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Modulus of continuity for polymer fluctuations and weight profiles in Poissonian last passage percolation

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Abstract

In last passage percolation models, the energy of a path is maximized over all directed paths with given endpoints in a random environment, and the maximizing paths are called *geodesics*. The geodesics and their energy can be scaled so that transformed geodesics cross unit distance and have fluctuations and scaled energy of unit order. Here we consider Poissonian last passage percolation, a model lying in the KPZ universality class, and refer to scaled geodesics as *polymers* and their scaled energies as *weights*. Polymers may be viewed as random functions of the vertical coordinate and, when they are, we show that they have modulus of continuity whose order is at most $t^{2/3}\big(\log t^{-1}\big)^{1/3}$. The power of one-third in the logarithm may be expected to be sharp and in a related problem we show that it is: among polymers in the unit box whose endpoints have vertical separation t (and a horizontal separation of the same order), the maximum transversal fluctuation has order $t^{2/3}\big(\log t^{-1}\big)^{1/3}$. Regarding the orthogonal direction, in which growth occurs, we show that, when one endpoint of the polymer is fixed at $(0,0)$ and the other is varied vertically over $(0,z)$, $z \in [1,2]$, the resulting random weight profile has sharp modulus of continuity of order $t^{1/3}\big(\log t^{-1}\big)^{2/3}$. In this way, we identify exponent pairs of $(2/3, 1/3)$ and $(1/3, 2/3)$ in power law and polylogarithmic correction, respectively for polymer fluctuation, and polymer weight under vertical endpoint perturbation. The two exponent pairs describe [9, 10, 8] the fluctuation of the boundary separating two phases in subcritical planar random cluster models.

Keywords: Poissonian last passage; modulus of continuity; fluctuations; weight profile.
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1 Introduction and main results

In 1986, Kardar, Parisi, and Zhang [14] predicted universal scaling behaviour for many planar random growth processes, including first and last passage percolation as well as corner growth processes, though rigorous validation has been subsequently provided for only a handful of them. In such models, fluctuation in the direction of growth is governed by an exponent of one-third, with this fluctuation enduring on a scale governed by an exponent of two-thirds in the orthogonal, or transversal, direction.

Poissonian last passage percolation illustrates these effects. We will define it shortly, since it is our object of study; briefly, the model specifies a growth process whose height at a given moment is the maximum number of points (or the *energy*) obtainable in a directed path through a planar Poisson point process. Baik, Deift and Johansson [2] established the $n^{1/3}$-order fluctuation of the maximum number of Poisson points on an increasing path from $(0,0)$ to (n,n), deriving the GUE Tracy-Widom distributional limit of the scaled energy. Later Johansson [13] proved the transversal fluctuation exponent of two-thirds in this model. These are exactly solvable models, for which certain exact distributional formulas are available, and the derivations of these formulas typically employ deep machinery from algebraic combinatorics or random matrix theory. It is interesting to study geometric properties of universal KPZ objects by approaches that, while they are reliant on certain integrable inputs, are probabilistic in flavour: for example, [5], [4] and [3] are recent results and applications concerning geometric properties of last passage percolation paths.

It is rigorously understood, then, that last passage percolation paths experience fluctuation in their energy and transversal fluctuation governed by scaling exponents of one-third and two-thirds. It is very natural to view such paths via the lens of scaled coordinates, in which transversal fluctuation and path energy have unit order. We will be more precise very shortly, when suitable notation has been introduced, but for now we mention that our aim in this article is to refine rigorous understanding of the magnitude and geometry of fluctuation in last passage percolation paths. We shall call the scaled geodesics *polymers*, and refer to the scaled energy as *weight*. We will see that polylogarithmic corrections to the scaled laws implied by the exponents of one-third and two-thirds arise when we consider natural geometric problems concerning the weights and the maximum fluctuation among polymers in a unit order region. The techniques for verifying our claims will employ geometric and probabilistic tools rather than principally integrable ones, since problems involving maxima as both endpoints of a last passage percolation path are varied are not usually amenable to integrable techniques. We will draw on the integrable approach in a way that, while essential, is limited to a simple aspect of this theory, namely by applying bounds on the upper and lower tails of the fluctuation of point-to-point polymer weights; the needed results will be recalled in Section 2.

1.1 Model definition and main results

Let Π be a homogeneous rate one Poisson point process (PPP) on $\mathbb{R}^2$. We introduce a partial order on $\mathbb{R}^2$: $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 \le x_2$ and $y_1 \le y_2$. For $u \preceq v$, $u, v \in \mathbb{R}^2$, an increasing path γ from u to v is a piecewise affine path, viewed as a subset of $\mathbb{R}^2$, that joins points $u = u_0 \preceq u_1 \preceq u_2 \preceq \ldots \preceq u_k = v$ such that $u_i \in \Pi$ for $i \in [\![1, k-1]\!]$. Here and later, $[\![a, b]\!]$ for $a, b \in \mathbb{Z}$ with $a \le b$ denotes the integer interval $\{a, \cdots, b\}$. Also let $|\gamma|$ denote the *energy* of γ, namely the number of points in $\Pi \setminus \{v\}$ that lie on γ; (the last vertex is excluded from the definition of energy so that the sum of the energies of two paths equals the energy of the concatenated path, as we will see in Section 3.1). Then we define the last passage time from u to v, denoted by X_u^v, to be the maximum of $|\gamma|$ as γ varies over all increasing paths from u to v. Any such maximizing path is called a geodesic. There may be several such, but if Γ_u^v denotes any one of them, we have

$$X_u^v = |\Gamma_u^v| \,. \tag{1.1}$$

Note that, in this notation, the starting and ending points of the geodesic, u and v, are assigned subscript and superscript placements. We will often use this convention, including in the case of the scaled coordinates that we will introduce momentarily.

When $u \preceq v$, any geodesic from u to v may be viewed as a function of its horizontal coordinate, since it contains a vertical line segment with probability zero. The operations of maximum and minimum may be applied to any pair of such geodesics, and the results are also geodesics. For this reason, we may speak unambiguously of $\Gamma_u^{\leftarrow;v}$, the uppermost geodesic between u and v, and of $\Gamma_u^{\rightarrow;v}$, the lowermost geodesic between u and v. (The notation $\leftarrow$ and $\rightarrow$ is compatible with these two paths being equally well described as the leftmost and rightmost geodesics. This choice of notation also anticipates the form of these paths when viewed in the scaled coordinates that we are about to introduce.) When the endpoints are $(0,0)$ and (n,n), we will call these geodesics $\Gamma_n^{\leftarrow}$ and $\Gamma_n^{\rightarrow}$.

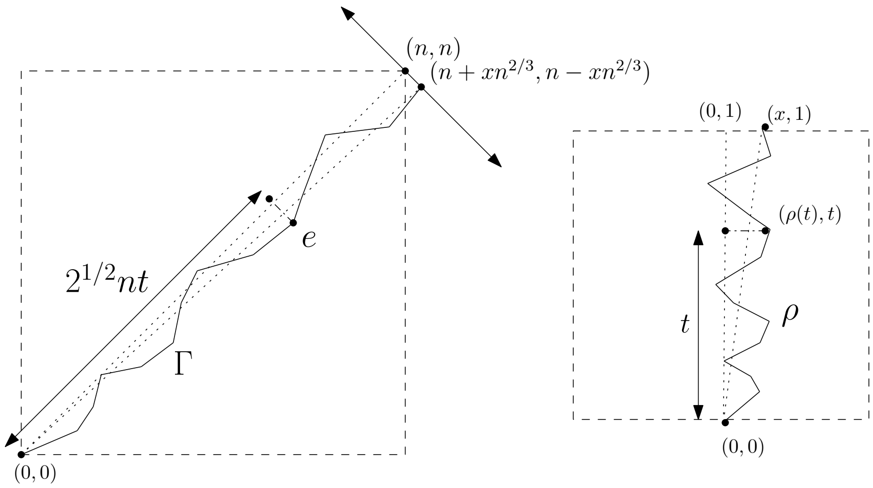

Figure 1: The scaling map T_n applied to the left figure produces the figure on the right. The point e in the geodesic Γ is the preimage of the point $(\rho(t),t)$ in the polymer ρ.

1.1.1 Introducing scaled coordinates

We rotate the plane about the origin counterclockwise by 45 degrees, squeeze the vertical coordinate by a factor $2^{1/2}n$ and the horizontal one by $2^{1/2}n^{2/3}$, thus setting

$$T_n : (x,y) \mapsto \Big(2^{-1}n^{-2/3}(x-y), 2^{-1}n^{-1}(x+y)\Big) . \tag{1.2}$$

The horizontal line at vertical coordinate t is the image under T_n of the anti-diagonal line through (nt,nt). It is easy to see that, for $(x,t) \in \mathbb{R}^2$, $T_n^{-1}(x,t) = (nt + xn^{2/3}, nt - xn^{2/3})$.

Paths that are the image of geodesics under T_n will be called *polymers*; we might say n-polymers, but the suppressed parameter will always be n. Geodesics from $(0,0)$ to (n,n) transform to polymers $(0,0)$ to $(0,1)$. Figure 1 depicts a geodesic Γ and its image polymer ρ. The polymer between planar points u and v that is the image of the uppermost geodesic given the preimage endpoints will be denoted by $\rho_{n;u}^{\leftarrow;v}$, and, naturally enough, called the leftmost polymer from u to v. The rightmost polymer from u to v is the image of the corresponding lowermost geodesic and will be denoted by $\rho_{n;u}^{\rightarrow;v}$. The simpler notation $\rho_n^{\leftarrow}$ and $\rho_n^{\rightarrow}$ will be adopted when $u = (0,0)$ and $v = (0,1)$. When $u = (x_1,t_1), v = (x_2,t_2)$, with $x_1,x_2,t_1,t_2 \in \mathbb{R}$, $t_1 < t_2$, such that $T_n^{-1}(x_1,t_1) \preceq T_n^{-1}(x_2,t_2)$, we will, when it is convenient, regard any polymer ρ from u to v as a function of its vertical coordinate: that is, for $t \in [t_1,t_2]$, $\rho(t)$ will denote the unique point such that $(\rho(t),t) \in \rho$. (This definition makes sense since an increasing path can intersect any anti-diagonal at most once.) We regard the vertical coordinate as time, as the t-notation suggests, and will sometimes refer to the interval $[t_1,t_2]$ as the *lifetime* of the polymer. In particular, when $t_1 = 0$ and $t_2 = 1$, writing $C[0,1]$ for the space of continuous real-valued functions on $[0,1]$ (equipped for later purposes with the topology of uniform convergence), we may thus view $\rho = \{\rho(t)\}_{t\in[0,1]}$ as an element of $C[0,1]$.

1.1.2 Condition for existence of polymers

For $u = (x_1,t_1), v = (x_2,t_2)$ with $x_1,x_2,t_1,t_2 \in \mathbb{R}$, $t_1 < t_2$, we have that $T_n^{-1}(u) = (nt_1{+}x_1n^{2/3}, nt_1{-}x_1n^{2/3})$ and $T_n^{-1}(v) = (nt_2{+}x_2n^{2/3}, nt_2{-}x_2n^{2/3})$. Thus $T_n^{-1}(u) \preceq T_n^{-1}(v)$ if and only if $|x_1 - x_2| < n^{1/3}(t_2 - t_1)$. Indeed, we will write $u \overset{n}{\preceq} v$ to mean that

$|x_1 - x_2| < n^{1/3}(t_2 - t_1)$; this condition ensures that polymers exist between the endpoints u and v.

The first of our three main results shows that polymers, viewed as functions of the vertical coordinate, enjoy modulus of continuity of order $t^{2/3}\big(\log t^{-1}\big)^{1/3}$.

Theorem 1.1. *(a) The sequence $\{\rho_n^{\leftarrow}\}_{n\in\mathbb{N}}$ is tight in $(C[0,1], \|\cdot\|_\infty)$.*

(b) There exists a constant $C > 0$ such that, for the weak limit $\rho_*^{\leftarrow}$ of any weakly converging subsequence of $\{\rho_n^{\leftarrow}\}_{n\in\mathbb{N}}$, almost surely,

$$\limsup_{t\searrow 0} \sup_{0\le z\le 1-t} t^{-2/3}\big(\log t^{-1}\big)^{-1/3}|\rho_*^{\leftarrow}(z+t) - \rho_*^{\leftarrow}(z)| \le C\,. \tag{1.3}$$

The same result holds for the rightmost polymer.

Note that the constant C does not depend on the choice of the weakly converging subsequence.

The exponent pair $(2/3, 1/3)$ for power law and polylogarithmic correction is thus demonstrated to hold in an upper bound on polymer fluctuation. We believe that a lower bound holds as well, in the sense that the limit infimum counterpart to (1.3) is positive. A polymer is an object specified by a global constraint, and it by no means clearly enjoys independence properties as it traverses disjoint regions, even though the underlying Poisson randomness does. In order to demonstrate the polymer fluctuation lower bound, this subtlety would have to be addressed. We choose instead to demonstrate that the exponent pair $(2/3, 1/3)$ describes polymer fluctuation by proving a lower bound of this form for the maximum fluctuation witnessed among a natural class of short polymers in a unit region. This alternative formulation offers a greater supply of independent randomness.

Indeed, we now specify a notion of *maximum transversal fluctuation* over a collection of short polymers. Fix any two points $u = (x_1, t_1), v = (x_2, t_2)$ such that $t_2 > t_1$. Let $\Phi_{n;u}^v$ denote the set of all polymers ρ from u to v. Let ℓ_u^v denote the planar line segment that joins u and v; extending an abuse of notation that we have already made, we write $\ell_u^v(t)$ for the unique point such that $(\ell_u^v(t), t) \in \ell_u^v$, where $t \in [t_1, t_2]$. Then, for any polymer ρ, the transversal fluctuation $\mathrm{TF}(\rho)$ of ρ is specified to be

$$\mathrm{TF}(\rho) := \sup_{t\in[t_1,t_2]} |\rho(t) - \ell_u^v(t)|, \tag{1.4}$$

and the transversal fluctuation between the points u and v to be

$$\mathrm{TF}_{n;u}^v := \max_{\rho\in\Phi_{n;u}^v} \mathrm{TF}(\rho) = \max\left\{\mathrm{TF}(\rho_{n;u}^{\leftarrow;v}), \mathrm{TF}(\rho_{n;u}^{\rightarrow;v})\right\}. \tag{1.5}$$

Now fix some large constant $\psi > 0$. Then, for any fixed parameter $t \in (0,1]$ and any $n \in \mathbb{N}, n > \psi^3$, we define the set of *admissible endpoint pairs*

$$\begin{aligned}\mathrm{AEP}_n(t) = \mathrm{AEP}_{n,\psi}(t) := \Big\{((x_1,t_1),(x_2,t_2)) : t_2 - t_1 \in (0,t]\,, \left|\frac{x_2 - x_1}{t_2 - t_1}\right| \le \psi,\\ x_1, x_2 \in [-1,1]\,, t_1, t_2 \in [0,1]\Big\}.\end{aligned} \tag{1.6}$$

Observe that $\frac{x_2 - x_1}{t_2 - t_1}$ denotes the reciprocal of the slope of the line joining (x_1, t_1) and (x_2, t_2). Since $n > \psi^3$,

$$|x_2 - x_1|n^{2/3} \le \psi(t_2 - t_1)n^{2/3} < (t_2 - t_1)n\,.$$

Recalling the notation at the start of Subsection 1.1.2, we thus have $(x_1, t_1) \overset{n}{\preceq} (x_2, t_2)$, so that polymers do exist between such endpoint pairs.

We then define

$$\mathrm{MTF}_n(t) = \mathrm{MTF}_{n,\psi}(t) := \sup\left\{ \mathrm{TF}^{v}_{n;u} : (u,v) \in \mathrm{AEP}_{n,\psi}(t) \right\}, \tag{1.7}$$

so that $\mathrm{MTF}_n(t)$ is the *maximum transversal fluctuation* over polymers between all endpoint pairs at vertical distance at most t such that the slope of the interpolating line segment is bounded away from being horizontal; (we suppress the parameter ψ in the notation). Our second theorem demonstrates that the exponent pair $(2/3, 1/3)$ governs this maximum traversal fluctuation.

Theorem 1.2. *There exist ψ-determined constants $0 < c < C < \infty$ such that*

$$\liminf_n \, \mathbb{P}\left(t^{-2/3}\big(\log t^{-1}\big)^{-1/3} \mathrm{MTF}_n(t) \in [c, C] \right) \to 1 \quad \textit{as} \;\; t \searrow 0 \, .$$

1.1.3 Scaled energies are called weights

It is natural to scale the energy of a geodesic when we view the geodesic as a polymer after scaling. Scaled energy will be called *weight* and specified so that it is of unit order for polymers that cross unit-order distances. For $t_1 < t_2$, let $t_{1,2}$ denote $t_2 - t_1$; (this is a notation that we will often use). Let $(x,t_1), (y,t_2) \in \mathbb{R}^2$ be such that $|x-y| < t_{1,2} n^{1/3}$. (This condition ensures that $(x,t_1) \overset{n}{\preceq} (y,t_2)$, so that polymers exist between this pair of points.) Since $T_n^{-1}((x,t_1)) = (nt_1 + xn^{2/3}, nt_1 - xn^{2/3})$ and $T_n^{-1}((y,t_2)) = (nt_2 + yn^{2/3}, nt_2 - yn^{2/3})$, it is natural to define the scaled energies, which we call *weights*, in the following way. Define

$$\mathsf{W}_{n;(x,t_1)}^{(y,t_2)} = n^{-1/3}\left(X_{(nt_1+n^{2/3}x,\, nt_1 - n^{2/3}x)}^{(nt_2+n^{2/3}y,\, nt_2 - n^{2/3}y)} \; - \; 2nt_{1,2} \right). \tag{1.8}$$

Because of translation invariance of the underlying Poisson point process, $t_{1,2}$ is a far more relevant parameter than t_1 or t_2. The notation on the left-hand side of (1.8) is characteristic of our presentation in this article: a scaled object is being denoted, with planar points $(\cdot,\cdot)$ in the subscript and superscript indicating starting and ending points.

1.1.4 A continuous modification of the weight function

For the statement of our third theorem, we prefer to make an adjustment to the polymer weight to cope with a minor problem concerning discontinuity of geodesic energy under endpoint perturbation. For $n \in \mathbb{N}$, define $X_n : [1,2] \mapsto [0,\infty)$,

$$X_n(t) := X_{(0,0)}^{(nt,nt)} \, .$$

Observe that $X_n(t)$ is integer-valued, non-decreasing, right continuous and has almost surely a finite number of jump discontinuities. Let $d_0 = 1$ and $d_m = 2$. Record in increasing order the points of discontinuity of X_n as a list $\big(d_1, d_2, \cdots, d_{m-1}\big)$. We specify a *modified* and continuous form of the function X_n by linearly interpolating it between these points of discontinuity, setting

$$X_n^{\mathrm{mod}}(t) := X_n(d_i) + (t-d_i)(d_{i+1}-d_i)^{-1}\big(X_n(d_{i+1}) - X_n(d_i)\big), \text{ for } t \in [d_i, d_{i+1}],$$

for $i = 1, 2, \cdots, m-1$. Because almost surely no two points in a planar Poisson point process share either their horizontal or vertical coordinate, $X_n(d_{i+1}) - X_n(d_i) = 1$ for all i. Thus, for all $t \in [1,2]$,

$$X_n(t) \le X_n^{\mathrm{mod}}(t) \le X_n(t) + 1 \, . \tag{1.9}$$

Now define the *modified weight* function $\mathsf{Wgt}_n : [1,2] \mapsto \mathbb{R}$ for polymers from $(0,0)$ to $(0,\cdot)$:

$$\mathsf{Wgt}_n(t) := n^{-1/3}\left(X_n^{\mathrm{mod}}(t) - 2nt\right) \, . \tag{1.10}$$

Because of (1.9),

$$\left| \mathsf{Wgt}_n(t) - \mathsf{W}^{(0,t)}_{n;(0,0)} \right| \le n^{-1/3} \,. \tag{1.11}$$

By construction, Wgt_n sending $t \in [1,2]$ to $\mathsf{Wgt}_n(t)$ is an element of $C[1,2]$, the space of continuous functions on $[1,2]$; (similarly to before, this space will be equipped with the topology of uniform convergence).

Our third main result demonstrates that the exponent pair $(1/3, 2/3)$ offers a description of the modulus of continuity of polymer weight when one endpoint is varied vertically.

Theorem 1.3. *The sequence $\{\mathsf{Wgt}_n\}_{n\in\mathbb{N}}$ is tight in $(C[1,2], \|\cdot\|_\infty)$. There exist constants $0 < c < C < \infty$ such that, for the weak limit Wgt_* of any weakly converging subsequence of $\{\mathsf{Wgt}_n\}_{n\in\mathbb{N}}$, almost surely*

$$\begin{aligned} c \ &\le \ \liminf_{t\searrow 0} \sup_{1\le z\le 2-t} t^{-1/3}\big(\log t^{-1}\big)^{-2/3} \left|\mathsf{Wgt}_*(z+t) - \mathsf{Wgt}_*(z)\right| \\ &\le \ \limsup_{t\searrow 0} \sup_{1\le z\le 2-t} t^{-1/3}\big(\log t^{-1}\big)^{-2/3} \left|\mathsf{Wgt}_*(z+t) - \mathsf{Wgt}_*(z)\right| \ \le \ C \,. \end{aligned} \tag{1.12}$$

Note that, as in Theorem 1.1, the constants c and C do not depend on the choice of weak limit point or converging subsequence.

Beyond these three theorems, we present a proposition, which is needed for the proof of Theorem 1.2 and which may have an independent interest. That the maximum fluctuation of any geodesic joining $(0,0)$ and (n,n) around the interpolating line is of the order $n^{2/3}$ was first shown in [13]. We first state Johansson's result using scaled coordinates. Observe from (1.4) that, for any polymer ρ between $(0,0)$ and $(0,1)$, $\mathrm{TF}(\rho) = \sup_{y\in[0,1]} |\rho(y)|$. Recall that $\Phi^{(0,t_2)}_{n;(0,t_1)}$ is the set of all polymers from $(0,t_1)$ to $(0,t_2)$, and define

$$\xi := \inf\left\{\theta > 0 : \lim_n \mathbb{P}\left(\max\left\{\mathrm{TF}(\rho) : \rho \in \Phi^{(0,1)}_{n;(0,0)}\right\} \ge n^{\theta - 2/3}\right) = 0\right\} .$$

Johansson [13] proved that $\xi = 2/3$. This value appears in the scaled coordinates in (1.2). His result is an upper bound on the maximum fluctuation from the diagonal of the geodesic joining $(0,0)$ and (n,n) whose order is $n^{2/3+o(1)}$. That this fluctuation has probability at most e^{-ck} of exceeding $kn^{2/3}$ has been obtained in [5, Theorem 11.1 and Corollary 11.7]; the concerned proof may be straightforwardly varied to obtain an upper bound of the form e^{-ck^3}, and later we will state and prove the result in such a form: see Theorem 2.6. Our next proposition is the matching lower bound, stated using scaled coordinates. We adopt such coordinates throughout because they offer a coherent notation for the central aims of this paper, but in the present case it is worth noting the simple expression of the result in unscaled terms: it is with probability at least e^{-ck^3} that the maximum fluctuation from the diagonal of the geodesics joining $(0,0)$ and (n,n) exceeds $kn^{2/3}$.

Proposition 1.4. *There exist positive constants c^*, n_0, s_0 and α_0 such that, for all t_1, t_2 with $t_{1,2} = t_2 - t_1 > 0$ and all $nt_{1,2} \ge n_0$ and $s \in [s_0, \alpha_0 (nt_{1,2})^{1/3}]$,*

$$\mathbb{P}\left(\min\left\{\mathrm{TF}(\rho) : \rho \in \Phi^{(0,t_2)}_{n;(0,t_1)}\right\} \ge s t_{1,2}^{2/3}\right) \ \ge \ \exp\left\{ - c^* s^3 \right\} .$$

1.2 A few words about the proofs

The main ingredients in the proofs of Theorem 1.1 and Theorem 1.2 are tail estimates on polymer weight arising from integrable probability (and certain ramifications thereof) assembled in Section 2, and a *polymer ordering* property elaborated in Lemma 3.2 that propagates control on polymer fluctuation among polymers whose endpoints lie in a discrete mesh to all polymers in the region of this mesh. The basic tools in the proof of the upper bound in Theorem 1.3 and that of Proposition 1.4 are surgical techniques and comparisons of the weights of polymers, and are reminiscent of the techniques developed and extensively used in [5] and [4].

The proofs in this paper depend on the moderate deviation estimates of the point-to-point energies proved in [15, Theorem 1.3] and [16, Theorem 1.2], which are recalled here in Theorem 2.2 and Theorem 2.3. As we record shortly in Section 2, further inputs that we use from [5] and [4], namely [5, Propositions 10.1 and 10.5], [5, Theorem 11.1] and [4, Theorem 2], are results that are themselves derived from the same integrable input point-to-point estimates of Theorem 2.2. It is thus plausible that our results concerning modulus of continuity may be proved for other models in the KPZ universality class that enjoy the same point-to-point estimates.

1.3 Phase separation and KPZ

Certain random models manifest the scaling exponents of KPZ universality and some of its qualitative features, without exhibiting the richness of behaviour of models in this class. For example, the least convex majorant of the stochastic process $\mathbb{R} \mapsto \mathbb{R} : x \mapsto B(x) - t^{-1}x^2$ is comprised of planar line segments, or facets, the largest of which in a compact region has length of order $t^{2/3+o(1)}$ when $t > 0$ is high; and the typical deviation of the process from its majorant scales as $t^{1/3+o(1)}$.

Some such models form a testing ground for KPZ conjectures. Phase separation concerns the form of the boundary of a droplet of one substance suspended in another. When supercritical bond percolation on $\mathbb{Z}^2$ is conditioned on the cluster (or droplet) containing the origin being finite and large, namely of finite size at least n^2, with n high, the interface at the boundary of this cluster is expected to exhibit KPZ scaling characteristics, with the scaling parameter n playing a comparable role to t in the preceding example. Indeed, the papers [9, 10, 8], which develop the study made in [1, 18], a surrogate of this interface, expressed in terms of the random cluster model, was investigated. The maximum length of the facets that comprise the boundary of the interface's convex hull was proved to typically have the order $n^{2/3}\big(\log n\big)^{1/3}$, while the maximum local roughness, namely the maximum distance from a point on the interface to the convex hull boundary, was shown to be of the order of $n^{1/3}\big(\log n\big)^{2/3}$.

Viewed in this light, the present article validates for the KPZ universality class the implied predictions: that exponent pairs of $(1/3, 2/3)$ and $(2/3, 1/3)$ for power-law and logarithmic-power govern maximal polymer weight change under vertical endpoint displacement and maximal transversal polymer fluctuation.

In a natural sense, these two exponent pairs are accompanied by a third, namely $(1/2, 1/2)$, for interface regularity. In the example of parabolically curved Brownian motion, $x \to B(x) - x^2 t^{-1}$, the modulus of continuity of the process on $[-1, 1]$ is easily seen to have the form $s^{1/2}\big(\log s^{-1}\big)^{1/2}$, up to a random constant, and uniformly in $t \geq 1$. In KPZ, this assertion finds a counterpart when it is made for the Airy_2 process, which offers a limiting description in scaled coordinates of the weight of polymers of given lifetime with first endpoint fixed. This assertion has been proved in [11, Theorem 1.11(1)]. Recently, for a very broad class of initial data, the polymer weight profile was shown in [12, Theorem 1.3] to have a modulus of continuity of the order of $s^{1/2}\big(\log s^{-1}\big)^{2/3}$, uniformly in the scaling parameter and the initial condition. The present article and [12] derive different modulus of continuity results for polymer weight profiles. In [12], the weight profiles that are considered may be called 'spatial', in the sense that the variation of the polymer endpoint is *horizontal*. In contrast, Theorem 1.3 addresses 'temporal' weight profiles, where the variation in polymer endpoint is instead *vertical*. The two articles share a perspective of employing probabilistic and geometric techniques that harness limited integrable inputs, but those techniques are rather different: in [12], the tools concern resampling associated to the *Brownian Gibbs* property enjoyed by the Airy line ensemble, while, for Theorem 1.3, the key tools are surgeries on polymers allied with prelimiting moderate deviation estimates.

1.4 Organization

We continue with two sections that offer basic general tools. The first, Section 2, provides useful estimates including the basic integrable input Theorem 2.2 concerning the tail of polymer weights. Then, in Section 3, we state and prove the polymer ordering lemmas and some other basic results, which are essential tools in the proofs of the main theorems.

The next four sections, 4 – 7, contain the main proofs. Consecutively, these sections are devoted to proving:

- the polymer Hölder continuity upper bound Theorem 1.1;
- the modulus of continuity for maximum transversal fluctuation over short polymers, Theorem 1.2, subject to assuming Proposition 1.4;
- Hölder continuity for the polymer weight profile, Theorem 1.3;
- and the lower bound on transversal polymer fluctuation, Proposition 1.4.

A partial list of the main pieces of notation used in the paper with a brief summary of their meanings is provided in Appendix A for the convenience of the reader. Finally, Appendix B reviews the instances of our use in this paper of the results in [5], and provides summaries of the corresponding proofs in the notation of the scaled coordinates.

We will stick to scaled coordinates in the results' statements and, except in Section 2, in their proofs. A bridge between scaled coordinates and the original ones is offered in this next section, in whose proofs we use the scaling map T_n from (1.2) and weight function W from (1.8) to transfer unscaled results to their scaled counterparts.

2 Scalings and estimates: input results and their adaptations

In this section, we assemble the results that we will quote in our arguments. Most of these results were derived in terms of unscaled coordinates in [5] and [4]. Point-to-point estimates of last passage percolation energies were used crucially in [5] to resolve the *slow bond* conjecture, and in [4] to show the coalescence of nearby geodesics, and those estimates will be employed in this paper as well. The concerned results will either be recalled from [5] and [4] or proved in this section: in each case, the underlying integrable input is the pair of tail estimates concerning point-to-point polymer weights stated here in Theorem 2.2 and 2.3.

We state results in scaled coordinates – valuable we believe for grasping the putatively KPZ universal behaviour at stake – and the proofs explain how to obtain these statements from their unscaled and largely already available counterparts. The transformation from unscaled to scaled uses the definitions of the scaling map in (1.2) and the weight in (1.8).

First we observe some simple relations enjoyed by polymers and weights.

The scaling principle. Because of translation invariance and the definition (1.2), it is easy to see that for any $x, y, t_1, t_2 \in \mathbb{R}$ with $t_{1,2} = t_2 - t_1 > 0$ and $(x, t_1) \overset{n}{\preceq} (y, t_2)$ (see Subsection 1.1.2),

$$\begin{aligned} &\left\{ \rho_{n;(x,t_1)}^{\leftarrow;(y,t_2)}(t_1 + \theta t_{1,2}) - \ell_{(x,t_1)}^{(y,t_2)}(t_1 + \theta t_{1,2}) : \theta \in [0,1] \right\} \\ \overset{d}{=} \quad &\left\{ t_{1,2}^{2/3} \left(\rho_{nt_{1,2};(0,0)}^{\leftarrow;((y-x)t_{1,2}^{-2/3},1)}(\theta) - \ell_{(0,0)}^{((y-x)t_{1,2}^{-2/3},1)}(\theta) \right) : \theta \in [0,1] \right\} . \end{aligned} \qquad (2.1)$$

Here, as before, ℓ_u^v denotes the planar line segment joining u and v. The same statement holds for the rightmost polymers as well. Here and throughout $\overset{d}{=}$ denotes that the two random quantities on either side have the same distribution. We will sometimes call the displayed assertion the *scaling principle*. The use of the subscripts and superscripts in (2.1) is consistent with our notation for denoting starting and ending points in scaled coordinates. The formula is however perhaps a little difficult to parse and we offer a few words of interpretation. Take $\theta \in (0,1)$ given, though the distributional equality holds at a process level. Then (2.1) asserts that the fluctuation of the n-polymer from (x,t_1) to (y,t_2) relative to the line that interpolates these endpoints, when a fraction θ of this polymer's lifetime has passed, is equal in law to the $t_{1,2}^{2/3}$-th multiple of the analogous fluctuation for the $nt_{1,2}$-polymer from $(0,0)$ to $((y-x)t_{1,2}^{-2/3},1)$, at the same stage of life.

Also by translation invariance and the definition of weight in (1.8), it follows that

$$t_{1,2}^{-1/3}\mathsf{W}_{n;(x,t_1)}^{(y,t_2)} \overset{d}{=} \mathsf{W}_{nt_{1,2};(0,0)}^{((y-x)t_{1,2}^{-2/3},1)}. \tag{2.2}$$

Boldface notation for applying results. In our proofs, we will naturally often be applying tools such as those stated in this section. Sometimes the notation of the tool and of the context of the application will be in conflict. To alleviate this conflict, we will use boldface notation when we specify the values of the parameters of a given tool in terms of quantities in the context of the application. We will first use this notational device shortly, in one of the upcoming proofs.

The next theorem, which was proved in [2], indicates a basic aspect of the role of scaled coordinates, though in fact we will never use the result.

Theorem 2.1. *As $n \to \infty$,*

$$\mathsf{W}_{n;(0,0)}^{(0,1)} \Rightarrow F_{TW},$$

where the convergence is in distribution and F_{TW} denotes the GUE Tracy-Widom distribution.

For a definition of the GUE Tracy-Widom distribution, also called the F_2 distribution, see [2].

The next two results, concerning moderate deviations for the polymer weight, are the only inputs from integrable probability used in this paper. The first follows immediately from [15, Theorem 1.3], [16, Theorem 1.2] and (2.2); the second from [15, Theorem 1.3] and the same identity (2.2).

Theorem 2.2. *There exist positive constants c, s_0 and n_0 such that, for all $t_1 < t_2$ with $nt_{1,2} > n_0$ and $s > s_0$,*

$$\mathbb{P}\left(t_{1,2}^{-1/3}\mathsf{W}_{n;(0,t_1)}^{(0,t_2)} \geq s\right) \leq e^{-cs^{3/2}},$$

and

$$\mathbb{P}\left(t_{1,2}^{-1/3}\mathsf{W}_{n;(0,t_1)}^{(0,t_2)} \leq -s\right) \leq e^{-cs^{3/2}}.$$

Theorem 2.3. *There exist constants $c_2, s_0, n_0 > 0$ such that, for all $t_1 < t_2$ with $nt_{1,2} > n_0$ and $s > s_0$,*

$$\mathbb{P}\left(t_{1,2}^{-1/3}\mathsf{W}_{n;(0,t_1)}^{(0,t_2)} \geq s\right) \geq e^{-c_2 s^{3/2}}.$$

We shall need not just tail bounds for weights of point-to-point polymers, but uniform tail bounds on polymer weights whose endpoints vary over fixed unit order intervals. The unscaled version of this theorem was proved in [5, Propositions 10.1 and 10.5], which has been reviewed and summarized here in Appendix B and makes essential use of Theorem 2.2.

Theorem 2.4. *There exist $C, c \in (0,\infty), C_0 \in (1,\infty)$ and $n_1 \in \mathbb{N}$ such that, for all $t_1 < t_2$ with $nt_{1,2} \geq n_1$, $s \in [0, 10(nt_{1,2})^{2/3}]$, $A = C_0^{-1} s^{1/4} n^{1/6} t_{1,2}^{5/6}$, and I and J intervals of length at most $t_{1,2}^{2/3}$ that are contained in $[-A, A]$,*

$$\mathbb{P}\left(\sup_{x \in I, y \in J} \left| t_{1,2}^{-1/3} \mathsf{W}_{n;(x,t_1)}^{(y,t_2)} + t_{1,2}^{-4/3}(x-y)^2 \right| > s \right) \leq C \exp\left\{ -cs^{3/2} \right\}.$$

Proof. Observe that $|x-y| < 2C_0^{-1} s^{1/4} n^{1/6} t_{1,2}^{5/6} < 2^{-1} n^{1/3} t_{1,2}$ for $C_0 > 2^2 \cdot 10^{1/4}$ since $s \leq 10(nt_{1,2})^{2/3}$. This ensures that $\mathsf{W}_{n;(x,t_1)}^{(y,t_2)}$ is well defined.

First we prove the theorem when $t_1 = 0$ and $t_2 = 1$. At the end we prove Theorem 2.4 for general $t_1 < t_2$. Thus we first show that for n large enough, and I and J intervals of at most unit length contained in the interval of length $2C_0^{-1} s^{1/4} n^{1/6}$ centred at the origin,

$$\mathbb{P}\left(\sup_{x \in I, y \in J} \left| \mathsf{W}_{n;(x,0)}^{(y,1)} + (x-y)^2 \right| > s \right) \leq C \exp\left\{ -cs^{3/2} \right\}. \tag{2.3}$$

Let $u = (xn^{2/3}, -xn^{2/3})$ and $v = (n + yn^{2/3}, n - yn^{2/3})$. To begin with, observe that due to the scaling invariance of the underlying Poisson process, we have

$$X_u^v \overset{d}{=} X_{(0,0)}^{(n_*,n_*)}, \tag{2.4}$$

where $n_* = n\left(1 - n^{-2/3}(y-x)^2\right)^{1/2}$. Using a simple binomial expansion giving $|(1-z)^{1/2} - (1 - 2^{-1}z)| \leq C_1 z^2$ for some constant $C_1 > 0$ and all $z \in (-1,1)$, we get that

$$n^{-1/3} \left| 2n_* - (2n - (x-y)^2 n^{1/3}) \right| \leq C_1 n^{-2/3}(x-y)^4.$$

Since $|x-y| \leq 2C_0^{-1} s^{1/4} n^{1/6}$ by our assumption,

$$C_1 n^{-2/3}(x-y)^4 \leq 2^4 C_0^{-4} C_1 s < 2^{-1} s$$

for $C_0 > 2^{5/4} C_1^{1/4}$. Since $n_* < n$, using the definition of the weight function in (1.8),

$$\begin{aligned} \left\{ \left| \mathsf{W}_{n;(x,0)}^{(y,1)} + (x-y)^2 \right| > s \right\} \quad &\subseteq \quad \left\{ n^{-1/3} \left| X_u^v - 2n_* \right| > 2^{-1}s \right\} \\ &\subseteq \quad \left\{ n_*^{-1/3} \left| X_u^v - 2n_* \right| > 2^{-1}s \right\} \overset{d}{=} \left\{ \left| \mathsf{W}_{n_*;(0,0)}^{(0,1)} \right| > 2^{-1}s \right\}, \end{aligned}$$

where the last equality in distribution follows from (2.4). Now set $n_1 = 2n_0$. Next, using Theorem 2.2, it follows that for all $n > n_1$ (a bound which, in turn, ensures that the condition $n_* > n_0$ is satisfied, since $|x-y| < 2^{-1}n^{1/3}$ guarantees that $n_* > 2^{-1}n$), $s > s_0$ and $x \in I, y \in J$,

$$\mathbb{P}\left(\left| \mathsf{W}_{n;(x,0)}^{(y,1)} + (x-y)^2 \right| > s \right) \leq e^{-cs^{3/2}}, \tag{2.5}$$

where c, s_0, n_0 are as defined in Theorem 2.2. Then we use Proposition B.1 to get (2.3).

We now make a first use of the boldface notation for applying results specified at the beginning of Section 2. For general $t_1 < t_2$, set $\boldsymbol{n} = nt_{1,2}, \boldsymbol{x} = xt_{1,2}^{-2/3}, \boldsymbol{y} = yt_{1,2}^{-2/3}, \boldsymbol{I} = t_{1,2}^{-2/3} I, \boldsymbol{J} = t_{1,2}^{-2/3} J$ and $\boldsymbol{s} = s$ in (2.3). Recall that the boldface variables are those of Theorem 2.4 and that these are written in terms of non-boldface parameters specified by the present context. From the hypothesis of Theorem 2.4, $\boldsymbol{I}$ and $\boldsymbol{J}$ are intervals of at most unit length contained in $[-\boldsymbol{n}^{1/6}, \boldsymbol{n}^{1/6}]$. Thus, applying (2.3) and using the scaling principle (2.2), we get Theorem 2.4. □

The unscaled transversal fluctuations for paths between $(0,0)$ and (n,n) around the diagonal were shown to be of the order $n^{2/3+o(1)}$ with high probability in [13]. More precise estimates were established in [5]. However, the unscaled fluctuation of the geodesic at the point (r,r) for any $r \leq n$ is only of the order $r^{2/3}$. This is the content of the next theorem which in essence is the scaled version of [4, Theorem 2] adapted for Poissonian LPP. In the proof of [4, Theorem 2], Theorem 2.4 is again essential, applied at several scales alongside a union bound. Recall that, for $u, v \in \mathbb{R}^2$, $\Phi^v_{n;u}$ is the set of all polymers from u to v, and ℓ^v_u is the straight line joining u and v.

Theorem 2.5. *There exist positive constants n_0, s_1, c such that for all $x, y, t_1, t_2 \in \mathbb{R}$ with $t_{1,2} = t_2 - t_1 > 0$ and $|x - y| \leq 2^{-1} n^{1/3} t_{1,2}$ and for all $nt_{1,2} \geq n_0, s \geq s_1$ and $t \in [t_1, t_2]$,*

$$\mathbb{P}\left(\max\left\{\left|\rho(t) - \ell^{(y,t_2)}_{(x,t_1)}(t)\right| : \rho \in \Phi^{(y,t_2)}_{n;(x,t_1)}\right\} \geq s\Big((t-t_1) \wedge (t_2 - t)\Big)^{2/3}\right) \leq 2e^{-cs^3}. \quad (2.6)$$

Here $a \wedge b$ denotes $\min\{a, b\}$.

Proof of Theorem 2.5. First we prove the theorem when $t_1 = 0$, $t_2 = 1$, and $x = 0$. Observe that in this case it is enough to bound the probabilities of the events

$$\left\{\left|\rho^{\leftarrow;(y,1)}_{n;(0,0)}(t) - \ell^{(y,1)}_{(0,0)}(t)\right| \geq s\Big(t \wedge (1-t)\Big)^{2/3}\right\} \text{ and}$$

$$\left\{\left|\rho^{\rightarrow;(y,1)}_{n;(0,0)}(t) - \ell^{(y,1)}_{(0,0)}(t)\right| \geq s\Big(t \wedge (1-t)\Big)^{2/3}\right\},$$

and use a union bound to obtain (2.6).

We first prove an upper bound for the probability of the first of these two events. Also, first assume that $t \in [0, 2^{-1}]$. To prove the bound in this case, we move to unscaled coordinates, and use [4, Theorem 2].

To this end, let $\Gamma := \Gamma^{\leftarrow;(n+yn^{2/3}, n-yn^{2/3})}_{(0,0)}$ be the leftmost geodesic, and $\mathcal{S}$ the straight line from $(0,0)$ to $(n + yn^{2/3}, n - yn^{2/3})$. For $r \in [0, n + yn^{2/3}]$, let $\Gamma(r)$ and $\mathcal{S}(r)$ be such that $(r, \Gamma(r)) \in \Gamma$ and $(r, \mathcal{S}(r)) \in \mathcal{S}$. Now, for $r = nt$,

$$\begin{aligned} & \left\{\left|\rho^{\leftarrow;(y,1)}_{n;(0,0)}(t) - \ell^{(y,1)}_{(0,0)}(t)\right| \geq st^{2/3}\right\} \\ = \quad & \left\{\left|n^{2/3}\rho^{\leftarrow;(y,1)}_{n;(0,0)}(rn^{-1}) - n^{2/3}\ell^{(y,1)}_{(0,0)}(rn^{-1})\right| \geq sr^{2/3}\right\} \\ \subseteq \quad & \left\{|\Gamma(r') - \mathcal{S}(r')| \geq sr^{2/3}\right\} =: \mathsf{B}\,, \end{aligned} \quad (2.7)$$

where r' is such that the anti-diagonal line passing through (r,r) intersects $\mathcal{S}$ at $(r', \mathcal{S}(r'))$. The last inclusion follows from the definition of the scaling map T_n in (1.2). Since $|y| \leq 2^{-1} n^{1/3}$, $2^{-1} r \leq r' \leq 2r$. Thus,

$$\mathsf{B} \subseteq \left\{|\Gamma(r') - \mathcal{S}(r')| \geq 2^{-1} s (r')^{2/3}\right\} =: \mathsf{C}\,. \quad (2.8)$$

Thus it is enough to bound the probability of the event C. This local fluctuation estimate for the leftmost geodesic in (2.9) was proved for exponential directed last passage percolation in [4, Theorem 2 and Corollary 2.4]. The proof goes through verbatim for the leftmost (and also the rightmost) geodesic in Poissonian last passage percolation. Moreover, the refined bounds of Theorem 2.4 give corresponding improvements for Poissonian LPP: see [4, Remark 1.5]. This gives that, for some positive constants n_0, r_0, s_0, and for $n \geq n_0, r' \geq r'_0$ and $s \geq s_0$,

$$\mathbb{P}(\mathsf{C}) \leq e^{-cs^3}\,. \quad (2.9)$$

However, observe that (2.9) holds only when $r' \geq r_0'$. Now assume $r' \leq r_0'$, so that $r \leq r_0$, where $r_0 = 2r_0'$. Let the anti-diagonal passing through (r,r) intersect the geodesic Γ at v and the line $\mathcal{S}$ at w. Clearly $\|v-(r,r)\|_2 \leq 2^{1/2}r$. Also, since $|y| \leq 2^{-1}n^{1/3}$,

$$\|w-(r,r)\|_2 = 2^{1/2}|y|rn^{-1/3} \leq r\,.$$

Thus, with $r = nt \leq r_0$,

$$\left|\rho^{\leftarrow;(y,1)}_{n;(0,0)}(t) - \ell^{(y,1)}_{(0,0)}(t)\right| = 2^{-1/2}n^{-2/3}\|v-w\|_2 \leq 2^{-1}(2^{1/2}+1)n^{-2/3}r \leq 2r_0^{1/3}t^{2/3}\,.$$

Define $s_1 = \max\{s_0, 2r_0^{1/3}\}$. Then for $n \geq n_0, s \geq s_1$ and $t \in [0,2^{-1}]$,

$$\mathbb{P}\left(\left|\rho^{\leftarrow;(y,1)}_{n;(0,0)}(t) - \ell^{(y,1)}_{(0,0)}(t)\right| \geq st^{2/3}\right) \leq e^{-cs^3}\,.$$

For $t \in [2^{-1},1]$, we consider the reversed polymer and translate it by $-y$ so that its starting point is $(0,0)$, that is, $\rho'(v) = \rho^{\leftarrow;(y,1)}_{n;(0,0)}(1-v) - y$ for $v \in [0,1]$. Now we follow the same arguments as above to get the bound for the probability of the event

$$\left\{\left|\rho^{\leftarrow;(y,1)}_{n;(0,0)}(t) - \ell^{(y,1)}_{(0,0)}(t)\right| \geq s\Big(t \wedge (1-t)\Big)^{2/3}\right\}\,.$$

Since the same arguments work for the rightmost polymer $\rho^{\rightarrow;(y,1)}_{n;(0,0)}$, we get for $n \geq n_0, s \geq s_1$ and all $t \in [0,1]$,

$$\mathbb{P}\left(\max\left\{\left|\rho(t) - \ell^{(y,1)}_{(0,0)}(t)\right| : \rho \in \Phi^{(y,1)}_{n;(0,0)}\right\} \geq s\Big(t \wedge (1-t)\Big)^{2/3}\right) \leq 2e^{-cs^3}. \tag{2.10}$$

Now for general $t_1 < t_2$, set $\boldsymbol{n} = nt_{1,2}, \boldsymbol{y} = (y-x)t_{1,2}^{-2/3}, \boldsymbol{s} = s$ and $\boldsymbol{t} = t_{1,2}^{-1}(t-t_1)$. Then from the hypothesis of Theorem 2.5, $|\boldsymbol{y}| \leq 2^{-1}\boldsymbol{n}^{1/3}$ since $|y-x| \leq 2^{-1}n^{1/3}t_{1,2}$. Thus applying (2.10) and using the scaling principle (2.1), we obtain the theorem. □

The following theorem bounds the transversal fluctuation of polymers; (recall the definitions in (1.4) and (1.5)).

Theorem 2.6. *There exist positive constants c, n_0 and k_0 such that, for $t \in (0,1]$, $k \geq k_0$ and $n \geq n_0t^{-1}$,*

$$\mathbb{P}\left(\mathrm{TF}^{(0,t)}_{n;(0,0)} \geq kt^{2/3}\right) \leq 2e^{-ck^3}.$$

Proof. Because of (1.5), it is enough to bound the probabilities of $\left\{\mathrm{TF}\big(\rho^{\leftarrow;(0,t)}_{n;(0,0)}\big) \geq kt^{2/3}\right\}$ and $\left\{\mathrm{TF}\big(\rho^{\rightarrow;(0,t)}_{n;(0,0)}\big) \geq kt^{2/3}\right\}$ and use a union bound. We bound only the first event, the arguments for the second event being the same. For simplicity of notation, let us denote $\rho^{\leftarrow;(0,t)}_{n;(0,0)}$ by ρ. From Theorem 2.5, setting $\boldsymbol{n} = n, \boldsymbol{t_1} = 0, \boldsymbol{t_2} = t, \boldsymbol{x} = \boldsymbol{y} = 0, \boldsymbol{t} = 2^{-1}t$ and $\boldsymbol{s} = 2^{2/3}k$, we see that there exist constants $c > 0$ and $n_0, k_0 > 0$ such that, for all $k \geq k_0$ and $nt \geq n_0$,

$$\mathbb{P}\left(\left|\rho(2^{-1}t)\right| > kt^{2/3}\right) \leq e^{-ck^3}\,. \tag{2.11}$$

From here, using Proposition B.3, we get

$$\mathbb{P}\left(\mathrm{TF}\big(\rho^{\leftarrow;(0,t)}_{n;(0,0)}\big) \geq kt^{2/3}\right) \leq e^{-ck^3}\,. \tag{2.12}$$

□

3 Basic tools

Fundamental facts about ordering and concatenation of polymers will be used repeatedly in the proofs of the main theorems.

3.1 Polymer concatenation and superadditivity of weights

Let $n \in \mathbb{N}$ and $(x, t_1), (y, t_2) \in \mathbb{R}^2$ with $t_1 < t_2$ and $|x - y| < n^{1/3}(t_2 - t_1)$. (This condition ensures that $(x, t_1) \overset{n}{\preceq} (y, t_2)$, see Subsection 1.1.2.) Let $u = T_n^{-1}(x, t_1)$ and $v = T_n^{-1}(y, t_2)$ and let ζ be an increasing path from u to v. Let $\gamma = T_n(\zeta)$. We call γ an *n-path*. We shall often consider γ as a subset of $\mathbb{R}^2$, and call (x, t_1) its *starting point* and (y, t_2) its *ending point*. Moreover, similarly to the definition of the weight of a polymer in (1.8), we define the weight of an n-path as

$$n^{-1/3} \left(|\zeta| - 2nt_{1,2} \right) , \tag{3.1}$$

where $|\zeta|$ denotes the energy of ζ, that is, the number of points in $\Pi \setminus \{v\}$ that lie on ζ.

Now, let $(x, t_1), (y, t_2), (z, t_3) \in \mathbb{R}^2$ be such that $t_1 < t_2 < t_3$, $|x - y| < n^{1/3}(t_2 - t_1)$ and $|y - z| < n^{1/3}(t_3 - t_2)$, so that there exist polymers from (x, t_1) to (y, t_2); and from (y, t_2) to (z, t_3). Let ρ_1 be any polymer from (x, t_1) to (y, t_2), and ρ_2 any polymer from (y, t_2) to (z, t_3). The union of these two subsets of $\mathbb{R}^2$ is an n-path from (x, t_1) to (z, t_3). We call this n-path the concatenation of ρ_1 and ρ_2 and denote it by $\rho_1 \circ \rho_2$. The weight of $\rho_1 \circ \rho_2$ is $\mathsf{W}_{n;(x,t_1)}^{(y,t_2)} + \mathsf{W}_{n;(y,t_2)}^{(z,t_3)}$. This additivity is the reason that the endpoint v was excluded from the definition of path energy in Section 1.1.

Again, let $n \in \mathbb{N}$ and $(x, t_1), (y, t_2), (z, t_3) \in \mathbb{R}^2$ be such that $t_1 < t_2 < t_3$ and $|x - y| < n^{1/3}(t_2 - t_1)$ and $|y - z| < n^{1/3}(t_3 - t_2)$. Then

$$\mathsf{W}_{n;(x,t_1)}^{(z,t_3)} \geq \mathsf{W}_{n;(x,t_1)}^{(y,t_2)} + \mathsf{W}_{n;(y,t_2)}^{(z,t_3)} . \tag{3.2}$$

Indeed, taking a polymer ρ_1 from (x, t_1) to (y, t_2) and a polymer ρ_2 from (y, t_2) to (z, t_3), the weight of $\rho_1 \circ \rho_2$ is a lower bound on $\mathsf{W}_{n;(x,t_1)}^{(z,t_3)}$.

3.2 Polymer ordering lemmas

The first lemma roughly says that if two polymers intersect at two points during their lifetimes, then they are identical between these points.

Lemma 3.1. *Let $n \in \mathbb{N}$ and $(x_1, t_1), (x_2, t_2), (y_1, s_1), (y_2, s_2) \in \mathbb{R}^2$ and $t, s \in \mathbb{R}$ be such that $t_1 < t < s < s_1$, $t_2 < t < s < s_2$, $|x_1 - y_1| < n^{1/3}(s_1 - t_1)$ and $|x_2 - y_2| < n^{1/3}(s_2 - t_2)$. Suppose that $\rho_{n;(x_1,t_1)}^{\leftarrow;(y_1,s_1)}$ and $\rho_{n;(x_2,t_2)}^{\leftarrow,(y_2,s_2)}$ intersect at two points $z_1 = (x, t)$ and $z_2 = (y, s)$. Then $\rho_{n;(x_1,t_1)}^{\leftarrow;(y_1,s_1)}$ and $\rho_{n;(x_2,t_2)}^{\leftarrow,(y_2,s_2)}$ are identical between t and s. The same statement holds for the rightmost polymers.*

To simplify notation in the proof, we write $\rho_1 = \rho_{n;(x_1,t_1)}^{\leftarrow;(y_1,s_1)}$ and $\rho_2 = \rho_{n;(x_2,t_2)}^{\leftarrow,(y_2,s_2)}$.

Proof of Lemma 3.1. First, for any polymer ρ, call a point $u \in \rho$ a *Poisson* point of ρ if $T_n^{-1}(u) \in \Pi \cap \Gamma$, where Γ is the geodesic $T_n^{-1}(\rho)$ and Π is the underlying unit rate Poisson point process. Also, for $r_1, r_2 \in \rho$, let $\rho[r_1, r_2]$ denote the part of the polymer between the points r_1 and r_2, and let $\#\rho[r_1, r_2]$ denote the number of Poisson points that lie in $\rho[r_1, r_2]$. We first claim that $\#\rho_1[z_1, z_2] = \#\rho_2[z_1, z_2]$ where z_1 and z_2 appear in the lemma's statement. For, if not, without loss of generality assume that $\#\rho_1[z_1, z_2] < \#\rho_2[z_1, z_2]$ and let u_1 and v_1 be the Poisson points of ρ_1 immediately before z_1 and immediately after z_2; and let u_2 and v_2 be the Poisson points of ρ_2 immediately after z_1 and immediately before z_2: see Figure 2. Then joining u_1 to u_2 and v_1 to v_2 (shown in the figure by dashed lines), one gets an alternative path ρ' between (x_1, t_1) and (y_1, s_1) that has more Poisson points than ρ_1, thereby contradicting that ρ_1 is a polymer between (x_1, t_1) and (y_1, s_1). Thus, $\#\rho_1[z_1, z_2] = \#\rho_2[z_1, z_2]$. Since both ρ_1 and ρ_2 are leftmost polymers between their respective endpoints, we see that $\rho_1[z_1, z_2] = \rho_2[z_1, z_2]$. This proves the lemma. □

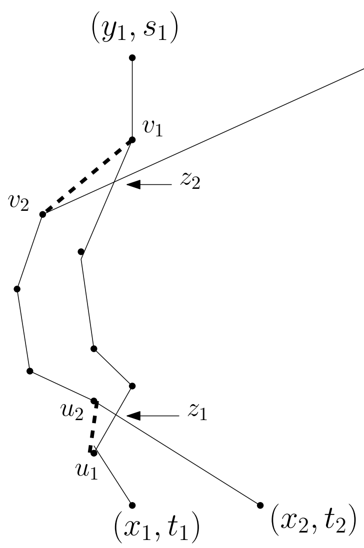

Figure 2: This illustrates Lemma 3.2. The points of the underlying Poisson process lying on a polymer are marked by dots, and the polymer is obtained by linearly interpolating between the points. The figure shows that both the paths cannot be leftmost polymers between their respective endpoints, since by joining the dashed lines, one obtains an alternative increasing path where the Poisson points between the intersecting points z_1 and z_2 in the two polymers are interchanged.

The next result roughly says that two polymers that begin and end at the same heights, with the endpoints of one to the right of the other's, cannot cross during their shared lifetime.

Lemma 3.2 (Polymer Ordering)**.** *Fix* $n \in \mathbb{N}$. *Consider points* $(x_1, t_1), (x_2, t_1), (y_1, t_2), (y_2, t_2) \in \mathbb{R}^2$ *such that* $t_1 < t_2$, $x_1 \leq x_2$, $y_1 \leq y_2$, $|x_1 - y_1| < n^{1/3}(t_2 - t_1)$ *and* $|x_2 - y_2| < n^{1/3}(t_2 - t_1)$. *Then* $\rho^{\leftarrow;(y_1,t_2)}_{n;(x_1,t_1)}(t) \leq \rho^{\leftarrow;(y_2,t_2)}_{n;(x_2,t_1)}(t)$ *and* $\rho^{\rightarrow;(y_1,t_2)}_{n;(x_1,t_1)}(t) \leq \rho^{\rightarrow;(y_2,t_2)}_{n;(x_2,t_1)}(t)$ *for all* $t \in [t_1, t_2]$.

Let $\rho_1 = \rho^{\leftarrow;(y_1,t_2)}_{n;(x_1,t_1)}$ and $\rho_2 = \rho^{\rightarrow;(y_1,t_2)}_{n;(x_1,t_1)}$.

Proof of Lemma 3.2. Supposing otherwise, there exists $z = (x, y) \in \rho_2$ such that $x < \rho_1(y)$. But then there exist $z_1, z_2 \in \rho_1 \cap \rho_2$ straddling the point z. By Lemma 3.1, $\rho_1[z_1, z_2] = \rho_2[z_1, z_2]$, and hence $z \in \rho_1 \cap \rho_2$, a contradiction. □

By ordering, a polymer whose endpoints are straddled between those of a pair of polymers becomes sandwiched between those polymers.

Corollary 3.3. *Fix* $n \in \mathbb{N}$. *Consider the points* $(x_1, t_1), (x_2, t_1), (x_3, t_1), (y_1, t_2), (y_2, t_2)$ *and* $(y_3, t_2) \in \mathbb{R}^2$ *such that* $t_1 < t_2$, $x_1 \leq x_2 \leq x_3$, $y_1 \leq y_2 \leq y_3$ *and* $|x_i - y_i| < n^{1/3}(t_2 - t_1)$ *for* $i = 1, 2, 3$. *Let* $t \in (t_1, t_2)$. *Let* $\rho_i = \rho^{\leftarrow;(y_i,t_2)}_{n;(x_i,t_1)}$ *for* $i = 1, 2, 3$. *Then*

$$|\rho_2(t) - \rho_2(t_1)| \leq \max_{i \in \{1,3\}} |\rho_i(t) - \rho_i(t_1)| + \max_{i \in \{1,3\}} |x_i - x_2| .$$

The same result holds for rightmost polymers.

Proof. By Lemma 3.2,

$$\rho_1(t) \leq \rho_2(t) \leq \rho_3(t) .$$

The result now follows immediately. □

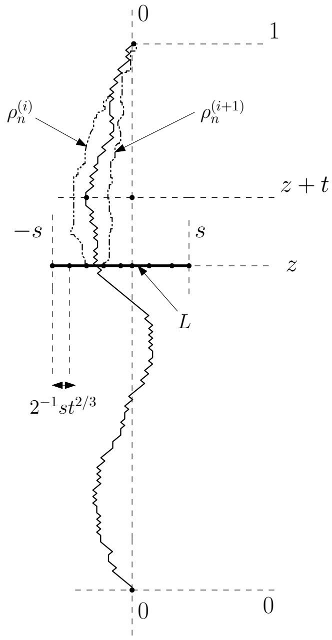

Figure 3: The proof of Proposition 4.1 is illustrated here. We mark the line segment L with a number of equally spaced points. As the leftmost polymer from $(0,0)$ to $(0,1)$ passes between two such points on the line L, it is, in view of polymer ordering, sandwiched between the two leftmost polymers, shown as dotted lines, originating from those points and ending at $(0,1)$. Hence it is sufficient to bound the fluctuations of the polymers originating from these equally spaced points on L.

4 Exponent pair $(2/3, 1/3)$ for a single polymer: Proof of Theorem 1.1

In this section, we show that the sequence $\left\{\rho_n^{\leftarrow} : n \in \mathbb{N}\right\}$ of leftmost n-polymers from $(0,0)$ to $(0,1)$ is tight, and any weak limit is Hölder $2/3-$-continuous with a polylogarithmic correction of order $1/3$. The main two ingredients in this proof are the local regularity estimate Theorem 2.5 and the polymer ordering Lemma 3.2. First, we bound the fluctuation of the polymer near any given point $z \in [0,1]$.

Proposition 4.1. *There exist positive constants n_0, s_1 and c such that, for all $n \geq n_0, s \geq s_1$, $z \in [0,1]$ and $0 \leq t \leq 1-z$,*

$$\mathbb{P}\left(|\rho_n^{\leftarrow}(z+t) - \rho_n^{\leftarrow}(z)| \geq st^{2/3}\right) \leq 10t^{-2/3}e^{-cs^3}. \tag{4.1}$$

The same statement holds for $\rho_n^{\rightarrow}$.

As we now explain, the proposition will be proved by reducing to the case that $z = 0$, when the result follows from Theorem 2.5. For any fixed $z \in (0,1)$, Theorem 2.5 again guarantees that the polymer $\rho_n^{\leftarrow}$ is at distance at most s from the point $(0,z)$ with probability at least $1 - e^{-cs^3}$. We break the horizontal line segment of length $2s$ centred at $(0,z)$ into a sequence of consecutive intervals of length $2^{-1}st^{2/3}$, and consider the leftmost polymers starting from each of these endpoints and ending at $(0,1)$, as in Figure 3. Due to the Corollary 3.3 of the polymer ordering Lemma 3.2, a big fluctuation of $\rho_n^{\leftarrow}$ between times z and $z+t$ creates a big fluctuation for one of the polymers starting from these deterministic endpoints. The probability of the latter fluctuations is controlled via Theorem 2.5 and since the number of these polymers is of the order of $t^{-2/3}$, a union bound gives (4.1).

Proof of Proposition 4.1. First observe that for $s > (nt)^{1/3}$, the probability in (4.1) is zero by the definition of the scaling map T_n in (1.2) and the geodesics being increasing paths. Hence we assume that $s \le (nt)^{1/3}$.

Fix $s \le (nt)^{1/3}$ and $z \in [0,1]$. For $t \ge 8^{-3}$,

$$\left\{|\rho_n^{\leftarrow}(z+t) - \rho_n^{\leftarrow}(z)| \ge st^{2/3}\right\} \subseteq \left\{|\rho_n^{\leftarrow}(z+t) - \rho_n^{\leftarrow}(z)| \ge 8^{-2}s\right\} \subseteq \left\{\mathrm{TF}^{(0,1)}_{n;(0,0)} \ge 2^{-1}8^{-2}s\right\},$$

where $\mathrm{TF}^{(0,1)}_{n;(0,0)}$ is defined in (1.5). Hence, applying Theorem 2.6 with the parameter specifications $\boldsymbol{t} = 1$ and $\boldsymbol{k} = 2^{-1}8^{-2}s$, we get that (4.1) holds for all n, s large enough. Hence we assume that $t \le 8^{-3}$. Also, let us assume for now that $z \in [0, 2^{-1}]$.

Let L be the line segment $[-s,s] \times \{z\}$. Let E be the event that $\rho_n^{\leftarrow}$ passes through L. By Theorem 2.5 with $\boldsymbol{n} = n, \boldsymbol{t} = z, \boldsymbol{x} = 0, \boldsymbol{y} = 0, \boldsymbol{s} = s, \boldsymbol{t_1} = 0$ and $\boldsymbol{t_2} = 1$, we have that, for $n \ge n_1$ and $s \ge s_1$,

$$\mathbb{P}(\mathsf{E}) \ge 1 - 2e^{-cs^3}.$$

Now, we divide L into $\lceil 4t^{-2/3}\rceil$-many adjacent intervals of length at most $2^{-1}st^{2/3}$, and let $(x_i, z), i = 0,1,2,\cdots,\lceil 4t^{-2/3}\rceil$ be the endpoints of these intervals, i.e.,

$$x_i = -s + 2^{-1}ist^{2/3} \quad \text{for } i = 0,1,2,\cdots,\lceil 4t^{-2/3}\rceil.$$

Let $\rho_n^{(i)} := \rho^{\leftarrow;(0,1)}_{n;(x_i,z)}$ be the leftmost polymer from (x_i,z) to $(0,1)$.

By Corollary 3.3, on E,

$$|\rho_n^{\leftarrow}(z+t) - \rho_n^{\leftarrow}(z)| \le \max_{i \in [\![0,\lceil 4t^{-2/3}\rceil]\!]} \left|\rho_n^{(i)}(z+t) - \rho_n^{(i)}(z)\right| + 2^{-1}st^{2/3}. \tag{4.2}$$

Also, for any fixed $i \in [\![0,\lceil 4t^{-2/3}\rceil]\!]$, let $\ell^{(i)} = \ell^{(0,1)}_{(x_i,z)}$ be the straight line segment joining (x_i,z) and $(0,1)$. Then, since $z \in [0,2^{-1}]$ and $t \le 8^{-3}$, for any $i \in 0,1,2,\cdots,\lceil 4t^{-2/3}\rceil$,

$$\left|\ell^{(i)}(z) - \ell^{(i)}(z+t)\right| \le \frac{st}{1-z} \le 2st \le 4^{-1}st^{2/3}.$$

Since $\rho_n^{(i)}(z) = \ell^{(i)}(z) = x_i$,

$$\begin{aligned} \left|\rho_n^{(i)}(z+t) - \rho_n^{(i)}(z)\right| &\le \left|\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)\right| + |\ell^{(i)}(z+t) - \ell^{(i)}(z)| \\ &\le \left|\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)\right| + 4^{-1}st^{2/3}. \end{aligned}$$

Thus, on the event E, by (4.2),

$$|\rho_n^{\leftarrow}(z+t) - \rho_n^{\leftarrow}(z)| \le \max_{i \in [\![0,\lceil 4t^{-2/3}\rceil]\!]} \left|\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)\right| + \tfrac{3}{4}st^{2/3}.$$

From here, it follows by taking a union bound that

$$\begin{aligned} & \mathbb{P}\left(|\rho_n^{\leftarrow}(z+t) - \rho_n^{\leftarrow}(z)| \ge st^{2/3}\right) \\ \le\ & \mathbb{P}(\mathsf{E}^c) + \sum_{i=0}^{\lceil 4t^{-2/3}\rceil} \mathbb{P}\left(\left|\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)\right| \ge 4^{-1}st^{2/3}\right) \\ \le\ & 10t^{-2/3}e^{-cs^3}, \end{aligned}$$

for some absolute positive constant c and all $n \geq 2n_0$. Here the last inequality follows by applying Theorem 2.5 to each of the polymers $\rho^{(i)}$. For given i, set the parameters $\boldsymbol{n} = n, \boldsymbol{t_1} = z, \boldsymbol{t_2} = 1, \boldsymbol{t} = t + z, \boldsymbol{x} = -s + 2^{-1}ist^{2/3}, \boldsymbol{y} = 0$ and $\boldsymbol{s} = 4^{-1}s$. Since $z \in [0, 2^{-1}]$ and $s \leq (nt)^{1/3}$, we have that $|\boldsymbol{x} - \boldsymbol{y}| \leq s \leq n^{1/3}t^{1/3} \leq 8^{-1}n^{1/3} \leq 4^{-1}\mathbf{n^{1/3}t_{1,2}}$. Thus one can apply Theorem 2.5 to get the above inequality for all $\boldsymbol{nt_{1,2}} \geq 2^{-1}n \geq n_0$.

For $z \in [2^{-1}, 1]$, define the reversed polymer $\widehat{\rho}_n^{\leftarrow}$ by $\widehat{\rho}_n^{\leftarrow}(a) = \rho_n^{\leftarrow}(1-a)$ for $a \in [0,1]$, and follow the above argument. $\square$

Next we show the tightness of the members of the sequence $\{\rho_n^{\leftarrow}\}_{n\in\mathbb{N}}$ as elements in the space $(C[0,1], \|\cdot\|_\infty)$. We prove that Proposition 4.1 guarantees that Kolmogorov-Chentsov's tightness criterion is satisfied.

Proof of Theorem 1.1(a). Fix $n \geq n_0$ and any $\lambda > 0$. Fix $t \in (0,1]$ small enough that $\lambda t^{-2/3} \geq s_1$, where n_0 and s_1 are as in Proposition 4.1. Also fix some $M \in \mathbb{N}$ large enough that $2M - 2/3 > 1$. Then it follows from Proposition 4.1 that for any $z, z' \in [0,1]$ with $|z - z'| = t$,

$$\begin{aligned}&\mathbb{P}\left(|\rho_n^{\leftarrow}(z') - \rho_n^{\leftarrow}(z)| \geq \lambda\right)\\ &\quad\leq 10t^{-2/3}e^{-c(\lambda^3 t^{-2})} \leq K_M\lambda^{-3M}t^{2M-2/3} = K_M\lambda^{-3M}|z'-z|^{2M-2/3},\end{aligned} \tag{4.3}$$

where $K_M := \sup_{x\geq 0} x^M e^{-cx} < \infty$. Since $2M - 2/3 > 1$, by Kolmogorov-Chentsov's tightness criterion (see for example [7, Theorem 8.1.3]), it follows that the sequence $\{\rho_n^{\leftarrow}\}_{n\in\mathbb{N}}$ is tight in $(C[0,1], \|\cdot\|_\infty)$. $\square$

4.1 Modulus of continuity

Here we prove Theorem 1.1(b), thus finding the modulus of continuity for any weak limit of a weakly converging subsequence of $\{\rho_n^{\leftarrow}\}_{n\in\mathbb{N}}$. We will follow the arguments used to derive the Kolmogorov continuity criterion, where one infers Hölder continuity of a stochastic process from moment bounds on the difference of the process between pairs of times. Thus we introduce the set of dyadic rationals

$$D = \bigcup_{i=0}^{\infty} 2^{-i}\mathbb{Z}\,.$$

Next is the first step towards proving the modulus of continuity.

Lemma 4.2. *Let $\rho_*^{\leftarrow}$ be the weak limit of a weakly converging subsequence of $\{\rho_n^{\leftarrow}\}_{n\in\mathbb{N}}$. Then there exists a universal positive constant C (not depending on the particular weak limit $\rho_*^{\leftarrow}$) such that, almost surely, for some random $m_0(\omega) \in \mathbb{N}$ and for all $s, t \in D \cap [0,1]$ with $|t - s| \leq 2^{-m_0(\omega)}$,*

$$|\rho_*^{\leftarrow}(t) - \rho_*^{\leftarrow}(s)| \leq C(t-s)^{2/3}\left(\log(t-s)^{-1}\right)^{1/3}.$$

Proof. For $m \in \mathbb{N}$, let S_m be the set of all intervals of the form $[j2^{-m}, (j+1)2^{-m}]$, for $j \in \{0, 1, 2, \cdots, 2^m - 1\}$. Fix $c_0 > (\frac{5}{3c})^{1/3}$, where c is the constant in Proposition 4.1.

Writing $\Rightarrow$ for convergence in distribution, let $\{\rho_{n_k}^{\leftarrow}\}_{k\in\mathbb{N}}$ be a subsequence of $\{\rho_n^{\leftarrow}\}_{n\in\mathbb{N}}$ such that $\rho_{n_k}^{\leftarrow} \Rightarrow \rho_*^{\leftarrow}$ as random variables in $(C[0,1], \|\cdot\|_\infty)$. Since for $a, b \in [0,1]$, the map $\tau_{a,b}$ defined by $(C[0,1], \|\cdot\|_\infty) \mapsto (\mathbb{R}, |\cdot|) : f \mapsto |f(a) - f(b)|$ is continuous,

$$\mathsf{U} := \bigcup \left\{\tau^{-1}_{(j+1)2^{-m}, j2^{-m}}\left(c_0 2^{-\frac{2m}{3}}\left(\log 2^m\right)^{1/3}, \infty\right) : j = 0, 1, \cdots, 2^m - 1\right\}$$

is an open set. Thus, by the Portmanteau theorem,

$$\begin{aligned}
& \mathbb{P}\left(\sup_{j\in\{0,1,\cdots,2^m-1\}} |\rho_*^{\leftarrow}((j+1)2^{-m}) - \rho_*^{\leftarrow}(j2^{-m})| > c_0 2^{-\frac{2m}{3}} \left(\log 2^m\right)^{1/3}\right) \qquad (4.4) \\
\leq\ & \liminf_k \mathbb{P}\left(\sup_{j\in\{0,1,\cdots,2^m-1\}} |\rho_{n_k}^{\leftarrow}((j+1)2^{-m}) - \rho_{n_k}^{\leftarrow}(j2^{-m})| > c_0 2^{-\frac{2m}{3}} \left(\log 2^m\right)^{1/3}\right) \\
\leq\ & \limsup_n \mathbb{P}\left(\sup_{j\in\{0,1,\cdots,2^m-1\}} |\rho_{n}^{\leftarrow}((j+1)2^{-m}) - \rho_{n}^{\leftarrow}(j2^{-m})| > c_0 2^{-\frac{2m}{3}} \left(\log 2^m\right)^{1/3}\right).
\end{aligned}$$

Now, for all m large enough that $\left(\log 2^m\right)^{1/3} \geq s_1$, where s_1 is as in Proposition 4.1, and all $n \geq n_0$, applying Proposition 4.1 and a union bound,

$$\begin{aligned}
& \mathbb{P}\left(\sup_{j\in\{0,1,\cdots,2^m-1\}} |\rho_{n}^{\leftarrow}((j+1)2^{-m}) - \rho_{n}^{\leftarrow}(j2^{-m})| > c_0 2^{-\frac{2m}{3}} \left(\log 2^m\right)^{1/3}\right) \\
\leq\ & 10 \cdot 2^m \left(\frac{1}{2^m}\right)^{c_0^3 c - 2/3} \leq 10 \left(\frac{1}{2^m}\right)^{c_0^3 c - 5/3}.
\end{aligned}$$

Hence, from (4.4),

$$\mathbb{P}\left(\sup_{j\in\{0,1,\cdots,2^m-1\}} |\rho_*^{\leftarrow}((j+1)2^{-m}) - \rho_*^{\leftarrow}(j2^{-m})| > c_0 2^{-\frac{2m}{3}} \left(\log 2^m\right)^{1/3}\right) \leq 10 \cdot 2^{-m(c_0^3 c - 5/3)}.$$

As the right hand side is summable in m (by the choice of c_0 made at the beginning of the proof), the Borel-Cantelli lemma implies that there exists a null set N_0, such that, for each $\omega \notin N_0$, there is some $m_0(\omega)$ for which $m \geq m_0(\omega)$ entails that

$$|\rho_*^{\leftarrow}(t) - \rho_*^{\leftarrow}(s)| \leq c_0 (t-s)^{2/3} \left(\log (t-s)^{-1}\right)^{1/3} \quad \text{for all } [s,t] \in S_m. \qquad (4.5)$$

Now, let $\omega \notin N_0$ and $s, t \in D \cap [0,1]$ be such that $|s-t| \leq 2^{-m_0(\omega)}$. Let $m = m(s,t)$ be the greatest integer such that $|s-t| \leq 2^{-m}$; then clearly, $m \geq m_0(\omega)$. Also, consider the binary expansions of s and t:

$$s = s_0 + \sum_{j>m} \sigma_j 2^{-j}, \quad t = t_0 + \sum_{j>m} \tau_j 2^{-j},$$

where $\sigma_j, \tau_j \in \{0,1\}$, and each of the sequences is eventually zero. Either $s_0 = t_0$ or $[s_0, t_0] \in S_m$. Moreover, for $n \geq 1$, let

$$s_n = s_0 + \sum_{m<j\leq m+n} \sigma_j 2^{-j}.$$

Then, for $n \geq 1$, either $s_n = s_{n-1}$ or $[s_{n-1}, s_n] \in S_{m+n}$. Since $m \geq m_0(\omega)$, by (4.5),

$$|\rho_*^{\leftarrow}(t_0)(\omega) - \rho_*^{\leftarrow}(s_0)(\omega)| \leq c_0 2^{-\frac{2m}{3}} \left(\log 2^m\right)^{1/3}.$$

Also,

$$\begin{aligned}
|\rho_*^{\leftarrow}(s)(\omega) - \rho_*^{\leftarrow}(s_0)(\omega)| &\leq \sum_{n=1}^{\infty} |\rho_*^{\leftarrow}(s_n)(\omega) - \rho_*^{\leftarrow}(s_{n-1})(\omega)| \\
&\leq \sum_{n=1}^{\infty} c_0 2^{-\frac{2(m+n)}{3}} \left(\log 2^{m+n}\right)^{1/3} \\
&\leq C_1 2^{-\frac{2(m+1)}{3}} \left(\log 2^{m+1}\right)^{1/3},
\end{aligned}$$

and similarly

$$|\rho_*^{\leftarrow}(t)(\omega) - \rho_*^{\leftarrow}(t_0)(\omega)| \le C_2 2^{-\frac{2(m+1)}{3}} \left(\log 2^{m+1}\right)^{1/3},$$

for some absolute constants C_1 and C_2. Hence,

$$|\rho_*^{\leftarrow}(t) - \rho_*^{\leftarrow}(s)| \le |\rho_*^{\leftarrow}(t) - \rho_*^{\leftarrow}(t_0)| + |\rho_*^{\leftarrow}(t_0) - \rho_*^{\leftarrow}(s_0)| + |\rho_*^{\leftarrow}(s) - \rho_*^{\leftarrow}(s_0)| \le C 2^{-\frac{2m}{3}} \left(\log 2^m\right)^{1/3}.$$

Since by definition $2^{-m-1} \le |s-t| \le 2^{-m}$, the result follows. □

Proof of Theorem 1.1(b). For any $s, t \in [0,1]$ satisfying $s < t$ and $|s-t| \le 2^{-m_0(\omega)}$, choose $s_k, t_k \in D \cap [s,t]$ such that $s_k \searrow s$ and $t_k \nearrow t$. Then, since $|s_k - t_k| \le |s-t| \le 2^{-m_0(\omega)}$, by Lemma 4.2,

$$|\rho_*^{\leftarrow}(t_k) - \rho_*^{\leftarrow}(s_k)| \le C(t_k - s_k)^{2/3} \left(\log(t_k - s_k)^{-1}\right)^{1/3}.$$

Since $\rho_*^{\leftarrow}(t_k)(\omega) \to \rho_*^{\leftarrow}(\omega)$ and $\rho_*^{\leftarrow}(s_k)(\omega) \to \rho_*^{\leftarrow}(s)(\omega)$, the theorem follows by taking the limit as $k \to \infty$. The same argument applies without any change for the rightmost polymers as well. □

5 Exponent pair $(2/3, 1/3)$ for maximum fluctuation over short polymers: Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. It is the upper bound that is the more subtle. Recall the notation of transversal fluctuations from (1.4) and (1.5), $\mathrm{AEP}_n(t)$ from (1.6) and $\mathrm{MTF}_n(t)$ from (1.7).

Here is the idea behind the proof. Proposition 1.4 offers a lower bound on the transversal fluctuation of a polymer between two given points. By considering order-t^{-1} endpoint pairs with disjoint intervening lifetimes of length t, we obtain a collection of independent opportunities for the fluctuation lower bound to occur. By tuning the probability of the individual event to have order t, at least one among the constituent events typically does occur, and the lower bound in Theorem 1.2 follows.

On the other hand, suppose that a big swing in the unit order region happens between a certain endpoint pair, with an intervening duration, or height difference, of order t. Members of the endpoint pair may be exceptional locations when viewed as functions of the underlying Poisson point field, both in horizontal and vertical coordinate. Thus, the upper bound in Theorem 1.2 does not follow directly from a union bound of a given endpoint estimate over elements in a discrete mesh, since such a mesh may not capture the exceptional endpoints. However, polymer ordering forces exceptional behaviour to become typical and to occur between an endpoint pair in a discrete mesh. To see this, assume that the original polymer between exceptional endpoints makes a big left swing. (Figure 4 illustrates the argument.) We take a discrete mesh endpoint pair whose lifetime includes that of the original polymer but has the same order t, and whose lower and upper points lie to the left of the original endpoint locations, about halfway between these and the leftmost coordinate visited by the original polymer. Then we consider the leftmost mesh polymer at the beginning and ending times of the original polymer. If the mesh polymer is to the right of the original polymer at any of these endpoints, then the mesh polymer has already made a big rightward swing at one of these endpoints. If, on the other hand, the mesh polymer is to the left of the original polymer at both the endpoints of the original polymer, then by polymer ordering Lemma 3.2, the mesh polymer cannot cross the original polymer during the latter's lifetime. Hence the big left swing of the original polymer forces a significant left swing for the mesh polymer as well.

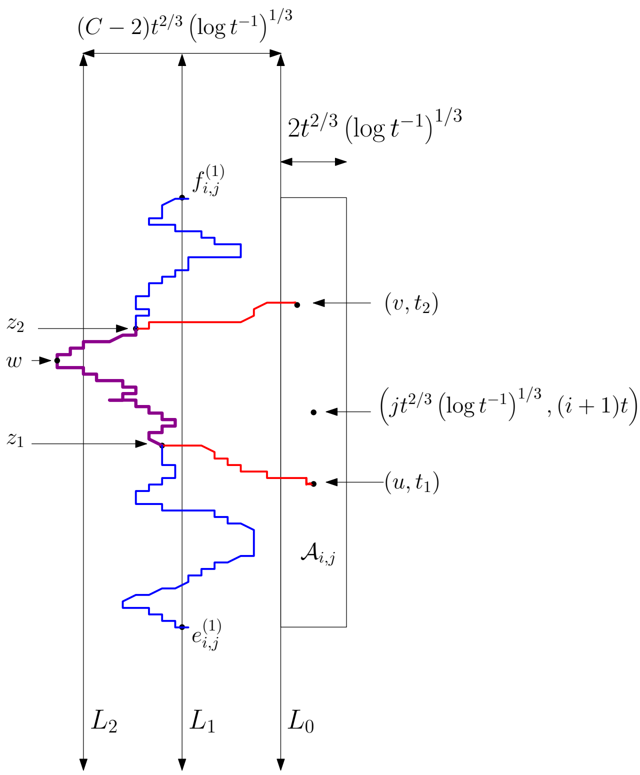

Figure 4: The figure illustrates the proof of the upper bound in Theorem 1.2. If the leftmost polymer between (u,t_1) and (v,t_2) (shown in red) makes a huge leftward fluctuation and the leftmost polymer between points $e^{(1)}_{i,j}$ and $f^{(1)}_{i,j}$ (shown in blue) is to the left of u and v at t_1 and t_2 respectively, then the blue polymer stays to the left of the red polymer between times t_1 and t_2 by polymer ordering. Thus the big left fluctuation transmits from the red to the blue polymer. If, however, the blue polymer reaches to the right of either u or v, then it creates a big right fluctuation for the blue polymer. Thus by bounding the fluctuations of a small number of polymers between deterministic endpoints, one can bound the fluctuation between all admissible endpoint pairs.

Proof of Theorem 1.2. The lower bound follows in a straightforward way from Proposition 1.4. For any $t \in (0,1)$ and $i \in \left\{0,1,2,\cdots,\left[t^{-1}\right]-1\right\}$, define

$$\mathsf{F}_{i,t,n} = \left\{ \mathrm{TF}^{(0,(i+1)t}_{n;(0,it)} \geq ct^{2/3}\left(\log t^{-1}\right)^{1/3} \right\}.$$

For given such (t,i), we apply Proposition 1.4 with parameter settings $\boldsymbol{n}=n, \boldsymbol{t_1}= it, \boldsymbol{t_2}=(i+1)t$ and $\boldsymbol{s}=c(\log t^{-1})^{1/3}$, to find that, when $c(\log t^{-1})^{1/3} \geq s_0$ and $n \geq \max\{\alpha_0^{-3}c^3t^{-1}\log t^{-1}, n_0t^{-1}\}$,

$$\mathbb{P}(\mathsf{F}_{i,t,n}) \geq e^{-c^*c^3\log t^{-1}} = t^{c^*c^3}\,,$$

where the proposition specifies the quantities α_0, n_0 and s_0.

Thus, for all $t \leq e^{-(c^{-1}s_0)^3}$ and $i \in \left\{0,1,2,\cdots,\left[t^{-1}\right]-1\right\}$,

$$\liminf_n t^{-1}\mathbb{P}(\mathsf{F}_{i,t,n}) = \liminf_n t^{-1}\mathbb{P}(\mathsf{F}_{0,t,n}) \geq t^{c^*c^3-1}\,.$$

By choosing $c>0$ small enough that $c^*c^3<1$, one has $\liminf_n t^{-1}\mathbb{P}(\mathsf{F}_{0,t,n}) \to \infty$ as $t \searrow 0$. For such $c>0$, using the definition (1.7) of $\mathrm{MTF}_n(t)$ and independence of the events $\mathsf{F}_{i,t,n}$ for $i \in \{0,1,2,\cdots,[t^{-1}]-1\}$,

$$\mathbb{P}\left(\mathrm{MTF}_n(t)t^{-2/3}\big(\log t^{-1}\big)^{-1/3} < c\right) \leq \mathbb{P}\left(\bigcap_{i=0}^{[t^{-1}]-1} \mathsf{F}^c_{i,t,n}\right) = \prod_{i=0}^{[t^{-1}]-1} \mathbb{P}\left(\mathsf{F}^c_{i,t,n}\right) .$$

Thus,

$$\begin{aligned} &\limsup_n \mathbb{P}\left(\mathrm{MTF}_n(t)t^{-2/3}\big(\log t^{-1}\big)^{-1/3} < c\right) \\ \leq\ & \limsup_n \left(1-\mathbb{P}\big(\mathsf{F}_{0,t,n}\big)\right)^{[t^{-1}]} \leq \limsup_n \exp\left\{-\left[t^{-1}\right]\mathbb{P}(\mathsf{F}_{0,n})\right\} \to 0\,, \end{aligned}$$

the latter convergence as $t \searrow 0$.

Now we show the upper bound. Fix $t \in (0,1]$ small enough that $\psi t \leq t^{2/3}$, where the parameter ψ appears in the definition (1.6) of $\mathrm{AEP}_n(t)$.

For any $i = 0,1,2,\ldots,\lceil t^{-1}\rceil$ and $j \in \left[\!\left[-\left\lceil t^{-2/3}\big(\log t^{-1}\big)^{-1/3}\right\rceil, \left\lceil t^{-2/3}\big(\log t^{-1}\big)^{-1/3}\right\rceil\right]\!\right]$, define the rectangle $\mathcal{A}_{i,j}$ with lower-left corner $\left((j-1)t^{2/3}\big(\log t^{-1}\big)^{1/3}, it\right)$, width $2t^{2/3}\big(\log t^{-1}\big)^{1/3}$ and height $2t$. Figure 4 illustrates this rectangle and the arguments that follow.

Let $C > 0$ be an even integer whose value will later be specified. For such i,j as above, define planar points

$$\begin{aligned} e^{(1)}_{i,j} &:= \left((j-2^{-1}C)t^{2/3}\big(\log t^{-1}\big)^{1/3}, it\right), \quad f^{(1)}_{i,j} := \left((j-2^{-1}C)t^{2/3}\big(\log t^{-1}\big)^{1/3}, (i+2)t\right), \\ e^{(2)}_{i,j} &:= \left((j+2^{-1}C)t^{2/3}\big(\log t^{-1}\big)^{1/3}, it\right), \quad f^{(2)}_{i,j} := \left((j+2^{-1}C)t^{2/3}\big(\log t^{-1}\big)^{1/3}, (i+2)t\right). \end{aligned}$$

Then we claim that, whatever the value of $C > 0$,

$$\begin{aligned} \mathsf{B}_{i,j} := \left\{\sup\left\{\mathrm{TF}^{(x_2,y_2)}_{n;(x_1,y_1)} : (x_1,y_1),(x_2,y_2) \in \mathcal{A}_{i,j}, y_2 > y_1\right\} > Ct^{2/3}\big(\log t^{-1}\big)^{1/3}\right\} \\ \subseteq \mathsf{D}^{(1)}_{i,j} \cup \mathsf{D}^{(2)}_{i,j}, \quad (5.1) \end{aligned}$$

where

$$\mathsf{D}^{(1)}_{i,j} := \left\{\mathrm{TF}^{f^{(1)}_{i,j}}_{n;e^{(1)}_{i,j}} \geq (2^{-1}C-1)t^{2/3}\big(\log t^{-1}\big)^{1/3}\right\}$$

and

$$\mathsf{D}^{(2)}_{i,j} := \left\{\mathrm{TF}^{f^{(2)}_{i,j}}_{n;e^{(2)}_{i,j}} \geq (2^{-1}C-1)t^{2/3}\big(\log t^{-1}\big)^{1/3}\right\}.$$

To see (5.1), define the vertical lines:

$$L_2 = \left\{x = (j-C+1)t^{2/3}\big(\log t^{-1}\big)^{1/3}\right\} \text{ and } L'_2 = \left\{x = (j+C-1)t^{2/3}\big(\log t^{-1}\big)^{1/3}\right\}.$$

Then, on the event $\mathsf{B}_{i,j}$, there exists a pair of points $(u,t_1),(v,t_2) \in \mathcal{A}_{i,j}$ such that either $\rho^{\leftarrow;(v,t_2)}_{n;(u,t_1)}$ intersects L_2 or $\rho^{\rightarrow;(v,t_2)}_{n;(u,t_1)}$ intersects L'_2. We now show that, when $\rho^{\leftarrow;(v,t_2)}_{n;(u,t_1)}$ intersects L_2, the event $\mathsf{D}^{(1)}_{i,j}$ occurs. Let

$$\rho := \rho^{\leftarrow;f^{(1)}_{i,j}}_{n;e^{(1)}_{i,j}} .$$

Let $\ell^{(1)}_{i,j}$ be the line segment joining $e^{(1)}_{i,j}$ and $f^{(1)}_{i,j}$. If $\rho(t_1) > u$, then

$$\begin{aligned} \rho(t_1) - \ell^{(1)}_{i,j}(t_1) &\geq (j-1)t^{2/3}\big(\log t^{-1}\big)^{1/3} - (j-2^{-1}C)t^{2/3}\big(\log t^{-1}\big)^{1/3} \\ &\geq (2^{-1}C-1)t^{2/3}\big(\log t^{-1}\big)^{1/3}, \end{aligned}$$

and thus $\mathsf{D}^{(1)}_{i,j}$ holds. Similarly, if $\rho(t_2) > v$, then $\mathsf{D}^{(1)}_{i,j}$ holds. Now assume that $\rho(t_1) < u$ and $\rho(t_2) < v$. Polymer ordering Lemma 3.2 then implies that $\rho(t) \leq \rho^{\leftarrow;(v,t_2)}_{n;(u,t_1)}(t)$ for all $t \in [t_1, t_2]$. Thus ρ intersects L_2 as well, and hence $\mathsf{D}^{(1)}_{i,j}$ occurs.

By similar reasoning, we see that, when $\rho^{\rightarrow;(v,t_2)}_{n;(u,t_1)}$ intersects L_2', the event $\mathsf{D}^{(2)}_{i,j}$ occurs. We have proved (5.1).

For any compatible pair of points $(u,v) \in \mathrm{AEP}_n(t)$, there exists a pair (i,j) for which $u, v \in \mathcal{A}_{i,j}$; here we use $\psi t \leq t^{2/3}$. Hence,

$$\begin{aligned} & \Big\{ t^{-2/3} \big(\log t^{-1}\big)^{-1/3} \mathrm{MTF}_n(t) > C \Big\} \\ \subseteq & \bigcup \Big\{ \mathsf{B}_{i,j} : i \in [\![0, \lceil t^{-1} \rceil]\!],\, j \in \Big[\!\!\Big[- \Big\lceil t^{-2/3} \big(\log t^{-1}\big)^{-1/3} \Big\rceil , \Big\lceil t^{-2/3} \big(\log t^{-1}\big)^{-1/3} \Big\rceil \Big]\!\!\Big] \Big\} \\ \subseteq & \bigcup \Big\{ \mathsf{D}^{(1)}_{i,j} \cup \mathsf{D}^{(2)}_{i,j} : i \in [\![0, \lceil t^{-1} \rceil]\!], j \in \Big[\!\!\Big[- \Big\lceil t^{-2/3} \big(\log t^{-1}\big)^{-1/3} \Big\rceil , \Big\lceil t^{-2/3} \big(\log t^{-1}\big)^{-1/3} \Big\rceil \Big]\!\!\Big] \Big\} , \end{aligned}$$

where (5.1) was used in the latter inclusion.

Thus, with c, k_0, n_0 as in the statement of Theorem 2.6, for any fixed t small enough that $\log t^{-1} \geq 2^2 k_0^3$, and all $n \geq n_0 (2t)^{-1}$, we have by a union bound and the translation invariance of the environment,

$$\begin{aligned} & \mathbb{P}\left(t^{-2/3} \big(\log t^{-1}\big)^{-1/3} \mathrm{MTF}_n(t) > C \right) \\ \leq \quad & \big(2t^{-2/3} \big(\log t^{-1}\big)^{-1/3} + 2\big)(t^{-1} + 2) \mathbb{P}\left(\mathrm{TF}^{(0,2t)}_{n;(0,0)} > (2^{-1}C - 1) t^{2/3} \big(\log t^{-1}\big)^{1/3} \right) \\ \leq \quad & 2(t^{-2/3} + 1)(t^{-1} + 2) \exp\left\{ -c(2^{-1}C - 1)^3 \log t^{-1} \right\} \leq 8 \cdot t^{c(C/2-1)^3 - 5/3} . \end{aligned}$$

Here the second inequality follows from Theorem 2.6 with $\boldsymbol{t} = 2t, \boldsymbol{k} = 2^{-2/3}(2^{-1}C - 1)\big(\log t^{-1}\big)^{1/3}$ and $\boldsymbol{n} = n$ being the parameter settings. The assumptions $\log t^{-1} \geq 2^2 k_0^3$, and $n \geq n_0 (2t)^{-1}$ ensure that $\boldsymbol{n} \geq n_0 \boldsymbol{t}^{-1}$ and $\boldsymbol{k} \geq k_0$ for any $C \geq 2$.

Finally, choosing C large enough that $c\,(C/2 - 1)^3 > 5/3$, we learn that

$$\mathbb{P}\left(t^{-2/3} \big(\log t^{-1}\big)^{-1/3} \mathrm{MTF}_n(t) > C \right) \to 0 \quad \text{as} \quad t \searrow 0 \,,$$

whenever $n = n(t)$ verifies $n \geq n_0 (2t)^{-1}$.

This completes the proof of Theorem 1.2. $\square$

6 Exponent pair $(1/3, 2/3)$ for polymer weight: Proof of Theorem 1.3

A lemma and two propositions will lead to the proof of Theorem 1.3 on the Hölder continuity of $[1,2] \mapsto \mathbb{R} : t \mapsto \mathsf{Wgt}_n(t)$, the polymer weight profile under vertical displacement.

Lemma 6.1. *There exist positive constants n_0, r_0, s_0, c_0 such that, for all $n \geq n_0$, $z \in [1,2]$, $t \in [r_0 n^{-1}, 2 - z]$ and $s \in [s_0, 10(nt)^{2/3}]$,*

$$\mathbb{P}\left(|\mathsf{Wgt}_n(z+t) - \mathsf{Wgt}_n(z)| \geq s t^{1/3} \right) \leq 5 e^{-c_0 s^{3/2}} \,.$$

We postpone the proof to Section 6.1 and first see how the lemma implies the upper bound in Theorem 1.3. This bound follows from Lemma 6.1 similarly to how Theorem 1.1 is derived from Proposition 4.1.

Proposition 6.2. *The sequence $\{\mathsf{Wgt}_n\}_{n \in \mathbb{N}}$ is tight in $(C[1,2], \|\cdot\|_\infty)$. Moreover, if Wgt_* is the weak limit of a weakly converging subsequence of $\{\mathsf{Wgt}_n\}_{n \in \mathbb{N}}$, then there exists a* *positive constant C not depending on the particular weak limit Wgt_* such that, almost surely,*

$$\limsup_{t\searrow 0} \sup_{1\le z\le 2-t} |\mathsf{Wgt}_*(z+t)-\mathsf{Wgt}_*(z)|\, t^{-1/3}\big(\log t^{-1}\big)^{-2/3} \le C\,. \tag{6.1}$$

Lemma 6.1 holds only for $t\in[\max\{r_0 n^{-1}, 10^{-3/2}s^{3/2}n^{-1}\}, 2-z]$ for some fixed constant $r_0>0$, and not for all $t\in[0,1-z]$, as was the case in Proposition 4.1. Hence, we directly show tightness in the following proof instead of applying Kolmogorov-Chentsov's tightness criterion.

Proof of Proposition 6.2. To show the first statement, concerning tightness, we follow the proof of the tightness criterion used to derive [6, Theorem 12.3]. To this end, it is enough to show that, for given $\varepsilon,\eta>0$, there exist $\delta\in[0,1]$, which we may harmlessly suppose to verify $\delta^{-1}\in\mathbb{N}$, and $N_0\in\mathbb{N}$ such that, for all $n\ge N_0$,

$$\sum_{j<\delta^{-1}} \mathbb{P}\left(\sup_{j\delta\le u\le (j+1)\delta} |\mathsf{Wgt}_n(1+u)-\mathsf{Wgt}_n(1+j\delta)|\ge\varepsilon\right)<\eta\,. \tag{6.2}$$

Assume then that $\varepsilon,\eta>0$ are given small constants. For the time being, fix some $\delta>0$ small to be chosen later (depending on ε and η).

Now fix any $M>1$. For any $z_1,z_2\in[1,2]$ such that $|z_1-z_2|=10^{-1}\varepsilon n^{-2/3}$, set $t=|z_1-z_2|$. For all $\lambda\in[0,\varepsilon]$, clearly $\lambda t^{-1/3}\le 10(nt)^{2/3}$. Hence, choosing $s=\lambda t^{-1/3}$ in Lemma 6.1, one gets, for all n large enough,

$$\mathbb{P}\Big(|\mathsf{Wgt}_n(z_1)-\mathsf{Wgt}_n(z_2)|\ge\lambda\Big)\le K_M\lambda^{-3M}|z_1-z_2|^M, \tag{6.3}$$

for some constant K_M depending only on M.

To establish tightness, the general strategy is to bound the distribution of the maximum of certain fluctuations. To achieve this, we crucially use the bound in (6.3) together with the inequality in [6, Theorem 12.2] that bounds the maximum of partial sums. To this end, fix $j<\delta^{-1}$, and break the interval $[j\delta,(j+1)\delta]$ into $\lceil\delta\beta^{-1}\rceil$-many subintervals of length $\beta:=10^{-1}\varepsilon n^{-2/3}$ each, and follow the proof of the inequality in [6, Theorem 12.2] to obtain

$$\mathbb{P}\left(\max_{0\le i\le\lceil\delta\beta^{-1}\rceil} |\mathsf{Wgt}_n(1+j\delta+i\beta)-\mathsf{Wgt}_n(1+j\delta)|\ge\frac{\varepsilon}{2}\right)\le K'_M\varepsilon^{-3M}\delta^M, \tag{6.4}$$

for some appropriate constant K'_M depending only on M. Note that by [6, Theorem 12.2] it directly follows that if (6.3) holds for all $\lambda>0$, then (6.4) holds for all $\varepsilon>0$. However, in our case (6.3) holds for all $\lambda\in[0,\varepsilon]$, instead of all $\lambda>0$. Hence, we resort to the proof of [6, Theorem 12.2] which shows that if for some fixed $\varepsilon>0$, (6.3) holds for all $\lambda\in[0,\varepsilon]$, then (6.4) holds for that particular ε.

Now, fix any $i\in[\![0,\lceil\delta\beta^{-1}\rceil-1]\!]$. For any $u\in[j\delta+i\beta, j\delta+(i+1)\beta]$, it clearly follows from the definition (1.8),

$$\mathsf{W}_{n;(0,1+u)}^{(0,1+j\delta+(i+1)\beta)}\ge -2n^{2/3}(1+j\delta+(i+1)\beta-(1+u))\ge -2n^{2/3}\beta\,,\ \text{and}$$

$$\mathsf{W}_{n;(0,1+j\delta+i\beta)}^{(0,1+u)}\ge -2n^{2/3}(1+u-(1+j\delta+i\beta))\ge -2n^{2/3}\beta\,.$$

Thus, for any $u\in[j\delta+i\beta, j\delta+(i+1)\beta]$, by superaddivity of polymer weights described in (3.2),

$$\mathsf{W}_{n;(0,0)}^{(0,1+j\delta+i\beta)}-2n^{2/3}\beta\le \mathsf{W}_{n;(0,0)}^{(0,1+j\delta+i\beta)}+\mathsf{W}_{n;(0,1+j\delta+i\beta)}^{(0,1+u)}\le \mathsf{W}_{n;(0,0)}^{(0,1+u)}\ \text{and}$$

$$\mathsf{W}_{n;(0,0)}^{(0,1+u)}\le \mathsf{W}_{n;(0,0)}^{(0,1+j\delta+(i+1)\beta)}-\mathsf{W}_{n;(0,1+u)}^{(0,1+j\delta+(i+1)\beta)}\le \mathsf{W}_{n;(0,0)}^{(0,1+j\delta+(i+1)\beta)}+2n^{2/3}\beta\,.$$

This, together with (1.11), imply that for any $i \in [\![0, \lceil \delta\beta^{-1} \rceil - 1]\!]$ and $u \in [j\delta + i\beta, j\delta + (i+1)\beta]$,

$$n^{1/3} \left|\mathsf{Wgt}_n(1+u) - \mathsf{Wgt}_n(1+j\delta)\right| \le 2n\beta + 2+ \tag{6.5}$$
$$n^{1/3} \max\Big\{ \left|\mathsf{Wgt}_n(1+j\delta+i\beta) - \mathsf{Wgt}_n(1+j\delta)\right|, \left|\mathsf{Wgt}_n(1+j\delta+(i+1)\beta) - \mathsf{Wgt}_n(1+j\delta)\right| \Big\}.$$

Since $2n\beta = 5^{-1}\varepsilon n^{1/3}$, for all n large enough that $2n^{-1/3} \le \varepsilon/5$, (6.4) and (6.5) imply

$$\begin{aligned} & \mathbb{P}\left(\sup_{j\delta \le u \le (j+1)\delta} \left|\mathsf{Wgt}_n(1+u) - \mathsf{Wgt}_n(1+j\delta)\right| \ge \varepsilon \right) \\ \le \quad & \mathbb{P}\left(\max_{0 \le i \le \lceil \delta\beta^{-1} \rceil} \left|\mathsf{Wgt}_n(1+j\delta+i\beta) - \mathsf{Wgt}_n(1+j\delta)\right| \ge \frac{\varepsilon}{2} \right) \le K'_M \varepsilon^{-3M} \delta^M. \end{aligned}$$

Thus, by choosing δ small enough that $K'_M \varepsilon^{-3M} \delta^{M-1} < \eta$, we obtain (6.2), and hence tightness.

To show (6.1), we follow the proof of Theorem 1.1(b). Let n_0, r_0, s_0 and c_0 be as in Lemma 6.1. For any fixed $m \in \mathbb{N}$ such that $c_1 \left(\log 2^m\right)^{2/3} \ge s_0$, and any $j \in \{0, 1, 2, \cdots, 2^m - 1\}$, and all $n \ge \max\{r_0 2^m, 10^{-3/2} c_1^{3/2} 2^m \log 2^m\}$, by applying Lemma 6.1 with the parameters $\boldsymbol{n} = n, \boldsymbol{t} = 2^{-m}$ and $\boldsymbol{s} = c_1 \left(\log 2^m\right)^{2/3}$, it follows that

$$\mathbb{P}\left(\left|\mathsf{Wgt}_n(1+(j+1)2^{-m}) - \mathsf{Wgt}_n(1+j2^{-m})\right| > c_1 2^{-\frac{m}{3}} \left(\log 2^m\right)^{2/3} \right) \le 5 \cdot 2^{-m(c_0 c_1^{3/2})}.$$

Now, observe that (4.4) in the proof of Lemma 4.2 carries over verbatim to the present case. By choosing c_1 high enough that $c_0 c_1^{3/2} > 1$, and exactly imitating the rest of the proof of Lemma 4.2 followed by the proof of Theorem 1.1(b), we complete the proof of Proposition 6.2. ☐

Turning to prove the lower bound in (1.12), we restate it now.

Proposition 6.3. *There exists a constant $c > 0$ such that, almost surely,*

$$\liminf_{t \searrow 0} \sup_{1 \le z \le 2-t} t^{-1/3} \big(\log t^{-1}\big)^{-2/3} \left|\mathsf{Wgt}_*(z+t) - \mathsf{Wgt}_*(z)\right| \ge c\,.$$

This result will follow directly from weight superadditivity, i.e. $\mathsf{W}_{n;(0,0)}^{(0,1+z+t)} - \mathsf{W}_{n;(0,0)}^{(0,1+z)} \ge \mathsf{W}_{n;(0,1+z)}^{(0,1+z+t)}$ for $z, t > 0$, control on weight with given endpoints via Theorem 2.3, independence in disjoint strips, and the weight $\mathsf{W}_{n;(0,1+z)}^{(0,1+z+t)}$ depending on the configuration in the strip delimited by the lines $y = 1 + z$ and $y = 1 + z + t$. The proof is reminiscent of an argument for a similar statement made for Brownian motion: see the proof on page 362 of Exercise 1.7 in the book [17].

Proof of Proposition 6.3. We need to show that, for some constant $c > 0$, almost surely, there exists $\varepsilon > 0$ such that, for all $0 < t < \varepsilon$ and some $z \in [1, 2-t]$,

$$\left|\mathsf{Wgt}_*(z+t) - \mathsf{Wgt}_*(z)\right| \ge c t^{1/3} \big(\log t^{-1}\big)^{2/3}.$$

Let $c > 0$ satisfy $2^{3/2} c_2 c^{3/2} < 1$, where c_2 arises from Theorem 2.3. For integers $n, m \ge 1$ and $k \in \{0, 1, 2, \cdots, m-1\}$, we define the events

$$\mathsf{A}_{k,m,n} = \Big\{ \mathsf{Wgt}_n\left(1+(k+1)m^{-1}\right) - \mathsf{Wgt}_n\left(1+km^{-1}\right) \ge c m^{-1/3} \left(\log m\right)^{2/3} \Big\}$$

and

$$\mathsf{A}_{k,m} = \left\{ \mathsf{Wgt}_* \left(1 + (k+1)m^{-1}\right) - \mathsf{Wgt}_* \left(1 + km^{-1}\right) \geq cm^{-1/3} \left(\log m\right)^{2/3} \right\} .$$

Also let

$$\mathsf{B}_{k,m,n} = \left\{ \mathsf{W}_{n;(0,1+km^{-1})}^{(0,1+(k+1)m^{-1})} \geq cm^{-1/3} \left(\log m\right)^{2/3} + 2n^{-1/3} \right\} .$$

Let n_0, s_0 and c_2 be as in Theorem 2.3, and let m_0 be large enough that $2c(\log m_0)^{2/3} \geq \max\{s_0, 4n_0^{-1/3}\}$. Then from Theorem 2.3 with parameter settings $\boldsymbol{t_1} = 1 + km^{-1}, \boldsymbol{t_2} = 1 + (k+1)m^{-1}, \boldsymbol{t_{1,2}} = m^{-1}, \boldsymbol{n} = n$ and $\boldsymbol{s} = 2c(\log m)^{2/3}$, for all $m \geq m_0$ and $n \geq n_0 m$,

$$\mathbb{P}(\mathsf{B}_{0,m,n}) \geq \mathbb{P}\left(\mathsf{W}_{n;(0,1+km^{-1})}^{(0,1+(k+1)m^{-1})} \geq 2cm^{-1/3} \left(\log m\right)^{2/3} \right) \geq e^{-2^{3/2} c_2 c^{3/2} \log m} = m^{-2^{3/2} c_2 c^{3/2}} . \tag{6.6}$$

Here the first inequality follows because

$$cm^{-1/3} \left(\log m\right)^{2/3} \geq cm^{-1/3} \left(\log m_0\right)^{2/3} \geq 2n_0^{-1/3} m^{-1/3} \geq 2n^{-1/3}$$

for $m \geq m_0$, $2c(\log m_0)^{2/3} \geq 4n_0^{-1/3}$ and $n \geq n_0 m$.

Now $\mathsf{B}_{k,m,n}$ are i.i.d. random variables for $k \in \{0, 1, 2, \cdots, m-1\}$ as the weights of polymers over disjoint regions are independent. Also using $\mathsf{W}_{n;(0,0)}^{(0,1+(k+1)m^{-1})} - \mathsf{W}_{n;(0,0)}^{(0,1+km^{-1})} \geq \mathsf{W}_{n;(0,1+km^{-1})}^{(0,1+(k+1)m^{-1})}$ by superadditivity of polymer weights, together with (1.11), we get that $\mathsf{B}_{k,m,m} \subseteq \mathsf{A}_{k,m,n}$. Thus, using (6.6), for all $m \geq m_0$ and $n \geq n_0 m$,

$$\begin{aligned} \mathbb{P}\left(\bigcap_{k=0}^{m-1} \mathsf{A}_{k,m,n}^c \right) &\leq \mathbb{P}\left(\bigcap_{k=0}^{m-1} \mathsf{B}_{k,m,n}^c \right) = \left(1 - \mathbb{P}(\mathsf{B}_{0,m,n})\right)^m \\ &\leq \exp\left\{-m\mathbb{P}(\mathsf{B}_{0,m,n})\right\} \leq \exp\left\{ -m^{1-2^{3/2} c_2 c^{3/2}} \right\}, \end{aligned} \tag{6.7}$$

where we use that $1 - x \leq e^{-x}$ for all $x \geq 0$.

Next, similarly to the first part of the proof of Lemma 4.2, let $\{\mathsf{Wgt}_{n_r}\}_r$ be a subsequence of $\{\mathsf{Wgt}_n\}_n$ such that $\mathsf{Wgt}_{n_r} \Rightarrow \mathsf{Wgt}_*$ as random variables in $(C[1,2], \|\cdot\|_\infty)$ (where $\Rightarrow$ denotes convergence in distribution). Since for $a, b \in [1,2]$, the map $T_{a,b}$ defined by $(C[1,2], \|\cdot\|_\infty) \mapsto (\mathbb{R}, |\cdot|) : f \mapsto f(a) - f(b)$ is continuous, the set

$$\mathsf{U} := \bigcap \left\{ T_{1+(k+1)m^{-1}, 1+km^{-1}}^{-1} \left(-\infty, cm^{-1/3} \big(\log m\big)^{2/3} \right) : k = 0, 1, \cdots, m-1 \right\}$$

is open. Thus, by the Portmanteau theorem,

$$\mathbb{P}\left(\bigcap_{k=0}^{m-1} \mathsf{A}_{k,m}^c \right) \leq \liminf_r \mathbb{P}\left(\bigcap_{k=0}^{m-1} \mathsf{A}_{k,m,n_r}^c \right) \leq \limsup_n \mathbb{P}\left(\bigcap_{k=0}^{m-1} \mathsf{A}_{k,m,n}^c \right) .$$

From here, using (6.7) and that our given choice of the constant c ensures $2^{3/2} c_2 c^{3/2} < 1$, we get

$$\sum_{m=m_0}^{\infty} \mathbb{P}\left(\bigcap_{k=0}^{m-1} \mathsf{A}_{k,m}^c \right) \leq \sum_{m=m_0}^{\infty} \limsup_n \mathbb{P}\left(\bigcap_{k=0}^{m-1} \mathsf{A}_{k,m,n}^c \right) \leq \sum_{m=m_0}^{\infty} \exp\left\{ -m^{1-2^{3/2} c_2 c^{3/2}} \right\} < \infty.$$

Hence, using the Borel-Cantelli lemma, almost surely there exists $M_0 \in \mathbb{N}$ such that for all $m \geq M_0$, one has some $k_m \leq m-1$ with $z = 1 + k_m m^{-1}$ satisfying

$$\left| \mathsf{Wgt}_*(z + m^{-1}) - \mathsf{Wgt}_*(z) \right| \geq cm^{-1/3} \left(\log m\right)^{2/3} .$$

Let $\varepsilon = M_0^{-1}$. Also let M_0^{-1} be small enough in the sense of Proposition 6.2: namely, almost surely for all $t \in [0, M_0^{-1}]$, $\sup_{1 \le z \le 2-t} |\mathsf{Wgt}_*(z+t) - \mathsf{Wgt}_*(z)| t^{-1/3} \big(\log t^{-1}\big)^{-2/3} \le 2C$. Then, for any given $t \in [0, \varepsilon]$, let m be such that $(m+1)^{-1} < t \le m^{-1}$. Then for $z = 1 + k_m m^{-1}$,

$$\begin{aligned} & \left|\mathsf{Wgt}_*(z+t) - \mathsf{Wgt}_*(z)\right| \\ \ge \; & \left|\mathsf{Wgt}_*\left(z+m^{-1}\right) - \mathsf{Wgt}_*(z)\right| - \left|\mathsf{Wgt}_*(z+t) - \mathsf{Wgt}_*\left(z+m^{-1}\right)\right| \\ \ge \; & c m^{-1/3} (\log m)^{2/3} - 2C \left(m^{-1} - (m+1)^{-1}\right)^{1/3} \left(\log\left(m^{-1} - (m+1)^{-1}\right)^{-1}\right)^{2/3}. \end{aligned}$$

As the second term decays much faster than the first, choosing M_0 large enough so that the second term is smaller that $2^{-1} c m^{-1/3} (\log m)^{2/3}$ gives the result. ☐

Proof of Theorem 1.3. This result follows from Proposition 6.2 and Proposition 6.3. ☐

6.1 Upper bound on polymer weight fluctuation: Proof of Lemma 6.1

In this subsection, we complete the proof of Theorem 1.3. The remaining element, Lemma 6.1, will be derived from Lemmas 6.4 and 6.5.

Lemma 6.4. *There exist positive constants s_0, r_0 and c_0 such that for $s \ge s_0$, $z \in [1,2]$ and $t \in [r_0 n^{-1}, 2-z]$,*

$$\mathbb{P}\left(\mathsf{W}_{n;(0,0)}^{(0,z)} \ge \mathsf{W}_{n;(0,0)}^{(0,z+t)} + s t^{1/3}\right) \le e^{-c_0 s^{3/2}}.$$

Proof. Using $\mathsf{W}_{n;(0,0)}^{(0,z+t)} \ge \mathsf{W}_{n;(0,0)}^{(0,z)} + \mathsf{W}_{n;(0,z)}^{(0,z+t)}$, we see that, for $nt \ge r_0$ and $s \ge s_0$,

$$\mathbb{P}\left(\mathsf{W}_{n;(0,0)}^{(0,z)} \ge \mathsf{W}_{n;(0,0)}^{(0,z+t)} + s t^{1/3}\right) \le \mathbb{P}\big(\mathsf{W}_{n;(0,z)}^{(0,z+t)} \le -s t^{1/3}\big) \le e^{-c s^{3/2}},$$

where the latter inequality follows from the moderate deviation estimate Theorem 2.2, with $\boldsymbol{t_1} = z, \boldsymbol{t_2} = z+t, \boldsymbol{n} = n$ and $\boldsymbol{s} = s$, and setting r_0 and s_0 to equal n_0 and s_0 respectively from the statement of Theorem 2.2. ☐

Next is the more subtle of the two constituents of Lemma 6.1.

Lemma 6.5. *There exist positive constants n_0, s_2, r_1 and c_0 such that, for $n \ge n_0$, $t \in [r_1 n^{-1}, 2-z]$, $s \in [s_2, 10(nt)^{2/3}]$ and $z \in [1,2]$,*

$$\mathbb{P}\left(\mathsf{W}_{n;(0,0)}^{(0,z+t)} \ge \mathsf{W}_{n;(0,0)}^{(0,z)} + s t^{1/3}\right) \le 4\, e^{-c_0 s^{3/2}}. \tag{6.8}$$

This proof is reminiscent of arguments used in [5] and [4]. We first explain the basic idea, which is illustrated in Figure 5. A path may be formed from $(0,0)$ to $(0,z)$ by following the route of a polymer from $(0,0)$ to $(0,z+t)$ until its location, $(U, z-t)$ say, at height $z-t$; and then following a polymer from $(U, z-t)$ to $(0,z)$. The discrepancy in weight between the original polymer, from $(0,0)$ to $(0,z+t)$, and the newly formed path, from $(0,0)$ to $(0,z)$, is equal to the difference in weights between the polymer from $(U, z-t)$ to $(0, z+t)$ and that from $(U, z-t)$ to $(0,z)$. The latter two polymers have duration of order t; Theorem 2.4 may then show that their weights have order $t^{1/3}$. Thus, the weight difference $\mathsf{W}_{n;(0,0)}^{(0,z+t)} - \mathsf{W}_{n;(0,0)}^{(0,z)}$, which is at most the discrepancy we are considering, is seen to be unlikely to exceed order $t^{1/3}$.

Proof of Lemma 6.5. To implement this idea, we will consider, for definiteness, the leftmost polymer from $(0,0)$ to $(z+t, 0)$, namely $\rho_{n;(0,0)}^{\leftarrow;(z+t,0)}$. In accordance with the notation in the plan, we will set $U = \rho_{n;(0,0)}^{\leftarrow;(z+t,0)}(z-t)$.

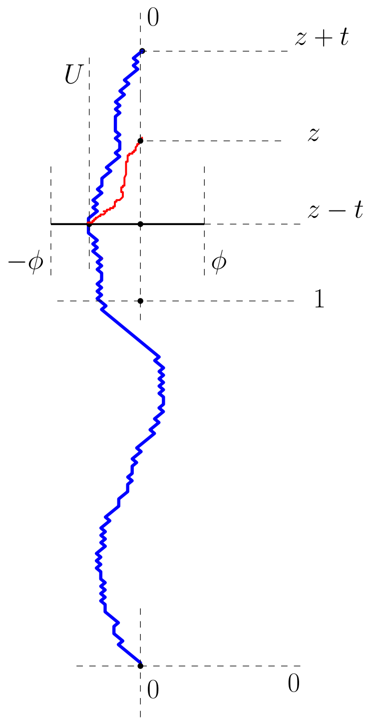

Figure 5: When the thick blue polymer $\rho_{n;(0,0)}^{\leftarrow;(z+t,0)}$ crosses height $z-t$ without immoderately high fluctuation, it may be diverted via the red polymer to form a path of comparable weight from $(0,0)$ to $(z,0)$.

The height-$(z-t)$ polymer location U typically has order $t^{2/3}$. The plan will run into trouble if U is atypically high, because then the two short polymers running to $(0,z+t)$ and $(0,z)$ from $(U,z-t)$ will have large negative weights dictated by parabolic curvature.

To cope with this difficulty, we introduce a *good* event G,

$$\mathsf{G} = \{|U| \le \phi\} ,$$

specified in terms of a parameter ϕ that is set equal to $D^{-1}s^{1/2}(2t)^{2/3}$. Here, the constant D is chosen to be $2^{2/3}10^{1/2}C_0$, with C_0 given by Theorem 2.4. In view of Theorem 2.5, this choice of ϕ ensures that the event G fails to occur with probability of order $\exp\{-\Theta(1)s^{3/2}\}$. (The appearance of the factor of D^{-1} in ϕ is a detail concerning values of s in Lemma 6.5 close to the maximum value $10(nt)^{2/3}$.)

Indeed, applying Theorem 2.5 with $\boldsymbol{n}=n, \boldsymbol{t_1}=0, \boldsymbol{t_2}=z+t, \boldsymbol{t}=z-t, \boldsymbol{x}=0, \boldsymbol{y}=0$ and $\boldsymbol{s}=D^{-1}s^{1/2}$, we find that, when $n \ge n_0$ (a bound which ensures that the hypothesis that $\boldsymbol{nt_{1,2}} \ge n_0$ is met) and $s \ge s_1$,

$$\mathbb{P}(\mathsf{G}^c) \le 2\exp\{-cD^{-3}s^{3/2}\}, \tag{6.9}$$

where the positive constants c and s_1 are provided by the theorem being applied.

When G occurs,

$$|U| \le D^{-1}s^{1/2}(2t)^{2/3} \le D^{-1}2^{2/3}10^{1/2}tn^{1/3} < tn^{1/3} ,$$

because $s \le 10(nt)^{2/3}$, $D = 2^{2/3}10^{1/2}C_0$ and $C_0 > 1$. As we saw in Subsection 1.1.2, it is this bound on $|U|$ that ensures the existence of polymers between $(U,z-t)$ and $(0,z)$. By superadditivity of polymer weights, we thus have

$$\mathsf{W}_{n;(0,0)}^{(0,z)} \geq \mathsf{W}_{n;(0,0)}^{(U,z-t)} + \mathsf{W}_{n;(U,z-t)}^{(0,z)} \,.$$

Thus, when G occurs,

$$\begin{aligned}
\mathsf{W}_{n;(0,0)}^{(0,z+t)} - \mathsf{W}_{n;(0,0)}^{(0,z)} &\leq \mathsf{W}_{n;(0,0)}^{(0,z+t)} - \mathsf{W}_{n;(0,0)}^{(U,z-t)} - \mathsf{W}_{n;(U,z-t)}^{(0,z)} \\
&= \mathsf{W}_{n;(U,z-t)}^{(0,z+t)} - \mathsf{W}_{n;(U,z-t)}^{(0,z)} \leq \sup_{x\in[-\phi,\phi]} \Big(\mathsf{W}_{n;(x,z-t)}^{(0,z+t)} - \mathsf{W}_{n;(x,z-t)}^{(0,z)} \Big) \,,
\end{aligned}$$

where the equality is dependent on the definition of U and the final inequality on the occurrence of G. We see then that

$$\begin{aligned}
&\mathbb{P}\left(\mathsf{G} \cap \left\{ \mathsf{W}_{n;(0,0)}^{(0,z+t)} \geq \mathsf{W}_{n;(0,0)}^{(0,z)} + st^{1/3} \right\}\right) \\
&\leq \mathbb{P}\left(\sup_{x\in[-\phi,\phi]} \Big(\mathsf{W}_{n;(x,z-t)}^{(0,z+t)} - \mathsf{W}_{n;(x,z-t)}^{(0,z)} \Big) \geq st^{1/3} \right) \\
&\leq \mathbb{P}\left(\sup_{x\in[-\phi,\phi]} \Big| \mathsf{W}_{n;(x,z-t)}^{(0,z+t)} \Big| > 2^{-1} st^{1/3} \right) + \mathbb{P}\left(\sup_{x\in[-\phi,\phi]} \Big| \mathsf{W}_{n;(x,z-t)}^{(0,z)} \Big| > 2^{-1} st^{1/3} \right) . \qquad (6.10)
\end{aligned}$$

The latter two probabilities will each be bounded above by a union bound over several applications of Theorem 2.4. Addressing the first of these probabilities to begin with, we set parameters for a given application of the theorem, taking $\boldsymbol{I}$ to be a given interval of length at most $t^{2/3}$ contained in $[-\phi,\phi]$ and $\boldsymbol{J} = \{0\}$, and also setting $\boldsymbol{n} = n, \boldsymbol{t_1} = z-t, \boldsymbol{t_2} = z$ and $\boldsymbol{s} = 4^{-1}s$.

The theorem's hypothesis concerning inclusion for the interval $\mathbf{I}$ (and $\mathbf{J}$) is ensured because

$$|x| \leq D^{-1} s^{1/2} (2t)^{2/3} \leq 2^{2/3} 10^{1/4} D^{-1} s^{1/4} n^{1/6} t^{5/6} < C_0^{-1} s^{1/4} n^{1/6} t^{5/6} \,,$$

for $x \in [-\phi,\phi]$, where here we use $s \leq 10(nt)^{2/3}$ and $D = 2^{2/3}10^{1/2}C_0 > 2^{2/3}10^{1/4}C_0$.

In these applications of Theorem 2.4, the parabolic curvature term inside the supremum, $t^{-4/3}x^2$, is at most $t^{-4/3}\phi^2$. It is thus also at most $s/4$, because $\phi = D^{-1}s^{1/2}(2t)^{2/3}$ and $D \geq 2^{5/3}$.

Thus, dividing $[-\phi,\phi]$ into $\lceil 2^{5/3} D^{-1} s^{1/2} \rceil$-many consecutive intervals of length at most $t^{2/3}$, we are indeed able to apply Theorem 2.4 and a union bound, finding that, for $n_1 \in \mathbb{N}$ and $C, c > 0$ the constants furnished by the theorem, and for $nt \geq n_1$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x\in[-\phi,\phi]} \Big| \mathsf{W}_{n;(x,z-t)}^{(0,z)} \Big| > 2^{-1} st^{1/3} \right) \\
&\leq \mathbb{P}\left(\sup_{x\in[-\phi,\phi]} \Big| t^{-1/3} \mathsf{W}_{n;(x,z-t)}^{(0,z)} + t^{-4/3}x^2 \Big| > 4^{-1}s \right) \leq \lceil 2^{5/3} D^{-1} s^{1/2} \rceil C e^{-cs^{3/2}} \leq e^{-c's^{3/2}} \,,
\end{aligned}$$

for $c' = 2^{-1}c$ and $s \geq s_0$ where s_0 is chosen in such a way that $e^{2^{-1}cs_0^{3/2}} \geq C\lceil 2^{5/3} D^{-1} s_0^{1/2} \rceil$.

The second probability in (6.10) is bounded above by similar means. Several applications of Theorem 2.4 will be made. In a given application, the parameters $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{n}$ and $\boldsymbol{s}$ are chosen as before, but we now set $\boldsymbol{t_1} = z-t$ and $\boldsymbol{t_2} = z+t$, so that $\boldsymbol{t_{1,2}}$ equals $2t$, rather than t. The curvature term $(2t)^{-4/3}x^2$ is bounded above by $(2t)^{-4/3}\phi^2$, a smaller bound than before, so that the preceding bound of $s/4$ remains valid. The condition for inclusion for the intervals $\boldsymbol{I}$ (and $\boldsymbol{J}$), namely $\phi \leq C_0^{-1} s^{1/4} n^{1/6} (2t)^{5/6}$, is weaker than it was previously and is thus satisfied. Hence, using Theorem 2.4 and a union bound, we find that, for all $n \geq 2^{-1}n_1 t^{-1}$,

$$\mathbb{P}\left(\sup_{x\in[-\phi,\phi]}\left|\mathsf{W}_{n;(x,z-t)}^{(0,z+t)}\right| > 2^{-1}st^{1/3}\right) \leq e^{-c's^{3/2}}\,,$$

for $s \geq s_0$.

Combining (6.9) and (6.10) with the two bounds just derived, we obtain Lemma 6.5 by taking $c_0 > 0$ to be less than $\min\{cD^{-3}, c'\}$, s_2 to be suitably greater than $\max\{s_0, s_1\}$, and $r_1 = 2^{-1}n_1$. □

Proof of Lemma 6.1. This follows immediately using (1.11) and from Lemmas 6.4 and 6.5 and a union bound. □

7 Lower bound on transversal fluctuation: Proof of Proposition 1.4

In this last section we shall prove the lower bound on the transversal fluctuation of the polymer, the corresponding upper bound of which was proved in [5, Theorem 11.1] (and is stated here, with the optimal exponent in the bound, as Theorem 2.6). In fact, Proposition 1.4 does slightly more than just providing a corresponding lower bound on the quantity whose upper bound is proved in Theorem 2.6. Indeed, in Proposition 1.4, one takes the minimum over the transversal fluctuations of all the polymers between two fixed points, and not just the transversal fluctuation of the leftmost one. The proof of Proposition 1.4 crucially uses the polymer weight lower tail Theorem 2.3. We also fix the constant α_0 in this Proposition 1.4 as $\alpha_0 = C_0^{-2}3^{-5/3}10^{-1/2}$, where C_0 is as in Theorem 2.4. This choice of α_0 ensures that the condition in the hypothesis of Theorem 2.4 is met whenever it is applied.

Proof of Proposition 1.4. We prove the proposition for $t_1 = 0$ and $t_2 = 1$. The case for general $t_1 < t_2$ follows readily using the scaling principle (2.1).

A box is a subset of $\mathbb{R}^2$ of the form $[a,b]\times[r_1,r_2]$, where $a \leq b$ and $r_1 \leq r_2$. Any box has a lower and an upper side, namely $[a,b]\times\{r_1\}$ and $[a,b]\times\{r_2\}$.

The key box for the proof is Strip, now specified to be $[-s,s]\times[0,1]$. Proposition 1.4 is, after all, a lower bound on the probability that there exists a polymer between $(0,0)$ and $(0,1)$ that escapes Strip.

We divide Strip into three further boxes, writing Mid for the box $[-s,s]\times[1/3,2/3]$, and South and North for the boxes obtained from Mid by vertical translations of $-1/3$ and $1/3$. We further set West to be the box obtained from Mid by a horizontal translation of $-2s$. See Figure 6.

Recall that, when (x,t_1) and (y,t_2) verify $n^{1/3}t_{1,2} \geq |y-x|$, we denote the polymer weight with this pair of endpoints by $\mathsf{W}_{n;(x,t_1)}^{(y,t_2)}$. We now use a set theoretic notational convention to refer in similar terms to the set of weights of polymers between two collections of endpoint locations. Indeed, let I and J be compact real intervals. We will write

$$\mathsf{W}_{n;(I,t_1)}^{(J,t_2)} = \left\{\mathsf{W}_{n;(x,t_1)}^{(y,t_2)} : x\in I, y\in J\right\};$$

we will ensure that whenever this notation is used, $(x,t_1) \overset{n}{\preceq} (y,t_2)$ for all $x\in I$ and $y \in J$ in the sense of Subsection 1.1.2. When an interval is a singleton, $I = \{x\}$ say, we write (x,t_1) instead of $(\{x\},t_1)$ when using this notation.

To any box B and $s\in\mathbb{R}$, we define the event $\mathrm{High}(B,s)$ that the weight of some path that is contained in B with starting point in the lower side of B and ending point in the upper side of B is at least s.

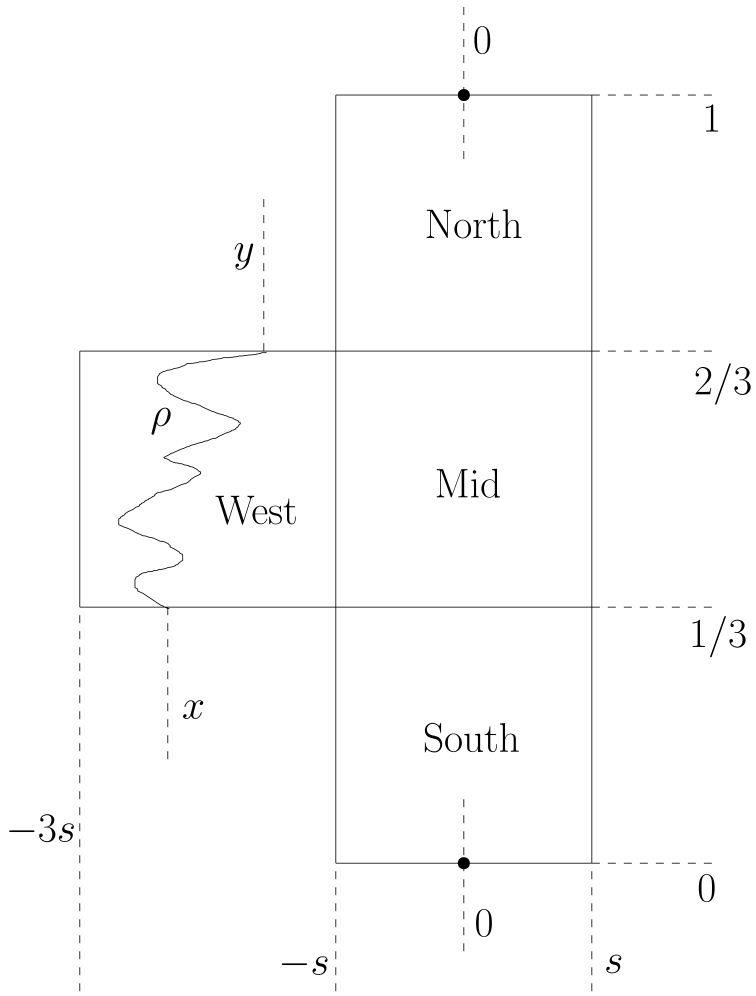

Figure 6: In Case High, the high weight path ρ is extended to form a path from $(0,0)$ to $(0,1)$ whose weight exceeds that of any path between these points that remains in $\mathrm{Strip} = \mathrm{North} \cup \mathrm{Mid} \cup \mathrm{South}$.

Our approach to proving Proposition 1.4 gives a central role to the event $\mathrm{High}(\mathrm{Mid}, 300s^2)$. It may be expected that the order of probability of this event is $\exp\big\{ - \Theta(1)s^3 \big\}$, but we do not attempt to prove this. Rather, we analyse two cases, called *High* and *Low*, according to the value of the event's probability.

We will quantify the notion of high or low probability for $\mathrm{High}(\mathrm{Mid}, 300s^2)$ in terms of the decay rate for a very high weight polymer between $(0,0)$ and $(0,1)$. Indeed, noting from Theorem 2.3 that there exists $C > 0$ such that, for $s \geq s_0$,

$$\mathbb{P}\Big(\mathsf{W}_{n;(0,0)}^{(0,1)} \geq 1000s^2\Big) \geq \exp\big\{ - Cs^3 \big\} \, , \tag{7.1}$$

we declare that Case High occurs if

$$\mathbb{P}\big(\mathrm{High}(\mathrm{Mid}, 300s^2)\big) \geq \exp\big\{ - 2Cs^3 \big\} \, ;$$

Case Low occurs when Case High does not.

In order to analyse Case High, we introduce a *favourable* event F. The event is specified as the intersection of the following events:

- $\mathsf{G}_1 = \Big\{ \inf \mathsf{W}_{n;(0,0)}^{([-3s,-s],1/3)} \geq -50s^2 \Big\}$;
- $\mathsf{G}_2 = \Big\{ \inf \mathsf{W}_{n;([-3s,-s],2/3)}^{(0,1)} \geq -50s^2 \Big\}$;
- $\mathsf{G}_3 = \Big\{ \sup \mathsf{W}_{n;(0,0)}^{([-s,s],1/3)} \leq 50s^2 \Big\}$;
- $\mathsf{G}_4 = \Big\{ \sup \mathsf{W}_{n;([-s,s],2/3)}^{(0,1)} \leq 50s^2 \Big\}$;
- and G_5 is the event that $\mathrm{High}(\mathrm{Mid}, 50s^2)$ does not occur.

Thus, the occurrence of F forces the absence of any high weight path inside Mid that crosses this box from its lower to its upper side, while also ensuring that any polymer connecting $(0,0)$ (or $(0,1)$) to the lower (or upper) sides of Mid and West is not of very low weight. We claim that F is a high probability event, proving this by applying Theorem 2.4. Indeed, for the events G_1 and G_3 entailed by F, we make several applications of Theorem 2.4. For a given application, we consider the parameter settings $\boldsymbol{n} = n, \boldsymbol{t_1} = 0, \boldsymbol{t_2} = 1/3, \boldsymbol{s} = 10s^2, \boldsymbol{I} = \{0\}$ and

$$\boldsymbol{J} = \big[-3s + (i-1)3^{-2/3}, \max\{-3s + i3^{-2/3}, s\} \big]$$

for some $i \in \{1, 2, \cdots, \lceil 4 \cdot 3^{2/3} s \rceil\}$. The condition on inclusion for the intervals $\boldsymbol{I}$ and $\boldsymbol{J}$ is satisfied since for $y \in [-3s, s]$,

$$|y| \leq 3s \leq \boldsymbol{s}^{1/2} \leq 10^{1/4}\alpha_0^{1/2}\boldsymbol{n}^{1/6}\boldsymbol{s}^{1/4} \leq 3^{5/6}10^{1/4}\alpha_0^{1/2}\boldsymbol{n}^{1/6}\boldsymbol{s}^{1/4}\boldsymbol{t_{1,2}}^{5/6} = C_0^{-1}\boldsymbol{n}^{1/6}\boldsymbol{s}^{1/4}\boldsymbol{t_{1,2}}^{5/6} ,$$

where we use that $s \leq \alpha_0 n^{1/3}$ and our given choice of α_0 has been made so that $\alpha_0 = C_0^{-2} 3^{-5/3} 10^{-1/2}$. Also the parabolic curvature inside the supremum is

$$\sup_{y \in [-3s,s]} 3^{4/3} y^2 \leq 3^{4/3} \cdot 3^2 s^2 < 40s^2 .$$

Thus, dividing $[-3s, s]$ into $\lceil 4 \cdot 3^{2/3} s \rceil$-many intervals of length at most $3^{-2/3}$ and using Theorem 2.4 and a union bound, it follows that, for s large enough and $n \geq 3n_0$,

$$\mathbb{P}(\mathsf{G}_1^c \cup \mathsf{G}_3^c) \leq \mathbb{P}\left(\sup_{y \in [-3s,s]} \left| 3^{1/3} \mathsf{W}_{n;(0,0)}^{(y,1/3)} + 3^{4/3} y^2 \right| > 10s^2 \right) \leq \lceil 4 \cdot 3^{2/3} s \rceil C e^{-cs^3} \leq 6^{-1} .$$

Similarly for the events G_2 and G_4, in a given application of Theorem 2.4, we set the parameters $\boldsymbol{n} = n, \boldsymbol{t_1} = 2/3, \boldsymbol{t_2} = 1, \boldsymbol{s} = 10s^2, \boldsymbol{I} = [-3s+(i-1)3^{-2/3}, \max\{-3s+i3^{-2/3}, s\}]$ and $\boldsymbol{J} = \{0\}$, for some $i \in \{1, 2, \cdots, \lceil 4 \cdot 3^{2/3} s \rceil\}$. The condition on the inclusion for the intervals $\boldsymbol{I}$ and $\boldsymbol{J}$ is ensured exactly in the same way as before, and the parabolic curvature is bounded above by $40s^2$. Hence, using Theorem 2.4 and a union bound, it follows that, for s large enough and $n \geq 3n_0$,

$$\mathbb{P}(\mathsf{G}_2^c \cup \mathsf{G}_4^c) \leq \lceil 4 \cdot 3^{2/3} s \rceil C e^{-cs^3} \leq 6^{-1} .$$

Finally, for G_5, observe that, since paths between two fixed endpoints constrained to stay in a box have smaller weight than does the polymer between these endpoints, we can again use Theorem 2.4. For a given application of Theorem 2.4, take $\boldsymbol{n} = n, \boldsymbol{t_1} = 1/3, \boldsymbol{t_2} = 2/3, \boldsymbol{s} = 40s^2, \boldsymbol{I} = [-s + (i-1)3^{-2/3}, \max\{-s + i3^{-2/3}, s\}]$ and $\boldsymbol{J} = [-s + (j-1)3^{-2/3}, \max\{-s + j3^{-2/3}, s\}]$ for $i \in \{1, 2, \cdots, \lceil 2 \cdot 3^{2/3} s \rceil\}$ and $j \in \{1, 2, \cdots, \lceil 2 \cdot 3^{2/3} s \rceil\}$. As before, the condition on inclusion for $\boldsymbol{I}$ and $\boldsymbol{J}$ is satisfied, and the parabolic curvature is at most $3^{4/3} s^2$, which is less than $10s^2$. Thus, applying Theorem 2.4 and a union bound, we find that, for $n \geq 3n_0$ and s large,

$$\mathbb{P}(\mathsf{G}_5^c) \leq \mathbb{P}\left(\sup \mathsf{W}_{n;([-s,s],1/3)}^{([-s,s],2/3)} > 50s^2 \right) \leq \lceil 2 \cdot 3^{2/3} s \rceil^2 C e^{-cs^3} \leq 6^{-1} .$$

Thus we have $\mathbb{P}(\mathsf{F}) \geq 1/2$ by a union bound. In Case High, we also have

$$\mathbb{P}\big(\mathrm{High}(\mathrm{West}, 300s^2)\big) \geq \exp\big\{ - 2Cs^3 \big\} \, ,$$

because West is a translate of Mid. Since the interior of West is disjoint from the regions that dictate the occurrence of F, we see that

$$\mathbb{P}\Big(\mathrm{High}(\mathrm{West}, 300s^2) \cap \mathsf{F}\Big) \geq 2^{-1} \exp\big\{ - 2Cs^3 \big\}. \tag{7.2}$$

When $\mathrm{High}(\mathrm{West}, 300s^2) \cap \mathsf{F}$ occurs, a high weight path connecting $(0,0)$ to $(0,1)$ may be formed by running it through West. Indeed, and as Figure 6 depicts, let ρ denote a polymer running across, and contained in, West, whose weight is at least $300s^2$. If $x, y \in [-3s, -s]$ are such that $(x, 1/3)$ and $(y, 2/3)$ are ρ's endpoints, then the path $\rho_{n;(0,0)}^{\leftarrow;(x,1/3)} \circ \rho \circ \rho_{n;(y,2/3)}^{\leftarrow,(0,1)}$ connects $(0,0)$ to $(0,1)$ and has weight at least $-50s^2 + 300s^2 - 50s^2$, in view of the first two conditions that specify F.

On the other hand, the final three conditions specifying F ensure that, when this event occurs, any path from $(0,0)$ to $(0,1)$ whose x-coordinate never exceeds s in absolute value has weight at most $50s^2 + 50s^2 + 50s^2$; indeed, the weight of any such path may be represented as a sum of the weights of the three subpaths formed by cutting the path at heights one-third and two-thirds.

We thus find that, on $\mathrm{High}(\mathrm{West}, 300s^2) \cap \mathsf{F}$, any path from $(0,0)$ to $(0,1)$ that remains in Strip has weight at most $150s^2$; at the same time, a path of weight at least $200s^2$ connects these two points. Thus, we see that any polymer from $(0,0)$ to $(0,1)$ has maximum transversal fluctuation at least s in this event. By (7.2), we find that

$$\mathbb{P}\left(\min\Big\{ \mathrm{TF}(\rho) : \rho \in \Phi_{n;(0,0)}^{(0,1)} \Big\} \geq s \right) \, \geq \, 2^{-1} \exp\big\{ - 2Cs^3 \big\} \, . \tag{7.3}$$

Suppose now instead that Case Low holds. We will argue that

$$\mathbb{P}\Big(\mathsf{W}_{n;(0,0)}^{(0,1)} \geq 1000s^2 \, , \, \neg \, \mathrm{High}\big([-s,s] \times [0,1], 900s^2 \big) \Big) \geq 2^{-1} \exp\big\{ - Cs^3 \big\} \, , \tag{7.4}$$

where $\neg \, A$ denotes the complement of the event A. Before we do so, we show that the event on this left-hand side entails that any polymer from $(0,0)$ to $(0,1)$ must leave the strip $[-s,s] \times [0,1]$; thus, (7.3) holds in Case Low, even when the factor of 2 is omitted from the right-hand exponential. When the last left-hand event occurs, any path from $(0,0)$ to $(0,1)$ that remains in the strip has weight at most $900s^2$. At the same time, the weight of any polymer from $(0,0)$ to $(0,1)$ is at least $1000s^2$. It is thus impossible for any polymer to remain in the strip.

To derive (7.4), note that, because North and South are translates of Mid, Case Low entails that

$$\mathbb{P}\Big(\mathrm{High}(\mathrm{South}, 300s^2) \cup \mathrm{High}(\mathrm{Mid}, 300s^2) \cup \mathrm{High}(\mathrm{North}, 300s^2) \Big) < 3 \exp\big\{ - 2Cs^3 \big\} \, .$$

The bound (7.1) then yields (7.4), since $3 \exp\big\{ - 2Cs^3 \big\} \leq 2^{-1} \exp\big\{ - Cs^3 \big\}$ for all s large enough.

The bound (7.3) has been derived in both of the cases, so that proof of Proposition 1.4 is complete. □

A Glossary of notation

Our use of scaled coordinates leads to a certain amount of notation appearing in the article. Each line in the table below recalls a piece of notation; records a summarizing phrase; and indicates the page at which the notation is first used.

energy	the value assigned to an unscaled increasing path by Poissonian LPP	3
geodesic	a path of maximum energy given its endpoints	3
X_u^v	maximum energy among all paths between u and v in unscaled notation	3
T_n	the scaling map	4
polymer	the image of a geodesic under the scaling map	4
$\rho_{n;u}^{\leftarrow;v}$	the leftmost polymer between planar points u and v	4
$u \underset{n}{\preceq} v$	u and v are such that a polymer starting at u and ending at v exists	4
$\Phi_{n;u}^v$	the set of all polymers between planar points u and v	5
ℓ_u^v	the line segment joining the planar points u and v	5
$\mathrm{TF}(\rho)$	transversal fluctuation of polymer ρ from the line segment joining its endpoints	5
$\mathrm{AEP}_n(t)$	the set of all admissible endpoint pairs at vertical distance at most t	5
$\mathrm{MTF}_n(t)$	maximum transversal fluctuation over polymers between endpoints in $\mathrm{AEP}_n(t)$	6
weight	the scaled energy	6
$t_{1,2}$	the difference $t_2 - t_1$; the lifetime of a given polymer in most applications	6
$\mathsf{W}_{n;(x,t_1)}^{(y,t_2)}$	the weight of the polymer $\rho_{n;(x,t_1)}^{\leftarrow;(y,t_2)}$	6
$\mathsf{Wgt}_n(\cdot)$	a continuous modification of the weight function for polymers from $(0,0)$ to $(0,\cdot)$	6

B Proof summaries for a few inputs

Our arguments invoke on some occasions proofs and results from [5]. The latter article presents intricate arguments in its proof of the slow bond conjecture, and it remains under peer review. The results from [5] that we use are the uniform deviation estimates [5, Propositions 10.1 and 10.5], and the transversal fluctuation estimate [5, Theorem 11.1]. In this appendix, we review the instances of our use of these outside arguments, and provide summaries of the concerned proofs in the notation of scaled coordinates.

The first instance is the bound (2.3) on the maximum weight of a polymer crossing a unit-order region that appears in the proof of Proposition 2.2. Here, we restate the bound (2.3) as Proposition B.1; and then explain its proof.

Proposition B.1. *There exist $C, c \in (0,\infty)$, $C_0 \in (1,\infty)$ and $n_1 \in \mathbb{N}$ such that, for $n \geq n_1$, $s \in [0, 10n^{2/3}]$, and I and J intervals of at most unit length that are contained in the interval of length $2C_0^{-1}s^{1/4}n^{1/6}$ centred at the origin,*

$$\mathbb{P}\left(\sup_{x\in I, y\in J}\left|\mathsf{W}_{n;(x,0)}^{(y,1)} + (x-y)^2\right| > s\right) \leq C\exp\left\{-cs^{3/2}\right\}.$$

The argument that we will give for Proposition B.1 mimics that of [5, Propositions 10.1 and 10.5]. It will be logically complete, except for the use of what is little more than a simplifying notational device. That is, we will take $I = J = [-1,1]$ in explaining the argument for the proposition.

Proof of Proposition B.1 with $I = J = [-1,1]$. The next presented proposition is sufficient to prove this assertion. Indeed, Proposition B.2(2) and (3) imply it. ☐

Proposition B.2. *1. There exist $H, h \in (0,\infty)$ and $n_1 \in \mathbb{N}$ such that, for $n \geq n_1$ and $R > 0$, the probability that the condition*

$$\inf_{x\in[-1,1]} \mathsf{W}_{n;(0,0)}^{(x,1)} \leq -R$$

is satisfied is at most $H \exp\big\{ -hR^{3/2}\big\}$.

2. *This assertion holds also for the condition* $\inf_{x,y\in[-1,1]} \mathsf{W}_{n;(x,0)}^{(y,1)} \leq -R$*;*
3. *and likewise for the condition* $\sup_{x,y\in[-1,1]} \mathsf{W}_{n;(x,0)}^{(y,1)} \geq R$.

Proof. (1). Let $r_0 \in \mathbb{N}$ be the largest integer r such that $2^r \leq nn_1^{-1}$, where n_1 is as in (2.5). We define a rooted binary tree embedded in the plane. The tree's root is $(0,0)$ and the tree has graphical height r_0. The root's children are $(-1/2,1,2)$ and $(1/2,1/2)$. Height advanced by one-half from generation zero to generation one; with each later generation, height will advance by one-half of the value of the preceding advance. The children of a given vertex v are located by moving from v in the directions $(-1,1)$ and $(1,1)$. For example, the root's grandchildren share the height $3/4$ and are located at $-3/4$, $-1/4$, $1/4$ and $3/4$. Indeed, vertices of generation $k \in \mathbb{N}$ number 2^k and are elements of $\Lambda_k \times \big\{1-2^{-k}\big\} = \big\{\gamma_k(j) = (\lambda_k(j), 1-2^{-k}) : j \in [\![0, 2^k-1]\!]\big\}$, where $\Lambda_k = \big\{ -1 + (1+2j)2^{-k} : j \in [\![0,2^k-1]\!]\big\}$.

The tree's edges take the form $\big(\gamma_k(j), \gamma_{k+1}(j+\ell)\big)$ where $k \in \{1,2,\ldots,r_0-1\}$, $j \in [\![0,2^k-1]\!]$ and $\ell \in \{0,1\}$.

For $D > 0$, an edge $\big((x,1-2^{-k}),(y,1-2^{-k-1})\big)$ connecting generations k and $k+1$ is called D-typical if

$$2^{(k+1)/3}\Big|\mathsf{W}_{n;(x,1-2^{-k})}(y,1-2^{-k-1})\Big| \leq D(k+1)^{2/3}\,. \tag{B.1}$$

The left-hand quantity is a normalized weight – it is random but of unit order, satisfying tail bounds that are uniform over edges in the tree. Indeed, the scaling principle (2.2) and (2.5), together with $n2^{-(k+1)} \geq n_1$, imply that the condition (B.1) fails with probability at most $C\exp\big\{-cD^{3/2}(k+1)\big\}$. Observe here that the parabolic curvature term takes the form $t_{1,2}^{-4/3}(x-y)^2 = t_{1,2}^{-4/3}\cdot t_{1,2}^2 = t_{1,2}^{2/3} \leq 1$ for all k; hence we ignore it.

We now specify the event $\mathsf{Typical} = \mathsf{Typical}(D)$ that every edge in the tree is D-typical. We see that

$$\mathbb{P}\big(\neg\,\mathsf{Typical}\big) \;\leq\; \sum_{k=0}^{\infty} 2^k C\exp\big\{-cD^{3/2}(k+1)\big\}\,,$$

whose right-hand side is at most $2C\exp\big\{-2^{-1}cD^{3/2}\big\}$ provided that $D \geq (\log 2)^{2/3}c^{-2/3}$ is high enough. Henceforth, we fix such a value of D.

Let $x \in [-1,1]$. Then let $j \in [\![0,2^k-1]\!]$ be such that $x \in [\lambda_{r_0}(j) - 2^{-r_0}, \lambda_{r_0}(j) + 2^{-r_0}]$. The planar point $\gamma_{r_0}(j)$ is the vertex in the tree closest to $(x,1)$. We will consider the journey between $(0,0)$ and $(x,1)$ offered by visiting consecutive vertices in the tree between the root and $\gamma_{r_0}(j)$, along a path to be called P; and then a final segment between the latter vertex and $(x,1)$. The latter segment is a perturbation of a tree-based route to $(x,1)$ which is imposed so that the high n condition $n2^{-(k+1)} \geq n_1$ is consistently verified. That is, let e_0 denote the line segment joining $(x,1)$ and $\gamma_{r_0}(j)$; and write $E(P)$ for the edge-set of the path P. Note thus that

$$\mathsf{W}_{n;(0,0)}^{(x,1)} \geq \omega(e_0) + \sum_{e\in E(P)} \omega(e)\,, \tag{B.2}$$

where $\omega(e)$ denotes the weight $\mathsf{W}_{n;(u,s)}^{(v,t)}$ of the polymer that travels between the endpoints (x,s) and (y,t) of the edge e. Indeed, as we have noted, the right-hand side of (B.2) is the weight of an n-path that interpolates $(0,0)$ and $(x,1)$.

Clearly, by (1.8) and from the choice of r_0, it follows that

$$\omega(e_0) \geq -2n^{-1/3}n2^{-r_0} \geq -4n_1n^{-1/3} \geq -1\,,$$

for all n large enough.

We see then that, when $\mathsf{Typical}$ occurs, $\mathsf{W}^{(x,1)}_{n;(0,0)} \geq \omega(e_0) + \sum_{e \in E(P)} \omega(e) \geq -D\kappa$, where here we set $\kappa = 1 + \sum_{k=0}^{\infty} (k+1)^{2/3} 2^{-(k+1)/3}$. Invoking the derived bound on $\mathbb{P}\big(\neg\, \mathsf{Typical}\big)$, we find that

$$\mathbb{P}\Big(\inf_{x \in [-1,1]} \mathsf{W}^{(x,1)}_{n;(0,0)} \leq -\kappa D \Big) \leq 2C \exp\big\{ -2^{-1} c D^{3/2} \big\}\,.$$

Setting $R = D$, $h = 2^{-1} c \kappa^{-3/2}$ and $H \geq 2C$ high enough to permit any choice of $R > 0$, we obtain Proposition B.2(1).

(2). Let $x, y \in [-1,1]$ and note that $\mathsf{W}^{(y,1)}_{n;(x,-1)} \geq \mathsf{W}^{(0,0)}_{n;(x,-1)} + \mathsf{W}^{(y,1)}_{n;(0,0)}$. Since $\mathsf{W}^{(0,0)}_{n;(x,-1)}$ has the law of $\mathsf{W}^{(-x,1)}_{n;(0,0)}$, two applications of Proposition B.2(1) and a union bound yield Proposition B.2(2), up to a relabelling of the constants H and h.

(3). Note that the occurrence of the condition

$$\sup_{x,y \in [-1,1]} \mathsf{W}^{(y,1)}_{n;(x,-1)} > R\,,$$

alongside the conditions

$$\min\left\{ \inf_{x \in [-1,1]} \mathsf{W}^{(x,-1)}_{n;(0,-2)}\,,\ \inf_{x \in [-1,1]} \mathsf{W}^{(0,2)}_{n;(y,1)} \right\} \geq -R/4\,, \tag{B.3}$$

entails that

$$\mathsf{W}^{(0,2)}_{n;(0,-2)} > R/2\,. \tag{B.4}$$

Bounds on the failure probabilities of the two conditions (B.3) arise by applying Proposition B.2(2) in light of simple scaling properties. The one-point control given by Theorem 2.2 provides an upper bound on the probability of (B.4). Thus we obtain Proposition B.2(3). □

The final instance of an input from [5] is the bound (2.12) on the transversal fluctuation of the leftmost polymer, which we state here as Proposition B.3. The proof follows [5, Theorem 11.1], and uses a chaining argument with repeated applications of (2.11) at the boundaries of dyadic subintervals and the polymer ordering Lemma 3.2.

Proposition B.3. *Let $\rho^{\leftarrow;(0,t)}_{n;(0,0)}$ be denoted by ρ. Then there exist positive constants c, n_0 and k_0 such that for $t \in (0,1]$, $k \geq k_0$ and $n \geq n_0 t^{-1}$,*

$$\mathbb{P}\left(\sup_{r \in [0,t]} |\rho(r)| > k t^{2/3} \right) \leq e^{-ck^3}.$$

Proof. We prove the proposition for $t = 1$. From here, the argument for general $t \in (0,1]$ follows easily using the scaling principle (2.1). Moreover, by symmetry, the proposition (for $t = 1$) follows if we prove

$$\mathbb{P}\left(\sup_{r \in [0,1]} \rho(r) > k \right) \leq 2^{-1} e^{-ck^3}. \tag{B.5}$$

To this end, fix $s > 2n_0$ to be chosen appropriately later; where $n_0 \geq 1$ is as in (2.11). Let j_0 be an integer such that $n_0 \leq 2^{-j_0} n \leq s$. For $j = 1, 2, \ldots, j_0$, let $S_j = \{\ell 2^{-j} : \ell = 0, 1, 2, \ldots, 2^j\}$. Let A_j denote the event that for all $x \in S_j$,

$$\rho(x) \leq s \sum_{i=0}^{j-1} 2^{-i/10}\,.$$

Let $s_j := s\sum_{i=0}^{j-1} 2^{-i/10}$. First, we show that for all $j \geq 1$, $\mathbb{P}(A_j^c \cap A_{j-1}) \leq 2^{-j}e^{-cs^3}$, where we use the convention that $A_0 = \Omega$ denotes whole set. For $j=1$, this follows directly from (2.11). To see this for $j>1$, fix $1 < j \leq j_0$ and $h \in [\![0, 2^{j-1}-1]\!]$. Let $\rho_{j,h}$ be the leftmost n-polymer between $(-s_{j-1}, h2^{-(j-1)})$ and $(-s_{j-1}, (h+1)2^{-(j-1)})$. Then, using (2.11), the scaling principle (2.1), and the observation that

$$s_j - s_{j-1} = s2^{-(j-1)/10} \geq s2^{(j-1)/3}2^{-2(j-1)/3}\,,$$

we get

$$\mathbb{P}\left(\rho_{j,h}\left((2h+1)2^{-j}\right) \geq s_j\right) \leq e^{-c2^{-(j-1)}s^3}\,.$$

Summing the above over all $h \in [\![0, 2^{j-1}-1]\!]$, we have by polymer ordering Lemma 3.2 that

$$\mathbb{P}(A_j^c \cap A_{j-1}) \leq 2^{-(j+1)}e^{-cs^3}\,,$$

for s large enough. If A denotes the event that A_j holds for each $1 \leq j \leq j_0$, then

$$\mathbb{P}(A^c) \leq \sum_{j=1}^{j_0} \mathbb{P}(A_j^c \cap A_{j-1}) \leq 2^{-1}e^{-cs^3}\,.$$

Finally, when the event A occurs, for any $x \in [x_1, x_2]$, where x_1, x_2 are consecutive elements of S_{j_0}, by polymer ordering Lemma 3.2 and our choice of j_0,

$$\rho(x) \leq \max\{\rho(x_1) - x_1, \rho(x_2) - x_2\} + 2^{-j_0}n \leq s_{j_0} + s < \mu s\,,$$

where we set $\mu = 1 + \sum_{i=0}^{\infty} 2^{-i/10}$. We now set s, choosing it to equal $\mu^{-1}k$, where k is given by the statement of the proposition. Thus, we obtain (B.5). □

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