - S_r^\nu > 0\} = \{\tau_0(t) > r\}$, by the monotonicity of the subordinator S^ν. Recalling Lemma 2.13, Remark 3.1-(i) and the integrability of $\tau_0(t)$, we can apply Dynkin formula [18, Corollary of Theorem 5.1] to the Feller process in Theorem 2.9-(ii) with respect to its first exit from the open set $(0, T] \times \Omega$ to obtain

$$\mathbf{E} \left[\int_0^{\tau_0(t)} \mathcal{L}^\nu \phi_0 \left(B_{S_r^\nu}^x \right) \, dr \right] + \phi_0(x) = \mathbf{E} \left[\phi_0 \left(B_{S_{\tau_0(t)}^\nu}^x \right) \right],$$

and this proves that u can be written as (3.2). $\qquad \square$

We now give another definition of solution as the pointwise limit of solutions in the domain of the generator. This allows us to drop the compatibility condition on the data in Lemma 3.3.

Definition 3.4. *Let $g \in L^\infty((0, T) \times \Omega)$ and let $\phi_0 \in Dom(\mathcal{L}^\nu)$. Then u is a generalised solution to (3.1) if*

$$u = \lim_{n \to \infty} u_n, \quad \text{pointwise on } (0, T] \times \Omega,$$

where $\{u_n\}_{n \geq 1}$ is the sequence of solutions in the domain of the generator to (3.1) for a respective sequence of forcing terms $\{g_n\}_{n \geq 1} \subset C_\infty([0, T] \times \Omega)$ such that $g_n(0) = \mathcal{L}^\nu \phi_0$ for all $n \geq 1$, $g_n \to g$ bpw a.e.

Theorem 3.5. *Assume (H0), (H1) and let $g \in L^\infty((0, T) \times \Omega)$ and $\phi_0 \in Dom(\mathcal{L}^\nu)$. Then there exist a unique generalised solution to (3.1). Moreover the generalised solution allows the stochastic representation (3.2).*

Proof. As Ω is either open or the closure of an open set, by the theory of mollifiers we can take a sequence $\{g_n\}_{n \geq 1}$ as in Definition 3.4. Then the respective solution in the domain of the generator u_n allows the representation (3.2) for $g \equiv g_n$, by Lemma 3.3-(ii). Fix $(t, x) \in (0, T] \times \Omega$. By assumption (H1), Remark 2.5-(i), Remark 3.1-(i), and independence of S_r^ν and B_r^x, we can rewrite the second term in (3.2) as

$$F(g_n) := \int_0^\infty \left(\int_\Omega \int_0^\infty \mathbb{1}_{\{t-s>0\}} g_n(t-s, y) p_s^\Omega(x, y) \, p_r^\nu(s) \, ds \, dy \right) dr$$
$$= \int_\Omega \int_0^\infty g_n(t-s, y) \left(\mathbb{1}_{\{t-s>0\}} p_s^\Omega(x, y) \int_0^\infty p_r^\nu(s) \, dr \right) ds \, dy.$$

Then, by DCT, $F(g_n) \to F(g)$ as $n \to \infty$, using the dominating function

$$(s, y) \mapsto \sup_n \|g_n\|_{C_\infty([0, T] \times \Omega)} \mathbb{1}_{\{t>s\}} p_s^\Omega(x, y) \int_0^\infty p_r^\nu(s) \, dr,$$

given that $F(|g_n|) \leq \sup_n \|g_n\|_{C_\infty([0, T] \times \Omega)} \mathbf{E}[\tau_0(t)]$. Hence a generalised solution exists and it allows the stochastic representation (3.2). Conclude observing that independence of the approximating sequence proves uniqueness. $\qquad \square$

Remark 3.6.

(i) By definition, a sequence u_n of solutions in the domain of the generator converges pointwise to the generalised solution u on $(0, T] \times \Omega$. Moreover, by the stochastic representation (3.2),

$$\sup_n \|u_n\|_{C_\infty([0,T] \times \Omega)} \le \|\phi_0\|_{C_\infty(\Omega)} + \sup_n \|g_n\|_{C_\infty([0,T] \times \Omega)} \mathbf{E}[\tau_0(T)] < \infty,$$

where each g_n is the data of the solution in the domain of the generator u_n. Therefore $u_n \to u$ bpw on $[0, T] \times \Omega$.

(ii) We assumed (H1) in Theorem 3.5 to treat L^∞ data. But if we were to assume continuous data (or the closure of $C_\infty((0, T] \times \Omega)$ with respect to bpw convergence [10, page 1]), then Theorem 3.5 would still hold by changing "bpw a.e." with "bpw" in Definition 3.4. In this case we would also not need the subordinator S^ν to allow a density. These conditions would still allow to define and establish existence of weak solutions on appropriate spaces with the same strategy of Section 4. But we considered such technical treatment beyond the scope of this work, which aims to present a clear and concrete treatment of the new formulation of the CEE (1.1) with initial conditions on the past.

We now show that the fundamental solution that defines (1.3) allows a density with respect to Lebesgue measure. Then we conclude this section with several examples of concrete densities for solutions to (1.1).

Lemma 3.7. *Assume (H0). Then for each $t > 0$, the random variable $S^\nu_{\tau_0(t)} - t$ allows a density supported on $(0, \infty)$, and we can write the density for almost every $r \in (0, \infty)$ as*

$$p^{\nu, \tau_0(t)}(r) = \int_0^t \nu(y + r) \int_0^\infty p_s^\nu(t - y) \, ds \, dy.$$

Proof. This follows, for example, by performing the proof of [17, Proposition 3.13] in the simpler setting without the spatial process. $\square$

Lemma 3.8. *Assume (H0) and (H1). Suppose $\phi \in L^\infty((-\infty, 0) \times \Omega)$ and $g \in L^\infty((0, \infty) \times \Omega)$. Then for $t > 0$, $x \in \Omega$*

$$\mathbf{E}\left[\phi\left(t - S^\nu_{\tau_0(t)}, B^x_{S^\nu_{\tau_0(t)}}\right) \mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right] = \int_0^\infty \int_\Omega \phi(-r, y) \left(p_{t+r}^\Omega(x, y) p^{\nu, \tau_0(t)}(r)\right) dy \, dr,$$

(3.5)

and

$$\mathbf{E}\left[\int_0^{\tau_0(t) \wedge \tau_0(\tau_\Omega(x))} g\left(t - S^\nu_r, B^x_{S^\nu_r}\right) dr\right] = \int_\Omega \int_0^t g(t - s, y) \left(p_s^\Omega(x, y) \int_0^\infty p_r^\nu(s) \, dr\right) ds \, dy.$$

Proof. Extend ϕ and g to 0 on the appropriate cemetery state. Then, proceeding as in Remark 3.1-(i) and then, using independence between $S^\nu_{\tau_0(t)}$ and B^x_r along with Lemma 3.7, we obtain

$$\mathbf{E}\left[\phi\left(t - S^\nu_{\tau_0(t)}, B^x_{S^\nu_{\tau_0(t)}}\right) \mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right] = \mathbf{E}\left[\phi\left(t - S^\nu_{\tau_0(t)}, B^x_{S^\nu_{\tau_0(t)}}\right)\right]$$

$$= \int_0^\infty \int_\Omega \phi(-r, y) \left(p_{t+r}^\Omega(x, y) p^{\nu, \tau_0(t)}(r) \, dy\right) dr.$$

The inhomogeneous term is treated similarly and we omit the proof. $\square$

Corollary 3.9. *Assume (H0) and (H1). Let $f_n, f \in L^\infty((0, \infty) \times \Omega)$, $\phi_n, \phi \in L^\infty((-\infty, 0) \times \Omega)$, for $n \in \mathbb{N}$, such that $f_n \to f$ and $\phi_n \to \phi$ bpw a.e. as $n \to \infty$.*
   Then, as $n \to \infty$

$$u_n \to u \quad \text{bpw a.e. on } (-\infty, T) \times \Omega,$$

where u_n is defined as (1.3) for $f \equiv f_n$, $\phi \equiv \phi_n$ on $(0, T) \times \Omega$, and as ϕ_n on $(-\infty, 0) \times \Omega$, and u is defined as (1.3) for $f \equiv f$, $\phi \equiv \phi$ on $(0, T) \times \Omega$, and as ϕ on $(-\infty, 0) \times \Omega$.

Proof. This is a straightforward application of DCT given Lemma 3.8 and $\mathbf{E}[\tau_0(T)] < \infty$. $\qquad\square$

Example 3.10. We list some examples of solutions (1.3). These examples show that the space-time coupling can lead to closed form solution kernels. Also note that if $\phi = \phi_0$ does not depend on time, then (3.5) equals

$$\mathbf{E}\left[\phi_0\left(B^x_{S^\nu_{\tau_0(t)}}\right) \mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right] = \int_\Omega \phi_0(y) \left(\int_0^\infty p^\Omega_{t+r}(x, y) p^{\nu, \tau_0(t)}(r)\, dr\right) dy.$$

(i) If $S^\nu = S^\alpha$, the α-stable subordinator, $\alpha \in (0, 1)$, then [24, Formula (5.12)]

$$p^{\nu, \tau_0(t)}(r) = \int_0^t \frac{(y+r)^{-1-\alpha}}{|\Gamma(-\alpha)|} \frac{(t-y)^{\alpha-1}}{\Gamma(\alpha)}\, dy = t^\alpha \frac{r^{-\alpha}(t+r)^{-1}}{\Gamma(\alpha)\Gamma(1-\alpha)},$$

and if in addition B is a d-dimensional Brownian motion, then

$$\mathbf{E}\left[\phi\left(t - S^\alpha_{\tau_0(t)}, B^x_{S^\alpha_{\tau_0(t)}}\right)\right] = \int_0^\infty \int_{\mathbb{R}^d} \phi(-r, y) \left(e^{\frac{-|x-y|^2}{4(t+r)}} \frac{c_{d,\alpha} t^\alpha}{r^\alpha(t+r)^{d/2+1}}\right) dy\, dr,$$

where $c_{d,\alpha} = \sin(\pi\alpha)/(2^d \pi^{d/2+1})$, and of course $\mathcal{L} = \Delta$, the d-dimensional Laplacian. Moreover

$$\mathbf{E}\left[\int_0^{\tau_0(t)} g\left(t - S^\alpha_r, B^x_{S^\alpha_r}\right) dr\right] = \int_{\mathbb{R}^d} \int_0^t g(t-s, y) \left(\frac{e^{\frac{-|x-y|^2}{4s}}}{(4\pi s)^{d/2}} \frac{s^{\alpha-1}}{\Gamma(\alpha)}\right) ds\, dy.$$

(ii) If instead B is a killed 1-d Brownian motion for $\Omega = (0, \pi)$, then for $t > 0$, $x \in (0, \pi)$

$$\mathbf{E}\left[\phi\left(t - S^\alpha_{\tau_0(t)}, B^x_{S^\alpha_{\tau_0(t)}}\right) \mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right]$$
$$= \int_0^\infty \int_0^\pi \phi(-r, y) \left(\left(\sum_{n=1}^\infty e^{-n^2(t+r)} \sin(nx) \sin(ny)\right) \frac{c_\alpha t^\alpha}{r^\alpha(t+r)}\right) dy\, dr,$$

where $c_\alpha = 2\sin(\alpha\pi)/\pi^2$, and $\{n^2, \sqrt{2/\pi}\sin(n\cdot)\}_{n \in \mathbb{N}}$ are the eigenvalues-eigenfunctions of the Dirichlet Laplacian $\mathcal{L} = \Delta_\Omega$ [15].

(iii) If now B is the subordination of the above killed Brownian motion by an independent β-stable Lévy subordinator [9, 8], so that

$$\mathcal{L}u(x) = -(-\Delta_\Omega)^\beta u(x) = \frac{1}{|\Gamma(-\beta)|} \int_0^\infty \left(e^{r\Delta_\Omega} u(x) - u(x)\right) \frac{dr}{r^{1+\beta}}, \quad \beta \in (0, 1),$$

then, using the Laplace transform of the β-stable Lévy subordinator, for $t > 0$, $x \in (0, \pi)$,

$$\mathbf{E}\left[\phi\left(t - S^\alpha_{\tau_0(t)}, B^x_{S^\alpha_{\tau_0(t)}}\right) \mathbb{1}_{\{\tau_0(t) < \tau_0(\tau_\Omega(x))\}}\right]$$
$$= \int_0^\infty \int_0^\pi \phi(-r, y) \left(\left(\sum_{n=1}^\infty e^{-n^{2\beta}(t+r)} \sin(nx) \sin(ny)\right) \frac{c_\alpha t^\alpha}{r^\alpha(t+r)}\right) dy\, dr.$$

(iv) If B is the reflection at 0 of a 1-d Brownian motion, then $\Omega = [0, \infty)$, $\mathcal{L} = \partial_x^2$ with Neumann boundary condition on $(0, T] \times \{0\}$, and for t, $x > 0$ and $c_{d,\alpha}$ as in (i)

$$\mathbf{E}\left[\phi\left(B^x_{S^\alpha_{\tau_0(t)}}\right)\right] = \int_0^\infty \phi(y) \left(\int_0^\infty \left(e^{\frac{-|x-y|^2}{4(t+r)}} + e^{\frac{-|x+y|^2}{4(t+r)}}\right) \frac{c_{d,\alpha} t^\alpha}{r^\alpha (t+r)^{d/2+1}} \, dr\right) dy.$$

4 Weak solutions

In this section we prove that the stochastic representation (1.3) is a weak solution for problem (1.1), under the stronger assumption (H1′) on the spatial semigroup $e^{\mathcal{L}}$. We recall that Ω is open and bounded under (H1′), we introduce the notation

$$\langle f, g \rangle = \int_{-\infty}^T \int_\Omega f(t,x) g(t,x) \, dx \, dt,$$

and we define the adjoint operator of H^ν as

$$H^{\nu,*}\varphi(t,x) := \int_0^\infty \left(e^{r\mathcal{L}}\varphi(t+r,x) - \varphi(t,x)\right) \nu(r) \, dr.$$

For our notion of weak solution we need the pairing $\langle u, H^{\nu,*}\varphi \rangle$ to be well defined for some test function φ (see Definition 4.7), and we want to allow constant-in-time data ϕ, so that the solution u will be in $L^\infty((-\infty, T) \times \Omega)$. Moreover, recalling that a generalised solution u is characterised by the existence of a sequence $u_n \to u$ bpw a.e., we want to be able to show $\langle u_n, H^{\nu,*}\varphi \rangle \to \langle u, H^{\nu,*}\varphi \rangle$. And so, to guarantee a well defined pairing and access dominated convergence arguments, we now prove that $H^{\nu,*}\varphi \in L^1((-\infty, T) \times \Omega)$.

Remark 4.1. Recall from Theorem 2.9 that $\mathcal{H}^\nu$ and $\mathcal{H}_0^\nu$ denote abstract generators, meanwhile H^ν and H_0^ν denote pointwise defined operators.

Lemma 4.2. *Assume (H0) and (H1′). If $\varphi = pq \in C_\infty^1(-\infty, T] \cdot Dom(\mathcal{L})$ is such that $p, \partial_t p \in L^1(\mathbb{R})$, then*

$$(t,x) \mapsto \int_0^\infty \left|e^{r\mathcal{L}}\varphi(t+r,x) - \varphi(t,x)\right| \nu(r) \, dr \in L^1((-\infty, T) \times \Omega),$$

and in particular $H^{\nu,*}\varphi \in L^1((-\infty, T) \times \Omega)$.

Proof. We rewrite

$$H^{\nu,*}\varphi(t,x) = \int_0^\infty e^{r\mathcal{L}}q(x)\left(p(t+r) - p(t)\right)\nu(r)\,dr + p(t)\int_0^\infty \left(e^{r\mathcal{L}}q(x) - q(x)\right)\nu(r)\,dr$$

$$=: (I + II)(t,x).$$

Then, with inequalities holding up to a constant

$$\int_{\mathbb{R}\times\Omega} |I(t,x)| \, dx \, dt \le \|q\|_{C(\overline{\Omega})} \int_{\mathbb{R}} \left|\int_0^\infty \left(p(t+r) - p(t)\right)\nu(r)\,dr\right| dt$$

$$\le \|q\|_{C(\overline{\Omega})}\left(\|p\|_{L^1(\mathbb{R})} + \|\partial_t p\|_{L^1(\mathbb{R})}\right),$$

where we used [17, Lemma 4.3] in the second inequality. Considering II,

$$\int_{\mathbb{R}\times\Omega} |II(t,x)| \, dx \, dt \le \|p\|_{L^1(\mathbb{R})} \int_\Omega \left|\int_0^\infty \left(e^{r\mathcal{L}}q(x) - q(x)\right)\nu(r)\,dr\right| dx$$

$$\le \|p\|_{L^1(\mathbb{R})} \int_\Omega \int_0^\infty \left(\left(r\|\mathcal{L}q\|_{C(\overline{\Omega})}\right) \wedge \|2q\|_{C(\overline{\Omega})}\right)\nu(r)\,dr\,dx,$$

which is finite using (H0) and that Ω is bounded. $\qquad\square$

Proposition 4.3. *Assume (H0) and (H1′). Let $u \in L^\infty((-\infty, T] \times \Omega)$ such that $u \in Dom(\mathcal{H}_0)$ if restricted to $t \geq 0$. Then for every $\varphi \in C^1_c(0, T) \cdot Dom(\mathcal{L})$*

$$\langle H^\nu u, \varphi \rangle = \langle u, H^{\nu,*} \varphi \rangle. \tag{4.1}$$

Proof. Let $k > 0$ such that $\varphi(t) = 0$ for every $t \leq k$. Note that for each $(t, x) \in (k, T] \times \Omega$ we have the bound for a.e. $r > 0$

$$\left| e^{r\mathcal{H}} u(t, x) - u(t, x) \right| = \left| e^{r\mathcal{H}_0} u(t, x) - u(t, x) \right| \mathbb{1}_{\{r \leq k\}} + \left| e^{r\mathcal{L}} u(t - r, x) - u(t, x) \right| \mathbb{1}_{\{r > k\}}$$
$$\leq r \|\mathcal{H}_0 u\|_{C_\infty([0,T] \times \Omega)} \mathbb{1}_{\{r \leq k\}} + 2 \|u\|_{L^\infty((-\infty, T] \times \Omega)} \mathbb{1}_{\{r > k\}},$$

and so $H^\nu u$ (defined in (2.6)) is well defined for each $(t, x) \in (k, T] \times \Omega$ and bounded on the same set. And so the left hand side in (4.1) is well defined recalling that $\varphi \in L^1(\mathbb{R} \times \Omega)$. To conclude we compute

$$\langle H^\nu u, \varphi \rangle = \int_{-\infty}^{T} \int_{\Omega} \left(\int_0^\infty \left(e^{r\mathcal{H}} u(t, x) - u(t, x) \right) \nu(r) \, dr \right) \varphi(t, x) \, dx \, dt$$
$$= \lim_{\epsilon \downarrow 0} \left(\int_{-\infty}^{T} \int_{\Omega} \left(\int_\epsilon^\infty e^{r\mathcal{H}} u(t, x) \nu(r) \, dr \right) \varphi(t, x) \, dx \, dt \right.$$
$$\left. - \int_{-\infty}^{T} \int_{\Omega} \left(\int_\epsilon^\infty \varphi(t, x) \nu(r) \, dr \right) u(t, x) \, dx \, dt \right)$$
$$= \lim_{\epsilon \downarrow 0} \left(\int_{-\infty}^{T} \int_{\Omega} u(t, y) \left(\int_\epsilon^\infty e^{r\mathcal{L}} \varphi(t + r, y) \nu(r) \, dr \right) dy \, dt \right.$$
$$\left. - \int_{-\infty}^{T} \int_{\Omega} \left(\int_\epsilon^\infty \varphi(t, y) \nu(r) \, dr \right) u(t, y) \, dy \, dt \right)$$
$$= \int_{-\infty}^{T} \int_{\Omega} u(t, y) \left(\int_0^\infty \left(e^{r\mathcal{L}} \varphi(t + r, y) - \varphi(t, y) \right) \nu(r) \, dr \right) dy \, dt$$
$$= \langle u, H^{\nu,*} \varphi \rangle,$$

where we use DCT in the second identity, for the third identity we use (2.2), Fubini's Theorem and $\varphi(t + r) = 0$ for $t \geq T - r$, and for the fourth identity we use DCT, thanks to Lemma 4.2 and $u \in L^\infty((-\infty, T) \times \Omega)$. □

Our approximation procedure, in the proof of Theorem 4.8, will be carried out using the following assumption on the approximating data.

(H2) Let ϕ be a linear combination of functions in

$$C^1_\infty(-\infty, 0] \cap \{f'(0-) = 0\} \cdot Dom(\mathcal{L}).$$

If ϕ satisfies (H2), then it satisfies (H2′) below, as a consequence of $C^1_\infty(-\infty, T] \cdot Dom(\mathcal{L}) \subset Dom(\mathcal{H})$. We use (H2′) to apply Dynkin formula in the next lemma.

(H2′) The function $\phi : (-\infty, 0] \times \Omega \to \mathbb{R}$ is such that the extension of ϕ to $\phi(0)$ on $(0, T] \times \Omega$ satisfies $\phi \in Dom(\mathcal{H}) \subset Dom(\mathcal{H}^\nu)$.

Remark 4.4. The functions satisfying (H2) are dense in $L^\infty((-\infty, 0) \times \Omega)$ with respect to bpw a.e. convergence. To prove it, for $Dom(\mathcal{L}) \subset C_\infty(\Omega)$ one can use the Stone-Weierstrass strategy in [38, Appendix II] to show that the functions satisfying (H2) are uniformly dense in $C_\infty((-\infty, 0) \times \Omega)$, which in turn is bpw a.e. dense in $L^\infty((-\infty, 0) \times \Omega)$. If instead $Dom(\mathcal{L})$ is dense in $C_\infty(\overline{\Omega})$, then the same strategy holds by showing that the functions satisfying (H2) are uniformly dense in $C_\infty((-\infty, 0) \times \overline{\Omega})$.

For the next two lemmas the domain Ω and the semigroup $e^{\mathcal{L}}$ only need to be as in Definition 2.1.

Lemma 4.5. *Assume (H0) and (H2). Let $g = f + f_\phi$, for $f \in L^\infty((0,T) \times \Omega)$, and*

$$f_\phi(t,x) := \int_t^\infty \left(e^{r\mathcal{H}}\phi(t,x) - e^{r\mathcal{L}}\phi(0,x) \right) \nu(r)\, dr, \quad (t,x) \in (0,T] \times \Omega. \qquad (4.2)$$

Then $f_\phi \in C_\infty([0,T] \times \Omega)$, and (3.2) for $g \equiv f + f_\phi$, $\phi_0 \equiv \phi(0)$, equals (1.3) for f, ϕ.

Proof. Extend ϕ to $\phi(0)$ on $(0,T]$. Observe that for $t > 0$

$$\begin{aligned}
f_\phi(t,x) &= \int_0^\infty \left(e^{r\mathcal{H}}\phi(t,x) - e^{r\mathcal{L}}\phi(0,x) \right) \nu(r)\, dr \\
&= \int_0^\infty \left(e^{r\mathcal{H}}\phi(t,x) - \phi(0,x) \right) \nu(r)\, dr + \int_0^\infty \left(\phi(0,x) - e^{r\mathcal{L}}\phi(0,x) \right) \nu(r)\, dr \\
&= H^\nu \phi(t,x) - \mathcal{L}^\nu \phi(0,x),
\end{aligned}$$

where $H^\nu \phi \in C_\infty([0,T] \times \Omega)$ by (H2$'$) and Theorem 2.9-(i), and $\mathcal{L}^\nu \phi$ is a linear combination of elements in $C_\infty(\Omega)$ by (H2) and $\mathrm{Dom}(\mathcal{L}) \subset \mathrm{Dom}(\mathcal{L}^\nu)$. Rearranging, we also proved that for $t > 0$

$$f_\phi + \mathcal{L}^\nu \phi = H^\nu \phi = \mathcal{H}^\nu \phi. \qquad (4.3)$$

The same argument at the end of the proof of Lemma 3.3 allows to apply Dynkin formula [18, Corollary of Theorem 5.1] to obtain for each $t > 0$ and $x \in \Omega$

$$\mathbf{E}\left[\phi\left(0, B^x_{S^\nu_{\tau_0(t)}}\right) \right] - \phi(t,x) = \mathbf{E}\left[\int_0^{\tau_0(t)} \mathcal{L}^\nu \phi\left(t - S^\nu_r, B^x_{S^\nu_r}\right) dr \right], \qquad (4.4)$$

where we used $\phi(t) = \phi(0)$ on $(0,T]$ and $\phi(0) \in \mathrm{Dom}(\mathcal{L}) \subset \mathrm{Dom}(\mathcal{L}^\nu)$. We conclude by justifying the following equalities for each $t > 0$ and $x \in \Omega$,

$$\begin{aligned}
\mathbf{E}\left[\phi\left(t - S^\nu_{\tau_0(t)}, B^x_{S^\nu_{\tau_0(t)}}\right) \right] &= \mathbf{E}\left[\int_0^{\tau_0(t)} \mathcal{H}^\nu \phi\left(t - S^\nu_r, B^x_{S^\nu_r}\right) dr \right] + \phi(t,x) \\
&= \mathbf{E}\left[\int_0^{\tau_0(t)} (f_\phi + \mathcal{L}^\nu \phi)\left(t - S^\nu_r, B^x_{S^\nu_r}\right) dr \right] + \phi(t,x) \\
&= \mathbf{E}\left[\int_0^{\tau_0(t)} f_\phi\left(t - S^\nu_r, B^x_{S^\nu_r}\right) dr \right] + \phi(t,x) \\
&\quad + \mathbf{E}\left[\phi\left(0, B^x_{S^\nu_{\tau_0(t)}}\right) \right] - \phi(t,x).
\end{aligned}$$

The first equality holds by Dynkin formula [18, Corollary of Theorem 5.1] combining Theorem 2.9-(i) and (H2$'$); the second equality holds by (4.3); the third equality holds by (4.4). $\qquad \square$

The following simple lemma shows how to recover the operator H^ν for an initial condition ϕ from H_0^ν plus the correct forcing term.

Lemma 4.6. *Assume (H0). Let $u \in \mathrm{Dom}(\mathcal{H}_0)$ and denote by $\tilde{u}$ its extension to $\phi \in L^\infty((-\infty, 0) \times \Omega)$ for $t < 0$ and write $u(0) = \phi(0)$. Then $H^\nu \tilde{u} = H_0^\nu u + f_\phi$ for $t > 0$, where f_ϕ is defined as in (4.2).*

Proof. Recalling (2.6), simply compute for $t > 0$, $x \in \Omega$

$$
\begin{aligned}
H^\nu \tilde{u}(t,x) &= \int_0^\infty \left(e^{s\mathcal{H}} \tilde{u}(t,x) - \tilde{u}(t,x) \right) \nu(r) \, dr \\
&= \int_0^t \left(e^{r\mathcal{H}} u(t,x) - u(t,x) \right) \nu(r) \, dr + \int_t^\infty \left(e^{r\mathcal{H}} \phi(t,x) - u(t,x) \right) \nu(r) \, dr \\
&= \int_0^t \left(e^{r\mathcal{H}} u(t,x) - u(t,x) \right) \nu(r) \, dr + \int_t^\infty \left(e^{r\mathcal{L}} u(0,x) - u(t,x) \right) \nu(r) \, dr \\
&\quad + \int_t^\infty \left(e^{r\mathcal{H}} \phi(t,x) - e^{r\mathcal{L}} \phi(0,x) \right) \nu(r) \, dr \\
&= H_0^\nu u(t,x) + f_\phi(t,x). \qquad \qquad \square
\end{aligned}
$$

We are now ready to define our weak solution for problem (1.1) and to show that the stochastic representation (1.3) is indeed a weak solution.

Definition 4.7. *For given $f \in L^\infty((0,T) \times \Omega)$ and $\phi \in L^\infty((-\infty,0) \times \Omega)$, a function u is said to be a* weak solution *to (1.1) if $u \in L^\infty((-\infty,T) \times \Omega)$ and*

$$
\begin{cases}
\langle u, H^{\nu,*}\varphi \rangle = \langle -f, \varphi \rangle, & \text{for } \varphi \in C_c^1(0,T) \cdot Dom(\mathcal{L}), \\
u = \phi, & \text{a.e. on } (-\infty,0) \times \Omega.
\end{cases}
\tag{4.5}
$$

Theorem 4.8. *Assume (H0) and (H1'), and let $f \in L^\infty((0,T) \times \Omega)$ and $\phi \in L^\infty((-\infty,0) \times \Omega)$. Then the function defined in (1.3) is a weak solution to (1.1).*

Proof. We assume that $e^{\mathcal{L}}$ acts on $C_\infty(\Omega)$ (the proof for $e^{\mathcal{L}}$ acting on $C_\infty(\overline{\Omega})$ is essentially identical[1], and we omit it). Note that in each step we may redefine the notation u, $\tilde{u}$, u_n, $\tilde{u}_n$, f, f_n, ϕ and ϕ_n. Also, we assume in the first two steps that ϕ satisfies (H2).

Step 1) Let $u \in \text{Dom}(\mathcal{H}_0^\nu)$ be the unique solution in the domain of the generator to problem (3.1) for $g \equiv f + f_\phi$ and $\phi_0 \equiv \phi(0)$, where $f_\phi \in C_\infty([0,T] \times \Omega)$ by Lemma 4.5, and some $f \in C_\infty([0,T] \times \Omega)$ such that $f(0) = -f_\phi(0) - \mathcal{L}^\nu \phi(0)$. This implies that for any $\varphi \in C_c^1(0,T) \cdot \text{Dom}(\mathcal{L})$

$$
\langle \mathcal{H}_0^\nu u + f_\phi, \varphi \rangle = \langle -f, \varphi \rangle.
\tag{4.6}
$$

By Theorem 2.9-(iv) we are guaranteed that $u - \phi(0) \in \text{Dom}(\mathcal{H}_0^{\nu,\text{kill}})$. Then, by Theorem 2.9-(iii), we can pick $\{\hat{u}_n\}_{n \geq 1} \subset \text{Dom}(\mathcal{H}_0^{\text{kill}})$ such that $\hat{u}_n \to u - \phi(0)$ and

$$
H_0^\nu \hat{u}_n = \mathcal{H}_0^\nu \hat{u}_n = \mathcal{H}_0^{\nu,\text{kill}} \hat{u}_n \to \mathcal{H}_0^{\nu,\text{kill}}(u - \phi(0)) = \mathcal{H}_0^\nu(u - \phi(0)),
$$

with both convergences in $C_\infty([0,T] \times \Omega)$ as $n \to \infty$. Then, $u_n := \hat{u}_n + \phi(0) \to u$ with $u_n(0) = \phi(0)$ for all n, and

$$
H_0^\nu u_n = \mathcal{H}_0^\nu \hat{u}_n + \mathcal{H}_0^\nu \phi(0) \to \mathcal{H}_0^\nu(u - \phi(0)) + \mathcal{H}_0^\nu \phi(0) = \mathcal{H}_0^\nu u,
\tag{4.7}
$$

with both convergences in $C_\infty([0,T] \times \Omega)$ as $n \to \infty$, where we used Lemma 2.13 and the linearity of $\mathcal{H}_0^\nu$. Define the extension of u as

$$
\tilde{u} := \begin{cases} u, & t > 0, \\ \phi, & t \leq 0. \end{cases}
\tag{4.8}
$$

Then, for every $\varphi \in C_c^1(0,T) \cdot \text{Dom}(\mathcal{L})$, we can apply DCT as $n \to \infty$ to obtain

$$
\langle -f, \varphi \rangle \leftarrow \langle H_0^\nu u_n + f_\phi, \varphi \rangle = \langle H^\nu \tilde{u}_n, \varphi \rangle = \langle \tilde{u}_n, H^{\nu,*}\varphi \rangle \to \langle \tilde{u}, H^{\nu,*}\varphi \rangle,
$$

---

[1] The only differences are in Step 1, where the Banach space for $\mathcal{H}_0^\nu$ is $C_\infty([0,T] \times \overline{\Omega})$, and in Step 2, where the sequence $\{f_n\}_{n \in \mathbb{N}}$ will have to be selected from $C_\infty([0,T] \times \overline{\Omega})$.

where we use (4.7) and (4.6) in the first convergence, Lemma 4.6 with $u_n \in \text{Dom}(\mathcal{H}_0)$ in the first equality, Proposition 4.3 in the second equality with (H2) and $u_n \in \text{Dom}(\mathcal{H}_0)$, and Lemma 4.2 with $\tilde{u}_n \to \tilde{u}$ uniformly on $(-\infty, T] \times \Omega$ for the second convergence, where $\tilde{u}_n, \tilde{u}$ are respectively the extensions of u_n, u to ϕ as defined in (4.8).

Step 2) For $f \in L^\infty((0, T) \times \Omega)$, let u be the generalised solution to problem (3.1) for $g \equiv f + f_\phi$ and $\phi_0 \equiv \phi(0)$. Now pick a sequence $\{f_n\}_{n \geq 1} \subset C_\infty([0, T] \times \Omega)$ such that $f_n \to f$ bpw a.e., and $f_n(0) = -f_\phi(0) - \mathcal{L}^\nu \phi(0)$ for each $n \in \mathbb{N}$. Then the respective solutions in the domain of the generator u_n converge bpw to u, by Remark 3.6-(i). And so for every $\varphi \in C_c^1(0, T) \cdot \text{Dom}(\mathcal{L})$

$$\langle -f, \varphi \rangle \leftarrow \langle -f_n, \varphi \rangle = \langle \tilde{u}_n, H^{\nu, *}\varphi \rangle \to \langle \tilde{u}, H^{\nu, *}\varphi \rangle,$$

where we can apply DCT in the second convergence thanks to Lemma 4.2, and the equality holds by Step 1, where again the functions are extended to ϕ as in (4.8).

Step 3) Let $\phi \in L^\infty((-\infty, 0) \times \Omega)$ and $f \in L^\infty((0, T) \times \Omega)$ and denote by u the function defined in (1.3) for such ϕ and f and $t > 0$, and denote by $\tilde{u}$ the extension of u to ϕ for $t < 0$. By Remark 4.4 we can take $\phi_n \to \phi$ bpw a.e., and ϕ_n satisfies (H2) for each $n \in \mathbb{N}$. Denote by $\tilde{u}_n$ the extension of u_n to ϕ_n as in (4.8), where u_n is the generalised solution to problem (3.1) for $g \equiv f + f_{\phi_n}$ and $\phi_0 \equiv \phi_n(0)$. Then, by Lemma 4.5 combined with the representation (3.2) of each u_n, we can apply Corollary 3.9 to obtain as $n \to \infty$

$$\tilde{u}_n \to \tilde{u} \quad \text{bpw a.e. on } (-\infty, T] \times \Omega.$$

Then, for every $\varphi \in C_c^1(0, T) \cdot \text{Dom}(\mathcal{L})$,

$$\langle -f, \varphi \rangle = \langle \tilde{u}_n, H^{\nu, *}\varphi \rangle \to \langle \tilde{u}, H^{\nu, *}\varphi \rangle, \quad \text{as } n \to \infty,$$

where we use Step 2 for the equality and we use Lemma 4.2 to apply DCT, and we are done. $\square$

References

[1] Allen, M.: Uniqueness for weak solutions of parabolic equations with a fractional time derivative. New developments in the analysis of nonlocal operators, Contemp. Math., vol. 723, Amer. Math. Soc., Providence, RI, 2019, pp. 137–148. MR-3916701

[2] Baeumer, B., Kovács, M., Meerschaert, M. M., Schilling, R. and Straka, P.: Reflected spectrally negative stable processes and their governing equations. *Trans. Amer. Math. Soc.* **368** (2016), no. 1, 227–248. MR-3413862

[3] Baeumer, B., Meerschaert, M. M. and Mortensen, J.: Space-time fractional derivative operators. *Proc. Amer. Math. Soc.* **133** (2005), no. 8, 2273–2282. MR-2138870

[4] Baeumer, B. and Straka, P.: Fokker-Planck and Kolmogorov backward equations for continuous time random walk scaling limits. *Proc. Amer. Math. Soc.* **145** (2017), no. 1, 399–412. MR-3565391

[5] Becker-Kern, P., Meerschaert, M. M. and Scheffler, H.-P.: Limit theorems for coupled continuous time random walks. *Ann. Probab.* **32** (2004), no. 1B, 730–756. MR-2039941

[6] Bertoin, J.: Lévy processes. Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR-1406564

[7] Bhattacharya, R. N. and Waymire, E. C.: Stochastic processes with applications. Classics in Applied Mathematics, vol. 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009. MR-3396216

[8] Bogdan, K., Byczkowski, T., Kulczycki, T., Ryznar, M., Song, R. and Vondraček, Z.: Potential analysis of stable processes and its extensions. Lecture Notes in Mathematics, vol. 1980, Springer-Verlag, Berlin, 2009. MR-2569321

[9] Bonforte, M. and Vázquez, J. L.: Fractional nonlinear degenerate diffusion equations on bounded domains part I. Existence, uniqueness and upper bounds. *Nonlinear Anal.* **131** (2016), 363–398. MR-3427986

[10] Böttcher, B., Schilling, R. and Wang, J.: Lévy matters III. Lecture Notes in Mathematics, vol. 2099, Springer, Cham, 2013. MR-3156646

[11] Chen, Z.-Q. and Fukushima, M.: Symmetric Markov processes, time change, and boundary theory. London Mathematical Society Monographs Series, vol. 35, Princeton University Press, Princeton, NJ, 2012. MR-2849840

[12] Chen, Z.-Q., Kim, P. and Song, R.: Two-sided heat kernel estimates for censored stable-like processes. *Probab. Theory Related Fields* **146** (2010), no. 3-4, 361–399. MR-2574732

[13] Chen, Z.-Q., Meerschaert, M. M. and Nane, E.: Space-time fractional diffusion on bounded domains. *J. Math. Anal. Appl.* **393** (2012), no. 2, 479–488. MR-2921690

[14] Chung, K. L.: Doubly-Feller process with multiplicative functional. Seminar on stochastic processes, 1985 (Gainesville, Fla., 1985), Progr. Probab. Statist., vol. 12, Birkhäuser Boston, Boston, MA, 1986, pp. 63–78. MR-896735

[15] Davies, E. B.: Heat kernels and spectral theory. Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989. MR-990239

[16] Diethelm, K.: The analysis of fractional differential equations. Lecture Notes in Mathematics, vol. 2004, Springer-Verlag, Berlin, 2010. MR-2680847

[17] Du, Q., Toniazzi, L. and Zhou, Z.: Stochastic representation of solution to nonlocal-in-time diffusion. *Stochastic Process. Appl.* **130** (2020), no. 4, 2058–2085. MR-4074694

[18] Dynkin, E. B.: Markov processes. Vols. I, II. Die Grundlehren der Mathematischen Wissenschaften, Bände 121, vol. 122, Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965. MR-0193671

[19] Froemberg, D., Schmiedeberg, M., Barkai, E. and Zaburdaev, V.: Asymptotic densities of ballistic Lévy walks. *Phys. Rev. E* (3) **91** (2015), no. 2, 022131, 9. MR-3418589

[20] Grzywny, T. and Szczypkowski, K.: Heat kernels of non-symmetric Lévy-type operators. *J. Differential Equations* **267** (2019), no. 10, 6004–6064. MR-3996792

[21] Gyrya, P. and Saloff-Coste, L.: Neumann and Dirichlet heat kernels in inner uniform domains. *Astérisque* (2011), no. 336, viii+144. MR-2807275

[22] Hawkes, J.: Potential theory of Lévy processes. *Proc. London Math. Soc.* (3) **38** (1979), no. 2, 335–352. MR-531166

[23] Jacob, N.: Pseudo differential operators and Markov processes. Vol. I. Imperial College Press, London, 2001. MR-1873235

[24] Jurlewicz, A., Kern, P., Meerschaert, M. M. and Scheffler, H.-P.: Fractional governing equations for coupled random walks. *Comput. Math. Appl.* **64** (2012), no. 10, 3021–3036. MR-2989331

[25] Jurlewicz, A., Meerschaert, M. M. and Scheffler, H.-P.: Cluster continuous time random walks. *Studia Math.* **205** (2011), no. 1, 13–30. MR-2822501

[26] Jurlewicz, A., Wyłomańska, A. and Żebrowski, P.: Coupled continuous-time random walk approach to the Rachev–Rüschendorf model for financial data. *Phys. A* **388** (2009), no. 4, 407–418.

[27] Kühn, F.: Lévy matters VI. Lecture Notes in Mathematics, vol. 2187, Springer, Cham, 2017. MR-3701414

[28] Kyprianou, A. E., Osojnik, A. and Shardlow, T.: Unbiased 'walk-on-spheres' Monte Carlo methods for the fractional Laplacian. *IMA J. Numer. Anal.* **38** (2018), no. 3, 1550–1578. MR-3829169

[29] Liao, M.: The Dirichlet problem of a discontinuous Markov process. *Acta Math. Sinica* (N.S.) **5** (1989), no. 1, 9–15. MR-998383

[30] Magdziarz, M. and Teuerle, M.: Asymptotic properties and numerical simulation of multidimensional Lévy walks. *Commun. Nonlinear Sci. Numer. Simul.* **20** (2015), no. 2, 489–505. MR-3251510

[31] Magdziarz, M. and Zorawik, T.: Densities of Lévy walks and the corresponding fractional equations. arXiv:1504.05835 (2015).

[32] Meerschaert, M. M. and Scheffler, H.-P.: Triangular array limits for continuous time random walks. *Stochastic Process. Appl.* **118** (2008), no. 9, 1606–1633. MR-2442372

[33] Meerschaert, M. M. and Sikorskii, A.: Stochastic models for fractional calculus. De Gruyter Studies in Mathematics, vol. 43, Walter de Gruyter & Co., Berlin, 2012. MR-2884383

[34] Sato, K.-I.: Lévy processes and infinitely divisible distributions. Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 2013. MR-3185174

[35] Stinga, P. R. and Torrea, J. L.: Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation. *SIAM J. Math. Anal.* **49** (2017), no. 5, 3893–3924. MR-3709888

[36] Straka, P. and Henry, B. I.: Lagging and leading coupled continuous time random walks, renewal times and their joint limits. *Stochastic Process. Appl.* **121** (2011), no. 2, 324–336. MR-2746178

[37] Stroock, D. W.: Diffusion semigroups corresponding to uniformly elliptic divergence form operators. Séminaire de Probabilités, XXII, Lecture Notes in Math., vol. 1321, Springer, Berlin, 1988, pp. 316–347. MR-960535

[38] Toniazzi, L.: Stochastic classical solutions for space-time fractional evolution equations on a bounded domain. *J. Math. Anal. Appl.* **469** (2019), no. 2, 594–622. MR-3860441

[39] Weron, K., Jurlewicz, A. and Magdziarz, M.: Havriliak-Negami response in the framework of the continuous-time random walk. *Acta Phys. Pol. B* **36** (2005), no. 5, 1855–1868.

[40] Weron, K., Jurlewicz, A., Magdziarz, M., Weron, A. and Trzmiel, J.: Overshooting and undershooting subordination scenario for fractional two-power-law relaxation responses. *Physical Review E* **81** (2010), no. 4, 041123.

[41] Weron, K., Stanislavsky, A., Jurlewicz, A., Meerschaert, M. M. and Scheffler, H.-P.: Clustered continuous-time random walks: diffusion and relaxation consequences. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **468** (2012), no. 2142, 1615–1628. MR-2923838

[42] Zaburdaev, V., Denisov, S. and Klafter, J.: Lévy walks. *Rev. Modern Phys.* **87** (2015), no. 2, 483–530. MR-3403266

[43] Zaburdaev, V.: Random walk model with waiting times depending on the preceding jump length. *J. Stat. Phys.* **123** (2006), no. 4, 871–881. MR-2255084

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards

- Free for authors, free for readers

- Quick publication (no backlog)

- Secure publication (LOCKSS[1])

- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4] , ProjectEuclid[5]

- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)

- Submit your best articles to EJP-ECP

- Choose EJP-ECP over for-profit journals

---

[1]LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2]EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
[3]IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4]BS: Bernoulli Society http://www.bernoulli-society.org/
[5]Project Euclid: https://projecteuclid.org/
[6]IMS Open Access Fund: http://www.imstat.org/publications/open.htm