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Sublinear preferential attachment combined with a growing number of choices*

Yury Malyshkin†

Abstract

We prove almost sure convergence of the maximum degree in an evolving graph model combining a growing number of local choices with sublinear preferential attachment. At each step in the growth of the graph, a new vertex is introduced. Then we draw a random number of edges from it to existing vertices, chosen independently by the following rule. For each edge, we consider a sample of the growing size of vertices chosen with probabilities proportional to a sublinear function of their degrees. Then the new vertex attaches to the vertex with the highest degree from the sample. Depending on the growth rate of the sample and the sublinear function, the maximum degree could be of sublinear order, of linear order, or having almost all edges drawing to it. The proof uses various stochastic approximation processes and a large deviation approach.

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1 Introduction

Preferential attachment graphs are used to model different complex networks that exhibit certain properties, in particular power law degree distribution. The standard preferential attachment graph, introduced in [BA99] by Barabási and Albert, is constructed by following way. We start with some initial graph G_0 on n_0 vertices $v_{1-n_0}, ..., v_0$. Then, the graph G_{n+1} is built from G_n by adding a new vertex v_{n+1} and drawing m edges from it to already existing vertices $Y_1^n, ..., Y_m^n \in \{v_{1-n_0}, ..., v_n\}$ chosen independently from each other with probabilities proportion to their degrees, i.e.

$$\mathbb{P}(Y_i^n = v_j) = \frac{\deg_{G_n} v_j}{\sum_{k=1-n_0}^{n} \deg_{G_n} v_k}.$$

For this model, many of it properties have been studied (see, e.g. [Hof16], section 8). In the present paper we are interested in degree distribution and maximum degree of a modification of this model. Since the asymptotic degree distribution of a preferential attachment graph does not depend on the initial graph, for simplification of the formulas

*The present work was funded by a grant from the Russian Science Foundation (project No. 19-71-00043).
†Tver State University, Moscow Institute of Physics and Technology. E-mail: yury.malyshkin@mail.ru

it is usually suggested that we start with the graph, that consists of a single edge, i.e. G_1 consists of vertices v_0, v_1 and m edges between them.

There are different ways to generalize and modify the standard preferential attachment model. One of them is to use an increasing weight function $w(x)$, so the vertices are chosen with probability proportional to this function of their degree:

$$\mathbb{P}(Y_i^n = v_j) = \frac{w(\deg_{G_n} v_j)}{\sum_{k=1-n_0}^{n} w(\deg_{G_n} v_k)}.$$

The linear case was studied in [Mór02, Mór05], where Móri proved that for $w(x) = x + c$, $c > -1$ and $m = 1$ ($m = 1$ was considered for simplification) the degree distribution follows power law with power $-(3+c)$ and maximum degree is of order $n^{\frac{1}{2+c}}$. The case of a nonlinear weighted function of a form $w(x) = x^\alpha$ (with $\alpha > 0$) was studied in [Ath08]. For the sublinear case ($\alpha < 1$), the degree distribution has exponential tails and the maximum degree is of order $(\ln n)^b$ for some $b > 0$ and for the superlinear case ($\alpha > 1$), the degree distribution is degenerate and the maximum degree is asymptotically n.

The other way to generalize the model is the addition of choice ([KR14, MP15]). In this case, when a new vertex added to the graph, it first selects a sample of vertices and then attaches to one of them in accordance with some rule. There were considered different types of this rule, for example in [MP14, MP15, HJ16, Mal18] authors used rule based on the degree of the vertices and in [HJY20, GLY19] location-based choice has been used. In all these models the size of the sample is not growing with the number of vertices in the graph (in [Mal18] the sample is of random size while in the rest the sample is of constant size). The effect of the choice (from the sample of d independently chosen vertices) appears to be somewhat similar to the effect of the nonlinear weight function. In the case of min choice, as was shown in [MP15], the maximum degree asymptotically $\ln\ln n/\ln d$ and for max choice and linear weight function maximum degree could be made both of sublinear and linear order (depending on parameters d and c, see [Mal18]), and in the case of meek choice (when one choose vertex with s-th highest degree, see [HJ16]) linear and logarithmic (but not sublinear) maximal degree could be obtained. In the present paper, we study a combination of sublinear weight function with the max choice from the sample of the growing size. We will show that both sublinear and linear maximum degree is possible in this case. Note that to compensate for the effect of sublinear weight function we would need the sample of sublinear size.

Let us describe our model. Fix $\alpha \in (0,1)$, $\gamma \in (0,1)$ and $c_d > 0$. Let us define $d_n = \lfloor c_d n^\gamma \rfloor$ ($\lfloor x \rfloor$ means integer part of x, we will always suggest the integer part when the variable should be an integer but it equals to noninteger term), $n \in \mathbb{N}$. Let us consider i.i.d. random variables $m, \{m_n\}_{n\in\mathbb{N}}$ with values in N, such that $\mathbb{E}m^2 < \infty$. We would consider a sequence of random graphs G_n, $n \in \mathbb{Z}_+$, that builds by the following inductive rule. We start with the initial graph G_1 that consists of two vertices v_0 and v_1 and m_1 edges between them. Then on $n+1$-th step we add a new vertex v_{n+1} and draw m_{n+1} edges from it to vertices $Y_n^1, ..., Y_n^{m_{n+1}}$ chosen from $V(G_n)$ by the following rule. For each $i \in N$, $1 \le i \le m_{n+1}$, we independently (given G_n) choose vertices $X_n^{i,1}, ..., X_n^{i,d_{n+1}}$ from $V(G_n)$ with probabilities proportional to their degree in power α:

$$\mathbb{P}(X_n^i = v_j) = \frac{(\deg_{G_n} v_j)^\alpha}{\sum_{k=0}^{n} (\deg_{G_n} v_k)^\alpha}.$$

Then Y_n^i would be the vertex with the highest degree among $X_n^{i,1}, ..., X_n^{i,d_n}$, in case of a tie we choose vertex in accordance with a fair coin toss.

We would be interested in the number of vertices of fixed degree and the maximum degree of the graph. Let $N_n(k)$ be the number of vertices of degree k in the graph

G_n and M_n be the maximum degree of vertices in G_n. Then the total weight D_n of all vertices in G_n is

$$D_n := \sum_{i=0}^{n} \left(\deg_{G_n} v_i\right)^{\alpha} = \sum_{k=1}^{\infty} N_n(k)k^{\alpha}.$$

Note that for each n this sun contains only a finite number of nonzero terms. Let $D_n(k)$ be its k-th partial sum, i.e.

$$D_n(k) := \sum_{j=1}^{k} N_n(j)j^{\alpha}.$$

Let us formulate our main results.

Theorem 1.1. *Let $\mathbb{P}(m > c) > 0$ for any $c > 0$ and $\mathbb{E}m^2 < \infty$. Then*

$$\frac{N_k(n)}{n} \to \mathbb{P}(m = k) \ a.s.$$

In particular

$$\frac{D(n)}{n} \to \sum_{k=1}^{\infty} k^{\alpha}\mathbb{P}(m = k) = \mathbb{E}m^{\alpha} \ a.s. \tag{1.1}$$

Theorem 1.2. *Let $\mathbb{P}(m = k) \le ck^{-\beta}$ for some $\beta > 1 + \frac{1-\alpha}{\gamma}$ and constant $c > 0$. Let $\mathbb{E}m^2 < \infty$ (if $\beta > 3$ it automatically holds). Then*

1. *If $\alpha + \gamma < 1$, then $\frac{M(n)}{n^{\frac{\gamma}{1-\alpha}}} \to x^*$, where $x^* = \left(\frac{\mathbb{E}m(1-\alpha)c_d}{\gamma\mathbb{E}m^{\alpha}}\right)^{\frac{1}{1-\alpha}}$ a.s.*
2. *If $\alpha+\gamma = 1$, then $\frac{M(n)}{n} \to \rho^*$ a.s., where ρ^* is a unique positive root of $1-e^{\frac{c_d x^{\alpha}}{\mathbb{E}m^{\alpha}}} - \frac{x}{\mathbb{E}m}$*
3. *If $\alpha + \gamma > 1$, then $\frac{M(n)}{n} \to \mathbb{E}m$ a.s.*

Theorem 1.1 shows that degrees of most vertices do not change after their appearance. It happens, as would be proven in section 2, due to the increasing size of the sample, which results in vertices with a relatively high degree to be present in the sample with high probability. In other words, the new vertex with high probability connects to the vertices whose degree exceeds a certain growing level. Theorem 1.2 shows how new edges could be accumulated among vertices with high degrees. In the case $\alpha + \gamma > 1$ almost all edges would be drawn toward a single vertex, whose degree asymptotically equals to $(\mathbb{E}m)n$, while in the case $\alpha + \gamma < 1$ edges would be drawn to vertices with degrees up to $x^* n^{\frac{\gamma}{1-\alpha}}$. If we consider m to have power-law distribution then such a combination of max choice with sublinear weighted function would result in vertices with high degrees to follow different exponent then vertices with relatively small degrees up to existing of the condensation for $\alpha + \gamma \ge 1$.

There are two ways to increase the maximum degree of the graph. First, we could add a new vertex with a degree higher than the degree of already existing vertices. To prevent that, we put conditions on the tails of m_n which would provide that with high probability $m_n \le M(n)$ for all large enough n. Second, we could increase the maximum degree by drawing edges to the vertex with the maximum degree. Given G_n, the probability to draw an edge to the vertex with a maximum degree equals

$$p_n := \mathbb{P}\left(\deg_{G_n} Y_n^i = M(n)\right) = \left(1 - \left(1 - \frac{(M(n))^{\alpha}L(n)}{D_n}\right)^{d_n}\right),$$

where $L(n)$ is the number of vertices of degree $M(n)$. We would prove Theorem 1.2 by first proving the lower bound for the maximal degree and then using it to prove matching upper bound. To do so, we need lower and upper bounds on the evolution of $M(n)$. For the lower bound, we get

$$\mathbb{E}(M(n+1)-M(n)|\mathcal{F}_n) \geq \mathbb{E}m\left(1-\left(1-\frac{(M(n))^\alpha}{D_n}\right)^{d_n}\right), \tag{1.2}$$

where $\mathcal{F}_n$ is sigma-algebra that corresponds to G_n.

For the upper bound, we need to count in the impact of m_n. Note that due to conditions on m there are $c>0$ and small enough $c_m \in \left(0, \min\{\frac{\gamma}{1-\alpha}, 1\}\right)$, such that

$$\mathbb{P}\left(m > n^{\min\{\frac{\gamma}{1-\alpha},1\}-c_m}\right) < cn^{-1-c_m}.$$

Hence with high probability $m_n < n^{\min\{\frac{\gamma}{1-\alpha},1\}-c_m}$ for all large enought n. Also,

$$\mathbb{E}\left(m|\, m < n^{\min\{\frac{\gamma}{1-\alpha},1\}-c_m}\right) = \mathbb{E}m\left(1+O\left(\frac{1}{n^{c_m}}\right)\right)\left(1+O\left(\frac{1}{n^{\min\{\frac{\gamma}{1-\alpha},1\}-c_m}}\right)\right).$$

Hence on the event $M(n) > n^{\min\{\frac{\gamma}{1-\alpha},1\}-c_m}$ (which holds for all large enough n with high probability due to the lower bound) for small enough $\delta > 0$ we get

$$\begin{aligned}&\mathbb{E}\left(M(n+1)-M(n)|\,\mathcal{F}_n, m_n < n^{\min\{\frac{\gamma}{1-\alpha},1\}-c_m} < M(n)\right)\\ &\leq \mathbb{E}m\left(1+O\left(\frac{1}{n^{c_m}}\right)\right)\left(1-\left(1-\frac{(M(n))^\alpha L(n)}{D_n}\right)^{d_n}\right).\end{aligned} \tag{1.3}$$

We will use stochastic approximation techniques to prove almost sure convergence in the linear case. Note that stochastic approximation is widely used to prove almost sure convergence for linear order of maximal degree (see, for example, [MP14, HJ16, HJY20]), while due to required conditions it usually could not be applied to prove sublinear order of the maximum degree. For the linear case, in contrast with some previous works on models with choice (in particular, [MP14, Mal18]), due to nonconvexity of the weight function, we do not use persistent hub argument (see, for example, [Gal16]) and instead use auxiliary stochastic approximation processes to separately get matching lower and upper bounds for the maximum degree.

Let us give a short description of the stochastic approximation approach (for more details see, for example, [Chen03, Pem07]) that we use to prove our results. Process $Z(n)$ is a stochastic approximation process if it could be written as

$$Z(n+1)-Z(n) = \gamma_n\left(F(Z(n)) + E_n + R_n\right)$$

where γ_n, E_n and R_n satisfy the following condition. γ_n is not random and $\sum_{n=1}^\infty \gamma_n > \infty$, $\sum_{n=1}^\infty (\gamma_n)^2 < \infty$, usually one puts $\gamma_n = \frac{1}{n}$ or $\gamma_n = \frac{1}{n+1}$. The function $F(x)$ continues with isolated roots and represents the dependence of the increment of the process from its current state. Often the process $Z(n)$ belongs to some interval $[a,b]$, and therefore the function is considered only on this interval as well. In our case we would consider $Z(n) = \frac{M(n)}{n}$ and hence $a=0$. The term E_n is $\mathcal{F}_n$-measurable where $\mathcal{F}_n$ is the natural filtration of $Z(n)$, $\mathbb{E}(E_n|\mathcal{F}_n) = 0$ and $\mathbb{E}((E_n)^2|\mathcal{F}_n) < c$ for some fixed constant c. Usually one put $E_n = \frac{1}{\gamma_n}(Z(n+1) - \mathbb{E}(Z(n+1)|\mathcal{F}_n))$ and therefore the function $F(x)$ could be found from representation $\mathbb{E}(Z(n+1)-Z(n)|\mathcal{F}_n) = \gamma_n(F(Z(n)) + R_n)$ where R_n is a small error term that satisfies $\sum_{n=1}^\infty \gamma_n |R_n| < \infty$ almost surely. Note that conditions on E_n are the ones that break in the sublinear case due to multiplication on the term $\frac{1}{\gamma_n}$ that turns to infinity. If necessary conditions are met, the process will almost surely converge to the zero set of $F(x)$. In our case, the function $F(x)$ would have two roots,

0 and a positive stable root (x^* is a stable zero if $F(x)$ changes sign from $+$ to $-$ when approaching it, an unstable zero if it changes from $-$ to $+$). So, to prove almost sure convergence to the positive root, we would need to prove non-convergence to 0.

In the sublinear case, we would use a different approach, including large deviation estimates. Let us give some outline of this approach. If we consider the degree of certain vertex or the maximal degree of the graph, on step n its increase could be represented as the sum of m_n conditionally independent Bernoulli random variables. Therefore, under certain conditions, we could estimate the evolution of the degree by the sum of independent (given the condition) Bernoulli random variables. Then we could consider their expectations and use large deviation results to ensure that the process does not deviate far from its expectation. We would use the following standard large deviations result on Bernoulli random variables

Lemma 1.3. *Let $\eta_1, ..., \eta_n$ be i.i.d. bernoulli variables with parameter p. Let $S_n = \sum_{i=1}^n \eta_i$. Then for any $\delta > 0$ there are constants C and $c = c(\delta) > 0$, such that for all $n \in \mathbb{N}$, $a \in \mathbb{N}$ and any $p \in (0,1)$*

$$\mathbb{P}(|S_n - pn| \geq \delta pn) \leq Ce^{-cpn}.$$

This is the standard multiplicative Chernoff bound.

Proof approach and organization

In section 2 we prove the strong law of large numbers for the number of vertices of fixed degree and almost sure converges for the total weight of the graph. We would later use it to simplify formulas for stochastic approximation argument.

In section 3 we prove Theorem 1.2 in the case $\alpha + \gamma \geq 1$. To do so, we use (1.2) to get a linear lower bound for the maximum degree. Then, due to the total degree of the graph being linear, the number of vertices with degrees above linear level is bounded by a constant and hence a simple argument provides that $L(n) = 1$ with high probability and therefore we would get almost sure convergence for the case $\alpha + \gamma \geq 1$.

In section 4 we provide the proof of Theorem 1.2 for the case $\alpha + \gamma < 1$. We first use (1.2) to get lower bound for maximum degree of the graph. Then we would use a large deviation approach towards the possible rate of growth of a fixed vertex to show that with high probability degrees of all vertices do not grow faster than the given rate.

2 The number of vertices of fixed degree

In this section, we provide proof of Theorem 1.1.

Proof. Note that, since $\mathbb{P}(m > c) > 0$ for any $c > 0$, the number of vertices with a degree higher than c with high probability is of order n for any c. Therefore for any $k \in \mathbb{N}$ there is a constant $C_k > 0$, such that $D_n - D_n(k) \geq C_k n$ with high probability. Since for any $\epsilon > 0$ with high probability $D_n \leq (2+\epsilon)n\mathbb{E}m$, we get that there is a constant $c_k \in (0,1)$, such that with high probability $\frac{D_n(k)}{D_n} \leq c_k$. Hence, with high probability

$$\mathbb{E}\left(\mathbf{1}\{\deg Y_n^i = k\}|\mathcal{F}_n\right) = \left(\sum_{j=1}^{k} \frac{N_n(j)j^\alpha}{D_n}\right)^{d_n} - \left(\sum_{j=1}^{k-1} \frac{N_n(j)j^\alpha}{D_n}\right)^{d_n}$$
$$= \exp\left\{d_n \ln\left(\frac{D_n(k)}{D_n}\right)\right\} - \exp\left\{d_n \ln\left(\frac{D_n(k-1)}{D_n}\right)\right\} \leq \exp\{d_n \ln(c_k)\} \to 0$$

as $n \to \infty$. Therefore, almost all vertices with degree k do not have edges drawn into them, which results in the first statement of the theorem.

Now let us get the second statement. Note that the sum of degrees of all vertices at time n is $2\sum_{i=1}^n m_i$. Hence

$$D_n - D_n(k) \le \sup_{\substack{x_1,\dots,x_j,\, j\in\mathbb{N}:\\ x_1+\dots+x_j\le 2\sum_{i=1}^n m_i,\, x_l\ge k, 1\le l\le j}} \sum_{i=1}^j x_j^\alpha.$$

Note that for each $b > a > 0$ and $j \le b/a$

$$\sup_{\substack{x_1,\dots,x_j,:\\ x_1+\dots+x_j\le b,\, x_l\ge a, 1\le l\le j}} \sum_{i=1}^j x_j^\alpha$$

is a standard concave maximization (convex minimization) problem. By the method of Lagrange multipliers, it could be easily shown that the maximum could only be achieved under conditions of the type $x_i = a$, $i = 1,\dots,l$ and $x_i = \frac{b-la}{j-l}$, $i = l+1,\dots,j$ for some $0 \le l \le j$. Applying these conditions (and using that $b/a \ge j$) we would get for each l

$$\sum_{i=1}^l a^\alpha + \sum_{i=l+1}^j \left(\frac{b-la}{j-l}\right)^\alpha = la^\alpha + (j-l)\left(\frac{b-la}{j-l}\right)^\alpha$$
$$= a^\alpha\left(l + (j-l)\left(\frac{b/a-l}{j-l}\right)^\alpha\right) \le a^\alpha\left(l + (j-l)\left(\frac{b/a-l}{j-l}\right)\right) = a^{\alpha-1}b.$$

Hence

$$\sup_{\substack{x_1,\dots,x_j:\\ x_1+\dots+x_j\le b,\, x_j\ge a}} \sum_{i=1}^j x_j^\alpha \le a^{\alpha-1}b.$$

Applying this estimate with $a = k$ and $b = 2\sum_{i=1}^n m_i$ we get that

$$D_n - D_n(k) \le 2k^{\alpha-1}\sum_{i=1}^n m_i.$$

Therefore ($D_n(k) \le D_n$ by definition)

$$\sum_{j=1}^k \frac{N_n(j)}{n} j^\alpha \le \frac{D_n}{n} \le \sum_{j=1}^k \frac{N_n(j)}{n} j^\alpha + \frac{2\sum_{i=1}^n m_i}{n} k^{\alpha-1}.$$

Due to the strong law of large numbers $\frac{\sum_{i=1}^n m_i}{n} \to \mathbb{E}m$ a.s. as $n\to\infty$. Hence, if we take the limit $n\to\infty$ while taking into account that $\frac{N_n(j)}{n} \to \mathbb{P}(m=j)$ almost surely, we get that

$$\sum_{j=1}^k \mathbb{P}(m=j)j^\alpha \le \lim_{n\to\infty}\frac{D_n}{n} \le \sum_{j=1}^k \mathbb{P}(m=j)j^\alpha + 2\mathbb{E}(m)k^{\alpha-1}$$

almost surely. Note that we could take the limit since the sum is finite. Then by taking limit $k\to\infty$, we get that the left and the right side of last equation converges (since $\mathbb{E}(m^2) < \infty$) to the same limit, which means that $\lim_{n\to\infty}\frac{D_n}{n}$ exists and

$$\lim_{n\to\infty}\frac{D_n}{n} = \sum_{j=1}^\infty \mathbb{P}(m=j)j^\alpha = \mathbb{E}(m^\alpha) \quad \text{a.s.},$$

which conclude the proof. □

3 The maximum degree: case $\alpha + \gamma \geq 1$

First, we provide an estimate of the maximum degree from below using stochastic approximation and formula (1.2). Note that, due to Theorem 1.1, for any $\epsilon > 0$ probability of event $\mathcal{A}_\epsilon(n) = \{\forall l \geq n : |\frac{D_l}{l\mathbb{E}m^\alpha} - 1| < \epsilon\}$ turns to 1 as n turns to ∞. Define $\mathcal{B}_\epsilon(n) = \{|\frac{D_n}{n\mathbb{E}m^\alpha} - 1| < \epsilon\}$. Fix $n_0 \in \mathbb{N}$. For all $n \geq n_0$ we get

$$\mathbf{1}\{\mathcal{B}_\epsilon(n)\}\mathbb{E}(M(n+1) - M(n)|\mathcal{F}_n) \geq \mathbf{1}\{\mathcal{B}_\epsilon(n)\}\mathbb{E}m\left(1 - \left(1 - \frac{(M(n))^\alpha}{D_n}\right)^{d_n}\right)$$
$$\geq \mathbf{1}\{\mathcal{B}_\epsilon(n)\}\left(1-\exp\left\{-\frac{d_n(M(n))^\alpha}{D_n}\right\}\right) \geq \mathbf{1}\{\mathcal{B}_\epsilon(n)\}\left(1-\exp\left\{-\frac{c_d\,(M(n)/n)^\alpha}{\mathbb{E}m^\alpha(1+\epsilon)n^{1-\gamma-\alpha}}\right\}\right)$$
$$\geq \mathbf{1}\{\mathcal{B}_\epsilon(n)\}\left(1 - \exp\left\{-\frac{c_d}{\mathbb{E}m^\alpha(1+\epsilon)}\left(\frac{M(n)}{n}\right)^\alpha\right\}\right).$$

Hence for any $n_0 \in \mathbb{N}$ and $\epsilon > 0$ exists $\mathcal{F}_n$-measurable $A_\epsilon(n) = A_\epsilon(n, n_0)$ such that $A_\epsilon(n_0) = M(n_0)$,

$$\mathbb{E}\left(A_\epsilon(n+1) - A_\epsilon(n)|F_n\right) := \mathbb{E}m\left(1 - \exp\left\{-\frac{c_d}{\mathbb{E}m^\alpha(1+\epsilon)}\left(\frac{A_\epsilon(n)}{n}\right)^\alpha\right\}\right)$$

and $A_\epsilon(n) \leq M(n)$ on $\mathcal{A}_\epsilon(n_0)$. Consider $B_\epsilon(n) := A_\epsilon(n)/n$. Then

$$\mathbb{E}(B_\epsilon(n+1) - B_\epsilon(n)|\mathcal{F}_n) = \frac{\mathbb{E}m}{n+1}\left(1 - \exp\left\{-\frac{c_d}{\mathbb{E}m^\alpha(1+\epsilon)}(B_\epsilon(n))^\alpha\right\} - \frac{B_\epsilon(n)}{\mathbb{E}m}\right).$$

Note that the function $g_\epsilon(x) = 1 - e^{-\frac{c_d x^\alpha}{\mathbb{E}m^\alpha(1+\epsilon)}} - \frac{x}{\mathbb{E}m}$ has two roots in $[0, \mathbb{E}m]$, 0 and the stable root $\rho^*_\epsilon \in (0,1)$. Also, $|n(B_\epsilon(n+1) - B_\epsilon(n))| \leq m_{n+1}$ and hence

$$\mathbb{E}((n(B_\epsilon(n+1) - B_\epsilon(n)))^2|\mathcal{F}_n) \leq \mathbb{E}m^2 < \infty.$$

Let us show non-convergence of $B_\epsilon(n)$ to 0. We get that

$$\mathbb{E}\left(\frac{A_\epsilon(n+1)}{A_\epsilon(n)}\middle|F_n\right) = 1 + \frac{\mathbb{E}m}{n}\frac{\left(1-\exp\left\{-\frac{c_d}{\mathbb{E}m^\alpha(1+\epsilon)}\left(\frac{A_\epsilon(n)}{n}\right)^\alpha\right\}\right)}{A_\epsilon(n)/n} = 1 + \frac{\mathbb{E}m}{n}h_\epsilon\left(\frac{A_\epsilon(n)}{n}\right),$$

where $h_\epsilon(x) = \left(1 - e^{-\frac{c_d x^\alpha}{\mathbb{E}m^\alpha(1+\epsilon)}}\right)/x$. Since $h_\epsilon(x) \to \infty$ as $x \to 0$ there are $\delta > 0$ and $\theta > 0$, such that for $x < \delta$ we get $(\mathbb{E}m)h_\epsilon(x) > 1 + \theta$. If $A_\epsilon(n)/n$ converges to 0 with positive probability, then $\mathbb{P}(\forall n \geq N\ A_\epsilon(n)/n < \delta, \mathcal{A}_\epsilon(n)) > q$ for some $q > 0$ and $N \geq n_0$. Hence for $n \geq N$

$$\mathbf{1}\{\mathcal{B}_\epsilon(n), A_\epsilon(n)/n < \delta\}\mathbb{E}(A_\epsilon(n+1)|\mathcal{F}_n) \geq \left(1 + \frac{1+\theta}{n}\right)\mathbf{1}\{\mathcal{B}_\epsilon(n), A_\epsilon(n)/n < \delta\}\mathbb{E}A_\epsilon(n).$$

Therefore

$$\mathbf{1}\{\forall n \geq N\ A_\epsilon(n)/n < \delta, \mathcal{A}_\epsilon(n)\}A_\epsilon(n) \geq \mathbf{1}\{\forall n \geq N\ A_\epsilon(n)/n < \delta, \mathcal{A}_\epsilon(n)\}A_\epsilon(N)\prod_{k=N}^{n-1}\left(1 + \frac{1+\theta}{k}\right).$$

As the result, we get that on the event $\{\forall n \geq N\ A_\epsilon(n)/n < \delta, \mathcal{A}_\epsilon(n)\}$, which has separated from 0 (by q) probability to occur, $A_\epsilon(n)$ grows (in terms of expectations) as $n^{1+\theta}$ for all $n \geq N$. But on this event $A_\epsilon(n) \leq M(n)$ and hence $EM(n)$ grows at least as fast as $qn^{1+\theta}$, which is impossible since there are only $\Theta(n)$ edges in the graph. Therefore $B_\epsilon(n) \to \rho^*_\epsilon$ a.s. as $n \to \infty$. As result, for $\alpha + \gamma \geq 1$ we get that $\liminf \frac{M(n)}{n} \geq \rho^*_\epsilon$

on $\mathcal{A}_\epsilon(n_0)$. Hence $\liminf \frac{M(n)}{n} \geq \rho^*$ with high probability where ρ^* is a unique root of function $g(x) = 1 - e^{-\frac{c_d x^\alpha}{\mathbb{E}m^\alpha}} - \frac{x}{\mathbb{E}m}$ in $(0,1)$.

Moreover, for $\alpha+\gamma > 1$ we get that

$$\mathbb{E}(M(n+1) - M(n)|\mathcal{F}_n) \geq \mathbb{E}m\left(1 - \exp\left\{-\frac{c_d\,(\rho^*)^\alpha}{\mathbb{E}m^\alpha + \epsilon}(1+o(1))n^{\gamma+\alpha-1}\right\}\right) \to \mathbb{E}m$$

almost surely, and therefore $\frac{M(n)}{n} \to \mathbb{E}m$ almost surely.

Now let us prove the matching upper bound for the case $\alpha+\gamma = 1$. Note that since $\liminf_{n\to\infty} \frac{M(n)}{n} \geq \rho^* > 0$, we get that with high probability $L(n) \leq \frac{\mathbb{E}m}{\rho^*}$. Therefore $\frac{L(n)(M(n))^\alpha}{D_n} = O(n^{\alpha-1})$ with high probability. Hence from formula (1.3) similar to the lower bound we would get

$$\begin{aligned}
\frac{\mathbb{E}(M(n+1) - M(n)|\mathcal{F}_n)}{\mathbb{E}m(1+O(n^{-c_m}))} &\leq 1 - \exp\left\{-\frac{c_d L(n)}{\mathbb{E}m^\alpha - \epsilon}\left(\frac{M(n)}{n}\right)^\alpha (1+O(n^{\alpha-1}))\right\} \\
&\leq \mathbf{1}\{L(n) > 1\} + \mathbf{1}\{L(n) = 1\}\left(1 - \exp\left\{-\frac{c_d}{\mathbb{E}m^\alpha - \epsilon}\left(\frac{M(n)}{n}\right)^\alpha (1+O(n^{\alpha-1}))\right\}\right) \\
&\leq \mathbf{1}\{L(n) > 1\} + \left(1 - \exp\left\{-\frac{c_d}{\mathbb{E}m^\alpha - \epsilon}\left(\frac{M(n)}{n}\right)^\alpha (1+O(n^{\alpha-1}))\right\}\right).
\end{aligned}$$

Note that the probability to increase the degree of the vertex with a degree higher the $\rho^* n$ is bound from below by some constant. Also, for two vertices with different degrees probability to increase degree is higher for the vertex with a higher degree. Therefore from standard estimates on the probability of the return of random walk to the origin, we get that for any pair of vertices with degrees higher then $\rho^* n$ the probability that they have the same degree is at time n is $O(n^{-1/2})$. Hence $\mathbb{P}(L(n) > 1) = O(n^{-1/2})$ and therefore

$$\sum_{n=1}^{\infty} \frac{\mathbf{1}\{L(n) > 1\}}{n} < \infty \quad \text{a.s.}$$

so the term $\mathbf{1}\{L(n) > 1\}$ satisfy condition on error term $R(n)$ of stochastic approximation. Hence by stochastic approximation argument (as in the proof of the lower bound), we would get that $\limsup_{n\to\infty} \frac{M(n)}{n} \leq \rho^*_{-\epsilon}$ on $\mathcal{A}_\epsilon(n)$, where $\rho^*_{-\epsilon}$ is the root of the function $1 - e^{-\frac{c_d x^\alpha}{\mathbb{E}m^\alpha - \epsilon}} - \frac{x}{\mathbb{E}m}$ in $(0, \mathbb{E}m)$, and hence $\limsup_{n\to\infty} \frac{M(n)}{n} \leq \rho^*$.

4 The maximum degree: case $\alpha + \gamma < 1$

In this section, we prove Theorem 1.2 in the case $\alpha+\gamma < 1$. To do so we would estimate the process $M(n)$ by sums of independent Bernoulli random variables and use large deviations (Lemma 1.3).

Let us consider $n_0 \in \mathbb{N}$, $\epsilon > 0$ (ϵ depends on δ, which would be introduced later). Note that due to condition on m, there are $y > 0$, such that with high probability (as $n_0 \to \infty$) we have $m_n \leq n^{\frac{\gamma}{1-\alpha} - y}$ for all $n > n_0$. Moreover, for all $n \geq n_0$ $\mathbb{E}(m_n | m_n \leq n^{\frac{\gamma}{1-\alpha} - y}) = \mathbb{E}m\left(1 + O\left(\frac{1}{n^c}\right)\right)$ for some constant $c > 0$. Also, due to convergence $\frac{\sum_{i=1}^n m_i}{n} \to \mathbb{E}m$ we get that for any $c_1 > 0$, $c_2 > 0$ with high probability (as $n_0 \to \infty$) for all $n > n_0$ and $k \in \mathbb{N}$ we have

$$\left|\sum_{i=n+1}^{n+k} m_i - k\mathbb{E}m\right| \leq c_1(n+k) + c_2 k\mathbb{E}m. \tag{4.1}$$

Also, $\frac{(\mathbb{E}m^\alpha)n}{1+\epsilon} < D_n < \frac{(\mathbb{E}m^\alpha)n}{1-\epsilon}$ for all $n \geq n_0$ with high probability (as $n_0 \to \infty$). Let $\mathcal{C}_{n_0}$ be the event that the above conditions hold. For the rest of this section, we would consider all random variables on event $\mathcal{C}_{n_0}$ (we would omit $\mathbf{1}\{\mathcal{C}_{n_0}\}$ in formulas). Since we get estimates with accuracy up to $1+\epsilon$ term we would omit terms of type $(1+O(n^{-c}))$, which could occur from taking integer part or conditional expectation.

Let us estimate p_n from below. Recall that $d_n = c_d n^{\gamma}$. Hence

$$\begin{aligned} p_n &\geq \left(1-\exp\left\{-\frac{d_n(M(n))^{\alpha}}{D_n}\right\}\right) \\ &\geq \left(\frac{c_d(1-\epsilon)(M(n))^{\alpha}}{(\mathbb{E}m^{\alpha})n^{1-\gamma}} + O\left(\left(\frac{(M(n))^{\alpha}}{n^{1-\gamma}}\right)^2\right)\right) \\ &\geq \left(\frac{c_d(1-2\epsilon)(M(n))^{\alpha}}{(\mathbb{E}m^{\alpha})n^{1-\gamma}}\right) \end{aligned}$$

for large enough (and non-random) n_0.

Let us consider $\delta > 0$ and $\sigma > 0$. First we prove that for any N there is $n > N$, such that $M(n) > (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}$. Note that

$$\begin{aligned} \mathbf{1}\{M(n) \leq (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}\}\mathbb{E}\left(\frac{M(n+1)}{M(n)}\middle| \mathcal{F}_n\right) &= \mathbf{1}\{M(n) \leq (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}\}\left(1+\frac{p_n \mathbb{E}m}{M(n)}\right) \\ \geq \mathbf{1}\{M(n) \leq (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}\}&\left(1+\frac{c_d(1-2\epsilon)\mathbb{E}m}{(\mathbb{E}m^{\alpha})n^{\frac{\mathbb{E}m(1-\alpha)c_d}{\gamma \mathbb{E}m^{\alpha}}}(1-\delta)^{1-\alpha}}\right) \\ = \mathbf{1}\{M(n) \leq (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}\}&\left(1+\frac{\gamma}{1-\alpha}\frac{(1-2\epsilon)}{(1-\delta)^{1-\alpha}}\frac{1}{n}\right) \end{aligned}$$

Hence, if we choose $\epsilon > 0$ such that $\frac{(1-2\epsilon)}{(1-\delta)^{1-\alpha}} > 1$, on event $\{M(n) \leq (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}\}$ for $n > N$ we would get (as in the proof of non-convergence $B_\epsilon(n)$ to 0) that $M(n)$ would grow faster than $n^{\frac{\gamma}{1-\alpha}}$ and hence such event holds with probability 0. Therefore, for any N there is $n > N$, such that $M(n) > (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}$.

Note that (for large enough n) if $M(n) > (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}}$ than $M(n+1) > (1-\delta-\sigma)x^*(n+1)^{\frac{\gamma}{1-\alpha}}$. Let us introduce stopping time

$$\pi_n := \inf\{t \geq n : \left|M(t) - (1-\delta-\sigma)x^* t^{\frac{\gamma}{1-\alpha}}\right| > \sigma x^* t^{\frac{\gamma}{1-\alpha}}\}.$$

Let us estimate probability that $M(\pi_n) > (1-\delta)x^*(\pi_n)^{\frac{\gamma}{1-\alpha}}$ if at time n process $M(n) > (1-\delta-\sigma)x^* n^{\frac{\gamma}{1-\alpha}}$. In particular, that would mean that $M(t) > (1-\delta-2\sigma)x^* t^{\frac{\gamma}{1-\alpha}}$ for all $n \leq t \leq \pi_n$.

To do so, first note that

$$\begin{aligned} \mathbf{1}\{t < \pi_n\}p_t &\geq \mathbf{1}\{t < \min\{\tau_n, \pi_n\}\}\left(\frac{c_d(1-2\epsilon)((1-\delta)x^*)^{\alpha}}{(\mathbb{E}m^{\alpha})} t^{\frac{\gamma}{1-\alpha}-1}\right) \\ &= \mathbf{1}\{t < \pi_n\}\left(\frac{\gamma(1-2\epsilon)(1-\delta-2\sigma)^{\alpha}x^*}{(1-\alpha)\mathbb{E}m} t^{\frac{\gamma}{1-\alpha}-1}\right). \end{aligned}$$

Let us consider moments $t_i = (1+i\sigma)n$. Note that if $M(t_i) > (1-\delta-\sigma)x^* t_i^{\frac{\gamma}{1-\alpha}}$, then $M(t) > (1-\delta-2\sigma)x^* t^{\frac{\gamma}{1-\alpha}}$ for all $t \in [t_i, t_{i+1}]$. For $t \in [t_i, t_{i+1}]$ on event $\{t < \pi_n\}$ we get that

$$\begin{aligned} p_t &\geq \left(\frac{\gamma(1-2\epsilon)(1-\delta-2\sigma)^{\alpha}x^*}{(1-\alpha)\mathbb{E}m} t_{i+1}^{\frac{\gamma}{1-\alpha}-1}\right) \\ \geq \hat{p}_i &:= \left(\frac{\gamma(1-2\epsilon)(1-\delta-2\sigma)^{\alpha}x^*}{(1-\alpha)\mathbb{E}m}\left((1+\sigma)(1+i\sigma)\right)^{\frac{\gamma}{1-\alpha}-1} n^{\frac{\gamma}{1-\alpha}-1}\right) \end{aligned}$$

Hence, due to (4.1), on $\mathcal{C}_{n_0} \cup \{t < \pi_n\}$ difference $M(t_{i+1}) - M(t_i)$ could be estimated from below by the sum of $(\sigma\mathbb{E}m - c_1(1+\sigma) - c_2\sigma\mathbb{E}m)n$ Bernoulli random variables with parameter $\hat{p}_i$. Due to lemma 1.3 we get that

$$\mathbb{P}\left(M(t_{i+1}) - M(t_i) < (1-\sigma)\sigma\left(1 - \frac{c_1(1+\sigma)}{\sigma\mathbb{E}m} - c_2\right)(1-2\epsilon)(1-\delta-2\sigma)^\alpha (1+\sigma)^{\frac{\gamma}{1-\alpha}-1} \right.$$
$$\left. \times \frac{\gamma}{1-\alpha} x^*(1+i\sigma)^{\frac{\gamma}{1-\alpha}-1} n^{\frac{\gamma}{1-\alpha}}, \mathcal{C}_{n_0} \cup \{t_{i+1} < \pi_n\}\right) < e^{-cn^{\frac{\gamma}{1-\alpha}}}$$

for some $c > 0$. Note that since $\alpha < 1$, for each $\delta > 0$ we could choose small enough $c_1, c_2, \sigma, \epsilon > 0$, such that

$$(1-\sigma)(1 - \frac{c_1(1+\sigma)}{\sigma\mathbb{E}m} - c_2)(1-2\epsilon)(1-\delta-2\sigma)^\alpha (1+\sigma)^{\frac{\gamma}{1-\alpha}-1} > 1 - \delta + \sigma.$$

Let us fix such parameters. Let us consider events

$$\mathcal{E}_i := \left\{ M(t_{i+1}) - M(t_i) < \sigma(1-\delta+\sigma)\frac{\gamma}{1-\alpha} x^*(1+i\sigma)^{\frac{\gamma}{1-\alpha}-1} n^{\frac{\gamma}{1-\alpha}} \right\}.$$

Then $\mathbb{P}(\mathcal{E}_i, \mathcal{C}_{n_0} \cup \{t_{i+1} < \pi_n\}) < e^{-cn^{\frac{\gamma}{1-\alpha}}}$. Therefore for any $j \in \mathbb{N}$

$$\mathbb{P}(\exists 1 \le i \le j : \mathcal{E}_i, \mathcal{C}_{n_0} \cup \{t_{j+1} < \pi_n\}) < je^{-cn^{\frac{\gamma}{1-\alpha}}}.$$

Note that if for all $i \le j$

$$M(t_{i+1}) - M(t_i) \ge \sigma(1-\delta+\sigma)\frac{\gamma}{1-\alpha} x^*(1+i\sigma)^{\frac{\gamma}{1-\alpha}-1} n^{\frac{\gamma}{1-\alpha}}$$

then, if $(1+(j+1)\sigma)^{\frac{\gamma}{1-\alpha}} - 1 > 1$, which holds for large enought j, we get

$$\begin{aligned} M(t_{j+1}) - M(n) &\ge \sigma(1-\delta+\sigma)\frac{\gamma}{1-\alpha} x^* n^{\frac{\gamma}{1-\alpha}} \sum_{i=0}^{j} (1+i\sigma)^{\frac{\gamma}{1-\alpha}-1} \\ &\ge (1-\delta+\sigma)x^* n^{\frac{\gamma}{1-\alpha}} \left((1+(j+1)\sigma)^{\frac{\gamma}{1-\alpha}} - 1\right) \\ &\ge (1-\delta)x^* n^{\frac{\gamma}{1-\alpha}} \left((1+(j+1)\sigma)^{\frac{\gamma}{1-\alpha}} - 1\right) + \sigma x^* n^{\frac{\gamma}{1-\alpha}} \\ &= (1-\delta)x^*(t_{j+1})^{\frac{\gamma}{1-\alpha}} - (1-\delta-\sigma)x^* n^{\frac{\gamma}{1-\alpha}} \end{aligned}$$

and hence (under suggestion $M(n) > (1-\delta-\sigma)x^* n^{\frac{\gamma}{1-\alpha}}$)

$$M(t_{j+1}) \ge (1-\delta)x^*(t_{j+1})^{\frac{\gamma}{1-\alpha}},$$

which imply that $t_{j+1} \ge \pi_n$. Also, in this case $M(\pi_n) > (1-\delta)x^*(\pi_n)^{\frac{\gamma}{1-\alpha}}$. Therefore,

$$\mathbb{P}\left(M(\pi_n) < (1-\delta-2\sigma)x^*(\pi_n)^{\frac{\gamma}{1-\alpha}}, \mathcal{C}_{n_0}\right) \le je^{-cn^{\frac{\gamma}{1-\alpha}}}.$$

Since $je^{-cn^{\frac{\gamma}{1-\alpha}}}$ form convergence series this estimate implies the lower bound.

Now, let us prove matching upper bound. We will use arguments similar to the lower bound. The main difference is that instead of considering the upper bound for the maximum degree we obtain the upper bound for the degree of the given vertex. Since we get exponential bound for probability, additional summing over all vertices (that present at a given time) does not affect the convergence of the series and resulting bound.

Let us consider n_0 and for $n \ge n_0$ estimate from above (on $\mathcal{C}_{n_0}$) the condition probability $p_n(v)$ to draw an edge to a single vertex. Note that such probability is increasing under the condition that there are no vertices with a higher degree in the sample, which achieved for vertices with the highest degree. For condition probability q_n to draw an edge to a certain vertex with the highest degree we get

$$q_n = \frac{1}{L_n}\left(1-\left(1-\frac{L_n(M(n))^\alpha}{D_n}\right)^{d_n}\right) \le \frac{1}{L_n}\frac{L_n(M(n))^\alpha d_n}{D_n} = \frac{(M(n))^\alpha d_n}{D_n}.$$

Hence for any vertex v we have

$$p_n(v) \le \frac{(\deg_n(v))^\alpha d_n}{D_n} \le (1+\epsilon)\frac{c_d(\deg_n(v))^\alpha}{(\mathbb{E}m^\alpha)n^{1-\gamma}}.$$

Therefore, similar to the lower bound, for any N there is $n>N$, such that

$$\deg_n(v) \le (1+\delta)x^* n^{\frac{\gamma}{1-\alpha}}.$$

If we once again consider stopping time

$$\rho_n(v) := \inf\left\{t \ge n : \left|\deg_t(v) - (1+\delta+\sigma)x^* k^{\frac{\gamma}{1-\alpha}}\right| > \sigma x^* t^{\frac{\gamma}{1-\alpha}}\right\}$$

we would get that

$$\mathbf{1}\{t<\rho_n(v)\}p_t(v) \le \mathbf{1}\{t<\rho_n(v)\}\left(\frac{\gamma(1+\epsilon)(1+\delta+2\sigma)^\alpha x^*}{(1-\alpha)\mathbb{E}m} t^{\frac{\gamma}{1-\alpha}-1}\right).$$

On $\mathcal{C}_{n_0} \cup \{t<\rho_n(n)\}$ the difference $\deg_{t_{i+1}}(v) - \deg_{t_i}(v)$ could be estimated from above by the sum of $(\sigma\mathbb{E}m + c_1(1+\sigma) + c_2\sigma\mathbb{E}m)n$ Bernoulli random variables with parameter $p_t(v)$. Therefore, for $t \in [t_i, t_{i+1}]$, $t_i = (1+i\sigma)n$, we get

$$p_t(v) \le \left(\frac{\gamma(1+\epsilon)(1+\delta+2\sigma)^\alpha x^*}{(1-\alpha)\mathbb{E}m}\left((1-\sigma)(1+(i+1)\sigma)\right)^{\frac{\gamma}{1-\alpha}-1} n^{\frac{\gamma}{1-\alpha}-1}\right)$$

and hence

$$\begin{aligned}\mathbb{P}\Bigg(&\deg_{t_{i+1}}(v)-\deg_{t_i}(v) > (1+\sigma)\sigma\left(1+\frac{c_1(1+\sigma)}{\sigma\mathbb{E}m}+c_2\right)(1+\epsilon)(1+\delta+2\sigma)^\alpha(1-\sigma)^{\frac{\gamma}{1-\alpha}-1} \\ &\times \frac{\gamma}{1-\alpha}x^*(1+(i+1)\sigma)^{\frac{\gamma}{1-\alpha}-1}n^{\frac{\gamma}{1-\alpha}}, \mathcal{C}_{n_0}\cup\{t_{i+1}<\rho_n(v)\}\Bigg) < e^{-cn^{\frac{\gamma}{1-\alpha}}}\end{aligned}$$

for some $c>0$. Therefore, similar to the lower bound, let us consider events

$$\mathcal{E}_i(v) := \left\{\deg_{t_{i+1}}(v) - \deg_{t_i}(v) > \sigma(1+\delta-\sigma)\frac{\gamma}{1-\alpha}x^*(1+(i+1)\sigma)^{\frac{\gamma}{1-\alpha}-1}n^{\frac{\gamma}{1-\alpha}}\right\},$$

and for each $\delta>0$ we choose small enough $c_1, c_2, \sigma, \epsilon > 0$, such that for any j

$$\mathbb{P}(\exists 1 \le i \le j : \mathcal{E}_i(v), \mathcal{C}_{n_0} \cup \{t_{j+1} < \rho_n(v)\}) < je^{-cn^{\frac{\gamma}{1-\alpha}}}.$$

Hence for large enough (that depends on other parameters but not on n) j

$$\mathbb{P}\Big(\deg_{\pi_n}(v) > (1+\delta+2\sigma)x^*(\pi_n)^{\frac{\gamma}{1-\alpha}}, \mathcal{C}_{n_0}, \pi_n < (1+(j+1)\sigma)n\Big) \le je^{-cn^{\frac{\gamma}{1-\alpha}}}.$$

Therefore

$$\begin{aligned}\mathbb{P}\Big(\exists v \le (1+(j+1)\sigma)n : \deg_{\pi_n}(v) > (1+\delta+2\sigma)x^*(\pi_n)^{\frac{\gamma}{1-\alpha}}, \mathcal{C}_{n_0}, \pi_n < (1+(j+1)\sigma)n\Big) \\ \le j(1+(j+1)\sigma)ne^{-cn^{\frac{\gamma}{1-\alpha}}}.\end{aligned}$$

Since $j(1+(j+1)\sigma)ne^{-cn^{\frac{\gamma}{1-\alpha}}}$ forms convergent series this estimate implies the upper bound.

5 Discussion

In the present work, we put conditions on m to insure that with high probability a new vertex would not have the maximum degree. It could be interesting to see what happens if we weaken such conditions, for example, if parameters γ and α would still affect asymptotic of the maximum degree.

The other modification of the model is to consider the combination of the min choice with a superlinear function. It is not clear if the power-law type of maximum degree could be achieved in this case. For example, in [HJ16] for a meek choice (when we choose vertex with s-th highest degree for $s > 1$) was shown that the maximum degree could be either of linear order or of $(\ln n)^b$ order with no power-law type of behavior.

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