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The Pareto record frontier*

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Abstract

For i.i.d. d-dimensional observations $X^{(1)}, X^{(2)}, \dots$ with independent Exponential(1) coordinates, consider the boundary (relative to the closed positive orthant), or "frontier", F_n of the closed Pareto record-setting (RS) region

$$\mathrm{RS}_n := \{0 \le x \in \mathbb{R}^d : x \not\prec X^{(i)} \text{ for all } 1 \le i \le n\}$$

at time n, where $0 \le x$ means that $0 \le x_j$ for $1 \le j \le d$ and $x \prec y$ means that $x_j < y_j$ for $1 \le j \le d$. With $x_+ := \sum_{j=1}^d x_j$, let

$$F_n^- := \min\{x_+ : x \in F_n\} \quad \text{and} \quad F_n^+ := \max\{x_+ : x \in F_n\},$$

and define the *width* of F_n as

$$W_n := F_n^+ - F_n^-.$$

We describe typical and almost sure behavior of the processes F^+, F^-, and W. In particular, we show that $F_n^+ \sim \ln n \sim F_n^-$ almost surely and that $W_n / \ln \ln n$ converges in probability to $d-1$; and for $d \ge 2$ we show that, almost surely, the set of limit points of the sequence $W_n / \ln \ln n$ is the interval $[d-1, d]$.

We also obtain modifications of our results that are important in connection with efficient simulation of Pareto records. Let T_m denote the time that the mth record is set. We show that $F_{T_m}^+ \sim (d!m)^{1/d} \sim F_{T_m}^-$ almost surely and that $W_{T_m} / \ln m$ converges in probability to $1 - d^{-1}$; and for $d \ge 2$ we show that, almost surely, the sequence $W_{T_m} / \ln m$ has $\liminf$ equal to $1 - d^{-1}$ and $\limsup$ equal to 1.

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1 Introduction, background, and main results

The study of univariate records is very well developed ([1] being a classical reference), but that of multivariate records less well so, in part because there are many ways one can formulate the latter concept. See [6], and the references therein, and [1, Chap. 8] for background.

This paper is mainly about the stochastic process (F_n), where F_n is the boundary, or "frontier", for *Pareto records* (otherwise known as *nondominated records* or *weak records*; consult Definitions 1.1–1.2) in general dimension d when the observed sequence of points $X^{(1)}, X^{(2)}, \ldots$ are assumed (as they are throughout the paper) to be i.i.d. (independent and identically distributed) copies of a d-dimensional random vector X with independent Exponential(1) coordinates X_j.

Theoretical investigation leading to the results in this paper were spurred by empirical observations whose generation is discussed briefly in Section 5 (see especially Figure 3) and in detail in [5] and began with the simple result of Theorem 1.4.

Notation: Throughout this paper we abbreviate the kth iterate of natural logarithm $\ln$ by L_k and L_1 by L, and we write $x_+ := \sum_{j=1}^{d} x_j$ for the sum of coordinates of the d-dimensional vector $x = (x_1, \ldots, x_d)$.

Unless otherwise specifically noted, all the results of this paper hold for any dimension $d \geq 1$.

1.1 Pareto records and the record-setting region

We begin with some definitions. Write $x \prec y$ (respectively, $x \leq y$) to mean that $x_j < y_j$ (resp., $x_j \leq y_j$) for $1 \leq j \leq d$. (We caution that, with this convention, $\leq$ is weaker than $\preceq$, the latter meaning "$\prec$ or $=$"; indeed, $(0,0) \leq (0,1)$ but we have neither $(0,0) \prec (0,1)$ nor $(0,0) = (0,1)$. This distinction will matter little in this paper, since the probability that any coordinate of an observation is repeated or vanishes is 0, but the distinction is important in [5].) The notation $x \succ y$ means $y \prec x$, and $x \geq y$ means $y \leq x$.

Definition 1.1. *(a) We say that $X^{(k)}$ is a* (Pareto) record *(or that it* sets *a record at time k) if $X^{(k)} \not\prec X^{(i)}$ for all $1 \leq i < k$.*

(b) If $1 \leq k \leq n$, we say that $X^{(k)}$ is a current record *(or* remaining record*, or* maximum*) at time n if $X^{(k)} \not\prec X^{(i)}$ for all $1 \leq i \leq n$.*

(c) If $1 \leq k \leq n$, we say that $X^{(k)}$ is a broken record *at time n if it is a record but not a current record, that is, if $X^{(k)} \not\prec X^{(i)}$ for all $1 \leq i < k$ but $X^{(k)} \prec X^{(\ell)}$ for some $k < \ell \leq n$; in that case, the observation corresponding to the smallest such ℓ is said to* break *or* kill *the record $X^{(k)}$.*

For $n \geq 1$ (or $n \geq 0$, with the obvious conventions) let R_n denote the number of records $X^{(k)}$ with $1 \leq k \leq n$, let r_n denote the number of remaining records at time n, and let $\beta_n := R_n - r_n$ denote the number of broken records. Note that R_n and β_n are nondecreasing in n, but the same is not true for r_n. For dimension $d \geq 2$, by standard consideration of concomitants [that is, by considering the d-dimensional sequence $X^{(1)}, \ldots, X^{(n)}$ sorted from largest to smallest value of (say) last coordinate] we see that $r_n(d)$ (that is, r_n for dimension d, with similar notation used here for R_n) has, for each n, the same (univariate) distribution as $R_n(d-1)$; note, however, the same equality in distribution does *not* hold for the stochastic processes $r(d)$ and $R(d-1)$.

Definition 1.2. *(a) The* record-setting region *at time n is the (random) closed set of points*

$$RS_n := \{x \in \mathbb{R}^d : 0 \leq x \not\prec X^{(i)} \text{ for all } 1 \leq i \leq n\}.$$

(b) We call the (topological) boundary of RS_n (relative to the closed positive orthant determined by the origin) its frontier *and denote it by F_n.*

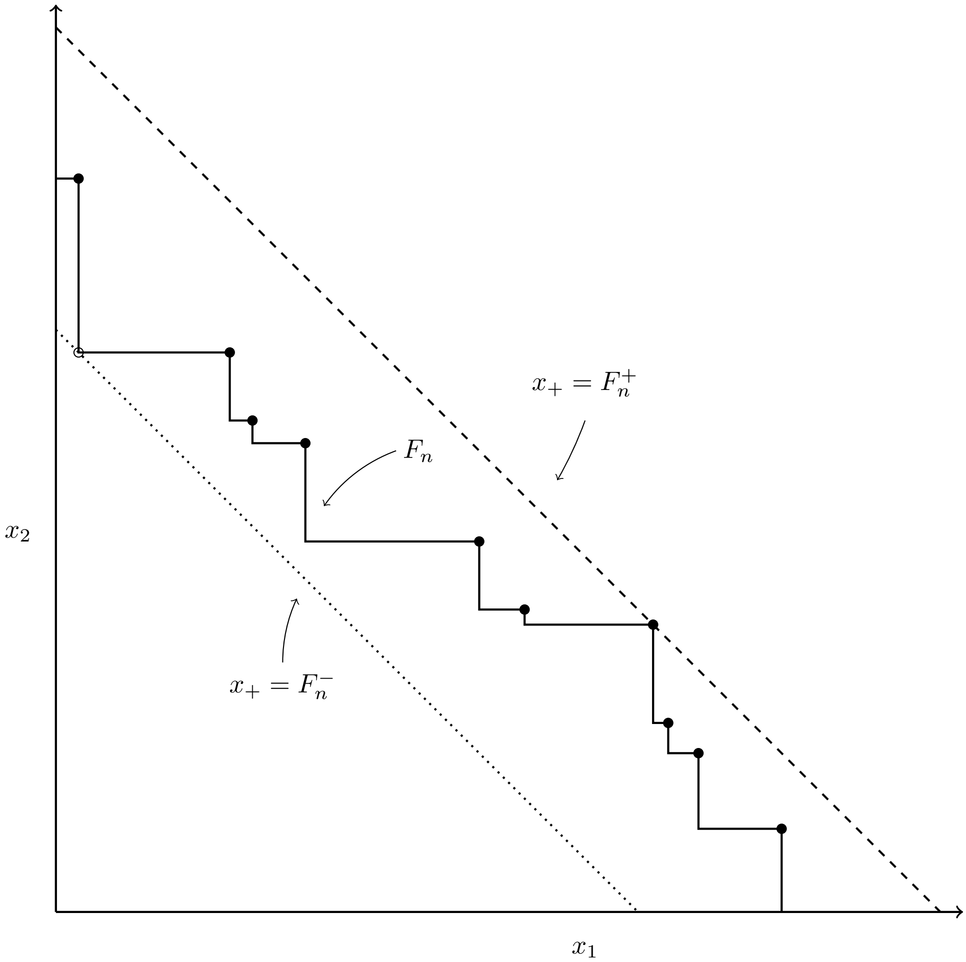

Figure 1: Record frontier F_n based on n observations resulting in 10 current records (shown as solid points). The values $F_n^- = \min\{x_+ : x \in F_n\}$ and $F_n^+ = \max\{x_+ : x \in F_n\}$ determine two hyperplanes $x_+ = F_n^-$ and $x_+ = F_n^+$. A new observation sets a record if and only if it falls in the region to the upper right of F_n.

Remark 1.3. The terminology in Definition 1.2(a) is natural since the next observation $X^{(n+1)}$ sets a record if and only if it falls in the record-setting region. Note that

$$\mathrm{RS}_n = \{x \in \mathbb{R}^d : 0 \le x \not\prec X^{(i)} \text{ for all } 1 \le i \le n \\ \text{such that } X^{(i)} \text{ is a current record at time } n\},$$

and that the current records at time n all belong to RS_n but lie on its frontier. Observe also that F_n is a closed subset of RS_n. Because this paper makes heavy use of the classical probabilistic notion of boundary-crossing probabilities, to avoid confusion we have chosen to use the term “frontier” for F_n, rather than “boundary”, in Definition 1.2(b).

1.2 The record-setting frontier

Our first result shows that deviations of the sum of coordinates for a generic current record at time n from $\mathrm{L}\, n$ are typically of constant order. Observe that the conditional distribution of $X_+^{(k)}$ given that $X^{(k)}$ is a current record at time n doesn’t depend on $k \in \{1, \dots, n\}$; in particular, it’s the conditional distribution of $X_+^{(n)}$ given that $X^{(n)}$ sets a record. Let Y_n be a random variable with that distribution. Let G denote a random variable with the standard Gumbel distribution (i.e., distribution function $x \mapsto e^{-e^{-x}}$, $x \in \mathbb{R}$), and write $\xrightarrow{\mathcal{L}}$ for convergence in law (i.e., in distribution)

Theorem 1.4. *We have*

$$Y_n - \mathrm{L}\, n \xrightarrow{\mathcal{L}} G.$$

Proof. This is quite elementary. Let p_n denote the probability that $X^{(n)}$ sets a record. Fix $n \geq 2$ for the moment. For $x \succ 0$ we have

$$\begin{aligned}
&\mathbb{P}(X^{(n)} \in dx \,|\, X^{(n)} \not\prec X^{(i)} \text{ for all } 1 \leq i \leq n) \\
&= p_n^{-1}\, \mathbb{P}(X^{(n)} \in dx,\, X^{(n)} \not\prec X^{(i)} \text{ for all } 1 \leq i \leq n) \\
&= p_n^{-1}\, \mathbb{P}(X^{(n)} \in dx,\, x \not\prec X^{(i)} \text{ for all } 1 \leq i \leq n-1) \\
&= p_n^{-1}\, \mathbb{P}(X^{(n)} \in dx)\, \mathbb{P}(x \not\prec X^{(i)} \text{ for all } 1 \leq i \leq n-1) \\
&= p_n^{-1} e^{-x_+}[1 - \mathbb{P}(x \prec X^{(1)})]^{n-1}\, dx = p_n^{-1} e^{-x_+}(1 - e^{-x_+})^{n-1}\, dx,
\end{aligned}$$

and so the conditional density depends on x only through x_+. It follows that the density $f_n(y)$ of Y_n satisfies

$$f_n(y) = p_n^{-1} \frac{y^{d-1}}{(d-1)!} e^{-y}(1 - e^{-y})^{n-1}, \quad y > 0.$$

Using the well-known asymptotic equivalence $p_n \sim n^{-1}(\mathrm{L}\, n)^{d-1}/(d-1)!$ as $n \to \infty$ [see (4.5) below], it is easy to check that, for each fixed $z \in \mathbb{R}$, the density of $Y_n - \mathrm{L}\, n$ at z converges to the standard Gumbel density $e^{-z} e^{-e^{-z}}$ as $n \to \infty$. The claimed result thus follows from Scheffé's theorem (e.g., [4, Thm. 16.12]), which shows that there is in fact convergence in total variation. □

This paper primarily concerns the stochastic process (F_n), and specifically its "width" as defined next (see Figure 1).

Definition 1.5. *Recall that F_n denotes the frontier of RS$_n$, and let*

$$F_n^- := \min\{x_+ : x \in F_n\} \quad \textit{and} \quad F_n^+ := \max\{x_+ : x \in F_n\}. \tag{1.1}$$

We define the width *of F_n as*

$$W_n := F_n^+ - F_n^-. \tag{1.2}$$

Very roughly put, what we will see in this paper is that, unlike Y_n of Theorem 1.4, deviations of F_n^+ from $\mathrm{L}\, n$ are exactly of order $\mathrm{L}_2\, n$; on the other hand, we will see that deviations of F_n^- from $\mathrm{L}\, n$ are of smaller order than $\mathrm{L}_2\, n$. It will follow that the width of the frontier is exactly of order $\mathrm{L}_2\, n$.

We next make some simple observations about the quantities appearing in Definition 1.5 that will prove fundamentally useful to our development.

Lemma 1.6 (characterization of F_n^+)**.** *We have*

$$F_n^+ = \max\{X_+^{(k)} : 1 \leq k \leq n\},$$

which is nondecreasing in n.

Proof. The current records at time n all belong to F_n, and broken records and non-records all have coordinate-sums (strictly) smaller than some current record. Thus $F_n^+ \geq \max\{X_+^{(k)} : 1 \leq k \leq n\}$. Conversely, if $x \in F_n$, then $x \preceq X^{(i)}$ for some i; it follows that $F_n^+ \leq \max\{X_+^{(k)} : 1 \leq k \leq n\}$. □

Lemma 1.7 (two upper bounds on F_n^-)**.**
(a) *Define*

$$B_n^+(j) := \max\{X_j^{(i)} : 1 \le i \le n\}.$$

Then

$$F_n^- \le \min_{1 \le j \le d} B_n^+(j).$$

(b) *Let* $1 \le m \le n$. *Define*

$$B_{m,n} := m^{\text{th}}\text{-largest value among } X_+^{(k)} \text{ with } 1 \le k \le n.$$

Then, over the event $\{r_n \ge m\}$ *that there are at least* m *remaining records at time* n*, we have*

$$F_n^- \le B_{m,n}.$$

(c) *The processes* F^-, $\min_{1 \le j \le d} B^+(j)$*, and* $B_{m,\cdot}$ *(for any* m*) all have nondecreasing sample paths.*

Proof. (a) For $j = 1, \dots, d$, let $i_j \in \{1, \dots, n\}$ denote the almost surely unique index such that

$$X_j^{(i_j)} = \max\{X_j^{(i)} : 1 \le i \le n\}.$$

Let $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ denote the jth coordinate vector. We claim that the points $Y^{(j)} := X^{(i_j)} e_j$ with $j = 1, \dots, d$ all belong to F_n (in fact, to $F_n \cap \mathrm{RS}_n$), and then the inequality is immediate. To prove the claim, note that all of the points $Y^{(j)}$ belong to RS_n [because $Y_j^{(j)} = X_j^{(i_j)}$ and hence $Y^{(j)} \not\prec X^{(i_j)}$] but also to F_n [because $Y^{(j)} \le X^{(i_j)}$].

(b) Over the event $\{r_n \ge m\}$, F_n^- is certainly at most the mth-largest sum of coordinates of remaining records, which is in turn at most $B_{m,n}$.

(c) The asserted monotonicity is clear for the bounding processes. The asserted monotonicity of F^- follows easily from the observation that $F_{n+1} \subseteq \mathrm{RS}_{n+1} \subseteq \mathrm{RS}_n$. □

It seems difficult to study the processes F^+ and F^- bivariately, so we draw all our conclusions about the width process W by studying F^+ and F^- univariately (that is, separately) and using $W = F^+ - F^-$. The behavior of F^+ is well known from classical extreme value theory and is reviewed in Section 2. Conclusions about F^- will be drawn from (i) the upper-bounding processes in Lemma 1.7(a)–(b) together with classical extreme value theory for those bounding processes and (ii) a rather nontrivial lower bound developed in Section 3.

1.3 Main results

We next present the main results of our paper. What the results show, in various precise senses, is that F_n^+ and F_n^- both concentrate near $\mathrm{L}\, n$, with deviations that are $O(\mathrm{L}_2\, n)$, from which it follows of course that $W_n = O(\mathrm{L}_2\, n)$. But for $d \ge 2$ we show more, namely, that $\mathrm{L}_2\, n$ is the exact scale for W_n, that is, that $W_n = \Theta(\mathrm{L}_2\, n)$. We can even narrow things down further: $W_n / \mathrm{L}_2\, n \to d - 1$ in probability for each $d \ge 1$, with an almost sure $\liminf$ equal to $d - 1$ and an almost sure $\limsup$ equal to d.

Here are our main results for arbitrary but fixed dimension $d \ge 1$. We consider both convergence in probability (typical behavior) and almost sure largest and smallest deviations from $\mathrm{L}\, n$ (top and bottom boundary-behavior, respectively) for large n.

Theorem 1.8 (Kiefer [7])**.** *Consider the process* F^+ *defined at* (1.1).

(a) Typical behavior of F^+:

$$F_n^+ - [\mathrm{L}\, n + (d-1)\, \mathrm{L}_2\, n - \mathrm{L}((d-1)!)] \xrightarrow{\mathcal{L}} G.$$

(b) Top boundaries for F^+:

$$\mathbb{P}(F_n^+ \geq \mathrm{L}\, n + c\, \mathrm{L}_2\, n \text{ i.o.}) = \begin{cases} 1 & \text{if } c \leq d; \\ 0 & \text{if } c > d. \end{cases}$$

(c) Bottom boundaries for F^+:

$$\mathbb{P}(F_n^+ \leq \mathrm{L}\, n + (d-1)\, \mathrm{L}_2\, n - \mathrm{L}_3\, n - \mathrm{L}((d-1)!) + c \text{ i.o.}) = \begin{cases} 1 & \text{if } c \geq 0; \\ 0 & \text{if } c < 0. \end{cases}$$

Theorem 1.8 gives rise immediately to the following succinct corollary.

Corollary 1.9 (Kiefer [7]). *Consider the process F^+ defined at* (1.1).

(a) Typical behavior of F^+:

$$\frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} \xrightarrow{\mathrm{P}} d - 1.$$

(b) Almost sure behavior for F^+:

$$\liminf \frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} = d - 1 < d = \limsup \frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} \text{ a.s.}$$

Remark 1.10. In fact, one can show rather simply from Corollary 1.9(b) and the fact that F^+ has nondecreasing sample paths that the set (call it Λ) of limit points of the sequence $(F_n^+ - \mathrm{L}\, n)/\, \mathrm{L}_2\, n$ is almost surely the closed interval $[d-1, d]$. Here is a sketch of the proof. The set Λ is closed, so we need only show that Λ is dense in $[d-1, d]$, which clearly follows if we can show that

$$\limsup_{n \to \infty} \left[\frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} - \frac{F_{n+1}^+ - \mathrm{L}(n+1)}{\mathrm{L}_2(n+1)} \right] \leq 0 \text{ a.s.}, \tag{1.3}$$

the roughly stated idea being that then (a.s.) the sequence $(F_n^+ - \mathrm{L}\, n)/\, \mathrm{L}_2\, n$ "can't leap downward over any interval i.o." in its infinitely many downward moves from its $\limsup$ to its $\liminf$. To prove (1.3), we first bound F_{n+1}^+ from below by F_n^+, then express the resulting difference with a common denominator, and finally use the consequence $F_n^+ \sim \mathrm{L}\, n$ a.s. of Corollary 1.9(b) to find

$$\begin{aligned} &\frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} - \frac{F_{n+1}^+ - \mathrm{L}(n+1)}{\mathrm{L}_2(n+1)} \\ &\leq \frac{(1+o(1))(n\, \mathrm{L}\, n)^{-1} F_n^+ + (1+o(1)) n^{-1}\, \mathrm{L}_2\, n}{(1+o(1))(\mathrm{L}_2\, n)^2} \sim n^{-1} (\mathrm{L}_2\, n)^{-1} = o(1) \text{ a.s.} \end{aligned}$$

as $n \to \infty$.

Remark 1.11. Our Theorem 1.8 formalizes and improves upon related computations in Bai et al. [3, Secs. 1 and 3.2] who, for the limited purpose of proving a central limit theorem reviewed in Theorem 4.1(a) below, "observe that nearly all maxima occur in a thin strip sandwiched between [the] two parallel hyper-planes"

$$x_+ = \mathrm{L}\, n - \mathrm{L}_3\, n - \mathrm{L}[4(d-1)] \quad \text{and} \quad x_+ = \mathrm{L}\, n + 4(d-1)\, \mathrm{L}_2\, n.$$

Our results for F^- show that the deviations of F_n^- from $\mathrm{L}\, n$ are almost surely negligible on a scale of $\mathrm{L}_2\, n$.

Theorem 1.12. *Consider the process F^- defined at* (1.1)*.*

(a) Typical behavior of F^-:

$$\mathbb{P}(F_n^- \le \mathrm{L}\, n - 3\, \mathrm{L}_3\, n) \to 0$$

and

$$\mathbb{P}(F_n^- \ge \mathrm{L}\, n + c_n) \to 0 \text{ if } c_n \to \infty.$$

(b) Top outer boundaries for F^-: *If $d \ge 2$, then*

$$\mathbb{P}(F_n^- \ge \mathrm{L}\, n + c\, \mathrm{L}_2\, n \text{ i.o.}) = 0 \text{ if } c > 0.$$

(c1) A bottom outer boundary for F^- on the scale of $\mathrm{L}_3\, n$:

$$\mathbb{P}(F_n^- \le \mathrm{L}\, n - 3\, \mathrm{L}_3\, n \text{ i.o.}) = 0.$$

(c2) A bottom inner boundary for F^- on the scale of $\mathrm{L}_3\, n$:

$$\mathbb{P}(F_n^- \le \mathrm{L}\, n - \mathrm{L}_3\, n \text{ i.o.}) = 1.$$

Theorem 1.12 gives rise immediately to the following succinct corollary.

Corollary 1.13. *Consider the process F^- defined at* (1.1)*.*

(a) Typical behavior of F^-:

$$\frac{F_n^- - \mathrm{L}\, n}{\mathrm{L}_2\, n} \xrightarrow{\mathrm{P}} 0.$$

(b) Almost sure behavior for F^-: *If $d \ge 2$, then*

$$\lim \frac{F_n^- - \mathrm{L}\, n}{\mathrm{L}_2\, n} = 0 \text{ a.s.}$$

We come now to our main focus, the process W. The results in Theorem 1.14 follow directly from Corollaries 1.9 and 1.13.

Theorem 1.14. *Consider the process W defined at* (1.2)*.*

(a) Typical behavior of W:

$$\frac{W_n}{\mathrm{L}_2\, n} \xrightarrow{\mathrm{P}} d - 1.$$

(b) Almost sure behavior for W: *If $d \ge 2$, then*

$$\liminf \frac{W_n}{\mathrm{L}_2\, n} = d - 1 < d = \limsup \frac{W_n}{\mathrm{L}_2\, n} \text{ a.s.},$$

and, in particular,

$$W_n = \Theta(\mathrm{L}_2\, n) \text{ a.s.}$$

Remark 1.15. (a) When $d = 1$, at each time $n \ge 1$ there is exactly one current record, $F_n^+ = F_n^-$ is the value of that record, RS_n is the closed interval $[F_n^+, \infty)$, and $W_n = 0$.

(b) Using Remark 1.10, Theorem 1.14(b) can be strengthened to the conclusion that the set of limit points of the sequence $W_n / \mathrm{L}_2\, n$ is almost surely the closed interval $[d-1, d]$.

(c) Theorem 1.14(b) has the following immediate corollary. If, for some positive integer d_0, processes $W(d)$ corresponding to dimension d, $d = d_0, d_0 + 1, \dots$, are defined on a common probability space (regardless of any dependence among the processes), then

$$\lim_{d \to \infty} \limsup_{n \to \infty} \frac{W_n(d)}{(d-1)\, \mathrm{L}_2\, n} = 1 = \lim_{d \to \infty} \liminf_{n \to \infty} \frac{W_n(d)}{(d-1)\, \mathrm{L}_2\, n} \text{ a.s.} \tag{1.4}$$

That is, roughly speaking, for time n large relative to large dimension d, the width $W_n(d)$ almost surely concentrates near $(d-1)\,\mathrm{L}_2\,n$.

(d) We could have used d in the denominators of (1.4), but we chose $d-1$ because of Theorem 1.14(a). A remark of a somewhat similar flavor as (b) for convergence in probability is the following. If, for some integer $d_0 \geq 2$, processes $W(d)$ corresponding to dimension d, $d = 2, \ldots, d_0$, are defined on a common probability space (regardless of any dependence among the processes), then

$$\max_{2 \leq d \leq d_0} \left| \frac{W_n(d)}{(d-1)\,\mathrm{L}_2\,n} - 1 \right| \xrightarrow{\mathrm{P}} 0.$$

We have not investigated whether this result might extend to dimension d_0 growing with n.

1.4 Outline of paper

The stochastic process F^+ is studied in Section 2, where we prove Theorem 1.8. We treat the process F^- in Section 3, where we prove Theorem 1.12. In Section 4 we assess asymptotic behavior of the record counts R_n, r_n, and β_n introduced following Definition 1.1 as preparation for Section 5, where we produce versions of our main results concerning the record-setting frontier process F when time is measured in the number of records (rather than observations $X^{(i)}$) generated.

2 The process F^+

This section is devoted to the proof of Theorem 1.8 concerning the process F^+ defined at (1.1). In light of the characterization provided by Lemma 1.6, Theorem 1.8 follows from results of [7]. Kiefer is concerned with behavior of the law of the iterated logarithm type for the empirical distribution function and sample p_n-quantiles for a sequence of independent uniform$(0,1)$ random variables, with $p_n > 0$ and $p_n \downarrow 0$, but notes that his results "may easily be translated into results for general laws." Since we are concerned here with a sequence $X_+^{(1)}, X_+^{(2)}, \ldots$ from the Gamma$(d,1)$ distribution and with (only) the $p_n = 1/n$ *upper* quantile, for completeness and the reader's convenience we distill Kiefer's proof(s) for our special case.

Proof of Theorem 1.8. (a) This is elementary. We have

$$\begin{aligned}
&\mathbb{P}(F_n^+ - [\mathrm{L}\,n + (d-1)\,\mathrm{L}_2\,n - \mathrm{L}((d-1)!)] \leq x) \\
&\quad = \left[\mathbb{P}\left(X_+^{(1)} - [\mathrm{L}\,n + (d-1)\,\mathrm{L}_2\,n - \mathrm{L}((d-1)!)] \leq x\right)\right]^n \\
&\quad = \left[\mathbb{P}\left(X_+^{(1)} \leq \mathrm{L}\,n + (d-1)\,\mathrm{L}_2\,n - \mathrm{L}((d-1)!) + x\right)\right]^n \\
&\quad = \left(1 - \sum_{j=0}^{d-1} e^{-\lambda}\frac{\lambda^j}{j!}\right)^n = \left[1 - (1+o(1))e^{-\lambda}\frac{\lambda^{d-1}}{(d-1)!}\right]^n \\
&\quad = \left[1 - (1+o(1))n^{-1}e^{-x}\right]^n \to e^{-e^{-x}} = \mathbb{P}(G \leq x),
\end{aligned}$$

where $\lambda := \mathrm{L}\,n + (d-1)\,\mathrm{L}_2\,n - \mathrm{L}((d-1)!) + x$.

(b) Kiefer describes two proofs. The first proof observes, for any sequence $b_n \to \infty$ which is ultimately monotone nondecreasing, that

$$\{F_n^+ > b_n \text{ i.o.}\} = \{X_+^{(n)} > b_n \text{ i.o.}\}$$

and applies the Borel–Cantelli lemmas to the sequence of independent events $\{X_+^{(n)} > b_n\}$ with $b_n \equiv \mathrm{L}\,n + c\,\mathrm{L}_2\,n$. The second proof exploits the nondecreasingness of the sample paths of the process $F^+_{\cdot} = B_{1,\cdot}$ noted in Lemma 1.7 and proceeds as follows. If (b_n) is ultimately monotone nondecreasing and (n_j) is any strictly increasing sequence of positive integers, then

$$\{F^+_{n_j,n_{j+1}} \ge b_{n_{j+1}} \text{ i.o.}(j)\} \subseteq \{F^+_n \ge b_n \text{ i.o.}(n)\} \subseteq \{F^+_{n_{j+1}} \ge b_{n_j} \text{ i.o.}(j)\},$$

where we note that the random variables

$$F^+_{n_j,n_{j+1}} \equiv \max\{X^{(k)}_+ : n_j < k \le n_{j+1}\} \tag{2.1}$$

are independent. Now choose $b_n \equiv \mathrm{L}\,n + c\,\mathrm{L}_2\,n$ and $n_j \equiv 2^j$ and apply the Borel–Cantelli lemmas.

(c) For the case $c < 0$ of outer-class bottom boundaries, we start with the observation that if (b_n) is ultimately monotone nondecreasing and (n_j) is any strictly increasing sequence of positive integers, then

$$\{F^+_n \le b_n \text{ i.o.}(n)\} \subseteq \{F^+_{n_j} \le b_{n_{j+1}} \text{ i.o.}(j)\}.$$

We then choose $b_n \equiv \mathrm{L}\,n + (d-1)\,\mathrm{L}_2\,n - \mathrm{L}_3\,n - \mathrm{L}((d-1)!) + c$ with $c < 0$ and $n_j \equiv \lfloor e^{|c|j/2} \rfloor$ and apply the first Borel–Cantelli lemma.

For the case $c \ge 0$ of inner-class bottom boundaries, we start with the observation that if (b_n) is ultimately monotone nondecreasing and (n_j) is any strictly increasing sequence of positive integers, then, recalling the definition (2.1),

$$\begin{aligned}\{F^+_{n_j} \le b_{n_{j+1}} \text{ a.a.}(j)\} &\cap \{F^+_{n_j,n_{j+1}} \le b_{n_{j+1}} \text{ i.o.}(j)\} \\ &\subseteq \{F^+_{n_{j+1}} \le b_{n_{j+1}} \text{ i.o.}(j)\} \subseteq \{F^+_n \le b_n \text{ i.o.}(n)\}.\end{aligned}$$

We then choose $b_n \equiv \mathrm{L}\,n + (d-1)\,\mathrm{L}_2\,n - \mathrm{L}_3\,n - \mathrm{L}((d-1)!) + c$ with $c \ge 0$ and $n_j \equiv \lfloor e^{\alpha j\,\mathrm{L}\,j} \rfloor$ with $\alpha > 1$ and apply the first Borel–Cantelli lemma to the events $\{F^+_{n_j} > b_{n_{j+1}}\}$ and the second Borel–Cantelli lemma to the independent events $\{F^+_{n_j,n_{j+1}} \le b_{n_{j+1}}\}$. □

3 The process F^-

3.1 Towards a stochastic lower bound on F^-_n

To prove Theorem 1.12 we need a stochastic lower bound on F^-_n to complement the upper bound of Lemma 1.7. For this we use the definitions of the frontier F_n and the closed record-setting region RS_n to argue as follows. For $x \in \mathbb{R}^d$, let

$$O^+_x := \{y \in \mathbb{R}^d : y \succ x\}$$

denote the open positive orthant determined by x. For any set $S \subseteq \mathbb{R}^d$, let $N_n(S)$ denote the number of observations $X^{(i)}$ with $1 \le i \le n$ that fall in S. Then

$$\begin{aligned}\{F^-_n \le b\} &= \{x_+ \le b \text{ for some } x \in F_n\} = \{x_+ \le b \text{ for some } x \in \mathrm{RS}_n\} \\ &= \{x_+ \le b \text{ for some } x \ge 0 \text{ satisfying } x \not\prec X^{(i)} \text{ for all } 1 \le i \le n\} \\ &= \bigcup_{x \ge 0:\, x_+ \le b} \{N_n(O^+_x) = 0\} \\ &= \bigcup_{x \ge 0:\, x_+ = b} \{N_n(O^+_x) = 0\}. \end{aligned} \tag{3.1}$$

The difficulty with upper-bounding the probability of this event is of course that the last union is uncountable. In the next subsection we produce a geometric lemma whose application effectively bounds the uncountable union by a finite union.

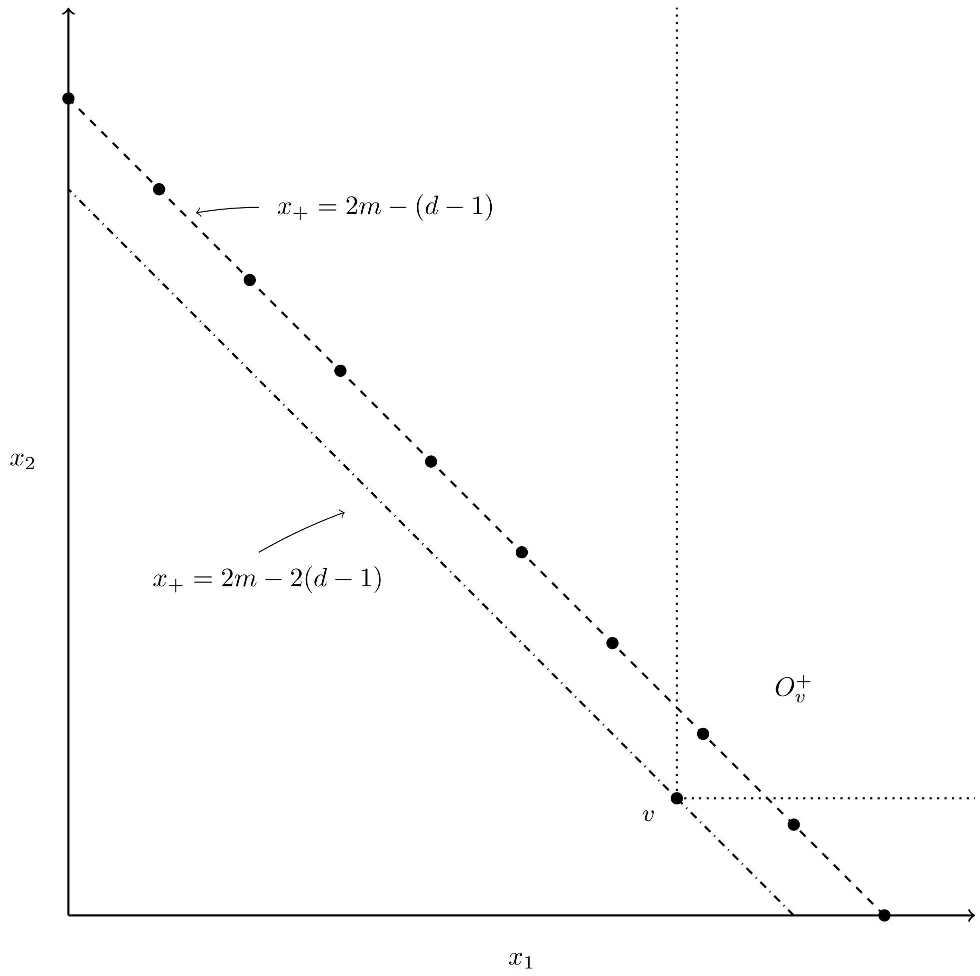

Figure 2: Geometric lemma illustrated for $d = 2$. Given v with $v_+ = 2m - 2(d-1)$, the orthant O_v^+ determined by v must contain a point i with integer coordinates on the hyperplane $x^+ = 2m - (d-1)$.

3.2 A geometric lemma

Consider the (uncountable) union of positive orthants whose vertices lie on the hyperplane $x_+ = 2m - 2(d-1)$ in $\mathbb{R}^d$, where $m \geq d-1$ is an integer. We can also form a finite union of positive orthants whose vertices lie on the hyperplane $x+ = 2m - (d-1)$ situated a bit further from the origin. Our key geometric lemma guarantees that the uncountable union contains the finite union (see Figure 2).

Lemma 3.1. *Given a positive integer $m \geq d-1$, and $0 \leq x \in \mathbb{R}^d$ with*

$$x_+ = 2m - 2(d-1), \tag{3.2}$$

there exists $0 \leq i \in \mathbb{Z}^d$ with

$$i_+ = 2m - (d-1) \tag{3.3}$$

such that

$$O_i^+ \subseteq O_x^+. \tag{3.4}$$

Proof. We need to prove the existence of $0 \le i \in \mathbb{Z}^d$ satisfying (3.3) and (3.4) (i.e., $x \le i$). The frugal choice $0 \le i' \in \mathbb{Z}^d$ defined by

$$i'_j := \lceil x_j \rceil, \quad j = 1, \dots, d,$$

satisfies (3.4) but not necessarily (3.3). However, using (3.2) we observe that i'_+ is at least the integer

$$x_+ = 2m - 2(d-1)$$

and strictly less than the integer $2m-2(d-1)+d = 2m-(d-2)$, i.e., is at most $2m-(d-1)$. Thus we need only (arbitrarily) "sweeten" (i.e., add 1 to) precisely $2m-(d-1)-i'_+ \in \mathbb{Z} \cap [0, d-1]$ of the entries i'_j to obtain i with the desired properties. ☐

3.3 A stochastic lower bound on F_n^-

Let $0 \le b < \mathrm{L}\, n$. Returning to (3.1), we now see from Lemma 3.1 with $t = \mathrm{L}\, n \ge 0$ and

$$m = \left\lceil \frac{(d-1)\,\mathrm{L}\, n}{\mathrm{L}\, n - b} \right\rceil \ge d - 1,$$

together with homogeneity [$O^+_{cy} = c\, O^+_y$ for $0 \le y \in \mathbb{R}^d$ and $0 \le c \in \mathbb{R}^1$], that

$$\begin{aligned}
\{F_n^- \le b\} &= \bigcup_{x \ge 0:\, x_+ = b} \{N_n(O^+_x) = 0\} \\
&\subseteq \bigcup_{x \ge 0:\, x_+ = \left(1 - \frac{d-1}{m}\right) t} \{N_n(O^+_x) = 0\} \\
&\subseteq \bigcup_{0 \le i \in \mathbb{Z}^d:\, i_+ = 2m-(d-1)} \left\{ N_n\left(O^+_{\frac{t}{2m} i}\right) = 0 \right\},
\end{aligned}$$

and so by finite subadditivity

$$\mathbb{P}(F_n^- \le b) \le \sum_{0 \le i \in \mathbb{Z}^d:\, i_+ = 2m-(d-1)} \mathbb{P}\left(N_n\left(O^+_{\frac{t}{2m} i}\right) = 0 \right).$$

But

$$\begin{aligned}
\mathbb{P}\left(N_n\left(O^+_{\frac{t}{2m} i}\right) = 0 \right) &= \mathbb{P}\left(X \notin O^+_{\frac{t}{2m} i} \right)^n = \left[1 - \mathbb{P}\left(X \in O^+_{\frac{t}{2m} i} \right) \right]^n \\
&= \left[1 - \exp\left(-\tfrac{t}{2m} i_+ \right) \right]^n \\
&= \left[1 - \exp\left\{ -\left(1 - \tfrac{d-1}{2m}\right) \mathrm{L}\, n \right\} \right]^n \\
&= \left[1 - n^{-\left(1 - \frac{d-1}{2m}\right)} \right]^n \\
&\le \exp\left(-n^{\frac{d-1}{2m}} \right).
\end{aligned}$$

Since the cardinality of $\{0 \le i \in \mathbb{Z}^d : i_+ = 2m-(d-1)\}$ equals

$$\binom{2m}{d-1} \le \frac{(2m)^{d-1}}{(d-1)!}$$

we conclude that

$$\begin{aligned}
\mathbb{P}(F_n^- \le b) &\le \frac{(2m)^{d-1}}{(d-1)!} \exp\left(-n^{\frac{d-1}{2m}} \right) \\
&\le (1 + o(1)) \frac{[2(d-1)]^{d-1}}{(d-1)!} \left(1 - \frac{b}{\mathrm{L}\, n} \right)^{-(d-1)} \exp\left[-\exp\left\{ (1 + o(1)) \tfrac{1}{2} (\mathrm{L}\, n - b) \right\} \right],
\end{aligned}$$

where the last inequality holds assuming that $b = b_n = (1 + o(1)) \mathrm{L}\, n$ as $n \to \infty$.

We summarize and simplify the bound we have derived in the next proposition, where we assume further that $\mathrm{L}\, n - b_n \to \infty$. The bound is the key to the proof of the first assertion in Theorem 1.12(a) and of Theorem 1.12(c1).

Proposition 3.2 (Stochastic lower bound on F_n^-). *Let* $0 \le b_n < \mathrm{L}\, n$ *with* $b_n = (1 - o(1)) \mathrm{L}\, n$ *and* $\mathrm{L}\, n - b_n \to \infty$. *Then*

$$\mathbb{P}(F_n^- \le b_n) \le (\mathrm{L}\, n)^{d-1} \exp\left[- \exp\left\{(1 + o(1)) \tfrac{1}{2} (\mathrm{L}\, n - b_n)\right\}\right]. \qquad \square$$

3.4 Proof of Theorem 1.12

In this subsection we prove Theorem 1.12, part by part in the order (a), (c1), (c2), (b).

Proof of Theorem 1.12(a). The second assertion in Theorem 1.12(a) follows from the case $d = 1$ of Theorem 1.8(a) since, according to Lemma 1.7(a), we have

$$F_n^- \le \min_{1 \le j \le d} B_n^+(j) \le B_n^+(1), \tag{3.5}$$

where we recall the definition

$$B_n^+(j) := \max\{X_j^{(i)} : 1 \le i \le n\}.$$

The first assertion follows from part (c1), proved next. $\square$

Proof of Theorem 1.12(c1). As noted in Lemma 1.7, the process F^- has nondecreasing sample paths. From this it follows that if (b_n) is (ultimately) monotone nondecreasing and (n_j) is any strictly increasing sequence of positive integers, then

$$\{F_n^- \le b_n \text{ i.o.}(n)\} \subseteq \{F_{n_j}^- \le b_{n_{j+1}} \text{ i.o.}(j)\}.$$

To complete the proof, we choose $b_n \equiv \mathrm{L}\, n - 3 \mathrm{L}_3\, n$ and $n_j \equiv 2^j$, bound $\mathbb{P}(F_{n_j}^- \le b_{n_{j+1}})$ using Proposition 3.2, and apply the first Borel–Cantelli lemma.

Here are the details. Since $\mathrm{L}\, n_j = j\, \mathrm{L}\, 2$ and

$$b_{n_{j+1}} = (j + 1) \mathrm{L}\, 2 - 3 \mathrm{L}_2[(j + 1) \mathrm{L}\, 2] = j\, \mathrm{L}\, 2 - (1 + o(1)) 3 \mathrm{L}_2\, j,$$

the hypotheses of Proposition 3.2 are met and

$$\mathbb{P}(F_{n_j}^- \le b_{n_{j+1}}) \le (j\, \mathrm{L}\, 2)^{d-1} \exp\left[- \exp\left\{(1 + o(1)) \tfrac{3}{2} \mathrm{L}_2\, j\right\}\right] = \exp\left[-(\mathrm{L}\, j)^{(1+o(1))(3/2)}\right],$$

which is summable. $\square$

Remark 3.3. We chose the constant 3 as the coefficient of $-\mathrm{L}_3\, n$ in parts (a) and (c1) of Theorem 1.12 for convenience. As the proof shows, we could have used any constant larger than 2.

Proof of Theorem 1.12(c2). This follows immediately from the case $d = 1$ of Theorem 1.8(c) using the aforementioned bound (3.5). $\square$

There remains only the proof of Theorem 1.12(b). For that we need first the following almost sure lower bound on r_n, which is of interest in its own right.

Theorem 3.4. *Assume* $d \ge 2$. *Let* r_n *denote the number of remaining records at time* n. *Then*

$$\liminf \frac{r_n}{(\mathrm{L}\, n)/(d\, \mathrm{L}_2\, n)} \ge 1 \text{ a.s.}$$

Proof. Fix $\epsilon > 0$. From Corollary 1.9(b) with $d = 1$ it follows that almost surely

$$\liminf \frac{B_n^+(1) - \mathrm{L}\, n}{\mathrm{L}_2\, n} = 0$$

and hence $B_n^+(1) \geq \mathrm{L}\, n - \epsilon\, \mathrm{L}_2\, n$ a.a. Additionally, from the now-established Corollary 1.9(b) and Theorem 1.12(c1), it follows that almost surely

$$\limsup \frac{W_n}{\mathrm{L}_2\, n} \leq d$$

and hence $W_n / \mathrm{L}_2\, n \leq (1 + \epsilon) d$ a.a.

Label the remaining records in (a.s. strictly) increasing order of first coordinate as $Z^{(1)}, \ldots, Z^{(r_n)}$, and define $Z^{(0)} := Y^{(2)}$ as defined in the proof of Lemma 1.7(a). Note in particular that the points $Z^{(i)}$ with $0 \leq i \leq r_n$ all belong to F_n, that $Z_1^{(0)} = Y_1^{(2)} = 0$, and that $Z_1^{(r_n)} = B_n^+(1)$. Therefore,

$$\mathrm{L}\, n - \epsilon\, \mathrm{L}_2\, n \leq B_n^+(1) = Z_1^{(r_n)} - Z_1^{(0)} = \sum_{i=1}^{r_n} \left(Z_1^{(i)} - Z_1^{(i-1)} \right) \leq r_n W_n \leq (1 + \epsilon)\, d\, r_n\, \mathrm{L}_2\, n$$

for all large n, almost surely. The desired result follows. □

Proof of Theorem 1.12(b). In light of Theorem 3.4 and Lemma 1.7(b), it is sufficient that for each fixed positive integer m we have

$$\mathbb{P} \left(B_{m,n} \geq \mathrm{L}\, n + \frac{a}{m}\, \mathrm{L}_2\, n \text{ i.o.} \right) = 0 \tag{3.6}$$

if $a > 1$. But (3.6) is known from [7, Thm. 1, see esp. (3.1)]. □

4 Record counts

Knowledge about the record counts R_n, r_n, and β_n discussed in Section 1 is interesting in its own right, and knowledge about R_n will be needed in the next section.

4.1 Typical behavior

In this subsection we review a known central limit theorem (CLT) of Berry–Esseen type for r_n and use it to derive easily CLTs for R_n and β_n. Here are the results. Complicated but explicit forms are known for the constants $\gamma_{d,j}$ appearing in the variance expressions.

Theorem 4.1 (Bai et al. [3, 2]). *Let Φ denote the standard normal distribution function.*

(a) *Let $d \geq 2$. Then there exist constants $\gamma_{d,j}$ with $\gamma_{d,0} \geq 1/(d-1)! > 0$ such that the number r_n of remaining records at time n satisfies*

$$\mathbb{E}\, r_n = (\mathrm{L}\, n)^{d-1} \sum_{j=0}^{d-1} \frac{(-1)^j \Gamma^{(j)}(1)}{j!(d-1-j)!} (\mathrm{L}\, n)^{-j} + O(n^{-1} (\mathrm{L}\, n)^{d-1}) \sim \frac{(\mathrm{L}\, n)^{d-1}}{(d-1)!},$$

$$\operatorname{Var} r_n = (\mathrm{L}\, n)^{d-1} \sum_{j=0}^{d-1} \gamma_{d,j} (\mathrm{L}\, n)^{-j} + O(n^{-1} (\mathrm{L}\, n)^{2d-2}) \sim \gamma_{d,0} (\mathrm{L}\, n)^{d-1},$$

and

$$\sup_x \left| \mathbb{P} \left(\frac{r_n - \mathbb{E}\, r_n}{\sqrt{\operatorname{Var} r_n}} < x \right) - \Phi(x) \right| = O((\mathrm{L}\, n)^{-(d-1)/4} (\mathrm{L}_2\, n)^d).$$

(b) *Let $d \geq 1$. Then the number R_n of records set through time n satisfies*

$$\mathbb{E}\, R_n = (\mathrm{L}\, n)^d \sum_{j=0}^{d} \frac{(-1)^j \Gamma^{(j)}(1)}{j!(d-j)!} (\mathrm{L}\, n)^{-j} + O(n^{-1} (\mathrm{L}\, n)^d) \sim \frac{(\mathrm{L}\, n)^d}{d!},$$

$$\operatorname{Var} R_n = (\mathrm{L}\, n)^d \sum_{j=0}^{d} \gamma_{d+1,j} (\mathrm{L}\, n)^{-j} + O(n^{-1} (\mathrm{L}\, n)^{2d}) \sim \gamma_{d+1,0} (\mathrm{L}\, n)^d,$$

and

$$\sup_x \left| \mathbb{P}\left(\frac{R_n - \mathbb{E}\, R_n}{\sqrt{\operatorname{Var} R_n}} < x \right) - \Phi(x) \right| = O((\mathrm{L}\, n)^{-d/4} (\mathrm{L}_2\, n)^{d+1}).$$

(c) *Let $d \geq 1$. Then the number $\beta_n = R_n - r_n$ of broken records at time n satisfies*

$$\mathbb{E}\, \beta_n = (\mathrm{L}\, n)^d \left[\frac{1}{d!} + \sum_{j=1}^{d} \frac{(-1)^j [\Gamma^{(j)}(1) + j \Gamma^{(j-1)}(1)]}{j!(d-j)!} (\mathrm{L}\, n)^{-j} \right] + O(n^{-1} (\mathrm{L}\, n)^d) \sim \frac{(\mathrm{L}\, n)^d}{d!},$$

$$\operatorname{Var} \beta_n = \gamma_{d+1,0} (\mathrm{L}\, n)^d [1 + O((\mathrm{L}\, n)^{-1/2})],$$

and the central limit theorem

$$\frac{\beta_n - \mathbb{E}\, \beta_n}{\sqrt{\operatorname{Var} \beta_n}} \text{ converges in law to standard normal.}$$

Proof. Part (a) is known from [3]: their eq. (8) for $\mathbb{E}\, r_n$, their Theorem 1 for $\operatorname{Var} r_n$, their eq. (13)—and the main theorem of [2]—for the stated lower bound on $\gamma_{d,0}$, and their Theorem 2 for the CLT.

Part (b) follows immediately from part (a) by use of concomitants. (Recall the discussion concerning concomitants preceding Definition 1.2.)

For $d = 1$, part (c) follows from part (b) because $r_n = 1$ for $n \geq 1$. For $d \geq 2$, part (c) follows from parts (a) and (b); for $\operatorname{Var} \beta_n$ we use the triangle inequality for L^2-norm after centering by means, and for the CLT we use the CLT of part (b) together with Slutsky's theorem. □

We have not attempted to find further terms in the asymptotic expansion for $\operatorname{Var} \beta_n$ nor a Berry–Esseen theorem for β_n.

4.2 Almost sure behavior

We next establish a sufficient condition for a top boundary for the absolute centered process $(|R_n - \mathbb{E}\, R_n|)$ to be of outer class, and derive from that condition strong-law concentration for R about its mean function. We also establish analogous results for the processes β and r.

Theorem 4.2. *Let $d \geq 1$.*

(a) *If $\epsilon > 0$, then*

$$\mathbb{P}\left(|R_n - \mathbb{E}\, R_n| \geq (\mathrm{L}\, n)^{\frac{3d}{4} + \epsilon} \text{ i.o.} \right) = 0.$$

As a consequence,

$$\frac{R_n}{\mathbb{E}\, R_n} \xrightarrow{\text{a.s.}} 1.$$

(b) *If $\epsilon > 0$, then*

$$\mathbb{P}\left(|\beta_n - \mathbb{E}\, \beta_n| \geq (\mathrm{L}\, n)^{\frac{3d}{4} + \epsilon} \text{ i.o.} \right) = 0.$$

As a consequence,

$$\frac{\beta_n}{\mathbb{E}\,\beta_n} \xrightarrow{\text{a.s.}} 1.$$

(c) *If* $\epsilon > 0$*, then*

$$\mathbb{P}\left(|r_n - \mathbb{E}\,r_n| \geq (\mathrm{L}\,n)^{\frac{3d}{4}+\epsilon} \text{ i.o.}\right) = 0.$$

As a consequence, if $d \geq 5$ *then*

$$\frac{r_n}{\mathbb{E}\,r_n} \xrightarrow{\text{a.s.}} 1.$$

Proof. (a) Since $\mathbb{E}\,R_n \sim (\mathrm{L}\,n)^d/d!$ by Theorem 4.1(b), the second assertion is indeed an immediate consequence of the first. To prove the first assertion, we establish

$$\mathbb{P}\left(R_n \geq \mathbb{E}\,R_n + (\mathrm{L}\,n)^{\frac{3d}{4}+\epsilon} \text{ i.o.}\right) = 0 \tag{4.1}$$

and

$$\mathbb{P}\left(R_n \leq \mathbb{E}\,R_n - (\mathrm{L}\,n)^{\frac{3d}{4}+\epsilon} \text{ i.o.}\right) = 0. \tag{4.2}$$

To prove (4.1) we exploit the nondecreasingness of the sample paths of the process R. If (b_n) is ultimately monotone nondecreasing and (n_j) is any strictly increasing sequence of positive integers, then

$$\{R_n \geq b_n \text{ i.o.}(n)\} \subseteq \{R_{n_{j+1}} \geq b_{n_j} \text{ i.o.}(j)\}. \tag{4.3}$$

Now choose $b_n \equiv \mathbb{E}\,R_n + (\mathrm{L}\,n)^{\frac{3d}{4}+\epsilon}$ (which is clearly nondecreasing) and $n_j \equiv \lfloor e^{j^{2/d}} \rfloor$. Observe for large j that $\mathrm{L}\,n_j = j^{2/d} + O(e^{-j^{2/d}})$, and hence from Theorem 4.1(b) that

$$\begin{aligned}
\mathbb{E}\,R_{n_j} &= (\mathrm{L}\,n_j)^d \sum_{k=0}^{d} \frac{(-1)^k \Gamma^{(k)}(1)}{k!(d-k)!} (\mathrm{L}\,n_j)^{-k} + o(1) \\
&= j^2 \sum_{k=0}^{d} \frac{(-1)^k \Gamma^{(k)}(1)}{k!(d-k)!} j^{-2k/d} + o(1) \sim \frac{j^2}{d!}, \\
\mathbb{E}\,R_{n_{j+1}} &= (j+1)^2 \sum_{k=0}^{d} \frac{(-1)^k \Gamma^{(k)}(1)}{k!(d-k)!} (j+1)^{-2k/d} + o(1) \\
&= [1 + O(j^{-1})] j^2 \sum_{k=0}^{d} \frac{(-1)^k \Gamma^{(k)}(1)}{k!(d-k)!} j^{-2k/d} + o(1) \\
&= \mathbb{E}\,R_{n_j} + O(j^{-1}\,\mathbb{E}\,R_{n_j}) + o(1) = \mathbb{E}\,R_{n_j} + O(j).
\end{aligned}$$

Observe also that

$$b_{n_j} - \mathbb{E}\,R_{n_j} = (\mathrm{L}\,n_j)^{\frac{3d}{4}+\epsilon} \sim j^{\frac{3}{2}+\frac{2}{d}\epsilon};$$

As a consequence of these two observations,

$$b_{n_j} - \mathbb{E}\,R_{n_{j+1}} = (b_{n_j} - \mathbb{E}\,R_{n_j}) - (\mathbb{E}\,R_{n_{j+1}} - \mathbb{E}\,R_{n_j}) \sim j^{\frac{3}{2}+\frac{2}{d}\epsilon} > 0.$$

Further, from Theorem 4.1(b) we have

$$\mathrm{Var}\,R_{n_{j+1}} \sim \gamma_{d+1,0} (\mathrm{L}\,n_{j+1})^d = \Theta(j^2).$$

Hence, by Chebyshev's inequality,

$$\mathbb{P}(R_{n_{j+1}} \geq b_{n_j}) \leq (b_{n_j} - \mathbb{E}\,R_{n_{j+1}})^{-2}\,\mathrm{Var}\,R_{n_{j+1}} = \Theta(j^{-(1+\frac{4}{d}\epsilon)}),$$

which is summable. The first Borel–Cantelli lemma now implies that

$$\mathbb{P}(R_{n_{j+1}} \geq b_{n_j} \text{ i.o.}(j)) = 0,$$

and then (4.3) yields the desired (4.1).

The proof of (4.2) is similar and again uses the nondecreasingness of the sample paths of R. If (b_n) is ultimately monotone nondecreasing and (n_j) is any strictly increasing sequence of positive integers, then

$$\{R_n \leq b_n \text{ i.o.}(n)\} \subseteq \{R_{n_j} \leq b_{n_{j+1}} \text{ i.o.}(j)\}. \tag{4.4}$$

Now choose $b_n \equiv \mathbb{E}\, R_n - (\mathrm{L}\, n)^{\frac{3d}{4}+\epsilon}$ and, again, $n_j \equiv \lfloor e^{j^{2/d}} \rfloor$. The sequence (b_n) is ultimately monotone nondecreasing because it is known (e.g., [3]) that

$$\mathbb{E}\, R_n - \mathbb{E}\, R_{n-1} = \mathbb{P}(X^{(n)} \text{ sets a record}) = n^{-1}\, \mathbb{E}\, r_n \sim n^{-1} \frac{(\mathrm{L}\, n)^{d-1}}{(d-1)!}, \tag{4.5}$$

while also

$$(\mathrm{L}\, n)^{\frac{3d}{4}+\epsilon} - [\mathrm{L}(n-1)]^{\frac{3d}{4}+\epsilon} \sim (\tfrac{3d}{4}+\epsilon) n^{-1} (\mathrm{L}\, n)^{\frac{3d}{4}-1+\epsilon} = o(n^{-1}(\mathrm{L}\, n)^{d-1}),$$

provided $\epsilon < d/4$ (which we may assume without loss of generality), whence

$$b_n - b_{n-1} \sim n^{-1} \frac{(\mathrm{L}\, n)^{d-1}}{(d-1)!} > 0.$$

Proceeding as for (4.1), by Chebyshev's inequality we have

$$P(R_{n_j} \leq b_{n_{j+1}}) \leq (\mathbb{E}\, R_{n_j} - b_{n_{j+1}})^{-2} \operatorname{Var} R_{n_j} = \Theta(j^{-(1+\frac{4}{d}\epsilon)}),$$

which is summable. The first Borel–Cantelli lemma now implies that

$$\mathbb{P}(R_{n_j} \leq b_{n_{j+1}} \text{ i.o.}(j)) = 0,$$

and then (4.4) yields the desired (4.2).

(b) For $d = 1$, part (b) follows from part (a) because $r_n = 1$ for $n \geq 1$, so we assume $d \geq 2$. The sample paths of β, like those of R, are nondecreasing. Thus, in precisely the same fashion that part (a) is proved using the mean and variance results from Theorem 4.1(b), so one can prove part (b) using the mean and variance results from Theorem 4.1(c). A key technical detail in establishing the analogue of (4.2) for the process β is this analogue of (4.5) [which follows immediately from (4.5) by use of concomitants]:

$$\begin{aligned}
\mathbb{E}\, \beta_n - \mathbb{E}\, \beta_{n-1} &= (\mathbb{E}\, R_n - \mathbb{E}\, R_{n-1}) - (\mathbb{E}\, r_n - \mathbb{E}\, r_{n-1}) \\
&= (1+o(1)) n^{-1} \frac{(\mathrm{L}\, n)^{d-1}}{(d-1)!} - (1+o(1)) n^{-1} \frac{(\mathrm{L}\, n)^{d-2}}{(d-2)!} \\
&\sim n^{-1} \frac{(\mathrm{L}\, n)^{d-1}}{(d-1)!}.
\end{aligned}$$

(c) We obtain part(c) by subtraction from parts (a)–(b):

$$\begin{aligned}
&\mathbb{P}\left(|r_n - \mathbb{E}\, r_n| \geq (\mathrm{L}\, n)^{\frac{3d}{4}+\epsilon} \text{ i.o.}\right) \\
&= \mathbb{P}\left(|(R_n - \mathbb{E}\, R_n) - (\beta_n - \mathbb{E}\, \beta_n)| \geq (\mathrm{L}\, n)^{\frac{3d}{4}+\epsilon} \text{ i.o.}\right) \\
&\leq \mathbb{P}\left(|R_n - \mathbb{E}\, R_n| \geq \tfrac{1}{2}(\mathrm{L}\, n)^{\frac{3d}{4}+\epsilon} \text{ i.o.}\right) + \mathbb{P}\left(|\beta_n - \mathbb{E}\, \beta_n| \geq \tfrac{1}{2}(\mathrm{L}\, n)^{\frac{3d}{4}+\epsilon} \text{ i.o.}\right) \\
&\leq \mathbb{P}\left(|R_n - \mathbb{E}\, R_n| \geq (\mathrm{L}\, n)^{\frac{3d}{4}+\frac{\epsilon}{2}} \text{ i.o.}\right) + \mathbb{P}\left(|\beta_n - \mathbb{E}\, \beta_n| \geq (\mathrm{L}\, n)^{\frac{3d}{4}+\frac{\epsilon}{2}} \text{ i.o.}\right) = 0.
\end{aligned}$$

This gives the first assertion. Since $\mathbb{E}\, r_n \sim (\mathrm{L}\, n)^{d-1}/(d-1)!$ by Theorem 4.1(a), the second assertion is indeed an immediate consequence of the first provided $3d/4 < d-1$, i.e., $d \geq 5$. □

Remark 4.3. (a) In the proof of Theorem 4.2(a) we utilized Chebyshev's inequality. Use of normal tail probabilities would give a sharper result, except that the error estimate in the Berry–Esseen theorem of Theorem 4.1(b) is insufficiently sharp for that.

(b) For $d = 2, 3, 4$ we conjecture on the basis of simulations discussed in Example 5.2 that the second conclusion

$$r_n / \mathbb{E}\, r_n \xrightarrow{\text{a.s.}} 1,$$

i.e.,

$$r_n/(\mathrm{L}\, n)^{d-1} \xrightarrow{\text{a.s.}} 1/(d-1)!, \tag{4.6}$$

of Theorem 4.2(c) remains true. We do at least know from the first assertion in Theorem 4.2(c) that for any $\epsilon > 0$ we have

$$r_n = O((\mathrm{L}\, n)^{\frac{3d}{4}+\epsilon}) \text{ a.s.} \tag{4.7}$$

In dimension $d = 2$ we can come close to (4.6), or at least to showing that $r_n = \Theta(\mathrm{L}\, n)$ a.s. Indeed, we can combine the representation of the distribution of r_n as a Poisson-binomial sum with a Chernoff bound and the first Borel–Cantelli lemma to show that $r_n = O(\mathrm{L}\, n)$ a.s., and Theorem 3.4 gives $r_n = \Omega((\mathrm{L}\, n)/(\mathrm{L}_2\, n))$ a.s.

5 Time change

It is natural to wonder about the appearance of the record-setting frontier (even in dimension 2) when many observations, or (equivalently) many records, have been generated. Figure 3 displays the record-setting frontier for one trial after 10,000 bivariate records had been generated, at which point results such as those in Section 1 suggest themselves. According to Theorem 4.1(b) [or Proposition 5.1(a2)], had this been done naively, by generating observations $X^{(i)}$ and waiting for new records to be set, it would have taken roughly 10^{61} observations to obtain 10,000 records. Instead, only the records were generated, using the importance-sampling scheme described and analyzed in [5].

The record-setting region process (RS_n), and therefore also the frontier process (F_n) we have studied in earlier sections, is adapted to the natural filtration for the process $C = (C_n)_{n \geq 0}$, where $C_n = (C_n^{(1)}, \ldots, C_n^{(r_n)})$ is the r_n-tuple of remaining records at time n in order of creation. Let $T_0 = 0$, and for $m \geq 1$ let T_m denote the mth record-creation epoch; note that C remains constant over each of the time-intervals $[T_{m-1}, T_m)$, $m \geq 1$. Fill and Naiman [5] don't simulate the i.i.d. observations process $X^{(1)}, X^{(2)}, \ldots$ (that is, they don't work in "observations-time"), but rather simulate the process $\widetilde{C} = (\widetilde{C}_m)_{m \geq 0}$, where $\widetilde{C}_m := C_{T_m}$ [and hence the processes $(\widetilde{RS}_m := \mathrm{RS}_{T_m})$ and $(\widetilde{F}_m := F_{T_m})$] (that is, they work in "records-time"). The following goal thus naturally arises: Translate results about C to results about $\widetilde{C}$.

The keys to doing so are (i) monotonicity of the sample paths of various processes of interest (such as F^+ and F^-) and (ii) the switching relation

$$\{T_m \leq n\} = \{R_n \geq m\}. \tag{5.1}$$

The switching relation enables us to obtain information about the record-creation times T_m from the records-counts Theorems 4.1(b) and 4.2(a). The following proposition is not the most elaborate result which can be obtained in such fashion, but it will suffice for our purposes.

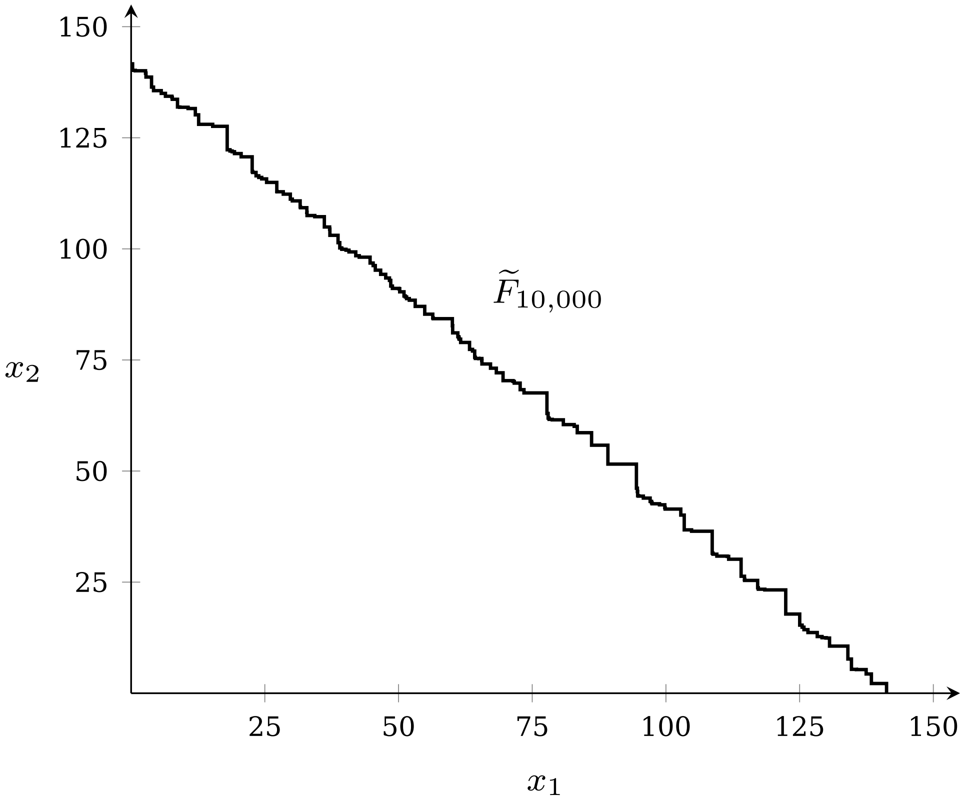

Figure 3: Record frontier (denoted $\widetilde{F}_{10,000}$) after 10,000 records generated using the importance-sampling algorithm described in [5].

Proposition 5.1. *Let T_m denote the m^{th} epoch at which a record is set, and let γ denote the Euler–Mascheroni constant.*

(a) Typical behavior as $m \to \infty$:

(a1) *If $d = 1$, then*

$$\frac{\mathrm{L}\, T_m - (m - \gamma)}{m^{1/2}} \xrightarrow{\mathcal{L}} \text{standard normal.}$$

(a2) *If $d = 2$, then*

$$\frac{\mathrm{L}\, T_m - [(2m)^{1/2} - \gamma]}{\left(\frac{\pi^2}{6} + \frac{1}{2}\right)^{1/2}} \xrightarrow{\mathcal{L}} \text{standard normal.}$$

(a3) *If $d \geq 3$, then*

$$\mathrm{L}\, T_m - [(d!m)^{1/d} - \gamma] \xrightarrow{\mathrm{P}} 0.$$

(b) Almost sure behavior as $m \to \infty$:

(b1) *For every $d \geq 1$ we have*

$$\frac{\mathrm{L}\, T_m}{(d!m)^{1/d}} \xrightarrow{\text{a.s.}} 1.$$

(b2) *If $d \geq 5$, then*

$$\mathrm{L}\, T_m - [(d!m)^{1/d} - \gamma] \xrightarrow{\text{a.s.}} 0.$$

Proof. Fix $d \geq 1$.

(a) Given $\epsilon > 0$, by the switching relation (5.1) and Theorem 4.1(b) we have

$$\begin{aligned}\mathbb{P}(\mathrm{L}\,T_m - [(d!m)^{1/d} - \gamma] > \epsilon) &= \mathbb{P}(T_m > \exp[(d!m)^{1/d} - \gamma + \epsilon]) \\ &= \mathbb{P}(T_m > n) = \mathbb{P}(R_n < m) = \Phi\left(\frac{m - \mathbb{E}\,R_n}{\sqrt{\operatorname{Var} R_n}}\right) + o(1)\end{aligned} \tag{5.2}$$

as $m \to \infty$, where $0 \leq \epsilon_m = o(1)$ is chosen as small as possible to make $n \equiv n_m := \exp[(d!m)^{1/d} - \gamma + \epsilon - \epsilon_m]$ an integer. But $\mathrm{L}\,n = (d!m)^{1/d} - \gamma + \epsilon - o(1)$, so

$$(\mathrm{L}\,n)^d = d!m[1 - (1 + o(1))(\gamma - \epsilon)d(d!m)^{-1/d}] \quad \text{and} \quad (\mathrm{L}\,n)^{-1} \sim (d!m)^{-1/d},$$

and hence by Theorem 4.1(b)

$$\begin{aligned}\mathbb{E}\,R_n &= \frac{(\mathrm{L}\,n)^d}{d!}[1 + (1 + o(1))\gamma d(\mathrm{L}\,n)^{-1}] \\ &= m[1 - (1 + o(1))(\gamma - \epsilon)d(d!m)^{-1/d}][1 + (1 + o(1))\gamma d(d!m)^{-1/d}] \\ &= m[1 + (1 + o(1))\epsilon d(d!m)^{-1/d}] = m + (1 + o(1))\epsilon d(d!)^{-1/d}m^{(d-1)/d}\end{aligned}$$

and

$$\sqrt{\operatorname{Var} R_n} \sim \sqrt{\gamma_{d+1,0}}(\mathrm{L}\,n)^{d/2} \sim (\gamma_{d+1,0}\, d!\, m)^{1/2} = \Theta(m^{1/2}).$$

Thus $(m - \mathbb{E}\,R_n)/\sqrt{\operatorname{Var} R_n}$ is negative and of magnitude $\Theta(m^{\frac{d-1}{d} - \frac{1}{2}})$.

(a3) If $d \geq 3$, it follows that the probability (5.2) tends to 0, and similarly

$$\mathbb{P}(\mathrm{L}\,T_m - [(d!m)^{1/d} - \gamma] \leq -\epsilon) \to 0,$$

yielding the claimed convergence in probability.

(a2) If $d = 2$, then the same calculations show that for any real x we have

$$\mathbb{P}(\mathrm{L}\,T_m - [(2m)^{1/2} - \gamma] > x) = \Phi\left(-\gamma_{3,0}^{-1/2}x\right) + o(1),$$

yielding the claimed CLT, since from [3], $\gamma_{3,0} = \frac{\pi^2}{6} + \frac{1}{2}$.

(a1) If $d = 1$, then the same calculations show that for any real x we have

$$\mathbb{P}(\mathrm{L}\,T_m - [m - \gamma] > x) = \Phi\left(-(1 + o(1))\frac{x}{(\gamma_{2,0}\, m)^{1/2}}\right) + o(1),$$

yielding the claimed CLT, since $\gamma_{2,0} = 1$.

(b1) This follows readily from the conclusion $R_n / \mathbb{E}\,R_n \xrightarrow{\text{a.s.}} 1$ of Theorem 4.2(a) by first recalling from Theorem 4.1(b) that $\mathbb{E}\,R_n \sim (\mathrm{L}\,n)^d/d!$; then setting $n = T_m$, noting $R_{T_m} = m$; and finally taking $-d^{-1}$ powers.

(b2) According to Theorem 4.2, if $\epsilon > 0$ then as $n \to \infty$ we a.s. have

$$R_n = \rho_n + O((\mathrm{L}\,n)^{\frac{3d}{4} + \epsilon}),$$

where ρ is the mean function for R. In particular, setting $n = T_m$, as $m \to \infty$ we a.s. have

$$m = \rho_{T_m} + O((\mathrm{L}\,T_m)^{\frac{3d}{4} + \epsilon}).$$

If $d \geq 5$, then $d - 1 > (3d)/4$ and thus [from Theorem 4.1(b)] almost surely

$$m = \frac{(\mathrm{L}\,T_m)^d}{d!}[1 + (1 + o(1))\gamma d(\mathrm{L}\,T_m)^{-1}],$$

which implies

$$(d!m)^{1/d} = (\mathrm{L}\,T_m)[1 + (1 + o(1))\gamma(\mathrm{L}\,T_m)^{-1}] = \mathrm{L}\,T_m + \gamma + o(1),$$

as desired. □

Example 5.2. Here is a first illustration of the usefulness of Proposition 5.1 in connection with the simulations of records discussed at the outset of this section. Define $\tilde{r}_m := r_{T_m}$. From these simulations it is reasonable to conjecture that

$$\frac{\tilde{r}_m}{(d!m)^{(d-1)/d}} \xrightarrow{\text{a.s.}} \frac{1}{(d-1)!} \text{ as } m \to \infty. \tag{5.3}$$

But we now show that the records-time conjecture (5.3) is in fact equivalent to the observations-time conjecture (4.6)—and therefore both conjectures are [by Theorem 4.2(c) and the expected value asymptotics in Theorem 4.1(a)] true at least for $d \geq 5$.

Indeed, (5.3) follows immediately from (4.6) by substitution of T_m for n and use of Proposition 5.1(b1). To sketch a proof of the converse, consider the ratio on the left in (4.6) for $T_m \leq n < T_{m+1}$. For the numerator of the ratio, note that $r_n = r_{T_m}$. Use $T_m \leq n < T_{m+1}$ in the denominator to get upper and lower bounds on the ratio, and then use Proposition 5.1(b1) to relate the upper and lower bounds on the ratio in (4.6) to the ratio in (5.3).

We can now translate results of Section 1 from observations-time to records-time (the main goal of this section being to translate Theorem 1.14 about frontier width in this fashion), but [because of the limitation of Proposition 5.1(b2)] we only know how to translate some of our almost sure results when $d \geq 5$.

Theorem 5.3. *Consider the process $\widetilde{F}^+$ defined by $\widetilde{F}^+_m := F^+_{T_m}$.*

(a) Typical behavior of $\widetilde{F}^+$:

(a1) *For any $d \geq 2$ we have*

$$\frac{\widetilde{F}^+_m - (d!m)^{1/d}}{\mathrm{L}\, m} \xrightarrow{\mathrm{P}} 1 - d^{-1}.$$

(a2) *If $d \geq 3$ we have the following convergence in law to Gumbel:*

$$\widetilde{F}^+_m - [(d!m)^{1/d} + (1 - d^{-1})\,\mathrm{L}\, m + \mathrm{L}\, d - d^{-1}\,\mathrm{L}(d!) - \gamma] \xrightarrow{\mathcal{L}} G.$$

(b) Almost sure behavior for $\widetilde{F}^+$:

(b1) *For any $d \geq 1$ we have*

$$\widetilde{F}^+_m \sim (d!m)^{1/d} \text{ a.s.}$$

(b2) *If $d \geq 5$, then*

$$\liminf \frac{\widetilde{F}^+_m - (d!m)^{1/d}}{\mathrm{L}\, m} = 1 - d^{-1} < 1 = \limsup \frac{\widetilde{F}^+_m - (d!m)^{1/d}}{\mathrm{L}\, m} \text{ a.s.}$$

Proof. (a2) Assume that $d \geq 3$ and let

$$\widetilde{G}_m := \widetilde{F}^+_m - [(d!m)^{1/d} + (1 - d^{-1})\,\mathrm{L}\, m + \mathrm{L}\, d - d^{-1}\,\mathrm{L}(d!) - \gamma].$$

Given $x \in \mathbb{R}$ and $\epsilon > 0$, we will show that

$$\mathbb{P}(\widetilde{G}_m \leq x) \geq \mathbb{P}(G \leq x - \epsilon) - o(1), \tag{5.4}$$

and a similar proof establishes $\mathbb{P}(\widetilde{G}_m \leq x) \leq \mathbb{P}(G \leq x + \epsilon) + o(1)$. Letting $m \to \infty$ and then $\epsilon \downarrow 0$ completes the proof of (a2), and (a1) is a simple consequence.

We now prove (5.4). By Proposition 5.1(a3) and nondecreasingness of the sample paths of F^+, we have

$$\mathbb{P}(\widetilde{G}_m \leq x) \geq \mathbb{P}\left(F^+_n \leq x + (d!m)^{1/d} + (1 - d^{-1})\,\mathrm{L}\, m + \mathrm{L}\, d - d^{-1}\,\mathrm{L}(d!) - \gamma\right) - o(1),$$

where $n \equiv n_m = \lfloor \exp[(d!m)^{1/d} - \gamma + \epsilon] \rfloor$. Observe that

$$\mathrm{L}\, n = (d!m)^{1/d} - \gamma + \epsilon - o(1) \quad \text{and} \quad \mathrm{L}_2\, n = d^{-1}[\mathrm{L}\, m + \mathrm{L}(d!)] + o(1),$$

and so

$$\begin{aligned} &\mathrm{L}\, n + (d-1)\,\mathrm{L}_2\, n - L((d-1)!) \\ &\quad = (d!m)^{1/d} + (1 - d^{-1})\,\mathrm{L}\, m + \mathrm{L}\, d - d^{-1}\,\mathrm{L}(d!) - \gamma + \epsilon - o(1). \end{aligned}$$

Thus, making use of Theorem 1.8(a), we arrive at

$$\begin{aligned} \mathbb{P}(\widetilde{G}_m \le x) &\ge \mathbb{P}\Big(F_n^+ - [\mathrm{L}\, n + (d-1)\,\mathrm{L}_2\, n - L((d-1)!] \le x - \epsilon + o(1)\Big) - o(1) \\ &= \mathbb{P}(G \le x - \epsilon) - o(1), \end{aligned}$$

as desired.

(a1) We have already proved (a1) for $d \ge 3$. A similar proof establishes (a1) if $d = 2$.

(b1) By Corollary 1.9(b) and Proposition 5.1(b1), the following asymptotic equivalences hold a.s.:

$$\widetilde{F}_m^+ = F_{T_m}^+ \sim \mathrm{L}\, T_m \sim (d!m)^{1/d}.$$

(b2) One checks easily for $b \ge 0$ that $(b - \mathrm{L}\, n)/\,\mathrm{L}_2\, n$ decreases for $n \ge 15$, and so $(F_n^+ - \mathrm{L}\, n)/\,\mathrm{L}_2\, n$ decreases over each of the time-intervals $[T_{m-1}, T_m)$ with m large. (It is sufficient to choose $m \ge 16$.) It follows that

$$\limsup_{n \to \infty} \frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} = \limsup_{m \to \infty} \frac{\widetilde{F}_m^+ - \mathrm{L}\, T_m}{\mathrm{L}_2\, T_m} \tag{5.5}$$

and

$$\begin{aligned} \liminf_{n \to \infty} \frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} &= \liminf_{m \to \infty} \frac{F_{T_m - 1}^+ - \mathrm{L}(T_m - 1)}{\mathrm{L}_2(T_m - 1)} \\ &= \liminf_{m \to \infty} \frac{\widetilde{F}_{m-1}^+ - \mathrm{L}(T_m - 1)}{\mathrm{L}_2(T_m - 1)} \\ &= \liminf_{m \to \infty} \frac{\widetilde{F}_m^+ - \mathrm{L}\, T_{m+1} + o(1)}{\mathrm{L}_2\, T_{m+1} - o(1)}. \end{aligned}$$

But, by Proposition 5.1(b2), almost surely

$$\mathrm{L}\, T_{m+1} = [d!(m+1)]^{1/d} - \gamma + o(1) = d!m^{1/d} + O(1)$$

and hence

$$\mathrm{L}_2\, T_{m+1} = d^{-1}\,\mathrm{L}\, m + O(1),$$

whence

$$\begin{aligned} \liminf_{n \to \infty} \frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} &= \liminf_{m \to \infty} \frac{\widetilde{F}_m^+ - \mathrm{L}\, T_{m+1} + o(1)}{\mathrm{L}_2\, T_{m+1} - o(1)} \\ &= \liminf_{m \to \infty} \frac{\widetilde{F}_m^+ - d!m^{1/d} + O(1)}{d^{-1}\,\mathrm{L}\, m + O(1)} \\ &= d \liminf_{m \to \infty} \frac{\widetilde{F}_m^+ - d!m^{1/d}}{\mathrm{L}\, m}; \end{aligned}$$

similarly, by (5.5),

$$\limsup_{n \to \infty} \frac{F_n^+ - \mathrm{L}\, n}{\mathrm{L}_2\, n} = d \limsup_{m \to \infty} \frac{\widetilde{F}_m^+ - d!m^{1/d}}{\mathrm{L}\, m}.$$

The desired result now follows from Corollary 1.9(b). □

Remark 5.4. In the same manner as Remark 1.10, one can show that the set of limit points of the sequence $[\widetilde{F}^+_m - (d!m)^{1/d}]/\operatorname{L} m$ is for $d \geq 5$ almost surely the closed interval $[1-d^{-1}, 1]$.

Theorem 5.5. *Consider the process $\widetilde{F}^-$ defined by $\widetilde{F}^-_m := F^-_{T_m}$.*

(a) Typical behavior of $\widetilde{F}^-$: *If $d \geq 2$, then*

$$\mathbb{P}(\widetilde{F}^-_m \leq (d!m)^{1/d} - 3\operatorname{L}_2 m) \to 0$$

and

$$\mathbb{P}(\widetilde{F}^-_m \geq (d!m)^{1/d} + c_m) \to 0 \text{ if } c_m \to \infty.$$

As a consequence,

$$\frac{\widetilde{F}^-_m - (d!m)^{1/d}}{\operatorname{L} m} \xrightarrow{\mathrm{P}} 0.$$

(b) Almost sure behavior for $\widetilde{F}^-$: *If $d \geq 5$, then*

$$\lim \frac{\widetilde{F}^-_m - (d!m)^{1/d}}{\operatorname{L} m} = 0 \text{ a.s.}$$

Proof. (a) Recalling Remark 3.3 to provide some flexibility, part (a) follows from Theorem 1.12(a) in much the same way that Theorem 5.3(a) followed from Theorem 1.8(a) [and Corollary 1.9(a)]. In the interest of brevity, we omit the routine details.

(b) In the same way that Theorem 5.3(b) followed from Corollary 1.9(b), so part (b) follows from Corollary 1.13(b). □

We come finally to our main focus of this section, the process $\widetilde{W}$.

Theorem 5.6. *Consider the process $\widetilde{W}$ defined by $\widetilde{W}_m := W_{T_m}$.*

(a) Typical behavior of $\widetilde{W}$: *For every $d \geq 1$ we have*

$$\frac{\widetilde{W}_m}{\operatorname{L} m} \xrightarrow{\mathrm{P}} 1 - d^{-1}.$$

(b) Almost sure behavior for $\widetilde{W}$: *If $d \geq 2$, then*

$$\liminf \frac{\widetilde{W}_m}{\operatorname{L} m} = 1 - d^{-1} < 1 = \limsup \frac{\widetilde{W}_m}{\operatorname{L} m} \text{ a.s.}$$

and, in particular,

$$\widetilde{W}_m = \Theta(\operatorname{L} m) \text{ a.s.}$$

Proof. Part (a), and part (b) for $d \geq 5$, follow immediately by subtraction from the two preceding theorems about $\widetilde{F}^+$ and $\widetilde{F}^-$ [and by the triviality of part (a) for $d = 1$]. We next present an argument that establishes part (b) for *all* $d \geq 2$.

In the proofs of Theorems 5.3(b) and 5.5(b), the only use of the assumption $d \geq 5$ is in the application of Proposition 5.1(b2). From the computations prior to the application together with application of Proposition 5.1(b1) for the denominators, we almost surely have

$$\limsup_{m\to\infty} \frac{\widetilde{F}^+_m - \operatorname{L} T_m}{\operatorname{L} m} = 1, \qquad \liminf_{m\to\infty} \frac{\widetilde{F}^+_m - \operatorname{L} T_{m+1}}{\operatorname{L} m} = 1 - d^{-1}, \tag{5.6}$$
$$\limsup_{m\to\infty} \frac{\widetilde{F}^-_m - \operatorname{L} T_m}{\operatorname{L} m} = 0, \qquad \liminf_{m\to\infty} \frac{\widetilde{F}^-_m - \operatorname{L} T_{m+1}}{\operatorname{L} m} = 0.$$

From the two results here about $\widetilde{F}^-$, it follows quickly using the monotonicity of the paths of F^- that a.s.

$$\lim_{m\to\infty} \frac{\widetilde{F}^-_m - \mathrm{L}\, T_m}{\mathrm{L}\, m} = 0, \qquad \lim_{m\to\infty} \frac{\widetilde{F}^-_m - \mathrm{L}\, T_{m+1}}{\mathrm{L}\, m} = 0. \tag{5.7}$$

Now subtract the equations in (5.7) from the corresponding equations in (5.6) to complete the proof of part (b). □

Remark 5.7. (a) Using Remark 5.4, for $d \geq 5$ Theorem 5.6(b) can be strengthened to the conclusion that the set of limit points of the sequence $\widetilde{W}_m / \mathrm{L}\, m$ is almost surely the closed interval $[1 - d^{-1}, 1]$. We have not investigated whether this result can be extended to $d = 2, 3, 4$.

(b) Equation (5.7) has the independently interesting corollary that

$$\mathrm{L}\, T_{m+1} - \mathrm{L}\, T_m = o(\mathrm{L}\, m) \text{ a.s.} \tag{5.8}$$

for $d \geq 2$. For $d = 1$, it follows from the last sentence in [1, Sec. 2.5] that

$$\mathrm{L}\, T_{m+1} - \mathrm{L}\, T_m = O((m\, \mathrm{L}_2\, m)^{1/2}).$$

For $d \geq 5$ we can prove the stronger [than (5.8)] result that

$$\mathrm{L}\, T_{m+1} - \mathrm{L}\, T_m = O(m^{-(\frac{1}{4} - \frac{1}{d} - \epsilon)}) \text{ a.s.} \tag{5.9}$$

for any $\epsilon > 0$. Indeed, define the function $f \equiv f_d$ by

$$f(t) := \sum_{j=0}^{\lceil d/4 \rceil - 1} \frac{(-1)^{j-1} \Gamma^{(j)}(t)}{j!(d-j)!} t^{d-j}.$$

Then, setting $n = T_m$ in Theorem 4.2(b), with ρ defined as the mean function for R we almost surely have

$$\begin{aligned} m &= \rho_{T_m} + O((\mathrm{L}\, T_m)^{\frac{3d}{4} + d\epsilon}) \\ &= \rho_{T_m} + O(m^{\frac{3}{4} + \epsilon}) \ \text{ by Proposition 5.1(b1)} \\ &= f(\mathrm{L}\, T_m) + O(m^{\frac{3}{4} + \epsilon}) \ \text{ by Theorem 4.1(b).} \end{aligned}$$

Thus for some $I_m \in [T_m, T_{m+1}]$ we almost surely have

$$\begin{aligned} 1 &= f(\mathrm{L}\, T_{m+1}) - f(\mathrm{L}\, T_m) + O(m^{\frac{3}{4} + \epsilon}) \\ &= f'(\mathrm{L}\, I_m)(\mathrm{L}\, T_{m+1} - \mathrm{L}\, T_m) + O(m^{\frac{3}{4} + \epsilon}) \\ &\qquad \text{by the mean value theorem} \\ &= (1 + o(1)) \frac{1}{(d-1)!} (\mathrm{L}\, I_m)^{d-1} (\mathrm{L}\, T_{m+1} - \mathrm{L}\, T_m) + O(m^{\frac{3}{4} + \epsilon}) \\ &= (1 + o(1)) \frac{1}{(d-1)!} (d! m)^{1 - \frac{1}{d}} (\mathrm{L}\, T_{m+1} - \mathrm{L}\, T_m) + O(m^{\frac{3}{4} + \epsilon}) \\ &\qquad \text{by Proposition 5.1(b1),} \end{aligned}$$

yielding (5.9).

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