Representations of the Vertex Reinforced Jump Process as a mixture of Markov processes on $\mathbb{Z}^d$ and infinite trees

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Abstract

This paper concerns the Vertex Reinforced Jump Process (VRJP) and its representations as a Markov process in random environment. In [21], it was shown that the VRJP on finite graphs, under a certain time rescaling, has the distribution of a mixture of Markov jump processes. This representation was extended to infinite graphs in [23], by introducing a random potential β. In this paper, we show that all possible representations of the VRJP as a mixture of Markov processes can be expressed in a similar form as in [23], using the random field β and harmonic functions for an associated operator H_β. This allows to show that the VRJP on $\mathbb{Z}^d$ (with certain initial conditions) has a unique representation, by proving that an associated Martin boundary is trivial. Moreover, on infinite trees, we construct a family of representations, that are all different when the VRJP is transient and the tree is d-regular (with $d \geq 3$).

1 Introduction

This paper concerns the Vertex Reinforced Jump Process (or VRJP) on infinite graphs and its representations as a Markov process in a random environment. In particular, we are interested in knowing if the VRJP admits several different representations, and what form they can take.

Let $\mathcal{G} = (V, E)$ be a non-directed locally finite graph, *i.e.* each vertex $i \in V$ has finite degree. For $i, j \in V$, we write $i \sim j$ if i and j are neighbors, *i.e.* if $\{i, j\} \in E$. We endow $\mathcal{G}$ with positive conductances $(W_e)_{e \in E}$, and denote $W_{i,j} = \mathbb{1}_{\{i,j\} \in E} W_{\{i,j\}}$. The VRJP on $\mathcal{G}$, with respect to W, is the self-interacting random process $(Y_s)_{s \in \mathbb{R}_+}$ on V defined as

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follows: the process starts at some vertex $i_0 \in V$ at time 0, and conditionally on the past at time s, jumps to a neighbor j of $i = Y_s$ at rate

$$W_{i,j}L_j(s), \text{ where } L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u=j\}}du.$$

In other words, as the local time $\int_0^s \mathbb{1}_{\{Y_u=i\}}du$ spent by the process at i increases, the vertex i becomes more attractive. This process was introduced in [8].

In [21], Sabot and Tarrès introduced a time change for the VRJP, by defining the increasing function $D(s) = \sum_{i \in V}(L_i(s)^2 - 1)$, and taking $(Z_t)_{t \geq 0} = (Y_{D^{-1}(t)})_{t \geq 0}$. On finite graphs, this time-changed VRJP Z started at a vertex i_0 is then a mixture of Markov processes, in the following sense: there exists a random field $(u_{i_0}(i))_{i \in V}$, whose distribution is explicit, such that the law of Z is the same as that of a Markov process in a random environment given by jump rates

$$\frac{1}{2}W_{i,j}e^{u_{i_0}(j)-u_{i_0}(i)}$$

from i to j. The idea behind this time change is that the VRJP $(Y_s)_{s \geq 0}$ jumps faster and faster as the vertices become more attractive, and that the time change D is such that $(Z_t)_{t \geq 0}$ has more stationary jumping times, which is necessary for it to be a mixture of Markov processes.

They also showed that the VRJP was related to another self-interacting process, the Edge Reinforced Random Walk (or ERRW), introduced in [6] by Coppersmith and Diaconis. On finite graphs, thanks to a de Finetti type theorem for Markov chains (see [10]), it can be seen as a mixture of Markov chains. This interpretation of the ERRW as a mixture of random walks was studied in [16], [17], [13], [14], [3]. The link between VRJP and ERRW proven in [21] gives an explicit representation of the ERRW as a mixture of random walks on finite graphs.

Finally, the explicit distribution of the random field $(u_{i_0}(i))_{i \in V}$ is related to a statistical mechanics model: the supersymmetric hyperbolic sigma model. It was studied by Disertori, Spencer and Zirnbauer in [11] and [12], in which they showed localization and delocalization theorems for the field u_{i_0}. This provided results on the recurrence and transience of the VRJP and ERRW on lattices $\mathbb{Z}^d$.

In [20], Sabot, Tarrès and Zeng showed that the distributions of fields u_{i_0} can be coupled for $i_0 \in V$, using a potential $\beta = (\beta_i)_{i \in V}$ on V, and a random Schrödinger operator associated with β. Let us denote by $H_\beta = 2\beta - W$ the random Schrödinger operator, i.e. the $|V| \times |V|$ symmetrical matrix such that $(H_\beta)_{i,j} = 2\beta_i \mathbb{1}_{i=j} - W_{i,j}$ for $i, j \in V$. Moreover, we define by $G = (H_\beta)^{-1}$ the associated Green function. Then u_{i_0} can be defined by

$$e^{u_{i_0}(i)} = \frac{G(i_0, i)}{G(i_0, i_0)}$$

for $i_0, i \in V$.

This representation using the β field allows a generalization to infinite graphs: in [23], Sabot and Zeng used a similar potential β on infinite graphs to show that the VRJP is still a mixture of Markov processes. If we still denote by $H_\beta = 2\beta - W$ the operator associated with β, we can define the Green function $\hat{G} = (H_\beta)^{-1}$ in a certain sense. Moreover, there exists ψ, a H_β-harmonic function on V (i.e. $H_\beta\psi = 0$), obtained as the limit of a martingale. Then if we define $G(i, j) = \hat{G}(i, j) + \frac{1}{2\gamma}\psi(i)\psi(j)$, where γ is a random Gamma variable independent from β, the time-changed VRJP (Z_t) is still a mixture of Markov processes, with jump rates from i to j given by

$$\frac{1}{2}W_{i,j}\frac{G(i_0, j)}{G(i_0, i)}$$

The term $\frac{1}{2\gamma}$ corresponds to a boundary term. Indeed, to show the result for infinite graphs, the VRJP is first studied on finite subgraphs, endowed with a wired boundary condition. This representation also gave results for the ERRW on infinite graphs. In [18], Poudevigne used a coupling of potentials β on graphs with different weights W to show monotonicity results, which gave the existence of a phase transition between recurrence and transience of the VRJP.

In the case of infinite trees, there is another representation of (Z_t) as a mixture of Markov processes. This representation is obtained by using free boundary conditions on restrictions of the tree, since the representation of the VRJP on finite trees has a simpler expression. The particular structure of the tree already gave results for the ERRW (see [16]) and the VRJP (see [9], [4]). We show that in some cases, the representation of the VRJP obtained this way on the tree differs from the one defined in [23]. This raises the question of the classification of all possible representations of the VRJP as a mixture of Markov processes. This issue is related to the behavior of the VRJP at infinity, which was also studied by Merkl, Rolles and Tarrès in [15], using the point of view of random interlacements.

In this paper, we give several partial answers to the question of the classification of representations of the VRJP. We first show that any such representation can be expressed in the same form as before, using a β field, *i.e.* the random jump rates are given by

$$\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)},$$

where $G(i_0,i) = \hat{G}(i_0,i) + h(i)$, with h a random H_β-harmonic function.

In the case where the graph is the lattice $\mathbb{Z}^d$, this allows us to show that for certain initial conductances W, there is only one representation of the VRJP as a mixture of Markov processes. This is true for strong reinforcement (*i.e.* small W), since the VRJP is recurrent, but also for weak reinforcement (*i.e.* large W). In this last case, we use a local limit theorem for random walks in random environment to show that the only H_β-harmonic functions are constants, by proving that the associated Martin boundary is trivial.

In the case where the graph is an infinite tree, we already know of two different representations of the VRJP. Using new boundary conditions, we construct a family of representations, that are all different if the tree is regular enough.

2 Statement of the results

2.1 Previous results

Let $\mathcal{G} = (V, E)$ be a finite connected non-directed graph, endowed with conductances $(W_e)_{e \in E}$. We describe $(W_e)_{e \in E}$ with a matrix $(W_{i,j})_{i,j \in V}$, where

$$W_{i,j} = \begin{cases} W_{\{i,j\}} & \text{if } \{i,j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

In [21], Sabot and Tarrès proved that the time-changed VRJP on $\mathcal{G}$ with respect to W could be represented as a mixture of Markov processes, *i.e.* as a random walk in random environment. With Zeng, they showed in [20] that this environment could be related to a random Schrödinger operator H_β, constructed from a random potential $\beta = (\beta_i)_{i \in V}$, in the following way.

For $\beta \in \mathbb{R}^V$, we will denote by $H_\beta = 2\beta - W$ the $|V| \times |V|$ symmetrical matrix such that $(H_\beta)_{i,j} = 2\beta_i \mathbb{1}_{i=j} - W_{i,j}$ for $i, j \in V$. Let us define the set $\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, H_\beta > 0\}$, where $H_\beta > 0$ means that the matrix H_β is positive definite. Note that if $\beta \in \mathcal{D}_V^W$, then

$\beta_i > 0$ for all $i \in V$. The following proposition describes the probability distribution of the random potential that will be used to represent the VRJP.

Proposition 2.1 (Theorem 1 in [20], Lemma 4 in [23])**.**

(i) *Let $\mathcal{G} = (V, E)$ be a finite connected graph, endowed with conductances W, and let $\eta \in \mathbb{R}_+^V$. We define by $\nu_V^{W,\eta}$ the measure on $(\mathcal{D}_V^W, \mathcal{B}(\mathcal{D}_V^W))$ such that*

$$\nu_V^{W,\eta}(d\beta) = \left(\frac{2}{\pi}\right)^{\frac{|V|}{2}} e^{-\frac{1}{2}(\langle \mathbb{1}, H_\beta \mathbb{1}\rangle + \langle \eta, (H_\beta)^{-1}\eta\rangle)} e^{\langle \eta, \mathbb{1}\rangle} \frac{\prod_{i \in V} d\beta_i}{\sqrt{\det(H_\beta)}}.$$

Then $\nu_V^{W,\eta}$ is a probability distribution. Its Laplace transform is

$$\int e^{-\langle \lambda, \beta\rangle} \nu_V^{W,\eta}(d\beta) = e^{-\sum_{i \in V} \eta_i(\sqrt{1+\lambda_i}-1) - \sum_{i \sim j} W_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1)} \prod_{i \in V} \frac{1}{\sqrt{1+\lambda_i}},$$

for $\lambda \in \mathbb{R}_+^V$. When $\eta = 0$, we will write $\nu_V^W = \nu_V^{W,0}$.

(ii) *Let us denote by $d_\mathcal{G}$ the graph distance in $\mathcal{G}$. Under $\nu_V^{W,\eta}(d\beta)$, if $V_1, V_2 \subset V$ are such that $d_\mathcal{G}(V_1, V_2) \geq 2$, then $(\beta_i)_{i \in V_1}$ and $(\beta_j)_{j \in V_2}$ are independent. We will say that the potential with distribution ν_V^W is 1-dependent.*

Let $\mathcal{C}_r(\mathbb{R}_+, V)$ be the space of right-continuous functions from $\mathbb{R}_+$ to V. This will be the space of trajectories of the random processes we will study in this paper. These processes will be described by probability distributions on $\mathcal{C}_r(\mathbb{R}_+, V)$. Let us denote by (Z_t) the canonical process in $\mathcal{C}_r(\mathbb{R}_+, V)$, where $Z_t(\omega) = \omega(t)$ for $\omega \in \mathcal{C}_r(\mathbb{R}_+, V)$. Moreover for $i_0 \in V$, let $\mathbb{P}^{VRJP(i_0)}$ denote the distribution of the time-changed VRJP on $(\mathcal{G}, W)$, in the exchangeable time scale described in the introduction. The following theorem describes how to represent this process as a mixture of Markov processes, using a random environment that can be constructed from the β field under $\nu_V^W(d\beta)$.

Theorem 2.2 (Theorem 2 in [21], Theorem 3 in [20])**.** *Let $\mathcal{G} = (V, E)$ be a finite graph, endowed with conductances W. We fix a vertex $i_0 \in V$. For $\beta \in \mathcal{D}_V^W$, we denote by $G = (H_\beta)^{-1}$ the Green function associated with β, and by $\mathbb{P}_x^{\beta, i_0}$ the distribution of the Markov jump process started at $x \in V$, with jump rate from i to j given by $\frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$.*

Then for all $i_0 \in V$, the time-changed VRJP on $(\mathcal{G}, W)$, started at i_0, is a mixture of these Markov jump processes under the distribution $\nu_V^W(d\beta)$. In other words,

$$\mathbb{P}^{VRJP(W, i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{\beta, i_0}[\cdot] \nu_V^W(d\beta).$$

An interesting property of the distribution ν_V^W is its behavior with respect to restriction. For $\beta \in \mathbb{R}^V$ and $V_1, V_2 \subset V$, let us denote $\beta_{V_1} = (\beta_i)_{i \in V_1}$, and $W_{V_1, V_2} = (W_{i,j})_{i \in V_1, j \in V_2}$.

Proposition 2.3 (Lemma 4 in [23])**.** *Let us fix $U \subset V$, and set $\hat{\eta}_i = \sum_{j \in U^c} W_{i,j}$ for $i \in U$, i.e. $\hat{\eta} = W_{U, U^c} \mathbb{1}_{U^c}$. Then under $\nu_V^W(d\beta)$, β_U is distributed according to $\nu_U^{W_{U,U}, \hat{\eta}}$.*

Hence under $\nu_V^W(d\beta)$, the distribution of β_U depends only on the weights of edges inside U, and coming out of U. This is useful to define the β field on infinite graphs.

Let now $\mathcal{G} = (V, E)$ be an infinite connected non-directed graph, that is locally finite, i.e. each vertex $v \in V$ has finite degree. We endow $\mathcal{G}$ with conductances W. To study the associated VRJP, we want to define an analogue of the β field on $\mathcal{G}$. In [23], Sabot and Zeng did this by using a *wired* boundary condition, defined as follows.

Let $(V_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite connected subsets of V, such that

$$\bigcup_{n \in \mathbb{N}} V_n = V.$$

For $n \in \mathbb{N}$, we introduce a new vertex δ_n, and define a new graph $\mathcal{G}^{(n)} = (\tilde{V}^{(n)}, \tilde{E}^{(n)})$, where

$$\tilde{V}^{(n)} = V_n \cup \{\delta_n\}$$
$$\text{and } \tilde{E}^{(n)} = \{\{i,j\} \in E, i, j \in V_n\} \cup \{\{i, \delta_n\}, i \in V_n \text{ and } \exists j \notin V_n, i \sim j\}.$$

The graph $\mathcal{G}^{(n)}$ is called the restriction of $\mathcal{G}$ to V_n with *wired* boundary condition. We endow this graph with the conductances $\tilde{W}^{(n)}$ defined by $\tilde{W}_{i,j}^{(n)} = W_{i,j}$ if $i, j \in V_n$, and $\tilde{W}_{i,\delta_n}^{(n)} = \sum_{j \sim i, j \notin V_n} W_{i,j}$.

For all $n \in \mathbb{N}$, let $(\beta_i^{(n)})_{i \in \tilde{V}^{(n)}}$ be a random potential on the graph $\mathcal{G}^{(n)}$ distributed according to $\nu_{\tilde{V}^{(n)}}^{\tilde{W}^{(n)}}$. Then from Proposition 2.3, we know that the restriction $\beta_{V_n}^{(n)}$ is distributed according to $\nu_{V_n}^{W^{(n)}, \eta^{(n)}}$, where $W^{(n)} = W_{V_n, V_n}$ and $\eta^{(n)} := \tilde{W}_{V_n, \{\delta_n\}}^{(n)} = W_{V_n, V_n^c} \mathbb{1}_{V_n^c}$. In fact, for a fixed $n \in \mathbb{N}$ and any $n' \geq n$, the restrictions $\beta_{V_n}^{(n')}$ have the same distribution $\nu_{V_n}^{W^{(n)}, \eta^{(n)}}$. By Kolmogorov extension theorem, this allows the construction of a distribution ν_V^W for infinite V.

For $\beta \in \mathbb{R}^V$, let us still denote by $H_\beta = 2\beta - W$ the Schrödinger operator associated with $(\beta_i)_{i \in V}$, *i.e.* for all $f \in \mathbb{R}^V$ and $i \in V$, $(H_\beta f)_i = 2\beta_i f_i - \sum_{j \sim i} W_{i,j} f_j$. We also define the set $\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V, (H_\beta)_{U,U} > 0 \text{ for all finite subset } U \text{ of } V\}$, where $(H_\beta)_{U,U} > 0$ means that the symmetrical matrix $(H_\beta)_{U,U}$ is positive definite.

Proposition 2.4 (Proposition 1 in [23]). *Let $\mathcal{G} = (V, E)$ be an infinite locally finite graph, endowed with conductances W. There exists a unique probability distribution ν_V^W on $\mathcal{D}_V^W$ such that under $\nu_V^W(d\beta)$, for all finite subset $U \subset V$, $\beta_U \sim \nu_U^{W_{U,U}, \eta}$ where $\eta = W_{U,U^c} \mathbb{1}_{U^c}$. Its Laplace transform is*

$$\int e^{-\langle \lambda, \beta \rangle} \nu_V^W(d\beta) = e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+\lambda_i}\sqrt{1+\lambda_j} - 1)} \prod_{i \in V} \frac{1}{\sqrt{1+\lambda_i}}$$

for $\lambda \in \mathbb{R}_+^V$ with finite support.

The *wired* boundary condition is not only useful to define ν_V^W on infinite graphs, but also to link this distribution to representations of the VRJP, by applying Theorem 2.2 to the graph $\mathcal{G}^{(n)}$. Indeed from Proposition 2.4, for any $n \in \mathbb{N}$, under $\nu_V^W(d\beta)$ we have $\beta_{V_n} \sim \nu_{V_n}^{W^{(n)}, \eta^{(n)}}$. Hence, from Proposition 2.3, we can extend β_{V_n} into a potential $\beta^{(n)} \sim \nu_{\tilde{V}^{(n)}}^{\tilde{W}^{(n)}}$ such that $\beta_{V_n}^{(n)} = \beta_{V_n}$. We denote $H_\beta^{(n)} = 2\beta^{(n)} - \tilde{W}^{(n)}$ and $G^{(n)} = (H_\beta^{(n)})^{-1}$. From Theorem 2.2, we know that $G^{(n)}$ gives a representation of the VRJP on $\mathcal{G}^{(n)}$.

Definition 2.5.

(i) *For $\beta \in \mathcal{D}_V^W$, let us define $\hat{G}^{(n)} : V \times V \to \mathbb{R}_+$ by $(\hat{G}^{(n)})_{V_n, V_n} = ((H_\beta)_{V_n, V_n})^{-1}$, and $\hat{G}^{(n)}(i,j) = 0$ if $i \notin V_n$ or $j \notin V_n$.*

(ii) *For $\beta \in \mathcal{D}_V^W$, let $\psi^{(n)} \in \mathbb{R}_+^{V_n}$ be defined by*

$$\begin{cases} (H_\beta \psi^{(n)})_{V_n} = 0 \\ \psi_{V_n^c}^{(n)} = 1. \end{cases}$$

Note that $\psi_{V_n}^{(n)} = (\hat{G}_{V_n, V_n}^{(n)}) \eta^{(n)}$.

It is possible, using a decomposition of the Green function as a sum over paths (see [23], or Proposition 3.4), to write

$$G^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + \psi^{(n)}(i) G^{(n)}(\delta_n, \delta_n) \psi^{(n)}(j)$$

for $i, j \in V_n$. Under $\nu_V^W(d\beta)$, $G^{(n)}(\delta_n, \delta_n)$ is independent of β_{V_n}, and $1/2G^{(n)}(\delta_n, \delta_n)$ is always distributed according to a Gamma$(1/2, 1)$ distribution (see Proposition 3.1 (ii)). The following theorem describes how taking $n \to \infty$ in this previous expression gives a representation of the VRJP on infinite graphs.

Theorem 2.6 (Theorem 1 in [23]).

(i) *Under $\nu_V^W(d\beta)$, for $i, j \in V$, the increasing sequence $\hat{G}^{(n)}(i, j)$ converges almost surely to a finite random variable $\hat{G}(i, j)$.*

(ii) *Let $\mathcal{F}_n$ be the σ-field generated by β_{V_n}. Then under $\nu_V^W(d\beta)$, for all $i \in V$, $\psi^{(n)}(i)$ is a nonnegative $(\mathcal{F}_n)$-martingale which converges almost surely to an integrable random variable $\psi(i)$. Moreover, ψ is H_β-harmonic on V, i.e. $H_\beta\psi(i) = 2\beta_i\psi(i) - \sum_{j\sim i} W_{i,j}\psi(j) = 0$ for $i \in V$.*

(iii) *From now on, we will denote $\nu_V^W(d\beta, d\gamma) = \nu_V^W(d\beta) \otimes \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}}e^{-\gamma}d\gamma$, where the density $\frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi\gamma}}e^{-\gamma}d\gamma$ is that of a Gamma$(1/2, 1)$ distribution.*

Let now $i_0 \in V$ be fixed. For $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$, we define

$$G(i, j) = \hat{G}(i, j) + \frac{1}{2\gamma}\psi(i)\psi(j),$$

and denote by $\mathbb{P}_x^{\beta, \gamma, i_0}$ the distribution of the Markov jump process started at $x \in V$, where the jump rate from i to j is $\frac{1}{2}W_{i,j}\frac{G(i_0, j)}{G(i_0, i)}$.

Then the time-changed VRJP on $(\mathcal{G}, W)$, started at i_0, is a mixture of these Markov jump processes under $\nu_V^W(d\beta, d\gamma)$, i.e.

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{\beta, \gamma, i_0}[\cdot]\nu_V^W(d\beta, d\gamma).$$

(iv) *For ν_V^W-almost all β and all $i_0 \in V$, we have:*

- *The Markov process with law $\mathbb{P}_x^{\beta, \gamma, i_0}$ is recurrent if and only if $\psi(i) = 0$ for all $i \in V$.*
- *The Markov process with law $\mathbb{P}_x^{\beta, \gamma, i_0}$ is transient if and only if $\psi(i) > 0$ for all $i \in V$.*

Note that for $i_0 \in V$ fixed, in this representation of the VRJP started at i_0, the β field cannot be expressed as a function of the random jump rates $\frac{W_{i,j}}{2}\frac{G(i_0, j)}{G(i_0, i)}$ that define the environment. However, we can define the $\tilde{\beta}$ field rooted at i_0, where $\tilde{\beta}_i$ is the rate of the exponential holding time at i for the associated Markov process.

Proposition 2.7. *For all $i \in V$, $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$, we define*

$$\tilde{\beta}_i = \sum_{j\sim i}\frac{W_{i,j}}{2}\frac{G(i_0, j)}{G(i_0, i)} = \beta_i - \mathbb{1}_{\{i=i_0\}}\frac{1}{2G(i_0, i_0)}.$$

Then under $\nu_V^W(d\beta, d\gamma)$, $1/2G(i_0, i_0)$ has distribution Gamma$(1/2, 1)$ and is independent from $\tilde{\beta}$. Moreover the Laplace transform of $\tilde{\beta}$ is

$$\int e^{-\langle\lambda, \tilde{\beta}\rangle}\nu_V^W(d\beta, d\gamma) = e^{-\sum_{i\sim j}W_{i,j}(\sqrt{1+\lambda_i}\sqrt{1+\lambda_j}-1)}\prod_{i\neq i_0}\frac{1}{\sqrt{1+\lambda_i}}$$

for $\lambda \in \mathbb{R}_+^V$ with finite support.

2.2 A common form for all representations

We still consider $\mathcal{G} = (V, E)$ to be an infinite connected graph, locally finite and endowed with conductances $(W_{i,j})_{i,j \in V}$. Thanks to Theorem 2.6, we already know that the time-changed VRJP with distribution $\mathbb{P}^{VRJP(i_0)}$ can be written as a mixture of Markov jump processes, using the distribution ν_V^W. We will refer to this as the standard representation. We are now interested in other possible random environments, that would represent the VRJP in the same sense, and whether they can be expressed in a form similar to the standard representation.

We will denote by $\mathcal{J}_V^E$ the set of jump rates on $\mathcal{G}$, i.e. the set of $(r_{i,j})_{i,j \in V} \in \mathbb{R}_+^{V \times V}$ such that $r_{i,j} = 0$ if $\{i, j\} \notin E$.

Definition 2.8. Let $\mathcal{R}(dr)$ be a probability distribution on $\mathcal{J}_V^E$. For $i_0 \in V$ fixed, we will say that $\mathcal{R}(dr)$ is the distribution of a random environment representing the time-changed VRJP started at i_0 if

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^r[\cdot] \mathcal{R}(dr),$$

where for $r \in \mathcal{J}_V^E$, $\mathbb{P}^r$ is the distribution of the Markov jump process with jump rate from i to j given by $r_{i,j}$. We will also say that $\mathcal{R}(dr)$ defines a representation of $\mathbb{P}^{VRJP(i_0)}$.

The following result tells us that in fact, any representation of the VRJP can be expressed in a similar form as the standard representation, using a β field as well as H_β-harmonic functions.

For $i \in V$ and $r \in \mathcal{J}_V^E$, we define $r_i = \sum_{j \sim i} r_{i,j}$.

Theorem 2.9. Let $i_0 \in V$ be fixed, and let $\mathcal{R}(dr)$ be the distribution of a random environment representing the time-changed VRJP with law $\mathbb{P}^{VRJP(i_0)}$. We write $\mathcal{R}(dr, d\gamma) = \mathcal{R}(dr) \otimes \frac{\mathbb{1}_{\{\gamma \geq 0\}}}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma$.

For $r \in \mathcal{J}_V^E$ and $\gamma > 0$, we define $\beta \in (\mathbb{R}_+)^V$ by $\beta_i = r_i + \mathbb{1}_{\{i=i_0\}}\gamma$ for $i \in V$. Then under $\mathcal{R}(dr, d\gamma)$, $\beta \sim \nu_V^W$, and there exists a random H_β-harmonic function $h : V \to \mathbb{R}_+$, such that for all $i \sim j$,

$$r_{i,j} = \frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)},$$

where $G(i_0, i) = \hat{G}(i_0, i) + h(i)$ for $i \in V$, and $\hat{G}$ is the function of β defined in Theorem 2.6.

In order to try and classify all representations of the VRJP, we now need to identify H_β-harmonic functions, and to determine which ones can appear in the expression of a representation, as in Theorem 2.9. Two interesting cases arise, depending on $(\mathcal{G}, W)$: when the VRJP is almost surely recurrent, or almost surely transient.

In the first case, we can use the law of large numbers to show that the representation of the VRJP as a mixture of Markov processes is unique.

Proposition 2.10. If $(\mathcal{G}, W)$ is such that the VRJP is almost surely recurrent, then the representation of the time-changed VRJP started at i_0 as a mixture of Markov processes is unique, i.e. if $\mathcal{R}(dr)$ and $\mathcal{R}'(dr)$ define representations of $\mathbb{P}^{VRJP(i_0)}$, then $\mathcal{R}(dr) = \mathcal{R}'(dr)$.

Note that in this case, according to Theorem 2.6 (iv), under $\nu_V^W(d\beta)$, we have a.s. $\psi(i) = 0$ for all $i \in V$, and the jump rates in the standard representation are given by $\frac{W_{i,j}}{2} \frac{\hat{G}(i_0,j)}{\hat{G}(i_0,i)}$. Therefore, the H_β-harmonic function associated with the unique representation (by Theorem 2.9) is $h \equiv 0$.

In the second case, i.e. when the VRJP is almost surely transient, we can introduce a random conductance model, associated with ψ.

Proposition 2.11. If $(\mathcal{G}, W)$ is such that the VRJP is almost surely transient, then under $\nu_V^W(d\beta)$:

(i) We have a.s. $\psi(i) > 0$ for all $i \in V$, where ψ is defined in Theorem 2.6.

(ii) *We define the random conductances $c_{i,j}^{\psi} = W_{i,j}\psi(i)\psi(j)$ for all $i, j \in V$. Then the associated reversible random walk is a.s. transient.*

(iii) *Let Δ^{ψ} be the discrete Laplacian associated with the random conductances $c_{i,j}^{\psi}$. Then a function $\varphi : V \to \mathbb{R}$ is Δ^{ψ}-harmonic if and only if $i \mapsto \psi(i)\varphi(i)$ is H_{β}-harmonic.*

Remark 2.12. The introduction of the operator Δ^{ψ} allows a more convenient expression of representations in the transient case. Indeed, if $\mathcal{R}(dr)$ defines a representation of $\mathbb{P}^{VRJP(i_0)}$, Theorem 2.9 allows us to construct a β field distributed according to ν_V^W, and to express the jump rates $r_{i,j}$ using β and a H_{β}-harmonic function h. According to Proposition 2.11 (iii), we have $h = \psi\varphi$, where φ is a Δ^{ψ}-harmonic function, *i.e.* harmonic for a transient random walk. As a result, φ can be expressed using the Martin boundary associated with Δ^{ψ}, as described below.

The notion of Martin boundary is a useful tool to represent harmonic functions with respect to a transient random walk on a graph $\mathcal{G} = (V, E)$. Indeed, V admits a boundary $\mathcal{M}$ so that $V \cup \mathcal{M}$ is compact for a certain topology, and there is a kernel $K : V \times \mathcal{M}$ so that any positive harmonic function h can be written as

$$h(x) = \int_{\mathcal{M}} K(x, \alpha)\mu^h(d\alpha)$$

for $x \in V$, where μ^h is a positive measure on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$. $\mathcal{M}$ is called the Martin boundary of V with respect to the random walk, and K is the Martin kernel, which is defined using the Green kernel associated with the random walk. For more details on Martin boundaries, see Section 3.3.

In order to study representations of the VRJP in the transient case, we want to describe Δ^{ψ}-harmonic functions, according to Remark 2.12. We will therefore need to identify the Martin boundary $\mathcal{M}^{\psi}$ associated with Δ^{ψ}. This will be possible when $\mathcal{G}$ is $\mathbb{Z}^d$, or an infinite tree.

2.3 Main results

2.3.1 Representations of the VRJP on $\mathbb{Z}^d$

Let us consider the case where $\mathcal{G}$ is the lattice $\mathbb{Z}^d$, *i.e.* $\mathcal{G} = (V, E)$ with

$$V = \mathbb{Z}^d \text{ and } E = E_d := \{\{x, y\}, |x - y| = 1\}$$

where $|x|$ is the Euclidean norm of x. Let us endow $\mathcal{G}$ with constant initial conductances W. Since this model is invariant by isometries of $\mathbb{Z}^d$, we will only consider the VRJP started at 0.

We can identify several situations in which the representation is unique. For $d = 2$, or if W is small enough, the VRJP is almost surely recurrent (see [19], and Corollary 1 in [21]), so that the representation of $\mathbb{P}^{VRJP(0)}$ is unique according to Proposition 2.10. For $d \geq 3$ and W large enough, the VRJP is almost surely transient (see Corollary 3 in [21]), hence we can introduce the operator Δ^{ψ} defined in Proposition 2.11. Since $(\mathcal{G}, W)$ is vertex transitive, from Proposition 3 of [23], under $\nu_V^W(d\beta)$, ψ is stationary and ergodic. This allows us to apply a local limit theorem for random walks in random conductances (from [1]), and show that the Martin boundary $\mathcal{M}^{\psi}$ associated with Δ^{ψ} is almost surely trivial for W large enough. These cases are regrouped in the following result.

Theorem 2.13. *Let $\mathcal{G}$ be the $\mathbb{Z}^d$ lattice, endowed with constant edge weights, i.e. $W_{i,j} = W > 0$ for all $i \sim j$. We consider representations of $\mathbb{P}^{VRJP(0)}$ as a mixture of Markov processes.*
Then:

- *If $d \in \{1, 2\}$, there is a unique representation of $\mathbb{P}^{VRJP(0)}$.*
- *If $d \geq 3$, there are constants $0 < \underline{W} < \overline{W}$ such that for $0 < W < \underline{W}$ or for $W > \overline{W}$, there is a unique representation of $\mathbb{P}^{VRJP(0)}$.*

2.3.2 A family of representations on infinite trees

Let us now consider the case where the graph is an infinite tree $\mathcal{T} = (T, E)$, that we assume to be locally finite, and endow with conductances W. In [5], Chen and Zeng described a representation of the time-changed VRJP with a different expression than the standard representation. Indeed, if $(T_n)_{n \in \mathbb{N}}$ is an increasing and exhausting sequence of finite connected subsets of T, the subgraphs $\mathcal{T}^{(n)} = (V_n, E_n)$ of $\mathcal{G}$ are finite trees (where $E_n = \{\{i, j\} \in E, i, j \in V_n\}$). These are called restrictions of $\mathcal{G}$ with *free* boundary conditions.

Moreover, on finite trees, Theorem 2.2 gives a representation of the VRJP where jump rates are independent. Therefore, a representation of the VRJP on $\mathcal{T}$ can be obtained from representations on $\mathcal{T}^{(n)}$, using independent jump rates.

Theorem 2.14 (Theorem 3 in [5]). *Let ϕ be an arbitrary root for $\mathcal{T}$. For all $i \in T \backslash \{\phi\}$, we denote by $\overleftarrow{i}$ the parent of i. Let also $(A_i)_{i \in T \backslash \{\phi\}}$ be independent random variables where A_i is an inverse Gaussian random variable with parameter $(W_{\overleftarrow{i}, i}, 1)$, i.e.*

$$\mathbb{P}[A_i \in ds] = \mathbb{1}_{s \geq 0} \sqrt{\frac{W_{\overleftarrow{i}, i}}{2 \pi s^3}} e^{-W_{\overleftarrow{i}, i} \frac{(s-1)^2}{2s}} ds.$$

Then the process with law $\mathbb{P}^{VRJP(\phi)}$ on $\mathcal{T}$ is a mixture of Markov jump processes, in which the jump rate from $\overleftarrow{i}$ to i is $\frac{1}{2} W_{\overleftarrow{i}, i} A_i$, and the jump rate from i to $\overleftarrow{i}$ is $\frac{1}{2} \frac{W_{\overleftarrow{i}, i}}{A_i}$, for all $i \in T \backslash \{\phi\}$.

In some cases, this representation is different from the standard representation.

Proposition 2.15. *Let $\mathcal{T} = (T, E)$ be an infinite d-regular tree with $d \geq 3$, i.e. such that each vertex in T has exactly d neighbors. We endow $\mathcal{T}$ with constant conductances W. Then for W large enough, the distribution of the random environment described in Theorem 2.14 is different from the distribution of the standard representation.*

We now know two ways of constructing representations of the VRJP on $\mathcal{T}$, that are associated with different boundary conditions on restrictions to finite graphs, and can have distinct distributions. This leads us to introduce new boundary conditions in order to construct a family of different representations of the VRJP, following the same method as for the standard representation.

Let us start by giving a few notations on trees. For all $x, y \in T$, we denote by $d(x, y)$ the graph distance between x and y, and by $[x, y]$ the unique shortest path between x and y:
$$[x, y] = \big(x = [x, y]_0, [x, y]_1, ..., [x, y]_{d(x,y)-1}, y = [x, y]_{d(x,y)} \big).$$
Note that any path σ from x to y necessarily crosses all vertices $[x, y]_k$ for $0 \leq k \leq d(x, y)$.

Let us fix an arbitrary root ϕ in $\mathcal{T}$. Then, for all $x \in T$, we denote by $|x| = d(\phi, x)$ the depth of the vertex $x \in T$. If $x \neq \phi$, we also denote by $\bar{x} = [\phi, x]_{|x|-1}$ the parent of x. Finally, for any $x \in T$, we define the set of x's children $S(x) = \{y \in T, x = \bar{y}\}$, and the set of its descendants $T_x = \{y \in T, \exists k \geq 0, [\phi, y]_k = x\}$.

For $x, y \in T$, we denote by $x \wedge y$ the "closest common ancestor" of x and y, i.e. $x \wedge y = [\phi, x]_{N_{x,y}}$ where $N_{x,y} = \max\{k \geq 0, [\phi, x]_k = [\phi, y]_k\}$. Note that we also have $x \wedge y = [x, y]_{k_0}$, where k_0 is such that $|[x, y]_{k_0}| = \min\{|[x, y]_k|, 0 \leq k \leq d(x, y)\}$.

For $n \in \mathbb{N}$, we denote by $D^{(n)} = \{x \in T, |x| = n\}$ the tree's nth generation. Let us then define $T^{(n)} = \bigcup_{0 \leq k \leq n} D^{(k)}$, as well as $E^{(n)} = \{\{i, j\} \in E, i, j \in T^{(n)}\}$. The restriction of

the tree to the first n generations, with *free* boundary conditions, is the graph $(T^{(n)}, E^{(n)})$, that we endow with the induced conductances $W^{(n)} = W_{T^{(n)}, T^{(n)}}$. For $x \in T$ and $n \geq |x|$, we also denote $T_x^{(n)} = T_x \cap T^{(n)}$ the set of descendants of x in $T^{(n)}$.

Finally, we define the set Ω of ends of $\mathcal{T}$, *i.e.* the set of infinite self-avoiding paths (or rays) in $\mathcal{T}$ starting at ϕ. For $x \in T$, we denote by Ω_x the subset of Ω corresponding to the branch T_x, *i.e.* the set of rays in $\mathcal{T}$ that cross x. Note that the Martin boundary associated with a transient walk on a tree is always Ω, which depends only on the geometry of the tree. This will be convenient to express Δ^ψ-harmonic functions, where Δ^ψ is the random Laplace operator introduced in Proposition 2.11.

With these definitions out of the way, let us present new boundary conditions on trees, and the associated representations for the VRJP.

In the construction of the standard representation, the *wired* boundary condition was defined by adding a single boundary point δ to a finite graph, where δ could be interpreted as a point at infinity for the graph. We will now introduce a variant of this boundary condition, by adding multiple boundary points, each being a point at infinity for a different branch of the tree.

Let us first fix a generation $m \geq 0$, and to each vertex $x \in D^{(m)}$, we associate a boundary point δ_x, that will be the point at infinity for T_x. We denote by $B_m = \{\delta_x, x \in D^{(m)}\}$ the boundary set associated to this generation. For all $n \geq m$, let us then define the graph

$$\mathcal{G}_m^{(n)} = \left(\tilde{T}_m^{(n)}, \tilde{E}_m^{(n)} \right), \text{ where } \tilde{T}_m^{(n)} = T^{(n)} \cup B_m$$

$$\text{and } \tilde{E}_m^{(n)} = E^{(n)} \cup \bigcup_{x \in D^{(m)}} \left\{ \{y, \delta_x\}, y \in T_x \cap D^{(n)} \right\}.$$
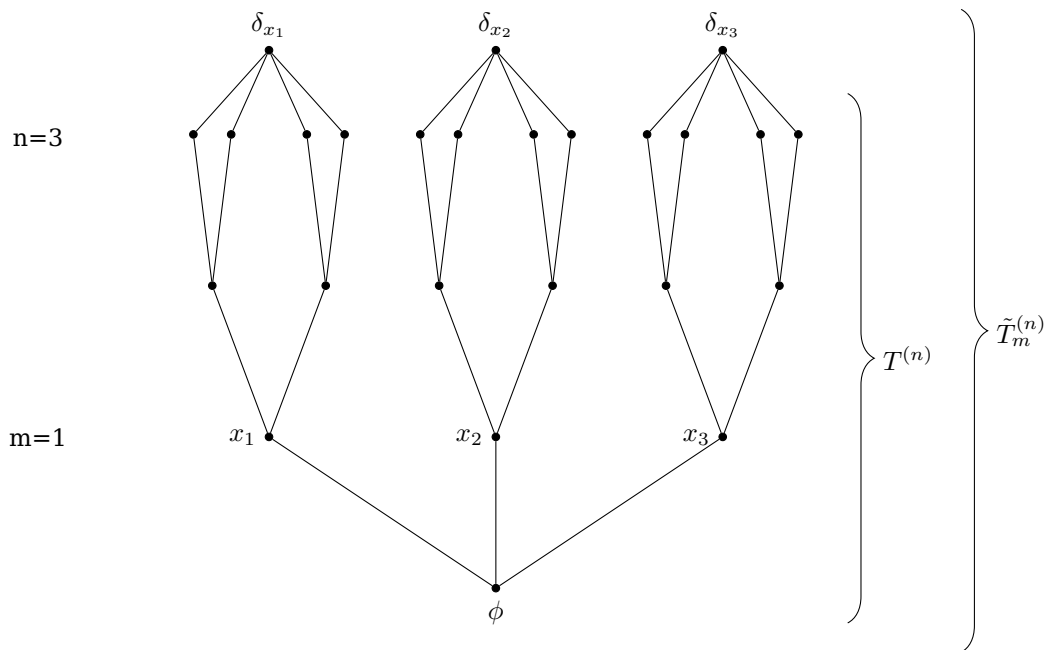
This graph is the restriction of T to $T^{(n)}$ with a variant of the *wired* boundary condition. Note that we get the standard *wired* boundary condition by taking $m = 0$. We endow $\mathcal{G}_m^{(n)}$ with the conductances $\tilde{W}_m^{(n)}$, defined for $e \in \tilde{E}^{(n)}$ by

$$\left(\tilde{W}_m^{(n)} \right)_e = \begin{cases} W_e^{(n)} = W_e & \text{if } e \in E^{(n)} \\ \sum_{j \in S(i)} W_{i,j} & \text{if } e = \{i, \delta_x\}, \text{ where } i \in T_x \cap D^{(n)}. \end{cases}$$

As with the *wired* boundary condition, these weights are defined so that for $n \geq m$, the weights coming out of $T^{(n)}$ are given by $W_{T^{(n)},(T^{(n)})^c} \mathbb{1}_{(T^{(n)})^c} = \eta^{(n)}$. This will allow for the compatibility of $\beta_m^{(n)}$ fields defined on $\mathcal{G}_m^{(n)}$ for $n \geq m$. Note that these weights do not depend on m, *i.e.* do not depend on the choice of the boundary condition.

For $\beta \in \mathcal{D}_T^W$, we still define $H_\beta = 2\beta - W$ and take $V_n = T^{(n)}$ for all $n \in \mathbb{N}$ in Definition 2.5. We then get $\hat{G}^{(n)} = ((H_\beta)_{T^{(n)},T^{(n)}})^{-1}$ and $\psi^{(n)} = \hat{G}^{(n)}\eta^{(n)}$, which converge ν_V^W-a.s. to $\hat{G}$ and ψ respectively, according to Theorem 2.6.

Moreover, for all $n \geq m$, under $\nu_T^W(d\beta)$, we can extend $\beta_{T^{(n)}}$ into a potential $\beta_m^{(n)} \sim \nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$ on $\mathcal{G}_m^{(n)}$ (see Lemma 6.1). Let us then denote by $G_m^{(n)} = (2\beta_m^{(n)} - \tilde{W}_m^{(n)})^{-1}$ the associated Green function. From Theorem 2.2, we know how to represent the time-changed VRJP on $\mathcal{G}_m^{(n)}$ using $G_m^{(n)}$. In order to obtain a result on the infinite tree $\mathcal{T}$, we will see that $G_m^{(n)}$ converges when $n \to \infty$, by writing it as a sum over paths in $\mathcal{G}_m^{(n)}$. Some new terms, defined below, appear in this expression.

Definition 2.16. *For $n \geq m \geq 0$, let $\chi_m^{(n)} \in \mathbb{R}_+^{T^{(n)} \times B_m}$ be defined by*

$$\begin{cases} (H_\beta \chi_m^{(n)}(\cdot, \delta_x))_{T^{(n)}} = 0, & \\ \chi_m^{(n)}(i, \delta_x) = 1 & \text{if } i \in T_x \backslash T_x^{(n)}, \\ \chi_m^{(n)}(i, \delta_x) = 0 & \text{if } i \in T \backslash (T_x \cup T^{(n)}), \end{cases}$$

for $x \in D^{(m)}$. Note that $(\chi_m^{(n)}(\cdot, \delta_x))_{T^{(n)}} = (\hat{G}^{(n)})_{T^{(n)},T^{(n)}} W_{T^{(n)},(T^{(n)})^c} \mathbb{1}_{T_x \backslash T_x^{(n)}}$.

Remark 2.17. *For $n \geq m$, $\chi_m^{(n)}$ is $\beta_{T^{(n)}}$-measurable, and for $x \in D^{(m)}$ and $y \in T^{(n)}$,*

$$\sum_{b \in B_m} \chi_m^{(n)}(y, b) = \sum_{x \in D^{(m)}} \chi_m^{(n)}(y, \delta_x) = \hat{G}^{(n)}(y, \cdot) W_{T^{(n)},(T^{(n)})^c} \mathbb{1}_{(T^{(n)})^c}$$

$$= \hat{G}^{(n)}(y, \cdot)\eta^{(n)} = \psi^{(n)}(y).$$

The decomposition of $G_m^{(n)}$ as a sum over paths in $\mathcal{G}_m^{(n)}$ gives the following expression (see Lemma 6.2): for $i, j \in T^{(n)}$,

$$G_m^{(n)}(i, j) = \hat{G}^{(n)}(i, j) + \sum_{x, x' \in D^{(m)}} \chi_m^{(n)}(i, \delta_x) G_m^{(n)}(\delta_x, \delta_{x'}) \chi_m^{(n)}(j, \delta_{x'}).$$

Once again, we will study the limit of this expression when $n \to \infty$, to obtain a representation of the VRJP on $(\mathcal{T}, W)$. However under $\nu_T^W(d\beta)$, contrary to $\psi^{(n)}$, $\chi_m^{(n)}(\cdot, \delta_x)$ is not a martingale when $m \neq 0$. Moreover, the term $(G_m^{(n)})_{B_m, B_m}$ is not independent of $\beta_{T^{(n)}}$ for $m \neq 0$. Therefore, we cannot use the same argument as in the proof of Theorem 2.6. However, we will still be able to show the almost sure convergence of $\chi_m^{(n)}$, using the structure of the tree and the associated Martin boundary, and deduce the convergence in distribution of $(G_m^{(n)})_{B_m, B_m}$ conditionally on β.

Let us give a few more details on the Martin boundary: we expect $\chi_m^{(n)}(\cdot, \delta_x)$ to converge to a H_β-harmonic function on T, for all $x \in D^{(m)}$ and ν_V^W-almost all β. When $\psi > 0$, we can introduce the operator Δ^ψ in order to study H_β-harmonic functions (see Proposition 2.11), thanks to the associated Martin boundary $\mathcal{M}^\psi$. Since the graph is a tree, the Martin boundary is equal to the set Ω of ends of $\mathcal{T}$, which is deterministic. Note that the boundary condition used to define $\mathcal{G}_m^{(n)}$ corresponds to identifying Ω_x to a single point δ_x, for all $x \in D^{(m)}$. We will see that the limit of $\chi_m^{(n)}(\cdot, \delta_x)$ can be expressed with the family of harmonic measures $(\mu_i^\psi)_{i \in T}$ defined, on the Martin boundary $\mathcal{M}^\psi = \Omega$, as the exit measure of the transient walk associated with Δ^ψ started at i (for more details, see Section 3.3). For $\beta \in \mathcal{D}_T^W$ such that $\psi \equiv 0$, we adopt the convention that μ_i^ψ is the null measure on Ω for all $x \in T$.

The following theorem states the almost sure convergence of $\chi_m^{(n)}$, and the existence of a family of representations constructed as previously described.

Theorem 2.18.

(i) For all $m \geq 0$, for $y \in T$ and $x \in D^{(m)}$, we have ν_T^W-almost surely $\chi_m^{(n)}(y, \delta_x) \to \psi(y)\mu_y^\psi(\Omega_x)$. For all $y \in T$, we define the measure $\chi(y, \cdot) = \psi(y)\mu_y^\psi(\cdot)$ on Ω.

(ii) Let $m \geq 0$ be fixed. For ν_T^W-almost all β, we define the $|B_m| \times |B_m|$ matrix $\check{C}_m$ by

$$(\check{C}_m)_{\delta_x, \delta_{x'}} = \begin{cases} 0 & \text{if } x = x', \\ \frac{\chi(x \wedge x', \Omega_x)\chi(x \wedge x', \Omega_{x'})}{\hat{G}(x \wedge x', x \wedge x')} & \text{otherwise.} \end{cases}$$

From now on, let us write: $\nu_{T,B_m}^W(d\beta, d\rho_m) = \nu_T^W(d\beta)\nu_{B_m}^{\check{C}_m}(d\rho_m)$.

For ν_T^W-almost all β and for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, we define $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$, as well as $\check{g}_m : \Omega^2 \to \mathbb{R}_+$ a locally constant function, such that for $x, x' \in D^{(m)}$ and $\omega \in \Omega_x$, $\tau \in \Omega_{x'}$, we have $\check{g}_m(\omega, \tau) = \check{G}_m(\delta_x, \delta_{x'})$. Finally, for ν_T^W-almost all β and for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, for $i, j \in T$, we define

$$G_m(i, j) = \hat{G}(i, j) + \int_{\Omega^2} \chi(i, d\omega)\chi(j, d\tau)\check{g}_m(\omega, \tau),$$

and denote by $\mathbb{P}_x^{\beta, \rho_m, i_0}$ the distribution of the Markov jump process started at $x \in V$, where the jump rate from i to j is $\frac{1}{2}W_{i,j}\frac{G_m(i_0, j)}{G_m(i_0, i)}$.

Then the process with law $\mathbb{P}^{VRJP(i_0)}$ is a mixture of these Markov jump processes, under the mixing measure $\nu_{T,B_m}^W(d\beta, d\rho_m)$, i.e.

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^{\beta, \rho_m, i_0}[\cdot]\nu_{T,B_m}^W(d\beta, d\rho_m).$$

(iii) The distribution under $\nu_{T,B_m}^W(d\beta, d\rho_m)$ of the jump rates $(\frac{1}{2}W_{i,j}\frac{G_m(i_0, j)}{G_m(i_0, i)})_{i \sim j}$ converges weakly to the distribution of jump rates in the representation described in Theorem 2.14.

Let us now consider the case where $\mathcal{T}$ is a d-regular tree, with $d \geq 3$, endowed with constant conductances, i.e. $W_e = W > 0$ for all $e \in E$. Then $(\mathcal{T}, W)$ is vertex transitive, and from Proposition 3 of [23], we know that under $\nu_T^W(d\beta)$, ψ is stationary and ergodic. Therefore, depending on d and W, we either have $\mathbb{P}[\forall i \in T, \psi(i) = 0] = 1$, or $\mathbb{P}[\forall i \in T, \psi(i) > 0] = 1$.

In the first case, from Theorem 2.6 (iv), this means that the VRJP is a.s. recurrent, and therefore admits a unique representation (see Proposition 2.10). Note that in Theorem 2.18, we have a.s. $G_m = \hat{G}$ for all $m \in \mathbb{N}$, so that all the corresponding representations are indeed equal. The following proposition describes the second case, i.e. when the VRJP is a.s. transient. According to a result from [9], this is true for large enough initial weights W.

Proposition 2.19. Let $\mathcal{T}$ be a d-regular tree, with $d \geq 3$, endowed with constant conductances W large enough so that the VRJP on $(\mathcal{T}, W)$ is a.s. transient. Then the representations of the VRJP given in Theorem 2.18 are different for distinct values of m, i.e. if $m \neq m'$, the distributions of jump rates $\left(\frac{1}{2}W_{i,j}\frac{G_m(i_0, j)}{G_m(i_0, i)}\right)_{i \sim j}$ under $\nu_{T,B_m}^W(d\beta, d\rho_m)$ and $\left(\frac{1}{2}W_{i,j}\frac{G_{m'}(i_0, j)}{G_{m'}(i_0, i)}\right)_{i \sim j}$ under $\nu_{T,B_{m'}}^W(d\beta, d\rho_{m'})$ are different for all $i_0 \in T$.

2.4 Open questions

A first question concerns the case of $\mathbb{Z}^d$ with constant conductances W: is it possible to show that the Martin boundary associated with Δ^ψ is a.s. trivial for any W such that the VRJP is transient? In this case, it would prove the uniqueness of the representation of the VRJP on $\mathbb{Z}^d$ for any constant initial conductances W.

Another question concerns a possible classification of all representations on trees using partitions of the Martin boundary. We have constructed a family of representations from different boundary conditions on the tree, corresponding to some finite partitions of the Martin boundary Ω, more precisely the partition $\Omega = \bigcup_{x \in D^{(m)}} \Omega_x$ for $m \in \mathbb{N}$. It should be possible to define more representations using the same method, with boundary conditions associated with other finite partitions of Ω, where each set in the partition can be written as a finite union of sets Ω_x. To generalize this, we can ask if it is possible to determine which partitions give us a valid representation, and whether all representations can be written in this form, or as a limit of such representations, as in Theorem 2.18 (iii).

2.5 Organization of the paper

Section 3 exposes some useful technical results concerning the β field, as well as basic definitions and properties of the Martin boundary. In Section 4, we prove how all representations of the VRJP have a common form, *i.e.* Theorem 2.9. We use these results in Section 5 to study the case of the graph $\mathbb{Z}^d$, and show Theorem 2.13 using a local limit theorem in random environment. In Section 6, we construct a family of new representations of the VRJP on infinite trees (Theorem 2.18). Section 7 presents several properties of this family, in particular that the representations are all different in the case of a regular tree (Proposition 2.19).

3 Technical prerequisite

3.1 The random potential β on finite graphs

Let $\mathcal{G} = (V, E)$ be a finite connected non-directed graph, endowed with conductances $(W_e)_{e \in E}$. Let us give some useful properties on the distribution ν_V^W.

Proposition 3.1 (Proposition 2, Theorem 3 in [20]). *For $\beta \in \mathcal{D}_V^W$, let $G = (H_\beta)^{-1}$ be the Green function associated with β. We define $F : V \times V \to \mathbb{R}$ by*

$$F(i,j) = \frac{G(i,j)}{G(j,j)}.$$

Then under $\nu_V^W(d\beta)$, for all $i_0 \in V$, we have the following properties:

(i) $(F(i, i_0))_{i \in V}$ is $(\beta_i)_{i \in V \setminus \{i_0\}}$-measurable.

(ii) If we denote $\gamma = \frac{1}{2G(i_0, i_0)}$, then γ is a Gamma random variable with parameter $(1/2, 1)$. Moreover, γ is independent of $(\beta_i)_{i \neq i_0}$, and therefore independent of $(F(i, i_0))_{i \in V}$.

This proposition explains the presence of γ in the expression on G in Theorem 2.6. Moreover, it allows us to prove Proposition 2.7, describing the distribution of the $\tilde{\beta}$ field.

Proof of Proposition 2.7. Let $\mathcal{G} = (V, E)$ be an infinite connected non-directed graph, and $(V_n)_{n \in \mathbb{N}}$ an increasing exhausting sequence of finite connected subsets of V. For $n \in \mathbb{N}$, let $\mathcal{G}^{(n)} = (\hat{V}^{(n)}, \tilde{E}^{(n)})$ be the restriction of $\mathcal{G}$ to V_n with *wired* boundary condition, endowed with conductances $\tilde{W}^{(n)}$, defined as in section 2.1. Moreover, for $n \in \mathbb{N}$, we still define $\hat{G}^{(n)}$ and $\psi^{(n)}$ as in Definition 2.5.

The proof of Theorem 2.6 (iii) uses the fact that under $\nu_V^W(d\beta, d\gamma)$, there exists a coupling of random fields $(\beta^{(n)})_{n\in\mathbb{N}}$, such that for all $n \in \mathbb{N}$: $\beta^{(n)} \sim \nu_{\tilde{V}^{(n)}}^{\tilde{W}^{(n)}}$; $\beta_{V_n}^{(n)} = \beta_{V_n}$; and $G^{(n)} = (2\beta^{(n)} - \tilde{W}^{(n)})^{-1}$, where for $i, j \in \tilde{V}^{(n)}$,

$$G^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + \frac{1}{2\gamma}\psi^{(n)}(i)\psi^{(n)}(j).$$

We can then apply Proposition 3.1 to $\beta^{(n)}$ at i_0: since $G^{(n)} = (2\beta^{(n)} - \tilde{W}^{(n)})^{-1}$, we have

$$\beta_i^{(n)} = \tilde{\beta}_i^{(n)} + \mathbb{1}_{\{i=i_0\}}\frac{1}{2G^{(n)}(i_0,i_0)}$$

for all $i \in \tilde{V}^{(n)}$, where

$$\tilde{\beta}_i^{(n)} = \sum_{j\sim i}\frac{\tilde{W}_{i,j}^{(n)}}{2}\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}.$$

According to Proposition 3.1, $1/2G^{(n)}(i_0,i_0)$ is a random Gamma$(1/2, 1)$ variable, which is independent of $(\beta_i^{(n)})_{i\in\tilde{V}^{(n)}\setminus\{i_0\}}$. Moreover, for $i \neq i_0$, $\tilde{\beta}_i^{(n)} = \beta_i^{(n)}$, and

$$\tilde{\beta}_{i_0}^{(n)} = \sum_{j\sim i_0}\frac{\tilde{W}_{i,j}^{(n)}}{2}\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i_0)} = \sum_{j\sim i_0}\frac{\tilde{W}_{i,j}^{(n)}}{2}F^{(n)}(j,i_0),$$

so that $\tilde{\beta}^{(n)}$ is $(\beta_i^{(n)})_{i\in\tilde{V}^{(n)}\setminus\{i_0\}}$-measurable, and therefore independent of $G^{(n)}(i_0,i_0)$.

Taking the limit when $n \to \infty$, we deduce that $1/2G(i_0,i_0)$ is a random Gamma$(1/2, 1)$ variable, independent from the $\tilde{\beta}$ field, where for $i \in V$,

$$\tilde{\beta}_i = \sum_{j\sim i}\frac{W_{i,j}}{2}\frac{G(i,j)}{G(i,i)} = \beta_i - \mathbb{1}_{\{i=i_0\}}\frac{1}{2G(i_0,i_0)}.$$

Since the Laplace transform of a Gamma$(1/2, 1)$ variable is, for $t \geq 0$,

$$\int e^{-t\gamma}\frac{\mathbb{1}_{\{\gamma>0\}}}{\sqrt{\pi\gamma}}e^{-\gamma}d\gamma = \frac{1}{\sqrt{1+t}},$$

and given the Laplace transform of ν_V^W in Proposition 2.4, we now know that the Laplace transform of $\tilde{\beta}$ is, for $\lambda \in \mathbb{R}_+^V$ with finite support,

$$\mathbb{E}[e^{-\langle\lambda,\tilde{\beta}\rangle}] = \frac{1}{\sqrt{1+\lambda_{i_0}}}e^{-\sum_{i\sim j}W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}\prod_{i\neq i_0}\frac{1}{\sqrt{1+l_i}}. \qquad \square$$

On finite graphs, the distribution ν_V^W, and more generally $\nu_V^{W,\eta}$ for $\eta \in \mathbb{R}_+^V$, behaves well with respect to restriction, as shown in the next proposition, which is a generalization of Proposition 2.3.

Proposition 3.2 (Lemma 4 in [23])**.** *Let us fix $U \subset V$ and $\eta \in (\mathbb{R}_+)^V$. Then, under $\nu_V^{W,\eta}(d\beta)$, we have:*

(i) *β_U is distributed according to $\nu_U^{W_{U,U},\hat{\eta}}$, where*

$$\hat{\eta} = \eta_U + W_{U,U^c}\mathbb{1}_{U^c}.$$

(ii) *Conditionally on β_U, β_{U^c} is distributed according to $\nu_{U^c}^{\check{W},\check{\eta}}$, where*

$$\check{W} = W_{U^c,U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1}W_{U,U^c} \text{ and } \check{\eta} = \eta_{U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1}\eta_U.$$

Proposition 2.3 is a direct consequence of (i), in the case where $\eta = 0$. Moreover, (ii) is useful to extend a potential $\beta_U \sim \nu_U^{W_{U,U},\hat{\eta}}$ where $\hat{\eta} = W_{U,U^c}\mathbb{1}_{U^c}$ into a potential $\beta_V \sim \nu_V^W$, using the distribution of β_{U^c} conditionally on β_U.

3.2 Green function and sums over paths

Let us still consider a finite connected non-directed graph $\mathcal{G} = (V, E)$ endowed with conductances W. For $\beta \in \mathcal{D}_V^W$, it will be useful to express the Green function $G = H_\beta^{-1}$ as a sum over paths in $\mathcal{G}$. We first introduce some notations for sets of paths.

Definition 3.3.

(i) For $i, j \in V$, we denote by $\mathcal{P}_{i,j}^V$ the set of paths σ from i to j in V, i.e. the set of finite sequences $\sigma = (\sigma_0, ... \sigma_l)$ in V, where $\sigma_0 = i$, $\sigma_l = j$ and $\sigma_k \sim \sigma_{k+1}$ for $0 \le k \le l - 1$. We denote by $|\sigma| = l$ the length of the path σ.

(ii) For $U \subset V$, $i \in U$ and $j \notin U$, we denote by $\overline{\mathcal{P}}_{i,j}^U$ the set of paths $\sigma \in \mathcal{P}_{i,j}^V$ such that $\sigma_k \in U$ for $0 \le k \le |\sigma| - 1$.

(iii) For $U \subset V$ and $i, j \in V$, we denote by $\mathcal{P}_{i,U,j}^V$ the set of paths $\sigma \in \mathcal{P}_{i,j}^V$ such that $\sigma_k \in U$ for some $k \in [\![0, |\sigma|]\!]$.

(iv) For $i, j \in V$ and $\sigma \in \mathcal{P}_{i,j}^V$, we define the following notations:

$$W_\sigma = \prod_{k=0}^{|\sigma|-1} W_{\sigma_k, \sigma_{k+1}}, (2\beta)_\sigma = \prod_{k=0}^{|\sigma|} 2\beta_{\sigma_k} \text{ and } (2\beta)_\sigma^- = \prod_{k=0}^{|\sigma|-1} 2\beta_{\sigma_k}.$$

We get the following expressions, in terms of sums over paths, for G and related quantities.

Proposition 3.4 (Proposition 6 in [20]). *Let $\beta \in \mathcal{D}_V^W$. Then:*

(i) *For $i, j \in V$,*

$$G(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}.$$

In particular, for $U \subset V$ we denote $\hat{G}^U = ((H_\beta)_{U,U})^{-1}$, then for $i, j \in U$, we obtain

$$\hat{G}^U(i, j) = \sum_{\sigma \in \mathcal{P}_{i,j}^U} \frac{W_\sigma}{(2\beta)_\sigma}.$$

(ii) *For $i, j \in V$,*

$$F(i, j) = \sum_{\sigma \in \overline{\mathcal{P}}_{i,j}^{V \setminus \{j\}}} \frac{W_\sigma}{(2\beta)_\sigma^-} = \sum_{z \sim j} \hat{G}^{V \setminus \{j\}}(i, z) W_{z,j}.$$

(iii) *For $U \subset V$ and $i, j \in U^c$,*

$$\sum_{\sigma \in \mathcal{P}_{i,U,j}^V} \frac{W_\sigma}{(2\beta)_\sigma^-} = \sum_{z_1, z_2 \in U} \left(\sum_{\sigma \in \overline{\mathcal{P}}_{i,z_1}^{U^c}} \frac{W_\sigma}{(2\beta)_\sigma^-} \right) G(z_1, z_2) \left(\sum_{\sigma \in \overline{\mathcal{P}}_{j,z_2}^{U^c}} \frac{W_\sigma}{(2\beta)_\sigma^-} \right).$$

In particular, if $U = \{z\}$, this becomes

$$\sum_{\sigma \in \mathcal{P}_{i,\{z\},j}^V} \frac{W_\sigma}{(2\beta)_\sigma} = F(i, z) G(z, z) F(j, z) = F(i, z) G(z, j).$$

Remark 3.5. If $\mathcal{G} = (V, E)$ is now an infinite graph, let $(V_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite connected subsets of V such that $V = \cup_{n \in \mathbb{N}} V_n$. For $\beta \in \mathcal{D}_V^W$ and $n \ge 0$,

we define $\hat{G}^{(n)} = \hat{G}^{V_n} = ((H_\beta)_{V_n, V_n})^{-1}$ as in Definition 2.5. Then from Proposition 3.4 (i), we get

$$\hat{G}^{(n)}(i,j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{V_n}} \frac{W_\sigma}{(2\beta)_\sigma}$$

for $n \geq 0$ and $i, j \in V_n$. From Theorem 2.6 (i), under $\nu_V^W(d\beta)$ the increasing sequence $(\hat{G}^{(n)}(i,j))_{n \in \mathbb{N}}$ converges almost surely to $\hat{G}(i,j)$. Hence, we get

$$\hat{G}(i,j) = \sum_{\sigma \in \mathcal{P}_{i,j}^V} \frac{W_\sigma}{(2\beta)_\sigma}$$

for $i, j \in V$.

Let us also define $\hat{F}^{(n)}(i,j) = \frac{\hat{G}^{(n)}(i,j)}{\hat{G}^{(n)}(j,j)}$ and $\hat{F}(i,j) = \frac{\hat{G}(i,j)}{\hat{G}(j,j)}$, for all $i, j \in V$ and $n \geq \max(|i|, |j|)$. Then, from Proposition 3.4 (ii) we have

$$\hat{F}^{(n)}(i,j) = \sum_{\sigma \in \overline{\mathcal{P}}_{i,j}^{V_n \setminus \{j\}}} \frac{W_\sigma}{(2\beta)_\sigma^-} \xrightarrow[n \to \infty]{} \hat{F}(i,j) = \sum_{\sigma \in \overline{\mathcal{P}}_{i,j}^{V \setminus \{j\}}} \frac{W_\sigma}{(2\beta)_\sigma^-},$$

where the convergence is true ν_V^W-almost surely.

3.3 Martin boundary and harmonic functions

Let us give more details about the theory of Martin boundaries. The following results can be found in Chapter IV of [24].

Let $\mathcal{G} = (V, E)$ be an infinite graph, we consider an irreducible random walk $(X_n)_{n \in \mathbb{N}}$ on $\mathcal{G}$, whose transition matrix is P, where $P_{i,j} = 0$ if $\{i, j\} \notin E$ (i.e. we assume that (X_n) is a nearest-neighbor random walk). Moreover, we assume that (X_n) is transient.

Let us denote by $\mathbb{P}_x$ the distribution of the random walk started at $x \in V$, and by g the associated Green function, i.e.

$$g(x,y) = \sum_{n \in \mathbb{N}} \mathbb{P}_x[X_n = y].$$

We also denote

$$f(x,y) = \mathbb{P}_x[\exists n \in \mathbb{N}, X_n = y] = \frac{g(x,y)}{g(y,y)}.$$

For all $y \in V$, $g(\cdot, y)$ is harmonic at any $x \in V \setminus \{y\}$, i.e. for all $x \neq y$, we have $g(x,y) = \sum_{z \sim x} P_{x,z} g(z,y)$. This is still true for $f(\cdot, y)$. The Martin Kernel, defined below using f, as well as the Martin boundary, will allow us to represent all positive harmonic functions for the random walk.

Definition 3.6. *Let us fix a reference point $\phi \in V$.*

(i) The Martin kernel is the function $K : V^2 \to \mathbb{R}_+$ defined by

$$K(x,y) = \frac{f(x,y)}{f(\phi,y)} = \frac{g(x,y)}{g(\phi,y)}.$$

(ii) The Martin compactification is the smallest compactification $\hat{V}$ of V, with respect to the discrete topology, so that $K(\cdot, \cdot)$ extends continuously to $V \times \hat{V}$. It is unique up to a homeomorphism. The Martin boundary is defined as $\mathcal{M} = \hat{V} \setminus V$.

For the proof that this compactification exists, see Chapter IV in [24].

In the following, we still denote by $K(\cdot, \cdot)$ the extension of the Martin kernel to $V \times \hat{V}$. For all $\alpha \in \mathcal{M}$, the function $K(\cdot, \alpha) : V \to \mathbb{R}_+$ is harmonic with respect to the random walk. Conversely, the following representation theorem states that all positive harmonic functions can be expressed using the Martin kernel.

Theorem 3.7. *Let us denote by $\mathcal{H}^+$ the set of positive harmonic functions on V. Then for all $h \in \mathcal{H}^+$, there is a Borel measure μ^h on $\mathcal{M}$ such that for all $x \in V$,*

$$h(x) = \int_{\mathcal{M}} K(x, \alpha) \mu^h(d\alpha).$$

Remark 3.8. If, for all $x \in V$ and for all sequences $(y_n)_{n \geq 1}$ going to infinity, we have $K(x, y_n) \to 1$, then the Martin boundary is trivial, *i.e.* reduced to a single point. According to Theorem 3.7, in this case, all positive harmonic functions are constant.

Since (X_n) is transient, we almost surely have $X_n \to \infty$, in the following sense: for all finite subset $U \subset V$, $\{n \in \mathbb{N}, X_n \in U\}$ is almost surely finite. Thanks to the Martin boundary, we can now describe this convergence more precisely.

Theorem 3.9. *For all $x \in V$, (X_n) converges $\mathbb{P}_x$-a.s. to a $\mathcal{M}$-valued random variable X_∞. The distribution of X_∞ under $\mathbb{P}_x$, denoted by μ_x, verifies*

$$\mu_x(B) = \int_B K(x, \alpha) \mu_\phi(d\alpha)$$

for all $B \subset \mathcal{M}$ and $x \in V$.

The space $(\mathcal{M}, \mathcal{B}(\mathcal{M}), (\mu_x)_{x \in V})$ is called Poisson boundary. Moreover, we call harmonic measures, or exiting measures, the family $(\mu_x)_{x \in V}$.

In the case where $\mathcal{T} = (T, E)$ is an infinite tree, the Martin compactification will coincide with another, which does not depend on the random walk defined by P, but simply on the geometry of the tree $\mathcal{T}$.

Definition 3.10. *Let us fix an arbitrary root ϕ for $\mathcal{T}$.*

(i) *We call infinite ray in T an infinite self-avoiding path starting at ϕ, i.e. a sequence $\omega = (\omega_k)_{k \in \mathbb{N}}$ of distinct vertices in T, such that $\omega_k \sim \omega_{k+1}$ for $k \in \mathbb{N}$ and $\omega_0 = \phi$. The set of infinite rays, also called the set of ends of T, is denoted by Ω.*

(ii) *If $\omega, \xi \in \Omega$, we denote $N_{\omega, \xi} = \max\{k \in \mathbb{N}, \omega_k = \xi_k\}$. We can also define, if $x \in T$, $N_{\omega, x} = \max\{k \leq |x|, \omega_k = [\phi, x]_k\}$. We then set $O_\omega^k = \{\xi \in \Omega, N_{\omega, \xi} \geq k\} \cup \{x \in T, N_{\omega, x} \geq k\}$.*

(iii) *We define the end topology on $T \cup \Omega$, which is discrete on T, and such that $(O_\omega^k)_{k \in \mathbb{N}}$ is a basis of neighborhoods at $\omega \in \Omega$.*

Recall also that for $x, y \in T$, we define $N_{x,y}$ in a similar way to Definition 3.10(ii), as the depth of the closest common ancestor to x and y: $N_{x,y} = \max\{k \geq 0, [\phi, x]_k = [\phi, y]_k\}$.

The following proposition introduces the end compactification, which will coincide with the Martin compactification on the tree, as stated in Theorem 3.12.

Proposition 3.11. *The end topology on $T \cup \Omega$ does not depend on the choice of ϕ, and is induced by the following metric:*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ e^{-N_{x,y}} & \text{otherwise,} \end{cases}$$

for $x, y \in T \cup \Omega$. Moreover $T \cup \Omega$ is compact, and called the end compactification.

Theorem 3.12.

(i) *Let (X_n) be a nearest-neighbor random walk on $\mathcal{T}$, that we assume to be transient. Then the Martin compactification coincides with the end compactification, and we can identify $\mathcal{M}$ to Ω, and set $\hat{T} = T \cup \Omega$.*

(ii) The Martin kernel on $T \times \hat{T}$ is locally constant, with

$$K(x, \omega) = K(x, x \wedge \omega), \text{ where } x \wedge \omega = \omega_{N_{\omega,x}}$$

for $x \in T, \omega \in \Omega$.

We also have an expression of harmonic measures μ_x on the tree. For $x \in T$, we denote by Ω_x the set of ends for the subtree T_x, i.e. $\Omega_x = \{\omega \in \Omega, \exists k \in \mathbb{N}, \omega_k = x\}$. Moreover, we denote by $U_x = T_x \backslash \{x\}$. Then:

Proposition 3.13. *For $x \neq \phi$ and $i \in T$,*

$$\mu_i(\Omega_x) = \mathbb{1}_{\{i \in U_x\}}(1 - f(i,x))) + f(i,x)\frac{1 - f(x,\check{x})}{1 - f(\check{x},x)f(x,\check{x})}.$$

Remark 3.14. From Carathéodory's extension theorem, this entirely describes the expression of μ_ϕ. From Theorem 3.9, we can then describe all harmonic measures using f.

4 Distributions of arbitrary representations

This section contains the proofs of the results from Section 2.2, stating that any representation of the time-changed VRJP can be expressed in a similar way to the standard representation.

4.1 A common expression for jump rates: proof of Theorem 2.9

Let $\mathcal{G} = (V, E)$ be a locally finite connected graph, endowed with conductances $(W_{i,j})_{i,j \in V}$ such that $W_{i,j} = W_{j,i} > 0$ if $\{i,j\} \in E$, and $W_{i,j} = 0$ otherwise. We still denote by $\mathbb{P}^{VRJP(i_0)}$ the law of the VRJP on $(\mathcal{G}, W)$, in the exchangeable time scale, started at $i_0 \in V$.

Let us first show that the distribution of the $\tilde{\beta}$ field (see Proposition 2.7) appears in all representations of the VRJP. Recall that for all $r \in \mathcal{J}_V^E$ and $i \in V$, we define $r_i = \sum_{j \sim i} r_{i,j}$.

Proposition 4.1. *Let $i_0 \in V$ be fixed, and let $\mathcal{R}(dr)$ be the distribution of a random environment representing $\mathbb{P}^{VRJP(i_0)}$, in the sense of Definition 2.8.*

Then under $\mathcal{R}(dr)$, $(r_i)_{i \in V}$ has the same distribution as the field $\tilde{\beta}$ rooted at i_0, i.e. its Laplace transform is

$$\int e^{-\langle \lambda, r \rangle} \mathcal{R}(dr) = e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+\lambda_i}\sqrt{1+\lambda_j}-1)} \prod_{i \neq i_0} \frac{1}{\sqrt{1+\lambda_i}},$$

for $\lambda \in \mathbb{R}_+^V$ with finite support.

Proof. Let $i_0 \in V$ be fixed, let $\mathcal{R}(dr)$ be the distribution of a random environment representing $\mathbb{P}^{VRJP(i_0)}$, i.e.

$$\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}_{i_0}^r[\cdot] \mathcal{R}(dr),$$

where $\mathbb{P}^r$ is the distribution of the Markov jump process with jump rate from i to j given by $r_{i,j}$.

Let us prove that under $\mathcal{R}(dr)$, $(r_i)_{i \in V}$ has the same distribution as the $\tilde{\beta}$ field from the standard representation.

Lemma 4.2. *There exists a random field $(u_i)_{i \in V} \in \mathbb{R}^V$ such that $\mathcal{R}$-almost surely, $r_{i,j} = \frac{W_{i,j}}{2}e^{u_j - u_i}$ for $i \sim j$.*

Remark 4.3. Since the random field $(u_i)_{i \in V}$ is defined up to an additive constant, we can set $u_{i_0} = 0$ a.s. without loss of generality.

Proof of Lemma 4.2. For $r \in \mathcal{J}_V^E$, let us define $t_{i,j} = \frac{2}{W_{i,j}} r_{i,j}$ for all $i \sim j$. Then to prove this lemma, it is enough to show that for any cycle $\sigma = (\sigma_0, ..., \sigma_n)$, we have $\mathcal{R}$-a.s. $t_\sigma := \prod_{k=0}^{n-1} t_{\sigma_k, \sigma_{k+1}} = 1$. Since $\mathcal{G}$ is connected, we only need to prove this for cycles σ such that $\sigma_0 = i_0$.

Recall that we denote by $(Z_t)_{t \geq 0}$ the canonical process on $\mathcal{C}_r(\mathbb{R}_+, V)$. Let $\mathbb{P}^{MJP}$ be the distribution of the Markov jump process with jump rates $\frac{1}{2} W_{i,j}$. Then, according to Theorem 3 from [22], for all $T \geq 0$ the law of $(Z_t)_{t \leq T}$ under $\mathbb{P}^{VRJP(i_0)}$ is absolutely continuous with respect to its law under $\mathbb{P}_{i_0}^{MJP}$, and its Radon-Nikodym derivative is

$$
\frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}},
$$

where $W_i = \sum_{j \sim i} W_{i,j}$, and $l_i = \int_0^T \mathbb{1}_{\{Z_t = i\}} dt$ is the local time at i.

Let σ be a cycle such that $\sigma_0 = \sigma_{|\sigma|} = i_0$, and let us denote by σ^n the nth concatenation of σ. Moreover, for $T \geq 0$, we define $(\tilde{Z})_T$ as the discrete path taken by the trajectory $(Z_t)_{t \leq T}$. Then we have, for $n \geq 1$ and $T \geq 0$,

$$
\mathbb{P}^{VRJP(i_0)}[(\tilde{Z})_T = \sigma^n] = \int \mathbb{1}_{\{(\tilde{z})_T = \sigma^n\}} \frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}} \mathbb{P}_{i_0}^{MJP}(dz).
$$

However, since the random environment $(r_{i,j})_{i \sim j}$ gives a representation of the VRJP as a mixture of Markov processes, we also have

$$
\mathbb{P}^{VRJP(i_0)}[(\tilde{Z})_T = \sigma^n] = \int \mathbb{P}_{i_0}^r[(\tilde{Z})_T = \sigma^n] \mathcal{R}(dr)
$$

$$
= \int \left(\int \mathbb{1}_{\{(\tilde{z})_T = \sigma^n\}} \frac{e^{-\sum_{i \in V} r_i l_i} \left(\prod_{k=0}^{|\sigma|-1} r_{\sigma_k, \sigma_{k+1}} \right)^n}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i} \left(\prod_{k=0}^{|\sigma|-1} \frac{1}{2} W_{\sigma_k, \sigma_{k+1}} \right)^n} \mathbb{P}_{i_0}^{MJP}(dz) \right) \mathcal{R}(dr)
$$

$$
= \int \mathbb{1}_{\{(\tilde{z})_T = \sigma^n\}} \int \frac{e^{-\sum_{i \in V} r_i l_i}}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i}} (t_\sigma)^n \mathcal{R}(dr) \mathbb{P}_{i_0}^{MJP}(dz)
$$

Let us fix $\varepsilon > 0$, and define the event $A_{\sigma,\varepsilon} = \{t_\sigma \geq 1 + \varepsilon\}$. Then we get

$$
\mathbb{P}^{VRJP(i_0)}[(\tilde{Z})_T = \sigma^n] \geq \int \mathbb{1}_{\{(\tilde{z})_T = \sigma^n\}} \int \mathbb{1}_{A_{\sigma,\varepsilon}} e^{-\sum_{i \in V} r_i l_i} (1+\varepsilon)^n \mathcal{R}(dr) \mathbb{P}_{i_0}^{MJP}(dz).
$$

Let $M > 0$ be such that under $\mathcal{R}(dr)$, $\mathbb{P}[A_{\sigma,\varepsilon} \cap B_M] \geq \mathbb{P}[A_{\sigma,\varepsilon}]/2$, where $B_M = \{\forall i \in V, r_i \leq M\}$. Note that $T = \sum_{i \in V} l_i$, so that

$$
\mathbb{P}^{VRJP(i_0)}[(\tilde{Z})_T = \sigma^n] \geq \int \mathbb{1}_{\{(\tilde{z})_T = \sigma^n\}} \int \mathbb{1}_{A_{\sigma,\varepsilon} \cap B_M} e^{-MT} (1+\varepsilon)^n \mathcal{R}(dr) \mathbb{P}_{i_0}^{MJP}(dz)
$$

$$
\geq \frac{e^{-MT}}{2} (1+\varepsilon)^n \mathbb{P}[A_{\sigma,\varepsilon}] \mathbb{P}_{i_0}^{MJP}[(\tilde{Z})_T = \sigma^n].
$$

On the other hand, we also have

$$
\mathbb{P}^{VRJP(i_0)}[(\tilde{Z})_T = \sigma^n] = \int \mathbb{1}_{\{(\tilde{z})_T = \sigma^n\}} \frac{e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i \in V} \frac{1}{2} W_i l_i}} \prod_{i \neq i_0} \frac{1}{\sqrt{1+l_i}} \mathbb{P}_{i_0}^{MJP}(dz)
$$

$$
\leq e^{M'T} \mathbb{P}_{i_0}^{MJP}[(\tilde{Z})_T = \sigma^n],
$$

where $M' = \max\{\frac{1}{2}W_{\sigma_k}, 0 \le k < |\sigma|\}$. Since $\mathbb{P}_{i_0}^{MJP}\big[(\tilde{Z})_T = \sigma^n\big] > 0$ for all $T > 0$ and $n \in \mathbb{N}$, we get

$$\frac{e^{-MT}}{2}(1+\varepsilon)^n \mathbb{P}[A_{\sigma,\varepsilon}] \le e^{M'T}.$$

Taking $n \to \infty$ for fixed $T > 0$ shows that $\mathbb{P}[A_{\sigma,\varepsilon}] = 0$. As a result, we have almost surely $t_\sigma \le 1$.

For $\varepsilon > 0$, we now set $A'_{\sigma,\varepsilon} = \{t_\sigma \le 1 - \varepsilon\}$. Using the same notations as before, and the fact that a.s. $t_\sigma \le 1$, we get

$$\mathbb{P}^{VRJP(i_0)}\big[(\tilde{Z})_T = \sigma^n\big] \le \int \mathbb{1}_{\{(\tilde{z})_T = \sigma^n\}} \int e^{M'T}\left(\mathbb{1}_{A'_{\sigma,\varepsilon}{}^c} + \mathbb{1}_{A'_{\sigma,\varepsilon}}(1-\varepsilon)^n\right) \mathcal{R}(dr)\mathbb{P}_{i_0}^{MJP}(dz)$$

$$\le e^{M'T}\left(\mathbb{P}[A'_{\sigma,\varepsilon}{}^c] + \mathbb{P}[A'_{\sigma,\varepsilon}](1-\varepsilon)^n\right)\mathbb{P}_{i_0}^{MJP}\big[(\tilde{Z})_T = \sigma^n\big].$$

On the other hand, on the event $\{(\tilde{Z})_T = \sigma^n\}$, we have $l_i \le T$ for all $i \in \{\sigma_k, 0 \le k < |\sigma|\}$ and $l_i = 0$ for all other $i \in V$. As a result, for such trajectories,

$$\frac{e^{-\sum_{i\sim j}W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i\in V}\frac{1}{2}W_i l_i}}\prod_{i\neq i_0}\frac{1}{\sqrt{1+l_i}} \ge \frac{e^{-M''T}}{(1+T)^{\frac{|\sigma|-1}{2}}},$$

where $M'' = \sum_{i,j\in\{\sigma_k\}}W_{i,j}$, so that,

$$\mathbb{P}^{VRJP(i_0)}\big[(\tilde{Z})_T = \sigma^n\big] \ge \frac{e^{-M''T}}{(1+T)^{\frac{|\sigma|-1}{2}}}\mathbb{P}_{i_0}^{MJP}\big[(\tilde{Z})_T = \sigma^n\big].$$

As before, this yields

$$e^{M'T}\left(\mathbb{P}[A'_{\sigma,\varepsilon}{}^c] + \mathbb{P}[A'_{\sigma,\varepsilon}](1-\varepsilon)^n\right) \ge \frac{e^{-M''T}}{(1+T)^{\frac{|\sigma|-1}{2}}}$$

for all $T > 0$ and $n \in \mathbb{N}$. Taking first $n \to \infty$, then $T \to 0$, we get that under $\mathcal{R}(dr)$, $\mathbb{P}[A'_{\sigma,\varepsilon}{}^c] = 1$. Therefore, we can conclude that $\mathcal{R}$-almost surely, $t_\sigma = 1$. $\square$

In order to identify the distribution of $(r_i)_{i\in V}$ under $\mathcal{R}(dr)$, we obtain their Laplace transform as the density of cyclic trajectories of $(Z_t)_{t\ge 0}$ under $\mathbb{P}^{VRJP(i_0)}$ with respect to $\mathbb{P}_{i_0}^{MJP}$. Indeed, given a cyclic trajectory $(z_t)_{t\ge 0}$ in $\mathcal{G}$, started at i_0, we denote by σ the associated cyclic path in $\mathcal{G}$, and $(l_i)_{i\in V}$ the local times, so that $T = \sum_{i\in V}l_i$, and $l_i > 0$ if and only if $i \in \{\sigma_k, 0 \le k < |\sigma|\}$. Then the Radon-Nikodym derivative at $(z_t)_{t\ge 0}$ of $\mathbb{P}^{VRJP(i_0)}$ with respect to $\mathbb{P}_{i_0}^{MJP}$ is almost surely

$$\frac{e^{-\sum_{i\sim j}W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}}{e^{-\sum_{i\in V}\frac{1}{2}W_i l_i}}\prod_{i\neq i_0}\frac{1}{\sqrt{1+l_i}},$$

but also

$$\int \frac{e^{-\sum_{i\in V}r_i l_i}}{e^{-\sum_{i\in V}\frac{1}{2}W_i l_i}}\mathcal{R}(dr)$$

since $t_\sigma = 1$ $\mathcal{R}$-almost surely. Therefore, for all finite connected subset U of V, and almost all $(l_i)_{i\in V} \in (\mathbb{R}_+^*)^U \times \{0\}^{V\setminus U}$, we have

$$e^{-\sum_{i\sim j}W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j}-1)}\prod_{i\neq i_0}\frac{1}{\sqrt{1+l_i}} = \int e^{-\sum_{i\in V}r_i l_i}\mathcal{R}(dr) = \mathbb{E}[e^{-\sum_{i\in V}r_i l_i}].$$

Since these are continuous functions of $(l_i)_{i\in V}$, this equality is true for all $(l_i)_{i\in V} \in \mathbb{R}_+^V$ with finite support. As a result, under $\mathcal{R}(dr)$, $(r_i)_{i\in V}$ has the same Laplace transform as the field $\tilde{\beta}$, associated with the standard representation of the VRJP started at i_0 (see Proposition 2.7), and therefore the same distribution. $\square$

Theorem 2.9 results easily from Proposition 4.1, since adding an independent Gamma variable to the previous $\tilde{\beta}$ field yields a potential with distribution ν_V^W. This provides the expression of the jump rates using the associated Green function $\hat{G}$ and a H_β-harmonic function.

Proof of Theorem 2.9. Let $i_0 \in V$ be fixed, and $\mathcal{R}(dr)$ be the distribution of a random environment representing $\mathbb{P}^{VRJP(i_0)}$. Thanks to Proposition 4.1, we know the distribution of $(r_i)_{i \in V}$ under $\mathcal{R}(dr)$. Note that the distribution of a Gamma$(1/2, 1)$ variable is $\frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma$, and that its Laplace transform is given by

$$\int e^{-t\gamma} \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma = \frac{1}{\sqrt{1+t}}$$

for $t \geq 0$. From now on, we denote $\mathcal{R}(dr, d\gamma) = \mathcal{R}(dr) \otimes \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma$, which is a distribution on $\mathcal{J}_V^E \times \mathbb{R}$.

For $r \in \mathcal{J}_V^E$ and $\gamma > 0$, we now define $(\beta_i)_{i \in V}$ by $\beta_i = r_i + \mathbb{1}_{\{i = i_0\}} \gamma$ for $i \in V$. Then under $\mathcal{R}(dr, d\gamma)$, the Laplace transform of β is, for $\lambda \in \mathbb{R}_+^V$ with finite support,

$$\int e^{-\langle \lambda, \beta \rangle} \mathcal{R}(dr) \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma = \int e^{-\sum_{i \in V} \lambda_i r_i} \mathcal{R}(dr) \int e^{-\lambda_{i_0} \gamma} \frac{\mathbb{1}_{\{\gamma > 0\}}}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma$$

$$= \frac{1}{\sqrt{1 + \lambda_{i_0}}} e^{-\sum_{i \sim j} W_{i,j}(\sqrt{1+l_i}\sqrt{1+l_j} - 1)} \prod_{i \neq i_0} \frac{1}{\sqrt{1 + l_i}},$$

i.e. β is distributed according to ν_V^W (see Proposition 2.4). We can then define $\mathcal{R}(dr, d\gamma)$-a.s. $\hat{G} : V \times V \to \mathbb{R}_+$ and $\psi : V \to \mathbb{R}_+$ thanks to Theorem 2.6. Moreover, by analogy with the standard representation, let $G(i_0, \cdot) : V \to \mathbb{R}_+$ be defined by:

$$G(i_0, i) = \frac{1}{2\gamma} e^{u_i},$$

where $(u_i)_{i \in V}$ was introduced in Lemma 4.2. This way, under $\mathcal{R}(dr, d\gamma)$, for all $i \neq j \in V$ the jump rate $r_{i,j}$ can be written as

$$r_{i,j} = \frac{W_{i,j}}{2} \frac{G(i_0, j)}{G(i_0, i)}.$$

Let us set $h(i) = G(i_0, i) - \hat{G}(i_0, i)$ for all $i \in V$. Then $H_\beta h = 0$. Indeed, for $i \neq i_0$, we have

$$2\beta_i G(i_0, i) - \sum_{j \sim i} W_{i,j} G(i_0, j) = 2r_i G(i_0, i) - \sum_{j \sim i} 2r_{i,j} G(i_0, i) = 0,$$

and for $i = i_0$,

$$2\beta_{i_0} G(i_0, i_0) - \sum_{j \sim i_0} W_{i_0,j} G(i_0, j) = \frac{r_{i_0} + \gamma}{\gamma} - \sum_{j \sim i_0} \frac{r_{i_0,j}}{\gamma} = 1,$$

therefore for $i \in V$, $H_\beta G(i_0, i) = \mathbb{1}_{\{i = i_0\}} = H_\beta \hat{G}(i_0, i)$.

Let us finally show that $h(i) \geq 0$ for all $i \in V$. We define $\tau_{i_0}^+ = \inf\{k \geq 1, \tilde{Z}_k = i_0\}$, where $(\tilde{Z}_k)_{k \in \mathbb{N}}$ is the discrete version of $(Z_t)_{t \geq 0}$. Then for all $i \neq i_0$,

$$\mathbb{P}_i^r[\tau_{i_0}^+ < \infty] = \sum_{\sigma \in \overline{\mathcal{P}}_{i,i_0}^{V \setminus \{i_0\}}} \mathbb{P}_i^r[(\tilde{Z}_0, ..., \tilde{Z}_{|\sigma|}) = (\sigma_0, ..., \sigma_{|\sigma|})]$$

$$= \sum_{\sigma \in \overline{\mathcal{P}}_{i,i_0}^{V \setminus \{i_0\}}} \prod_{k=0}^{|\sigma|-1} \frac{r_{\sigma_k, \sigma_{k+1}}}{r_{\sigma_k}} = \sum_{\sigma \in \overline{\mathcal{P}}_{i,i_0}^{V \setminus \{i_0\}}} \prod_{k=0}^{|\sigma|-1} \frac{W_{\sigma_k, \sigma_{k+1}}}{2\beta_{\sigma_k}} \frac{G(i_0, \sigma_{k+1})}{G(i_0, \sigma_k)}$$

$$= \frac{G(i_0, i_0)}{G(i_0, i)} \sum_{\sigma \in \overline{\mathcal{P}}_{i,i_0}^{V \setminus \{i_0\}}} \frac{W_\sigma}{(2\beta)_{\overline{\sigma}}} = \frac{G(i_0, i_0)}{G(i_0, i)} \frac{\hat{G}(i_0, i)}{\hat{G}(i_0, i_0)},$$

from Proposition 3.4. Therefore, we have $G(i_0, i) \leq \frac{G(i_0,i_0)}{\hat{G}(i_0,i_0)} \hat{G}(i_0, i)$. Moreover, thanks to the Markov property we have

$$
\begin{aligned}
\mathbb{P}^r_{i_0}[\tau^+_{i_0} < \infty] &= \sum_{j \sim i_0} \mathbb{P}^r_{i_0}[\tilde{Z}_1 = j] \mathbb{P}^r_j[\tau^+_{i_0} < \infty] = \sum_{j \sim i_0} \frac{r_{i_0,j}}{r_{i_0}} \frac{G(i_0,i_0)}{G(i_0,j)} \frac{\hat{G}(i_0,j)}{\hat{G}(i_0,i_0)} \\
&= \sum_{j \sim i_0} \frac{W_{i_0,j}}{2(\beta_{i_0} - \gamma)} \frac{\hat{G}(i_0,j)}{\hat{G}(i_0,i_0)} = \frac{1}{2\beta_{i_0} - \frac{1}{G(i_0,i_0)}} \sum_{j \sim i_0} W_{i_0,j} \frac{\hat{G}(i_0,j)}{\hat{G}(i_0,i_0)} \\
&= \frac{2\beta_{i_0} - \frac{1}{\hat{G}(i_0,i_0)}}{2\beta_{i_0} - \frac{1}{G(i_0,i_0)}},
\end{aligned}
$$

therefore we have $G(i_0, i_0) \geq \hat{G}(i_0, i_0)$, as well as $G(i_0, i) \geq \hat{G}(i_0, i)$ for all $i \in V$.

As a result, $G(i_0, \cdot)$ can be written, for all $i \in V$, as

$$
G(i_0, i) = \hat{G}(i_0, i) + h(i),
$$

where $h : V \to \mathbb{R}_+$ is a non-negative H_β-harmonic function. $\qquad \square$

4.2 The recurrent and transient cases: proofs of Propositions 2.10 and 2.11

We now consider two particular cases for the weighted graph $(\mathcal{G}, W)$. The first one is when the VRJP on $(\mathcal{G}, W)$ is a.s. transient, in which case the representation is unique, as proved below.

Proof of Proposition 2.10. We assume that $(\mathcal{G}, W)$ is such that the VRJP is almost surely recurrent.

Let $(r_{i,j})_{i \sim j}$ be fixed jump rates on V, such that the associated Markov chain is recurrent. We denote by $\mathbb{P}^r_{i_0}$ its distribution when started at i_0. Note that under $\mathbb{P}^r_{i_0}$, the time spent at a vertex i before jumping is an exponential variable with parameter r_i, and the probability to then jump to a specific neighbor j is $\frac{r_{i,j}}{r_i}$.

Let us then define the following functions of the trajectory (Z_t): for $i \in V$ and $n \geq 1$, we define $\delta t_i^{(n)}$ as the time spent by (Z_t) at the vertex i during its nth visit to i, and $v_i^{(n)}$ the neighbor of i towards which the process jumps after its nth visit to i. Under $\mathbb{P}^r_{i_0}$, since the process is recurrent, these random variables are well-defined for all $i \in V$ and $n \geq 1$. Moreover, the sequences $(\delta t_i^{(n)})_{n \geq 1}$ and $(v_i^{(n)})_{n \geq 1}$ are independent, so thanks to the law of large numbers, we have almost surely

$$
\overline{\delta t_i} := \lim_{n \to \infty} \frac{\sum_{k=1}^n \delta t_i^{(k)}}{n} = \frac{1}{r_i} \text{ and } \overline{p}_{i,j} := \lim_{n \to \infty} \frac{\sum_{k=1}^n \mathbb{1}_{\{v_i^{(k)} = j\}}}{n} = \frac{r_{i,j}}{r_i},
$$

for all $i, j \in V$.

Let now $\mathcal{R}(dr)$ be the distribution of a random environment representing the VRJP on $(\mathcal{G}, W)$, *i.e.* $\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}^r_{i_0}[\cdot] \mathcal{R}(dr)$. Since the VRJP is a.s. recurrent, then under $\mathcal{R}(dr)$, $\mathbb{P}^r_{i_0}$ is a.s. the distribution of a recurrent Markov chain. Moreover, under $\mathbb{P}^{VRJP(i_0)}$, $\overline{\delta t_i}$ and $\overline{p}_{i,j}$ are a.s. well-defined for all $i, j \in V$, and $\left(\frac{\overline{p}_{i,j}}{\overline{\delta t_i}} \right)_{i \sim j}$ is distributed according to $\mathcal{R}$. Since these functions of the trajectory do not depend on the chosen representation, the distribution $\mathcal{R}$ is uniquely determined. $\qquad \square$

The second interesting case is when the VRJP is a.s. transient. In this case we introduce a random conductance model associated with ψ, which defines a transient random walk. We can then relate H_β-harmonic functions to harmonic functions for this walk. This will be useful to study the H_β-harmonic function appearing in the expression of any representation, according to Theorem 2.9.

Proof of Proposition 2.11. Let $(\mathcal{G}, W)$ be such that the VRJP is almost surely transient. Since, according to Theorem 2.6 (iii), we have $\mathbb{P}^{VRJP(i_0)}[\cdot] = \int \mathbb{P}^{\beta, \gamma, i_0}_{i_0}[\cdot] \nu^W_V(d\beta, d\gamma)$, then under $\nu^W_V(d\beta, d\gamma)$, the Markov process with distribution $\mathbb{P}^{\beta, \gamma, i_0}_{i_0}$ is a.s. transient. From Theorem 2.6 (iv), this means that under $\nu^W_V(d\beta)$, we have a.s. $\psi(i) > 0$ for all $i \in V$, which proves *(i)*.

Let us now consider the random conductance model with conductances $c^\psi_{i,j} = W_{i,j}\psi(i)\psi(j)$. We denote by $\pi^\psi_i = \sum_{j \sim i} c^\psi_{i,j}$ the corresponding invariant measure, where $\pi^\psi_i = \psi(i) \sum_{j \sim i} W_{i,j}\psi(j) = 2\beta_i \psi(i)^2$ since ψ is H_β-harmonic. Let $\mathbb{P}^\psi$ be the distribution of the associated random walk, whose transition probability from i to j is

$$p^\psi_{i,j} = \frac{c^\psi_{i,j}}{\pi^\psi_i} = \frac{W_{i,j}\psi(j)}{2\beta_i \psi(i)}$$

for $i, j \in V$. Moreover, let us denote by g^ψ the Green kernel associated with $\mathbb{P}^\psi$, defined for $i, j \in V$ as $g^\psi(i,j) = \sum_{k \in \mathbb{N}} \mathbb{P}^\psi_i[X_k = j]$, where $(X_k)_{k \in \mathbb{N}}$ denotes the canonical process on $V^{\mathbb{N}}$. Then we have

$$g^\psi(i,j) = \sum_{k \in \mathbb{N}} \sum_{\sigma \in \mathcal{P}^V_{i,j}, |\sigma|=k} \mathbb{P}^\psi_i[(X_0, ..., X_k) = \sigma] = \sum_{\sigma \in \mathcal{P}^V_{i,j}} \prod_{k=0}^{|\sigma|-1} \frac{W_{\sigma_k, \sigma_{k+1}}\psi(\sigma_{k+1})}{2\beta_{\sigma_k}\psi(\sigma_k)}$$

$$= \frac{\psi(j)}{\psi(i)} \sum_{\sigma \in \mathcal{P}^V_{i,j}} \frac{W_\sigma}{(2\beta)^{-}_\sigma} = \frac{\psi(j)}{\psi(i)} 2\beta_j \hat{G}(i,j),$$

where under $\nu^W_V(d\beta)$, $\hat{G}(i,j)$ is a.s. finite for all $i, j \in V$, from Theorem 2.6 (i). As a result, we have almost surely $g^\psi(i,j) < \infty$, therefore the random walk $\mathbb{P}^\psi$ is transient almost surely, proving *(ii)*.

Let $\Delta^\psi = (p^\psi_{i,j} - \mathbb{1}_{\{i=j\}})_{i,j \in V}$ be the discrete Laplacian associated with $\mathbb{P}^\psi$. We will say that a function $\varphi : V \to \mathbb{R}$ is Δ^ψ-harmonic if $(\Delta^\psi \varphi)(i) = \left(\sum_{j \sim i} p^\psi_{i,j}\varphi(j)\right) - \varphi(i) = 0$ for all $i \in V$. Therefore, a function φ is Δ^ψ-harmonic if and only if for any $i \in V$,

$$2\beta_i \psi(i)\varphi(i) - \sum_{j \sim i} W_{i,j}\psi(j)\varphi(j) = 0,$$

i.e. if and only if $\psi\varphi$ is H_β-harmonic, which concludes the proof of *(iii)*. $\qquad \square$

5   Representations of the VRJP on $\mathbb{Z}^d$: proof of Theorem 2.13

Let us now consider the case where $\mathcal{G} = (V, E)$ is the $\mathbb{Z}^d$ lattice, endowed with constant edge weights, *i.e.* $W_{i,j} = W > 0$ for all $i \sim j$. For $x \in \mathbb{R}^d$, we will denote by $|x|$ its Euclidean norm. We fix $i_0 = 0$.

The aim of this section is to prove Theorem 2.13, concerning the uniqueness of the representation on $(\mathbb{Z}^d, W)$. Let us first distinguish two regimes for the weighted graph: when the VRJP is a.s. recurrent and when it is a.s. transient.

5.1   Recurrence and transience of the VRJP on $\mathbb{Z}^d$

For $d = 2$, the VRJP on $(\mathcal{G}, W)$ is a.s. recurrent for all $W > 0$, according to Theorem 2 in [19]. Therefore, the representation of $\mathbb{P}^{VRJP(0)}$ as a mixture of Markov jump processes is unique (see Proposition 2.10). If $d \geq 3$, Corollary 1 in [21] tells us that for small enough W, the VRJP is a.s. recurrent, in which case the representation of $\mathbb{P}^{VRJP(0)}$ is once again unique. Let us now show that for large enough W, even though the VRJP is almost surely transient, the representation is still unique.

From Corollary 3 in [21], we know that for W large enough, the VRJP is a.s. transient. From now on, we consider such W. Then thanks to Proposition 2.11, under $\nu_V^W(d\beta)$ we have a.s. $\psi(i) > 0$ for all $i \in V$. Moreover, we can define the Markov operator Δ^ψ and, for $h : V \to \mathbb{R}_+$, h is H_β-harmonic if and only if h/ψ is Δ^ψ-harmonic. In light of Remark 2.12, in order to show that the representation of the VRJP is unique, we need to show that the only positive Δ^ψ-harmonic functions are constants, *i.e.* that the Martin boundary $\mathcal{M}^\psi$ associated with Δ^ψ is almost surely trivial. To do this, we will need a local limit theorem in random environment, found in [1].

5.2 Local limit theorem for random walk in random conductances

Let us consider a random conductances model on $\mathcal{G} = (\mathbb{Z}^d, E_d)$, with $d \geq 2$. Let $\mathbb{P}$ be a distribution on the set of conductances $(\mathbb{R}_+^*)^{E_d}$, such that under $\mathbb{P}(d\omega)$, we have a.s. $0 < \omega_{i,j} < \infty$ for all $i \sim j$. For $\omega \in (\mathbb{R}_+^*)^{E_d}$, let P^ω be the distribution of the continuous-time constant speed random walk associated with ω. This is the Markov jump process with jump rate from i to j given by $\frac{\omega_{i,j}}{\pi_i^\omega}$, where $\pi_i^\omega = \sum_{j \sim i} \omega_{i,j}$. This way under P^ω, the holding time of $(Z_t)_{t \geq 0}$ at each point is an exponential variable of parameter 1, which justifies the term "constant speed". Finally, we denote by q^ω the heat kernel, *i.e.* the transition density of the walk with respect to π^ω: for $x, y \in \mathbb{Z}^d$ and $t \geq 0$,

$$q^\omega(t, x, y) = \frac{P_x^\omega[Z_t = y]}{\pi_y^\omega}.$$

The following theorem from [1] is a local limit theorem for q^ω, under ergodicity and integrability assumptions.

Theorem 5.1 (Theorem 1.11 in [1])**.** *Let us assume that $\mathbb{P}(d\omega)$ is stationary and ergodic with respect to translations of $\mathbb{Z}^d$, and that there exist $p, q \in (1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that $\mathbb{E}[\omega_{i,j}^p] < \infty$ and $\mathbb{E}[\omega_{i,j}^{-q}] < \infty$ for all $i \sim j$.*

Then for $0 < T_1 < T_2$ and $K > 0$, we have $\mathbb{P}$-a.s.

$$\lim_{n \to \infty} \sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} \left| n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x) \right| = 0,$$

where $\lfloor nx \rfloor = (\lfloor nx_1 \rfloor, ..., \lfloor nx_d \rfloor)$, $a = 1/\mathbb{E}[\pi_0^\omega]$ and k_t is the Gaussian heat kernel with some deterministic covariance matrix Σ^2, i.e.

$$k_t(x) = \frac{1}{\sqrt{(2\pi t)^d \det(\Sigma^2)}} e^{-\frac{x^t (\Sigma^2)^{-1} x}{2t}}.$$

Remark 5.2. If the distribution $\mathbb{P}(d\omega)$ is also invariant with respect to all permutations of coordinates in $\mathbb{Z}^d$, then the law of the limiting Brownian motion must be as well. Therefore its deterministic covariance matrix has the form $\Sigma^2 = \sigma^2 I_d$, where $\sigma^2 > 0$.

This also provides a local limit theorem for the Green kernel g^ω, defined for $\omega \in (\mathbb{R}_+^*)^{E_d}$ and $x, y \in \mathbb{Z}^d$ by

$$g^\omega(x, y) = \int_0^\infty q^\omega(t, x, y) dt.$$

This result was also mentioned in [1], we give here the details for the proof of a slightly stronger result[1], that insures the uniform convergence for x in an annulus.

Theorem 5.3 (Variant of Theorem 1.14 in [1])**.** *For $d \geq 3$, under the assumptions of Theorem 5.1, we have $\mathbb{P}$-a.s.*

$$\lim_{n \to \infty} \sup_{1 \leq |x| \leq 2} |n^{d-2} g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)| = 0,$$

---

[1] I would like to thank Sebastian Andres for his help regarding the details of this proof.

where g_{BM} is the Green kernel associated with the Brownian motion with covariance matrix Σ^2, i.e.

$$g_{BM}(0, x) = \int_0^\infty k_t(x)dt = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2} \det(\Sigma^2)^{1/2}} (x^t (\Sigma^2)^{-1} x)^{1-d/2}.$$

Proof. This result is obtained by integrating in Theorem 5.1. Moreover, we will need the following bounds on q^ω, which are true almost surely.

Firstly, Theorem 1.6 in [2] gives a short-range bound, which also applies to k_t: $\mathbb{P}$-a.s. there are constants $C, c_1, c_2 > 0$ such that for $t \geq Cn|x|$,

$$q^\omega(t, 0, \lfloor nx \rfloor) \leq c_1 t^{-d/2} e^{-\frac{c_2(n|x|)^2}{t}},$$

and for all $t \geq 0$,

$$k_t(x) \leq c_1 t^{-d/2} e^{-\frac{c_2|x|^2}{t}}.$$

Now, for a long-range bound: according to the proof of Theorem 10 from [7], for any $\lambda > 0$, and any $t \geq 0$, $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, we have

$$q^\omega(t, 0, \lfloor nx \rfloor) \leq \frac{1}{\sqrt{\pi_0^\omega \pi_{\lfloor nx \rfloor}^\omega}} \exp\left(d_{\mathcal{G}}(0, \lfloor nx \rfloor) \left(-\lambda + (\cosh(\lambda) - 1) \frac{t}{d_{\mathcal{G}}(0, \lfloor nx \rfloor)} \right) \right),$$

where $d_{\mathcal{G}}$ denotes the graph distance on $\mathbb{Z}^d$. Therefore, for $|x| \geq 1$ and $t \leq 2Cn$, and for n large enough so that $d_{\mathcal{G}}(0, \lfloor nx \rfloor) \geq n/2$, we have

$$\frac{t}{d_{\mathcal{G}}(0, \lfloor nx \rfloor)} \leq \frac{2Cn}{n/2} = 4C.$$

As a result, there exists $\lambda_0 > 0$ small enough so that for such x, n and t,

$$-\lambda_0 + (\cosh(\lambda_0) - 1) \frac{t}{d_{\mathcal{G}}(0, \lfloor nx \rfloor)} \leq -\lambda_0 + 4C(\cosh(\lambda_0) - 1) < 0.$$

Hence there are constants $c_3 > 0$ and $N_0 \in \mathbb{N}$ such that for $|x| \geq 1$, $n \geq N_0$ and $t \leq 2Cn$, we have

$$q^\omega(t, 0, \lfloor nx \rfloor) \leq \frac{1}{\sqrt{\pi_0^\omega \pi_{\lfloor nx \rfloor}^\omega}} e^{-c_3 n|x|}.$$

Note that the integrability assumption implies that $\mathbb{E}[\rho_0^\omega] < \infty$, where $\rho_i^\omega = \sum_{l \sim i} \frac{1}{\omega_{i,l}}$. Therefore, for $|x| \leq 2$,

$$\frac{1}{\pi_{\lfloor nx \rfloor}^\omega} \leq \rho_{\lfloor nx \rfloor}^\omega \leq \sum_{|y| \leq 2n} \rho_y^\omega, \text{ as well as } \frac{1}{\pi_0^\omega} \leq \sum_{|y| \leq 2n} \rho_y^\omega,$$

and thanks to the ergodic theorem, there exist $\mathbb{P}$-a.s. constants $c_4 > 0$ and $N_1 \geq N_0$, such that for $n \geq N_1$,

$$\sum_{|y| \leq 2n} \rho_y^\omega \leq c_4 (2n)^d \mathbb{E}[\rho_0^\omega].$$

This finally yields the following bound: for $1 \leq |x| \leq 2$, $n \geq N_1$ and $t \geq 2Cn$, we have $\mathbb{P}$-a.s.

$$q^\omega(t, 0, \lfloor nx \rfloor) \leq c_5 n^d e^{-c_3 n|x|}.$$

Using these bounds, we can now show that for $n \geq N_1$ and $1 \leq |x| \leq 2$, we have $\mathbb{P}$-a.s.

$$|n^{d-2}g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)| = \left| n^d \int_0^\infty q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt - a \int_0^\infty k_t(x) dt \right|$$

$$\leq n^d \int_0^{2C/n} q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt + n^d \int_{2C/n}^{T_1} q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt + a \int_0^{T_1} k_t(x) dt$$

$$+ \int_{T_1}^{T_2} |n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)| dt + n^d \int_{T_2}^\infty q^\omega(n^2 t, 0, \lfloor nx \rfloor) dt + a \int_{T_2}^\infty k_t(x) dt$$

$$\leq C' n^{2d-1} e^{-c_3 n} + (1+a) \int_0^{T_1} c_1 t^{-d/2} e^{-c_2/t} dt + (1+a) \int_{T_2}^\infty c_1 t^{-d/2} e^{-c_2/t} dt$$

$$+ (T_2 - T_1) \sup_{|x| \leq 2} \sup_{t \in [T_1, T_2]} |n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)|.$$

Let $\varepsilon > 0$. Since $t \mapsto c_1 t^{-d/2} e^{-c_2/t}$ is integrable on $(0, \infty)$, we can fix $T_1, T_2 > 0$ independently of x such that

$$\int_0^{T_1} c_1 t^{-d/2} e^{-c_2/t} dt + \int_{T_2}^\infty c_1 t^{-d/2} e^{-c_2/t} dt < \frac{\varepsilon}{2(1+a)}.$$

Then

$$\sup_{1 \leq |x| \leq 2} |n^{d-2} g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)|$$

$$\leq (T_2 - T_1) \sup_{|x| \leq 2} \sup_{t \in [T_1, T_2]} |n^d q^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)| + C' n^{2d-1} e^{-c_3 n} + \frac{\varepsilon}{2},$$

so that from Theorem 5.1, there exists $N \geq N_1$ independent of x such that for $n \geq N$,

$$\sup_{1 \leq |x| \leq 2} |n^{d-2} g^\omega(0, \lfloor nx \rfloor) - a g_{BM}(0, x)| \leq \varepsilon,$$

which is true $\mathbb{P}$-almost surely. $\qquad\square$

Remark 5.4. Let us fix conductances $\omega \in (\mathbb{R}_+^*)^{E_d}$. We denote by $(\tilde{Z}_n)_{n \in \mathbb{N}}$ the discrete version of $(Z_t)_{t \geq 0}$. Then, for $x, y \in \mathbb{Z}^d$,

$$g^\omega(x, y) = \int_0^\infty \frac{P_x^\omega[Z_t = y]}{\pi_y^\omega} dt = \frac{1}{\pi_y^\omega} E_x^\omega \left[\int_0^\infty \mathbb{1}_{\{Z_t = y\}} dt \right]$$

$$= \frac{1}{\pi_y^\omega} E_x^\omega \left[\sum_{n=0}^\infty \mathbb{1}_{\{\tilde{Z}_n = y\}} \right] = \frac{1}{\pi_y^\omega} \sum_{n=0}^\infty P_x^\omega[\tilde{Z}_n = y],$$

where $\sum_{n=0}^\infty P_x^\omega[\tilde{Z}_n = \cdot]$ is the Green kernel associated with $(\tilde{Z}_n)_{n \in \mathbb{N}}$ under P_x^ω. Indeed, since under P_x^ω the holding time of Z at each point is an exponential variable of parameter 1, the expected time spent by $(Z_t)_{t \geq 0}$ at y is exactly the expected number of visits of y by $(\tilde{Z}_n)_{n \in \mathbb{N}}$.

5.3 Martin boundary associated with Δ^ψ

We return to the VRJP on $\mathbb{Z}^d$, $d \geq 3$, with constant initial conductances W large enough so that the VRJP is almost surely transient. From Proposition 2.11, under $\nu_V^W(d\beta)$, we then have a.s. $\psi(i) > 0$ for all $i \in V$. Moreover, the random conductance model associated with conductances $c_{i,j}^\psi = W_{i,j} \psi(i) \psi(j)$ defines almost surely a transient

random walk. We still denote by Δ^ψ the discrete Laplacian, and define $\pi_i^\psi = \sum_{j \sim i} c_{i,j}^\psi = 2\beta_i \psi(i)^2$, as well as g^ψ the corresponding Green kernel:

$$g^\psi(x,y) = \sum_{k=0}^{\infty} \mathbb{P}_x^\psi[X_k = y] = \frac{\psi(j)}{\psi(i)} 2\beta_j \hat{G}(i,j).$$

We want to identify the Martin boundary $\mathcal{M}^\psi$ associated with Δ^ψ, by studying the behavior at infinity of the Martin kernel K^ψ, defined by

$$K^\psi(x,y) = \frac{g^\psi(x,y)}{g^\psi(0,y)}$$

for all $x, y \in \mathbb{Z}^d$. In order to do this, we will use Theorem 5.3.

Proposition 5.5. *There exists $\overline{W} > 0$ such that for $W > \overline{W}$, under $\nu_V^W(d\beta)$, the Martin boundary $\mathcal{M}^\psi$ is almost surely trivial.*

Proof. Note that under $\nu_V^W(d\beta)$, the distribution of the random conductances $c_{i,j}^\psi$ is stationary and ergodic with respect to all isometries of $\mathbb{Z}^d$, thanks to Proposition 3 of [23]. Moreover, for W large enough, the integrability assumption of Theorem 5.3 will be verified.

Lemma 5.6. *Consider the graph $\mathcal{G} = (V = \mathbb{Z}^d, E = E_d)$, with $d \geq 3$, with constant initial conductances W. Then for all $p \geq 1$, there exists $W_p > 0$ such that for $W > W_p$, the VRJP on $(\mathcal{G}, W)$ is transient a.s., and for all $i \sim j$, under $\nu_V^W(d\beta)$ we have*

$$\mathbb{E}[(\psi(i)\psi(j))^p] < \infty \text{ and } \mathbb{E}[(\psi(i)\psi(j))^{-p}] < \infty.$$

Proof. The proof is the same as for Lemma 7(i) in [23], which states the above result in the case $p = 1$. It uses a delocalization theorem for the supersymmetric hyperbolic sigma model, from [12]. Let us give an outline of the proof.

For $n \in \mathbb{N}$, define $V_n = [\![-n, n]\!]^d$, and let $\mathcal{G}^{(n)}$ be the restriction of V to V_n with *wired* boundary conditions, *i.e.* $\mathcal{G}^{(n)} = (V_n \cup \{\delta_n\}, \tilde{E}^{(n)})$ (see section 2.1 for more details). Recall that for $i \in V_n$, $\psi(i)$ was defined in Theorem 2.6 as the limit of $\psi^{(n)}(i) = \frac{G^{(n)}(\delta_n, i)}{G^{(n)}(\delta_n, \delta_n)}$, where $G^{(n)}$ is the Green function associated with a $\beta^{(n)}$ potential on the graph $(\mathcal{G}^{(n)}, \tilde{W}^{(n)})$. Then $\psi^{(n)}(i) = e^{u_{\delta_n}^{(n)}(i)}$, where $u_{\delta_n}^{(n)}$ has the same distribution as in the supersymmetric hyperbolic sigma model on $(\mathcal{G}^{(n)}, \tilde{W}^{(n)})$, rooted at δ_n.

This model was studied in [11] and [12] by Disertori, Spencer and Zirnbauer. In particular, Theorem 1 of [12] is a delocalization result for the model, and states that the fluctuations of $u_0^{(n)}$ (rooted at 0) are uniformly bounded in n: for any $m > 0$, for all $W \geq m^8$, for all $n \in \mathbb{N}$ and $x, y \in V_n$,

$$\mathbb{E}\left[\cosh^m\left(u_0^{(n)}(x) - u_0^{(n)}(y)\right)\right] \leq 2.$$

In the proof of Lemma 7 (i) in [23], Sabot and Zeng showed that this is still true when rooting the model at δ_n, and that this implies the existence of $W_1 > 0$ such that for $W \geq W_1$, we have

$$\mathbb{E}[\psi(i)\psi(j)] < \infty \text{ and } \mathbb{E}[(\psi(i)\psi(j))^{-1}] < \infty.$$

Since the delocalization result is true for arbitrarily large m, the same proof can be adapted to show Lemma 5.6. □

We define $\overline{W} = W_{d+1}$. From now on, we assume that $W > \overline{W}$, so that thanks to Lemma 5.6, under $\nu_V^W(d\beta)$, for all $i \sim j$ we have

$$\mathbb{E}[(c_{i,j}^\psi)^{d+1}] < \infty \text{ and } \mathbb{E}[(c_{i,j}^\psi)^{-(d+1)}] < \infty.$$

Therefore, according to Theorem 5.3 and Remarks 5.2 and 5.4, there exists $\sigma^2 > 0$ such that ν_V^W-almost surely,

$$\sup_{1 \leq |z| \leq 2} |n^{d-2}(\pi_{\lfloor nz \rfloor}^\psi)^{-1} g^\psi(0, \lfloor nz \rfloor) - a g_{BM}(0, z)| \xrightarrow[n \to \infty]{} 0,$$

where $a = 1/\mathbb{E}[\pi_0^\psi]$ and g_{BM} is the Green kernel associated with a Brownian motion with covariance matrix $\sigma^2 I_d$, i.e.

$$g_{BM}(0, z) = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}\sigma^2} |z|^{2-d}.$$

Using this result, we have ν_V^W-almost surely: $g^\psi(0, y_n) \sim_{n \to \infty} a\pi_{y_n}^\psi g_{BM}(0, y_n)$, for any sequence $(y_n)_{n \geq 1}$ such that $|y_n| \to \infty$. Indeed, for such a sequence (y_n), let us define $m_n = \lfloor |y_n| \rfloor$ and $z_n = y_n/m_n$. Then, since $1 \leq |z_n| \leq 2$ for all $n \geq 1$, we have

$$\left| \frac{g^\psi(0, y_n)}{a\pi_{y_n}^\psi g_{BM}(0, y_n)} - 1 \right| = \left| \frac{(\pi_{\lfloor m_n z_n \rfloor}^\psi)^{-1} g^\psi(0, m_n z_n)}{m_n^{2-d} a g_{BM}(0, z_n)} - 1 \right|$$

$$= \left| \frac{m_n^{d-2}(\pi_{\lfloor m_n z_n \rfloor}^\psi)^{-1} g^\psi(0, \lfloor m_n z_n \rfloor) - a g_{BM}(0, z_n)}{a g_{BM}(0, z_n)} \right|$$

$$\leq \frac{\sup_{1 \leq |z| \leq 2} |m_n^{d-2}(\pi_{\lfloor m_n z \rfloor}^\psi)^{-1} g^\psi(0, \lfloor m_n z \rfloor) - a g_{BM}(0, z)|}{a \inf_{1 \leq |z| \leq 2} g_{BM}(0, z)} \xrightarrow[n \to \infty]{} 0$$

ν_V^W-almost surely.

Moreover, for $x \in \mathbb{Z}^d$ fixed, let ψ^x be the translated function defined by $\psi^x(y) = \psi(y - x)$. Then ψ^x and ψ have the same distribution under $\nu_V^W(d\beta)$, therefore we have ν_V^W-a.s., for all $(y_n)_{n \geq 1}$ such that $|y_n| \to \infty$,

$$g^\psi(x, y_n) = g^{\psi^x}(0, y_n - x) \sim_{n \to \infty} a\pi_{y_n}^\psi g_{BM}(0, y_n - x),$$

since $|y_n - x| \to \infty$ and $\pi_{y_n - x}^{\psi^x} = \pi_{y_n}^\psi$. Let us denote by A_x the ν_V^W-almost sure event where this is true. Since $\mathbb{Z}^d$ is denumerable, $\bigcap_{x \in \mathbb{Z}^d} A_x$ is still ν_V^W-almost sure. Therefore, we have ν_V^W-a.s. that for all $x \in \mathbb{Z}^d$, for all $(y_n)_{n \geq 1}$ such that $|y_n| \to \infty$,

$$K^\psi(x, y_n) \sim_{n \to \infty} \frac{a\pi_{y_n}^\psi g_{BM}(0, y_n - x)}{a\pi_{y_n}^\psi g_{BM}(0, y_n)} = \frac{|y_n - x|^{2-d}}{|y_n|^{2-d}} \xrightarrow[n \to \infty]{} 1.$$

As a result, from Remark 3.8, the Martin boundary associated with Δ^ψ is ν_V^W-a.s. trivial. $\square$

Let $\mathcal{R}(dr)$ be the distribution of an environment representing $\mathbb{P}^{VRJP(0)}$ on $\mathbb{Z}^d$ endowed with constant initial conductances $W > \overline{W}$.

For $r \in \mathcal{J}_V^E$ and $\gamma > 0$, we define β by $\beta_i = r_i + \mathbb{1}_{\{i=0\}}\gamma$. According to Theorem 2.9, under $\mathcal{R}(dr, d\gamma)$ we then have $\beta \sim \nu_V^W$. We define $\hat{G}$ and ψ as functions of β, as in Theorem 2.6, and we can write

$$r_{i,j} = \frac{W_{i,j}}{2} \frac{G(0, j)}{G(0, i)},$$

where $G(0, i) = \hat{G}(0, i) + h(i)$ for all $i \in \mathbb{Z}^d$, with h a H_β-harmonic function. Since W is large enough so that under $\nu_V^W(d\beta)$, $\psi(i) > 0$ for all $i \in \mathbb{Z}^d$ almost surely, the operator Δ^ψ is well-defined, and h/ψ is Δ^ψ-harmonic. However, according to Proposition 5.5 the Martin boundary associated with Δ^ψ is ν_V^W-a.s. trivial, therefore positive Δ^ψ-harmonic

functions are almost surely constant. As a result, there is a nonnegative random variable g such that for all $i \in \mathbb{Z}^d$, we have $\mathcal{R}$-a.s.

$$G(0,i) = \hat{G}(0,i) + g\psi(i).$$

In particular, $g = (G(0,0) - \hat{G}(0,0))/\psi(0)$, so g can be written as a function of $(\beta, \frac{1}{2G(0,0)})$, and therefore has a function of $((r_i)_{i \in \mathbb{Z}^d}, \gamma)$. Since according to Proposition 4.1, under $\mathcal{R}(dr, d\gamma)$ the distribution of $((r_i)_{i \in \mathbb{Z}^d}, \gamma)$ does not depend on the chosen representation $\mathcal{R}$, this shows that the distribution of the jump rates $r_{i,j} = \frac{W_{i,j}}{2} \frac{G(0,j)}{G(0,i)}$ is uniquely determined, *i.e.* that the representation is unique.

Remark 5.7. Note that we can identify the distribution of g using the standard representation. This shows that under $\mathcal{R}(dr), d\gamma$, we have $g = \psi(0)/2\gamma'$, where γ' is a Gamma$(1/2, 1)$ random variable independent from $(\beta_i)_{i \in \mathbb{Z}^d}$.

6 Construction of new representations on trees

Let now $\mathcal{T} = (T, E)$ be an infinite tree, that is locally finite. We fix an arbitrary root ϕ, and endow the edges of $\mathcal{T}$ with positive weights $(W_e)_{e \in E}$. The aim of this section is to prove points (i) and (ii) of Theorem 2.18, which introduces a family of new representations for the VRJP on infinite trees.

We will proceed as in the construction of the standard representation, by considering the VRJP and the associated β potential on finite restrictions of the tree, with the new boundary conditions introduced in Section 2.3.2. Theorem 2.2 provides an expression for the representation on these finite graphs $\mathcal{G}_m^{(n)}$, and we will show that the associated jump rates converge to a representation on the infinite graph.

6.1 Representation of the VRJP on $\mathcal{G}_m^{(n)}$

Fix $m \in \mathbb{N}$, and recall the definition of $\mathcal{G}_m^{(n)}$: to each vertex $x \in D^{(m)}$, we associate a boundary point δ_x, and define the boundary $B_m = \{\delta_x, x \in D^{(m)}\}$; then for all $n \geq m$, we define

$$\mathcal{G}_m^{(n)} = \left(\tilde{T}_m^{(n)}, \tilde{E}_m^{(n)} \right), \text{ where } \tilde{T}_m^{(n)} = T^{(n)} \cup B_m$$

$$\text{and } \tilde{E}_m^{(n)} = E^{(n)} \cup \bigcup_{x \in D^{(m)}} \left\{ \{y, \delta_x\}, y \in T_x \cap D^{(n)} \right\}.$$

We endow this graph with edge weights $\tilde{W}_m^{(n)}$, where

$$\left(\tilde{W}_m^{(n)} \right)_e = \begin{cases} W_e^{(n)} = W_e & \text{for } e \in E^{(n)} \\ \sum_{j \in S(i)} W_{i,j} & \text{for } e = \{i, \delta_x\}, \text{ where } i \in T_x \cap D^{(n)}. \end{cases}$$

According to Theorem 2.2, the mixing measure for the VRJP on $(\mathcal{G}_m^{(n)}, \tilde{W}_m^{(n)})$ can be expressed using a potential $\beta_m^{(n)}$, with distribution $\nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$. As with the standard representation, we will construct a coupling of such potentials for all $n \in \mathbb{N}$, using the potential defined on the whole infinite tree with distribution ν_V^W.

For $\beta \in \mathcal{D}_T^W$, we still define $H_\beta = 2\beta - W$. For $n \in \mathbb{N}$, let us take $V_n = T^{(n)}$ in Definition 2.5, so that we get $\hat{G}^{(n)} = \hat{G}^{T^{(n)}} = ((H_\beta)_{T^{(n)}, T^{(n)}})^{-1}$ and $\psi^{(n)} = \hat{G}^{(n)} \eta^{(n)}$.

Lemma 6.1. Fix $n \geq m$. For $\beta \in \mathcal{D}_T^W$, and $\beta' \in (\mathbb{R}_+^*)^{B_m}$, we define the $|B_m| \times |B_m|$ matrix

$$\check{W}_m^{(n)} = (\tilde{W}_m^{(n)})_{B_m, T^{(n)}} \, \hat{G}^{(n)} \, (\tilde{W}_m^{(n)})_{T^{(n)}, B_m},$$

as well as the potential $\beta_m^{(n)} \in (\mathbb{R}_+^*)^{\tilde{T}_m^{(n)}}$ by $(\beta_m^{(n)})_{T^{(n)}} = \beta_{T^{(n)}}$, and $(\beta_m^{(n)})_{B_m} = \beta'$.

Then under $\nu_T^W(d\beta)\nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$, the potential $\beta_m^{(n)}$ is distributed according to $\nu_{\tilde{T}_m^{(n)}}^{\tilde{W}_m^{(n)}}$.

Proof. According to Proposition 3.2, we have $\beta_m^{(n)} \sim \nu_{\check{T}_m^{(n)}}^{\check{W}_m^{(n)}}$ if and only if: the restriction $(\beta_m^{(n)})_{T^{(n)}}$ is distributed according to $\nu_{T^{(n)}}^{\hat{W}_m^{(n)},\hat{\eta}_m^{(n)}}$, and conditionally on $(\beta_m^{(n)})_{T^{(n)}}$, the restriction $(\beta_m^{(n)})_{B_m}$ is distributed according to $\nu_{B_m}^{\check{W}_m^{(n)}}$, with the following notations:

$$\hat{W}_m^{(n)} = (\tilde{W}_m^{(n)})_{T^{(n)},T^{(n)}}, \hat{\eta}_m^{(n)} = (\tilde{W}_m^{(n)})_{T^{(n)},B_m}\mathbb{1}_{B_m}$$
$$\text{and } \check{W}_m^{(n)} = (\tilde{W}_m^{(n)})_{B_m,B_m} + (\tilde{W}_m^{(n)})_{B_m,T^{(n)}}((H_\beta^{(n)})_{T^{(n)},T^{(n)}})^{-1}(\tilde{W}_m^{(n)})_{T^{(n)},B_m},$$

where we define $H_\beta^{(n)} = 2\beta_m^{(n)} - \tilde{W}_m^{(n)}$ as the Schrödinger operator associated with $\beta_m^{(n)}$.

Let us first simplify the expressions of these parameters. Given the definition of $\tilde{W}_m^{(n)}$, note that we have $\hat{W}_m^{(n)} = (\tilde{W}_m^{(n)})_{T^{(n)},T^{(n)}} = W^{(n)}$. Moreover, for $i \in T^{(n)}$,

$$(\hat{\eta}_m^{(n)})_i = \sum_{b \in B}(\tilde{W}_m^{(n)})_{i,b} = \mathbb{1}_{i \in D^{(n)}}\sum_{j \in S(i)}W_{i,j} = W_{\{i\},D^{(n+1)}}\mathbb{1}_{D^{(n+1)}} = \eta_i^{(n)}.$$

Finally, $\check{W}_m^{(n)}$ has the same expression as in the statement of the lemma: indeed, $(\tilde{W}_m^{(n)})_{B_m,B_m} = 0$ and

$$(H_\beta^{(n)})_{T^{(n)},T^{(n)}} = 2(\beta_m^{(n)})_{T^{(n)}} - (\tilde{W}_m^{(n)})_{T^{(n)},T^{(n)}} = 2\beta_{T^{(n)}} - W^{(n)} = (H_\beta)_{T^{(n)},T^{(n)}},$$

therefore we have

$$\check{W}_m^{(n)} = (\tilde{W}_m^{(n)})_{B_m,B_m} + (\tilde{W}_m^{(n)})_{B_m,T^{(n)}}((H_\beta^{(n)})_{T^{(n)},T^{(n)}})^{-1}(\tilde{W}_m^{(n)})_{T^{(n)},B_m}$$
$$= (\tilde{W}_m^{(n)})_{B_m,T^{(n)}}\hat{G}^{(n)}(\tilde{W}_m^{(n)})_{T^{(n)},B_m}.$$

Let us now show that under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, $\beta_m^{(n)}$ has the right distribution. From Proposition 2.4, under $\nu_T^W(d\beta)$, we have $\beta_{T^{(n)}} \sim \nu_{T^{(n)}}^{W^{(n)},\eta^{(n)}}$, *i.e.* $(\beta_m^{(n)})_{T^{(n)}} \sim \nu_{T^{(n)}}^{\hat{W}_m^{(n)},\hat{\eta}_m^{(n)}}$. Moreover, conditionally on $(\beta_m^{(n)})_{T^{(n)}}$, we have $\beta' \sim \nu_{B_m}^{\check{W}_m^{(n)}}$, *i.e.* $(\beta_m^{(n)})_{B_m} \sim \nu_{B_m}^{\check{W}_m^{(n)}}$, which concludes the proof. □

From now on, we consider $\beta \in \mathcal{D}_T^W$ and $\beta' \in (\mathbb{R}_+^*)^{B_m}$, as well as the potential $\beta_m^{(n)}$ defined as in Lemma 6.1 for $n \geq m$. Let us denote by $G_m^{(n)} = (H_\beta^{(n)})^{-1} = (2\beta_m^{(n)} - \tilde{W}_m^{(n)})^{-1}$ the Green function associated with $\beta_m^{(n)}$ on $\mathcal{G}_m^{(n)}$. Then, from Theorem 2.2, we know that the law of the time-changed VRJP on $(\mathcal{G}_m^{(n)}, \tilde{W}_m^{(n)})$, started at $i_0 \in T^{(n)}$, is a mixture of Markov jump processes under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, where the jump rate from i to j is $\frac{1}{2}(\tilde{W}_m^{(n)})_{i,j}\frac{G_m^{(n)}(i_0,j)}{G_m^{(n)}(i_0,i)}$.

In order to obtain a representation on the infinite tree $\mathcal{T}$, we will need to show that the Green function $(G_m^{(n)}(i,j))_{i,j \in T}$ converges in distribution when $n \to \infty$. To show this convergence, we give another expression of $G_m^{(n)}$, involving $\hat{G}^{(n)}$ and the function $\chi_m^{(n)}$ introduced in Definition 2.16.

Lemma 6.2. *For $i,j \in T$, let $n_0 \geq m$ be such that $i,j \in T^{(n_0)}$. Then for $n \geq n_0$,*

$$G_m^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + \sum_{b,b' \in B_m}\chi_m^{(n)}(i,b)G_m^{(n)}(b,b')\chi_m^{(n)}(j,b').$$

Proof. We show this result by expressing $G_m^{(n)}$ as a sum over paths in $\mathcal{T}$. We will use notations and results presented in Section 3.2.

For $n \geq n_0$, by applying Proposition 3.4 (i) to $\beta_m^{(n)}$ in the graph $\mathcal{G}_m^{(n)}$, we get

$$G_m^{(n)}(i,j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{\tilde{T}_m^{(n)}}}\frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma}.$$

This sum over paths can be decomposed as follows: a path $\sigma \in \mathcal{P}_{i,j}^{\tilde{T}_m^{(n)}}$ can either hit some vertex in B_m, in which case $\sigma \in \mathcal{P}_{i,B_m,j}^{\tilde{T}_m^{(n)}}$, or never hit any vertex in B_m, in which case $\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}$. As a result, we have $\mathcal{P}_{i,j}^{\tilde{T}_m^{(n)}} = \mathcal{P}_{i,j}^{T^{(n)}} \cup \mathcal{P}_{i,B_m,j}^{\tilde{T}_m^{(n)}}$, where $\mathcal{P}_{i,j}^{T^{(n)}} \cap \mathcal{P}_{i,B,j}^{\tilde{T}_m^{(n)}} = \emptyset$, so

$$
G_m^{(n)}(i,j) = \sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} + \sum_{\sigma \in \mathcal{P}_{i,B_m,j}^{\tilde{T}_m^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma}
$$

$$
= \sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} + \sum_{b,b' \in B_m} \left(\sum_{\sigma \in \overline{\mathcal{P}}_{i,b}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma^-} \right) G_m^{(n)}(b,b') \left(\sum_{\sigma \in \overline{\mathcal{P}}_{j,b'}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma^-} \right),
$$

from Proposition 3.4. Note that for any $\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}$, $\frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} = \frac{W_\sigma}{(2\beta)_\sigma}$, since $(\tilde{W}_m^{(n)})_{T^{(n)},T^{(n)}} = W_{T^{(n)},T^{(n)}}$ and $(\beta_m^{(n)})_{T^{(n)}} = \beta_{T^{(n)}}$. As a result,

$$
\sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} = \sum_{\sigma \in \mathcal{P}_{i,j}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} = \hat{G}^{(n)}(i,j).
$$

Moreover, note that for $x \in D^{(m)}$ and $y \in T^{(n)}$,

$$
\sum_{\sigma \in \overline{\mathcal{P}}_{y,\delta_x}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma^-} = \sum_{z \sim \delta_x} \left(\sum_{\sigma_1 \in \mathcal{P}_{y,z}^{T^{(n)}}} \frac{(\tilde{W}_m^{(n)})_{\sigma_1}}{(2\beta_m^{(n)})_{\sigma_1}} \right) (\tilde{W}_m^{(n)})_{z,\delta_x}
$$

$$
= \sum_{z \in T_x \cap D^{(n)}} \hat{G}^{(n)}(y,z) \sum_{z' \in S(z)} W_{z,z'}
$$

$$
= \hat{G}^{(n)}(y,\cdot) W_{T^{(n)},(T^{(n)})^c} \mathbb{1}_{T_x \setminus T_x^{(n)}} = \chi_m^{(n)}(y,\delta_x),
$$

from Definition 2.16. As a result, we have

$$
G_m^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + \sum_{b,b' \in B_m} \chi_m^{(n)}(i,b) G_m^{(n)}(b,b') \chi_m^{(n)}(j,b'). \qquad \square
$$

We will show that the distribution of $G_m^{(n)}$ under $\nu_T^W(d\beta) \nu_{B_m}^{\tilde{W}_m^{(n)}}(d\beta')$ converges when $n \to \infty$. From Theorem 2.6 (i), we already know that $\hat{G}^{(n)}(i,j)$ converges a.s. to $\hat{G}(i,j)$, let us now study the respective limits of $\chi_m^{(n)}$ and $(G_m^{(n)})_{B_m,B_m}$.

6.2 Convergence of $\chi_m^{(n)}$: proof of Theorem 2.18 (i)

Let us still consider a fixed generation $m \in \mathbb{N}$. We will show that $\chi_m^{(n)}(i,\delta_x)$ converges a.s. for all $x \in D^{(m)}$ and $i \in T$. Moreover, the limit has a simple expression in terms of the harmonic measures associated with the random Markov operator Δ^ψ. We will first describe the Martin boundary associated with Δ^ψ, and the harmonic measures $(\mu_i^\psi)_{i \in T}$.

First, let us fix $\beta \in \mathcal{D}_T^W$, and consider the function ψ defined in Theorem 2.6. We either have $\psi(i) > 0$ for all $i \in T$, or $\psi \equiv 0$. In the first case, we can define the conductances $(c_{i,j}^\psi)_{i \sim j}$ as in Proposition 2.11, as well as the corresponding Markov operator Δ^ψ. Recall that a function h is Δ^ψ-harmonic if and only if ψh is H_β-harmonic. The associated random walk is transient, since the associated Green function $g = g^\psi$ is given by

$$
g^\psi(i,j) = \frac{\psi(j)}{\psi(i)} 2\beta_j \hat{G}(i,j),
$$

for $i,j \in T$. This allows us to apply results regarding the Martin boundary of a tree.

From Theorem 3.12, the Martin boundary $\mathcal{M}^\psi$ associated with Δ^ψ is the set Ω of ends of T. Note that it does not depend on β. We also get the Martin kernel $K = K^\psi$: for $x \in T$ and $\omega \in \Omega$,

$$K(x, \omega) = K(x, x \wedge \omega) = \frac{f(x, x \wedge \omega)}{f(\phi, x \wedge \omega)} = \frac{\psi(\phi)\hat{F}(x, x \wedge \omega)}{\psi(x)\hat{F}(\phi, x \wedge \omega)},$$

where $f(i, j) = \frac{g(i,j)}{g(j,j)} = \frac{\psi(j)}{\psi(i)}\hat{F}(i, j)$ for $i, j \in T$. Moreover, we denote by $(\mu_i^\psi)_{i \in T}$ the associated family of harmonic measures on Ω. From Proposition 3.13, we have, for $i, x \in T$,

$$\mu_i^\psi(\Omega_x) = \mathbb{1}_{\{i \in U_x\}}(1 - f(i, x)) + f(i, x)\frac{1 - f(x, \bar{x})}{1 - f(x, \bar{x})f(\bar{x}, x)}.$$

Note that we have only defined $(\mu_y^\psi)_{y \in T}$ for $\beta \in \mathcal{D}_T^W$ such that $\psi > 0$. In the other case, we adopt the convention that μ_y^ψ is the null measure on Ω for all $y \in T$.

Let us now show that almost surely, for all $x \in D^{(m)}$ and $i \in T$, $\chi_m^{(n)}(i, \delta_x)$ converges to $\psi(i)\mu_i^\psi(\Omega_x)$.

Proof of Theorem 2.18 (i). From Theorem 2.6, we know that $\nu_T^W(d\beta)$-almost surely, for all $i, j \in T$, $\hat{G}^{(n)}(i, j)$ converges to $\hat{G}(i, j)$ and $\psi^{(n)}(i)$ converges to $\psi(i)$. Let $\beta \in \mathcal{D}_T^W$ be such that these convergences hold. Let us show that for such β, and for all $x \in D^{(m)}$ and $i \in T$, $\chi_m^{(n)}(i, \delta_x)$ converges to $\psi(i)\mu_i^\psi(\Omega_x)$, and we will have shown that this convergence holds ν_T^W-almost surely.

If β is such that $\psi \equiv 0$, we know that for all $i \in T$, $x \in D^{(m)}$, $0 \leq \chi_m^{(n)}(i, \delta_x) \leq \psi^{(n)}(i)$ from Remark 2.17, so $\chi_m^{(n)}(i, \delta_x) \xrightarrow[n \to \infty]{} 0 = \psi(i)\mu_i^\psi(\Omega_x)$. We now assume that β is such that $\psi(i) > 0$ for all $i \in T$.

Let us fix $i \in T$ and $x \in D^{(m)}$. Recall that for $n \geq \max(|i|, m)$,

$$\chi_m^{(n)}(i, \delta_x) = \sum_{y \in T_x \cap D^{(n)}} \hat{G}^{(n)}(i, y)\eta_y^{(n)} = \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{i,y}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma}\right)\eta_y^{(n)}.$$

Let us decompose the paths $\sigma \in \mathcal{P}_{i,y}^{T^{(n)}}$, in order to write $\chi_m^{(n)}(i, \delta_x)$ as a function of $\hat{F}^{(n)}$ and $\psi^{(n)}$. We will distinguish two cases.

The first case is when $i \notin U_x = T_x \backslash \{x\}$. Then for all $y \in T_x \cap D^{(n)}$, any path from i to y in $T^{(n)}$ necessarily visits x, i.e. $\mathcal{P}_{i,y}^{T^{(n)}} = \mathcal{P}_{i,\{x\},y}^{T^{(n)}}$. Therefore, from Proposition 3.4 (iii), $\hat{G}^{(n)}(i, y) = \hat{F}^{(n)}(i, x)\hat{G}^{(n)}(x, y)$, and

$$\chi_m^{(n)}(i, \delta_x) = \hat{F}^{(n)}(i, x) \sum_{y \in T_x \cap D^{(n)}} \hat{G}^{(n)}(x, y)\eta_y^{(n)}.$$

Let us express $\hat{G}^{(n)}(x, y)$ in a more convenient way.

Lemma 6.3. *For $y \in T_x \cap D^{(n)}$, we have*

$$\hat{G}^{(n)}(x, y) = \frac{1}{1 - \hat{F}^{(n)}(x, \bar{x})\hat{F}^{(n)}(\bar{x}, x)} \sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma}$$

Proof. In order to decompose $\hat{G}^{(n)}(x, y)$, let us introduce $c_x(\sigma)$, defined as the number of times the path σ crosses the directed edge $(x, \bar{x})$, *i.e.*

$$c_x(\sigma) = \#\{k \in [\![0, |\sigma| - 1]\!], (\sigma_k, \sigma_{k+1}) = (x, \bar{x})\}.$$

Then we have

$$\hat{G}^{(n)}(x,y) = \sum_{C \in \mathbb{N}} \sum_{\substack{\sigma \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma) = C}} \frac{W_\sigma}{(2\beta)_\sigma}.$$

If $\sigma \in \mathcal{P}_{x,y}^{T^{(n)}}$ is such that $c_x(\sigma) = C \geq 1$, then it has to visit $\breve{x}$ at least once. As a result, σ can be written as the concatenation of a path $\sigma_1 \in \overline{\mathcal{P}}_{x,\breve{x}}^{T^{(n)} \setminus \{\breve{x}\}}$ with a path $\sigma_1' \in \mathcal{P}_{\breve{x},y}^{T^{(n)}}$ such that $c_x(\sigma_1') = C - 1$. Since $\breve{x} \notin U_x$, the path σ_1' has to visit x, so it can be written as the concatenation of a path $\sigma_2 \in \overline{\mathcal{P}}_{\breve{x},x}^{T^{(n)} \setminus \{x\}}$ with a path $\sigma_3 \in \mathcal{P}_{x,y}^{T^{(n)}}$ such that $c_x(\sigma_3) = C - 1$. Therefore, for all $C \geq 1$,

$$\sum_{\substack{\sigma \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma) = C}} \frac{W_\sigma}{(2\beta)_\sigma} = \left(\sum_{\sigma_1 \in \overline{\mathcal{P}}_{x,\breve{x}}^{T^{(n)} \setminus \{\breve{x}\}}} \frac{W_{\sigma_1}}{(2\beta)_{\sigma_1}} \right) \left(\sum_{\sigma_2 \in \overline{\mathcal{P}}_{\breve{x},x}^{T^{(n)} \setminus \{x\}}} \frac{W_{\sigma_2}}{(2\beta)_{\sigma_2}} \right) \left(\sum_{\substack{\sigma_3 \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma_3) = C-1}} \frac{W_{\sigma_3}}{(2\beta)_{\sigma_3}} \right)$$

$$= \hat{F}^{(n)}(x,\breve{x}) \hat{F}^{(n)}(\breve{x},x) \sum_{\substack{\sigma \in \mathcal{P}_{x,y}^{T^{(n)}} \\ c_x(\sigma) = C-1}} \frac{W_\sigma}{(2\beta)_\sigma}.$$

Moreover, note that the paths $\sigma \in \mathcal{P}_{x,y}^{T^{(n)}}$ such that $c_x(\sigma) = 0$ are those that stay in the subtree $T_x^{(n)}$, *i.e.* the set $\mathcal{P}_{x,y}^{T_x^{(n)}}$. By induction, we get:

$$\hat{G}^{(n)}(x,y) = \sum_{C \in \mathbb{N}} \left(\hat{F}^{(n)}(x,\breve{x}) \hat{F}^{(n)}(\breve{x},x) \right)^C \sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma}.$$

Since $\hat{G}^{(n)}(x,y) < \infty$, we have $\hat{F}^{(n)}(x,\breve{x}) \hat{F}^{(n)}(\breve{x},x) < 1$, which gives the expected result. $\square$

From Lemma 6.3, we get

$$\chi_m^{(n)}(i,\delta_x) = \hat{F}^{(n)}(i,x) \frac{1}{1 - \hat{F}^{(n)}(x,\breve{x}) \hat{F}^{(n)}(\breve{x},x)} \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}.$$

In order to express this last sum, recall that

$$\psi^{(n)}(x) = \sum_{y \in D^{(n)}} \hat{G}^{(n)}(x,y) \eta_y^{(n)}$$

$$= \sum_{y \in D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x,\{\breve{x}\},y}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} + \sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)},$$

where we have separated the paths that go from x to y by visiting $\breve{x}$, and those that stay in $T_x^{(n)}$, since $\mathcal{P}_{x,y}^{T^{(n)}} = \mathcal{P}_{x,\{\breve{x}\},y}^{T^{(n)}} \cup \mathcal{P}_{x,y}^{T_x^{(n)}}$. From Proposition 3.4 (iii), we have

$$\sum_{\sigma \in \mathcal{P}_{x,\{\breve{x}\},y}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} = \hat{F}^{(n)}(x,\breve{x}) \hat{G}^{(n)}(\breve{x},y).$$

Moreover, if $y \in D^{(n)} \backslash T_x$, then $\mathcal{P}_{x,y}^{T_x^{(n)}}$ is empty. As a result, we get

$$\sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)} = \psi^{(n)}(x) - \hat{F}^{(n)}(x, \breve{x}) \psi^{(n)}(\breve{x}),$$

which finally gives

$$\chi_m^{(n)}(i, \delta_x) = \hat{F}^{(n)}(i, x) \frac{\psi^{(n)}(x) - \hat{F}^{(n)}(x, \breve{x}) \psi^{(n)}(\breve{x})}{1 - \hat{F}^{(n)}(x, \breve{x}) \hat{F}^{(n)}(\breve{x}, x)}.$$

In the second case, *i.e.* if $i \in U_x$, then for $y \in T_x \cap D^{(n)}$, there are paths from i to y in $T^{(n)}$ that do not visit x. More precisely, we have the following partition: $\mathcal{P}_{i,y}^{T^{(n)}} = \mathcal{P}_{i,\{x\},y}^{T^{(n)}} \cup \mathcal{P}_{i,y}^{U_x^{(n)}}$, where $U_x^{(n)} = U_x \cap T^{(n)}$. As a result, we have

$$\chi_m^{(n)}(i, \delta_x) = \hat{F}^{(n)}(i, x) \frac{\psi^{(n)}(x) - \hat{F}^{(n)}(x, \breve{x}) \psi^{(n)}(\breve{x})}{1 - \hat{F}^{(n)}(x, \breve{x}) \hat{F}^{(n)}(\breve{x}, x)} + \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \cup \mathcal{P}_{i,y}^{U_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}.$$

In the same way we did above, we can show that

$$\sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \cup \mathcal{P}_{i,y}^{U_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)} = \psi^{(n)}(i) - \hat{F}^{(n)}(i, x) \psi^{(n)}(x).$$

In conclusion, we have established the following:

$$\chi_m^{(n)}(i, \delta_x) = \mathbb{1}_{\{i \in U_x\}} (\psi^{(n)}(i) - \hat{F}^{(n)}(i, x) \psi^{(n)}(x)) + \hat{F}^{(n)}(i, x) \frac{\psi^{(n)}(x) - \hat{F}^{(n)}(x, \breve{x}) \psi^{(n)}(\breve{x})}{1 - \hat{F}^{(n)}(x, \breve{x}) \hat{F}^{(n)}(\breve{x}, x)}$$

$$= \psi^{(n)}(i) \left(\mathbb{1}_{\{i \in U_x\}} (1 - f^{(n)}(i, x)) + f^{(n)}(i, x) \frac{1 - f^{(n)}(x, \breve{x})}{1 - f^{(n)}(x, \breve{x}) f^{(n)}(\breve{x}, x)} \right),$$

where

$$f^{(n)}(i, j) = \frac{\psi^{(n)}(j)}{\psi^{(n)}(i)} \hat{F}^{(n)}(i, j) \xrightarrow[n \to \infty]{} \frac{\psi(j)}{\psi(i)} \hat{F}(i, j) = f(i, j)$$

for all $i, j \in T$. As a result, we finally have

$$\chi_m^{(n)}(i, \delta_x) \xrightarrow[n \to \infty]{} \psi(i) \left(\mathbb{1}_{\{i \in U_x\}} (1 - f(i, x)) + f(i, x) \frac{1 - f(x, \breve{x})}{1 - f(x, \breve{x}) f(\breve{x}, x)} \right) = \psi(i) \mu_i^\psi(\Omega_x). \quad \square$$

Let us also define, for all $i \in T$, the measure $\chi(i, \cdot) = \psi(i) \mu_i^\psi$. Note that $\chi(i, \cdot)$ is absolutely continuous with respect to $\chi(\phi, \cdot)$, and its Radon-Nikodym derivative is $\omega \mapsto \frac{\hat{F}(i, i \wedge \omega)}{\hat{F}(\phi, i \wedge \omega)}$. Moreover, for all $A \in \mathcal{B}(\Omega)$, $i \mapsto \mu_i^\psi(A)$ is Δ^ψ-harmonic, so $\chi(\cdot, A) : i \mapsto \psi(i) \mu_i^\psi(A)$ is H_β-harmonic.

6.3 Convergence to a representation on $\mathcal{T}$: proof of Theorem 2.18 (ii)

To show that the distribution of jump rates $\frac{1}{2}(\tilde{W}_m^{(n)})_{i,j} \frac{G_m^{(n)}(i_0, j)}{G_m^{(n)}(i_0, i)}$ converges when $n \to \infty$, it remains to be shown that $(G_m^{(n)})_{B_m, B_m}$ converges in distribution. We show this convergence conditionally on β, thanks to the fact that the parameters of the distribution are functions of $\chi_m^{(n)}$ and $\hat{G}^{(n)}$, and therefore converge almost surely.

Lemma 6.4. *For $\beta \in \mathcal{D}_T^W$, let us define the $|B_m| \times |B_m|$ matrix $\check{C}_m$ such that for $x, x' \in D^{(m)}$:*

$$(\check{C}_m)_{\delta_x, \delta_{x'}} = \begin{cases} \frac{\chi(x \wedge y, \Omega_x)\chi(x \wedge y, \Omega_{x'})}{\hat{G}(x \wedge y, x \wedge y)} & \text{if } x \neq x' \\ 0 & \text{if } x = x'. \end{cases}$$

Moreover, for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, we define $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$.

Then the distribution of $(G_m^{(n)})_{B_m, B_m}$ under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$ converges weakly, when $n \to \infty$, to the distribution of $\check{G}_m$ under $\nu_T^W(d\beta)\nu_{B_m}^{\check{C}_m}(d\rho_m)$, which we will also denote as $\nu_{T,B_m}^W(d\beta, d\rho_m)$.

Proof. We can write $(G_m^{(n)})_{B_m, B_m}$ as the inverse of a Schur complement. Indeed,

$$\begin{aligned}
(G_m^{(n)})_{B_m, B_m} &= (H_\beta^{(n)})_{B_m, B_m}^{-1} \\
&= \left((H_\beta^{(n)})_{B_m, B_m} - (\tilde{W}_m^{(n)})_{B_m, T^{(n)}}((H_\beta^{(n)})_{T^{(n)}, T^{(n)}})^{-1}(\tilde{W}_m^{(n)})_{T^{(n)}, B_m}\right)^{-1} \\
&= \left((2\beta_m^{(n)})_{B_m} - \check{W}_m^{(n)}\right)^{-1} = (\check{H}_\beta^{(n)})^{-1},
\end{aligned}$$

where $\check{H}_\beta^{(n)} = 2(\beta_m^{(n)})_{B_m} - \check{W}_m^{(n)}$. We apply the following change of variables: for $\beta' \in \mathcal{D}_{B_m}^{\check{W}_m^{(n)}}$ and $b \in B_m$, let us define $(\rho_m^{(n)})_b = (\beta_m^{(n)})_b - \frac{1}{2}(\check{W}_m^{(n)})_{b,b}$. Then $\check{H}_\beta^{(n)} = 2\rho_m^{(n)} - \check{C}_m^{(n)}$, where if $b, b' \in B_m$,

$$(\check{C}_m^{(n)})_{b,b'} = \begin{cases} (\check{W}_m^{(n)})_{b,b'} & \text{if } b \neq b' \\ 0 & \text{if } b = b'. \end{cases}$$

Under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, the vector $\rho_m^{(n)}$ is then distributed according to $\nu_{B_m}^{\check{C}_m^{(n)}}$ conditionally on β. Let us show that the matrix $\check{C}_m^{(n)}$ converges ν_T^W-almost surely, to prove that $\rho_m^{(n)}$ converges in distribution.

Let us fix $\beta \in \mathcal{D}_T^W$, as well as $x \neq y \in D^{(m)}$, and $i \sim \delta_x$, $j \sim \delta_y$. A path from i to j in $T^{(n)}$ necessarily crosses $x \wedge y$, since $i \in T_x$ and $j \in T_y$. Therefore, $\mathcal{P}_{i,j}^{T^{(n)}} = \mathcal{P}_{i,\{x \wedge y\},j}^{T^{(n)}}$, so

$$\begin{aligned}
(\check{C}_m^{(n)})_{\delta_x, \delta_y} &= \sum_{i \sim \delta_x, j \sim \delta_y} (\tilde{W}_m^{(n)})_{\delta_x, i}\hat{G}^{(n)}(i,j)(\tilde{W}_m^{(n)})_{j, \delta_y} \\
&= \sum_{i \sim \delta_x, j \sim \delta_y} (\tilde{W}_m^{(n)})_{\delta_x, i}\left(\sum_{\sigma \in \mathcal{P}_{i,\{x \wedge y\},j}^{T^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma}\right)(\tilde{W}_m^{(n)})_{j, \delta_y} \\
&= \sum_{i \sim \delta_x, j \sim \delta_y} (\tilde{W}_m^{(n)})_{\delta_x, i}\hat{F}^{(n)}(i, x \wedge y)(\tilde{W}_m^{(n)})_{\delta_y, j}\hat{F}^{(n)}(j, x \wedge y)\hat{G}^{(n)}(x \wedge y, x \wedge y) \\
&= \frac{1}{\hat{G}^{(n)}(x \wedge y, x \wedge y)}\left(\sum_{i \sim \delta_x}(\tilde{W}_m^{(n)})_{\delta_x, i}\hat{G}^{(n)}(i, x \wedge y)\right)\left(\sum_{j \sim \delta_y}(\tilde{W}_m^{(n)})_{\delta_y, j}\hat{G}^{(n)}(j, x \wedge y)\right) \\
&= \frac{\chi_m^{(n)}(x \wedge y, \delta_x)\chi_m^{(n)}(x \wedge y, \delta_y)}{\hat{G}^{(n)}(x \wedge y, x \wedge y)},
\end{aligned}$$

and $(\check{C}_m^{(n)})_{\delta_x, \delta_x} = 0$. Since $\chi_m^{(n)}$ converges to χ_m ν_T^W-almost surely, the matrix $\check{C}_m^{(n)}$ also converges to $\check{C}_m$.

Under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, and conditionally on β, $\rho_m^{(n)}$ is distributed according to $\nu_{B_m}^{\check{C}_m^{(n)}}$, which ν_T^W-almost surely converges weakly to $\nu_{B_m}^{\check{C}_m}$ by Lévy's theorem. Therefore,

the distribution of $(G_m^{(n)})_{B_m, B_m} = (2\rho_m^{(n)} - \check{C}_m^{(n)})^{-1}$ under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$ converges weakly to the distribution of $\check{G}_m := (2\rho_m - \check{C}_m)^{-1}$ under $\nu_T^W(d\beta)\nu_{B_m}^{\check{C}_m}(d\rho_m)$. $\qquad\square$

We now have all the ingredients needed to show that the representations on restrictions $\mathcal{G}_m^{(n)}$ converge, when $n \to \infty$, to a new representation on the infinite graph.

Proof of Theorem 2.18 (ii). Thanks to Lemma 6.2, for $0 \le m \le n$ and $i, j \in T^{(n)}$ we have

$$G_m^{(n)}(i,j) = \hat{G}^{(n)}(i,j) + \sum_{b, b' \in B_m} \chi_m^{(n)}(i,b) G_m^{(n)}(b,b') \chi_m^{(n)}(j,b'),$$

where under $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, $G_m^{(n)} = (2\beta_m^{(n)} - \tilde{W}_m^{(n)})^{-1}$ provides a representation on the VRJP on $\mathcal{G}_m^{(n)}$, from Theorem 2.2 and Lemma 6.1. We have shown that when $n \to \infty$, $\chi_m^{(n)}$ converges almost surely, and $(\check{G}_m^{(n)})_{B_m, B_m}$ converges in distribution. As a result for all $i, j \in T$, the distribution of $G_m^{(n)}(i,j)$ converges weakly to the distribution of $G_m(i,j)$ under $\nu_{T, B_m}^W(d\beta, d\rho_m)$, where

$$G_m(i,j) = \hat{G}(i,j) + \sum_{x, x' \in D^{(m)}} \chi(i, \Omega_x) \chi(j, \Omega_y) \check{G}_m(\delta_x, \delta_y).$$

The second term in $G_m(i,j)$ can be rewritten as an integral on Ω^2. Indeed, let us define $\check{g}_m : \Omega^2 \to \mathbb{R}$ in the following way: for $\omega, \tau \in \Omega$, if $x, y \in D^{(m)}$ are such that $\omega \in \Omega_x$ and $\tau \in \Omega_y$, we set $\check{g}_m(\omega, \tau) = \check{G}_m(\delta_x, \delta_y)$. With these new notations, we can now write, for $i, j \in T$,

$$G_m(i,j) = \hat{G}(i,j) + \chi_m(i, \cdot) \check{G}_m{}^t (\chi_m(j, \cdot))$$
$$= \hat{G}(i,j) + \int_{\Omega^2} \chi(i, d\omega) \chi(j, d\tau) \check{g}_m(\omega, \tau).$$

For $\beta \in \mathcal{D}_T^W$, $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$, and all $i_0, i, j \in T$, we denote $r_{i,j}^{\beta, \rho_m, i_0} = \frac{W_{i,j}}{2} \frac{G_m(i_0, j)}{G_m(i_0, i)}$. To prove Theorem 2.18 (ii), we have to see that the process with distribution $\mathbb{P}^{VRJP(i_0)}$ is a mixture of Markov processes $\mathbb{P}_{i_0}^{\beta, \rho_m, i_0}$ under $\nu_{T, B_m}^W(d\beta, d\rho_m)$. The proof is the same than that of Theorem 2.6 (iii) (see [23]). It consists in studying trajectories of the time-changed VRJP, stopped when they leave a finite subgraph included in $T^{(n)}$. They can be considered as trajectories of the time-changed VRJP on $\mathcal{G}_m^{(n)}$, and represented using $G_m^{(n)}$ thanks to Theorem 2.2. Taking the limit in distribution when $n \to \infty$ then gives the result. Note that the proof needs an argument of uniform integrability on the family $\left(\frac{G_m^{(n)}(i_0, j)}{G_m^{(n)}(i_0, i)}\right)_{n \ge m}$ for all $i, j \in T$, which is given by Proposition 7 and Corollary 2 from [23]. $\qquad\square$

7 The family of representations of the VRJP on infinite trees

In this last section, we show some properties of the family of representations on infinite trees, constructed in the previous section. The first one is Theorem 2.18 (iii), and states that this family converges with the one described in Theorem 2.14. Then we show that all these representations are distinct for a regular tree on which the VRJP is transient.

7.1 Convergence to another representation: proof of Theorem 2.18 (iii)

Let us show that the representations of the VRJP built with G_m converge in distribution when $m \to \infty$ to the representation described in Theorem 2.14, with independent jump rates. To show this, we use a tightness argument, based on the following lemma regarding the distribution ν_V^W.

Lemma 7.1. *Let $\mathcal{G} = (V, E)$ be a finite graph, endowed with conductances W. We denote $G = (H_\beta)^{-1}$ for $\beta \in \mathcal{D}_V^W$. Then for all $\eta \in \mathbb{R}_+^V$, under $\nu_V^W(d\beta)$, $\langle \eta, G\eta \rangle$ has the same distribution as $\frac{\langle \eta, \mathbb{1} \rangle^2}{2\gamma}$, where γ is a Gamma random variable with parameter $(1/2, 1)$.*

Proof. Let $\eta \in \mathbb{R}_+^V$ be fixed. We will compute the Laplace transform of $\langle \eta, G\eta \rangle$: for $\lambda \in \mathbb{R}_+$,

$$
\begin{aligned}
\mathbb{E}\left[e^{-\lambda \langle \eta, G\eta \rangle} \right] &= \int e^{-\lambda \langle \eta, (H_\beta)^{-1}\eta \rangle} \nu_V^W(d\beta) \\
&= e^{-\sqrt{2\lambda}\langle \eta, \mathbb{1} \rangle} \int e^{-\frac{1}{2}\langle \sqrt{2\lambda}\eta, (H_\beta)^{-1}\sqrt{2\lambda}\eta \rangle} e^{\langle \sqrt{2\lambda}\eta, \mathbb{1} \rangle} \nu_V^W(d\beta) \\
&= e^{-\sqrt{2\lambda}\langle \eta, \mathbb{1} \rangle} \int \nu_V^{W, \sqrt{2\lambda}\eta}(d\beta) = e^{-\sqrt{2\lambda}\langle \eta, \mathbb{1} \rangle},
\end{aligned}
$$

since $\nu_V^{W, \sqrt{2\lambda}\eta}$ is a probability measure. Let us now compute, for $\gamma \sim \mathrm{Gamma}(1/2, 1)$, the Laplace transform of $\frac{1}{2\gamma}$: for $\lambda \in \mathbb{R}_+$,

$$
\begin{aligned}
\mathbb{E}\left[e^{-\frac{\lambda}{2\gamma}} \right] &= \int \mathbb{1}_{u>0} \frac{e^{-u}}{\sqrt{\pi u}} e^{-\frac{\lambda}{2u}} du = \int_{v>0} \frac{1}{\sqrt{2\pi v^3}} e^{-\frac{1}{2}(\frac{1}{v}+2\lambda v)} dv \\
&= e^{-\sqrt{2\lambda}} \int_{v>0} \frac{1}{\sqrt{2\pi v^3}} e^{-\frac{2\lambda}{2v}(v-\frac{1}{\sqrt{2\lambda}})^2} dv,
\end{aligned}
$$

by taking $v = 1/2u$. Since $\mathbb{1}_{v>0} \frac{1}{\sqrt{2\pi v^3}} e^{-\frac{2\lambda}{2v}(v-\frac{1}{\sqrt{2\lambda}})^2} dv$ is the density of an Inverse Gaussian distribution with parameter $(1, 1/\sqrt{2\lambda})$, we finally get $\mathbb{E}[e^{-\frac{\lambda}{2\gamma}}] = e^{-\sqrt{2\lambda}}$. Therefore, for all $\lambda \geq 0$,

$$
\mathbb{E}\left[e^{-\lambda \langle \eta, G\eta \rangle} \right] = e^{-\sqrt{2\lambda \langle \eta, \mathbb{1} \rangle^2}} = \mathbb{E}\left[e^{-\frac{\lambda \langle \eta, \mathbb{1} \rangle^2}{2\gamma}} \right],
$$

which proves the result. $\qquad\square$

Proof of Theorem 2.18 (iii). We will use Lemma 7.1 to prove that the sequence of random jump rates $(r^{(m),\phi})_{m \in \mathbb{N}}$ is tight, then identify the only possible limit distribution for each converging subsequence, which will provide both the weak convergence and the expression of the limit distribution.

For $m \geq 0$ and $\beta \in \mathcal{D}_T^W$, let us define, for $i \in T$ and $m \geq 0$, the vector $\bar{\mu}_i^{(m)} \in \mathbb{R}^{B_m}$ by $(\bar{\mu}_i^{(m)})(\delta_x) = \mu_i^\psi(\Omega_x)$ for all $x \in D^{(m)}$. Then, for $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$ and $i, j \in T$,

$$
\begin{aligned}
\int_{\Omega^2} \chi(i, d\omega)\chi(j, d\tau)\check{g}_m(\omega, \tau) &= \psi(i)\psi(j) \sum_{x, x' \in D^{(m)}} \int_{\Omega_x \times \Omega_{x'}} \mu_i(d\omega)\mu_j(d\tau)\check{G}_m(\delta_x, \delta_{x'}) \\
&= \psi(i)\psi(j)\langle \bar{\mu}_i^{(m)}, \check{G}_m \bar{\mu}_j^{(m)} \rangle.
\end{aligned}
$$

We denote, for $m \geq 0$ and $i, j \in T$,

$$
a_{i,j}^{(m)} = \frac{1}{4}\langle \bar{\mu}_i^{(m)} + \bar{\mu}_j^{(m)}, \check{G}_m(\bar{\mu}_i^{(m)} + \bar{\mu}_j^{(m)}) \rangle,
$$

so that we have

$$
\begin{aligned}
G_m(i, j) &= \hat{G}(i, j) + \psi(i)\psi(j)\langle \bar{\mu}_i^{(m)}, \check{G}_m \bar{\mu}_j^{(m)} \rangle \\
&= \hat{G}(i, j) + \frac{\psi(i)\psi(j)}{2}\left(4a_{i,j}^{(m)} - a_i^{(m)} - a_j^{(m)} \right).
\end{aligned}
$$

Therefore, we can write $(G_m(i, j))_{i,j \in T} = \Phi\left((\hat{G}(i, j))_{i,j \in T}, (\psi(i))_{i \in T}, (a_{i,j}^{(m)})_{i,j \in T} \right)$, where Φ is a continuous function.

We will denote by $\bar{\nu}_T^W(d\beta, d\rho)$ the distribution of a coupling of $\nu_{T, B_m}^W(d\beta, d\rho_m)$ for all $m \geq 0$.

Lemma 7.2. *Let us set* $Z_m = \left((\hat{G}(i,j))_{i,j\in T}, (\psi(i))_{i\in T}, (a_{i,j}^{(m)})_{i,j\in T} \right)$ *for* $m \geq 0$, *which takes its values in* $\mathbb{R}^{T^2} \times \mathbb{R}^T \times \mathbb{R}^{T^2}$. *Then under* $\bar{\nu}_T^W(d\beta, d\rho)$, $(Z_m)_{m\geq 0}$ *is tight.*

Proof. For $\varepsilon > 0$, let K_ε be a compact subset of $\mathbb{R}$ such that $0 \in K_\varepsilon$ and

$$\mathbb{P}\left[\frac{1}{2\gamma} \in K_\varepsilon \right] > 1 - \varepsilon$$

when $\gamma \sim \text{Gamma}(1/2, 1)$. Let us now fix $m \geq 0$. Recall that $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$, where $\check{C}_m$ is a β-measurable matrix of conductances on B_m, and conditionally on β, ρ_m is distributed according to $\nu_{B_m}^{\check{C}_m}$. Therefore from Lemma 7.1, for $i, j \in T$, conditionally on β the random variable $\langle \bar{\mu}_i^{(m)} + \bar{\mu}_j^{(m)}, \check{G}_m \left(\mu_i^{(m)} + \bar{\mu}_j^{(m)} \right) \rangle$ has the same distribution as $\frac{(\mu_i^\psi(\Omega) + \mu_j^\psi(\Omega))^2}{2\gamma}$, where $\gamma \sim \text{Gamma}(1, 1/2)$. This implies that conditionally on $\{\psi \not\equiv 0\}$, $a_{i,j}^{(m)}$ has the same distribution as $\frac{1}{2\gamma}$, and conditionally on $\{\psi \equiv 0\}$, $a_{i,j}^{(m)} = 0$. As a result, for all $\varepsilon > 0$,

$$\mathbb{P}\left[a_{x,x'}^{(m)} \in K_\varepsilon \right] = \mathbb{P}[\psi \equiv 0] + \mathbb{P}[\psi \not\equiv 0]\, \mathbb{P}\left[\frac{1}{2\gamma} \in K_\varepsilon \right] > 1 - \varepsilon.$$

Let now $\left(\tilde{a}_k^{(m)} \right)_{k\in\mathbb{N}}$ be an enumeration of $\left(a_{i,j}^{(m)} \right)_{i,j\in T}$. Then for $\varepsilon > 0$,

$$\mathbb{P}\left[\forall k \in \mathbb{N}, \tilde{a}_k^{(m)} \in K_{2^{-n-1}\varepsilon} \right] \geq 1 - \sum_{k\in\mathbb{N}} 2^{-n-1}\varepsilon = 1 - \varepsilon,$$

where $\tilde{K}_\varepsilon = \prod_{k\in\mathbb{N}} K_{2^{-n-1}\varepsilon}$ is a compact subset of $\mathbb{R}^\mathbb{N}$. Moreover, the β-measurable random variable $\left((\hat{G}(i,j))_{i,j\in T}, (\psi(i))_{i\in T} \right)$ takes its values in $\mathbb{R}^{T^2} \times \mathbb{R}^T$, where T is countable. As a result, for all $\varepsilon > 0$, there is a compact subset $K_\varepsilon' \subset \mathbb{R}^{T^2} \times \mathbb{R}^T$ such that

$$\mathbb{P}\left[\left((\hat{G}(i,j))_{i,j\in T}, (\psi(i))_{i\in T} \right) \in K_\varepsilon' \right] > 1 - \varepsilon.$$

We can now conclude that for all $\varepsilon > 0$,

$$\mathbb{P}\left[\left((\hat{G}(i,j))_{i,j\in T}, (\psi(i))_{i\in T}, (a_{i,j}^{(m)})_{i,j\in T} \right) \in \tilde{K}_{\varepsilon/2} \times K_{\varepsilon/2}' \right] > 1 - \varepsilon,$$

where $\tilde{K}_{\varepsilon/2} \times K_{\varepsilon/2}'$ is compact, and does not depend on m. $\qquad\square$

As a result, there is an extraction $(m_k)_{k\in\mathbb{N}}$ such that $(Z_{m_k})_{k\in\mathbb{N}}$ converges in distribution under $\bar{\nu}_T^W(d\beta, d\rho)$. Since $G_{m_k} = \Phi(Z_{m_k})$ where Φ is continuous, $(G_{m_k}(i,j))_{i,j\in T}$ also converges in distribution under under $\bar{\nu}_T^W(d\beta, d\rho)$, as do the random jump rates $(r_{i,j}^{(m_k),\phi})_{i,j\in T}$. Let us show that the limit distribution of the environment does not depend on the extraction, which will mean that $((r_{i,j}^{(m),\phi})_{i,j\in T})_{m\in\mathbb{N}}$ converges in distribution, since it is tight.

Lemma 7.3. *For* $m \geq 1$ *and for all* $n \geq m$, *under the distribution* $\nu_T^W(d\beta)\nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$, *the random variables* $\left(\frac{G_m^{(n)}(\phi,i)}{G_m^{(n)}(\phi,\bar{i})} \right)_{i\in T^{(m)}\setminus\{\phi\}}$ *are independent inverse Gaussian variables, where* $\frac{G_m^{(n)}(\phi,i)}{G_m^{(n)}(\phi,\bar{i})}$ *has parameter* $(W_{\bar{i},i}, 1)$ *for* $i \in T^{(m)}\setminus\{\phi\}$.

Proof. Let us fix $1 \leq m \leq n$. For $i \in T^{(m)}\setminus\{\phi\}$, we denote $g_i = \frac{G_m^{(n)}(\phi,i)}{G_m^{(n)}(\phi,\bar{i})}$. Since $|i| \leq m$, any path in $\mathcal{G}_m^{(n)}$ from ϕ to i crosses $\overleftarrow{i}$, so from Proposition 3.4 (ii) and (iii),

$$g_i = \frac{G_m^{(n)}(\overleftarrow{i}, i)}{G_m^{(n)}(\overleftarrow{i}, \overleftarrow{i})} = \sum_{j\sim\bar{i}} W_{\bar{i},j} \sum_{\sigma \in \mathcal{P}_{j,i}^{\tilde{T}_m^{(n)}\setminus\{\bar{i}\}}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma}.$$

For $i \in T^{(m)} \backslash \{\phi\}$, let us denote by $\tilde{T}_i$ the connected component of i in $\tilde{T}_m^{(n)} \backslash \{\overleftarrow{i}\}$, i.e. $\tilde{T}_i = T_m^{(n)} \cup \{\delta_x, x \in D^{(m)} \cap T_i\}$, endowed with the restriction of $\tilde{W}_m^{(n)}$. This way, we get

$$g_i = W_{\overleftarrow{i},i} \sum_{\sigma \in \mathcal{P}_{i,i}^{\tilde{T}_i}} \frac{(\tilde{W}_m^{(n)})_\sigma}{(2\beta_m^{(n)})_\sigma} = W_{\overleftarrow{i},i} \left((H_\beta^{(n)})_{\tilde{T}_i, \tilde{T}_i} \right)^{-1} (i,i),$$

so g_i is $(\beta_m^{(n)})_{\tilde{T}_i}$-measurable.

To prove that $(g_i)_{i \in T^{(m)} \backslash \{\phi\}}$ are independent, it will be enough to see that for $i \in T^{(m)} \backslash \{\phi\}$, g_i is independent of $g_{U_i^{(m)}}$, and that for $x \in T^{(m-1)}$, the restrictions $(g_{\tilde{T}_i})_{i \in S(x)}$ are independent.

Writing $\left((H_\beta^{(n)})_{\tilde{T}_i, \tilde{T}_i} \right)^{-1} (i,i)$ as a Schur complement, we see that, if we set $\tilde{U}_i = \tilde{T}_i \backslash \{i\}$,

$$g_i = \frac{W_{\overleftarrow{i},i}}{2(\beta_m^{(n)})_i - \sum_{j,j' \in S(i)} W_{i,j} W_{i,j'} \left((H_\beta^{(n)})_{\tilde{U}_i, \tilde{U}_i} \right)^{-1} (j,j')} = \frac{W_{\overleftarrow{i},i}}{\check{H}_\beta^{\{i\}}}.$$

From Proposition 3.2, conditionally on $(\beta_m^{(n)})_{\tilde{U}_i}$, the distribution of $(\beta_m^{(n)})_i$ is given by

$$\sqrt{\frac{2}{\pi}} \frac{\mathbb{1}_{\check{H}_\beta^{\{i\}} > 0}}{\sqrt{\det(\check{H}_\beta^{\{i\}})}} e^{-\frac{1}{2} \left(\check{H}_\beta^{\{i\}} + W_{\overleftarrow{i},i}^2 (\check{H}_\beta^{\{i\}})^{-1} \right)} e^{W_{\overleftarrow{i},i}} d(\beta_m^{(n)})_i,$$

so by a change of variables, the distribution of g_i conditionally on $(\beta_m^{(n)})_{\tilde{U}_i}$ is

$$\mathbb{1}_{g_i > 0} \sqrt{\frac{W_{\overleftarrow{i},i}}{2\pi g_i^3}} e^{-\frac{W_{\overleftarrow{i},i}}{2g_i}(g_i-1)^2} dg_i,$$

i.e. $g_i \sim IG(W_{\overleftarrow{i},i}, 1)$. Since this distribution does not depend on $(\beta_m^{(n)})_{\tilde{U}_i}$, g_i is independent of $(\beta_m^{(n)})_{\tilde{U}_i}$. For all $j \in U_i^{(m)}$, $\tilde{T}_j \subset \tilde{U}_i$, so g_j is $(\beta_m^{(n)})_{\tilde{U}_i}$-measurable. Therefore, g_i is independent of $g_{U_i^{(m)}}$.

Moreover, for $x \in T^{(m-1)}$, the sets $(\tilde{T}_i)_{i \in S(x)}$ are all at distance 2 from one another in $\mathcal{G}_m^{(n)}$. Since $\beta^{(n)}$ is 1-dependent, the restrictions $(\beta_{\tilde{T}_i}^{(n)})_{i \in S(x)}$ are independent. For $j \in \tilde{T}_i$, we have $\tilde{T}_j \subset \tilde{T}_i$, so g_j is $\beta_{\tilde{T}_i}^{(n)}$-measurable. Therefore the restrictions $(g_{\tilde{T}_i})_{i \in S(x)}$ are independent, which concludes the proof. $\qquad \square$

We can now use Lemma 7.3, to show that any converging subsequence of $(G_m)_{m \in \mathbb{N}}$ has the same limit in distribution, which corresponds to the representation from Theorem 2.14.

For $m \geq 1$, the distribution of $G_m^{(n)}$ under $\nu_T^W(d\beta) \nu_{B_m}^{\check{W}_m^{(n)}}(d\beta')$ converges weakly to the distribution of G_m under $\nu_{T,B_m}^W(d\beta, d\rho_m)$. If we denote $g_i^{(m)} = \frac{G_m(\phi,i)}{G_m(\phi,\overleftarrow{i})}$ for $i \in T^{(m)} \backslash \{\phi\}$, and take the limit in Lemma 7.3, we get that under $\nu_{T,B_m}^W(d\beta, d\rho_m)$, $(g_i^{(m)})_{i \in T^{(m)} \backslash \{\phi\}}$ are independent, and $g_i^{(m)} \sim IG(W_{\overleftarrow{i},i}, 1)$ for $i \in T^{(m)} \backslash \{\phi\}$. Recall that the random environment associated with G_m is given by the following jump rates:

$$r_{\overleftarrow{i},i}^{\beta,\rho_m,\phi} = \frac{W_{\overleftarrow{i},i}}{2} \frac{G_m(\phi,i)}{G_m(\phi,\overleftarrow{i})} \text{ and } r_{i,\overleftarrow{i}}^{\beta,\rho_m,\phi} = \frac{W_{\overleftarrow{i},i}}{2} \frac{G_m(\phi,\overleftarrow{i})}{G_m(\phi,i)},$$

for all $i \in T \backslash \{\phi\}$, and $r_{i,j}^{\beta,\rho_m,\phi} = 0$ if $i \not\sim j$.

Let now $(m_k)_{k\in\mathbb{N}}$ be an extraction such that under $\bar{\nu}_T^W(d\beta, d\rho)$, $(r_{i,j}^{\beta,\rho_{m_k},\phi})_{i,j\in T}$ converges in distribution to a limit environment $(r_{i,j}^{(\infty),\phi})_{i,j\in T}$. Then, we have $r_{i,j}^{(\infty),\phi} = 0$ for $i \nsim j$. Moreover, let us set $g_i^{(\infty)} = \frac{2}{W_{\check{i},i}} r_{\check{i},i}^{(\infty),\phi}$ for $i \in T\backslash\{\phi\}$. Note that for all $m \in \mathbb{N}$, if k is such that $m_k \geq m$, we have $T^{(m)} \subset T^{(m_k)}$ so for all $i \in T^{(m)}\backslash\{\phi\}$,

$$r_{\check{i},i}^{\beta,\rho_{m_k},\phi} = \frac{W_{\check{i},i}}{2} g_i^{(m_k)} \text{ and } r_{i,\check{i}}^{\beta,\rho_{m_k},\phi} = \frac{W_{\check{i},i}}{2g_i^{(m_k)}}.$$

Taking $k \to \infty$, we get that $(g_i^{(m_k)})_{i\in T^{(m)}\backslash\{\phi\}}$ converges in distribution to $(g_i^{(\infty)})_{i\in T^{(m)}\backslash\{\phi\}}$, which implies that $(g_i^{(\infty)})_{i\in T^{(m)}\backslash\{\phi\}}$ are independent. Since this is true for all $m \geq 0$, $(g_i^{(\infty)})_{i\in T\backslash\{\phi\}}$ are independent. Moreover, for all $i \in T\backslash\{\phi\}$, $g_i^{(\infty)} \sim IG(W_{\check{i},i}, 1)$ and

$$r_{\check{i},i}^{(\infty),\phi} = \frac{W_{\check{i},i}}{2} g_i^{(\infty)} \text{ and } r_{i,\check{i}}^{(\infty),\phi} = \frac{W_{\check{i},i}}{2g_i^{(\infty)}}.$$

The random environment given by these jump rates is in fact the one described in Theorem 2.14, hence its distribution does not depend on the extraction $(m_k)_{k\in\mathbb{N}}$. Since the sequence of jump rates $((r_{i,j}^{\beta,\rho_m,\phi})_{i,j\in T})_{m\geq 1}$ is tight, this implies that under $\bar{\nu}_T^W(d\beta, d\rho)$, it converges in distribution to the random environment given in Theorem 2.14. $\square$

7.2 Distinct representations on a regular tree: proofs of Propositions 2.15 and 2.19

Let us start by proving that on regular trees where the VRJP is transient, the standard representation and the one given in Theorem 2.14 are different.

Proof of Proposition 2.15. Let $\mathcal{T} = (T, E)$ be a d-regular tree, where $d \geq 3$. It was shown in [9] that there exists a $\overline{W} > 0$ such that for $W > \overline{W}$, the VRJP on $\mathcal{T}$ endowed with constant conductances W is almost surely transient. Note that the VRJP is defined in a slightly different manner in [9], but it can be related to the definition used here, thanks to a time rescaling described in Appendix B of [20]. From now on, we take $W > \overline{W}$.

We consider jump rates $(r_{i,j})_{i\sim j}$ on the tree $\mathcal{T}$. Let ϕ be an arbitrary root for $\mathcal{T}$, and let $(i_k)_{k\geq 0}$ be an infinite self-avoiding path (or ray) in $\mathcal{T}$, such that for $k \geq 0$, $|i_k| = k$. Let us define $S_n = \prod_{k=1}^n \frac{2}{W} r_{i_{k-1},i_k}$. We will compare the distribution of S_n under two distribution of jump rates.

Let $\mathcal{R}_{ind}(dr)$ be the distribution of jump rates in the representation described in Theorem 2.14. Under $\mathcal{R}_{ind}(dr)$, we know that S_n has the distribution of $\prod_{i=1}^n A_{i_k}$, where A_{i_k} are independent inverse Gaussian variables with parameter $(W, 1)$. Note that $\mathbb{E}[A_1] = 1$, so by Jensen's inequality, $\mathbb{E}[\log(A_1)] < 0$. By the law of large numbers, we then have a.s. that $\sum_{k=1}^n \log(A_i) \xrightarrow[n\to\infty]{} -\infty$, so that $S_n \xrightarrow[n\to\infty]{a.s.} 0$.

Let now $\mathcal{R}_{st}(dr)$ be the distribution of jump rates in the standard representation of the VRJP started at $\phi = i_0$. Under $\mathcal{R}_{st}(dr)$, Theorem 2.6 tells us that S_n has the same distribution as

$$\prod_{k=1}^n \frac{G(i_0, i_k)}{G(i_0, i_{k-1})} = \frac{G(i_0, i_n)}{G(i_0, i_0)} = \frac{\hat{G}(i_0, i_n) + \frac{1}{2\gamma}\psi(i_0)\psi(i_n)}{G(i_0, i_0)}$$

under $\nu_V^W(d\beta, d\gamma)$, where according to Proposition 2.11, $\psi(i) > 0$ a.s. for all $i \in T$. Moreover, since the distribution of ψ under $\nu_V^W(d\beta)$ is stationary for the group of transformations of $\mathcal{T}$ (see Proposition 3 in [23]), $\psi(i_n)$ has the same distribution as $\psi(i_0)$ for all $n \in \mathbb{N}$, and cannot tend to 0 a.s. when $n \to \infty$. Therefore, neither can S_n under $\mathcal{R}_{st}(dr)$, which proves that $\mathcal{R}_{st}$ and $\mathcal{R}_{ind}$ are different. $\square$

We will now prove Proposition 2.19, *i.e.* that on a d-regular tree with $d \geq 3$, for constant W large enough so that the VRJP is transient, the representations in the family given by Theorem 2.18 are all different. In order to do this, we will compare the distribution of the random harmonic measures for each representation.

The following Proposition gives an expression for the harmonic measure of sets Ω_x, for $x \in T$. We will see how this expression behaves differently whether or not $|x| > m$.

Proposition 7.4. *Let $m \in \mathbb{N}$ be fixed. We also fix $\beta \in \mathcal{D}_T^W$ and $\rho_m \in \mathcal{D}_{B_m}^{\check{C}_m}$. We denote by $\mu_\phi^{\beta,\rho_m,\phi}$ the exiting measure of the transient Markov process $\mathbb{P}_\phi^{\beta,\rho_m,\phi}$ defined in Theorem 2.18. Then for $x \in T$,*

$$\mu_\phi^{\beta,\rho_m,\phi}(\Omega_x) = \frac{\int_{\Omega \times \Omega_x} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau)}{\int_{\Omega^2} \chi(\phi, d\omega) \check{g}_m(\omega, \tau) \chi(\phi, d\tau)}.$$

Proof. We denote $\mu = \mu_\phi^{\beta,\rho_m,\phi}$. Let g be the Green function associated with the discrete Markov chain associated with $\mathbb{P}^{\beta,\rho_m,\phi}$, *i.e.* with jump rates $r_{i,j}^{\beta,\rho_m,\phi} = \frac{W_{i,j}}{2} \frac{G_m(\phi,j)}{G_m(\phi,i)}$. Let us denote, for $i, j \in T$, $f(i,j) = \frac{g(i,j)}{g(j,j)}$. Then from Proposition 3.13, we get the following expression for $x \neq \phi$:

$$\mu(\Omega_x) = f(\phi, x) \frac{1 - f(x, \bar{x})}{1 - f(x, \bar{x}) f(\bar{x}, x)}.$$

For $i, j \in T$, we have

$$g(i,j) = \sum_{k \in \mathbb{N}} \mathbb{P}_i^{\beta,\rho_m,\phi}[X_k = j] = \sum_{\sigma \in \mathcal{P}_{i,j}^T} \prod_{k=0}^{|\sigma|-1} \frac{r_{\sigma_k,\sigma_{k+1}}^{\beta,\rho_m,\phi}}{\tilde{\beta}_{\sigma_k}} = \frac{G_m(\phi,j)}{G_m(\phi,i)} \sum_{\sigma \in \mathcal{P}_{i,j}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma},$$

where $\tilde{\beta}_i = \sum_{j \sim i} r_{i,j}^{\beta,\rho_m,\phi}$ are the rates of the corresponding holding times. Note that $\tilde{\beta}_i = \beta_i - \mathbb{1}_{\{i=\phi\}} \frac{1}{2G_m(\phi,\phi)}$ for $i \in T$. In particular, if a path σ never crosses ϕ, then $(2\tilde{\beta})_\sigma = (2\beta)_\sigma$.

Let us denote $\tilde{G}(i,j) = \sum_{\sigma \in \mathcal{P}_{i,j}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma}$ and $\tilde{F}(i,j) = \frac{\tilde{G}(i,j)}{\tilde{G}(j,j)}$, then

$$g(i,j) = \frac{G_m(\phi,j)}{G_m(\phi,i)} \tilde{G}(i,j) 2\tilde{\beta}_j, \text{ and } f(i,j) = \frac{G_m(\phi,j)}{G_m(\phi,i)} \frac{\tilde{G}(i,j)}{\tilde{G}(j,j)} = \frac{G_m(\phi,j)}{G_m(\phi,i)} \tilde{F}(i,j).$$

The expression for the measure becomes

$$\mu(\Omega_x) = \frac{G_m(\phi,x)}{G_m(\phi,\phi)} \frac{\tilde{G}(\phi,x)}{\tilde{G}(x,x)} \left(\frac{1 - \frac{G_m(\phi,\bar{x})}{G_m(\phi,x)} \tilde{F}(x,\bar{x})}{1 - \tilde{F}(x,\bar{x}) \tilde{F}(\bar{x},x)} \right)$$

$$= \frac{\tilde{G}(\phi,x)}{G_m(\phi,\phi)} \left(\frac{G_m(\phi,x) - G_m(\phi,\bar{x}) \tilde{F}(x,\bar{x})}{\tilde{G}(x,x) - \tilde{F}(x,\bar{x}) \tilde{G}(\bar{x},x)} \right).$$

Let us compute the following terms: firstly,

$$\tilde{G}(x,x) - \tilde{F}(x,\bar{x}) \tilde{G}(\bar{x},x) = \tilde{G}(x,x) - \tilde{F}(x,\bar{x}) \tilde{G}(\bar{x},\bar{x}) \tilde{F}(x,\bar{x})$$

$$= \sum_{\sigma \in \mathcal{P}_{x,x}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma} - \sum_{\sigma \in \mathcal{P}_{x,\{\bar{x}\},x}^T} \frac{W_\sigma}{(2\tilde{\beta})_\sigma} = \sum_{\sigma \in \mathcal{P}_{x,x}^{T_x}} \frac{W_\sigma}{(2\tilde{\beta})_\sigma}.$$

Indeed, paths from x to x that do not cross $\bar{x}$ have to stay in the connected component of x in $T \setminus \{\bar{x}\}$, which is T_x, *i.e.* $\mathcal{P}_{x,x}^T \setminus \mathcal{P}_{x,\{\bar{x}\},x}^T = \mathcal{P}_{x,x}^{T_x}$. Moreover, $\phi \notin T_x$, so for $\sigma \in \mathcal{P}_{x,x}^{T_x}$, $(2\tilde{\beta})_\sigma = (2\beta)_\sigma$. As a result,

$$\tilde{G}(x,x) - \tilde{F}(x,\bar{x}) \tilde{G}(\bar{x},x) = \sum_{\sigma \in \mathcal{P}_{x,x}^{T_x}} \frac{W_\sigma}{(2\beta)_\sigma} = \hat{G}^{T_x}(x,x).$$

Secondly,

$$G_m(\phi, x) - G_m(\phi, \breve{x})\tilde{F}(x, \breve{x}) = \hat{G}(\phi, x) - \hat{G}(\phi, \breve{x})\tilde{F}(x, \breve{x})$$
$$+ \int_{\Omega^2} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\left(\chi(x, d\tau) - \tilde{F}(x, \breve{x})\chi(\breve{x}, d\tau)\right).$$

Note that if $\sigma \in \overline{\mathcal{P}}_{x,\breve{x}}^{T\setminus\{\breve{x}\}}$, then $\sigma_k \in T_x$ for $k \leq |\sigma| - 1$, so $(2\tilde{\beta})_\sigma^- = (2\beta)_\sigma^-$. Therefore, $\tilde{F}(x, \breve{x}) = \hat{F}(x, \breve{x})$. Moreover, since $\mathcal{P}_{\phi,x}^T = \mathcal{P}_{\phi,\{\breve{x}\},x}^T$, we have

$$\hat{G}(\phi, x) - \hat{G}(\phi, \breve{x})\hat{F}(x, \breve{x}) = 0.$$

Recall also that the density of $\chi(x, \cdot)$ with respect to $\chi(\phi, \cdot)$ is $\tau \mapsto \frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)}$. As a result,

$$G_m(\phi, x) - G_m(\phi, \breve{x})\tilde{F}(x, \breve{x}) = \int_{\Omega^2} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\left(\chi(x, d\tau) - \hat{F}(x, \breve{x})\chi(\breve{x}, d\tau)\right)$$
$$= \int_{\Omega^2} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\chi(\phi, d\tau)\left(\frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)} - \hat{F}(x, \breve{x})\frac{\hat{F}(\breve{x}, \breve{x} \wedge \tau)}{\hat{F}(\phi, \breve{x} \wedge \tau)}\right).$$

For $\tau \notin \Omega_x$, $x \wedge \tau = \breve{x} \wedge \tau$ and paths from x to $\breve{x} \wedge \tau$ cross $\breve{x}$. Therefore,

$$\frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)} - \hat{F}(x, \breve{x})\frac{\hat{F}(\breve{x}, \breve{x} \wedge \tau)}{\hat{F}(\phi, \breve{x} \wedge \tau)} = \frac{\hat{F}(x, \breve{x} \wedge \tau) - \hat{F}(x, \breve{x})\hat{F}(\breve{x}, \breve{x} \wedge \tau)}{\hat{F}(\phi, \breve{x} \wedge \tau)} = 0.$$

For $\tau \in \Omega_x$, $x \wedge \tau = x$ and $\breve{x} \wedge \tau = \breve{x}$, so

$$\frac{\hat{F}(x, x \wedge \tau)}{\hat{F}(\phi, x \wedge \tau)} - \hat{F}(x, \breve{x})\frac{\hat{F}(\breve{x}, \breve{x} \wedge \tau)}{\hat{F}(\phi, \breve{x} \wedge \tau)} = \frac{\hat{G}(x, x)}{\hat{G}(\phi, x)} - \hat{F}(x, \breve{x})\frac{\hat{G}(\breve{x}, \breve{x})}{\hat{G}(\phi, \breve{x})}$$
$$= \frac{\hat{G}(x, x) - \hat{F}(x, \breve{x})\hat{G}(\breve{x}, \breve{x})\hat{F}(\breve{x}, x)}{\hat{G}(\phi, x)} = \frac{\hat{G}^{T_x}(x, x)}{\hat{G}(\phi, x)}.$$

As a result, we have

$$G_m(\phi, x) - G_m(\phi, \breve{x})\tilde{F}(x, \breve{x}) = \frac{\hat{G}^{T_x}(x, x)}{\hat{G}(\phi, x)} \int_{\Omega \times \Omega_x} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\chi(\phi, d\tau).$$

For $x \neq \phi$, we finally get

$$\mu(\Omega_x) = \frac{\tilde{G}(\phi, x)}{G_m(\phi, \phi)\hat{G}(\phi, x)} \int_{\Omega \times \Omega_x} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\chi(\phi, d\tau)$$
$$= \frac{\tilde{G}(\phi, \phi)}{\hat{G}(\phi, \phi)} \int_{\Omega \times \Omega_x} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\chi(\phi, d\tau)$$

since $\tilde{F}(x, \phi) = \hat{F}(x, \phi)$. Moreover, by summing over $x \in S(\phi)$, we have the same expression for $\mu(\Omega_\phi) = \mu(\Omega)$:

$$1 = \mu(\Omega) = \frac{\tilde{G}(\phi, \phi)}{G_m(\phi, \phi)\hat{G}(\phi, \phi)} \int_{\Omega^2} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\chi(\phi, d\tau).$$

As a result, for all $x \in T$,

$$\mu(\Omega_x) = \frac{\int_{\Omega \times \Omega_x} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\chi(\phi, d\tau)}{\int_{\Omega^2} \chi(\phi, d\omega)\breve{g}_m(\omega, \tau)\chi(\phi, d\tau)}. \qquad \square$$

Proof of Proposition 2.19. Let $\mathcal{T}$ be a d-regular tree, with $d \geq 3$, endowed with constant conductances W such that $\mathbb{P}[\forall i \in T, \psi(i) > 0] = 1$. Note that $(\mathcal{T}, W)$ is vertex transitive, so it is enough to show the proposition for $i_0 = \phi$. The following lemma is a consequence of the symmetries of $(\mathcal{T}, W)$, and guarantees that almost surely, the exiting measure gives weight to the whole boundary Ω.

Lemma 7.5. *Almost surely under $\nu_T^W(d\beta)$, for all $x \neq \phi$, $\chi(\phi, \Omega_x) > 0$.*

Proof. For all $x \neq \phi$, we define $\hat{\chi}_x = \psi(x) - \hat{F}(x, \breve{x})\psi(\breve{x})$. Then we have $\chi(\phi, \Omega_x) = \hat{F}(\phi, x) \frac{\hat{\chi}_x}{1 - \hat{F}(x,\breve{x})\hat{F}(\breve{x},x)}$, and $\mathbb{P}[\chi(\phi, \Omega_x) > 0] = \mathbb{P}[\hat{\chi}_x > 0]$.

Note that

$$\psi^{(n)}(x) - \hat{F}^{(n)}(x, \breve{x})\psi^{(n)}(\breve{x}) = \sum_{y \in T_x \cap D^{(n)}} \left(\sum_{\sigma \in \mathcal{P}_{x,y}^{T_x^{(n)}}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \eta_y^{(n)}$$

is β_{T_x}-measurable. Therefore, taking the limit when $n \to \infty$ shows that $\hat{\chi}_x$ is also β_{T_x}-measurable. As a result, given a fixed $m \leq 1$, the random variables $(\hat{\chi}_x)_{x \in D^{(m)}}$ are independent, since ν_V^W is 1-dependent, and have the same distribution, since ν_V^W is invariant under the group of automorphisms of $\mathcal{T}$.

Moreover, we have $\psi(\phi) = \sum_{y \in D^{(m)}} \hat{F}(\phi, y) \frac{\hat{\chi}_y}{1 - \hat{F}(y,\breve{y})\hat{F}(\breve{y},y)}$, so

$$\mathbb{P}[\psi(\phi) = 0] = \mathbb{P}[\forall y \in D^{(m)}, \hat{\chi}_y = 0] = \mathbb{P}[\hat{\chi}_x = 0]^{|D^{(m)}|}$$

for any $x \in D^{(m)}$. Since $\mathbb{P}[\psi(\phi) = 0] = 0$, we get $\mathbb{P}[\hat{\chi}_x = 0] = 0$ for all $x \in D^{(m)}$ and all $m \geq 1$, which implies that almost surely, for all $x \neq \phi$, $\chi(\phi, \Omega_x) > 0$. $\square$

Let us fix $m > m'$, and denote $\mu^{(m)} = \mu_\phi^{\beta, \rho_m, \phi}$ and $\mu^{(m')} = \mu_\phi^{\beta, \rho_{m'}, \phi}$. Let $x \in D^{(m)}$ be fixed, note that $x \neq \phi$. We define the following events:

$$A_x^{(m)} = \left\{ \frac{\mu^{(m)}(\Omega_x)}{\mu^{(m)}(\Omega_{\breve{x}})} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\breve{x}})} \right\} \text{ and } A_x^{(m')} = \left\{ \frac{\mu^{(m')}(\Omega_x)}{\mu^{(m')}(\Omega_{\breve{x}})} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\breve{x}})} \right\}.$$

Let us first show that the event $A_x^{(m)}$ is $r^{\beta, \rho_m, \phi}$-measurable. Note that the exiting measure $\mu^{(m)}$ is measurable with respect to the corresponding environment $r^{\beta, \rho_m, \phi}$. Moreover, for $i \neq \phi$, $\beta_i = \tilde{\beta}_i = \sum_{j \sim i} r_{i,j}^{\beta, \rho_m, \phi}$ is $r^{\beta, \rho_m, \phi}$-measurable. Therefore, we just have to show that $\frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\breve{x}})}$ is $\beta_{T \setminus \{\phi\}}$-measurable. Since $\chi(\phi, \Omega_x) = \hat{G}(\phi, x) \sum_{y \in S(x)} W_{x,y} \hat{\chi}_y$, we have

$$\frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\breve{x}})} = \frac{\hat{G}(\phi, x) \sum_{y \in S(x)} W_{x,y} \hat{\chi}_y}{\hat{G}(\phi, \breve{x}) \sum_{z \in S(\breve{x})} W_{\breve{x},z} \hat{\chi}_z} = \hat{F}(x, \breve{x}) \frac{\sum_{y \in S(x)} W_{x,y} \hat{\chi}_y}{\sum_{z \in S(\breve{x})} W_{\breve{x},z} \hat{\chi}_z},$$

which is $\beta_{U_{\breve{x}}}$-measurable and therefore $\beta_{T \setminus \{\phi\}}$-measurable. We can conclude that $A_x^{(m)}$ is $r^{\beta, \rho_m, \phi}$-measurable, and in the same way, $A_x^{(m')}$ is $r^{\beta, \rho_{m'}, \phi}$-measurable. We are now going to show that under $\nu_{T, B_{m'}}^W(d\beta, d\rho_{m'})$ we have $\mathbb{P}[A_x^{(m')}] = 1$, while under $\nu_{T, B_m}^W(d\beta, d\rho_m)$ we have $\mathbb{P}[A_x^{(m)}] = 0$. This will prove that the distributions of $r^{\beta, \rho_m, \phi}$ under $\nu_{T, B_m}^W(d\beta, d\rho_m)$ and $r^{\beta, \rho_{m'}, \phi}$ under $\nu_{T, B_{m'}}^W(d\beta, d\rho_{m'})$ are different.

Since $|x| = m > m'$, we have $|\breve{x}| \geq m'$, so there exists $z \in D^{(m')}$ such that $\breve{x} \in T_z$, i.e. $\Omega_{\breve{x}} \subset \Omega_z$. Then for all $\tau \in \Omega_{\breve{x}}$, $\int_\Omega \chi(\phi, d\omega) \breve{g}_{m'}(\omega, \tau) = \sum_{b \in B_{m'}} \chi_{m'}(\phi, b) \check{G}_{m'}(b, \delta_z)$. As a

result,

$$\frac{\mu^{(m')}(\Omega_x)}{\mu^{(m')}(\Omega_{\bar{x}})} = \frac{\int_{\Omega_x} \left(\sum_{b \in B_{m'}} \chi_{m'}(\phi, b) \check{G}_{m'}(b, \delta_z) \right) \chi(\phi, d\tau)}{\int_{\Omega_{\bar{x}}} \left(\sum_{b \in B_{m'}} \chi_{m'}(\phi, b) \check{G}_{m'}(b, \delta_z) \right) \chi(\phi, d\tau)} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})},$$

so $\mathbb{P}[A_x^{(m')}] = 1$ under $\nu_{T, B_{m'}}^W(d\beta, d\rho_{m'})$.

We will finally show that ν_{T, B_m}^W-almost surely, $\mathbb{P}[A_x^{(m)} | \beta] = 0$. Since $|x| = m$, we have

$$\mu^{(m)}(\Omega_x) = \chi_m(\phi, \delta_x) \sum_{b \in B_m} \check{G}_m(\delta_x, b) \chi_m(\phi, b),$$

$$\text{and } \mu^{(m)}(\Omega_{\bar{x}}) = \sum_{y \in S(\bar{x})} \chi_m(\phi, \delta_y) \sum_{b \in B_m} \check{G}_m(\delta_y, b) \chi_m(\phi, b).$$

Let us denote, for $y \in D^{(m)}$, $u_y = \sum_{b \in B_m} \check{G}_m(\delta_y, b) \chi_m(\phi, b)$. Then

$$A_x^{(m)} = \left\{ \frac{\mu^{(m)}(\Omega_x)}{\mu^{(m)}(\Omega_{\bar{x}})} = \frac{\chi(\phi, \Omega_x)}{\chi(\phi, \Omega_{\bar{x}})} \right\} = \left\{ \sum_{y \in S(\bar{x})} \frac{\chi(\phi, \Omega_y)}{\chi(\phi, \Omega_{\bar{x}})} u_y = u_x \right\} = \{u \in \ker(f_\beta)\},$$

where $f_\beta : (v_y)_{y \in D^{(m)}} \mapsto \sum_{y \in S(\bar{x})} \frac{\chi(\phi, \Omega_y)}{\chi(\phi, \Omega_{\bar{x}})} v_y - v_x$ is a linear form conditionally on β, which has almost surely rank 1 according to Lemma 7.5, so that $\ker(f_\beta)$ is a hyperplane of $\mathbb{R}^{|D^{(m)}|}$. Let us show that conditionally on β, the distribution of $(u_y)_{y \in D^{(m)}}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{|D^{(m)}|}$, and therefore $\mathbb{P}[A_x^{(m)} | \beta] = \mathbb{P}[u \in \ker(f_\beta) | \beta] = 0$.

Recall that $\check{G}_m = (2\rho_m - \check{C}_m)^{-1}$, where conditionally on β, ρ_m is distributed according to $\nu_{B_m}^{\check{C}_m}$, which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{|B_m|} = \mathbb{R}^{|D^{(m)}|}$. Let us define

$$\Phi : \begin{array}{ccc} \mathbb{R}^{|D^{(m)}|} & \longrightarrow & \mathbb{R}^{|D^{(m)}|} \\ \rho_m & \longmapsto & (u_y)_{y \in D^{(m)}} = \check{G}_m \chi_m(\phi, \cdot) \end{array}.$$

For all ρ_m such that $2\rho_m - \check{C}_m > 0$, Φ is differentiable, and its differential is

$$d_{\rho_m} \Phi(v) = -2\check{G}_m \text{diag}(v) \check{G}_m \chi_m(\phi, \cdot) = -2\check{G}_m \text{diag}(v) u,$$

which is invertible, with $(d_{\rho_m} \Phi)^{-1}(w) = \left(-\frac{(\check{G}_m^{-1} w)_y}{2u_y} \right)_{y \in D^{(m)}}$. Note that this is well-defined since $u_y > 0$ for all $y \in D^{(m)}$, thanks to Lemma 7.5. As a result, Φ is a local diffeomorphism. Therefore, the distribution of $u = \Phi(\rho_m)$, conditionally on β, admits a density with respect to the Lebesgue measure on $\mathbb{R}^{|D^{(m)}|}$. We deduce that almost surely, $\mathbb{P}[A_x^{(m)} | \beta] = \mathbb{P}[u \in \ker(f_\beta) | \beta] = 0$, and therefore $\mathbb{P}[A_x^{(m)}] = 0$, which concludes the proof. $\square$

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