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The infinite two-sided loop-erased random walk

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Abstract

The loop-erased random walk (LERW) in $\mathbb{Z}^d, d \geq 2$, is obtained by erasing loops chronologically from simple random walk. In this paper we show the existence of the two-sided LERW which can be considered as the distribution of the LERW as seen by a point in the "middle" of the path.

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1 Introduction

In this paper we establish the existence of the infinite two-sided loop-erased random walk (LERW). We start by explaining what this means. The (infinite, one-sided) LERW is the measure on non self-intersecting paths obtained by erasing loops chronologically from a simple random walk. This sentence can be taken literally for $d \geq 3$, but for $d = 2$ we need a little care. We will give a definition that is valid for $d \geq 2$ that is easily seen to be equivalent to the usual definition for $d \geq 3$. A simple random walk conditioned to never return to the origin is a simple random walk weighted by the Green's function (for $d \geq 3$) or the potential kernel (for $d = 2$).

- **Loop erasure**. Let S_j denote a simple random walk starting at the origin in $\mathbb{Z}^d, d \geq 2$, conditioned to never return to the origin. Let $\sigma_0 = 0$, and for $n > 0$, let $\sigma_n = \max\{j : S_j = S_{\sigma_{n-1}+1}\}$. Then the *(infinite, one-sided) loop-erased random walk (LERW)* $\hat{S}_n$ is defined by

$$\hat{S}_n = S_{\sigma_n} = S_{\sigma_{n-1}+1}.$$

For $d \geq 3$, one gets the same measure by taking a simple random walk without conditioning and defining $\sigma_0 = \max\{j : S_j = 0\}$. This is the original definition as in [5], but it is often useful to view this probability measure on infinite self-avoiding paths as a

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consistent collection of measures on finite paths. We say that $\eta = [\eta_0, \eta_1, \ldots, \eta_n]$ is a self-avoiding walk (SAW) if it is a nearest neighbor path with no self-intersections. We will reserve the term SAW for finite paths. The following two facts can be readily derived from the definition by considering the unique decomposition of a simple random walk ω path starting at the origin and never returning to the origin whose loop-erasure is $[\eta_0, \eta_1, \ldots]$ as

$$\omega = [\eta_0, \eta_1] \oplus l_1 \oplus [\eta_1, \eta_2] \oplus l_2 \cdots , \tag{1.1}$$

where l_j is a loop rooted at η_j that does not visit $\{\eta_0, \ldots, \eta_{j-1}\}$, and $\oplus$ denotes concatenation.

- **Laplacian random walk**. Suppose $\eta = [\eta_0, \ldots, \eta_n]$ is a SAW in $\mathbb{Z}^d$ starting at the origin. Then,

$$\mathbf{P}\left\{\hat{S}_{n+1} = z \mid [\hat{S}_0, \ldots, \hat{S}_n] = \eta\right\} = \frac{g_\eta(z)}{2d\,\mathrm{Es}_\eta(\eta_n)},$$

  where

$$\mathrm{Es}_\eta(\eta_n) = \Delta g_\eta(\eta_n) = \frac{1}{2d} \sum_{|z-\eta_n|=1} g_\eta(z)$$

  and

  - ($d \geq 3$) $g_\eta(z)$ is the probability that a random walk starting at z never visits η, that is, the unique function that is (discrete) harmonic on $\mathbb{Z}^d \setminus \eta$; vanishes on η; and has boundary value 1 at infinity.
  - ($d = 2$) $g_\eta(z)$ is the unique function that is (discrete) harmonic on $\mathbb{Z}^2 \setminus \eta$; vanishes on η; and satisfies

$$g_\eta(z) \sim \frac{2}{\pi} \log |z|, \quad z \to \infty.$$

- **Loop measure formulation**. If η is a SAW starting at the origin, then

$$\mathbf{P}\{[\hat{S}_0, \ldots, \hat{S}_n] = \eta\} = (2d)^{-n}\, G_0\, F_\eta\, \mathrm{Es}_\eta(\eta_n), \tag{1.2}$$

  where $G_0 = 1$ if $d = 2$ and $G_0 = G_{\mathbb{Z}^d}(0,0) < \infty$ if $d \geq 3$, and

$$F_\eta = \prod_{j=1}^{n} G_{A_j}(\eta_j, \eta_j).$$

  Here $A_j = \mathbb{Z}^d \setminus \{\eta_0, \ldots, \eta_{j-1}\}$, and $G_{A_j}(\cdot, \cdot)$ denotes the simple random walk Green's function in A_j. That is, $G_{A_j}(x, y)$ is the expected number of visits to y of a random walk starting at x killed upon leaving A_j. An alternative expression for F_η (see Section 2.3 for definitions) is

$$F_\eta = F_\eta(\mathbb{Z}^d \setminus \{0\}) = \exp\left\{ \sum_{\ell \subset \mathbb{Z}^d \setminus \{0\}, \ell \cap \eta \neq \emptyset} m(\ell) \right\},$$

  where m denotes the random walk loop measure. If $d \geq 3$, we can write

$$G_0\, F_\eta = F_\eta(\mathbb{Z}^d) = \exp\left\{ \sum_{\ell \subset \mathbb{Z}^d, \ell \cap \eta \neq \emptyset} m(\ell) \right\},$$

  but the right-hand side is infinite for $d = 2$.

What makes this paper a little complicated is that we do both the $d = 2$ and $d = 3$ cases simultaneously. The basic idea of the coupling argument is the same in both cases but the details about the LERW and the loop measures differ. In three dimensions the random walk is transient and this will give us some useful estimates. The random walk in two dimensions is recurrent so these estimates will not be available. However, planarity gives us another set of tools. Random walks can make loops about the origin and disconnect the origin from infinity. Also, there is the "Beurling estimate" that tells us that if we are close to any continuous path then there is a good chance that a random walk will hit it before going too far.

The consistent family of measures on (finite) SAWs given by (1.2) represents the probability that the LERW starts with η. This can be called the *one-sided* measure because the LERW continues on only one side of η. In this paper we will consider the two-sided measure which can be viewed as the distribution of the "middle" of a LERW. There are two versions of our result. One is in terms of LERW excursions as discussed in [9]. Suppose A is a simply connected subset of $\mathbb{Z}^d$ containing the origin and x, y are distinct points in ∂A. Consider simple random walks starting at x conditioned to enter A and then leave A for the first time at y at which time they are stopped. Erase loops, restrict to the event that the loop-erasure goes through the origin, and then normalize to make this a probability measure which we denote by $\lambda^{\#}_{A,x,y}$. It is supported on $\mathcal{A}(A; x, y)$, the set of SAWs from x to y in A going through the origin. If $\eta = [z_0, z_1, \ldots, z_m], \gamma = [\gamma_0, \gamma_1, \ldots, \gamma_k]$, are SAWs we write $\eta \prec \gamma$ if γ includes η in the sense that there exists j such that $\gamma_{j+i} = \eta_i, i = 0, 1, \ldots, m$. We write $\mathcal{A}_{A,x,y}(\eta)$ for the corresponding sets of SAWs that include η. While our main theorem discusses this measure, we will first study a slightly different measure.

Let

$$C_n = \{x \in \mathbb{Z}^d : |x| < e^n\},$$

with boundary $\partial C_n = \{x \in \mathbb{Z}^d : \operatorname{dist}(x, C_n) = 1\}$. Let $\mathcal{A}_n$ be the union of $\mathcal{A}(C_n; x, y)$ over all $x, y \in \partial C_n$. We write $\mathcal{A}_n(\eta)$ for the corresponding set of SAWs that include η. Note that $\mathcal{A}_n$ is $\mathcal{A}_n(\eta)$ for the trivial SAW $\eta = [0]$.

There is another way to describe the set $\mathcal{A}_n$. Let $\mathcal{W}_n$ denote the set of SAWs starting at the origin, ending at ∂C_n, and otherwise staying in C_n. Then we can also define $\mathcal{A}_n$ to be the set of ordered pairs $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{W}_n^2 = \mathcal{W}_n \times \mathcal{W}_n$ with $\eta^1 \cap \eta^2 = \{0\}$. This is essentially the same definition of $\mathcal{A}_n$ as above if we use the natural bijection given by $\eta \leftrightarrow (\eta^1)^R \oplus \eta^2$, where R denotes path reversal. We define μ_n to be the probability measure on $\mathcal{W}_n$ induced by the infinite LERW by stopping the path at the first visit to ∂C_n; we will also write μ_n for $\mu_n \times \mu_n$, the product measure on $\mathcal{W}_n^2$. We define a measure λ_n on $\mathcal{A}_n$ by stating that its Radon-Nikodym derivative with respect to μ_n is

$$1\{\boldsymbol{\eta} \in \mathcal{A}_n\} \exp\{-L_n(\boldsymbol{\eta})\},$$

where $L_n(\boldsymbol{\eta})$ denotes the loop measure (see Section 2.3) of loops in $C_n \setminus \{0\}$ that intersect both η^1 and η^2. For $d = 2$, we will restrict to loops that do not disconnect the origin from infinity.

If $k < n$ and $\boldsymbol{\eta} \in \mathcal{A}_k$, we let $\mathcal{A}_n(\boldsymbol{\eta})$ be the set of $\boldsymbol{\gamma} \in \mathcal{A}_n$ that include $\boldsymbol{\eta}$ (we write $\boldsymbol{\eta} \prec \boldsymbol{\gamma}$). Similarly, if $A \supset C_{n+1}$, we let $\mathcal{A}_{A,x,y}[\boldsymbol{\eta}]$ be the set of $\boldsymbol{\gamma} \in \mathcal{A}_n(A, x, y)$ that include $\boldsymbol{\eta}$. We can state our main theorem.

Theorem 1.1. *There exists $\alpha > 0$ such that the following holds. For every positive integer k and every $\boldsymbol{\eta} \in \mathcal{A}_k$, the limit*

$$p(\boldsymbol{\eta}) = \lim_{n\to\infty} \frac{\lambda_n[\mathcal{A}_n(\boldsymbol{\eta})]}{\lambda_n[\mathcal{A}_n]}$$

exists. In fact,

$$\lambda_n[\mathcal{A}_n(\boldsymbol{\eta})] = p(\boldsymbol{\eta})\, \lambda_n[\mathcal{A}_n] \, \left[1 + O(e^{\alpha(k-n)})\right]. \tag{1.3}$$

Moreover, if A is a simply connected set containing C_{n+1} and $x, y \in \partial A$ with $\mathcal{A}(A; x, y)$ nonempty, then

$$\lambda^{\#}_{A,x,y}[\mathcal{A}_{A,x,y}(\boldsymbol{\eta})] = p(\boldsymbol{\eta}) \, \left[1 + O(e^{\alpha(k-n)})\right]. \tag{1.4}$$

The statement (1.3) uses a convention that we will use throughout this paper. Any implicit constants arising from $O(\cdot)$ or $\asymp$ notations can depend on d but otherwise are assumed to be uniformly bounded over all the parameters including vertices in $\mathbb{Z}^d$, k, n, and $\boldsymbol{\eta} \in \mathcal{A}_k$. In other words, we can say that there exists c, α such that for all $k \leq n-1$ and all $\boldsymbol{\eta} \in \mathcal{A}_k$,

$$\left|\log\left(\frac{\lambda_n[\mathcal{A}_n(\boldsymbol{\eta})]}{p(\eta)\,\lambda_n[\mathcal{A}_n]}\right)\right| \leq c\, e^{\alpha(k-n)}.$$

We will only do the details of the proof for $d = 2$ and $d = 3$ for which the result is new. For $d = 4$, it can be derived from the construction of the two-sided walk in [13], and for $d \geq 5$, it is even easier. The proof we give for $d = 3$ can be adapted easily to $d > 3$; it uses transience of the random walk. The $d = 2$ case is similar but there are difficulties arising from recurrence of the random walk. These can be overcome by making use of planarity, and, in particular, the Beurling estimate and the disconnection exponent for two-dimensional random walks.

If we let p_k be p restricted to $\mathcal{A}_k$, then $\{p_k\}$ is a consistent family of probability measures and induces a probability measure on pairs of infinite self-avoiding paths starting at the origin that do not intersect (other than the initial point). We call this process the *two-sided infinite loop-erased random walk*. The result in this paper is different (and, frankly, easier) than questions about the scaling limit of loop-erased walk. For the one-sided case in $d = 2$, the scaling limit is now well understood as the Schramm-Loewner evolution with parameter $\kappa = 2$; see [1, 2, 12, 14] for some of the main papers here. The existence of a scaling limit in $d \geq 3$ was established in some sense by Kozma [4] but there is still much work to do in understanding the limit and the rate of convergence to the limit; see [17] for some work in this direction. Our methods are similar to those in [8, 15] where convergence to the measure on mutually non-intersecting Brownian motions is studied and in [16] where two-dimensional loop-erased walk is studied.

We now outline the paper. Most of the work is focused on (1.3); the final section will show how to derive (1.4) from this. We start by giving the notations that we will use and state the main theorem we will prove. There are two different ways to define "loop-erased walk stopped when it leaves the ball of radius r": one can either take a simple random walk and stop it when it leaves the ball and then erase loops or erase loops from the infinite random walk and then stop the loop-erased walk when it leaves the disk. It turns out easier to analyze the latter case first and this is where we focus our effort. We use the "loop measure" description of the LERW and we review relevant facts about the loop measure in Section 2.3. Here there is a difference between two and higher dimensions. In two dimensions, the measure of loops intersecting a finite set is infinite. However, this can be handled by splitting the set of loops into those that surround the origin and those that do not. Roughly speaking, the loops that surround the origin give the divergence in the loop measure; however, all such loops intersect all infinite self-avoiding paths starting at the origin, so this term cancels out.

To use the approach in [8, 15], one needs two “obvious” lemmas about random walks. Although proofs of these results appear several other places, we choose to give sketches here as well. The first is in Section 2.4 where it is shown that a random walk starting on the sphere of radius R about the origin, stopped when it reaches distance r from its starting point, conditioned to avoid some set contained in the ball of radius R, has a reasonable chance of ending up at a point distance $R + (r/2)$ away from the origin. If there is no conditioning, this follows from the central limit theorem, and the conditioning should just increase the probability (this is why it is “obvious”). It is very useful to know that one can find uniform bounds, uniform over R, r, the starting point of the walk, and the avoidance set. The second “obvious” fact is called a *separation lemma* which roughly states that loop-erased walks conditioned to avoid each other tend to stay far apart. Again, the key is to find a version of this that is uniform. The proof uses the same basic idea as the original proof in [6] as adapted for loop-erased walk [16, 14, 17]. Section 2.5 goes over facts about LERW including an important lemma that the LERW stopped about reach radius R is “independent up to constants” with the walk starting at the last visit to the disk of radius $2R$. The next subsection considers pairs of walks and sets up for Section 2.8 where the coupling is done. As has been done in several of the papers before, one chooses a large integer N and considers the probability measure on pairs of SAWs given by the LERWs weighted by a loop measure term. We then view this measure as giving transition probabilities for LERW stopped when it reaches a smaller radius n and this is the process that we couple.

2 The main theorem

2.1 Notation and main result

We list the notation that we will use.

- If $A \subset \mathbb{Z}^d$, we write $\hat{A} = A \setminus \{0\}$. In particular, $\hat{\mathbb{Z}}^d = \hat{\mathbb{Z}}^d \setminus \{0\}$. We write

$$\partial A = \{z \in \mathbb{Z}^2 : \mathrm{dist}(z, A) = 1\}, \quad \partial_i A = \partial(\mathbb{Z}^2 \setminus A) = \{z \in A : \mathrm{dist}(z, \mathbb{Z}^2 \setminus A) = 1\}.$$

- If $z \in \partial A$, we let $H_A(x, z)$ denote the Poisson kernel, that is, the probability that a simple random walk starting at x first visits $\mathbb{Z}^2 \setminus A$ at z. If $x \in \partial A$, then $H_A(x, z) = \delta(x - z)$. If w, z are distinct points in ∂A, we let $H_{\partial A}(w, z)$ denote the boundary Poisson kernel defined by

$$H_{\partial A}(w, z) = \frac{1}{2d} \sum_{x \in A, |w - x| = 1} H_A(x, z).$$

  A last-exit decomposition shows that if $x \in A$, then

$$H_A(x, z) = G_A(x, x)\, H_{\partial(A \setminus \{x\})}(x, z).$$

- If $n \geq 0$ (not necessarily an integer), let

$$C_n = \{z \in \mathbb{Z}^d : |z| < e^n\},$$

  be the discrete ball of radius e^n. Note that $C_0 = \{0\}$ and $\hat{C}_n = C_n \setminus C_0$.
- Let $\mathrm{e}_n = (e^n, 0, \ldots, 0)$ be the element of $\mathbb{R}^d$ with first component e^n and all other components equal to zero.
- If S is a simple random walk, then we write $\hat{S}$ for its (chronological) loop-erasure.
- If $\eta = [\eta_0, \ldots, \eta_j]$ is a SAW, we write $|\eta| = j$ for the number of steps in η. We call η_0 and η_j the *initial* and *terminal* points or vertices of η, respectively.

- $\mathcal{W}_n$ is the set of SAWs starting at the origin whose terminal point is in ∂C_n and all other vertices are in C_n.
- $\bar{\mathcal{W}}_n$ is the set of infinite self-avoiding paths whose initial point is in C_n and all other vertices are in $\mathbb{Z}^d \setminus C_n$. In particular, $\bar{\mathcal{W}}_0$ is the set of infinite self-avoiding paths starting at the origin.
- If $n \le m$, then $\bar{\mathcal{W}}_{n,m}$ is the set of SAWs satisfying: the initial point is in C_n, the terminal point is in ∂C_m, and all other vertices are in $C_m \setminus C_n$.
- We write $\eta \prec \tilde{\eta}$ if η is contained in $\tilde{\eta}$. If $\eta, \tilde{\eta}$ both start at 0, then this means that η is an initial segment of $\tilde{\eta}$.

If $\eta \in \bar{\mathcal{W}}_0$ and $n > 0$, there is a unique decomposition

$$\eta = \eta_n \oplus \eta^* \oplus \bar{\eta}_{n+1}, \tag{2.1}$$

where $\eta_n \in \mathcal{W}_n, \bar{\eta}_{n+1} \in \bar{\mathcal{W}}_{n+1}$. Similarly, if $n \le m-1$ and $\eta \in \mathcal{W}_m$, there is a unique decomposition

$$\eta = \eta_n \oplus \eta^* \oplus \eta_{n+1,m}, \tag{2.2}$$

where $\eta_n \in \mathcal{W}_n, \eta_{n+1,m} \in \bar{\mathcal{W}}_{n+1,m}$. In these decompositions η^* is a SAW starting at ∂C_n with terminal vertex in $\partial_i C_{n+1}$. In (2.2) we also need $\eta^* \subset C_m$.

We will also be considering pairs of paths. We will use bold-face notation for pairs of SAWs.

- $\mathcal{A}_n$ is the set of ordered pairs $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{W}_n^2 := \mathcal{W}_n \times \mathcal{W}_n$ such that $\eta^1 \cap \eta^2 = \{0\}$.

The notation gets a little cumbersome, but it useful to remember that bold-face $\boldsymbol{\gamma}, \boldsymbol{\eta}$ will always refer to ordered pairs of SAWs.

Recall in the introduction that we wrote $\mathcal{A}_n$ for the set of SAWs η whose initial and terminal vertices are in ∂C_n; all other vertices are in C_n; and that include the origin as a vertex. Indeed, there is a simple bijection to show that these are essentially the same set:

$$(\eta^1, \eta^2) \longleftrightarrow \eta = (\eta^1)^R \oplus \eta^2,$$

where R denotes the reversal of the walk.

- If $1 \le n \le m-1$, $\bar{\mathcal{A}}_{n,m}$ is the set of ordered pairs $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \bar{\mathcal{W}}_{n,m} \times \bar{\mathcal{W}}_{n,m}$ with $\eta^1 \cap \eta^2 = \emptyset$.
- We write $(\eta^1, \eta^2) \prec (\tilde{\eta}^1, \tilde{\eta}^2)$ if $\eta^1 \prec \tilde{\eta}^1$ and $\eta^2 \prec \tilde{\eta}^2$.

- Let μ_m denote the probability measure on $\mathcal{W}_m$ obtained by taking an infinite loop-erased random walk and truncating the path at the first visit to ∂C_m. As a slight abuse of notation, we also write μ_m for the product measure $\mu_m \times \mu_m$ on $\mathcal{W}_m^2$.
- If $\boldsymbol{\gamma} = (\gamma^1, \gamma^2) \in \mathcal{W}_m \times \mathcal{W}_m$, we define

  $$Q_m(\boldsymbol{\gamma}) = \exp\left\{-L_m(\gamma^1, \gamma^2)\right\} = 1\{\boldsymbol{\gamma} \in \mathcal{A}_m\} \exp\left\{-L_m(\gamma^1, \gamma^2)\right\},$$

  where:

  - $L_m(\gamma^1, \gamma^2) = \infty$ if $\boldsymbol{\gamma} \notin \mathcal{A}_m$,

  and if $\boldsymbol{\gamma} \in \mathcal{A}_m$,

- ($d \geq 3$) $L_m(\gamma^1, \gamma^2)$ denotes the loop measure (see Section 2.3) of the set of loops in $\hat{C}_m$ that intersect both γ^1 and γ^2.
- ($d = 2$) $L_m(\gamma^1, \gamma^2)$ denotes the loop measure of the set of loops in $\hat{C}_m$ that intersect both γ^1 and γ^2 and do not disconnect 0 from ∂C_m.

For $d = 2$, we will be ignoring loops in $\hat{C}_n$ that disconnect 0 from ∂C_n. The reason is that all such loops intersect all $\gamma \in \mathcal{W}_n$ and hence these loops have no effect on the probability distribution obtained by tilting by $e^{-(\text{loop term})}$. Restricting to loops in two dimensions that do not disconnect will give us estimates analogous to estimates in three dimensions obtained from transience of the random walk.

- Let
$$\lambda_m = \mathbf{E}_{\mu_m}\left[Q_m(\boldsymbol{\gamma})\right] = \sum_{\boldsymbol{\gamma} \in \mathcal{W}_m^2} \mu_m(\boldsymbol{\gamma})\, Q_m(\boldsymbol{\gamma}),$$
and if $n \leq m-1$ and $\boldsymbol{\eta} \in \mathcal{W}_n^2$, we let
$$\lambda_m(\boldsymbol{\eta}) = \sum_{\boldsymbol{\gamma} \in \mathcal{W}_m^2,\, \boldsymbol{\eta} \prec \boldsymbol{\gamma}} \mu_m(\boldsymbol{\gamma})\, Q_m(\boldsymbol{\gamma}).$$
Note that $\lambda_m(\boldsymbol{\eta})$ is nonzero only if $\boldsymbol{\eta} \in \mathcal{A}_n$ and that
$$\lambda_m = \sum_{\boldsymbol{\eta} \in \mathcal{A}_n} \lambda_m(\boldsymbol{\eta}).$$

To prove (1.3), it suffices to prove that for all $\boldsymbol{\eta} \in \mathcal{A}_n$ and $m \geq n+1$,
$$\frac{\lambda_{m+1}(\boldsymbol{\eta})}{\lambda_m(\boldsymbol{\eta})} = \frac{\lambda_{m+1}}{\lambda_m}\,\left[1 + O(e^{\alpha(n-m)})\right], \tag{2.3}$$
since this implies that
$$\frac{\lambda_{m+1}(\boldsymbol{\eta})}{\lambda_{m+1}} = \frac{\lambda_m(\boldsymbol{\eta})}{\lambda_m}\,\left[1 + O(e^{\alpha(n-m)})\right].$$
We concentrate on (2.3) and use a coupling argument to establish this. Let $\lambda_m^{\#}, \lambda_m^{\#}(\boldsymbol{\eta})$ denote the probability measures on $\mathcal{A}_m$ whose Radon-Nikodym derivative with respect to μ_n are
$$\frac{Q_m(\boldsymbol{\eta}')}{\lambda_m}, \quad \frac{Q_m(\boldsymbol{\eta}')\, 1\{\boldsymbol{\eta} \prec \boldsymbol{\eta}'\}}{\lambda_m(\boldsymbol{\eta})},$$
respectively. We show that we can couple $\mathcal{A}_m$-valued random variables with distributions $\lambda_m^{\#}$ and $\lambda_m^{\#}(\boldsymbol{\eta})$ on the same probability space so that, except for an event of small probability, the paths agree except for an initial part of the path. (It would be impossible to couple them so that the total paths agree since $\lambda_m^{\#}(\boldsymbol{\eta})$ is supported on pairs of walks that start with $\boldsymbol{\eta}$.)

2.2 Some results about two-dimensional walks

We will assume that the reader is acquainted with basic facts about simple random walk; we will use [9] as a reference. For $d \geq 3$, we will be using transience of the random walk; in particular, we will use the estimate that if $z \in \mathbb{Z}^d$, then the probability that a random walk starting at z gets within distance r of the origin is bounded above by $c(r/|z|)^{d-2}$. For $d = 2$, some of the important results are perhaps less known, so we will review them here. This subsection can be skipped at first reading and referred to as necessary.

In this subsection we let S_j denote a simple random walk, and

$$\rho_n = \min\{j : S_j \notin C_n\}.$$

We let $a(x)$ be the potential kernel in $\mathbb{Z}^2$; it can be described as the unique function that is harmonic on $\mathbb{Z}^2 \setminus \{0\}$; vanishes at the origin; and is asymptotic to $(2/\pi)\log|x|$. It is known [9, Theorem 4.4] that $a(x) = 1$ for $|x| = 1$ and

$$a(x) = \frac{2}{\pi}\,\log|x| + k_0 + O(|x|^{-2}), \quad |x| \to \infty, \tag{2.4}$$

for a known constant k_0 (whose value is not important to us). In particular,

$$a(x) = \frac{2n}{\pi} + k_0 + O(e^{-n}), \quad x \in \partial C_n.$$

More generally, if η is a SAW (or any finite set), we define $a_\eta(x)$ to be the unique function that is harmonic on on $\mathbb{Z}^2 \setminus \eta$; vanishes on η; and is asymptotic to $(2/\pi)\log|x|$ as $|x| \to \infty$. It is related to escape probabilities by

$$a_\eta(x) = \lim_{n\to\infty} \frac{2n}{\pi}\,\mathbf{P}^x\{S[0,\rho_n] \cap \eta = \emptyset\},$$

see [9, Proposition 6.4.7]. If η contains 0, we can write

$$a_\eta(x) = a(x) - \mathbf{E}^x[a(S_\tau)], \tag{2.5}$$

where $\tau = \tau_\eta = \min\{j \geq 0 : S_j \in \eta\}$. If $x \in \eta$, we write

$$\mathrm{Es}_\eta(x) = \lim_{n\to\infty} \frac{2n}{\pi}\,\mathbf{P}^x\{S[1,\rho_n] \cap \eta = \emptyset\} = \Delta a_\eta(x),$$

where Δ denotes the discrete Laplacian. The capacity of η, $\mathrm{cap}(\eta)$, is defined by

$$a_\eta(x) = \frac{2}{\pi}\,\log|x| - \mathrm{cap}(\eta) + o(1), \quad |x| \to \infty.$$

Random walk in $\mathbb{Z}^2$ conditioned to avoid η is the h-process obtained from the function $a_\eta(x)$. In other words, if x is in the unbounded component of $\mathbb{Z}^2 \setminus \eta$, then the transition probabilities are given by

$$p(x,y) = \frac{a_\eta(y)}{4\,a_\eta(x)}.$$

This process can also be started on the boundary of the unbounded component (that is, on points of η that are connected to infinity in $\mathbb{Z}^2 \setminus \eta$), by the same formula, replacing $a_\eta(x)$ with $\mathrm{Es}_\eta(x)$. It is immediate that this is a transient process that never returns to η after time 0. The case $\eta = [0]$, $a_\eta = a$ corresponds to random walk conditioned to never return to the origin.

Lemma 2.1. *There exists $c > 0$ such that the following holds. Let $0 < k < n$ and let η be a SAW intersecting both C_k and ∂C_n. Let S be a simple random walk starting at $x \in C_k$.*

- *(Beurling)*
$$\mathbf{P}^x\{S[0,\rho_n]\cap\eta=\emptyset\}\leq c\,e^{(k-n)/2}.$$
- *(Disconnection probability)*
$$\mathbf{P}^x\{0 \text{ is connected to } \partial C_n \text{ in } \mathbb{Z}^2\setminus S[0,\rho_n]\}\leq ce^{(k-n)/4}. \tag{2.6}$$

We have stated these estimates for random walk starting in C_k ending at ∂C_n. By reversing paths we get analogous statements for random walks starting in ∂C_n stopped upon reaching C_k.

Proof. The discrete Beurling estimate was first proved by Kesten in [3]; see also, [9, Theorem 6.8.1]. The fact that the disconnection probability satisfies a power law up to constants (with no logarithmic correction) with the Brownian disconnection exponent was proved in [10]. The value of the exponent was determined rigorously in [11]. □

The exponents $1/2$ and $1/4$ are known but they take some effort to prove, especially the latter one. For our main theorem, it would suffice that there is some exponent that satisfies these conditions and proving that is significantly easier; however, in order to avoid having extra arbitrary exponents, we will use the actual values.

The following is an easy corollary of (2.4), (2.5), and the Beurling estimate.

Lemma 2.2. *There exists $0<c<\infty$ such that the following is true. Suppose $\eta\in\mathcal{W}_n$.*

- *For $|z|>e^n$,*
$$a_\eta(z)\geq\frac{2}{\pi}\left[\log|z|-n\right]+O(e^{-n}).$$
- *For all $z\in C_n$,*
$$a_\eta(z)\leq c\,e^{-n/2}\,\left[\operatorname{dist}(z,\eta)\right]^{1/2}.$$
In particular, if $z\in C_k$ with $k<n$,
$$a_\eta(z)\leq c\,e^{(k-n)/2}.$$

Another simple idea that we will use is the following.

Lemma 2.3. *There exists $c<\infty$, such that the following hold if $d=2$ and $n,m\geq 1$.*

- *Let V be a connected subset of C_n of diameter at least $e^n/100$. If $z\in C_{n+1}$, then the probability that a random walk starting at z reaches ∂C_{n+m} without hitting V is less than c/m.*
- *Let V be a connected subset of $C_{n+m+1}\setminus C_{n+m}$ of diameter at least $e^{n+m}/100$. If $z\in\mathbb{Z}^2\setminus C_{n+m-1}$, then the probability that a random walk starting at z reaches ∂C_n without hitting V is less than c/m.*

Proof. We will do the first; the proof of the second is similar. The key fact is that there exist universal $0<c_1<c_2<\infty$ such that

- the probability that random walk starting at $z\in C_{n+1}$ hits V before reaching ∂C_{n+2} is greater than c_1 (this can be shown, say, by the invariance principle and a simple topological argument using planarity);

- the probability that a random walk starting at $w \in \partial C_{n+2}$ reaches ∂C_{n+m} before hitting C_{n+1} is less than c_2/m.

If q denotes the maximum over $z \in C_{n+1}$ of the probability of reaching ∂C_{n+m} before hitting V, we get the inequality

$$q \leq \frac{c_2}{m} + (1 - c_1)\, q. \qquad \square$$

We will also use the following estimate of the transience of two-dimensional random walk conditioned to never return to the origin.

Lemma 2.4. *Suppose $d = 2$ and $\tilde{S}_j$ is a simple random walk conditioned to never return to the origin. For $0 < r \leq 2$, if $z \in \partial C_{n+r}$, then as $n \to \infty$,*

$$\mathbf{P}^z\{\tilde{S}[0,\infty) \cap C_n = \emptyset\} = \frac{r}{n} + O\left(\frac{r}{n^2}\right).$$

Moreover, there exists $c < \infty$ such that if $\eta \in \mathcal{W}_n$,

$$\mathbf{P}^z\{\tilde{S}[0,\infty) \cap \eta = \emptyset\} \leq \frac{c\, r}{n}.$$

Proof. From the definition of an h-process, and the fact that the unconditioned walk reaches C_n with probability one, we see that

$$\frac{\min\{a(x) : x \in \partial_i C_n\}}{a(z)} \leq \mathbf{P}^z\{\tilde{S}[0,\infty) \cap C_n \neq \emptyset\} \leq \frac{\max\{a(x) : x \in \partial_i C_n\}}{a(z)}. \tag{2.7}$$

Using (2.4) we see that if $x \in \partial_i C_n$,

$$a(x) = \frac{2n}{\pi} + k_0 + O(e^{-n}),$$

and

$$a(z) = \frac{2(n+r)}{\pi} + k_0 + O(e^{-n}).$$

Therefore, both the left and right hand sides of (2.7) equal

$$1 - \frac{r + O(e^{-n})}{n + r + (\pi k_0/2)} = 1 - \frac{r}{n} + O\left(\frac{r}{n^2}\right).$$

This gives the first inequality and the second is done similarly to the previous lemma. $\square$

2.3 Loop measures

Here we review some facts about the random walk loop measure and its relation to LERW; for more details see, [9, Chapter 9]. We will consider the loop measures in $\hat{\mathbb{Z}}^d := \mathbb{Z}^d \setminus \{0\}$. (If we were considering only the transient case $d \geq 3$, it would be a little easier to consider the loop measure in $\mathbb{Z}^d$; however, for $d = 2$, we need to restrict to $\hat{\mathbb{Z}}^2$ and this also works for $d \geq 3$, so we will use this approach.) A rooted loop is a nearest neighbor path

$$l = [l_0, l_1, \ldots, l_{2n}]$$

in $\hat{\mathbb{Z}}^d$ with $n > 0$ and $l_0 = l_{2n}$. The *rooted loop measure* $\tilde{m}$ is the measure on rooted loops that assigns measure $[2n(2d)^{2n}]^{-1}$ to each loop of $2n$ steps. An unrooted loop ℓ is an equivalence class of loops under the relation

$$[l_0, l_1, \ldots, l_{2n}] \sim [l_1, \ldots, l_{2n}, l_1] \sim [l_2, l_3, \ldots, l_{2n}, l_1, l_2] \sim \cdots.$$

The *(unrooted) loop measure* m is the measure induced on unrooted loops by the rooted loop measure

$$m(\ell) = \sum_{l \in \ell} \tilde{m}(l).$$

If $B \subset A \subset \hat{\mathbb{Z}}^d$, we set

$$F_B(A) = \exp\left\{ \sum_{\ell \subset A, \ell \cap B \neq \emptyset} m(\ell) \right\} = \exp\left\{ \sum_{l \subset A, l \cap B \neq \emptyset} \tilde{m}(l) \right\}.$$

In other words, $\log F_B(A)$ is the (loop) measure of loops in A that intersect B. An equivalent definition (see, e.g., [9, Propositions 9.3.1, 9.3.2]) can be given by setting $B = \{y_1, \ldots, y_m\}, A_k = A \setminus \{y_1, \ldots, y_{k-1}\}$, in which case

$$F_B(A) = \prod_{k=1}^{m} G_{A_k}(y_k, y_k),$$

where G_{A_k} denotes the usual random walk Green's function on A_k. If $B \not\subset A$, we set $F_B(A) = F_{B \cap A}(A)$.

A *loop soup* is a Poissonian realization from the loop measure. In particular, if $\mathcal{L}$ is a set of loops, then the probability that the loop soup contains no loop from $\mathcal{L}$ is $\exp\{-m(\mathcal{L})\}$. When giving measures of sets $\mathcal{L}$ of loops, one can either give $m(\mathcal{L})$ or one can give the probability of at least one loop, $1 - e^{-m(\mathcal{L})}$, and for small $m(\mathcal{L})$ these are equal up to an error of order $O(m(\mathcal{L})^2)$ (Loops soups with various intensities are studied in relation to other models. In this paper we consider only the soup with intensity one which corresponds to the loop-erased walk.)

The next lemma gives a useful way to compute loop measures of certain sets of loops (see [9, Section 9.5]). If $x \in V$ and we wish to obtain a realization of the loop soup (with intensity 1) restricted to loops in V that intersect x we can do the following:

- Start a random walk S at x and stop it at the first time T that it exits V (for $d \geq 3$, this can be infinity).
- Let σ be be the last time before T that the walk is at x. This gives a loop $S[0, \sigma]$. We can decompose this as a finite union of loops that return to x only once.

The next lemma follows from this observation.

Lemma 2.5. *Suppose A, B are disjoint subsets of $\hat{\mathbb{Z}}^d$. Suppose A is finite and the points in A are ordered $A = \{x_1, x_2, \ldots, x_n\}$. Let $A_j = A \setminus \{x_1, \ldots, x_{j-1}\}$. For each j, let S be a simple random walk starting at x_j and let*

$$T^j = \max\{k : S_k = x_j, S[0, k] \cap \{0, x_1, \ldots, x_{j-1}\} = \emptyset\}.$$

Then the probability that a loop soup contains a loop in A_j that intersects both x_j and B is

$$\mathbf{P}^{x_j}\{S[0, T^j] \cap B \neq \emptyset\}.$$

In particular, the probability that the loop soup contains a loop that intersects both A and B is bounded above by

$$\sum_{j=1}^{n} \mathbf{P}^{x_j}\{S[0, T^j] \cap B \neq \emptyset\} \tag{2.8}$$

Note that $\mathbf{P}^{x_j}\{S[0, T^j] \cap B \neq \emptyset\}$ is the probability that a simple random walk starting at x_j reaches B and then returns to x_j without visiting $\{0, x_1, \ldots, x_{j-1}\}$. The bound (2.8) holds regardless of which ordering of the vertices of A is used. The proofs of the following two lemmas are similar to that in [9, Lemma 11.3.3]. We sketch the proofs.

Lemma 2.6. *There exists $c = c(d) < \infty$ such that for all $r, n \geq 0$, the probability that the loop soup contains a loop that intersects both C_n and $\mathbb{Z}^d \setminus C_{n+r}$ is bounded above by $c\, e^{r(2-d)}$.*

Proof. We assume $r \geq 2$ and $d \geq 3$ for the other cases are trivial. Let $B = \mathbb{Z}^d \setminus C_{n+r}$. We write $C_n = \{0, x_1, \ldots, x_N\}$ where the vertices are ordered so that $|x_j|$ is nondecreasing. Let $A_j = \mathbb{Z}^d \setminus \{x_1, \ldots, x_{j-1}\}$. If we start at $x = x_j \in \hat{C}_n$, then the probability that it reaches distance $2|x|$ from the origin without leaving A_j is $O(|x|^{-1})$; the probability that after it leaves C_{n+r} it returns to within distance $2|x|$ of the origin is $O(|x|^{d-2}\, e^{(n+r)(2-d)})$; and given that, the probability of hitting x before any point in A_j is $O(|x|^{1-d})$. Hence,

$$\mathbf{P}^{x_j}\{S^j[0, T^j] \cap B \neq \emptyset\} \leq c\, |x_j|^{-2}\, e^{-(n+r)(d-2)}.$$

Summing over $0 < |x| < e^n$, we get

$$\sum_{x_j \in C_n} \mathbf{P}^{x_j}\{S^j[0, T^j] \cap B \neq \emptyset\} \leq c\, e^{n(d-2)}\, e^{(n+r)(2-d)} \leq c\, e^{r(2-d)}. \qquad \square$$

Lemma 2.7. *If $d = 2$, there exist $c < \infty$ such that for all $r, n \geq 0$, the probability that the loop soup contains a loop that intersects both C_n and $\mathbb{Z}^d \setminus C_{n+r}$ and does not disconnect C_n from ∂C_{n+r} is bounded above by $c\, e^{-r/2}$.*

Proof. This is done similarly using (2.6). We will say just disconnecting for "disconnecting C_n from ∂C_{n+r}". Let $B = \mathbb{Z}^2 \setminus C_{n+r}$, and write $C_n = \{0, x_1, \ldots, x_N\}$ where the vertices are ordered so that $|x_j|$ is nondecreasing. Let $A_j = \mathbb{Z}^d \setminus \{x_0, \ldots, x_{j-1}\}$. If we start at $x = x_j \in C_n$, then the probability that it reaches distance $2|x|$ from the origin without leaving A_j is $O(|x|^{-1})$; given that, the probability that it reaches ∂C_n without leaving A_j is $O((n + 1 - \log|x|)^{-1})$; given this, the probability that it reaches ∂C_{n+r} without disconnecting is $O(e^{-r/4})$; given this, the probability that after it leaves C_{n+r} it returns to C_n without disconnecting is $O(e^{-r/4})$; given this, the probability to get within $2|x|$ without disconnecting is $O((n + 1 - \log|x|)^{-1})$; and given that, the probability of hitting x before any point in A_j is $O(|x|^{-1})$. Hence,

$$\mathbf{P}^{x_j}\{S^j[0, T^j] \cap B \neq \emptyset, \text{no disconnection}\} \leq c\, |x_j|^{-2}\, [n + 1 - \log|x|]^{-2}\, e^{-r/2}.$$

Summing over $0 < |x| < e^n$, we get

$$\sum_{x_j \in C_n} \mathbf{P}^{x_j}\{S^j[0, T^j] \cap B \neq \emptyset, \text{no disconnection}\} \leq c\, e^{-r/2}. \qquad \square$$

Lemma 2.8. *If $d \geq 3$, let $\mathcal{L}_n(\delta)$ denote the set of loops that intersect $C_{n+2} \setminus C_{n-2}$ and have diameter at least $\delta\, e^n$. For every $\delta > 0$, there exists $\epsilon > 0$, such that for every n, the probability that the loop soup contains no loop in $\mathcal{L}_n(\delta)$ is at least ϵ.*

Proof. This follows from Lemma 2.6 and a simple covering argument. $\square$

Lemma 2.9. *There exists $c < \infty$ such that the following holds. Suppose $d = 2$, n, m are positive integers, and V is a simply connected subset of $\mathbb{Z}^2$ with*

$$C_{n+m} \subset V, \quad C_{n+m+1} \not\subset V.$$

Let $K = K_{n,V}$ denotes the measure of the set of loops that lie in $\mathbb{Z}^2 \setminus C_{n-1}$ and intersect both C_n and ∂V. Then,

$$\left| K - \frac{1}{m} \right| \leq c\, \frac{me^{-n} + 1}{m^2}.$$

Proof. We first consider the case $V = C_{n+m}$. We order the elements of $\mathbb{Z}^2 = \{x_0 = 0, x_1, x_2, \ldots\}$ so that $|x_0| \leq |x_1| \leq |x_2| \leq \cdots$. We let $A_k = \{0, x_1, \ldots, x_k\}$ and define k_n by $C_n = A_{k_n}$ (so that $k_n \sim \pi\, e^{2n}$). Let $\theta_k = \pi \operatorname{cap}(A_k)/2$, and $g_k = \pi\, a_{A_k}/2$ which is the unique function that is 0 on A_k, discrete harmonic on $\mathbb{Z}^2 \setminus A_k$, and satisfies

$$g_k(z) = \log|z| - \theta_k + o(1), \quad |z| \to \infty.$$

If S_t denotes a simple random walk and

$$\rho_n = \min\{t : |S_t| \geq e^n\},$$

then

$$g_k(z) = \lim_{r\to\infty} r\, \mathbf{P}^z\{S[0,\rho_r] \cap A_k = \emptyset\}.$$

Using well-known estimates (see [9, Proposition 6.4.1]), we see that for $z \in C_{n+m+1} \setminus C_{n+m}$, $k_{n-1} \leq k \leq k_n$,

$$\lim_{r\to\infty} r\, \mathbf{P}^z\{S[0,\rho_r] \cap A_{k-1} = \emptyset\} = m + O(1).$$

If we write $\mathbf{P}, \mathbf{E}$ for $\mathbf{P}^{x_k}, \mathbf{E}^{x_k}$, we have

$$\begin{aligned} & g_{k-1}(x_k) \\ = \ & \lim_{r\to\infty} r\, \mathbf{P}\{S[0,\rho_r] \cap A_{k-1} = \emptyset\} \\ = \ & \lim_{r\to\infty} r\, \mathbf{P}\{S[0,\rho_{n+m}] \cap A_{k-1} = \emptyset\}\, \mathbf{P}\{S[0,\rho_r] \cap A_{k-1} = \emptyset \mid S[0,\rho_{n+m}] \cap A_{k-1} = \emptyset\} \\ = \ & \mathbf{P}\{S[0,\rho_{n+m}] \cap A_{k-1} = \emptyset\}\, [m + O(1)], \end{aligned}$$

and hence

$$\mathbf{P}\{S[0,\rho_{n+m}] \cap A_{k-1} = \emptyset\} = \frac{g_{k-1}(x_k)}{m}\ \left[1 + O(m^{-1})\right].$$

Note that

$$g_k(z) - g_{k-1}(z) = -\mathrm{hm}_{A_k}(z, x_k)\, g_{k-1}(x_k),$$

where hm denotes harmonic measure, that is, the hitting distribution of A_k starting at z. We will use the estimate (this follows from [9, Proposition 6.4.5]),

$$\mathrm{hm}_{A_k}(z, x_k) = \mathrm{hm}_{A_k}(\infty, x_k)\ \left[1 + O\left(me^{-m}\right)\right], \quad z \in C_{n+m+1} \setminus C_{n+m}.$$

(We believe the error is actually $O(e^{-m})$ but it would take a little more effort to prove and we do not need the stronger result.) Therefore,

$$\begin{aligned} \theta_k - \theta_{k-1} &= \lim_{z\to\infty} [g_{k-1}(z) - g_k(z)] \\ &= \mathrm{hm}_{A_k}(\infty, x_k)\, g_{k-1}(x_k) \\ &= \mathrm{hm}_{A_k}(z, x_k)\, g_{k-1}(x_k)\ \left[1 + O\left(me^{-m}\right)\right]. \end{aligned}$$

Using Lemma 2.5 we see that the probability that the loop soup contains a loop including x_k, lying in $\mathbb{Z}^2 \setminus A_{k-1}$, and also intersecting ∂C_{n+m} is equal to

$$\begin{aligned} & \mathbf{E}^{x_k}\left[\mathrm{hm}_{A_k}(S_{\rho_{n+m}}, x_k); S[0,\rho_{n+m}] \cap A_{k-1} = \emptyset\right] \\ = \ & \mathbf{P}\{S[0,\rho_{n+m}] \cap A_{k-1} = \emptyset\}\, \mathbf{E}^{x_k}\left[\mathrm{hm}_{A_k}(S_{\rho_{n+m}}, x_k) \mid S[0,\rho_{n+m}] \cap A_{k-1} = \emptyset\right] \\ = \ & \frac{g_{k-1}(x_k)}{m}\, \frac{\theta_k - \theta_{k-1}}{g_{k-1}(x_k)}\, [1 + O(m^{-1})] \\ = \ & \frac{\theta_k - \theta_{k-1}}{m}\, [1 + O(m^{-1})]. \end{aligned}$$

It follows that the measure of the set of loops that lie in $\mathbb{Z}^2 \setminus A_{k-1}$, contain x_k, and intersect ∂C_{n+m} is

$$\frac{\theta_k - \theta_{k-1}}{m}\,[1 + O(m^{-1})].$$

The capacities of C_{n-1} and C_n are well known up to a small error (see [9, Proposition 6.6.5]); indeed,

$$\theta_{k_n} = \theta_{k_{n-1}} + 1 + O(e^{-n}),$$

and hence

$$\sum_{j=k_{n-1}+1}^{n_k} \mathbf{E}^{x_j}\left[\mathrm{hm}_{A_k}(S_{\rho_{n+m}}, x_k); S[0, \rho_{n+m}] \cap A_{k-1} = \emptyset\right] = \frac{1 + O(e^{-n})}{m} + O\left(\frac{1}{m^2}\right).$$

For more general V, we use the fact that V is simply connected and $\partial V \cap C_{n+m+1} \neq \emptyset$ to see that the probability that a random walk starting in ∂C_{n+m} reaches ∂C_{n+1} without hitting ∂V is $O(m^{-1})$ (see Lemma 2.3). Arguing as above, we can see that the measure of the set of loops that lie in $\mathbb{Z}^2 \setminus A_{k-1}$, contain x_k, intersect ∂C_{n+m}, but do *not* intersect ∂V is $O(m^{-2})$. □

Lemma 2.10. *Suppose $d = 2$. There exists $c < \infty$ such that the following holds.*

- *Let A be a simply connected subset of $\mathbb{Z}^2$ with $e^{n+1} \leq \mathrm{dist}(0, \partial A) \leq e^{n+1} + 1$, and let $L = L_A$ denote the measure of loops in $\hat{\mathbb{Z}}^2$ that intersect both C_n and ∂A. Then,*

$$|L - \log n| \leq c.$$

- *For every $\delta > 0$, there exists $c_\delta < \infty$ such that the measure of loops in C_{n+1} that intersect $C_{n+1} \setminus C_n$; are of diameter at least $\delta\, e^n$; and do not disconnect 0 from ∂C_{n+1} is bounded above by c_δ.*

Proof.

- This follows from the previous lemma by summing.
- The measure of the set of loops in $\mathbb{Z}^2 \setminus C_{n-j}$ that intersect both C_{n-j+1} and $\mathbb{Z}^2 \setminus C_{n+1}$ and do not disconnect 0 from ∂C_n is $O(e^{-j/4})$. □

2.4 A lemma about simple random walk

Here we discuss a lemma about simple random walk that plays a crucial role in our analysis. It is very believable, but the important fact is that a constant can be chosen uniformly. We first state the result and gives some important corollaries. The $d = 2$ case was done in [16, Propositon 3.5] and the $d = 3$ case was proved in [17]. For completeness, we discuss the proof in the appendix. Here S_j denotes a simple random walk and $\mathbf{P}^x, \mathbf{E}^x$ denote probabilities and expectations assuming that $S_0 = x$.

Lemma 2.11. *There exists $c > 0$ such that the following is true.*

1. *Suppose $A' \subset C_n, z \in \partial C_n, A = A' \cup \{z\}$. Let $\tau = \tau_A = \min\{j \geq 1 : S_j \in A\}$ and $\sigma_r = \min\{j : |S_j - S_0| \geq r\}$. Then,*

$$\mathbf{P}^z\left\{|S_{\sigma_r}| \geq e^n + \frac{r}{2} \mid \sigma_r < \tau\right\} \geq c. \tag{2.9}$$

2. *Suppose $r < e^n/2$, $A' \subset \mathbb{Z}^d \setminus C_n, z \in \partial_i C_n$, $A = A' \cup \{z\}$. Then,*

$$\mathbf{P}^z\left\{|S_{\sigma_r}| \leq e^n - \frac{r}{2} \mid \sigma_r < \tau\right\} \geq c. \tag{2.10}$$

The proof strongly uses the fact that A' is in C_n (or $\mathbb{Z}^d \setminus C_n$) and the random walk is starting on the boundary of C_n. We will discuss the proof of (2.10) in Section A (this is the harder case), but we give some preliminary reductions here.

- It suffices to prove the lemma for n sufficiently large, for then the small n can be done on a case by case basis.
- Using the invariance principle, it suffices to consider $r \leq \delta_0 \, e^n$ for some $\delta_0 > 0$.
- Using the invariance principle, it suffices to establish (2.9) and (2.10) with $\frac{r}{2}$ replaced with ϵr for some $\epsilon > 0$.
- It suffices to find c_1 such that for n sufficiently large and $r \leq \delta_0 \, e^n$, (2.9) and (2.10) hold for some z_1 with $|z - z_1| \leq c_1$. Indeed, there is a positive probability (bounded uniformly from below) that a random walk starting at z reaches z_1 without visiting A.

We will derive a number of corollaries of this lemma.

Corollary 2.12. *Suppose $d \geq 3$. There exists $c > 0$ such that if we choose $r = e^{n-4}$ in part 1 of Lemma 2.11, then*

$$\mathrm{Es}_A(z) \geq c \, \mathbf{P}^z\{\sigma_r < \tau\}.$$

- *In particular, if $A_1', A_2' \subset C_n$ agree in the disk of radius e^{n-4} about z, then*

$$\mathrm{Es}_{A_1}(z) \asymp \mathrm{Es}_{A_2}(z).$$

- *If $A_1', A_2' \subset C_n$ and $A_1' \cap (C_n \setminus C_{n-j}) = A_2' \cap (C_n \setminus C_{n-j})$ for some $j \geq 1$, then*

$$\mathrm{Es}_{A_1}(z) = \mathrm{Es}_{A_2}(z) \, [1 + O(e^{j(2-d)})].$$

Proof. We write $\mathbf{P}$ for $\mathbf{P}^z$. The lemma tells us that

$$\mathbf{P}\{|S_{\sigma_r}| \geq e^n + e^{n-5} \mid \sigma_r < \tau\} \geq c.$$

Since $d \geq 3$, there exists c_1 such that the probability that a random walk starting at distance $e^n + e^{n-5}$ from the origin never returns to C_n is greater than c_1. Hence, there exists $c_2 < \infty$ such that

$$\mathbf{P}\{S[1,\infty) \cap A = \emptyset \mid \sigma_r < \tau\} \geq c_2.$$

For the final bullet note that $|\mathrm{Es}_{A_1}(z) - \mathrm{Es}_{A_2}(z)|$ is bounded above by $\mathbf{P}\{\sigma_r < \tau\}$ times the conditional probability given this that the random walk enters C_{n-j}. The latter probability is $O(e^{j(2-d)})$, and hence

$$|\mathrm{Es}_{A_1}(z) - \mathrm{Es}_{A_2}(z)| \leq c \, e^{j(2-d)} \, \mathbf{P}\{\sigma_r < \tau\} \asymp e^{j(2-d)} \, \mathrm{Es}_{A_1}(z). \qquad \square$$

We will give a similar result for $d = 2$, but we will put in an additional condition. We say that z is connected to 0 in A if there exists a SAW $\eta \in \mathcal{A}_n$ with terminal vertex z with $\eta \subset A$. We similarly can say that z is connected to infinity in an infinite set A.

Corollary 2.13. *Suppose $d = 2$. There exists $c > 0$ such that if we choose $r = e^{n-4}$ in part 1 of Lemma 2.11, then*

$$\mathrm{Es}_A(z) \geq c \, n^{-1} \, \mathbf{P}^z\{\sigma_r < \tau\}.$$

Moreover, if z is connected to 0 in A_1, A_2, the following hold.

- *If $A_1', A_2' \subset C_n$ agree in the disk of radius e^{n-4} about z, then*

$$\mathrm{Es}_{A_1}(z) \asymp \mathrm{Es}_{A_2}(z) \asymp n^{-1}\,\mathbf{P}^z\{\sigma_r < \tau\}.$$

- *If $A_1' \cap (C_n \setminus C_{n-j}) = A_2' \cap (C_n \setminus C_{n-j})$ for some $j \geq 1$, then*

$$\mathrm{Es}_{A_1}(z) = \mathrm{Es}_{A_2}(z)\,[1 + O(e^{-j})], \quad d = 2.$$

Proof. The proof is similar. For the first inequality we use Lemma 2.4 to see that

$$\mathbf{P}\{\tilde{S}[1,\infty) \cap A = \emptyset \mid \sigma_r < \tau\} \geq c_2\, n^{-1},$$

where $\tilde{S}$ is random walk conditioned to avoid the origin.

If z is connected to 0 in A_1, we can see from the Harnack inequality, Lemma 2.3, and the Beurling estimate that there exists uniform $0 < c_3 < c_4 < \infty$ such that

$$c_3 \leq a_{A_1}(w) \leq c_4, \quad w \in \partial C_{(1+r)n},$$
$$a_{A_1}(w) \leq c_4\, e^{-j/2}, \quad w \in C_{n-j},$$
$$a_{A_1}(w) \geq c_3, \quad w \notin C_{(1+r)n}.$$

Also, the probability that a random walk starting at z reaches C_{n-j} without returning to A is bounded above by $O(e^{-j/2})$; if it succeeds in doing this, there is at most a $O(e^{-j/2})$ probability that it returns to ∂C_n without hitting A. Hence, conditioned that a random walk avoids A_1, the probability that it hits C_{n-j} is $O(e^{-j})$ which implies that

$$\mathrm{Es}_{A_1 \cup C_{n-j}}(z) \geq \mathrm{Es}_{A_1}(z)\,[1 - O(e^{-j})],$$

and similarly for A_2. □

The following was given in the proofs but it is important enough to state it separately.

Corollary 2.14. *There exists $c < \infty$ such that if $\eta \in \mathcal{W}_n$ with terminal point z, then the probability that a simple random walk starting at z conditioned to never return to η enters C_{n-j} is less than ce^{-j}.*

Corollary 2.15. *There exists $c > 0$ such that if we choose $r = e^{n-4}$ in part 2 of Lemma 2.11, and $B = \mathbb{Z}^d \setminus A$, then for $d \geq 3$,*

$$H_B(0,z) \asymp e^{n(2-d)}\,\mathbf{P}^z\{\sigma_r < \tau\}.$$

If $d = 2$ and A contains a connected path of diameter e^{n-4} including z, then the same result is true.

Proof. By reversing paths, we see that

$$H_B(0,z) = \mathbf{E}^z\left[G_B(0, S_{\sigma_r}); \sigma_r < \tau\right].$$

For the upper bound, we use $G_B(0, S_{\sigma_r}) \leq c\, e^{n(2-d)}$, which for $d = 2$ requires the extra assumption. For the lower bound, we use

$$\mathbf{P}^z\{|S_{\sigma_r}| \leq e^n - e^{n-5} \mid \sigma_r < \tau\} \geq c,$$

and for $|x| \leq e^n - e^{n-5}$,

$$G_B(0,x) \geq G_{C_n}(0,x) \geq c\,|x|^{2-d}.$$ □

Corollary 2.16. *If $n \leq m-1$, $\eta \in \mathcal{W}_n$ with terminal point y, and $\bar\eta \in \bar{\mathcal{W}}_{n+1,m}$ with initial point w, then if $A = \mathbb{Z}^d \setminus (\eta \cup \bar\eta)$,*

$$H_{\partial A}(y,w) \asymp \begin{cases} \mathrm{Es}_\eta(y)\, H_{\mathbb{Z}^d \setminus \bar\eta}(0,w), & d \geq 3 \\ n\, \mathrm{Es}_\eta(y)\, H_{\mathbb{Z}^d \setminus \bar\eta}(0,w), & d = 2 \end{cases}.$$

Proof. Let $S, \tilde S$ be independent random walks starting at y, w, and let $\sigma_r, \tilde\sigma_r$ be the corresponding stopping times with $r = e^{n-4}$. Any random walk path ω from y to w in A can be decomposed as

$$\omega = \omega^- \oplus \tilde\omega \oplus \omega^+,$$

where ω^- is the walk stopped at the first time it reaches distance r from y, and ω^+ is the reversal of the reversed walk stopped at the first time it reaches distance r from z. Using this decomposition, we can see that for $d \geq 3$,

$$\begin{aligned} H_{\partial A}(y,w) &= \sum_{x,z} \mathbf{P}^y\{S_{\sigma_r} = x; \sigma_r < \tau\}\, \mathbf{P}^w\{S_{\tilde\sigma_r} = z; \tilde\sigma_r < \tilde\tau\}\, G_A(x,z) \\ &\asymp \mathrm{Es}_\eta(y)\, e^{n(2-d)}\, \mathbf{P}^w\{\tilde\sigma_r < \tilde\tau\} \\ &\asymp \mathrm{Es}_\eta(y)\, H_{\mathbb{Z}^d \setminus \bar\eta}(0,w). \end{aligned}$$

For $d = 2$, we need to replace $\mathrm{Es}_\eta(y)$ with $n\,\mathrm{Es}_\eta(y)$. ☐

Corollary 2.17. *If $n \leq m-1$, $\eta, \tilde\eta \in \mathcal{W}_n$ with terminal point y and such that $\eta \setminus C_{n-j} = \tilde\eta \setminus C_{n-j}$, and $\bar\eta \in \bar{\mathcal{W}}_{n+1,m}$ with initial point w, then if $A = \mathbb{Z}^d \setminus (\eta \cup \bar\eta)$, $\tilde A = \mathbb{Z}^d \setminus (\tilde\eta \cup \bar\eta)$,*

$$H_{\partial A}(y,w) = H_{\partial \tilde A}(y,w)\,[1 + O(e^{-j})].$$

Proof. We start as in the last proof with

$$H_{\partial A}(y,w) = \sum_{x,z} \mathbf{P}^y\{S_{\sigma_r} = x; \sigma_r < \tau\}\, \mathbf{P}^w\{S_{\tilde\sigma_r} = z; \tilde\sigma_r < \tilde\tau\}\, G_A(x,z),$$

and similarly for $\tilde A$ and then use

$$G_A(x,z) = G_{\tilde A}(x,z)\,[1 + O(e^{-j})].$$

For $d = 2$, this uses the Beurling estimate. ☐

There is a simple fact about the loop-erasing process that we will use. We state it as a proposition (so we can refer to it), but it is an easily verified property of the deterministic loop-erasing procedure.

Proposition 2.18. *Suppose S is a simple random walk starting at $x \in C_n$ and $n < k < m$. Suppose that*

- *After the first visit to ∂C_k, the walk never returns to C_n.*
- *After the first visit to ∂C_m, the walk never returns to C_k.*

Suppose that we stop the path some time after it reaches ∂C_m and erase loops. After doing this, we view the remainder of the random walk and continue loop-erasing (and hence perhaps erasing some of the loop-erasure we already have). Then

- *The intersection of the original and the new loop-erased paths with C_n are the same.*

Indeed, in order to erase a point x in the intersection of the loop-erasure and C_n, the random walk would have to visit a point on the random walk that was visited before the last visit to x. There is no point in $\mathbb{Z}^d \setminus C_k$ that satisfies this, and the random walk visits no point in C_k after it has reached ∂C_m.

This gives a general procedure to give lower bounds on the probabilities of certain events for the loop-erased walk.

- If $n < m$, then in the measure μ_m, given the initial segment $\eta_n \in \mathcal{W}_n$, the continuation is obtained by taking a simple random walk conditioned to avoid η_n and erasing loops.
- Using our lemma and its corollaries, there is a positive probability that this simple random walk will start by reaching radius $e^{n+(1/10)}$ without going more than distance $e^{n+(1/5)}$ from the starting point.
- Given that, we can consider random walk conditioned to avoid C_n which is conditioning on an event of positive probability uniformly bounded away from zero. For $d = 2$, we need to use random walk conditioned to avoid some $\eta \in C_n$ (see Lemmas 2.2 and 2.4).
- The loop-erased path is a subpath of the simple random walk path, so if we know the simple path stays in some set, then so does the loop-erased path.

There are many applications of this; we state one as a corollary here.

Corollary 2.19. *There exists $c > 0$ such that the following holds. Suppose $\eta \in \mathcal{W}_n$ with terminal point y and $\bar{\eta} \in \bar{\mathcal{W}}_{n+1,m}$ with initial point w, $A = \mathbb{Z}^d \setminus (\eta \cup \bar{\eta})$, and ω is a simple random walk excursion starting at y conditioned to leave A at w. Let*

$$V = V_n(y, w) = (C_{n+1} \setminus C_n) \cup \{w' : |w' - w| \le e^{n-3}\} \cup \{y' : |y - y'| \le e^{n-3}\}.$$

Then,

- *The probability that $\omega \subset V$ is at least c.*
- *If it is also known that $y, w \in \{x = (x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1 \ge |x|/2\}$, then the probability that $\omega \subset \{x = (x_1, x_2, x_3) \in V : x_1 \ge |x|/4\}$ is at least c.*

Obviously these results hold for the loop-erasure of ω as well.

2.5 Loop-erased random walk

In this section discuss facts about a single loop-erased random walk (LERW) in $\mathbb{Z}^d, d \ge 2$.

- If S_j is a simple random walk starting at the origin conditioned to never return to the origin with loop-erasure $\hat{S}_j$, we let

$$\rho_n = \min\{j : S_j \notin C_n\}, \quad \hat{\rho}_n = \max\{j : S_j \in C_n\},$$
$$T_n = \min\{j : \hat{S}_j \notin C_n\}, \quad \bar{T}_n = \max\{j : \hat{S}_j \in C_n\},$$
$$\bar{T}_{n,m} = \max\{j \le T_m : \hat{S}_j \in C_n\}.$$

  Note that $\bar{T}_n + 1 \ge T_n$.
- μ_n is the distribution of $\hat{S}[0, T_n]$. It is a probability measure supported on $\mathcal{W}_n$.
- If $n < m$, $\mu_{n,m}$ is the distribution of $\hat{S}[\bar{T}_{n,m}, T_m]$. It is a probability measure on $\bar{\mathcal{W}}_{n,m}$.
- In the next subsection we also write μ_n and $\mu_{n,m}$ for the product measures $\mu_n \times \mu_n$ on $\mathcal{W}_n^2$ and $\mu_{n,m} \times \mu_{n,m}$ on $\bar{\mathcal{W}}_{n,m}^2$.

We recall the following fact that follows from the decomposition (1.1). Let

$$G_0 = \begin{cases} G(0,0), & d \geq 3 \\ 1, & d = 2. \end{cases}$$

Proposition 2.20. *If $\eta \in \mathcal{W}_n$ with terminal point z, then*

$$\mu_n(\eta) = (2d)^{-|\eta|}\, F_\eta\, G_0\, \mathrm{Es}_\eta(z). \tag{2.11}$$

Moreover, the distribution of $\hat{S}[T_n,\infty]$ given $\hat{S}[0,T_n] = \eta$ is the same as that obtained as follows:

- *Take a simple random walk starting at z conditioned to never return to η.*
- *Erase the loops chronologically.*

As we have mentioned, there are two different ways to define "loop-erased random walk stopped at ∂C_n": one is as the loop-erasure of $S[0,\rho_n]$, and the other is as $\hat{S}[0,T_n]$. These measures are significantly different near the terminal point. However, considered as measures on $\mathcal{W}_{n-1}$ by truncation, they are comparable.

We prefer to consider μ_n, that is the distribution of $\hat{S}[0,T_n]$, because we know that the distribution of the remainder of the path can be obtained by erasing loops from a simple random walk starting at $\hat{S}(T_n)$ conditioned to never return to $\hat{S}[0,T_n]$. The estimates from Section 2.4 apply to the conditioned random walk and the loop-erasure is a subpath of the conditioned walk. As an example, Corollary 2.14 implies that for $d = 2,3$, there exists $c < \infty$ such that conditioned on $\hat{S}[0,T_n]$, the probability that $\hat{S}[T_n,\infty)$ intersects C_{n-j} is less than $c\, e^{-j}$.

If $d = 2$, we define κ_n by saying that $\log \kappa_n$ is the measure of loops in $\hat{\mathbb{Z}}^2$ that disconnect 0 from ∂C_n. (We do not require the loop to lie in $\hat{C}_n$.) In this case, if $\eta \in \mathcal{W}_n$, we can write

$$\mu_n(\eta) = (2d)^{-|\eta|}\, \kappa_n\, F^{*,n}_\eta\, \mathrm{Es}_\eta(z), \quad d = 2, \tag{2.12}$$

where $\log F^{*,n}_\eta$ is the measure of loops in $\hat{\mathbb{Z}}^2$ that intersect η but do not disconnect 0 from ∂C_n.

The distribution $\mu_{n,m}$ is a little complicated, but we will only need to know it up to uniform multiplicative constants.

Proposition 2.21. *If $n \leq m-1$, and $\eta \in \bar{\mathcal{W}}_{n,m}$ with initial point w and terminal point z, then*

$$\mu_{n,m}(\eta) \asymp (2d)^{-|\eta|}\, F_\eta\, \mathrm{Es}_\eta(z)\, H_{\partial(\hat{\mathbb{Z}}^d \setminus \eta)}(0,w).$$

Recall that $H_{\mathbb{Z}^d\setminus\eta}(0,w) = G_{\mathbb{Z}^d\setminus\eta}(0,0)\, H_{\partial(\hat{\mathbb{Z}}^d\setminus\eta)}(0,w)$. If $d \geq 3$, $G_{\mathbb{Z}^d\setminus\eta}(0,0) \asymp 1$. If $d = 2$, $G_{\mathbb{Z}^d\setminus\eta}(0,0) \asymp n$.

Proof. We start by writing the exact expression

$$\begin{aligned} \mu_{n,m}(\eta) &= \sum_{\eta'\oplus\eta\in\mathcal{W}_m} \mu_m(\eta'\oplus\eta) \\ &= (2d)^{-|\eta|}\, F_\eta \sum_{\eta'\oplus\eta\in\mathcal{W}_m} \mathrm{Es}_{\eta'\oplus\eta}(z)\, (2d)^{-|\eta'|}\, F_{\eta'}(\hat{\mathbb{Z}}^d \setminus \eta). \end{aligned}$$

For the upper bound, we use $\mathrm{Es}_{\eta'\oplus\eta}(z) \leq \mathrm{Es}_\eta(z)$ to see that

$$\mu_{n,m}(\eta) \leq (2d)^{-|\eta|}\, F_\eta\, \mathrm{Es}_\eta(z) \sum_{\eta'\oplus\eta\in\mathcal{W}_m} (2d)^{-|\eta'|}\, F_{\eta'}(\hat{\mathbb{Z}}^d \setminus \eta).$$

The last sum is larger if we remove the restriction that $\eta' \oplus \eta \subset C_m$ and write just

$$\sum_{\eta'} (2d)^{-|\eta'|}\, F_{\eta'}(\hat{\mathbb{Z}}^d \setminus \eta), \tag{2.13}$$

where the sum is over all SAWs η' starting at the origin, ending at w, and otherwise staying in $\mathbb{Z}^d \setminus \eta$. Using a decomposition similar to (1.1), we can see that $(2d)^{-|\eta'|}\, F_{\eta'}(\mathbb{Z}^d \setminus \eta)$ is exactly the probability that a random walk starting at 0 stopped upon reaching $\eta \cup \{0\}$ stops in finite time and the loop-erasure of the stopped walk is η'. Therefore the sum in (2.13) equals $H_{\partial(\hat{\mathbb{Z}}^d \setminus \eta)}(0, w)$.

For the lower bound we write

$$\mu_{n,m}(\eta) \geq (2d)^{-|\eta|}\, F_\eta \sum_{\eta' \oplus \eta \in \mathcal{W}_m,\ \eta' \subset C_{n+\frac{1}{2}}} \mathrm{Es}_{\eta' \oplus \eta}(z)\, (2d)^{-|\eta'|}\, F_{\eta'}(\hat{\mathbb{Z}}^d \setminus \eta).$$

Using Corollaries 2.12 and 2.13, we can see this is greater than a constant times

$$(2d)^{-|\eta|}\, F_\eta\, \mathrm{Es}_\eta(z) \sum_{\eta' \oplus \eta \in \mathcal{W}_m,\ \eta' \subset C_{n+\frac{1}{2}}} (2d)^{-|\eta'|}\, F_{\eta'}(\hat{\mathbb{Z}}^d \setminus \eta).$$

As in the last paragraph, we see that the sum equals $H_{\partial(\hat{C}_{n+\frac{1}{2}} \setminus \eta)}(0, z)$. We can use Corollary 2.15 to see that

$$H_{\partial(\hat{C}_{n+\frac{1}{2}} \setminus \eta)}(0, z) \geq c\, H_{\partial(\hat{\mathbb{Z}}^d \setminus \eta)}(0, z). \qquad \square$$

We will now focus on the decomposition (2.2) of $\eta \in \mathcal{W}_m$. The next lemma shows that η_n and $\eta_{n+1,m}$ are "independent up to multiplicative constant". A two-dimensional version of this result can be found in [16, Section 4.1].

Proposition 2.22. *If $n \leq m-2$, $\eta \in \mathcal{W}_n, \tilde{\eta} \in \mathcal{W}_{n+1,m}$, then*

$$\sum_{\eta^*} \mu_m(\eta \oplus \eta^* \oplus \tilde{\eta}) \asymp \mu_n(\eta)\, \mu_{n+1,m}(\tilde{\eta}). \tag{2.14}$$

Here the sum is over all SAWs η^* such that $\eta \oplus \eta^* \oplus \tilde{\eta} \in \mathcal{W}_m$.

Proof. We will write $\eta' = \eta \oplus \eta^* \oplus \tilde{\eta}$. Let y be the terminal point of η, and let w, z be the initial and terminal points of η', respectively. Let $A = \mathbb{Z}^d \setminus (\eta \cup \tilde{\eta})$, and note that

$$F_{\eta \oplus \eta^* \oplus \tilde{\eta}} = F_\eta\, F_{\tilde{\eta}}(\mathbb{Z}^d \setminus \eta)\, F_{\eta^*}(A).$$

Since $\eta \subset C_n \cup \partial C_n$ and $\tilde{\eta} \subset (\mathbb{Z}^d \setminus C_{n+1}) \cup \partial_i C_{n+1}$, it follows from Lemma 2.6 that $F_{\tilde{\eta}} \asymp F_{\tilde{\eta}}(\mathbb{Z}^d \setminus \eta)$ for $d \geq 3$, and by Lemma 2.10 we see that $F^{*,n}_{\tilde{\eta}} \asymp F_{\tilde{\eta}}(\mathbb{Z}^d \setminus \eta)$ for $d = 2$. Therefore the sum on the left-hand side of (2.14) is comparable to

$$F_\eta\, F_{\tilde{\eta}} (2d)^{-|\eta| - |\tilde{\eta}|} \sum_{\eta^*} (2d)^{-|\eta^*|}\, F_{\eta^*}(A)\, \mathrm{Es}_{\eta'}(z), \quad d \geq 3, \tag{2.15}$$

and similarly for $d = 2$ with $F_{\tilde{\eta}}$ replaced with $F^{*,n}_{\tilde{\eta}}$.

For an upper bound, we use $\mathrm{Es}_{\eta'}(z) \leq \mathrm{Es}_{\tilde{\eta}}(z)$ to bound the sum by

$$\mathrm{Es}_{\tilde{\eta}}(z) \sum_{\eta^*} (2d)^{-|\eta^*|}\, F_{\eta^*}(A),$$

where the sum is over all SAWs from y to w and otherwise in A. The sum therefore equals $H_{\partial A}(y, w)$.

For the lower bound we restrict the sum in (2.15) to η^* such that $\eta^* \subset C_{n+\frac{3}{2}}$. In that case, we use Corollary 2.12 or Corollary 2.13 to tell us that $\mathrm{Es}_{\eta'}(z) \asymp \mathrm{Es}_{\tilde\eta}(z)$ and hence the quantity in (2.15) is bounded below by a constant times

$$F_{\eta_n}\, F_{\tilde\eta}(2d)^{-|\eta_n|-|\tilde\eta|}\, \mathrm{Es}_{\tilde\eta}(z) \sum_{\eta^* \subset C_{n+\frac{3}{2}}} (2d)^{-|\eta^*|}\, F_{\eta^*}(A).$$

We also use the results of that section to tell us that

$$\sum_{\eta^*} (2d)^{-|\eta^*|}\, F_{\eta^*}(A) \geq c\, H_{\partial A}(y,w).$$

Therefore, using Corollary 2.16, we see that for $d \geq 3$,

$$\begin{aligned} \sum_{\eta^*} \mu_m(\eta \oplus \eta^* \oplus \tilde\eta) &\asymp F_\eta\, F_{\tilde\eta}(2d)^{-|\eta|-|\tilde\eta|}\, H_{\partial A}(y,w)\, \mathrm{Es}_{\overline{\eta}}(z) \\ &\asymp F_\eta\, \mathrm{Es}_\eta(y)\, (2d)^{-|\eta|}\, F_{\tilde\eta}(2d)^{-|\tilde\eta|}\, H_{\mathbb{Z}^d\setminus\tilde\eta}(0,w)\, \mathrm{Es}_{\overline{\eta}}(z), \end{aligned}$$

and similarly for $d = 2$ with $F_{\tilde\eta}$ replaced with $F^{*,n}_{\tilde\eta}$. If $d \geq 3$, then $H_{\mathbb{Z}^d\setminus\tilde\eta}(0,w) \asymp H_{\partial(\hat{\mathbb{Z}}^d\setminus\tilde\eta)}(0,w)$.

We now claim that for $d = 2$, $F_\eta \asymp n\, F^{*,n}_{\tilde\eta}$, in other words, the measure of loops that intersect η and also disconnect 0 from ∂C_n equals $\log n + O(1)$. Indeed, this follows from Lemma 2.10. We therefore get

$$F^{*,n}_{\tilde\eta}\, H_{\mathbb{Z}^d\setminus\tilde\eta}(0,w) \asymp n^{-1}\, F_{\tilde\eta}\, H_{\mathbb{Z}^d\setminus\tilde\eta}(0,w) \asymp F_{\tilde\eta}\, H_{\partial(\hat{\mathbb{Z}}^d\setminus\tilde\eta)}(0,w).$$

Hence, for all $d \geq 2$,

$$\sum_{\eta^*} \mu_m(\eta \oplus \eta^* \oplus \tilde\eta) \asymp F_\eta\, \mathrm{Es}_\eta(y)\, (2d)^{-|\eta|}\, F_{\tilde\eta}(2d)^{-|\tilde\eta|}\, H_{\partial(\hat{\mathbb{Z}}^d\setminus\tilde\eta)}(0,w)\, \mathrm{Es}_{\overline{\eta}}(z) \asymp \mu_n(\eta)\, \mu_{n,m}(\tilde\eta).$$

□

It is useful to view the measures μ_n as generating a Markov chain γ_n with state space

$$\mathcal{W} := \bigcup_{n=0}^{\infty} \mathcal{W}_n.$$

The transitions always go from $\mathcal{W}_n$ to $\mathcal{W}_{n+1}$, and are such that $\gamma_n \prec \gamma_{n+1}$. Using (2.11) we give the transitions by

$$\phi(\gamma_n,\gamma_{n+1}) := \frac{\mu_n(\gamma_{n+1})}{\mu_n(\gamma_n)} = (2d)^{-|\tilde\gamma|} F_{\tilde\gamma}(\mathbb{Z}^d \setminus \gamma_n)\, \frac{\mathrm{Es}_{\gamma_{n+1}}(z_{n+1})}{\mathrm{Es}_{\gamma_n}(z_n)}. \tag{2.16}$$

Here z_n, z_{n+1} are the terminal points of γ_n, γ_{n+1}, respectively, and we have written $\gamma_{n+1} = \gamma_n \oplus \tilde\gamma$.

2.6 Coupling a one-sided LERW

Before handling the case of pairs of walks, it is useful to consider the simpler question of coupling one-sided infinite LERW with different initial conditions. The one-sided LERW is a probability measure on $\bar{\mathcal{W}}_0$. For each n, we write $\eta \in \bar{\mathcal{W}}_0$ uniquely as $\eta = \eta_n \oplus \eta_n^*$ where $\eta_n \in \mathcal{W}_n$. We will write $\eta \sim_k \tilde\eta$ if in this decomposition $\eta_k^* = \tilde\eta_k^*$, that is, if the paths agree after their first visit to ∂C_k. We do not require that η_k and $\tilde\eta_k$ have the same number of steps. If $\eta_n, \tilde\eta_n \in \mathcal{W}_n$, we write $\eta_n =_j \tilde\eta_n$ if the paths agree from the first visit to ∂C_{n-j} onward.

Proposition 2.23. *There exist $0 < u, c < \infty$ such that if $\eta_n, \tilde{\eta}_n \in \mathcal{W}_n$, then we can couple $\eta, \tilde{\eta}$ on the same probability space such that*

- *The distribution of η is LERW conditioned to start with η_n.*
- *The distribution of $\tilde{\eta}$ is LERW conditioned to start with $\tilde{\eta}_n$.*
- *If J denotes the smallest integer k such that $\eta^* \sim_{n+k} \tilde{\eta}^*$, then $\mathbf{E}[e^{uJ}] \leq c$.*

We start with a preliminary lemma.

Lemma 2.24. *There exist $c' < \infty$ such that if $\eta_n, \tilde{\eta}_n \in \mathcal{W}_n$ with $\eta_n =_k \tilde{\eta}_n$ with then we can couple $\eta, \tilde{\eta}$ on the same probability space such that*

- *The distribution of η is LERW conditioned to start with η_n.*
- *The distribution of $\tilde{\eta}$ is LERW conditioned to start with $\tilde{\eta}_n$.*
- $\mathbf{P}\{\eta_n^* = \tilde{\eta}_n^*\} \geq 1 - c'\, e^{-k}$.

Moreover, if $k \geq 1$, then $\mathbf{P}\{\eta_n^* = \tilde{\eta}_n^*\} \geq 1/c$.

Proof. We assume $k \geq 1$, The distribution of $\eta_n^*, \tilde{\eta}_n^*$ given $\eta_n, \tilde{\eta}_n$, is that of the loop erasure of a random walk starting at the endpoint conditioned to avoid $\eta_n, \tilde{\eta}_n$, respectively. Lemmas 2.12 and 2.13 show that we can couple these conditioned random walks so that they agree up to an event of probability $O(e^{-k})$. □

Lemma 2.25. *For all $j \geq 3$, there exists $\delta = \delta_j > 0$ such that if $\eta_n, \tilde{\eta}_n \in \mathcal{W}_n$, then we can define $\eta, \tilde{\eta}$ on the same probability space such that*

- *The distribution of η is LERW conditioned to start with η_n.*
- *The distribution of $\tilde{\eta}$ is LERW conditioned to start with $\tilde{\eta}_n$.*
- *With probability at least δ,*

$$\eta_{n+2}^* = \tilde{\eta}_{n+2}^*, \quad \eta_{n+2}^* \setminus \eta_{n+j}^* \subset \{\mathbf{x} \in \mathbb{Z}^d \setminus C_{n+1} : x_1 \geq |\mathbf{x}|/10\}.$$

Proof. We let $\omega, \tilde{\omega}$ denote simple random walks conditioned to avoid $\eta_n, \tilde{\eta}_n$ respectively. We first let $\omega, \tilde{\omega}$ move independently until they reach ∂C_{n+1}. For each one there is a positive probability that the walk did not enter C_{n-1} and that the endpoint is within distance $e^{n+1}/20$ of $\mathbf{e}_{n+1}$. The distribution of the endpoint is comparable to harmonic measure, that is, comparable to $e^{-n(1-d)}$ for each point.

Given that $\omega, \tilde{\omega}$ have reached ∂C_{n+1}, there is a universal $\rho > 0$ such that the probability (conditioned that it avoids η_n or $\tilde{\eta}_n$) that the rest of the path avoids C_n is greater than ρ. We consider the set of paths with this property, and we can now couple $\omega, \tilde{\omega}$ with positive probability such that on this event, the distribution of the remainder of the path is random walk conditioned to avoid C_n. We will write ω^* for the future in the coupled walks. This is a walk starting on ∂C_{n+1} within distance $e^{n+1}/20$ of $\mathbf{e}_{n+1}$. We write σ_r for the first visit of ω^* to ∂C_{n+r}.

Consider the event that all the following holds.

- the walk reaches ∂C_{n+j+1} without leaving $\{\mathbf{x} \in \mathbb{Z}^d \setminus C_{n+1} : x_1 \geq |\mathbf{x}|/10\}\}$
- After this time it never returns to C_{n+j}.
- there is a cut time for $\omega^*[\sigma_0, \sigma_2]$ that occurs between time $\sigma_{4/3}$ and $\sigma_{5/3}$.
- $\omega^*[\sigma_{4/3}, \infty)$ never visits ∂C_{n+1}
- $\omega^*[\sigma_2, \infty)$ never visits $\partial C_{n+\frac{5}{3}}$

Under this event, the cut time is also a cut time for the entire path (with either η_n or $\tilde{\eta}_n$ as initial condition). Hence the loop erasure is the same after that point, and, in particular, the loop erasure after the the first visit of the loop erasure to ∂C_{n+2} is the same. All we need is that the probability of this event is bigger than some $\epsilon_j > 0$ and this is easy to verify. (We could get a lower bound for the probability in terms of j but we will not need it.) □

We can now describe the coupling. Let r be sufficiently large so that $c' \sum_{k \geq r-2} e^{-k} \leq 1/2$. Let

$$q_i = \sum_{k=ir}^{(i+1)r-1} c'\, e^{-k},$$

and note that

$$q_i \leq c'' e^{-ir}, \qquad \sum_{i=1}^{\infty} q_i \leq \frac{1}{2}.$$

We start with $(\eta_n, \tilde{\eta}_n)$ and we will recursively define $\gamma_m = \eta_{n+mr}, \tilde{\gamma}_m = \tilde{\eta}_{n+mr}$ for $m = 0, 1, \ldots$ and a nonnegative integer valued random variable K_m. At each stage we will have

- $\gamma_{m-1} \prec \gamma_m, \qquad \tilde{\gamma}_{m-1} \prec \tilde{\gamma}_m$,
- γ_m ($\tilde{\gamma}_m$) has the distribution of a LERW conditioned to start with η_n (resp., $\tilde{\eta}_n$) stopped at its first visit to ∂C_{n+mr}.
- If $K_m = k$, then $\gamma_m =_{kr-2} \tilde{\gamma}_m$.

Using the lemmas above, we can couple so that the following is true given $\gamma_m, \tilde{\gamma}_m$.

- If $K_m = k$, we can define $\gamma_{m+1}, \tilde{\gamma}_{m+1}$ such that, except for an event of probability at most q_k, $\gamma_{m+1} =_{r(k+1)-2} \tilde{\gamma}_{m+1}$. If the last equality holds, we set $K_{m+1} = k+1$. Otherwise, we set $K_m = 0$.
- If $K_m = 0$, then we can define $\gamma_m, \tilde{\gamma}_m$ so that with probability at least δ, $\gamma_m =_{r-2} \tilde{\gamma}_m$. On the event that this happens, we set $K_{m+1} = 1$. Otherwise, we set $K_{m+1} = 0$.

Let $T = \sup\{m : K_m = 0\}$. Note that $J \leq rT$, Then these assumptions imply (see following lemma) that there exists $u = r\beta > 0$ with $\mathbf{E}[e^{uJ}] \leq \mathbf{E}[e^{\beta T}] < \infty$.

Lemma 2.26. *Suppose $X_0 = 0, X_1, X_2, \ldots$ is a sequence of nonnegative integer random variables adapted to a filtration $\{\mathcal{F}_n\}$ such that for each n, $X_{n+1} = X_n + 1$ or $X_{n+1} = 0$. Suppose there exist $0 < \delta, c, \alpha < \infty$ such that for all n, j*

$$\mathbf{P}\{X_m > 0 \text{ for all } m > n \mid \mathcal{F}_n\} \geq \delta,$$
$$\mathbf{P}\{X_{n+1} = 0 \mid \mathcal{F}_n\} \leq c\, e^{-\alpha X_n}.$$

Let $T = \max\{n : X_n = 0\}$. Then there exists $\beta = \beta(\delta, c, \alpha) > 0$ such that $\mathbf{E}[e^{\beta T}] < \infty$.

Proof. Let $\sigma_0 = 0$ and $\sigma_k = \min\{n < \sigma_{k-1} : X_n = 0\}$. Then $\mathbf{P}\{\sigma_1 < \infty\} \leq 1 - \delta$ and by iterating $\mathbf{P}\{\sigma_k < \infty\} \leq (1-\delta)^k$. Hence $\mathbf{P}\{T < \infty\} = 1$ and we can write

$$\mathbf{E}[e^{\beta T}] = \sum_{k=0}^{\infty} \mathbf{E}\left[e^{\beta \sigma_k}; \sigma_k < \infty, \sigma_{k+1} = \infty\right] \leq \sum_{k=0}^{\infty} \mathbf{E}\left[e^{\beta \sigma_k}; \sigma_k < \infty\right].$$

Since $\mathbf{P}\{\sigma_1 = n\} \leq \mathbf{P}\{\sigma_1 = n \mid \sigma_1 n - 1\} \leq c\, e^{-\alpha(n-1)}$, for β sufficiently small,

$$\mathbf{E}\left[e^{\beta\, \sigma_1}; \sigma_1 < \infty\right] \leq 1 - \frac{\delta}{2}.$$

By iterating this, we see for $k \geq 1$, $\mathbf{E}\left[e^{\beta \sigma_k}; \sigma_k < \infty\right] \leq (1 - \frac{\delta}{2})^k$. □

2.7 Pairs of walks

If $n < m-1$, and $\eta \in \mathcal{W}_m$, we define $\eta_n, \eta^*, \eta_{n+1,m}$ by the decomposition (2.2),

$$\eta = \eta_n \oplus \eta^* \oplus \eta_{n+1,m},$$

where $\eta_n \in \mathcal{W}_n, \eta_{n+1,m} \in \mathcal{W}_{n+1,m}$ and η^* is the middle. If $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{W}_m^2$, we similarly write

$$\boldsymbol{\eta} = \boldsymbol{\eta}_n \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta}_{n+1,m}$$

where the decomposition is done separately on η^1, η^2. Proposition 2.22 implies that if $m \geq n+2$,

$$\sum_{\boldsymbol{\eta} = \boldsymbol{\eta}_n \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta}_{n+1,m}} \mu_m(\boldsymbol{\eta}) \asymp \mu_n(\boldsymbol{\eta}_n)\, \mu_{n+1,m}(\boldsymbol{\eta}_{n+1,m}), \tag{2.17}$$

- Recall that Q_n is defined on $\mathcal{W}_n^2$ by

  $$Q_n(\boldsymbol{\eta}) = 1\{\boldsymbol{\eta} \in \mathcal{A}_n\}\, e^{-L_n(\boldsymbol{\eta})} = e^{-L_n(\boldsymbol{\eta})},$$

  where $L_n(\boldsymbol{\eta})$ is the loop measure of loops in $\hat{C}_n$ that intersect both η^1 and η^2. If $d=2$, we only consider loops that do not disconnect 0 from ∂C_n. By definition, $L_n(\boldsymbol{\eta}) = -\infty$ if $\eta^1 \cap \eta^2 \neq \{0\}$. If $n \leq m-1$, we also view Q_n as defined on $\mathcal{W}_m^2$ by

  $$Q_n(\boldsymbol{\eta}_n \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta}_{n+1,m}) = Q_n(\boldsymbol{\eta}_n).$$

- We define $\bar{Q}_{n+1,m}$ on $\mathcal{W}_m^2$ by

  $$\bar{Q}_{n+1,m}(\boldsymbol{\eta}_n \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta}_{n+1,m}) = e^{-L_m(\boldsymbol{\eta}_{n+1,m})},$$

  where

  - $L_m(\boldsymbol{\eta}_{n+1,m}) = -\infty$ if $\eta^1_{n+1,m} \cap \eta^2_{n+1,m} \neq \emptyset$,

  and

  - $(d \geq 3)$ $L_m(\boldsymbol{\eta}_{n+1,m})$ is the loop measure of loops in $\hat{C}_m$ that intersect both $\eta^1_{n+1,m}$ and $\eta^2_{n+1,m}$,
  - $(d = 2)$ $L_m(\boldsymbol{\eta}_{n+1,m})$ is the loop measure of loops in $\hat{C}_m$ that intersect both $\eta^1_{n+1,m}$ and $\eta^2_{n+1,m}$ and do not disconnect 0 from ∂C_m.

- If $\boldsymbol{\eta}_n \prec \boldsymbol{\eta}$, we define

  $$\lambda_m(\boldsymbol{\eta} \mid \boldsymbol{\eta}_n) = \frac{\lambda_m(\boldsymbol{\eta})}{\lambda_n(\boldsymbol{\eta}_n)} = \frac{Q_m(\boldsymbol{\eta})\, \mu_m(\boldsymbol{\eta})}{Q_n(\boldsymbol{\eta}_n)\, \mu_n(\boldsymbol{\eta}_n)} \leq \frac{\mu_m(\boldsymbol{\eta})}{\mu_n(\boldsymbol{\eta}_n)} = \mu_m(\boldsymbol{\eta} \mid \boldsymbol{\eta}_n),$$

  so that

  $$\lambda_m(\boldsymbol{\eta}) = \lambda_n(\boldsymbol{\eta}_n)\, \lambda_m(\boldsymbol{\eta} \mid \boldsymbol{\eta}_n).$$

  If $n < m$ and $\boldsymbol{\eta} \in \mathcal{A}_n$, let

  $$\lambda_m(\boldsymbol{\eta}) = \sum_{\boldsymbol{\eta} \prec \boldsymbol{\eta}' \in \mathcal{A}_m} \lambda_m(\boldsymbol{\eta}') = \lambda_n(\boldsymbol{\eta}) \sum_{\boldsymbol{\eta} \prec \boldsymbol{\eta}' \in \mathcal{A}_m} \lambda_m(\boldsymbol{\eta}' \mid \boldsymbol{\eta}).$$

- Let $\mathcal{K}_{n,m}$ be the set of $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{A}_m$ with

  $$\boldsymbol{\eta}^* \subset C_{n+2} \setminus C_{n-1},$$
  $$\operatorname{dist}\left[\eta^{j,*}, \eta^{3-j}\right] \geq e^{n-3}, \quad j = 1, 2.$$

Proposition 2.27. *There exist* $0 < c_1 < c_2 < \infty$ *such that if* $n < m - 1$ *and* $\boldsymbol{\eta} \in \mathcal{A}_n$*,*

$$c_1 \, 1\{\boldsymbol{\eta} \in \mathcal{K}_{n,m}\} \, Q_n(\boldsymbol{\eta}) \, \bar{Q}_{n+1,m}(\boldsymbol{\eta}) \ \leq Q_m(\boldsymbol{\eta}) \leq c_2 \, Q_n(\boldsymbol{\eta}) \, \bar{Q}_{n+1,m}(\boldsymbol{\eta}). \tag{2.18}$$

Proof. For the upper bound we note that

$$\frac{Q_m(\boldsymbol{\eta})}{Q_n(\boldsymbol{\eta}) \, \bar{Q}_{n+1,m}(\boldsymbol{\eta})} \leq \exp\{L'\},$$

where L' is the measure of loops ℓ that intersect both η^1_n and η^2_n and also intersect both $\eta^1_{n+1,m}$ and $\eta^2_{n+1,m}$. For $d = 2$, it is also required that the loops not disconnect 0 from ∂C_m. In particular, such loops must intersect both C_n and ∂C_{n+1}. If $d \geq 3$, Lemma 2.6 tells us that the measure of such loops is uniformly bounded. Similarly, for $d = 2$, Lemma 2.10 tells us that the measure of such nondisconnecting loops is bounded.

For the lower bound, note that if $\boldsymbol{\eta} \in \mathcal{K}_{n,m}$, then

$$\frac{Q_m(\boldsymbol{\eta})}{Q_n(\boldsymbol{\eta}) \, \bar{Q}_{n+1,m}(\boldsymbol{\eta})} \geq \exp\{-L''\},$$

where L'' is the measure of loops ℓ that intersect $C_{n+2} \setminus C_{n-1}$ and are of diameter at least e^{n-3}; for $d = 2$, we also require the loops to be nondisconnecting. Again, Lemmas 2.6 and 2.10 give uniform upper bounds for L''. □

One of the most important tools in understanding λ_n is the separation lemma. This says the (almost obvious) fact that if two paths are conditioned to avoid each other then their endpoints tend to be far apart. There are many versions that can be used. We will define a particular separation event. The choice of $1/10$ is arbitrary but it is convenient to choose a fixed small number. We give the definition for $d = 3$; if $d = 2$, replace (x_1, x_2, x_3) with (x_1, x_2).

Definition 2.28.

- *If* $\eta \in \mathcal{W}_n$*, let* $I_n(\eta)$ *be the indicator function of the event*

$$\eta \cap (C_n \setminus C_{n-(1/10)}) \subset \{x = (x_1, x_2, x_3) : x_1 \geq e^{-1} \, |x|\}.$$

- *Let* Sep_n *be the set of* $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{A}_n$ *such that* $I_n(\eta^1) = 1$ *and* $I_n(-\eta^2) = 1$*.*
- *If* $n < m$ *and* $\eta' \in \mathcal{W}_{n,m}$*, let* $I_{n,m}(\eta')$ *be the indicator function of the event*

$$\eta' \cap (C_{n+(1/10)} \setminus C_n) \subset \{(x_1, x_2, x_3) : x_1 \geq e^{n-1}\}.$$

- *If* $n < m$*, let* $\mathit{Sep}_{n,m}$ *be the set of* $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \bar{\mathcal{W}}^2_{n,m}$ *such that* $I_{n,m}(\eta^1) = 1$ *and* $I_{n,m}(-\eta^2) = -1$*.*

We will have two separation lemmas. The first is stronger and deals with the endpoint of the beginning of the path. The second is not as strong (we could prove the stronger result but do not need it) and deals with the initial part of the final piece of the path. Various version of the separation lemma can be found in [16] and [14] in the two-dimensional case and [17] in the three-dimensional case. For completeness we include a proof of one version in the appendix.

Lemma 2.29 (Separation Lemma I)**.** *There exists* $c > 0$ *such that if* $2 \leq n \leq m - 1$*,* $\boldsymbol{\eta} \in \mathcal{W}_n$*, and*

$$\lambda^{\mathit{Sep}}_m(\boldsymbol{\eta}) = \lambda_n(\boldsymbol{\eta}) \sum_{\boldsymbol{\eta}' \in \mathit{Sep}_m, \boldsymbol{\eta} \prec \boldsymbol{\eta}'} \lambda_m(\boldsymbol{\eta}' \mid \boldsymbol{\eta}).$$

Then $\lambda^{\mathit{Sep}}_m(\boldsymbol{\eta}) \geq c \, \lambda_m(\boldsymbol{\eta})$*.*

Lemma 2.30 (Separation Lemma II). *There exists $c > 0$ such that if $n < m$, then*

$$\sum_{\boldsymbol{\eta}=\tilde{\boldsymbol{\eta}}\oplus\boldsymbol{\eta}'\in\mathcal{A}_m,\boldsymbol{\eta}'\in \mathit{Sep}_{n,m}} \mu(\boldsymbol{\eta})\,\bar{Q}_{n,m}(\boldsymbol{\eta}) \geq c \sum_{\boldsymbol{\eta}\in\mathcal{A}_m} \mu(\boldsymbol{\eta})\,\bar{Q}_{n,m}(\boldsymbol{\eta}).$$

We will now consider some easy consequences.

Proposition 2.31. *There exist constants $0 < c_1 < c_2 < \infty$ such that the following holds.*

1. *If $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathit{Sep}_n$, then $\lambda_{n+1}(\boldsymbol{\eta}) \geq c_1\,\lambda_n(\boldsymbol{\eta})$,*
2. *For every $n \geq 0$,*
$$c_1\,\lambda_n \leq \lambda_{n+1} \leq \lambda_n.$$
3. *If $n < m - 1$,*
$$\lambda_{n+1,m} \asymp \frac{\lambda_m}{\lambda_n}.$$
and, more generally, $\lambda_m(\boldsymbol{\eta}) \geq c_1\,\lambda_m/\lambda_n$ for $\boldsymbol{\eta} \in \mathit{Sep}_n$.
4. *If $\boldsymbol{\eta} \in \mathcal{A}_n$, $\lambda_m(\boldsymbol{\eta}) \leq c_2\,\lambda_n(\boldsymbol{\eta})\,(\lambda_m/\lambda_n)$.*

Proof.

1. We use Corollary 2.19 to see that with positive μ_{n+1} probability the extension of $\boldsymbol{\eta}$ will still be separated.
2. This follows from part 1 and the Separation Lemma I.
3. Here we use both Separation Lemma I and Separation Lemma II.
4. This is done similarly. □

Separation lemmas are key tools for many problems. They can be considered generalizations of "boundary Harnack principles". We will not discuss this, but just say that the idea is that if you have a bounded domain, start a process very near the boundary, and condition the process to not leave the domain in, say, one unit of time, then the process will get away from the boundary very quickly. Although the probability of escaping the boundary is small, the probability of staying near the boundary without exiting is of a smaller order of magnitude.

The analogue for us of being near the boundary is to say that pair of walks are close to each other near their terminal points. Once the paths "separate" somewhat, then there is a reasonable chance that they will stay separated.

2.8 Coupling the pairs of walks

In this section we fix a (large) integer N. Our goal is to couple the probability measures $\lambda_N^{\#}(\boldsymbol{\eta})$ and $\lambda_N^{\#}(\boldsymbol{\eta}')$ for different starting configurations $\boldsymbol{\eta}, \boldsymbol{\eta}'$. We use a coupling strategy similar to that in Section 2.6 although here we will only go up to level N rather than to infinity. We start by giving some notation; in order to make it easier to read, we will leave N implicit.

- Let $b_n = \lambda_{n,N}$. If $n \leq N$ and $\boldsymbol{\eta} \in \mathcal{A}_n$, let
$$b(\boldsymbol{\eta}) = \frac{\lambda_N(\boldsymbol{\eta})}{\lambda_n(\boldsymbol{\eta})} = \sum_{\boldsymbol{\eta}'\in\mathcal{A}_N,\boldsymbol{\eta}\prec\boldsymbol{\eta}'} \lambda_N(\boldsymbol{\eta}' \mid \boldsymbol{\eta}).$$
If $\boldsymbol{\eta} \in \mathcal{A}_N$, then $b(\boldsymbol{\eta}) = 1$. Note that if $n < N - 1$,
$$b(\boldsymbol{\eta}) = \sum_{\boldsymbol{\eta}'\in\mathcal{A}_{n+1},\boldsymbol{\eta}\prec\boldsymbol{\eta}'} \lambda(\boldsymbol{\eta}' \mid \boldsymbol{\eta})\,b(\boldsymbol{\eta}'). \tag{2.19}$$

- If $\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} \in \mathcal{A}_n$ we write $\boldsymbol{\eta} =_j \tilde{\boldsymbol{\eta}}$ if the paths agree from the first visit to ∂C_{n-j} onwards. In other words, if we write

$$\boldsymbol{\eta} = \boldsymbol{\eta}_{n-j} \oplus \boldsymbol{\eta}', \quad \tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}_{n-j} \oplus \tilde{\boldsymbol{\eta}}',$$

  with $\boldsymbol{\eta}_{n-j}, \tilde{\boldsymbol{\eta}}_{n-j} \in \mathcal{A}_{n-j}$, then $\boldsymbol{\eta}' = \tilde{\boldsymbol{\eta}}'$.

We will consider Markov chains $\gamma_n, n = k, k+1, \ldots, N$ taking values in $\mathcal{A} := \bigcup_{i=1}^{\infty} \mathcal{A}_i$ with the properties:

- For all n, $\gamma_n \in \mathcal{A}_n$. If $j < n$, then $\gamma_j \prec \gamma_n$.
- The transitions are given by

$$p(\gamma_n, \gamma_{n+1}) = \lambda(\gamma_{n+1} \mid \gamma_n) \, \frac{b(\gamma_{n+1})}{b(\gamma_n)} \leq \mu(\gamma_{n+1} \mid \gamma_n) \, \frac{b(\gamma_{n+1})}{b(\gamma_n)} \tag{2.20}$$

This formula assumes that $b(\gamma_n) > 0$, but we start with this condition at time k and hence with probability one this will hold for all n. Note that (2.19) shows that this is a well-defined transition probability. We recall that $b(\gamma_{n+1}) \leq c_2 \, b_{n+1}$ and if $\gamma_{n+1} \in \mathsf{Sep}_{n+1}$, then $b(\gamma_{n+1}) \geq c_1 \, b_{n+1}$

We now describe the coupling which will be a construction of ordered pairs $\gamma_n^* = (\gamma_n, \tilde{\gamma}_n)$ such that $\gamma_n, \tilde{\gamma}_n$ both follow the Markov chain although with different initial distributions on $\gamma_n, \tilde{\gamma}_n$. More precisely, we define a coupling $X_n = (\gamma_n, \tilde{\gamma}_n, J_n)$ to be random variables defined on the same probability space such that the following hold.

- $(\gamma_k, \gamma_{k+1}, \ldots, \gamma_N)$ is a Markov chain satisfying the transition probabilities given by (2.20).
- Similarly, $(\tilde{\gamma}_k, \tilde{\gamma}_{k+1}, \ldots, \tilde{\gamma}_N)$ is a Markov chain satisfying (2.20) although the initial distribution may be different.
- J_n is a nonnegative integer random variable with the property that if $J_n = j$, then

$$\gamma_n =_j \tilde{\gamma}_n, \quad b(\gamma_n) \geq e^{-j/4} \, b_n.$$

- $J_k = 0$, and for every $k \leq n < N$, either $J_{n+1} = J_n + 1$ or $J_{n+1} = 0$.

We can now state the main result.

Proposition 2.32. *There exist $\alpha > 0, c < \infty$ such that for all k, n and all $N \geq 2n + k$, for any initial distributions on $\gamma_k, \tilde{\gamma}_k$ we can find a coupling such that*

$$\mathbf{P}\{J_N \leq n\} \leq c \, e^{-\alpha n}.$$

In particular, except for an event of probability $O(e^{-\alpha n})$, $\gamma_N =_n \tilde{\gamma}_N$.

We collect some of the lemmas from previous sections here. These are the facts that we will need in this subsection. We have already done most of the work in establishing these results so we will be brief in our proof.

Lemma 2.33. *There exist $0 < c_1 < c_2 < \infty$ and $\beta > 0$ such that the following holds.*

1. *For any γ_n,*

$$\sum_{\boldsymbol{\eta} \cap C_{n-(j/2)} \neq \emptyset} \lambda(\gamma_n \oplus \boldsymbol{\eta} \mid \gamma_n) \leq c \, e^{-j/2}.$$

2. *If $\gamma_n =_j \tilde{\gamma}_n$, and $\boldsymbol{\eta} \cap C_{n-(j/2)} = \emptyset$, then*

$$\lambda(\gamma_n \oplus \boldsymbol{\eta} \mid \gamma_n) = \lambda(\tilde{\gamma}_n \oplus \boldsymbol{\eta} \mid \tilde{\gamma}_n) \, [1 + O(e^{-j/2})],$$

3. *If $\gamma_n =_j \tilde{\gamma}_n$, then*

$$|b(\gamma_n) - b(\tilde{\gamma}_n)| \leq c_2\, e^{-j/2}\, b_n.$$

In particular, if $b(\gamma_n) \geq e^{-j/4}\, b_n$, then

$$b(\gamma_n) = b(\tilde{\gamma}_n)\,[1 + O(e^{-j/4})],$$
$$\sum_{\eta \cap C_{n-(j/2)} \neq \emptyset} p(\gamma_n, \gamma_n \oplus \eta) \leq c\, e^{-j/4},$$

and if also $b(\gamma_n \oplus \eta) \geq e^{-(j+1)/4}\, b_{n+1}$, then

$$p(\gamma_n, \gamma_n \oplus \eta) = p(\tilde{\gamma}_n, \tilde{\gamma} \oplus \eta)\,[1 + O(e^{-j/4})].$$

4. *If $n \leq N-1$, then given γ_n, the probability that $\gamma_{n+1} \in Sep_{n+1}$ is at least c_1. In other words, for every γ_n,*

$$\sum_{\eta \in Sep_{n+1}} p(\gamma_n, \gamma_n \oplus \eta) \geq c_1. \tag{2.21}$$

In particular,

$$\sum_{\eta \in Sep_{n+1}} p(\gamma_n, \gamma_n \oplus \eta)\, b(\eta) \geq c\, b_{n+1}.$$
$$\sum_{\eta, b_{n+1}(\gamma_n \oplus \eta) \leq e^{-(j+1)/4}} p(\gamma_n, \gamma_n \oplus \eta) \leq c\, e^{-j/4}.$$

Proof.

1. Write $\eta = \eta^* \oplus \tilde{\eta}'$ where $\tilde{\eta}' \in \mathcal{A}_{n+1,N}$. and use

$$\lambda(\gamma_n \oplus \eta' \oplus \tilde{\eta} \mid \gamma_n) \leq \mu(\gamma_n \oplus \eta' \oplus \tilde{\eta} \mid \gamma_n)\, Q_{n+1,N}(\tilde{\eta}).$$

Using Corollary 2.15 we can see that given γ_n and $\tilde{\eta}$, the μ-probability that $\eta^* \cap C_{n-(j/2)} \neq \emptyset$ is $O(e^{-j/2})$. We then use (2.18).

2. This follows from a combination of Lemma 2.6 and Corollary 2.12.

3. We write

$$b(\gamma_n) = \sum_{\gamma_n \oplus \eta \in \mathcal{A}_N} \lambda_N(\gamma_n \oplus \eta \mid \gamma),$$

and similarly for $b(\gamma_n')$ and use the first two parts.

4. This follows from the separation lemma. □

Lemma 2.34. *For every $j > 0$, there exists $\delta_j > 0$ such that the following holds. Suppose $k \leq N - (j+2)$ and initial conditions $\gamma_k, \tilde{\gamma}_k$ are given. Then we can couple $\gamma^* = (\gamma, \tilde{\gamma})$ on the same probability space such that with probability at least δ_j, we have $I_{k+j+2}(\gamma) = 1$ and $\gamma_{k+j+2} =_j \tilde{\gamma}_{k+j+2}$.*

Note that $I_{k+j+2}(\gamma) = 1, \gamma_{k+j+2} =_j \tilde{\gamma}_{k+j+2}$ imply that $I_{k+j+2}(\tilde{\gamma}) = 1$. It will not be important to give estimates for δ_j in terms of j; the coupling works as long as $\delta_j > 0$ although the exponent α does depend on the actual values. The proof of this is similar to Lemma 2.25 although we first separate the paths. We do this so that the measure of loops that intersect the extensions of the paths will be bounded.

Proof. We fix j. In this proof all constants (implicit or explicit) or phrases like "with positive probability" mean that there exist constants that can be chosen uniformly over all k and all $\boldsymbol{\gamma}_k, \tilde{\boldsymbol{\gamma}}_k$ (although they may depend on j).

By (2.21), we can see that there is a positive probability that $\boldsymbol{\gamma}_{k+1}, \tilde{\boldsymbol{\gamma}}_{k+1} \in \mathsf{Sep}_{k+1}$. Let

$$\mathcal{C}^1 = \{x = (x_1, x_2, x_3) : x_1 \geq e^{-1} |x|\}, \quad \mathcal{C}^2 = \{x = (x_1, x_2, x_3) : x_1 \leq -e^{-1} |x|\}.$$

Recall that $\mu(\boldsymbol{\gamma}_j \mid \boldsymbol{\gamma}_{k+1})$ is obtained by taking simple random walks starting at the endpoint z^i_{k+1} of each γ^i_{k+1} conditioned to avoid γ^i_{k+1} and then erasing loops. Let us consider the first time that these random walks reach $\partial C_{k+(6/5)}$. Using Lemma 2.11 we can see that with positive probability the random walk avoiding γ^1_{k+1} stays in

$$\mathcal{C}^1_* := \{x = (x_1, x_2, x_3) : x_1 \geq e^{-2} |x|\},$$

and its terminal point is in $\mathcal{C}^1$. The same is true for the walk avoiding γ^2_{k+1} staying in $\mathcal{C}^2_* = \{x = (x_1, x_2, x_3) : x_1 \leq -e^{-2} |x|\}$ with terminal point in $\mathcal{C}^2$. Using the Harnack principle, we can see that the distribution of the terminal point in $\mathcal{C}^i$ is comparable to the harmonic measure from 0 of $\partial C_{k+(6/5)}$ which in turn is comparable to $e^{k(1-d)}$.

For a simple random walk S starting on $\mathcal{C}^i \cap \partial C_{k+(6/5)}$, conditioned to avoid γ^i, there is a positive probability that the following holds:

- The walk never visits C_{k+1}.
- The intersection of the walk with C_{k+j+2} is contained in $\mathcal{C}^i_*$.
- The intersection of the walk with ∂C_{k+j+2} is contained in $\mathcal{C}^i$.
- There is a time t with $S_t \in C_{k+(8/5)} \setminus C_{k+(7/5)}$ that is a cut point for the random walk. (See [7] for existence of cut points for random walks in $\mathbb{Z}^2$ and $\mathbb{Z}^3$).
- The walk never returns to $C_{k+(9/5)}$ after reaching ∂C_{k+2} for the first time.

Using this we can see that we can couple the conditional simple random walks avoiding $(\gamma^1_{k+1}, \gamma^2_{k+1})$ with those avoiding $(\tilde{\gamma}^1_{k+1}, \tilde{\gamma}^2_{k+1})$ such that with positive probability the paths agree from their first visit to $\partial C_{k+(6/5)}$ onward and they lie in the event described above. In particular, the loop-erasure of the paths are the same from the first visit to ∂C_{k+2} onwards. (See Lemma 2.18 and Corollary 2.19.) Also, the paths are sufficiently separated, and hence using Lemma 2.6, we can see that

$$\lambda(\boldsymbol{\gamma}_{k+j+2} \mid \boldsymbol{\gamma}_k) \geq c\, \mu(\boldsymbol{\gamma}_{k+j+2} \mid \boldsymbol{\gamma}_k),$$

and similarly for $\tilde{\boldsymbol{\gamma}}$. $\square$

Lemma 2.35. *There exists c_0 such that the following is true. Suppose that $m + j \leq n \leq N - 1$ and $\boldsymbol{\gamma}_n, \tilde{\boldsymbol{\gamma}}_n \in \mathcal{A}_n$ with $\boldsymbol{\gamma}_n =_j \tilde{\boldsymbol{\gamma}}_n$ and $b(\boldsymbol{\gamma}_n) \geq e^{-j/4}\, b_n$. Then we can couple $(\boldsymbol{\gamma}_{n+1}, \tilde{\boldsymbol{\gamma}}_{n+1})$ on the same probability space such that, except perhaps on an event of probability at most $c_0\, e^{-j/4}$,*

$$\boldsymbol{\gamma}_{n+1} =_{j+1} \tilde{\boldsymbol{\gamma}}_{n+1},$$
$$(\boldsymbol{\gamma}_{n+1} \setminus \boldsymbol{\gamma}_n) \cap C_{n-(j/2)} = \emptyset,$$
$$b(\boldsymbol{\gamma}_{n+1}) \geq e^{-(j+1)/4}\, b_n.$$

Proof. We write $\boldsymbol{\gamma} = (\gamma^1, \gamma^2), \tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}^1, \gamma^2)$. In the measure μ, the conditional distribution of the remainder of the paths is obtained by taking simple random walks starting at the terminal point conditioned to avoid the past and then erasing loops. Given the results so far, we can couple random walks conditioned to avoid (γ^1, γ^2) and $(\tilde{\gamma}^1, \tilde{\gamma}^2)$, except for an event of probability $O(e^{-j/2})$ they agree and stay in $\mathbb{Z}^d \setminus C_{n-(j/2)}$. This will also be true of their loop erasures.

We claim that, except perhaps on this exceptional set,

$$\frac{Q_{n+1}(\gamma \oplus \gamma')}{Q_n(\gamma)} = \frac{Q_{n+1}(\tilde{\gamma} \oplus \tilde{\gamma}')}{Q_n(\tilde{\gamma})} [1 + O(e^{-j/2})].$$

To see this we first see that any loop ℓ in C_{n+1} that intersects both $\gamma^1 \oplus (\gamma')^1$ and $\gamma^2 \oplus (\gamma')^2$, but does not intersect both γ^1 and γ^2 must intersect $\mathbb{Z}^3 \setminus C_{n-(j/2)}$. If the loop does not also intersect C_{n-j}, then this happens if and only if ℓ intersects both $\tilde{\gamma}^1 \oplus (\tilde{\gamma}')^1$ and $\tilde{\gamma}^2 \oplus (\tilde{\gamma}')^2$, but does not intersect both $\tilde{\gamma}^1$ and $\tilde{\gamma}^2$. The measure of the set of loops that intersect C_{n-j} and $\mathbb{Z}^3 \setminus C_{n-(j/2)}$ is $O(e^{-j/2})$. If $d = 2$, we only consider nondisconnecting loops. □

Proof of Proposition 2.32. Let $J_k = 0$. Let c_0 be as in the previous lemma, and choose $r \geq 3$ sufficiently large so that

$$\sum_{j=r-2}^{\infty} c_0 \, e^{-j/4} \leq \frac{1}{2}.$$

Choose ϵ so that for any $(\boldsymbol{\gamma}_i, \tilde{\boldsymbol{\gamma}}_i)$ with $i < N - r$, we can find a coupling $(\boldsymbol{\gamma}_{i+r}, \tilde{\boldsymbol{\gamma}}_{i+r})$ such that with probability at least 2ϵ we have $I(\boldsymbol{\gamma}_{i+r}) = 1$, $\boldsymbol{\gamma}_{i+r} =_{r-2} \tilde{\boldsymbol{\gamma}}_{i+r}$, and $b_{i+r}(\boldsymbol{\gamma}_{i+r}) \geq e^{-(i+r)/4}$. If the paths are coupled satisfying this we set $J_{k+1} = J_{k+2} = 0$ and $J_{k+2+j} = j, j = 1, 2, \ldots, r$; otherwise, we set $J_{k+j} = 0, j = 1, \ldots, r$.

Recursively, if we have seen $(\gamma_{k+jr}, \tilde{\gamma}_{k+jr}, J_{k+jr})$. We do the following.

- If $J_{k+jr} = 0$, we try to couple as above. If we succeed, then we set $J_{k+jr+1} = J_{k+jr+2} = 0$ and $J_{k+jr+s} = s - 2$ for $s = 3, 4, \ldots, r$. Note that for any $\boldsymbol{\gamma}_{k+jr}$, the probability that we will be able to couple is at least 2ϵ.
- If $J_{k+jr} = ir - 2 \geq r - 2$, we couple as in the last lemma for r consecutive levels. This will succeed except for an event of probability at most

$$c_0 \sum_{t=ir-2}^{(i+1)r-3} e^{-t/4}.$$

If we succeed, we set $J_{m+1} = J_m + 1$ for $m = k + jr, \ldots, k + (j+1)r - 1$; otherwise, we set $J_m = 0, m = k + jr + 1, \ldots, k + (j+1)r$.

We now finish as in the proof of Lemma 2.26. Assume $N \geq k + 2n$, and let $\sigma = \min\{j \geq 1 : J_{k+jr} = 0\}$. From the estimates above we see that $\mathbf{P}\{\sigma = \infty\} \geq \epsilon$ and

$$\mathbf{P}\{\sigma = j \mid \sigma < \infty\} \leq c \, e^{-rj/4}.$$

In particular, we can find $\alpha > 0$ such that

$$\mathbf{E}[e^{\alpha r \sigma} \mid \sigma < \infty] \leq 1 + \epsilon.$$

More generally if $\sigma_l = \min\{j > \sigma_{l-1} : J_{n+jr} = 0\}$, then

$$\mathbf{P}\{\sigma_l < \infty\} \leq (1 - \epsilon)^l,$$

and

$$\mathbf{E}[e^{\alpha r \sigma_l}; \sigma_l < \infty] \leq \mathbf{P}\{\sigma_l < \infty\} \, \mathbf{E}[e^{\alpha r \sigma_l} \mid \sigma_l < \infty] \leq (1 - \epsilon)^l \, (1 + \epsilon)^l = (1 - \epsilon^2)^l.$$

In particular, if $\bar\sigma = \max\{j : J_{k+jr} = 0\}$, then

$$\mathbf{E}[e^{\alpha r \bar\sigma}] \le \sum_{l=0}^{\infty} \mathbf{E}[e^{\alpha r \sigma_l}; \sigma_l < \infty] < \infty,$$

and hence

$$\mathbf{P}\{r\bar\sigma \ge n\} \le e^{-\alpha n} \mathbf{E}[e^{\alpha r \bar\sigma}] \le c\, e^{-\alpha n}. \qquad \square$$

The estimate (2.3) follows immediately since

$$\frac{\lambda_{n+1}}{\lambda_n} = \sum_{\gamma \in \mathcal{A}_n, \tilde{\gamma} \in \mathcal{A}_{n+1}, \gamma \prec \tilde{\gamma}} \lambda_n^{\#}(\gamma)\, \lambda_{n+1}(\tilde{\gamma} \mid \gamma),$$

$$\frac{\lambda_{n+1}[\boldsymbol{\eta}]}{\lambda_n[\boldsymbol{\eta}]} = \sum_{\gamma \in \mathcal{A}_n, \tilde{\gamma} \in \mathcal{A}_{n+1}, \gamma \prec \tilde{\gamma}} \tilde{\lambda}_n^{\#}(\gamma)\, \lambda_{n+1}(\tilde{\gamma} \mid \gamma),$$

where $\tilde{\lambda}_n^{\#}$ denote the normalized probability measure given $\boldsymbol{\eta}$. The sums on the right-hand side are greater than some absolute $c_1 > 0$ by the separation lemma. Also, $\lambda_{n+1}(\tilde{\gamma} \mid \gamma) \le \mu_{n+1}(\tilde{\gamma} \mid \gamma)$, so the sum over any set of paths $\boldsymbol{\gamma}$ of probability $O(e^{-\alpha j})$ is bounded by $O(e^{-\alpha j})$. Finally, as we have seen, if $\gamma =_j \gamma'$, then

$$\left| \sum_{\tilde{\gamma} \in \mathcal{A}_{n+1}, \gamma \prec \tilde{\gamma}} \lambda_{n+1}(\tilde{\gamma} \mid \gamma) - \sum_{\tilde{\gamma} \in \mathcal{A}_{n+1}, \gamma' \prec \tilde{\gamma}} \lambda_{n+1}(\tilde{\gamma} \mid \gamma') \right| \le c\, e^{-\alpha j}.$$

3 Proof of (1.4)

Suppose A is a simply connected subset of $\mathbb{Z}^d$ containing the origin and x, y are distinct points in ∂A, and, as before, $\hat{A} = A \setminus \{0\}$. Let $\mathcal{A}(A; x, y)$ denote the set of SAWs starting at x, ending at y, otherwise staying in A, and going through the origin. To avoid trivial cases, we assume that $\mathcal{A}(A; x, y)$ is non-empty. As in the case of $\mathcal{A}_n$, we can also view elements of $\mathcal{A}(A; x, y)$ as ordered pairs $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{W}_A^x \times \mathcal{W}_A^y$ with $\eta^1 \cap \eta^2 = \{0\}$. Here $\mathcal{W}_A^x$ denotes the set of SAWs starting at the origin, ending at x, and otherwise staying in A.

The loop-erased measure on a single path $\mathcal{W}_A^x$ is given by

$$\mu_{A,x}(\eta) = (2d)^{-|\eta|}\, F_\eta(\hat{A}).$$

This is the measure obtained by starting a simple random walk at the origin, stopping the path at the first visit to ∂A, restricting to the event that the terminal vertex is x and that there were no previous returns to the origin, and then erasing loops. It has total mass $H_{\partial \hat{A}}(0, x)$. (If we did not want to restrict to walks that do not return to 0 before leaving A, we would get the same expression with an extra factor of $G_A(0,0)$; this does not affect the probability distribution.)

If $A \supset C_{n+1}$ we can also view $\mu_{A,x}$ as a measure on $\mathcal{W}_n$ by considering the path stopped at its first visit to ∂C_n. The measure $\mu_{A,x}$ is mutually absolutely continuous with respect to μ_n on $\mathcal{W}_n$ (with constants uniform over all n, x, A and paths $\eta \in \mathcal{W}_n$). This would not be true for the measure on $\mathcal{W}_n$ obtained by stopping a simple random walk at ∂C_n and erasing loops.

We define the loop-erased measure $\lambda_{A,x,y}$ on $\mathcal{A}(A;x,y)$ to be absolutely continuous with respect to $\mu_{A,x} \times \mu_{A,y}$ on $\mathcal{W}_n^x \times \mathcal{W}_n^y$ with Radon-Nikodym derivative

$$Y(\boldsymbol{\eta}) = Y_{A,x,y}(\boldsymbol{\eta}) = \exp\left\{-L_A(\boldsymbol{\eta})\right\},$$

where $L_A(\boldsymbol{\eta}) = -\infty$ if $\boldsymbol{\eta} \notin \mathcal{A}(A;x,y)$, and

- ($d \geq 3$) $\log L_A(\boldsymbol{\eta})$ is the measure of the set of loops in $\hat{A}$ that intersect both η^1 and η^2.
- ($d = 2$) $\log L_A(\boldsymbol{\eta})$ is the measure of the set of loops in $\hat{A}$ that intersect both η^1 and η^2 and do not disconnect 0 from ∂A.

If we view the ordered pair $\boldsymbol{\eta}$ as a single SAW η from x to y, then this is the same as the weight

$$(2d)^{-|\eta|}\, F_\eta(\hat{A})\, \kappa_A, \tag{3.1}$$

where $\kappa_A = 1$ for $d \geq 3$ and if $d = 2$, $\log \kappa_A$ is the measure of loops in $\hat{A}$ that disconnect 0 from ∂A (all of which intersect both η^1 and η^2). We write $\lambda^{\#}_{A,x,y}$ for the corresponding probability measure. For $d = 2$, the probability measure is the same if we normalized the measure in (3.1).

Another way to describe the probability measure $\lambda^{\#}_{A,x,y}$ is as follows.

- Let ω^1, ω^2 be independent conditioned random walks (h-process) where the walk starts at x, y, respectively, stops when it reaches the origin, and is conditioned to reach the origin before returning to ∂A.
- Erase the loops from each walk separately and reverse the paths to get $\boldsymbol{\eta} = (\eta^1, \eta^2)$. We can write this path as the pair $\boldsymbol{\eta}$ or the single path $\eta = (\eta^1)^R \oplus \eta^2$.
- Tilt the measure by $\exp\{-L_A(\boldsymbol{\eta})\}/\rho_{A,x,y}$ where

$$\rho_{A,x,y} = \mathbf{E}\left[\exp\{-L_A(\boldsymbol{\eta})\}\right].$$

It follows from the definition that the probability measure $\lambda^{\#}_{A,x,y}$ satisfies the following "two-sided domain Markov property".

- Suppose η^1, η^2 are disjoint SAWs starting at x, y, respectively, and otherwise staying in $\hat{A}$, with terminal vertices x', y', respectively. Then in the measure $\lambda^{\#}_{A,x,y}$, conditioned that the SAW has the form

$$\eta = \eta^1 \oplus \eta^* \oplus \eta_2^R, \tag{3.2}$$

  the distribution of η^* is $\lambda^{\#}_{A\setminus(\eta^1\cup\eta^2),x',y'}$.

Let us describe our strategy. We will assume that A is a simply connected subset with $C_{n+1} \setminus A$ consisting of two points $x, y \in \partial_i C_{n+1}$. If we prove our main result in this case, it will hold more generally for $A \supset C_{n+1}$ using the two-sided domain Markov property by letting η^1, η^2 be the parts of the SAW stopped at the first visit to $\partial_i C_{n+1}$. Let us fix $k < n$, A a set as above with corresponding x, y, $\boldsymbol{\eta} \in \mathcal{A}_k$, and to ease notation we will leave some dependence on these parameters implicit. However, all constants, including implicit constants in $\asymp$ and $O(\cdot)$ notation will be uniform over all choices. Let $\mathcal{A}_A = \mathcal{A}(A;x,y)$, $\lambda_A = \lambda_{A,x,y}$, $\lambda = \lambda_n$, $\tilde{\lambda}$ is λ restricted to $\boldsymbol{\gamma}$ with $\boldsymbol{\eta} \prec \boldsymbol{\gamma}$, and $\lambda^{\#}$, $\tilde{\lambda}^{\#}$ the corresponding probability measures obtained by normalization. Let $\lambda^{\#}_A$ denote the probability measure on $\mathcal{A}_n$ obtained from λ_A by normalization and then truncating the paths so they are in $\mathcal{A}_n$. Proposition 2.32 states that we can define $(\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}})$ on the same probability space such that the marginal distribution of $\boldsymbol{\gamma}$ is $\lambda^{\#}$, the marginal distribution of $\tilde{\boldsymbol{\gamma}}$ is $\tilde{\lambda}^{\#}$, and, except perhaps on an event of probability $O(e^{-\alpha j})$, we have $\tilde{\boldsymbol{\gamma}} =_j \boldsymbol{\gamma}$ where $j = (n-k)/2$. Here α is an unknown positive constant which we may assume is less than $1/4$. We let

$$Z_A(\boldsymbol{\gamma}) = \frac{\lambda_A^\#(\boldsymbol{\gamma})}{\lambda^\#(\boldsymbol{\gamma})}, \quad \boldsymbol{\gamma} \in \mathcal{A}_n.$$

The main work will be to establish the following proposition.

Proposition 3.1. *There exist $c_2 < \infty$ such that if A, x, y are as above, then for all $\boldsymbol{\gamma} \in \mathcal{A}_n$,*

$$Z_A(\boldsymbol{\gamma}) \le c_2.$$

Moreover, if $\boldsymbol{\gamma} =_j \tilde{\boldsymbol{\gamma}}$,

$$|Z_A(\boldsymbol{\gamma}) - Z_A(\tilde{\boldsymbol{\gamma}})| \le c_2\, e^{-j/4}.$$

If $\boldsymbol{\gamma} \in \mathsf{Sep}_n$, then $Z_A(\boldsymbol{\gamma}) \asymp 1$. However, if the tips of γ^1, γ^2 are close, it is possible for $Z_A(\boldsymbol{\gamma})$ to be small.

The theorem follows almost immediately from the proposition as we now show. Let q denote a probability measure on $\mathcal{A}_n \times \mathcal{A}_n$ such that the marginal distributions are $\tilde{\lambda}^\#, \lambda^\#$, respectively and such that

$$q\{(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\gamma}) : \tilde{\boldsymbol{\gamma}} =_j \boldsymbol{\gamma}\} \ge 1 - c\, e^{-j\alpha}.$$

Note that $\tilde{\lambda}^\#$ is the same as $[\lambda^\#[\mathcal{A}_n(\boldsymbol{\eta})]]^{-1}\, \lambda^\#$, restricted to $\mathcal{A}_n(\boldsymbol{\eta})$, and hence if $Z = Z_A$,

$$\sum_{\boldsymbol{\eta} \prec \boldsymbol{\gamma}} \lambda_A^\#(\boldsymbol{\gamma}) = \sum_{\boldsymbol{\eta} \prec \boldsymbol{\gamma}} \lambda^\#(\boldsymbol{\gamma})\, Z(\boldsymbol{\gamma}) = \lambda^\#[\mathcal{A}_n(\boldsymbol{\eta})] \sum_{\boldsymbol{\gamma}} \tilde{\lambda}^\#(\boldsymbol{\gamma})\, Z(\boldsymbol{\gamma}).$$

Also, given Proposition 3.1,

$$\begin{aligned}
\sum_{\tilde{\boldsymbol{\gamma}}} \tilde{\lambda}^\#(\tilde{\boldsymbol{\gamma}})\, Z(\tilde{\boldsymbol{\gamma}}) &= \sum_{(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\gamma})} q(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\gamma})\, Z(\tilde{\boldsymbol{\gamma}}) \\
&= O(e^{-j\alpha}) + \sum_{(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\gamma})} q(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\gamma})\, Z(\boldsymbol{\gamma}) \\
&= O(e^{-j\alpha}) + \sum_{\boldsymbol{\gamma}} \lambda^\#(\boldsymbol{\gamma})\, Z(\boldsymbol{\gamma}) \\
&= O(e^{-j\alpha}) + \sum_{\boldsymbol{\gamma}} \lambda_A^\#(\boldsymbol{\gamma}) = 1 + O(e^{-j\alpha}).
\end{aligned}$$

Therefore,

$$\sum_{\boldsymbol{\eta} \prec \boldsymbol{\gamma}} \lambda_A^\#(\boldsymbol{\gamma}) = \lambda^\#[\mathcal{A}_n(\boldsymbol{\eta})]\, \left[1 + O(e^{-j\alpha})\right].$$

Proof of Proposition 3.1. We will first prove the result in the case where x and y are separated, say $|x - y| \ge e^{n-5}$. We write each $\hat{\gamma}^1 \in \mathcal{W}_A^x$ as

$$\hat{\gamma}^1 = \gamma^1 \oplus \eta^1,$$

where $\gamma^1 \in \mathcal{W}_n$. We do similarly for $\hat{\gamma}^2 \in \mathcal{W}_A^y$ and we also write

$$\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \oplus \boldsymbol{\eta}.$$

We write

$$L_A(\hat{\boldsymbol\gamma}) = L_n(\boldsymbol\gamma) + \tilde L(\hat{\boldsymbol\gamma}),$$

where $\tilde L(\hat{\boldsymbol\gamma}) = L_A(\hat{\boldsymbol\gamma}) - L_n(\boldsymbol\gamma)$. If $d = 2$, we restrict to nondisconnecting loops. We will also write $L_n(\boldsymbol\gamma) = -\infty$ if $\gamma^1 \cap \gamma^2 \neq \{0\}$, and $\tilde L(\hat{\boldsymbol\gamma}) = -\infty$ if $\hat\gamma^1 \cap \hat\gamma^2 \neq \{0\}$.

We write $\mu^\#$ for the measure on loop-erased walks obtained by taking an infinite loop-erased walk and stopping it at the first visit to ∂C_n. We write $\mu^\#_A$ for the measure obtained from the loop-erasure of a random walk from 0 conditioned to leave A at x or y (that is, we stop the walk at ∂A and then erase the loops).

As we have seen, we have

$$\lambda^\#(\boldsymbol\gamma) = a_n\, \mu^\#(\boldsymbol\gamma)\, \exp\{-L_n(\boldsymbol\gamma)\},$$

where

$$a_n^{-1} = \mathbf{E}\left[\exp\{-L_n\}\right],$$

and the expectation is with respect to $\mu^\#$. Similarly,

$$\lambda^\#_A(\boldsymbol\gamma) = b_A \sum_{\boldsymbol\eta} \mu^\#_A(\boldsymbol\gamma)\, \mu^\#_A(\boldsymbol\eta \mid \boldsymbol\gamma)\, \exp\left\{-[L_n(\boldsymbol\gamma) + \tilde L(\boldsymbol\gamma \oplus \boldsymbol\eta)]\right\}, \tag{3.3}$$

where $\log \tilde L(\boldsymbol\gamma \oplus \boldsymbol\eta)$ is the measure of loops in A that intersect both $\gamma^1 \oplus \eta^1$ and $\gamma^2 \oplus \eta^2$ but are not loops in C_n intersecting both γ^1 and γ^2, and

$$b_A^{-1} = \mathbf{E}_A\left[\exp\left\{-(L_n + \tilde L)\right\}\right].$$

Here the expectation is with respect to $\mu^\#_A$. Therefore,

$$Z_A(\boldsymbol\gamma) = \frac{a_n}{b_A}\, \frac{\mu^\#_A(\gamma^1)\, \mu^\#_A(\gamma^2)}{\mu^\#_n(\gamma^1)\, \mu^\#_n(\gamma^2)} \sum_{\boldsymbol\eta} \mu^\#_A(\boldsymbol\eta \mid \boldsymbol\gamma)\, \exp\left\{-\tilde L(\boldsymbol\gamma \oplus \boldsymbol\eta)]\right\}.$$

Recall that

$$\frac{\mu^\#_A(\gamma^1)}{\mu^\#_n(\gamma^1)} = \frac{H_{\partial(A\setminus\gamma^1)}(z,x)}{H_{\partial\hat A}(0,x)\, \mathrm{Es}_{\gamma^1}(z)\, \phi_A\, \phi_A(\gamma^1)},$$

where

- $(d \geq 3)$ $\phi_A = 1$ and $\log \phi_A(\gamma^1)$ is the measure of loops in $\hat{\mathbb{Z}}^d$ that intersect γ^1 but do not lie in A.
- $(d = 2)$ $\log \phi_A$ is the measure of loops in $\hat{\mathbb{Z}}^2$ that do not lie in A but disconnect 0 from ∂C_n and $\log \phi_A(\gamma^1)$ is the measure of nondisconnecting loops not in A that intersect γ^1.

Lemma 2.10 shows that $\phi_A \asymp n$ if $d = 2$, and hence we can use Corollaries 2.12 and 2.16 to conclude that

$$H_{\partial(A\setminus\gamma^1)}(z,x) \asymp H_{\partial A}(0,x)\, \mathrm{Es}_{\gamma^1}(z)\, \phi_A, \quad \phi_A(\gamma^1) \asymp 1,$$

and using Corollary 2.17, if $\gamma^1 =_j \tilde\gamma^1$,

$$\frac{\mu^\#_A(\gamma^1)}{\mu^\#_n(\gamma^1)} = \frac{\mu^\#_A(\tilde\gamma^1)}{\mu^\#_n(\tilde\gamma^1)}[1 + O(e^{-j/4})].$$

The same results hold for γ^2, and hence if $\boldsymbol\gamma =_j \tilde{\boldsymbol\gamma}$,

$$\frac{\mu^\#_A(\boldsymbol\gamma)}{\mu^\#_n(\boldsymbol\gamma)} = \frac{\mu^\#_A(\tilde{\boldsymbol\gamma})}{\mu^\#_n(\tilde{\boldsymbol\gamma})}[1 + O(e^{-j/4})]. \tag{3.4}$$

Since,

$$\sum_{\boldsymbol{\eta}} \mu_A^{\#}(\boldsymbol{\eta} \mid \boldsymbol{\gamma}) \exp\left\{-\tilde{L}(\boldsymbol{\gamma} \oplus \boldsymbol{\eta})]\right\} \leq 1,$$

we see that Z_A is uniformly bounded above. If $\boldsymbol{\gamma} =_j \tilde{\boldsymbol{\gamma}}$, then

$$\begin{aligned}
&\left|\sum_{\boldsymbol{\eta}} \mu_A^{\#}(\boldsymbol{\eta} \mid \boldsymbol{\gamma}) \exp\left\{-\tilde{L}(\boldsymbol{\gamma} \oplus \boldsymbol{\eta})\right\} - \sum_{\boldsymbol{\eta}} \mu_A^{\#}(\boldsymbol{\eta} \mid \tilde{\boldsymbol{\gamma}}) \exp\left\{-\tilde{L}(\tilde{\boldsymbol{\gamma}} \oplus \boldsymbol{\eta})\right\}\right| \\
&\quad \leq \sum_{\boldsymbol{\eta}} |\mu_A^{\#}(\boldsymbol{\eta} \mid \boldsymbol{\gamma}) - \mu_A^{\#}(\boldsymbol{\eta} \mid \tilde{\boldsymbol{\gamma}})| \\
&\qquad\qquad + \sum_{\boldsymbol{\eta}} \mu_A^{\#}(\boldsymbol{\eta} \mid \boldsymbol{\gamma}) \left|\exp\left\{-\tilde{L}(\boldsymbol{\gamma} \oplus \boldsymbol{\eta})\right\} - \exp\left\{-\tilde{L}(\tilde{\boldsymbol{\gamma}} \oplus \boldsymbol{\eta})\right\}\right|.
\end{aligned}$$

We use Corollary 2.17 to see that

$$\sum_{\boldsymbol{\eta}} |\mu_A^{\#}(\boldsymbol{\eta} \mid \boldsymbol{\gamma}) - \mu_A^{\#}(\boldsymbol{\eta} \mid \tilde{\boldsymbol{\gamma}})| \leq O(e^{-j}),$$

$$\sum_{\boldsymbol{\eta} \cap C_{n-(j/2)} \neq \emptyset} \mu_A^{\#}(\boldsymbol{\eta} \mid \boldsymbol{\gamma}) \leq O(e^{-j}),$$

and, again, if $\boldsymbol{\eta} \cap C_{n-(j/2)} = \emptyset$,

$$\left|\exp\left\{-\tilde{L}(\boldsymbol{\gamma} \oplus \boldsymbol{\eta})\right\} - \exp\left\{-\tilde{L}(\tilde{\boldsymbol{\gamma}} \oplus \boldsymbol{\eta})\right\}\right| \leq O(e^{-j/2}).$$

This establishes the proof when x and y are separated.

The proof when x and y are separated is the main part. If x and y are not separated, then we first separate x and y and then use the argument for separated points. This is straightforward using a separation lemma, but we discuss the details below.

When x and y are not separated, we first run paths from x and y until the paths get to $C_{n+(4/5)}$ and use the fact that there is a positive probability that the paths have separated. Indeed, we write each $\hat{\gamma}^1 \in \mathcal{W}_A^x$ as

$$\hat{\gamma}^1 = \gamma^1 \oplus \hat{\eta}^1 \oplus \eta^1,$$

where $\gamma^1 \in \mathcal{W}_n$ and η^1 starts at the last visit to $C_{n+(4/5)}$. We do similarly for $\hat{\gamma}^2 \in \mathcal{W}_A^y$ and we also write

$$\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \oplus \hat{\boldsymbol{\eta}} \oplus \boldsymbol{\eta}.$$

We partition loops that intersect both $\hat{\gamma}^1$ and $\hat{\gamma}^2$ into three sets:

- Loops in C_n that intersect both γ^1 and γ^2.
- Loops in A that intersect both η^1 and η^2
- All other loops.

If $d = 2$, we restrict to nondisconnecting loops, In that way we write

$$L_A(\hat{\boldsymbol{\gamma}}) = L_n(\boldsymbol{\gamma}) + L_A(\boldsymbol{\eta}) + \tilde{L}(\hat{\boldsymbol{\gamma}}).$$

As before, we write $L_n(\boldsymbol{\gamma}) = -\infty$ if $\gamma^1 \cap \gamma^2 \neq \{0\}$, $L_A(\boldsymbol{\eta}) = -\infty$ if $\eta^1 \cap \eta^2 \neq \emptyset$, and $\tilde{L}(\hat{\gamma}) = -\infty$ if $\hat{\gamma}^1 \cap \hat{\gamma}^2 \neq \{0\}$.

We write $\mu^{\#}$ for the measure on loop-erased walks obtained by taking an infinite loop-erased walk and stopping it at the first visit to ∂C_n. We write $\mu_A^{\#}$ for the measure obtained from the loop-erasure of a random walk from 0 conditioned to leave A at x or y (that is, we stop the walk at ∂A and the erase the loops).

By definition, we have

$$\lambda^{\#}(\boldsymbol{\gamma}) = a_n \, \mu^{\#}(\boldsymbol{\gamma}) \, \exp\{-L_n(\boldsymbol{\gamma})\},$$

where

$$a_n^{-1} = \mathbf{E}\left[\exp\{-L_n\}\right],$$

and the expectation is with respect to $\mu^{\#}$. Similarly,

$$\lambda_A^{\#}(\boldsymbol{\gamma}) = b_A \sum_{\boldsymbol{\eta}\oplus\hat{\boldsymbol{\gamma}}} \mu_A^{\#}(\boldsymbol{\gamma}\oplus\boldsymbol{\eta}\oplus\hat{\boldsymbol{\gamma}}) \, \exp\left\{-[L_n(\boldsymbol{\gamma}) + L_A(\boldsymbol{\eta}) + \tilde{L}(\tilde{\boldsymbol{\gamma}})]\right\}, \tag{3.5}$$

where

$$b_A^{-1} = \mathbf{E}_A\left[\exp\left\{-(L_n + L_A + \tilde{L})\right\}\right],$$

and the expectation is with respect to $\mu_A^{\#}$.

We will choose from $\mu_A^{\#}$ by choosing $\boldsymbol{\eta}$ first, then $\boldsymbol{\gamma}$, and then finally $\hat{\boldsymbol{\eta}}$. By doing this we can see that (3.5) can be written as

$$\lambda_A^{\#}(\boldsymbol{\gamma}) = b_A \sum_{\boldsymbol{\eta}} \sum_{\hat{\boldsymbol{\eta}}} \mu_A^{\#}(\boldsymbol{\eta}) \, \mu_A^{\#}(\boldsymbol{\gamma} \mid \boldsymbol{\eta}) \, \mu_A^{\#}(\hat{\boldsymbol{\eta}} \mid \boldsymbol{\gamma},\boldsymbol{\eta}) \, \exp\left\{-[L_n(\boldsymbol{\gamma}) + L_A(\boldsymbol{\eta}) + \tilde{L}(\tilde{\boldsymbol{\gamma}})]\right\},$$

which in turn can be written as

$$b_A \sum_{\boldsymbol{\eta}} \mu_A^{\#}(\boldsymbol{\eta}) \, \exp\{-L_A(\boldsymbol{\eta})\} \, \mu_A^{\#}(\boldsymbol{\gamma} \mid \boldsymbol{\eta}) \, \exp\{-L_n(\boldsymbol{\gamma})\} \, \Psi(\boldsymbol{\gamma},\boldsymbol{\eta}),$$

where

$$\Psi(\boldsymbol{\gamma},\boldsymbol{\eta}) = \sum_{\hat{\boldsymbol{\eta}}} \mu_A^{\#}(\hat{\boldsymbol{\eta}} \mid \boldsymbol{\gamma},\boldsymbol{\eta}) \, \exp\{-\tilde{L}(\boldsymbol{\gamma}\oplus\hat{\boldsymbol{\eta}}\oplus\boldsymbol{\eta})\} \leq 1.$$

Therefore,

$$\frac{\lambda_A^{\#}(\boldsymbol{\gamma})}{\lambda^{\#}(\boldsymbol{\gamma})} = \frac{b_A}{a_n} \sum_{\boldsymbol{\eta}} \mu_A^{\#}(\boldsymbol{\eta}) \, \exp\{-L_A(\boldsymbol{\eta})\} \, \frac{\mu_A^{\#}(\boldsymbol{\gamma} \mid \boldsymbol{\eta})}{\mu^{\#}(\boldsymbol{\gamma})} \, \Psi(\boldsymbol{\gamma},\boldsymbol{\eta}).$$

We will need to use the separation lemma on both the beginning and the ending of the path. There is a lot of arbitrariness in the definition of the separation event. We will not be specific here, but the important facts are that there there exists $c_1 > 0$ such that

$$\begin{gathered}
\Psi(\boldsymbol{\gamma},\boldsymbol{\eta}) \geq c_1, \quad \boldsymbol{\gamma},\boldsymbol{\eta} \in \mathsf{Sep}, \\
\sum_{\boldsymbol{\eta}\in\mathsf{Sep}} \mu_A^{\#}(\boldsymbol{\eta}) \, \exp\{-L_A(\boldsymbol{\eta})\} \geq c_1 \sum_{\boldsymbol{\eta}} \mu_A^{\#}(\boldsymbol{\eta}) \, \exp\{-L_A(\boldsymbol{\eta})\}, \\
\sum_{\boldsymbol{\gamma}^*\in\mathsf{Sep}} \mu_n^{\#}(\boldsymbol{\gamma}^*) \, \exp\{-L_n(\boldsymbol{\gamma}^*)\} \geq c_1 \sum_{\boldsymbol{\gamma}^*} \mu_n^{\#}(\boldsymbol{\gamma}^*) \, \exp\{-L_n(\boldsymbol{\gamma}^*)\}.
\end{gathered} \tag{3.6}$$

To be specific, we will say that $\boldsymbol{\eta} \in \mathsf{Sep}$ if each initial point (the vertex in $\partial C_{n+(4/5)}$) is distance at least $e^n/100$ from the other path in the pair. For $\boldsymbol{\gamma}^* \in \mathcal{A}_n$ we can use the definition of separation as before. Establishing (3.6) uses the same argument as in Proposition 2.31 — given $(\boldsymbol{\gamma},\boldsymbol{\eta})$ that are separated, the conditioned simple random walks whose loop erasure given $\hat{\boldsymbol{\eta}}$ have a positive probability of staying apart; hence so do the paths in $\hat{\boldsymbol{\eta}}$; and the measure of loops intersecting both is bounded uniformly above zero.

We now claim the following.

- If $\boldsymbol{\gamma} =_j \tilde{\boldsymbol{\gamma}}$, then

$$\left| \frac{\mu_A^{\#}(\boldsymbol{\gamma} \mid \boldsymbol{\eta})}{\mu^{\#}(\boldsymbol{\gamma})} - \frac{\mu_A^{\#}(\tilde{\boldsymbol{\gamma}} \mid \boldsymbol{\eta})}{\mu^{\#}(\tilde{\boldsymbol{\gamma}})} \right| \leq c\, e^{-j/4}. \tag{3.7}$$

We will consider the result without conditioning,

$$\left| \frac{\mu_A^{\#}(\boldsymbol{\gamma})}{\mu^{\#}(\boldsymbol{\gamma})} - \frac{\mu_A^{\#}(\tilde{\boldsymbol{\gamma}})}{\mu^{\#}(\tilde{\boldsymbol{\gamma}})} \right|$$

where the estimate should be uniform over all A containing $C_{n+(4/5)} \setminus \partial_i C_{n+(4/5)}$ and boundary points $x, y \in C_{n+(4/5)} \cap \partial A$. and all $x, y \in \partial A$. Since $\mu_A^{\#}(\gamma \mid \boldsymbol{\eta}) = \mu_{A \setminus \boldsymbol{\eta}}^{\#}(\gamma)$ (with appropriately chosen starting points), we get our estimate as in (3.4).

We have already noted that $\Psi(\boldsymbol{\gamma}, \boldsymbol{\eta}) \leq 1$. We now claim the following.

- If $\boldsymbol{\gamma} =_j \tilde{\boldsymbol{\gamma}}$, then

$$|\Psi(\boldsymbol{\gamma}, \boldsymbol{\eta}) - \Psi(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\eta})| \leq c\, e^{-j/4}.$$

To show this we first note again that

$$\sum_{\boldsymbol{\eta}^* \cap C_{n-(j/2)} \neq \emptyset} \mu_A^{\#}(\boldsymbol{\eta}^* \mid \boldsymbol{\gamma}, \boldsymbol{\eta}) \leq c\, e^{-j/4}.$$

Also, if $\boldsymbol{\eta}^* \cap C_{n-(j/2)} = \emptyset$, we have

$$\boldsymbol{\gamma} \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta} =_{j/2} \tilde{\boldsymbol{\gamma}} \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta},$$

in which case

$$|\tilde{L}(\boldsymbol{\gamma} \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta}) - \tilde{L}(\tilde{\boldsymbol{\gamma}} \oplus \boldsymbol{\eta}^* \oplus \boldsymbol{\eta})| \leq c\, e^{-j/4}. \qquad \square$$

A On the proof of Lemma 2.11

The proof of the result for $d = 2$ was done in [16] and it was adapted for $d = 3$ in [17]. We will not give the complete proof here, but we will sketch most of the argument using a different function than used before. Let us first consider Brownian motion, where the result is easier.

Let U denote the open cube

$$U = \left\{ x = (x^1, \ldots, x^d) \in \mathbb{R}^d : |x^j - 1| < 1 \right\},$$

and let $V = \partial U \cap \{x^1 = 1\}$. Let $g(x)$ be the the harmonic function on U with boundary value 1_V, that is, $g(x)$ is the probability that a Brownian motion starting at x exits U at V. By symmetry, we see that $g(0) = 1/2d$. Let

$$U^- = \{(x^1, \ldots, x^d) \in U : x^1 \leq 0\},$$

and let L be the line segment $L = \{(x^1, 0, \ldots, 0) \in U : 0 < x^1 < 1\}$. We will need several properties of g.

Lemma A.1.

1. *There exists $c < \infty$ such that $|x|/4d \leq g(x) - g(0) \leq c\,|x|$ for $x \in L$.*
2. *There exists $c_1 > 0$ such that if $x = (x^1, \tilde{x}) \in U^-$, then $g(x) \leq g(0) - c_1 |\tilde{x}|^2$.*
3. *There exists $\epsilon > 0$ such that if $U_\epsilon^- = \{x = (x^1, \tilde{x}) \in U : x^1 \leq \epsilon |\tilde{x}|^2\}$, then*

$$g(0) = \sup\{g(x) : x \in U_\epsilon^-\}. \tag{A.1}$$

Proof.

1. The upper bound follows from derivative estimates for harmonic functions, so we will only show the lower bound using a coupling argument. Let B_t be a Brownian motion starting at the origin and $W_t = B_t + x$ a Brownian motion starting at x. Let $T = \min\{t : B_t \in \partial U\}, T^x = \min\{t : W_t \in \partial U\}$. Note that $W_t - B_t = x$ for all t.

   - if $T = T^x$, then B_τ and W_τ cannot be exiting at V.
   - If $T < T^x$, then $B_T \notin V$. There is still a chance that $B_{T^x} \in V$.
   - If $T^x < T$, then $W_{T^x} \in V$ and $B_{T^x} = W_{T^x} - x$. By symmetry, we can see that

   $$\mathbf{P}\{T^x < T\} \geq \mathbf{P}\{B_T \in V\} = \frac{1}{2d}.$$

   Using the gambler's ruin on the first component, we can see that, given $T^x < T$, the probability that the first component of B_t will equal -1 before it equals 1 is $|x|/2$. In particular,

   $$\mathbf{P}\left\{B_T \notin V \mid B_{T^x} \in V\right\} \geq \frac{|x|}{2}.$$

   Therefore,

   $$g(x) - g(0) = \mathbf{P}\left\{B_{T^x} \in V, B_T \notin V\right\} \geq \frac{|x|}{4d}.$$

2. The previous argument can be used to see that $g(-\delta, x^2, \ldots, x^d) < g(0, x^2, \ldots, x^d)$ for $\delta > 0$ so it suffices to consider the maximum over $(0, x^2, \ldots, x^d)$. By symmetry, we can consider the same problem for the positive quadrant $Q = \{(x^1, \ldots, x^d) \in U : x^j > 0\}$ where we reflect the Brownian motion on the boundaries $\{x^j = 0\}$. We can couple reflected Brownian motions B_t starting at 0 and W_t at $x \in \{(0, x^2, \ldots, x^d) : 0 \leq x^j < 1\}$, so that all of the components of the latter Brownian motion are greater than those of the former Brownian motion and the first components always agree. This gives d independent coupled reflecting Brownian motions (the coupling is described in the next paragraph) with the following properties: the first components are the same and start at 0; for the others, one starts at 0, the other at x^j, and coupling occurs when the latter reaches $x^j/2$. In particular, the probability that the latter one reaches 1 before coupling is greater than $x^j/2$, and given that, the probability that the first one exits somewhere other than V is greater than $c|x^j|$. Using this we can see that

   $$\mathbf{P}\{T^x < T^0\} \geq c\,|x|^2.$$

   We now describe the coupling of one-dimensional reflected Brownian motions (Y_t, Z_t) with $Y_0 = 0, Z_0 = x \in (0,1)$. Let W_t be a Brownian motion starting at x and let $Z_t = |W_t|$. Let $\sigma_{x/2}, \sigma_1$ be the first times that $Z_t = x/2, Z_t = 1$, respectively. Let $Y_t = |W_t - x|$ for $t \leq \sigma_{x/2}$ and $Y_t = Z_t$ for $t \geq \sigma_{x/2}$, and note that Y_t is a reflected Brownian motion starting at the origin. Let T_Z, T_Y denote the first time greater than or equal to $\sigma_{x/2} \wedge \sigma_1$ at which $Z_t \in \{0,1\}, Y_t \in \{0,1\}$, respectively.

   - If $\sigma_{x/2} < \sigma_1$, then $T_Z = T_Y$.
   - If $\sigma_1 < \sigma_{x/2}$, $T_Z = \sigma_1$ and then $Y_{\sigma_1} = 1 - x$. Therefore, using the gambler's ruin estimate,

   $$\mathbf{P}\{Y_{T_Y} = 0 \mid \sigma_1 < \sigma_{x/2}\} = x.$$

   Therefore,

   $$\mathbf{P}\{Y_{T_Y} = 0\} = \mathbf{P}\{\sigma_1 < \sigma_{x/2}\}\,\mathbf{P}\{Y_{T_Y} = 0 \mid \sigma_1 < \sigma_{x/2}\} \geq \frac{x^2}{2}.$$

3. This follows from Part 2 and derivative estimates for positive harmonic functions. $\square$

Now suppose $A \subset \{x \in \mathbb{R}^d : |x| \geq 1\}$ and $|y| = 1$. Let U' be a rotated, dilated, and translated version of the set U above centered at y, rotated so that the inward radial direction of U' corresponds to the positive first component, and dilated by a factor δ where δ is sufficiently small so that the analogue of U_ϵ^- lies entirely in $\{|x| \geq 1\}$. Using (A.1), we can find a uniform δ depending only on the ϵ in (A.1). Let V' be the analogue of V and let f be the analogue of g, that is, $f(x)$ is the probability that a Brownian motion starting at x exits U' at V'. There exists $\delta' > 0$ such that all the point in V' have radius at most $1 - \delta'$. Let τ_A be the first time a Brownian motion hits A. Then if $r < 1$ and $z = ry \in U'$, we have $g(z) \geq g(x)$ for all $x \in A \cap U'$, and hence,

$$\mathbf{P}^z\{B_T \in V \mid T < \tau_A\} \geq \mathbf{P}^z\{B_T \in V\} \geq \frac{1}{2d},$$
$$\mathbf{P}^z\{B_T \in V \mid T > \tau_A\} \leq \mathbf{P}^z\{B_T \in V\}.$$

For random walk, we need to find the discrete analogue of the function g. Some work needs to be done because we do not want to require that the cube U be lined up with the lattice $\mathbb{Z}^d$; indeed, we want an estimate that is uniform over all rotations. Let U' be a rotation of the U above, with corresponding $V \subset \partial U'$, and g the harmonic function on U with boundary value 1_V. For each $r = 1/n > 0$, Let $K_r = \{x \in \mathbb{Z}^d : rx \in U'\}$. Let V_r be the subset of ∂K_r corresponding to nV. Let $g^*(x)$ be the discrete harmonic function on K_r with boundary value 1_{V_r}, that is, $g^*(x)$ is the probability that a random walk starting at x exits K_r at V_r.

Lemma A.2. *There exists $c < \infty$ such that for every rotation and every n,*

$$|g^*(x) - g(rx)| \leq \frac{c}{n} \quad \textit{if} \quad \operatorname{dist}(x, \partial V_r) \geq \frac{n}{10}.$$

Proof. We use Δ to denote the discrete Laplacian, let $\hat{g}(x) = g(rx)$ and recall that

$$\hat{g}(x) = g^*(x) - \sum_{y \in U_r} G_{U_r}(x, y)\, \Delta \hat{g}(y),$$

where Δ denotes the discrete Laplacian. Therefore, it suffices to show that

$$\sum_{y \in U_r} G_{U_r}(x, y)\, |\Delta \hat{g}(y)| \leq \frac{c}{n}, \quad \operatorname{dist}(x, \partial V_r) \geq \frac{n}{10}, \tag{A.2}$$

and to prove (A.2) it suffices to establish that

$$\sum_{y \in U_r,\, j-1 < \operatorname{dist}(y, \partial U_r) \leq j} G_{U_r}(x, y)\, |\Delta \hat{g}(y)| \leq \frac{c}{j^2 n}. \tag{A.3}$$

We will use the following fact that uses only the Taylor approximation of h and the derivative bounds for harmonic functions.

- There exists $c > 0$ such that if $R \geq 2$ and h is a (continuous) harmonic function on $\{x \in \mathbb{R}^d : |x| < R\}$, then

$$\Delta h(0) \leq \frac{c}{R^4} \sup_{|x| \leq R/2} |h(x) - h(0)|,$$

  where Δ denotes the discrete Laplacian.

We now split into two cases.

- Suppose that $\operatorname{dist}(y, V_r) \geq n/20$. In this case, the gambler's ruin estimate shows that
$$\hat{g}(y) \leq c\,\frac{\operatorname{dist}(y, \partial U)}{n}.$$
Therefore, if $j-1 < \operatorname{dist}(y, \partial U) \leq j$,
$$|\Delta \hat{g}(y)| \leq c\, j^{-4}\,\frac{j}{n} = \frac{c}{j^3 n}.$$
Also, using the gambler's ruin estimate, we can see that for all x,
$$\sum_{j-1<\operatorname{dist}(y,\partial U_r)\leq j} G(x,y) \leq c\, j.$$
Therefore,
$$\sum_{y\in U_r,\; j-1<\operatorname{dist}(y,\partial U_r)\leq j, \operatorname{dist}(y,V_r)\geq n/20} G_{U_r}(x,y)\,|\Delta \hat{g}(y)| \leq \frac{c}{j^2\, n}.$$
- Suppose that $\operatorname{dist}(y, V_r) = \delta \leq n/20$ and let $\delta' = \operatorname{dist}(y, \partial U_r \setminus V_r)$.
  - Suppose that $\delta < \delta'$. Then using the gambler's ruin estimate, we can see that there exists c such that for $|z-y| < \delta/2$,
  $$1 - \hat{g}(z) \leq c\,\delta/\delta'.$$
  Therefore,
  $$|\Delta \hat{g}(y)| \leq c\,\delta^{-4}\,(\delta/\delta') = \frac{c}{\delta^3\,\delta'}.$$
  - Suppose that $\delta' \leq \delta$. Then similarly that there exists c such that for $|z-y| < \delta/2$,
  $$\hat{g}(z) \leq c\,\delta'/\delta.$$
  Therefore,
  $$|\Delta \hat{g}(y)| \leq \frac{c}{\delta\,\delta'^{\,3}}.$$

  Using the gambler's ruin estimate (or similarly), we see that
$$G_{U_r}(x,y) \leq \frac{c}{n^{d-2}}\,\frac{\delta}{n}\,\frac{\delta'}{n} = \frac{c\,\delta\,\delta'}{n^d},$$
and hence if $j-1 < \operatorname{dist}(y, \partial U_r) \leq j$,
$$G_{U_r}(x,y)\,|\Delta \hat{g}(y)| \leq \frac{c}{(\delta\wedge\delta')^2\,n^d} \leq \frac{1}{j^2\,n^d}.$$
Using the fact that $\#\{x \in U_r : j-1 < \operatorname{dist}(y, \partial U_r) \leq j\} \leq c\,n^{d-1}$, we get that
$$\sum_{y\in U_r,\; j-1<\operatorname{dist}(y,\partial U_r)\leq j, \operatorname{dist}(y,V_r)\leq n/20} G_{U_r}(x,y)\,|\Delta \hat{g}(y)| \leq \frac{c}{j^2\, n}.$$

This establishes (A.3), and hence proves the lemma. □

Using this we establish that there exists a c_1 such that there exists x with $|x| \leq c_1$ and
$$g^*(x) \geq g^*(y), \quad y \in n\,U_\epsilon^-.$$
Therefore, for any $A \subset \mathbb{Z}^d \setminus C_n$, the probability that random walk starting at x leaves on the analogue of V' given that if avoids A is greater than the unconditioned probability which is greater than $(1/2d)$.

B Sketch of proof of Separation Lemma I

If $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{A}_n$, with terminal vertices z_1, z_2, we define the separation $\Delta(\boldsymbol{\eta}) = \Delta_n(\boldsymbol{\eta})$ to be the largest r such that

$$\mathrm{dist}(z_j, \eta^{3-j}) \geq r\, e^n, \quad j = 1, 2, .$$

- **Claim.** There exists $r_0, c > 0$ such that if $0 < r < r_0$ and $\boldsymbol{\eta} \in \mathcal{A}_n$ with $\Delta(\boldsymbol{\eta}) \geq r$, then

  $$\sum \lambda_{n+8r}(\boldsymbol{\eta}' \mid \boldsymbol{\eta}) \geq c.$$

  where the sum is over all $\boldsymbol{\eta}' = (\eta^1 \oplus \tilde{\eta}^1, \eta^2 \oplus \tilde{\eta}^2) \in \mathcal{A}_{n+8r}$ with

  $$\mathrm{diam}[\tilde{\eta}^i] \leq 16r\, e^n, \quad i = 1, 2,$$
  $$\mathrm{dist}(\tilde{\eta}^i, \eta^{3-i} \oplus \tilde{\eta}^{3-i}) \geq \frac{r}{2}\, e^n, \quad i = 1, 2,$$
  $$\Delta(\boldsymbol{\eta}') \geq 2r.$$

Indeed, using Proposition 2.20 and Lemma 2.11, we can see that

$$\sum \mu(\boldsymbol{\eta}' \mid \boldsymbol{\eta}) \geq c.$$

Also, any loop that intersects both $\eta^1 \oplus \tilde{\eta}^1$ and $\eta^2 \oplus \tilde{\eta}^2$ but does not intersect both η^1 and η^2, must be of diameter at least $(r/2)\, e^n$ and intersect either $\tilde{\eta}^1$ or $\tilde{\eta}^2$. Since the diameters of these curves are bounded by $16re^n$ we see (Lemma 2.6) that the loop measure of such curves is uniformly bounded. Hence $Q_{n+8r}(\boldsymbol{\eta}') \geq c\, Q_n(\boldsymbol{\eta})$.

- **Claim** There exist c_1, θ such that if $0 \leq s \leq 1/2$ and $\boldsymbol{\eta} \in \mathcal{A}_{n+s}$, then

  $$\sum_{\boldsymbol{\eta}' \in \mathsf{Sep}_{n+1}} \lambda(\boldsymbol{\eta}' \mid \boldsymbol{\eta}) \geq c_1\, \Delta(\boldsymbol{\eta})^{\theta}.$$

  This is obtained by repeated application of the previous claim.

- **Claim** There exist $\rho < 1$ such that if $\boldsymbol{\eta} \in \mathcal{A}_n$, then

  $$\sum_{\boldsymbol{\eta}' \in \mathcal{A}_{n+4r}, \Delta(\boldsymbol{\eta}') < r} \lambda(\boldsymbol{\eta}' \mid \boldsymbol{\eta}) \leq \rho.$$

Indeed, using Proposition 2.20 and Lemma 2.11, we can see that

$$\sum_{\boldsymbol{\eta}' \in \mathcal{A}_{n+4r}, \Delta(\boldsymbol{\eta}') < r} \mu(\boldsymbol{\eta}' \mid \boldsymbol{\eta}) \leq \rho.$$

We choose N sufficiently large so that

$$\sum_{j=N}^{\infty} j^2\, e^{-j} \leq \frac{1}{4}.$$

Let $u_j = 1/2$ for $j \leq N$ and for $j > N$,

$$u_j = u_{j-1} - j^2\, e^{-j}.$$

in particular, $1/4 \leq u_j \leq 1/2$ for all j. Let b_j be the infimum of

$$\frac{\lambda_{n+1}^{\mathsf{Sep}}(\boldsymbol{\eta})}{\lambda_{n+1}(\boldsymbol{\eta})},$$

where the infimum is over all $n \leq u \leq n + u_j$, and all $\boldsymbol{\eta} \in \mathcal{A}_{n+u}$ with

$$\Delta(\boldsymbol{\eta}) \geq e^{-j}.$$

Note that the result will hold with

$$c = \inf_j b_j,$$

so we need only show that the right-hand side is strictly positive. The result is obtained by establishing two facts:

1. There exist $\alpha > 0, c_0 > 0$ such that $b_j \geq c_0\, e^{-\alpha j}$. In particular, for each j, $b_j > 0$.
2. There exists a summable sequence δ_j such that for all j sufficiently large,

$$b_{j+1} \geq b_j\, [1 - \delta_j].$$

Indeed, if these hold and M is sufficiently large so that $\delta_j \leq 1/2$ for $j \geq M$, then

$$\inf_j b_j \geq b_M \prod_{j=M}^{\infty} (1 - \delta_j).$$

The proof proceeds now as in [14].

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