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Stability of overshoots of zero mean random walks

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Abstract

We prove that for a random walk on the real line whose increments have zero mean and are either integer-valued or spread out (i.e. the distributions of steps of the walk are eventually non-singular), the Markov chain of overshoots above a fixed level converges in total variation to its stationary distribution. We find the explicit form of this distribution heuristically and then prove its invariance using a time-reversal argument. If, in addition, the increments of the walk are in the domain of attraction of a non-one-sided α-stable law with index $\alpha \in (1, 2)$ (resp. have finite variance), we establish geometric (resp. uniform) ergodicity for the Markov chain of overshoots. All the convergence results above are also valid for the Markov chain obtained by sampling the walk at the entrance times into an interval.

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1 Introduction

Let $S = (S_n)_{n\geq 0}$ with $S_n = S_0 + X_1 + \ldots + X_n$ be a one-dimensional random walk with independent identically distributed (i.i.d.) increments $X_1, X_2, \ldots$ and the starting point S_0 that is a random variable independent with the increments. Assume that

$$\mathbb{E}|X_1| \in (0, \infty) \qquad \text{and} \qquad \mathbb{E}X_1 = 0, \tag{1.1}$$

which implies that $\limsup_{n\to\infty} S_n = -\liminf_{n\to\infty} S_n = \infty$ a.s. Define the *up-crossings times* of zero

$$T_0 := 0, \quad T_n := \inf\{k > T_{n-1} : S_{k-1} < 0, S_k \geq 0\}, \qquad n \in \mathbb{N}, \tag{1.2}$$

and let

$$O_n := S_{T_n}, \qquad U_n := S_{T_n - 1}, \qquad n \in \mathbb{N}, \tag{1.3}$$

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be the corresponding *overshoots* and *undershoots*; put $O_0 = U_0 := S_0$. The choice of zero is arbitrary and can be replaced by any fixed level. The sequence of overshoots $O = (O_n)_{n\geq 0}$ is a Markov chain. The sequence of undershoots $U = (U_n)_{n\geq 0}$ also forms a Markov chain. Both statements can be checked easily, although the latter one is less intuitive. We are mostly interested in the chain of overshoots, but our techniques also yield results for the chain of undershoots.

Under assumption (1.1), consider the law

$$\pi_+(dx) := \frac{2}{\mathbb{E}|X_1|} \mathbb{1}_{[0,\infty)}(x)\mathbb{P}(X_1 > x)\lambda(dx), \qquad x \in \mathcal{Z},$$

where $\mathcal{Z}$ is the *state space* of the walk S, defined as the minimal closed (in the topological sense) subgroup of $(\mathbb{R}, +)$ containing the topological support of the distribution of X_1, and λ is the Haar measure on $(\mathcal{Z}, +)$ normalized such that $\lambda([0, |x|) \cap \mathcal{Z}) = |x|$ for $x \in \mathcal{Z}$.

We will prove that the distribution π_+ is invariant for the Markov chain of overshoots O (Theorem 2.1). Our proof is based on a time reversal of the path of S between the up-crossings of the level zero. Since this proof gives no insight into the form of π_+, in Section 2.3 we present a heuristic argument which we used to find this invariant distribution. Invariance of π_+ is also established in our companion paper [14, Corollary 4.1] in a much more general setting using entirely different methods based on infinite ergodic theory; the proof presented here precedes the one in [14]. By [14, Corollary 4.2], the assumption in (1.1) implies that the law π_+ is the unique (up to multiplicative constant) locally finite Borel invariant measure of the chain of overshoots O on $\mathcal{Z}$. Moreover, we will see in Section 2.1 that assumption (1.1) is the weakest possible ensuring that O has an invariant distribution (i.e. probability measure).

The main goal of this paper is to study convergence of the Markov chain of overshoots O to its unique invariant law π_+. Our aim is to identify the conditions on the law of the increments of S under which the total variation distance between the law of O_n and π_+ converges to zero as $n \to \infty$ (Theorem 3.1) and study its rate of decay (Theorem 4.1). Since the chain O is in general neither weak Feller ([14, Remark 3.2]) nor ψ-irreducible (see Section 5), the total variation convergence requires additional smoothness assumptions on the distribution of increments of S. In particular, Theorem 3.1 holds if the distribution of X_1 is either *arithmetic* or *spread out*, which means respectively that either X_1 is supported on $d\mathbb{Z}$ for some $d > 0$ or the distribution of S_k is non-singular for some $k \geq 1$. The geometric rate of convergence in Theorem 4.1 is established under a further assumption that the law of X_1 is in the domain of attraction of a non-one-sided α-stable law with index $\alpha \in (1, 2)$. For increments with finite variance we get a stronger version with the geometric rate of convergence uniformly in the starting point of O. Section 5 concludes the paper by offering a conjecture about the weak convergence of the Markov chain O to π_+ without additional assumptions on the law of X_1 other than (1.1).

Our interest in the Markov chains of overshoots of random walks stems from their close connection to the local time of the random walk at level zero (see Perkins [15]) and their relevance to the study of asymptotics of the probability that the integrated random walk $(S_1 + \ldots + S_k)_{1\leq k\leq n}$ stays positive (see Vysotsky [23, 24]). A discussion on these applications and on further connections to a special class of Markov chains called *random walks with switch* at zero, is available in [14, Section 1.2]. Distributions of the same form as π_+ appear on many occasions, as discussed in Sections 2.1 and 2.2.

Finally, we note that our methods developed for establishing convergence of the chain O of overshoots above zero work without any changes for the Markov *chain of entrances* into the interval $[0, h]$ with any $h > 0$. In Section 5.1 we show that all the results for the chain O remain valid for this new chain, whose stationary distribution, given in (5.1) below, is unique and explicit; see also [14, Section 4].

2 Stationary distributions of overshoots

2.1 Setting and results

Consider the random walk $S = (S_n)_{n\geq 0}$ from Section 1, and define its version $S' = (S'_n)_{n\geq 0}$ with $S'_n := S_n - S_0$, which always starts at zero. We assume that S, as well as other random elements considered in this paper, are defined on a generic measurable space equipped with a variety of measures: a probability measure $\mathbb{P}$; the family of probability measures $\{\mathbb{P}_x\}_{x\in\mathbb{R}}$ given by $\mathbb{P}_x(S \in \cdot) = \mathbb{P}(x + S' \in \cdot)$ (satisfying $\mathbb{P}_x(S_0 = x) = 1$); and the measures of the form $\mathbb{P}_\mu(\cdot) := \int_{\mathbb{R}} \mathbb{P}_x(\cdot)\mu(dx)$, where μ is a Borel measure μ on $\mathbb{R}$. We do not necessarily assume that μ is a probability but we prefer to (ab)use the probabilistic notation $\mathbb{P}_\mu$ and the terms "law", "expectation", "random variable", etc., by which we actually mean the corresponding notions of general measure theory. Under the measure $\mathbb{P}_\mu$, the starting point S_0 of the random walk S follows the "law" μ. Denote by $\mathbb{E}$ and $\mathbb{E}_x$ the respective expectations under $\mathbb{P}$ and $\mathbb{P}_x$. All measures considered in the paper are Borel, that is defined on the corresponding Borel σ-algebras.

Recall that the state space $\mathcal{Z}$ of the random walk S was defined as the minimal closed subgroup of $(\mathbb{R}, +)$ containing the support of the distribution of X_1. Let us give a different representation for $\mathcal{Z}$ assuming throughout that X_1 is not degenerate. Denote

$$\mathcal{Z}_h := \begin{cases} \mathbb{R}, & \text{if } h = 0, \\ h\mathbb{Z}, & \text{if } h > 0. \end{cases}$$

We equip $\mathcal{Z}_h$ with the discrete (resp. Euclidean) topology if $h > 0$ (resp. $h = 0$). Note that any closed (in the topological sense) subgroups of $(\mathbb{R}, +)$ is of the form $(\mathcal{Z}_h, +)$ for some $h \geq 0$. Finally, denote $\mathcal{Z}_h^+ := \mathcal{Z}_h \cap [0, \infty)$ and $\mathcal{Z}_h^- := \mathcal{Z}_h \cap (-\infty, 0)$.

Define the *span* of the distribution of increments of S by

$$d := \sup\{h \in [0, \infty) : \mathbb{P}(X_1 \in \mathcal{Z}_h) = 1\}, \tag{2.1}$$

and note that $d \in [0, \infty)$ and $\mathcal{Z} = \mathcal{Z}_d$. We *always* assume that the random walk starts in $\mathcal{Z}_d$, hence $\mathbb{P}(S_0, S_1, \ldots \in \mathcal{Z}_d) = 1$. The distribution of increments of S is called *arithmetic* (with span d) if $d > 0$ and is called *non-arithmetic* if $d = 0$. We shall often use $d > 0$ and $d = 0$ as synonyms for arithmetic and non-arithmetic, respectively. Define the measure λ_d on $\mathcal{Z}_d$ as follows: for any $B \in \mathcal{B}(\mathcal{Z}_d)$, put

$$\lambda_d(B) := \begin{cases} \lambda_0(B), & \text{if } d = 0, \\ d \cdot \#B, & \text{if } d > 0, \end{cases}$$

where λ_0 denotes the Lebesgue measure on $\mathbb{R}$ and $\#$ denotes the number of elements in a set. Then λ_d is the normalized Haar measure on the additive group $\mathcal{Z}_d = \mathcal{Z}$, as defined in the Introduction. Furthermore, define the measures on $\mathcal{Z}_d$:

$$\lambda_d^+(dx) := \mathbb{1}_{\mathcal{Z}_d^+}(x)\lambda_d(dx) \quad \text{and} \quad \lambda_d^-(dx) := \mathbb{1}_{\mathcal{Z}_d^-}(x)\lambda_d(dx), \qquad x \in \mathcal{Z}_d,$$

and

$$\pi_+(dx) := c_1\mathbb{P}(X_1 > x)\lambda_d^+(dx) \quad \text{and} \quad \pi_-(dx) := c_1\mathbb{P}(X_1 \leq x)\lambda_d^-(dx), \qquad x \in \mathcal{Z}_d, \tag{2.2}$$

where $c_1 := 1$ if $\mathbb{E}|X_1| = \infty$ and $c_1 := 2/\mathbb{E}|X_1|$ if $\mathbb{E}|X_1| < \infty$. This extends the definition of π_+ given in the Introduction under assumption (1.1).

The classic trichotomy states that the (non-degenerate) random walk S either drifts to $+\infty$, drifts to $-\infty$, or *oscillates*; see Feller [7, Section XII.2]. By definition, the latter possibility means that $\limsup_{n\to\infty} S_n = \infty$ a.s. and $\liminf_{n\to\infty} S_n = -\infty$ a.s. It is known that S oscillates if and only if either $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1| \in (0,\infty)$ or $\mathbb{E}X_1$ does not exist, i.e. $\mathbb{E}X_1^+ = \mathbb{E}X_1^- = +\infty$, where $x^+ := \max\{x,0\}$ and $x^- := (-x)^+$ for a real x; cf. Feller [7, Theorems XII.2.1] and Kesten [11, Corollary 3]. In particular, S oscillates when it is *topologically recurrent* on $\mathcal{Z}_d$, meaning that $\mathbb{P}_0(S_n$ eventually returns to $G) = 1$ for every open neighbourhood $G \subset \mathcal{Z}_d$ of 0. Indeed, such random walks satisfy $\mathbb{P}_0(S_n \in G \text{ i.o.}) = 1$ for *every* non-empty open set $G \subset \mathcal{Z}_d$; see Revuz [16, Proposition 3.4].

Clearly, the assumption of oscillation is necessary and sufficient for S to cross a level infinitely often a.s., in which case the Markov chains of overshoots and undershoots of the zero level introduced in (1.2) and (1.3) are well-defined. Similarly, define the *down-crossing times* of the level zero

$$T_0^{\downarrow} := 0, \quad T_n^{\downarrow} := \inf\{k > T_{n-1}^{\downarrow} : S_{k-1} \geq 0, S_k < 0\}, \qquad n \in \mathbb{N}, \tag{2.3}$$

and the corresponding overshoots and undershoots at the down-crossings

$$O_n^{\downarrow} = S_{T_n^{\downarrow}}, \qquad U_n^{\downarrow} := S_{T_n^{\downarrow}-1}, \qquad n \in \mathbb{N}, \tag{2.4}$$

with $O_0^{\downarrow} = U_0^{\downarrow} := S_0$. The random sequences in (1.3) and (2.4) are defined on the event that all crossing times T_n are finite. Since S oscillates, this event occurs almost surely under $\mathbb{P}$ and under $\mathbb{P}_\mu$ with arbitrary measure μ on $\mathcal{Z}_d$.

The Markov chains of overshoots at up-crossings $O = (O_n)_{n\geq 0}$ and at down-crossings $O^{\downarrow} = (O_n^{\downarrow})_{n\geq 0}$ take values in $\mathcal{Z}_d^+$ and $\mathcal{Z}_d^-$, respectively. Both chains are started at $O_0 = O_0^{\downarrow} = S_0 \in \mathcal{Z}_d$. Note that there is asymmetry at zero. Namely, since $-\mathcal{Z}_d^+ \neq \mathcal{Z}_d^-$, the down-crossing times $T_n^{\downarrow}$ (resp. positions $O_n^{\downarrow}$ and $U_n^{\downarrow}$) need *not* be almost surely equal to the up-crossing times T_n (resp. positions $-O_n$ and $-U_n$) for the dual random walk $(-S_n)_{n\geq 0}$. We will be mostly concerned with the chain O, which for brevity will be called the chain of overshoots if there is no risk of confusion with $O^{\downarrow}$. We will also consider the sequence $(-U_n - d)_{n\geq 0}$, taking values in $\mathcal{Z}_d^+$, which is a Markov chain since so is U.

Theorem 2.1. *Let S be any random walk that oscillates. Then the measure π_+ is invariant for the Markov chains O and $(-U_n - d)_{n\geq 0}$ of overshoots and shifted sign-changed undershoots at up-crossings of the zero level, i.e. $\mathbb{P}_{\pi_+}(O_n \in \cdot) = \pi_+$ and $\mathbb{P}_{\pi_+}(-U_n - d \in \cdot) = \pi_+$ for all $n \in \mathbb{N}$. Similarly, π_- is an invariant measure for the chains $O^{\downarrow}$ and $(-U_n^{\downarrow} - d)_{n\geq 0}$.*

Remark 2.2. We will show in Section 2.4.1 below that the laws of overshoots and undershoots of the zero level at consecutive down- and up-crossings are related by

$$\mathbb{P}_{\pi_+}(O_1^{\downarrow} \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(O_1 \in \cdot) = \pi_+, \quad \mathbb{P}_{\pi_+}(-U_1^{\downarrow} - d \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(-U_1 - d \in \cdot) = \pi_+.$$

We will prove these results using an argument based on the time reversal of the path of S between the up-crossings of the level zero. Since this proof gives no insight about the form of π_+, we will also present a heuristic argument which we used to find this invariant distribution. After these results were obtained, we found an entirely different proof of Theorem 2.1, which is based on the methods of infinite ergodic theory and applies in a much more general setting; see our companion paper [14].

The assumption that the random walk S oscillates is the weakest possible to consider the Markov chains of overshoots and undershoots. By [14, Corollary 4.2], the chains of overshoots and undershoots of such random walks possess no other (up to multiplicative constants) locally finite invariant Borel measures, including the ones singular with respect to π_+ and π_-. Therefore, the probabilistic question of convergence of these chains to stationarity can be posed only if the measures π_+ and π_- in Theorem 2.1 have total mass one. This need not be the case in general since every non-degenerate symmetric random walk oscillates. However, by (2.2), both measures π_+ and π_- have finite mass if and only if $\mathbb{E}|X_1| \in (0,\infty)$, in which case the oscillation assumption forces $\mathbb{E}X_1 = 0$ and the equalities $\pi_+(\mathcal{Z}_d) = \pi_-(\mathcal{Z}_d) = 1$ follow. Thus, condition (1.1) is the weakest assumption under which convergence to stationarity of the chains of overshoots and undershoots can be stated.

Probability measures of the same form as π_+ and π_- appear as limit distributions for the following stochastic processes closely related to random walks. Assume that (1.1) holds. First, π_+ is the unique stationary distribution of the *reflected random walk* driven by an i.i.d. sequence with the common non-arithmetic distribution $\mathbb{P}(X_1 \in \cdot | X_1 > 0)$; see Feller [7, Section VI.11] and Knight [12]. This can be generalised as follows. Consider a time-homogenious Markov chain Y on $\mathcal{Z}_d$ whose first increment has distribution $\mathbb{P}(X_1 \in \cdot | X_1 < 0)$ for all starting points in $\mathcal{Z}_d^+$ and $\mathbb{P}(X_1 \in \cdot | X_1 > 0)$ for all starting points in $\mathcal{Z}_d^-$. This chain belongs to a special type of Markov chains which we call *random walks with switch at zero*. A stationary distribution for such chains was found by Borovkov [4], and we can use it show that $\frac{1}{2}\pi_+ + \frac{1}{2}\pi_-$ is an invariant distribution of Y. Second, π_+ is known as the stationary distribution, as well as the limit distribution, for the non-negative *residual lifetime* in a *renewal process* with inter-arrival times distributed according to $\mathbb{P}(X_1 \in \cdot | X_1 > 0)$; see Asmussen [1, Section V.3.3] or Gut [8, Theorem 2.6.2]. For random walks this limit property can be interpreted as follows.

Denote by H_1^- the first strict descending ladder height of the random walk S', i.e. the first strictly negative value of S'. Similarly, denote by H_1^+ the first strict increasing ladder height of S'. It is known that random variables H_1^+ and H_1^- are integrable if $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 < \infty$; see Feller [7, Sections XVIII.4 and 5]. When this is the case, by the results of renewal theory (e.g. by [8, Theorem 2.6.2] and (4.6) below), we have

$$\mathbb{P}_x(O_1^{\downarrow} \in dy) \xrightarrow[x\to\infty,\, x\in\mathcal{Z}_d]{d} \frac{1}{-\mathbb{E}H_1^-}\mathbb{P}(H_1^- \le y)\lambda_d^-(dy), \quad y \in \mathcal{Z}_d. \tag{2.5}$$

The r.h.s. of (2.5) is referred to as the distribution of the overshoot of the walk S above an “infinitely remote” level at $-\infty$. This distribution equals π_- defined for H_1^- instead of X_1. Similarly, π_+ corresponds to the non-strict overshoot of S above an “infinitely remote” level at $+\infty$, which is distributed as the strict overshoot above this level decreased by d:

$$\mathbb{P}_x(O_1 \in dy) \xrightarrow[x\to-\infty,\, x\in\mathcal{Z}_d]{d} \frac{1}{\mathbb{E}H_1^+}\mathbb{P}(H_1^+ > y)\lambda_d^+(dy), \quad y \in \mathcal{Z}_d. \tag{2.6}$$

2.2 An alternative representation for π_+ and π_-

Notice that the invariant measures π_+ and π_- in Theorem 2.1, which are probabilities if and only if $\mathbb{E}|X_1| \in (0,\infty)$, are defined only in terms of the tails of the distribution of increments of the random walk S. On the other hand, it is natural to expect that π_+ and π_- are closely related to the distributions of the overshoots above infinitely remote levels at $\pm\infty$ given by the r.h.s.’s of (2.5) and (2.6). In this section we give such a representation. Note that since the limit distributions of overshoots above infinite levels exist only for zero-mean random walks with finite variance, before we proved Theorem 2.1 it was not at all clear why the chain of overshoots should have a stationary distribution for walks with infinite variance.

Denote by $\tilde{H}_1^-$ the first non-strict (weak) descending ladder height of S', i.e. the first non-positive value of $(S'_n)_{n\ge1}$.

Lemma 2.3. *For any random walk S that oscillates, we have*

$$\pi_+ = c_1\mathbb{P}(\tilde{H}_1^- \neq 0)\big[\mathbb{P}(H_1^- \le x)\lambda_d^-(dx)\big] * \mathbb{P}(H_1^+ \in \cdot) \quad \text{on } \mathcal{Z}_d^+.$$

Similarly,

$$\pi_- = c_1\mathbb{P}(\tilde{H}_1^- \neq 0)\big[\mathbb{P}(H_1^+ > x)\lambda_d^+(dx)\big] * \mathbb{P}(H_1^- \in \cdot) \quad \text{on } \mathcal{Z}_d^-.$$

Remark 2.4. If $\mathbb{E}X_1^2 < \infty$ (which implies (1.1)), the first identity can be interpreted as

$$\pi_+(dy) = \mathbb{P}(R^- + H_1^+ \in dy | R^- + H_1^+ \geq 0),$$

where R^- is a random variable having the distribution of the overshoot of S above an "infinitely remote" level at $-\infty$ (given by the r.h.s. of (2.5)) and independent with H_1^+, and

$$\mathbb{P}(R^- + H_1^+ \geq 0) = -\frac{1}{c_1 \mathbb{P}(\tilde{H}_1^- \neq 0)\mathbb{E}H_1^-} = -\frac{1}{c_1 \mathbb{E}\tilde{H}_1^-}.$$

Combining this with the analogous probabilistic interpretation of π_- and in the case $d = 0$ using that any distribution function is continuous a.e. with respect to the Lebesgue measure λ_0 allows us to rewrite the above representations directly in terms of the random walk as follows.

Proposition 2.5. *For any random walk S satisfying $\mathbb{E}X_1 = 0$ and $0 < \mathbb{E}X_1^2 < \infty$, we have*

$$\mathbb{P}_x\Big(S_{T_1} \in \cdot \,\Big|\, S_{T_1^\downarrow} \geq S_{T_1^\downarrow + 1}, \dots, S_{T_1^\downarrow} \geq S_{T_1 - 1}\Big) \xrightarrow[x \to \infty,\, x \in \mathcal{Z}_d]{d} \pi_+.$$

and

$$\mathbb{P}_x\Big(S_{T_1^\downarrow} \in \cdot \,\Big|\, S_{T_1} \leq S_{T_1 + 1}, \dots, S_{T_1} \leq S_{T_1^\downarrow - 1}\Big) \xrightarrow[x \to -\infty,\, x \in \mathcal{Z}_d]{d} \pi_-.$$

We found the representations for π_+ and π_- in Lemma 2.3 by considering the overshoots above zero for the Markov chain of the so-called switching ladder heights, which is a particular example of random walks with switch at zero. This argument will be presented elsewhere. Here we give a different independent proof.

Proof of Lemma 2.3. From the Wiener–Hopf factorization (Feller [7, Chapter XII.3])

$$\mathbb{P}(X_1 \in \cdot) = \mathbb{P}(H_+ \in \cdot) + \mathbb{P}(\tilde{H}_1^- \in \cdot) - \mathbb{P}(H_1^+ \in \cdot) * \mathbb{P}(\tilde{H}_1^- \in \cdot) \tag{2.7}$$

it follows that for any $y \in \mathcal{Z}_d^+$,

$$\begin{aligned}\mathbb{P}(X_1 > y) &= \mathbb{P}(H_1^+ > y) - \int_{(y,\infty)} \mathbb{P}(\tilde{H}_1^- > y - z)\mathbb{P}(H_1^+ \in dz) \\ &= \int_{(y,\infty)} \mathbb{P}(\tilde{H}_1^- \leq y - z)\mathbb{P}(H_1^+ \in dz).\end{aligned}$$

Then from the identity $\mathbb{P}(\tilde{H}_1^- \leq u) = \mathbb{P}(\tilde{H}_1^- \neq 0)\mathbb{P}(H_1^- \leq u)$ for $u \in \mathcal{Z}_d^-$, we get

$$c_1\mathbb{P}(X_1 > y) = c_1\mathbb{P}(\tilde{H}_1^- \neq 0) \int_{\mathcal{Z}_d} \mathbb{P}(H_1^- \leq y - z)\mathbb{1}_{\mathcal{Z}_d^-}(y - z)\mathbb{P}(H_1^+ \in dz).$$

This establishes equality of densities with respect to λ_d of the measures given in the first assertion of Lemma 2.3. Indeed, the function on the l.h.s. is the density of π_+. For the r.h.s., notice that the integral has the form $\mathbb{E}f(y - H_1^+)$, where $f(x) := \mathbb{P}(H_1^- \leq x)\mathbb{1}_{\mathcal{Z}_d^-}(x)$ for $x \in \mathcal{Z}_d$. Recall a formula for the density of convolutions: for any random variable H supported on $\mathcal{Z}_d$ and any measure μ on $\mathcal{Z}_d$ with a bounded density g with respect to λ_d,

$$(\mu * \mathbb{P}(H \in \cdot))(dy) = [\mathbb{E}g(y - H)]\lambda_d(dy), \qquad y \in \mathcal{Z}_d. \tag{2.8}$$

This equality is evident for $d > 0$; for the absolutely continuous case $d = 0$, see e.g. Cohn [5, Proposition 10.1.12]. It remains to use (2.8) with $g = f$ and $H = H_1^+$.

The second assertion of Lemma 2.3 follows similarly. □

2.3 Derivation of π_+

Let us present a simple probabilistic argument that we used to *guess* the form of π_+. Assume that $\mathbb{E}X_1 = 0$, the variance of increments $\sigma^2 = \mathbb{E}X_1^2$ is finite and positive, and the random walk S is integer-valued and aperiodic, i.e. the distribution of $X_1 - a$ is arithmetic with span 1 for every $a \in \mathbb{Z}$. In this case $\mathcal{Z}_d^+ = \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Consider the number of up-crossings of the zero level by time n:

$$L_n^{\uparrow} := \sum_{i=0}^{n-1} \mathbb{1}(S_i < 0, S_{i+1} \geq 0) = \max\{k \geq 0 : T_k \leq n\}.$$

Assume that the chain O has an ergodic stationary distribution μ. Then by the ergodic theorem, for any $x, y \in \{z \in \mathbb{N}_0 : \mathbb{P}(X_1 > z) > 0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(O_i = y) = \lim_{n\to\infty} \frac{1}{L_n^{\uparrow}} \sum_{i=1}^{L_n^{\uparrow}} \mathbb{1}(O_i = y) = \mu(y), \quad \mathbb{P}_x\text{-a.s.} \tag{2.9}$$

On the other hand,

$$\begin{aligned}
\mathbb{E}_x\left[\frac{L_n^{\uparrow}}{\sqrt{n}} \cdot \frac{1}{L_n^{\uparrow}} \sum_{i=1}^{L_n^{\uparrow}} \mathbb{1}(O_i = y)\right] &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \mathbb{P}_x(S_i < 0, S_{i+1} = y) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \sum_{k=1}^{\infty} \mathbb{P}_x(S_i = -k)\mathbb{P}(X_1 = y + k) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(X_1 = y + k) \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \mathbb{P}_x(S_i = -k).
\end{aligned}$$

By the local central limit theorem, there exists a constant $c > 0$ such that for every integer i and $k \geq 1$ we have $\mathbb{P}_x(S_i = -k) \leq c/\sqrt{n}$, and also $\mathbb{P}_x(S_i = -k) \sim \frac{1}{\sqrt{2\pi i}\sigma}$ as $i \to \infty$. Hence from (2.9) and the dominated convergence theorem, we obtain

$$\mu(y) \lim_{n\to\infty} \mathbb{E}_x\left[\frac{L_n^{\uparrow}}{\sqrt{n}}\right] = \sum_{k=1}^{\infty} \mathbb{P}(X_1 = y + k) \Big(\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{1}{\sqrt{2\pi i}\sigma} \Big) = \sqrt{\frac{2}{\pi\sigma^2}} \mathbb{P}(X_1 > y).$$

Thus, $\mu = \pi_+$ in the special case considered above.

Therefore it is feasible that the distribution π_+ is stationary for the chain of overshoots O for general random walks but of course we need to prove this directly, and even for the case considered here.

2.4 Proof of Theorem 2.1

The main result of the section, Proposition 2.7 below, reveals a distributional symmetry hidden in the trajectory of an arbitrary oscillating random walk, which is key for the proof of Theorem 2.1.

Define new Markov transition kernels P and Q on $\mathcal{Z}_d$ as follows: for $x, y \in \mathcal{Z}_d$ let

$$P(x, dy) := \mathbb{P}_x(-U_1 - d \in dy), \quad Q(x, dy) := \mathbb{P}(X_1 - d \in dy + x | X_1 - d \geq x), \tag{2.10}$$

with the convention that $Q(x, dy) := \delta_0(dy)$ in the case when $\mathbb{P}(X_1 - d \geq x) = 0$; the choice of the delta measure is arbitrary and will not be relevant for what follows. The kernel P is defined in terms of the sign-changed first undershoot U_1, given in (1.3) above, which is shifted by d to ensure that $-U_1 - d$ may take value zero in the arithmetic case. The kernel Q corresponds to up-crossings of the zero level by the walk S. Clearly, for every $x \in \mathcal{Z}_d$, the transition probabilities $P(x, dy)$ and $Q(x, dy)$ are supported on $\mathcal{Z}_d^+$.

The sequences of overshoots $(O_n)_{n\geq 0}$ and shifted sign-changed undershoots $(-U_n - d)_{n\geq 0}$ are Markov chains with respective transition kernels PQ and QP. More precisely, for any probability measure μ on $\mathcal{Z}_d$ and any $n \in \mathbb{N}$,

$$\mathbb{P}_\mu(O_n \in dy) = [\mu(PQ)^n](dy), \quad \mathbb{P}_\mu(-U_n - d \in dy) = [\mu P(QP)^{n-1}](dy), \quad y \in \mathcal{Z}_d, \tag{2.11}$$

where $(QP)^0(x,dy) := \delta_x(dy)$, $(PQ)^n$ and $(QP)^{n-1}$ are the Chapman–Kolmogorov convolutions over $\mathcal{Z}_d$, and for any transition kernel T on $\mathcal{Z}_d$, we denoted $\mu T(dy) := \int_{\mathcal{Z}_d} T(z,dy)\mu(dz)$. A formal proof of the Markov property, which is not entirely obvious for the sequence of undershoots U, is given in a very general setting in [14, Lemma 2.1].

In the arithmetic case, we clearly have the equality

$$\lambda_d(dx)\mathbb{P}(X_1 - d \in dy + x) = \lambda_d(dy)\mathbb{P}(X_1 - d \in dx + y)$$

of measures on $\mathcal{Z}_d \times \mathcal{Z}_d$; we will also prove this identity for $d = 0$. Combined with the equality of measures $\mathbb{P}(X_1 - d \geq z)\lambda_d^+(dz) = \pi_+(dz)$ on $\mathcal{Z}_d$, this implies that the transition kernel Q is *reversible* with respect to π_+. Put differently, the *detailed balance* condition

$$\pi_+(dx)Q(x,dy) = \pi_+(dy)Q(y,dx), \qquad x,y \in \mathcal{Z}_d$$

holds true for the measures on $\mathcal{Z}_d \times \mathcal{Z}_d$ (which are supported on $\mathcal{Z}_d^+ \times \mathcal{Z}_d^+$). Surprisingly, the kernel P shares the same property. Put together, we have the following statement, which we will prove in full below in Section 2.4.1.

Proposition 2.6. *For any random walk S that oscillates, the kernels P and Q are reversible with respect to π_+.*

A direct corollary to this proposition is invariance of the measure π_+ for the Markov chains $(O_n)_{n\geq 0}$ and $(-U_n - d)_{n\geq 0}$, asserted by Theorem 2.1. A similar argument yields invariance of π_- for the chains $(O_n^\downarrow)_{n\geq 0}$ and $(-U_n^\downarrow - d)_{n\geq 0}$ (indeed, use (2.14) from Section 2.4.1 below and a kernel decomposition for these chains analogous to (2.11)). Thus, Theorem 2.1 follows from Proposition 2.6, which in turn is a direct corollary to Proposition 2.7 (see Section 2.4.1).

2.4.1 The time reversal argument

We now present a result concerning the entire trajectory of the random walk between up-crossings of the level zero. Our proof is based on a generalisation of the argument from Vysotsky [24, Lemma 1].

Recall from (1.2) (resp. (2.3)) that $T_m \geq 1$ (resp. $T_m^\downarrow \geq 1$) for any $m \in \mathbb{N}$ and any value of S_0.

Proposition 2.7. *For any random walk S that oscillates, for any $m \in \mathbb{N}$ we have*

$$\begin{aligned}&\mathbb{P}_{\pi_+}\big((S_0, S_1, \ldots, S_{T_m - 1}, 0, \ldots) \in \cdot\big)\\ &\qquad = \mathbb{P}_{\pi_+}\big((-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot\big)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&\mathbb{P}_{\pi_+}\big((S_0, S_1, \ldots, S_{T_m^\downarrow - 1}, 0, \ldots) \in \cdot\big)\\ &\qquad = \mathbb{P}_{\pi_-}\big((-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot\big).\end{aligned} \tag{2.13}$$

The choice of the value 0 in the random sequences in (2.12) and (2.13) is arbitrary and could be substituted by any constant. The constant is included in (2.12) and (2.13) to emphasise that the random vectors take values in an infinite dimensional space.

Furthermore, we stress that the equalities in (2.12) and (2.13) cease to hold if this constant value is substituted by the remaining part of the path of S. Note that (2.12) can be stated more elegantly as

$$\big(S_0, S_1, \ldots, S_{T_m-1}\big) \stackrel{d}{=} \big(-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d\big) \text{ under } \mathbb{P}_{\pi_+}.$$

Remark 2.8. Similarly, we have

$$\begin{aligned}\mathbb{P}_{\pi_-}&\big((S_0, S_1, \ldots, S_{T_m^\downarrow-1}, 0, \ldots) \in \cdot\big)\\ &= \mathbb{P}_{\pi_-}\big((-S_{T_m^\downarrow-1} - d, -S_{T_m^\downarrow-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot\big) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\mathbb{P}_{\pi_-}&\big((S_0, S_1, \ldots, S_{T_m-1}, 0, \ldots) \in \cdot\big)\\ &= \mathbb{P}_{\pi_+}\big((-S_{T_m^\downarrow-1} - d, -S_{T_m^\downarrow-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot\big). \end{aligned} \tag{2.15}$$

We first prove Proposition 2.6 and Remark 2.2, which are simple corollaries of Proposition 2.7.

Proof of Proposition 2.6. Reversibility of the P-kernel follows immediately by (2.12) with $m = 1$ since $U_1 = S_{T_1-1}$.

As explained above, reversibility of the Q-kernel follows from the equalities of measures

$$\lambda_d(dx)\mathbb{P}(X_1 - d \in dy + x) = \lambda_d(dy)\mathbb{P}(X_1 - d \in dx + y)$$

on $\mathcal{Z}_d \times \mathcal{Z}_d$ and $\mathbb{P}(X_1 - d \geq z)\lambda_d^+(dz) = \pi_+(dz)$ on $\mathcal{Z}_d$. The latter equality is trivial. The former one is equivalent to

$$\lambda_d(dx)\mathbb{P}(x + X_1 \in dy) = \lambda_d(dy)\mathbb{P}(y - X_1 \in dx), \qquad x, y \in \mathcal{Z}_d, \tag{2.16}$$

as follows from substituting y by $y - d$ using invariance of λ_d under shifts in $\mathcal{Z}_d$ and substituting x by $-x$ using the central symmetry of λ_d; cf. (2.24) below for the meaning of (2.16).

It suffices to check equality of measures (2.16) only for rectangular sets with Borel sides $A, B \subset \mathcal{Z}_d$. By Fubini's theorem and the mentioned shift invariance of λ_d,

$$\begin{aligned}\big[\lambda_d(dx)\mathbb{P}(x + X_1 \in dy)\big](A \times B) &= \int_{\mathcal{Z}_d^2} \mathbb{1}(x \in A, x + z \in B)\lambda_d(dx) \otimes \mathbb{P}(X_1 \in dz)\\ &= \int_{\mathcal{Z}_d} \lambda_d(A \cap (B - z))\mathbb{P}(X_1 \in dz)\\ &= \int_{\mathcal{Z}_d} \lambda_d\big(B \cap (A - z)\big)\mathbb{P}(-X_1 \in dz)\\ &= \big[\lambda_d(dx)\mathbb{P}(x - X_1 \in dy)\big](B \times A),\end{aligned}$$

where the last equality follows from the first two. This is exactly (2.16). □

Recall that Remark 2.2 asserts that

$$\mathbb{P}_{\pi_+}(O_1^\downarrow \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(O_1 \in \cdot) = \pi_+, \quad \mathbb{P}_{\pi_+}(-U_1^\downarrow - d \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(-U_1 - d \in \cdot) = \pi_+.$$

Proof of Remark 2.2. Fix $m = 1$. By (2.12), the random variables $-O_1^\downarrow - d = -S_{T_1^\downarrow} - d$ and $U_1^\downarrow = S_{T_1^\downarrow-1}$ have the same law under $\mathbb{P}_{\pi_+}$, hence $\mathbb{P}_{\pi_+}(O_1^\downarrow \in \cdot) = \mathbb{P}_{\pi_+}(-U_1^\downarrow - d \in \cdot)$. By (2.13), the law of $-U_1^\downarrow - d = -S_{T_1^\downarrow-1} - d$ under $\mathbb{P}_{\pi_+}$ is the same as the law of S_0 under $\mathbb{P}_{\pi_-}$, i.e. $\mathbb{P}_{\pi_+}(-U_1^\downarrow - d \in \cdot) = \pi_-$, and hence $\mathbb{P}_{\pi_+}(O_1^\downarrow \in \cdot) = \pi_-$. Similarly, by (2.14), we find $\mathbb{P}_{\pi_-}(O_1 \in \cdot) = \mathbb{P}_{\pi_-}(-U_1 - d \in \cdot)$. Finally, by (2.15), it holds $\mathbb{P}_{\pi_-}(-U_1 - d \in \cdot) = \pi_+$. □

We now prove the main statement of the section.

Proof of Proposition 2.7. Consider equality (2.12) in the case $m = 1$. Pick an arbitrary $k \in \mathbb{N}$ and define the time-reversal mapping $R_k : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ by

$$R_k(x_0, \dots, x_k) := (-x_k - d, \dots, -x_0 - d).$$

Introduce the random vector $K := (S_0, \dots, S_k)$ and note that (2.12) follows if we establish the equality of measures on $(\mathcal{Z}_d)^{k+1}$:

$$\mathbb{P}_{\pi_+}(K \in \cdot, T_1 = k+1) = \mathbb{P}_{\pi_+}(R_k(K) \in \cdot, T_1 = k+1). \tag{2.17}$$

Put

$$\tilde{\mathcal{Z}}_d^+ := \begin{cases} \mathcal{Z}_d^+ \setminus \{0\}, & \text{if } d = 0, \\ \mathcal{Z}_d^+, & \text{if } d > 0, \end{cases}$$

and denote $C_k := \cup_{i=0}^{k-1} (\tilde{\mathcal{Z}}_d^+)^i \times (\mathcal{Z}_d^-)^{k-1-i}$. Then $C'_k := \tilde{\mathcal{Z}}_d^+ \times C_k \times \mathcal{Z}_d^-$ is the set of sequences of length $k+1$ that start from $\tilde{\mathcal{Z}}_d^+$, down-cross the level zero exactly once, and in the non-arithmetic case have no zeroes. The general case $m \in \mathbb{N}$ follows analogously, with the only difference that the set C'_k should be defined to account for $2m-1$ crossings of the level zero.

Note that R_k is an invertible mapping on $\mathbb{R}^{k+1}$, and it is an involution. Further, $R_k(C_k) = C_k$ since $-\tilde{\mathcal{Z}}_d^+ - d = \mathcal{Z}_d^-$ in both cases $d = 0$ and $d > 0$. Similarly, $R_k(C'_k) = C'_k$, implying that $R_k(\mathbb{R}^{k+1} \setminus C'_k) = \mathbb{R}^{k+1} \setminus C'_k$. This gives

$$\mathbb{P}_{\pi_+}\big(R_k(K) \in \mathbb{R}^{k+1} \setminus C'_k, T_1 = k+1\big) = \mathbb{P}_{\pi_+}\big(K \in \mathbb{R}^{k+1} \setminus C'_k, T_1 = k+1\big) = 0. \tag{2.18}$$

The second equality is trivial in the arithmetic case. In the non-arithmetic case, it is due to the fact that K has density with respect to the Lebesgue measure on $\mathbb{R}^{k+1}$, which in turn holds true since in this case the measure π_+ has density with respect to the Lebesgue measure on $\mathbb{R}$.

By (2.18), if suffices to check equality (2.17) on rectangles of the form $B_0 \times B \times B_k$ with Borel sides $B_0 \subset \tilde{\mathcal{Z}}_d^+$, $B_k \subset \mathcal{Z}_d^-$ and $B \subset C_k$. Using the definition of π_+ and the fact that X_{k+1} is independent with K under $\mathbb{P}_{x_0}$ for every $x_0 \in \mathcal{Z}_d$, we obtain

$$\begin{aligned}
&\mathbb{P}_{\pi_+}\big(K \in B_0 \times B \times B_k, T_1 = k+1\big) \\
&= \int_{B_0} \mathbb{P}_{x_0}\big((S_1, \dots, S_k) \in B \times B_k, T_1 = k+1\big)\pi_+(dx_0) \\
&= \int_{B_0} \int_{B_k} \Big[\mathbb{P}_{x_0}\big((S_1, \dots, S_{k-1}) \in B, T_1 = k+1 \big| S_k = x_k\big)\mathbb{P}(X_1 > x_0)\Big] \mathbb{P}_{x_0}(S_k \in dx_k)\lambda_d(dx_0) \\
&= \int_{B_0 \times B_k} f_B(x_0, x_k)\mathbb{P}_{\lambda_d}((S_0, S_k) \in dx_0 \otimes dx_k),
\end{aligned} \tag{2.19}$$

where

$$f_B(x_0, x_k) := \mathbb{P}_{x_0}\big((S_1, \dots, S_{k-1}) \in B \big| S_k = x_k\big)\mathbb{P}(X_1 > x_0)\mathbb{P}(X_1 \ge -x_k)$$

for $(x_0, x_k) \in \tilde{\mathcal{Z}}_d^+ \times \mathcal{Z}_d^-$. Then we use equality (2.19) to get

$$\begin{aligned}
&\mathbb{P}_{\pi_+}\big(R_k(K) \in B_0 \times B \times B_k, T_1 = k+1\big) \\
&= \mathbb{P}_{\pi_+}\big(K \in (-B_k - d) \times R_{k-2}(B) \times (-B_0 - d), T_1 = k+1\big) \\
&= \int_{(-B_k - d) \times (-B_0 - d)} f_{R_{k-2}(B)}(x_0, x_k)\mathbb{P}_{\lambda_d}((S_0, S_k) \in dx_0 \otimes dx_k) \\
&= \int_{B_0 \times B_k} f_{R_{k-2}(B)}(R_1(x_0, x_k))\mathbb{P}_{\lambda_d}\big(R_1(S_0, S_k) \in dx_0 \otimes dx_k\big),
\end{aligned} \tag{2.20}$$

where in the last equality we used the change of variables formula, the fact that R_k is an involution, and the equality $(-B_k - d) \times (-B_0 - d) = R_1(B_0, B_k)$.

Let us simplify the integrand under the last integral in (2.20). We have

$$\begin{aligned}
&\mathbb{P}_{-x_k-d}\big((S_1,\dots,S_{k-1}) \in R_{k-2}(B)\big|S_k = -x_0 - d\big)\\
&= \mathbb{P}_0\big((S_1 - x_k - d,\dots,S_{k-1} - x_k - d) \in R_{k-2}(B)\big|S_k = x_k - x_0\big)\\
&= \mathbb{P}_0\big(R_{k-2}(S_1 - S_k - x_0 - d,\dots,S_{k-1} - S_k - x_0 - d) \in B\big|S_k = x_k - x_0\big)\\
&= \mathbb{P}_0\big((S_k - S_{k-1} + x_0,\dots,S_k - S_1 + x_0) \in B\big|S_k + x_0 = x_k\big).
\end{aligned}$$

The well-known duality principle for random walks states that the random vectors $(S_1,\dots,S_k)$ and $(S_k - S_{k-1},\dots,S_k - S_1, S_k)$ have the same law under $\mathbb{P}_0$. By a conditional version of this distributional identity, for every $x_0 \in \mathcal{Z}_d$ and $\mathbb{P}_{x_0}(S_k \in \cdot)$-a.e. $x_k \in \mathcal{Z}_d$,

$$\mathbb{P}_{-x_k-d}\big((S_1,\dots,S_{k-1}) \in R_{k-2}(B)\big|S_k = -x_0 - d\big) = \mathbb{P}_{x_0}\big((S_1,\dots,S_{k-1}) \in B\big|S_k = x_k\big). \tag{2.21}$$

By the definition of f_B, this gives

$$\begin{aligned}
f_{R_{k-2}(B)}(R_1(x_0,x_k)) &= f_{R_{k-2}(B)}(-x_k - d, -x_0 - d))\\
&= \mathbb{P}_{x_0}\big((S_1,\dots,S_{k-1}) \in B\big|S_k = x_k\big)\mathbb{P}(X_1 > -x_k - d)\mathbb{P}(X_1 \ge -x_0 - d).
\end{aligned}$$

Thus, using in the non-arithmetic case the fact that a distribution function can have at most countably many jumps, we get

$$f_B(x_0,x_k) = f_{R_{k-2}(B)}(R_1(x_0,x_k)), \qquad \mathbb{P}_{\lambda_d}((S_0,S_k) \in \cdot)\text{-a.e. } (x_0,x_k). \tag{2.22}$$

Hence we see from (2.19), (2.20), and (2.22) combined with (2.18), that equality (2.17) will follow once we show the following equality of measures on $\mathcal{Z}_d^+ \times \mathcal{Z}_d^-$:

$$\mathbb{P}_{\lambda_d}((S_0,S_k) \in \cdot) = \mathbb{P}_{\lambda_d}(R_1(S_0,S_k) \in \cdot). \tag{2.23}$$

By translation invariance of λ_d under shifts in $\mathcal{Z}_d$,

$$\mathbb{P}_{\lambda_d}(R_1(S_0,S_k) \in \cdot) = \mathbb{P}_{\lambda_d}((-S_k - d, -S_0 - d) \in \cdot) = \mathbb{P}_{\lambda_d}((-S_k, -S_0) \in \cdot),$$

and thus the claim (2.23) reduces to

$$\mathbb{P}_{\lambda_d}((S_0,S_k) \in \cdot) = \mathbb{P}_{\lambda_d}((-S_k,-S_0) \in \cdot), \tag{2.24}$$

which means that the random walk $-S$ is dual to S with respect to λ_d. To prove this property, note that by the shift invariance of λ_d under shifts in $\mathcal{Z}_d$, the equality (2.24) of measures on $\mathcal{Z}_d^2 = \mathcal{Z}_d \times \mathcal{Z}_d$ is equivalent to $\mathbb{P}_{\lambda_d}((S_0,S_k) \in \cdot) = \mathbb{P}_{\lambda_d}((S_0 - S_k', S_0) \in \cdot)$. This is exactly (2.16) with X_1 replaced by S_k'. Thus, (2.12) is proved for $m = 1$. As mentioned above, the general case $m \in \mathbb{N}$ follows analogously, with the only difference that the set C_k' shall account for $2m - 1$ crossings of the level zero.

Consider now (2.13). We need to prove that the law of $(S_0, S_1, \dots, S_{T_m^\downarrow - 1})$ under $\mathbb{P}_{\pi_+}$ equals the law of $(-S_{T_m - 1} - d, -S_{T_m - 2} - d, \dots, -S_0 - d)$ under $\mathbb{P}_{\pi_-}$. Similarly to the proof of (2.12), by the duality principle for random walks this reduces to the equality

$$\mathbb{P}_{\pi_+}((S_0,S_k) \in \cdot, S_{k+1} < 0) = \mathbb{P}_{\pi_-}(R_1(S_0,S_k) \in \cdot, S_{k+1} \ge 0)$$

of measures on $\tilde{\mathcal{Z}}_d^+ \times \tilde{\mathcal{Z}}_d^+$ for $k \in \mathbb{N}_0$. Use the definitions of π_+, π_-, R_1 to write this as

$$\begin{aligned}
&\mathbb{P}_{\lambda_d}((S_0,S_k) \in dx_0 \otimes dx_k)\mathbb{P}(X_1 > x_0)\mathbb{P}(X_1 < -x_k)\\
&= \mathbb{P}_{\lambda_d}(R_1(S_0,S_k) \in dx_0 \otimes dx_k)\mathbb{P}(X_1 \ge x_0 + d)\mathbb{P}(X_1 \le -x_k - d).
\end{aligned}$$

This equality holds by (2.23) and the fact that $\mathbb{P}(X_1 > x) = \mathbb{P}(X_1 \ge x + d)$ for λ_d-a.e. x. □

3 Convergence to the stationary distribution

For the rest of the paper we assume (1.1) and investigate convergence in total variation of the law of O_n to the probability distribution π_+ as $n \to \infty$.

In the non-arithmetic case convergence in the total variation norm requires additional assumptions on the law of the increments of S. We say that the distribution of the increment X_1 is *spread out* if $\mathbb{P}_0(S_k \in \cdot)$ is non-singular with respect to the Lebesgue measure for some $k \geq 1$. It is clear that this assumption is necessary for the total variation convergence to π_+ of the law of the chain O_n starting from a point. In fact, if this assumption is violated in the non-arithmetic case, then $\|\mathbb{P}_x(O_n \in \cdot) - \pi_+(\cdot)\|_{\mathrm{TV}} = 1$ for every $x \in \mathbb{R}$ and $n \geq 1$ since π_+ has density. In this sections we will show that that the spread out assumption is actually sufficient for the total variation convergence. Let us mention that spread out distributions arise often in the context of renewal theory, see Asmussen [1, Section VII].

Theorem 3.1. *Assume* (1.1) *and that the distribution of X_1 is either arithmetic or spread out. Then*

$$\lim_{n\to\infty} \|\mathbb{P}_x(O_n \in \cdot) - \pi_+(\cdot)\|_{\mathrm{TV}} = 0 \quad \textit{for all } x \in \mathcal{Z}_d.$$

A standard application of the dominated convergence theorem yields another proof of the fact (given in full generality by [14, Theorem 4.2]) that, under the assumptions of Theorem 3.1, π_+ is the unique stationary distribution of the chain $(O_n)_{n\geq 0}$ in the class of all probability laws on $\mathcal{Z}_d$, including the ones singular with respect to π_+.

The convergence in Theorem 3.1 may fail for every starting point $x \in \mathcal{Z}_0$ in the case of general non-arithmetic distributions of increments, e.g. for discrete non-arithmetic distributions, but π_+ remains the unique stationary distribution of O by [14, Corollary 4.2]. Therefore one may argue that the total variation metric is too fine for the study of convergence of the chain of overshoots for general zero mean random walks. It is feasible that the convergence holds in other metrics under less restrictive assumptions than those in Theorem 3.1 but we did not succeed in proving results of such type; see the discussion in Section 5 below.

It is well known that under the spread out assumption on the increments of a random walk, a successful coupling of the walks started at arbitrary distinct points $x, y \in \mathcal{Z}_0$ can be defined, implying in particular $\lim_{n\to\infty} \|\mathbb{P}_x(S_n \in \cdot) - \mathbb{P}_y(S_n \in \cdot)\|_{\mathrm{TV}} = 0$, see e.g. Theorem 6.1 of Chapter 3 in Thorisson [22]. However, this coupling yields only a shift-coupling [22, Section 3.1] of the chains of overshoots started at x and y. Thus only the Cesaro total variation convergence [22, Section 3.2] of O can be deduced from these results, which is weaker than the convergence stated in Theorem 3.1. Our proof of Theorem 3.1 rests on the crucial property of the Markov chain $(O_n)_{n\geq 0}$ stated below in Proposition 3.2, implying that a successful coupling of the chains of overshoots started at any distinct levels can be constructed for any span $d \in [0, \infty)$. We do not exhibit a coupling construction in this paper but instead apply Theorem 4 in Roberts and Rosenthal [18], which is established using this coupling.

For any measure μ on $\mathcal{Z}_d$, denote respectively by μ^a and μ^s its absolutely continuous and singular components with respect to λ_d. We will slightly abuse this notation for distributions of random variables and write, say, $\mathbb{P}^a_x(O_1 \in \cdot)$ instead of $(\mathbb{P}_x(O_1 \in \cdot))^a$. We reserve the term "density" to mean the density with respect to the Lebesgue measure λ_0 without referring to the measure. The set $\mathcal{X}_+ := [0, M_+) \cap \mathcal{Z}_d$, where $M_+ := \sup(\operatorname{supp}(X_1))$, is the actual state space of the Markov chain of overshoots: for any $x \in \mathcal{Z}_d$ and $n \in \mathbb{N}$ we have $\mathbb{P}_x(O_n \in \mathcal{X}_+) = 1$. Moreover, the equality $\pi_+(\mathcal{X}_+) = 1$ holds true.

Proposition 3.2. *Assume* (1.1) *and that the distribution of X_1 is either arithmetic or spread out. Then the measures $\mathbb{P}^a_x(O_1 \in \cdot)$ and $\pi_+(\cdot)$ are equivalent for any $x \in \mathcal{Z}_d$. Put* *differently, for any $x \in \mathcal{Z}_d$ there exists a version of the density $\frac{d}{d\lambda_d}\mathbb{P}^a_x(O_1 \in dy)$ that is strictly positive for all $y \in \mathcal{X}_+$.*

Proof of Theorem 3.1. Proposition 3.2 implies that with positive probability, the chain of overshoots visits in a single step any Borel set $A \subseteq \mathcal{Z}_d$ satisfying $\pi_+(A) > 0$. This means that the Markov chain $(O_n)_{n\geq 0}$ is π_+-irreducible and aperiodic in the sense of Meyn and Tweedie [13, Sections 4.2 and 5.4]. By Theorem 2.1 above, $(O_n)_{n\geq 0}$ has a stationary distribution π_+. Then Theorem 4 in Roberts and Rosenthal [18], which applies to ψ-irreducible aperiodic Markov chains with a stationary distribution on a general state space with a countably generated σ-algebra, implies the total variation convergence in Theorem 3.1 for π_+-a.e. $x \in \mathcal{Z}_d$.

Since $\mathbb{P}_x(O_1 \in \mathcal{X}_+) = 1$ for every $x \in \mathcal{Z}_d$, we will conclude the proof of Theorem 3.1 if we show that the non-convergence set

$$N := \Big\{x \in \mathcal{X}_+ : \limsup_{n\to\infty} \|\mathbb{P}_x(O_n \in \cdot) - \pi_+(\cdot)\|_{\text{TV}} > 0\Big\}$$

is empty. In the arithmetic case ($d > 0$) this is clear by the fact that every point of $\mathcal{X}_+$ has positive π_+-measure and $\pi_+(N) = 0$. In the non-arithmetic case ($d = 0$) first note that since the Borel σ-algebra on $\mathcal{X}_+$ is countably generated, the function $x \mapsto \|\mathbb{P}_x(O_n \in \cdot) - \pi_+(\cdot)\|_{\text{TV}}$ is measurable for every $n \in \mathbb{N}$ by Roberts and Rosenthal [17, Appendix], making the set N measurable. Thus the claim will follow by a standard application of the strong Markov property and the dominated convergence theorem if we show that the chain $(O_n)_{n\geq 0}$ hits the convergence set $\mathcal{X}_+ \setminus N$ with probability one when started in N. Put differently, we need to prove that $\mathbb{P}_x(O_n \in N, \forall n \in \mathbb{N}) = 0$ for every $x \in N$.

Since $\pi_+(N) = \lambda_0(N) = 0$ we have $\mathbb{P}_x(S_m \in N) = \mathbb{P}^s_x(S_m \in N)$ for all $m \in \mathbb{N}$. Hence,

$$\begin{aligned} &\mathbb{P}_x(O_n \in N, \forall n \in \mathbb{N}) \leq \liminf_{n\to\infty} \mathbb{P}_x(O_n \in N) \\ &\leq \liminf_{n\to\infty} \sum_{m=2n}^{\infty} \mathbb{P}_x(S_m \in N) \leq \liminf_{n\to\infty} \sum_{m=2n}^{\infty} \mathbb{P}^s_x(S_m \in \mathbb{R}), \end{aligned} \tag{3.1}$$

where in the second inequality we used the identity $O_n = S_{T_n}$ and the fact that $T_n \geq 2n$ for $x \geq 0$, cf. (1.2) and (1.3). By the definition of spread out distributions, we have $\mathbb{P}^s_x(S_k \in \mathbb{R}) = \mathbb{P}^s(S'_k \in \mathbb{R}) < 1$ for some $k \geq 1$. Then, using that the convolution of an absolutely continuous measure with any other measure is absolutely continuous, we get

$$\mathbb{P}^s(S'_m \in \mathbb{R}) = \Big(\big(\mathbb{P}^s(S'_k \in \cdot) + \mathbb{P}^a(S'_k \in \cdot)\big)^{*\lfloor \frac{m}{k} \rfloor} * \mathbb{P}(S'_{m-k\lfloor \frac{m}{k} \rfloor} \in \cdot)\Big)^s(\mathbb{R}) \leq \big(\mathbb{P}^s(S'_k \in \mathbb{R})\big)^{\lfloor \frac{m}{k} \rfloor}$$

for any integer $m \geq 1$, where $\lfloor c \rfloor$ denotes the largest non-negative integer smaller or equal to a $c \geq 0$. Hence the sequence $\mathbb{P}^s_x(S_m \in \mathbb{R})$, which equals $\mathbb{P}^s(S'_m \in \mathbb{R})$, decays exponentially fast to zero as $m \to \infty$, and it follows that the last bound in (3.1) is zero. □

Proof of Proposition 3.2. Pick any $x \in \mathcal{Z}_d$ and denote by y an arbitrary element in $\mathcal{X}_+$. Consider two cases.

<u>Arithmetic distributions.</u> We need to prove that $\mathbb{P}_x(O_1 = y) > 0$.

Since $y < M_+$, there exists a $z \in \mathcal{Z}_d^+$ such that $z > y$ and $\mathbb{P}(X_1 = z) > 0$. Furthermore, it follows from the definition of $\mathcal{Z}_d$ that there exists an integer $k \geq 1$ such that $\mathbb{P}_x(S_k = y - z) > 0$; see, e.g., Spitzer [20, Propositions 2.1 and 2.5]. Then

$$\mathbb{P}_x(O_1 = y) \geq \mathbb{P}_x(S_k = y - z, T_1 > k) \cdot \mathbb{P}(X_1 = z), \tag{3.2}$$

and it remains to show that the first factor on the r.h.s. is positive.

Denote by Sym(k) the symmetric group on the set $\{1,\dots,k\}$. For any permutation $\sigma \in$ Sym(k), define a new random walk $S(\sigma)=(S_n(\sigma))_{n\ge 0}$ by $S_n(\sigma):=S_0+X_{\sigma(1)}+\ldots+X_{\sigma(n)}$ for $1\le n\le k$ and $S_n(\sigma):=S_n$ for $n\ge k$. Denote by $T_1(\sigma)$ the first up-crossing time of the level zero by $S(\sigma)$ (cf. (1.2)), and let ξ be the number of negative terms among $X_1,\dots,X_k$.

Note that on the event $A_\sigma := \{\xi\ge 1, X_{\sigma(1)}<0,\dots,X_{\sigma(\xi)}<0\}\cup\{\xi=0\}$ the sequence $(S_n(\sigma))_{n\in\{\sigma(\xi),\dots,k\}}$ is non-decreasing (on $\{\xi=0\}$, we define $\sigma(\xi):=0$). Then, since $y-z<0$ and $S_k=S_k(\sigma)$, we have

$$\{S_k=y-z\}\cap A_\sigma \subset \{S_k(\sigma)=y-z, T_1(\sigma)>k\}. \tag{3.3}$$

Recall that the cardinality of Sym(k) is $k!$ and note that

$$\frac{1}{k!}\sum_{\sigma\in \text{Sym}(k)} \mathbb{1}(A_\sigma)=\mathbb{1}(\xi=0)+\mathbb{1}(\xi>0)\xi!(k-\xi)!/k! = 1/\binom{k}{\xi} \ge 1/\binom{k}{\lfloor k/2\rfloor}. \tag{3.4}$$

Since the laws of the random walks S an $S(\sigma)$ coincide for all $\sigma\in$ Sym(k), we get

$$\begin{aligned}\mathbb{P}_x(S_k=y-z, T_1>k) &= \frac{1}{k!}\mathbb{E}_x\Big[\sum_{\sigma\in\text{Sym}(k)}\mathbb{1}(S_k(\sigma)=y-z,T_1(\sigma)>k)\Big]\\ &\ge \frac{1}{k!}\mathbb{E}_x\Big[\mathbb{1}(S_k=y-z)\sum_{\sigma\in\text{Sym}(k)}\mathbb{1}(A_\sigma)\Big]\\ &\ge \mathbb{P}_x(S_k=y-z)/\binom{k}{\lfloor k/2\rfloor}>0, \end{aligned}\tag{3.5}$$

where the first inequality holds by (3.3) and the second by (3.4). Combined with (3.2), this proves $\mathbb{P}_x(O_1=y)>0$, and hence the proposition holds in the arithmetic case.

Spread out distributions. We say that measures μ and ν on $\mathcal{Z}_0=\mathbb{R}$ satisfy

$$\mu(du)\ge\nu(du) \text{ on an interval } I\subset\mathbb{R}$$

if $\mu(B)\ge\nu(B)$ for any Borel set $B\subset I$. Note that $\mu(du)\ge c\lambda_0(du)$ on I implies $\mu^a(du)\ge c\lambda_0(du)$ on I. In this case there exists a version of the density of μ^a which is bounded from below on I by the positive constant c.

Since the distribution of X_1 is spread out, there exist $\varepsilon_1,h>0$, an integer $k_1\ge 1$, and a real a such that $\mathbb{P}_0(S_{k_1}\in du)\ge\varepsilon_1\lambda_0(du)$ on $[a,a+2h]$; see the proof of Proposition 5.3.1 in Meyn and Tweedie [13]. By the Chung–Fuchs theorem, the zero mean random walk S is topologically recurrent. Hence for any $b\in\mathbb{R}$ (to be specified later) there exists $k_2=k_2(b-x)\in\mathbb{N}$ such that

$$\varepsilon_2=\varepsilon_2(b-x):=\mathbb{P}_x\big(S_{k_2}\in[b-a-h,b-a]\big)>0. \tag{3.6}$$

Let $k=k(b-x):=k_2(b-x)+k_1$. Then, for any $v\in[b-a-h,b-a]$, we have $\mathbb{P}_0(S_{k_1}\in du-v)\ge\varepsilon_1\lambda_0(du)$ on $[b,b+h]$, since $u-v\in[a,a+2h]$ and the Lebesgue measure λ_0 is invariant under translations. Hence on the interval $[b,b+h]$ it holds

$$\begin{aligned}\mathbb{P}_x(S_k\in du)&\ge\mathbb{P}_x(b-a-h\le S_{k_2}\le b-a, S_k\in du)\\ &=\int_{[b-a-h,b-a]}\mathbb{P}_x(S_{k_2}\in dv)\mathbb{P}_0(S_{k_1}\in du-v)\ge\varepsilon_1\varepsilon_2\lambda_0(du).\end{aligned}$$

In particular, the density of $\mathbb{P}^a_x(S_k\in\cdot)$ is bounded below by $\varepsilon_1\varepsilon_2$ on $[b,b+h]$.

Since $y<M_+$, we can choose $z>y$ such that $\varepsilon_3=\varepsilon_3(y):=\mathbb{P}(X_1\in[z,z+h/2])>0$. Set $b':=y-z-3h/4$ and $h':=\min(h,y-z-3h/4)$, and let $k':=k_2(b'-x)+k_1$ and

$\varepsilon_2' := \varepsilon_2(b' - x)$ as in (3.6). Then $\mathbb{P}_x(S_{k'} \in du) \geq \varepsilon_1\varepsilon_2'\lambda_0(du)$ on $[b', b' + h]$. Moreover, substituting the events $\{S_k = y - z\}$ and $\{S_k(\sigma) = y - z\}$, where $\sigma \in \mathrm{Sym}(k)$, in the proof of the arithmetic case above by $\{S_{k'} \in [b', b' + h')\}$ and $\{S_{k'}(\sigma) \in [b', b' + h')\}$, where $\sigma \in \mathrm{Sym}(k')$ and $b' + h' \leq 0$, yields a bound analogous to (3.5):

$$\mathbb{P}_x(S_{k'} \in du, T_1 > k') \geq \frac{\varepsilon_1\varepsilon_2'}{\binom{k'}{\lfloor k'/2 \rfloor}}\lambda_0(du) \quad \text{on } [b', b' + h'). \tag{3.7}$$

On the event $\{S_{k'} \in [b', b' + h'), T_1 > k', X_{k'+1} \geq z\}$ we have $O_1 = S_{k'} + X_{k'+1}$. The Markov property at k' implies that on the interval $[y - h/4, \min(z, y + h/4))$ we have

$$\begin{aligned}\mathbb{P}_x(O_1 \in dv) &\geq \int_{[z,z+h/2]} \mathbb{P}_x(S_{k'} \in dv - u, T_1 > k')\mathbb{P}(X_{k'+1} \in du)\\ &\geq \frac{\varepsilon_1\varepsilon_2'}{\binom{k'}{\lfloor k'/2 \rfloor}}\mathbb{P}(X_1 \in [z, z + h/2])\lambda_0(dv). \end{aligned} \tag{3.8}$$

The second inequality holds by (3.7) and the translation invariance of λ_0 since, for $v \in [y - h/4, \min(z, y + h/4))$ and $u \in [z, z + h/2]$, we have $v - u \in [b', b' + h')$. Since we can partition $\mathcal{X}_+$ by a countable subcollection of the intervals $\{[y - h/4, \min(z, y + h/4)) : y \in \mathcal{X}_+\}$, by (3.8) there exists a version of the density $p(x, \cdot)$ of $\mathbb{P}_x^a(O_1 \in \cdot)$ satisfying

$$p(x, y) \geq \frac{\varepsilon_1\varepsilon_2'\varepsilon_3}{\binom{k'}{\lfloor k'/2 \rfloor}} > 0 \qquad \text{for all } y \in \mathcal{X}_+. \tag{3.9}$$

□

4 Rate of convergence to the stationary distribution

In this section we present results on the rate of convergence in Theorem 3.1. We will use the following norm: for any function $f : \mathcal{Z}_d^+ \to [1, \infty)$, the *$f$-norm* of a signed measure μ on $\mathcal{Z}_d^+$ is

$$\|\mu\|_f := \sup_{g:|g|\leq f} \int_{\mathcal{Z}_d^+} g(x)\mu(dx).$$

In particular, for $f \equiv 1$ the following relationship with the total variation norm holds: $\|\mu\|_f = 2\|\mu\|_{\mathrm{TV}}$. Clearly, convergence in any f-norm is stronger than the total variation convergence. We will only need the V_γ-norms, where $V_\gamma(x) := 1 + x^\gamma$ with $\gamma \geq 0$.

Further, define the set of bivariate parameters

$$\mathcal{I} := \{(\alpha, \beta) : 1 < \alpha < 2, |\beta| < 1\}.$$

For a random variable X, we write $X \in \mathcal{D}(\alpha, \beta)$ for a pair $(\alpha, \beta) \in \mathcal{I}$ if the distribution of X belongs to the domain of attraction of a strictly stable law with the characteristic function

$$\chi_{\alpha,\beta}(t) = \exp\bigl(-c|t|^\alpha(1 - i\beta\,\mathrm{sign}(t)\tan(\pi\alpha/2))\bigr)$$

for some $c > 0$. Denote $\mathcal{D} := \cup_{(\alpha,\beta)\in\mathcal{I}}\mathcal{D}(\alpha, \beta)$. The quantity

$$p := 1/2 + (\pi\alpha)^{-1}\arctan(\beta\tan(\pi\alpha/2)),$$

which is called the *positivity parameter* of the stable law, ranges over the open interval $(1 - 1/\alpha, 1/\alpha)$ on $\mathcal{I}$, see in Bertoin [2, Section 8.1].

Theorem 4.1. *Assume* (1.1) *and that the distribution of X_1 is either arithmetic or spread out. In addition, assume either $\mathbb{E}X_1^2 < \infty$ with $\gamma \in \{0, 1\}$ or $X_1 \in \mathcal{D}$ with $\gamma \in (0, \min\{\alpha p, \alpha(1 - p)\})$. Then there exist constants $r \in (0, 1)$ and $c_1 > 0$ such that*

$$\|\mathbb{P}_x(O_n \in \cdot) - \pi_+(\cdot)\|_{V_\gamma} \leq c_1(1 + x^\gamma)r^n, \qquad x \in \mathcal{Z}_d^+. \tag{4.1}$$

Equation (4.1) with $\gamma = 0$ translates to a uniform (in x) convergence at a geometric rate in the total variation norm. Thus the chain of overshoots is *uniformly ergodic* if the increments have finite variance; see Meyn and Tweedie [13, Theorem 16.0.2].

Our proof of Theorem 4.1 rests on two statements. The first can be viewed as a uniform version of Proposition 3.2 stated in a slightly different form to avoid measurability issues.

Proposition 4.2. *Under the assumptions of Theorem 4.1, for any $K > 0$ in the case $X_1 \in \mathcal{D}$ and for $K = \infty$ in the case $\mathbb{E}X_1^2 < \infty$, there exists a measurable function $g_K : \mathcal{X}_+ \to (0, \infty)$ such that*

$$\mathbb{P}_x(O_1 \in B) \geq \int_B g_K(y)\lambda_d(dy) \qquad \textit{for all } x \in \mathcal{Z}_d^+ \cap [0, K) \textit{ and Borel sets } B \subset \mathcal{X}_+. \quad (4.2)$$

Remark 4.3. Note that (4.2) implies $\mathbb{P}_x(O_1 \in B) \geq \int_B g_K(y)\lambda_d(dy) > 0$ for any Borel set B with $\lambda_d(B) > 0$. In particular, every compact set $C \subset \mathcal{Z}_d^+$ with non-empty interior in $\mathcal{Z}_d^+$ (or the whole set $\mathcal{Z}_d^+$ in the finite variance case) is *small* with respect to the measure $g_{\mathrm{diam}(C)}(y)\mathbb{1}_{\mathcal{X}_+}(y)\lambda_d(dy)$; see Meyn and Tweedie [13, Section 5.2] for the definition of small sets. The proposition also yields that the Markov chain $(O_n)_{n\geq 0}$ is *strongly aperiodic* and satisfies the *minorization condition*, cf. respective Sections 5.4 and 5.1 in [13].

Remark 4.4. Our proof, based on Stone's local limit theorem, actually implies that the inequality in (4.2) with finite K is also valid for asymptotically stable distributions of increments with $1 < \alpha < 2, |\beta| = 1$ and with $\alpha = 2$. Moreover, it is plausible that (4.2) holds under assumptions of Proposition 3.2, i.e. without any assumptions on the tail behaviour of X_1 beyond (1.1).

Second, we need the following *geometric drift condition*. We will prove it using results of renewal theory.

Proposition 4.5. *Under the assumptions of Theorem 4.1, there exist constants $\rho \in (0, 1)$ and $L > 0$ such that*

$$\mathbb{E}_x O_1^\gamma \leq \rho x^\gamma + L, \qquad x \in \mathcal{Z}_d^+. \quad (4.3)$$

Put together, Propositions 4.2 and 4.5 imply Theorem 4.1 via Theorems 15.0.1 and 16.0.2 and Proposition 5.5.3 in Meyn and Tweedie [13].

Proof of Proposition 4.2. The case $X_1 \in \mathcal{D}(\alpha, \beta)$. In the arithmetic case the set $\mathcal{Z}_d^+ \cap [0, K)$ has a finite number of elements and the claim follows from Proposition 3.2. For spread out distributions we have $d = 0$, implying $\mathcal{Z}_0^+ \cap [0, K) = [0, K)$, and it is clearly sufficient to prove that there exist a measurable function $g_K : \mathcal{X}_+ \to (0, \infty)$ and for every x, a version $p(x, y)$ of the density of $\mathbb{P}_x^a(O_1 \in dy)$ such that

$$\inf_{x \in [0,K)} p(x, y) \geq g_K(y) > 0 \qquad \text{for all } y \in \mathcal{X}_+. \quad (4.4)$$

We will do this by refining the argument in the proof of Proposition 3.2.

Pick $y \in \mathcal{X}_+$ and consider the estimate in (3.9). Note that ε_1 does not depend on x and y while ε_3 depends on y only through the choice of $z > y$. By decomposing $\mathcal{X}_+$ into a pair-wise disjoint collection of countably many bounded half-open intervals and choosing the same z for all y in each of the intervals makes $y \mapsto \varepsilon_3(y) = \mathbb{P}(X_1 \in [z, z + h/2])$ a measurable function of y. Therefore it suffices to check that ε_2' can be bounded away from zero and k' can be bounded from above, both uniformly in $x \in [0, K)$ and y in each of the intervals in the partition of $\mathcal{X}_+$. These claims will follow once we establish a refined version of (3.6): for any compact interval I in $\mathbb{R}$ and $h > 0$, there exists an integer $m \geq 1$ such that

$$\inf_{x \in [0,K), u \in I} \mathbb{P}_x(S_m \in [u, u+h]) > 0. \tag{4.5}$$

Possibly the easiest way to prove (4.5) is to apply Stone's local limit theorem which holds for non-lattice asymptotically stable distributions [21, Corollary 1]: if the sequence $(b_n)_{n\geq 1}$ tending to infinity is such that S_n/b_n converges weakly to a strictly stable law with the characteristic function $\chi_{\alpha,\beta}$ given above, then

$$\mathbb{P}_x(S_n \in [u, u+h)) = \mathbb{P}_0(S_n \in [u-x, u-x+h)) = (hp_{\alpha,\beta}(0) + o(1))b_n^{-1}$$

as $n \to \infty$ uniformly in $x \in [0, K)$ and $u \in I$, where $p_{\alpha,\beta}$, the density of the stable law with the characteristic function $\chi_{\alpha,\beta}$, is strictly positive and continuous at 0 for $(\alpha, \beta) \in \mathcal{I}$. Hence the inequality in (4.5) holds for all n sufficiently large.

The case $\mathbb{E}X_1^2 < \infty$. Note that in this case the above proof implies (4.2) for any finite $K > 0$. In order to construct $g_\infty : (0, \infty) \to \mathcal{X}_+$, let $T_{(-\infty,L)} := \min\{n \geq 0 : S_n < L\}$ be the moment of the first entrance of the walk $(S_n)_{n\geq 0}$ to the half-line $(-\infty, L)$, where $L := d + 1 > 0$. For any Borel set B in $\mathcal{X}_+$ we have

$$\begin{aligned}\mathbb{P}_x(O_1 \in B) &= \int_{(-\infty,L)} \mathbb{P}_z(O_1 \in B)\mathbb{P}_x(S_{T_{(-\infty,L)}} \in dz)\\ &\geq \int_{(0,L)} \mathbb{P}_z(O_1 \in B)\mathbb{P}_x(S_{T_{(-\infty,L)}} \in dz)\\ &\geq \mathbb{P}_x\big(S_{T_{(-\infty,L)}} \in (0, L)\big) \int_B g_L(y)\lambda_d(dy)\end{aligned}$$

where g_L is the lower bound in (4.4) that corresponds to the interval $(0, L)$.

By the definition of $O_1^\downarrow$ in (2.4), we have

$$\mathbb{P}_x\big(S_{T_{(-\infty,L)}} \in (0, L)\big) = \mathbb{P}_{x-L}\big(O_1^\downarrow \in (-L, 0)\big).$$

By (2.5), under the assumption $\mathbb{E}X_1^2 < \infty$, $\mathbb{P}_x(O_1^\downarrow \in \cdot)$ converges weakly as $x \to \infty$ to a distribution which assigns positive mass to $(-L, 0)$. Hence there exist constants $c_0, K_0 > 0$ such that $\mathbb{P}_x(O_1 \in B) \geq c_0 \int_B g_L(y)\lambda_d(dy)$ for all $x \geq K_0$ and all Borel sets B in $\mathcal{X}_+$. The positive function $g_\infty := \min\{g_{K_0}, c_0 g_L\}$ satisfies the inequality in (4.2) for $K = \infty$. This concludes the proof of the proposition. $\square$

Proof of Proposition 4.5. We will need a representation of the overshoots O_1 and $O_1^\downarrow$ as residual lifetimes of renewal processes of ladder heights. The sequence of descending ladder heights $(H_k^-)_{k\geq 0}$ of the random walk S' (recall that $S'_0 = 0$) satisfies $H_0^- = 0$ and its increments $Y_k := H_k^- - H_{k-1}^-$ are negative i.i.d. random variables distributed as H_1^-, the first strictly negative value of S'. For any $x \geq 0$, denote by

$$R^-(x) := \sup\{H_k^- + x : k \geq 1, H_k^- < -x\}$$

the overshoot at the down-crossing of the level $-x$. Then, by definition (2.4), we have

$$O_1^\downarrow = R^-(S_0) \text{ on } \{S_0 \geq 0\}. \tag{4.6}$$

In particular, this implies (2.5) under the assumption $\mathbb{E}X_1^2 < \infty$.

Clearly, there is a similar representation for the overshoot O_1 at the first up-crossing:

$$O_1 = \tilde{R}^+(-S_0) \text{ on } \{S_0 < 0\},$$

where $\tilde{R}^+(x) := \inf\{H_k^+ - x : k \geq 1, H_k^+ \geq x\}$ is the non-negative residual lifetime at time $x > 0$ for the ascending ladder height process $(H_k^+)_{k\geq 1}$ of the random walk S'. The increments of this process are i.i.d. and have the same common distribution as H_1^+, the first strictly positive value of S'.

The case $X_1 \in \mathcal{D}(\alpha, \beta)$. We need to estimate $\mathbb{E}_x O_1^\gamma$ and we start with the following bounds. For any $x > 0$, denote $T(x) := \inf\{k \geq 1 : |H_k^-| > x\}$ and $T'(x) := \inf\{k \geq 1 : |Y_k| > x\}$ with the convention $\inf \emptyset := \infty$. By the assumption $|\beta| < 1$ the distribution of $|Y_1|$ has unbounded support, implying $T'(x) < \infty$ a.s. for any real x. Since $|R^-(x)| < |Y_{T(x)}| \leq |Y_{T'(x)}|$ a.s., we have

$$\mathbb{E}_x|O_1^\downarrow|^\gamma = \mathbb{E}|R^-(x)|^\gamma < \mathbb{E}|Y_{T(x)}|^\gamma \leq \mathbb{E}|Y_{T'(x)}|^\gamma.$$

Clearly, a similar estimate applies for $\mathbb{E}_{-x}O_1^\gamma$. Since the law of $|Y_{T'(x)}|$ equals that of $|H_1^-|$ conditioned to be greater than x, we have

$$\mathbb{E}_x|O_1^\downarrow|^\gamma \leq \frac{\mathbb{E}\big[|H_1^-|^\gamma \mathbb{1}_{\{|H_1^-|>x\}}\big]}{\mathbb{P}(|H_1^-| > x)} \quad \text{and} \quad \mathbb{E}_{-x}O_1^\gamma \leq \frac{\mathbb{E}\big[(H_1^+)^\gamma \mathbb{1}_{\{H_1^+\geq x\}}\big]}{\mathbb{P}(H_1^+ \geq x)}. \tag{4.7}$$

Note that the r.h.s.'s of these inequalities are monotone in x since $Y_{T'(x)}$ is non-decreasing in x a.s.

Recall that, since $|\beta| < 1$, we have $\alpha p < 1$ and $\alpha q < 1$, where p is the positivity parameter introduced in the beginning of the section and $q := 1 - p$. By Theorem 9 of Rogozin [19] we have $|H_1^-| \in \mathcal{D}(\alpha q, 1)$. Hence the renewal theorem of Dynkin [6, Theorem 3] applies to the residual lifetime process $(|R^-(x)|)_{x>0}$ and so the distributions $\mathbb{P}_x(-O_1^\downarrow/x \in \cdot)$ converge weakly as $x \to \infty$ to the distribution with the density

$$g_{\alpha q}(t) = \pi^{-1}\sin(\pi\alpha q)t^{-\alpha q}(1+t)^{-1}, \quad t > 0,$$

supported on the positive half-line. Recalling that $\gamma \in (0, \alpha q)$, we will obtain that

$$\lim_{x\to\infty} \frac{\mathbb{E}_x|O_1^\downarrow|^\gamma}{x^\gamma} = \int_0^\infty t^\gamma g_{\alpha q}(t)dt = \frac{\sin(\pi\alpha q)}{\sin(\pi(\alpha q - \gamma))} =: c_{\alpha,q}(\gamma) \tag{4.8}$$

once we check the uniform integrability of the distributions $\mathbb{P}_x(|O_1^\downarrow/x|^\gamma \in \cdot)$.

Consider the numerator in the first estimate in (4.7). Using Karamata's theorem (see Bingham et al. [3, Proposition 1.5.10]) and the fact that the tail probability $\mathbb{P}(|H_1^-| > x)$ is regularly varying at infinity with index $-\alpha q$, we get

$$\begin{aligned}\mathbb{E}\big[|H_1^-|^\gamma \mathbb{1}_{\{|H_1^-|>x\}}\big] &= \int_{(x,\infty)} t^\gamma \mathbb{P}(|H_1^-| \in dt) \\ &= x^\gamma \mathbb{P}(|H_1^-| > x) + \int_x^\infty \gamma t^{\gamma-1}\mathbb{P}(|H_1^-| > t)dt \\ &\sim \frac{\alpha q}{\alpha q - \gamma} x^\gamma \mathbb{P}(|H_1^-| > x)\end{aligned}$$

as $x \to \infty$. Hence, by (4.7), we have

$$\limsup_{x\to\infty} \mathbb{E}_x(|O_1^\downarrow|/x)^\gamma \leq \frac{\alpha q}{\alpha q - \gamma}.$$

Since the above computations work for any $\gamma \in (0, \alpha q)$, the $\limsup_{x\to\infty} \mathbb{E}_x(|O_1^\downarrow|/x)^{\gamma_0}$ is finite for any $\gamma_0 \in (\gamma, \alpha q)$. This yields the required uniform integrability, and (4.8) follows.

Further, consider $\mathbb{E}_{-x}O_1^\gamma = \mathbb{E}(\tilde{R}^+(x))^\gamma$. The result [6, Theorem 3] used above applies only to (positive) residual times $R^+(x) := \inf\{H_k^+ - x : k \geq 1, H_k^+ > x\}$, where $H_1^+ \in$

$\mathcal{D}(\alpha p, 1)$ by [19, Theorem 9]. However, it gives weak convergence of $R^+(x)/x$ as $x \to \infty$ to the distribution with density $g_{\alpha p}(t)$. This yields weak convergence of $\mathbb{P}_{-x}(O_1/x \in \cdot)$ to the same limit since $R^+(x) = \tilde{R}^+(x)$ on the event $\{R^+(x-1) > 1\}$ whose probability tends to 1 as $x \to \infty$. Then, recalling that $\gamma \in (0, \alpha p)$, we obtain an analogue of (4.8):

$$\lim_{y\to\infty} \frac{\mathbb{E}_{-y} O_1^\gamma}{y^\gamma} = \frac{\sin(\pi\alpha p)}{\sin(\pi(\alpha p - \gamma))} = c_{\alpha,p}(\gamma). \tag{4.9}$$

We now apply the strong Markov property of the random walk S at $T_1^\downarrow$: for any $R > 0$,

$$\mathbb{E}_x O_1^\gamma = \int_{(0,R]} \mathbb{E}_{-y} O_1^\gamma \cdot \mathbb{P}_x(O_1^\downarrow \in -dy) + \int_{(R,\infty)} [\mathbb{E}_{-y}(O_1/y)^\gamma] y^\gamma \mathbb{P}_x(O_1^\downarrow \in -dy). \tag{4.10}$$

The first term is bounded uniformly in x for any fixed R:

$$\begin{aligned} \int_{(0,R]} \mathbb{E}_{-y} O_1^\gamma \cdot \mathbb{P}_x(O_1^\downarrow \in -dy) &\le \sup_{0<y\le R} \mathbb{E}_{-y} O_1^\gamma \\ &\le \frac{\mathbb{E}\big[(H_1^+)^\gamma \mathbb{1}_{\{H_1^+ \ge R\}}\big]}{\mathbb{P}(H_1^+ \ge R)} \le \frac{\mathbb{E}\big[(H_1^+)^\gamma\big]}{\mathbb{P}(H_1^+ \ge R)} < \infty, \end{aligned} \tag{4.11}$$

where in the second inequality we used the second inequality in (4.7), whose r.h.s. is monotone. For the second term in (4.10), by (4.9), we make the expression $\mathbb{E}_{-y}(O_1/y)^\gamma$ in the integrand arbitrarily close to $c_{\alpha,p}(\gamma)$ by taking R sufficiently large. Finally, for any fixed R,

$$\limsup_{x\to\infty} \int_{(0,R]} y^\gamma \mathbb{P}_x(O_1^\downarrow \in -dy) \le \lim_{x\to\infty} R^\gamma \mathbb{P}_x(-O_1^\downarrow \le R) = 0.$$

Hence by (4.10) there exists a constant $C_R > 0$, such that $\left|\mathbb{E}_x O_1^\gamma - c_{\alpha,p}(\gamma)\mathbb{E}_x |O_1^\downarrow|^\gamma\right| \le C_R$ for all $x > 0$. By (4.8) and (4.9) we obtain

$$\lim_{x\to\infty} \frac{\mathbb{E}_x (O_1)^\gamma}{x^\gamma} = \frac{\sin(\pi\alpha q)}{\sin(\pi(\alpha q - \gamma))} \cdot \frac{\sin(\pi\alpha p)}{\sin(\pi(\alpha p - \gamma))} =: \rho_0. \tag{4.12}$$

Since $0 < \gamma < 1 < \alpha < 2$, the following implies $\rho_0 < 1$:

$$\begin{aligned} \sin(\pi\alpha q)\sin(\pi\alpha p) - \sin(\pi(\alpha q - \gamma))\sin(\pi(\alpha p - \gamma)) &= \frac{1}{2}\cos(\pi(\alpha - 2\gamma)) - \frac{1}{2}\cos(\pi\alpha) \\ &= \sin(\pi\gamma)\sin(\pi(\alpha - \gamma)) < 0. \end{aligned}$$

Thus the inequality in (4.3) holds for any $\rho \in (\rho_0, 1)$ since $\mathbb{E}_x O_1^\gamma$ is locally bounded by (4.10), (4.11), and the fact that for all R sufficiently large and any $K > 0$,

$$\begin{aligned} \sup_{0\le x\le K} \int_{(R,\infty)} [\mathbb{E}_{-y}(O_1/y)^\gamma] y^\gamma \mathbb{P}_x(O_1^\downarrow \in -dy) &\le (1 + c_{\alpha,p}(\gamma)) \sup_{0\le x\le K} \mathbb{E}_x |O_1^\downarrow|^\gamma \\ &\le \frac{(1 + c_{\alpha,p}(\gamma))\mathbb{E}|H_1^-|^\gamma}{\mathbb{P}(|H_1^-| > K)}, \end{aligned}$$

where we used (4.9) for the first inequality and (4.7) for the second one as we did in (4.11).

The case $\mathbb{E}X_1^2 < \infty$. The case $\gamma = 0$ is trivial so take $\gamma = 1$. It is well known that the ladder heights of random walks with finite variance of increments are integrable; see Feller [7, Sections XVIII.4 and 5]. Moreover, we have the following versions of (4.8) and (4.9):

$$\lim_{x\to\infty} \frac{\mathbb{E}_x |O_1^\downarrow|}{x} = \lim_{y\to\infty} \frac{\mathbb{E}_{-y} O_1}{y} = 0,$$

see Gut [8, Theorem 3.10.2]. The rest of the proof is exactly as in the first case: by (4.10), the value of the l.h.s. of (4.12) is now zero and $\mathbb{E}_x O_1$ is locally bounded. □

5 Concluding remarks

5.1 The entrance chain into an interval

The methods of this paper developed for establishing convergence of the chain O of overshoots above zero work without any changes for the Markov *chain of entrances* into the interval $[0,h]$ for any $h>0$, defined analogously to O (cf. (1.2) and (1.3)): put $O_n^{(h)} := S_{T_n^{(h)}}$ for $n \in \mathbb{N}_0$, where

$$T_0^{(h)} := 0, \quad T_n^{(h)} := \inf\{k > T_{n-1}^{(h)} : S_{k-1} \notin [0,h], S_k \in [0,h]\}, \qquad n \in \mathbb{N}.$$

By [14, Theorem 4.2], the Markov chain $O^{(h)} := (O_n^{(h)})_{n\ge 0}$ on $\mathcal{Z}_d$ has a unique stationary distribution given by

$$\pi_h(dx) := c_h \mathbb{1}_{[0,h]}(x)(1 - \mathbb{P}(x-h \le X_1 \le x))\lambda_d(dx), \qquad x \in \mathcal{Z}_d, \tag{5.1}$$

where $c_h > 0$ is a normalizing constant. The assertions of Theorems 3.1 and 4.1 remain valid if we replace O_n and π_+ respectively by $O_n^{(h)}$ and π_h, with $\gamma = 0$ in Theorem 4.1.

To see this, recall that the proof of Theorem 3.1 was based on Proposition 3.2 describing $\mathbb{P}_x(O_1 \in \cdot)$, which was actually used only for starting points x in $\mathcal{X}_+ = \operatorname{supp}(\pi_+)$, where $\mathcal{X}_+ = [0, M_+) \cap \mathcal{Z}_d$ with $M_+ = \sup(\operatorname{supp} X_1)$. For the chain $O^{(h)}$, we need to consider only $x \in \operatorname{supp}(\pi_h)$, where $\operatorname{supp}(\pi_h) = ([0,M_+) \cup (h + M_-, h]) \cap [0,h] \cap \mathcal{Z}_d$ with $M_- := \inf(\operatorname{supp} X_1)$. The case $x \in [0, M_+) \cap [0,h] \cap \mathcal{Z}_d$ (which gives the claim if $M_+ \ge h + d$) is actually covered in the proof of the proposition, where we can replace throughout O_1 by $O_1^{(h)}$ without any other changes. The remaining case $x \in (h + M_-, h] \cap [0,h] \cap \mathcal{Z}_d$ follows by considering the random walk $-S$. Finally, Theorem 4.1 immediately follows from Proposition 4.2, which is simply the uniform version of Proposition 3.2, and Proposition 4.5, which trivially holds with $L = h$.

5.2 Convergence of the chain of overshoots under minimal assumptions

By [14, Corollary 4.2], the probability law π_+ is the unique stationary distribution for the chain of overshoots O of any random walk satisfying (1.1). By Theorem 3.1, the laws of O_n converge to π_+ in the total variation distance for random walks with either arithmetic or spread out distributions of increments. Our intuition coming from renewal theory suggests that the following hypothesis is plausible.

Conjecture 5.1. *Under assumption* (1.1)*, we have* $\mathbb{P}_x(O_n \in \cdot) \xrightarrow{d} \pi_+$ *as* $n \to \infty$ *for any* $x \in \mathcal{Z}_0$*.*

Below we discuss the difficulties of proving convergence of O_n in other metrics on probability distributions under the minimal assumptions in (1.1). Let us start with two observations.

First, the total variation norm is clearly inappropriate since it requires the spread out assumption, as explained in the beginning of Section 3. Moreover, in the non-spread out non-arithmetic case the chain of overshoot is not ψ-irreducible and thus not Harris recurrent, placing it outside of the scope of the well-established classical convergence theory (see Meyn and Tweedie [13]). In fact, the spread out assumption on the distribution of X_1 is equivalent to ψ-irreducibility of O (and S, of course). To see this, recall that any ψ-irreducible Markov chain on $\mathbb{R}$ has a finite period p by Theorems 5.2.2 and 5.4.4 in Meyn and Tweedie [13]. Then, by Theorem 4 in Roberts and Rosenthal [18], which we used in the proof of Theorem 3.1, the ψ-irreducibility of O implies that the aperiodic chain $(O_{pn})_{n\ge 0}$ converges to π_+ in the total variation distance. But this can only be true when the distribution of X_1 is spread out.

Second, recall from Section 2.4 that stationarity of π_+ for the chain O can be established by factorizing the transition kernel of O into the Markov kernels P and Q, defined in (2.10), both having π_+ as their stationary distribution (see (2.11)). Unfortunately, this representation appears to be of a very limited use for studying the questions of convergence. In fact, the following example shows that the chain generated by Q may have an invariant distribution other that π_+, hence it may fail to converge to π_+ starting from an arbitrary point.

Example 5.2. Let X_1 satisfy $\mathbb{P}(X_1 = a|X_1 > 0) = 1$ for some $a > d$. Then for any $x \in (0, a)$ we have $Q(x, dy) = \delta_{a-x}(dy)$ and hence $\frac{1}{2}\delta_x + \frac{1}{2}\delta_{a-x}$ is a stationary distribution of Q. An analogous phenomenon occurs for any non-arithmetic distribution of X_1 whose restriction to $\mathcal{Z}_0^+$ is atomic with finitely many atoms.

The next candidate is convergence in $L^2(\pi_+)$. First of all, here we can work only with initial distributions (of $O_0 = S_0$) that are absolutely continuous with respect to π_+. Given that the transition operator of the chain of overshoots O is the product of two reversible transition operators (see Section 2.4 above), it is tempting to apply the methods of the theory of self-adjoint operators. We would need to show that either P or Q has a spectral gap. A plausible way to prove this is to check that the operator is compact, with 1 being an eigenvalue of multiplicity one, and that -1 is not an eigenvalue.

The operator Q appears to be more amenable for the analysis, but it seems that Q may be non-compact for a general distribution of increments. In addition, Example 5.2 above shows that 1 can be a multiple eigenvalue of Q, since the Q-chain can in general have more than one stationary distribution on $\mathcal{Z}_d^+$. We are not aware of any works that establish compactness of Markov transition operators on infinitely-dimensional functional spaces without assuming some form of absolute continuity (as in this paper with spread out distributions of increments).

Regarding the weak convergence of Markov chains, the only technique we are aware of is based on the so-called ε-coupling for continuous-time Markov chains; see Thorisson [22, Section 5.6]. This does not seem to be applicable in the non-arithmetic case: even though, for any distinct real values x_1 and x_2, the walks $x_1 + S'$ and $x_2 + S'$ enjoy a version of ε-coupling (see Thorisson [22, Theorem 2.7.1]), the level zero will be crossed at different times by the two walks making it hardly possible to deduce that the corresponding chains of overshoots are eventually only a small distance away from each other.

Our last candidate are Wasserstein-type metrics with a carefully chosen distance on $\mathcal{Z}_0^+ = [0, \infty)$. Here there is a promising approach, introduced by Hairer and Mattingly [9, 10], which works under a significantly relaxed version of the restrictive ψ-irreducibility assumption and allows one to prove convergence of Markov chains whose transition probabilities can even be mutually singular. Our problem with non-arithmetic distributions that are not spread out appears to be in this category, but we were unable to apply these ideas in our context because of the analytical intrectability of the transition kernel of the chain O.

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