Electron. Commun. Probab. **25** (2020), article no. 84, 1–14.
https://doi.org/10.1214/20-ECP364
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

On the boundary behavior of multi-type continuous-state branching processes with immigration*

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Abstract

In this article, we provide a sufficient condition for a continuous-state branching process with immigration (CBI process) to not hit its boundary, i.e. for non-extinction. Our result applies to arbitrary dimension $d \geq 1$ and is formulated in terms of an integrability condition for its immigration and branching mechanisms F and R. The proof is based on a comparison principle for multi-type CBI processes being compared to one-dimensional CBI processes, and then an application of an existing result for one-dimensional CBI processes. The same technique is also used to provide a sufficient condition for the transience of multi-type CBI processes.

Keywords: multi-type continuous-state branching process with immigration; extinction; transience; comparison principle.
MSC2020 subject classifications: 60G17; 60J25; 60J80.
Submitted to ECP on February 4, 2019, final version accepted on November 18, 2020.

1 Introduction

Continuous-state branching processes with immigration (abbreviated as CBI processes) form a class of time-homogeneous Markov processes with state space

$$\mathbb{R}_+^d = \{x \in \mathbb{R}^d \mid x_1, \dots, x_d \geq 0\}, \quad d \in \mathbb{N},$$

whose Laplace transform is an exponentially affine function of the initial state variable. In particular, CBI processes are affine processes in the sense of [7, Definition 2.6]. For $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$ each component x_i denotes the (continuous) number of individuals of type $i \in \{1, \dots, d\}$, while d is the number of types. Such processes have been first studied in [10] for single-type models ($d = 1$) in the diffusion case without immigration and in [23] for multi-types ($d \geq 1$) without immigration including models with jumps, see also [34] for a pioneering work in this direction. CBI processes arise as large population limits of Galton-Watson branching processes. Indeed, the single-type case was studied in [28, 20] without immigration and in [26, 29, 3] with immigration. Analogous results for

*Peng Jin is supported by the Guangdong Basic and Applied Basic Research Foundation (No. 2020A1515010436) and the STU Scientific Research Foundation for Talents (No. NTF18023).
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multi-type CBI processes have been then obtained in [24, 5]. An introduction, additional references and results for single-type CBI processes are given in [27, 33] and [30, Chapter 3], while a construction of multi-type CBI processes from a system of stochastic equations was studied in [2]. Let us mention that the CIR process, as well as several multi-factor extensions of it, form a particular class of examples closely related with applications in mathematical finance, see, e.g., [1] and [7] and the references therein.

Here and below $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product on $\mathbb{R}^d$ and $|\cdot|$ the induced norm. Below we describe the parameters of the multi-type CBI process.

Definition 1.1. *The tuple (c, β, B, ν, μ) is called admissible if*

- *(i)* $c = (c_1, \dots, c_d) \in \mathbb{R}^d_+$.
- *(ii)* $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d_+$.
- *(iii)* $B = (b_{kj})_{k,j \in \{1,\dots,d\}}$ *is such that* $b_{kj} \geq 0$ *for* $k, j \in \{1, \dots, d\}$ *with* $k \neq j$.
- *(iv)* ν *is a Borel measure on* $\mathbb{R}^d_+$ *satisfying* $\int_{\mathbb{R}^d_+} (1 \wedge |z|)\nu(dz) < \infty$ *and* $\nu(\{0\}) = 0$.
- *(vi)* $\mu = (\mu_1, \dots, \mu_d)$*, where, for each* $j \in \{1, \dots, d\}$*,* μ_j *is a Borel measure on* $\mathbb{R}^d_+$ *satisfying*

$$\int_{\mathbb{R}^d_+} \left(z_j \wedge z_j^2 + \sum_{k \in \{1,\dots,d\} \setminus \{j\}} z_k \right) \mu_j(dz) < \infty, \quad \mu_j(\{0\}) = 0. \tag{1.1}$$

Note that this definition is a special case of [7, Definition 2.6] in the sense that we consider here the state space $\mathbb{R}^d_+$, exclude killing of the process, and assume that the measures $\mu_1, \dots, \mu_d$ satisfy the additional moment condition for the big jumps $\int_{|z|>1} |z| \mu_j(dz) < \infty$ for $j = 1, \dots, d$. Let (c, β, B, ν, μ) be admissible parameters in the sense of Definition 1.1. It was shown in [7, Theorem 2.7] (see also [2, Theorem 2.4]) that there exists a unique Markov transition kernel $P_t(x, dy)$ with representation

$$\int_{\mathbb{R}^d_+} e^{-\langle \xi, y \rangle} P_t(x, dy) = \exp\left(-\langle x, v(t,\xi) \rangle - \int_0^t F(v(s,\xi)) ds \right), \quad x, \xi \in \mathbb{R}^d_+, \quad t \geq 0, \tag{1.2}$$

where, for any $\xi \in \mathbb{R}^d_+$, the continuously differentiable function $t \longmapsto v(t, \xi) \in \mathbb{R}^d_+$ is the unique locally bounded solution to the system of differential equations

$$\frac{\partial v(t,\xi)}{\partial t} = -R(v(t,\xi)), \quad v(0,\xi) = \xi. \tag{1.3}$$

Here F and R are of Lévy-Khinchine form

$$F(\xi) = \langle \beta, \xi \rangle + \int_{\mathbb{R}^d_+} \left(1 - e^{-\langle \xi, z \rangle} \right) \nu(dz),$$
$$R_j(\xi) = c_j \xi_j^2 - \langle B e_j, \xi \rangle + \int_{\mathbb{R}^d_+} \left(e^{-\langle \xi, z \rangle} - 1 + \xi_j z_j \right) \mu_j(dz), \qquad j \in \{1, \dots, d\},$$

and $e_1, \dots, e_d$ denote the canonical basis vectors in $\mathbb{R}^d$. The corresponding Markov process with transition kernel $P_t(x, dy)$ is called multi-type CBI process. Here, the function F is the so-called immigration mechanism and describes the immigration of individuals into the system from outside. Such immigration may be continuous described by a drift parameter β, but also discontinuous described by a Lévy subordinator with Lévy measure ν. The function R describes the branching mechanism, i.e., R_j is the branching mechanism where each individual of type j may produce new individuals of types $k \in \{1, \dots, d\}$ independently of all other individuals. Here $c = (c_1, \dots, c_d)$ denotes the classical local branching associated with a diffusion process, $Be_j = (b_{jk})_{k\in\{1,\dots,d\}}$ describes the continuous branching rates in term of a drift, and finally the Lévy process associated with the Lévy measure μ_j describes the branching of individuals produced by an individual of type j. The following is a particular case of [7], see also [2, Theorem 2.4].

Remark 1.2. For given admissible parameters (c, β, B, ν, μ) there exists a unique conservative Feller transition semigroup $(P_t)_{t\geq 0}$ acting on the Banach space of continuous functions vanishing at infinity with state space $\mathbb{R}_+^d$ such that its generator has core $C_c^\infty(\mathbb{R}_+^d)$ and is, for $f \in C_c^2(\mathbb{R}_+^d)$, given by

$$(Lf)(x) = \sum_{j=1}^d c_j x_j \frac{\partial^2 f(x)}{\partial x_j^2} + \langle \beta + Bx, (\nabla f)(x)\rangle + \int_{\mathbb{R}_+^d} (f(x+z) - f(x))\nu(dz) \tag{1.4}$$

$$+ \sum_{j=1}^d x_j \int_{\mathbb{R}_+^d} \left(f(x+z) - f(x) - z_j \frac{\partial f(x)}{\partial x_j}\right) \mu_j(dz). \tag{1.5}$$

The corresponding transition semigroup coincides with the one given by (1.2) and hence describes a multi-type CBI process with admissible parameters (c, β, B, ν, μ).

The smoothness of transition probabilities for one-dimensional CBI processes was recently studied in [6], where very precise results have been obtained. In [14] (see also [11] for related results) we have studied existence of transition densities for multi-type CBI processes. It was shown that, under appropriate conditions, such a density exists on the interior of its state space, i.e. on $\Gamma = \{x \in \mathbb{R}_+^d \mid x_1, \dots, x_d > 0\}$. In this work provide conditions under which the corresponding multi-type CBI process is supported on Γ, i.e. $\mathbb{P}[X(t) \in \Gamma, \quad t \geq 0] = 1$. Such property simply states that the population described by X does not get extinct. As a consequence, it has, under the conditions of [14] and those presented in this work, a density on the whole state space $\mathbb{R}_+^d$.

The boundary behavior for one-dimensional CBI processes was studied in [19], for two-dimensional diffusion processes in [34], and more recently in [4, 8, 12] where also recurrence and transience was studied. Let us also mention the work [35] where recurrence and transience for general Lévy driven OU-processes was characterized. Note that the class of CBI processes satisfying $c = 0$ and $\mu = 0$ form a particular class of Lévy driven OU-processes whose state-space is $\mathbb{R}_+^d$. In order to study the multi-dimensional case, we establish first in Section 2 a general comparison principle for multi-type CBI processes. Such comparison allows us to relate two CBI processes with different admissible parameters with respect to the classical order on $\mathbb{R}_+$. Based on this comparison result we show that each component of the multi-type CBI process dominates a one-dimensional CBI process obtained from the original multi-type CBI process by ignoring all possibilities that an individual of type $k \in \{1, \dots, d\}$ can create individuals of different types $j \neq k$. By assuming conditions sufficient for the smaller one-dimensional CBI processes to not hit zero, or converge to infinity, we obtain a similar result also for the original multi-type CBI process. At this point we implicitly use the results obtained in [12] and [8] for one-dimensional CBI processes.

At this point, we would like to mention that the property $\mathbb{P}[X(t) \in \Gamma, \quad t \geq 0] = 1$ proved in this work can be combined with the Besov regularity of the law of $X(t)$ (see [14]) to show that the multi-type CBI process has the strong Feller property. Assuming in addition mild additional conditions such that this CBI process has a unique invariant measure (see [22]), this method can be combined with the coupling argument from [16] to show that its transition probabilities converge in the total variation distance to the unique invariant measure. Such an approach was first established in [13] for the anisotropic stable JCIR process, see Example 3.8 for its definition. It is clear from the proofs given in [13] that they can also be adapted to general multi-type CBI processes satisfying the conditions imposed in this work and those given in [14]. Finally, let us mention that convergence in total variation for extensions of CBI processes have been studied by different techniques in [32, 18].

2 Comparison principle for multi-type CBI processes

The possibility to describe a multi-type CBI process as a unique $\mathbb{R}_+^d$-valued strong solution to a stochastic differential equation was studied in [2]. The main technical tool there was a comparison principle (see [2, Lemma 4.1]) for multi-type CBI processes with respect to the initial condition and the drift parameter β from the immigration mechanism. Below we provide an extension of this principle where the comparison can also be made with respect to jump measures μ and ν, respectively, from the branching and immigration mechanisms. Let (c, β, B, ν, μ) and $(c, \widetilde{\beta}, \widetilde{B}, \widetilde{\nu}, \widetilde{\mu})$ be admissible parameters satisfying

(A1) $\beta_k \geq \widetilde{\beta}_k$ and $b_{kk} = \widetilde{b}_{kk}$ for all $k = 1, \dots, d$, and $b_{kj} \geq \widetilde{b}_{kj}$ for all $k, j = 1, \dots, d$ with $k \neq j$,

(A2) $\nu \geq \widetilde{\nu}$, $\mu_k \geq \widetilde{\mu}_k$, and $\mu_k \circ \mathrm{pr}_k^{-1} = \widetilde{\mu}_k \circ \mathrm{pr}_k^{-1}$ holds for all $k = 1, \dots, d$, where $\mathrm{pr}_k(z) = z_k$ denotes the projection onto the k-th coordinate.

Condition (A1) imposes a comparison on the drift parameters while condition (A2) imposes a comparison on the state-dependent and state-independent jump measures. Since we can only compare terms with finite variation, the drift coefficients $B, \widetilde{B}$ and jump measures μ_j, $\widetilde{\mu}_j$, $j = 1, \dots, d$, are supposed to coincide on the diagonal.

Below we construct multi-type CBI processes X and Y with admissible parameters (c, β, B, ν, μ) and $(c, \widetilde{\beta}, \widetilde{B}, \widetilde{\nu}, \widetilde{\mu})$, respectively. These processes should be defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with the usual conditions and are obtained as the unique strong $\mathbb{R}_+^d$-valued solutions to a system of stochastic differential equations. Write $\mu_k = \widetilde{\mu}_k + (\mu_k - \widetilde{\mu}_k)$, $k = 1, \dots, d$, and note that $\mu_k - \widetilde{\mu}_k \geq 0$ is a Lévy measure satisfying (1.1). Hence we may consider the following objects defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$:

- A d-dimensional $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion $W = (W(t))_{t \geq 0}$.
- $(\mathcal{F}_t)_{t \geq 0}$-Poisson random measures $N_{\widetilde{\mu}_1}(ds, dz, dr), \dots, N_{\widetilde{\mu}_d}(ds, dz, dr)$ on $\mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+$ with compensators $\widehat{N}_{\widetilde{\mu}_j}(ds, dz, dr) = ds\widetilde{\mu}_j(dz)dr$, $j = 1, \dots, d$, and $(\mathcal{F}_t)_{t \geq 0}$-Poisson random measures $N_{\mu_1 - \widetilde{\mu}_1}(ds, dz, dr), \dots, N_{\mu_d - \widetilde{\mu}_d}(ds, dz, dr)$ on $\mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+$ with compensators $\widehat{N}_{\mu_j - \widetilde{\mu}_j}(ds, dz, dr) = ds\,(\mu_j(dz) - \widetilde{\mu}_j(dz))\,dr$, $j = 1, \dots, d$.
- $(\mathcal{F}_t)_{t \geq 0}$-Poisson random measures $N_{\widetilde{\nu}}$ and $N_{\nu - \widetilde{\nu}}$ on $\mathbb{R}_+ \times \mathbb{R}_+^d$ with compensators $\widehat{N}_{\widetilde{\nu}}(ds, dz) = ds\widetilde{\nu}(dz)$ and $\widehat{N}_{\nu - \widetilde{\nu}}(ds, dz) = ds(\nu(dz) - \widetilde{\nu}(dz))$.

The random objects $W, N_{\widetilde{\nu}}, N_{\nu - \widetilde{\nu}}, N_{\widetilde{\mu}_1}, \dots, N_{\widetilde{\mu}_d}, N_{\mu_1 - \widetilde{\mu}_1}, \dots, N_{\mu_d - \widetilde{\mu}_d}$ are supposed to be independent. Denote by $\widetilde{N}_\nu = N_\nu - \widehat{N}_\nu$, etc., the corresponding compensated Poisson random measures. For given $x, y \in \mathbb{R}_+^d$ we consider the following system of stochastic equations on the state space $\mathbb{R}_+^d$:

$$
\begin{aligned}
X_k(t) = x_k &+ \int_0^t \left(\beta_k + \sum_{j=1}^d b_{kj} X_j(s) \right) ds + \sqrt{2c_k} \int_0^t \sqrt{X_k(s)} dW_k(s) \\
&+ \int_0^t \int_{\mathbb{R}_+^d} z_k N_{\widetilde{\nu}}(ds, dz) + \int_0^t \int_{\mathbb{R}_+^d} z_k N_{\nu - \widetilde{\nu}}(ds, dz)
\end{aligned} \tag{2.1}
$$

$$
+\int_0^t\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+} z_k\mathbb{1}_{\{r\le X_k(s-)\}}\left(\widetilde{N}_{\widetilde{\mu}_k}(ds,dz,dr)+\widetilde{N}_{\mu_k-\widetilde{\mu}_k}(ds,dz,dr)\right)
$$

$$
+\sum_{j\neq k}\int_0^t\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+} z_k\mathbb{1}_{\{r\le X_j(s-)\}}\left(N_{\widetilde{\mu}_j}(ds,dz,dr)+N_{\mu_j-\widetilde{\mu}_j}(ds,dz,dr)\right)
$$

and

$$
Y_k(t)=y_k+\int_0^t\left(\widetilde{\beta}_k+\sum_{j=1}^d\widetilde{b}_{kj}Y_j(s)\right)ds+\sqrt{2c_k}\int_0^t\sqrt{Y_k(s)}dW_k(s) \tag{2.2}
$$

$$
+\int_0^t\int_{\mathbb{R}_+^d} z_kN_{\widetilde{\nu}}(ds,dz)+\int_0^t\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+} z_k\mathbb{1}_{\{r\le Y_k(s-)\}}\widetilde{N}_{\widetilde{\mu}_k}(ds,dz,dr)
$$

$$
+\sum_{j\neq k}\int_0^t\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+} z_k\mathbb{1}_{\{r\le Y_j(s-)\}}N_{\widetilde{\mu}_j}(ds,dz,dr)
$$

It is not difficult to verify that X and Y are multi-type CBI processes with the corresponding admissible parameters.

Proposition 2.1. *Let (c,β,B,ν,μ) and $(c,\widetilde{\beta},\widetilde{B},\widetilde{\nu},\widetilde{\mu})$ be admissible parameters satisfying (A1) and (A2). Then there exist unique $\mathbb{R}_+^d$-valued strong solutions to* (2.1) *and* (2.2)*. Moreover,* (2.1) *determines a CBI process with admissible parameters (c,β,B,ν,μ), and* (2.2) *determines a CBI process with admissible parameters $(c,\widetilde{\beta},\widetilde{B},\widetilde{\nu},\widetilde{\mu})$.*

Proof. Using the Itô formula and a simple computation shows that any solution to (2.1) is also a solution to the martingale problem $(L, C_c^\infty(\mathbb{R}_+^d),\delta_x)$ and any solution to (2.2) is a solution to the martingale problem $(\widetilde{L}, C_c^\infty(\mathbb{R}_+^d),\delta_y)$, see [9, Chapter 4] for the general theory of martingale problems. Here L denotes the generator of the conservative multi-type CBI process with admissible parameters (c,β,B,ν,μ) and $\widetilde{L}$ denotes the generator with admissible parameters $(c,\widetilde{\beta},\widetilde{B},\widetilde{\nu},\widetilde{\mu})$, respectively. Since $C_c^\infty(\mathbb{R}_+^d)$ is a core for the generators, it follows that the martingale problems are well-posed, see [9, Chapter 4, Proposition 1.7, Theorem 2.7, Theorem 4.4]. Hence (2.1) and (2.2) determine the corresponding CBI processes.

Note that existence of uniqueness $\mathbb{R}_+^d$-valued strong solutions to (2.1) and (2.2) was essentially shown in [2, Theorem 4.6], provided that the immigration measures $\nu,\widetilde{\nu}$ satisfy the additional moment condition

$$
\int_{|z|>1}|z|\left(\nu(dz)+\widetilde{\nu}(dz)\right)<\infty. \tag{2.3}
$$

However, since $\nu(\{|z|>1\}),\widetilde{\nu}(\{|z|>1\})<\infty$, a standard interlacing argument similar to [21, Proof of Propostion 9.1] or [17, Section 2] shows that existence and uniqueness to (2.1) and (2.2) for general immigration measures $\nu,\widetilde{\nu}$ is equivalent for those satisfying this additional moment condition. □

The following is our main result for this section.

Theorem 2.2. *Let (c,β,B,ν,μ) and $(c,\widetilde{\beta},\widetilde{B},\widetilde{\nu},\widetilde{\mu})$ be admissible parameters satisfying (A1) and (A2). Let X and Y be the unique strong solutions to* (2.1) *and* (2.2)*, respectively. If $x_k\ge y_k$ holds for all $k=1,\dots,d$, then $\mathbb{P}[X_k(t)\ge Y_k(t),\ \ \forall t\ge 0,\ \ k=1,\dots,d]=1$ holds.*

Proof. Define $\Delta_k(t) := Y_k(t) - X_k(t)$ and $\delta_k(r, s-) = \mathbb{1}_{\{r \le Y_k(s-)\}} - \mathbb{1}_{\{r \le X_k(s-)\}}$. Then $\Delta_k(0) = 0$ and we obtain, for each $k \in \{1, \dots, d\}$,

$$\begin{aligned}\Delta_k(t) = y_k - x_k &+ \int_0^t \left(\widetilde{\beta}_k - \beta_k + \sum_{j=1}^d \left(\widetilde{b}_{kj} Y_j(s) - b_{kj} X_j(s) \right) \right) ds \\ &+ \sqrt{2c_k} \int_0^t \left(\sqrt{Y_k(s)} - \sqrt{X_k(s)} \right) dW_k(s) - \int_0^t \int_{\mathbb{R}_+^d} z_k dN_{\nu - \widetilde{\nu}} \\ &+ \int_0^t \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} z_k \delta_k(r, s-) d\widetilde{N}_{\widetilde{\mu}_k} - \int_0^t \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \le X_k(s-)\}} d\widetilde{N}_{\mu_k - \widetilde{\mu}_k} \\ &+ \sum_{j \neq k} \int_0^t \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} z_k \delta_k(r, s-) dN_{\widetilde{\mu}_j} - \sum_{j \neq k} \int_0^t \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \le X_j(s-)\}} dN_{\mu_j - \widetilde{\mu}_j}.\end{aligned}$$

Let $\phi_m : \mathbb{R} \longrightarrow \mathbb{R}_+$ be twice continuously differentiable functions with the properties:

(i) $\phi_m(z) \nearrow z_+ := \max\{0, z\}$, as $m \to \infty$ for all $z \in \mathbb{R}$.
(ii) $\phi_m'(z) \in [0, 1]$ for all $m \in \mathbb{N}$ and $z \ge 0$.
(iii) $\phi_m'(z) = \phi_m(z) = 0$ for all $m \in \mathbb{N}$ and $z \le 0$.
(vi) $\phi_m''(x - y)|x - y| \le 2/m$ for all $m \in \mathbb{N}$ and $x, y \ge 0$.

The existence of such a sequence was shown in the proof of [31, Theorem 3.1]. Applying the Itô formula to $\phi_m(\Delta_k(t))$ gives

$$\phi_m(\Delta_k(t)) = \phi_m(y_k - x_k) + \sum_{n=1}^7 \int_0^t \mathcal{R}_{k,m}^n(s) ds + \mathcal{M}_{k,m}(t), \tag{2.4}$$

where $\mathcal{R}_{k,m}^1, \dots, \mathcal{R}_{k,m}^7$ are given by

$$\begin{aligned}\mathcal{R}_{k,m}^1(s) &= \phi_m'(\Delta_k(s)) \left(\widetilde{\beta}_k - \beta_k + \sum_{j=1}^d \left(\widetilde{b}_{kj} Y_j(s) - b_{kj} X_j(s) \right) \right), \\ \mathcal{R}_{k,m}^2(s) &= c_k \phi_m''(\Delta_k(s)) \left(\sqrt{Y_k(s)} - \sqrt{X_k(s)} \right)^2, \\ \mathcal{R}_{k,m}^3(s) &= \int_{\mathbb{R}_+^d} \left[\phi_m(\Delta_k(s) - z_k) - \phi_m(\Delta_k(s)) \right] (\nu(dz) - \widetilde{\nu}(dz)), \\ \mathcal{R}_{k,m}^4(s) &= \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} \Big[\phi_m(\Delta_k(s) + z_k \delta_k(r, s)) - \phi_m(\Delta_k(s)) - z_k \delta_k(r, s) \phi_m'(\Delta_k(s)) \Big] dr \widetilde{\mu}_k(dz), \\ \mathcal{R}_{k,m}^5(s) &= \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} \Big[\phi_m(\Delta_k(s) - z_k \mathbb{1}_{\{r \le X_k(s-)\}}) - \phi_m(\Delta_k(s)) \\ &\qquad + z_k \mathbb{1}_{\{r \le X_k(s-)\}} \phi_m'(\Delta_k(s)) \Big] dr (\mu_k(dz) - \widetilde{\mu}_k(dz)), \\ \mathcal{R}_{k,m}^6(s) &= \sum_{j \neq k} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} \left[\phi_m(\Delta_k(s) + z_k \delta_k(r, s)) - \phi_m(\Delta_k(s)) \right] dr \widetilde{\mu}_j(dz), \\ \mathcal{R}_{k,m}^7(s) &= \sum_{j \neq k} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} \left[\phi_m(\Delta_k(s) - z_k \mathbb{1}_{\{r \le X_j(s)\}}) - \phi_m(\Delta_k(s)) \right] dr (\mu_j(dz) - \widetilde{\mu}_j(dz)),\end{aligned}$$

$(\mathcal{M}_{k,m}(t))_{t \ge 0}$ is a local martingale and $\delta_k(r, s) = \mathbb{1}_{\{r \le Y_k(s)\}} - \mathbb{1}_{\{r \le X_k(s)\}}$. For $l \in \mathbb{N}$, define the stopping time $\tau_l = \inf\{t > 0 \mid \max_{i \in \{1, \dots, d\}} \max\{X_i(t), Y_i(t)\} > l\}$. Using the precise form of $\mathcal{M}_{k,m}$ given by Itô's formula combined with similar estimates to [2, Lemma 4.1], one can show that $(\mathcal{M}_{k,m}(t \wedge \tau_l))_{t\geq 0}$ is a martingale for any $l \in \mathbb{N}$. Next we will prove that there exists a constant $C > 0$ such that

$$\sum_{n=1}^{7} \mathcal{R}_{k,m}^{n}(s) \leq C \sum_{j=1}^{d} \Delta_j(s)_+ + \frac{C}{m}. \tag{2.5}$$

Taking then expectations in (2.4), using that $(\mathcal{M}_{k,m}(t \wedge \tau_l))_{t\geq 0}$ is a martingale and estimating as in (2.5) combined with $\phi_m(y_k - x_k) = 0$ by (iii), gives

$$\mathbb{E}[\phi_m(\Delta_k(t \wedge \tau_l))] \leq C \int_0^t \mathbb{E}\left[\sum_{j=1}^{d} \Delta_j(s \wedge \tau_l)_+\right] ds + \frac{Ct}{m}.$$

Letting $m \to \infty$, using property (i), and finally summing over $k = 1, \dots, d$ gives

$$\mathbb{E}\left[\sum_{j=1}^{d} \Delta_j(t \wedge \tau_l)_+\right] \leq C \int_0^t \mathbb{E}\left[\sum_{j=1}^{d} \Delta_j(s \wedge \tau_l)_+\right] ds.$$

Applying Gronwall lemma shows that, for any $l \in \mathbb{N}$, one has $\mathbb{E}\left[\sum_{j=1}^{d} \Delta_j(t \wedge \tau_l)_+\right] = 0$. Letting now $l \to \infty$ and using $\tau_l \to \infty$ a.s. (since X, Y have cádlág paths) yields $\sum_{j=1}^{d} \Delta_j(t)_+ = 0$ a.s. which proves the assertion.

Hence it remains to prove (2.5). In order to estimate $\mathcal{R}_{k,m}^{1}$ we use properties (ii), (iii), (A1), $\widetilde{\beta}_k - \beta_k \leq 0$, $\widetilde{b}_{kj} - b_{kj} \leq 0$ and $\widetilde{b}_{kj} \geq 0$ for $k \neq j$, and $X_j(s) \geq 0$ to obtain

$$\begin{aligned}
\mathcal{R}_{k,m}^{1}(s) &\leq \phi_m'(\Delta_k(s)) \left(b_{kk}\Delta_k(s) + \sum_{j\neq k} \widetilde{b}_{kj}\Delta_j(s) + \sum_{j\neq k} (\widetilde{b}_{kj} - b_{kj}) X_j(s) \right) \\
&\leq |b_{kk}|\Delta_k(s)_+ + \left(\sup_{j\neq k} \widetilde{b}_{kj} \right) \sum_{j\neq k} \Delta_j(s)_+.
\end{aligned}$$

For $\mathcal{R}_{k,m}^{2}$ we obtain from (iv) the estimate $\mathcal{R}_{k,m}^{2}(s) \leq \frac{2c_k}{m}$. Using (ii) we easily find that $\phi_m(\Delta_k(s) - z_k) - \phi_m(\Delta_k(s)) \leq 0$, for $z_k \geq 0$, and hence $\mathcal{R}_{k,m}^{3} \leq 0$ since $\nu(dz) \geq \widetilde{\nu}(dz)$. In order to estimate $\mathcal{R}_{k,m}^{4}(s)$ we first write $\mathcal{R}_{k,m}^{4}(s) = \mathcal{R}_{k,m}^{4,1}(s) + \mathcal{R}_{k,m}^{4,2}(s) + \mathcal{R}_{k,m}^{4,3}(s)$ with

$$\begin{aligned}
\mathcal{R}_{k,m}^{4,1}(s) &= \int_{|z|\leq 1} \int_{\mathbb{R}_+} \left[\phi_m(\Delta_k(s) + z_k\delta_k(r,s)) - \phi_m(\Delta_k(s)) - z_k\delta_k(r,s)\phi_m'(\Delta_k(s))\right] dr\widetilde{\mu}_k(dz), \\
\mathcal{R}_{k,m}^{4,2}(s) &= \int_{|z|>1} \int_{\mathbb{R}_+} \left[\phi_m(\Delta_k(s) + z_k\delta_k(r,s)) - \phi_m(\Delta_k(s))\right] dr\widetilde{\mu}_k(dz), \\
\mathcal{R}_{k,m}^{4,3}(s) &= -\int_{|z|>1} \int_{\mathbb{R}_+} z_k\delta_k(r,s)\phi_m'(\Delta_k(s)) dr\widetilde{\mu}_k(dz).
\end{aligned}$$

For the first term we use property (iv) so that, for each $y > 0$, $z \geq 0$ and $m \in \mathbb{N}$, there exists $\vartheta = \vartheta(y,z) \in [0,1]$ such that

$$\phi_m(y+z) - \phi_m(y) - \phi_m'(y)z = \phi_m''(y + \vartheta z)\frac{z^2}{2} \leq \frac{2z^2}{2m(y+\vartheta z)} \leq \frac{z^2}{my}.$$

Next observe that $\delta_k(r,s) > 0$ if and only if $\Delta_k(s) > 0$ and $r \in (X_k(s), Y_k(s)]$. Applying both observations to $\mathcal{R}_{k,m}^{4,1}(s)$ gives

$$\mathcal{R}_{k,m}^{4,1}(s) \leq \frac{\mathbb{1}_{\{\Delta_k(s)>0\}}}{m\Delta_k(s)} \int_{|z|\leq 1} \int_{\mathbb{R}_+} z_k^2 \delta_k(r,s)^2 dr\widetilde{\mu}_k(dz) \leq \frac{1}{m} \int_{|z|\leq 1} z_k^2 \widetilde{\mu}_k(dz),$$

where we have used $\int_{\mathbb{R}_+} \delta_k(r,s)^2 dr = \Delta_k(s)$ a.s. on $\{\Delta_k(s) > 0\}$. For $\mathcal{R}^{4,2}_{k,m}$ we use property (ii), so that

$$\mathcal{R}^{4,2}_{k,m}(s) \leq \mathbb{1}_{\{\Delta_k(s)>0\}}\Delta_k(s) \int_{|z|>1} \int_{\mathbb{R}_+} z_k \delta_k(r,s) \widetilde{\mu}_k(dz) dr \leq \Delta_k(s)_+ \int_{|z|>1} z_k \widetilde{\mu}_k(dz),$$

where we have also used $\int_{\mathbb{R}_+} \delta_k(r,s)dr = \Delta_k(s)$ a.s. on $\{\Delta_k(s) > 0\}$. For the last term we use again (ii) and similar arguments as above to find that $\mathcal{R}^{4,3}_{k,m}(s) \leq 0$. For $\mathcal{R}^5_{k,m}(s)$ we use the fact that $\mu_k \circ \mathrm{pr}_k^{-1} = \widetilde{\mu}_k \circ \mathrm{pr}_k^{-1}$ to conclude that $\mathcal{R}^5_{k,m}(s) = 0$. For $\mathcal{R}^6_{k,m}(s)$ we use property (ii) to find that

$$\mathcal{R}^6_{k,m}(s) \leq \mathbb{1}_{\{\Delta_k(s)>0\}} \sum_{j\neq k} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}_+} z_k \delta_k(r,s) dr \widetilde{\mu}_j(dz) \leq \sum_{j\neq k} \int_{\mathbb{R}^d_+} z_k \widetilde{\mu}_j(dz) \Delta_k(s)_+,$$

where we have used $\int_{\mathbb{R}_+} \delta_k(r,s)dr = \Delta_k(s)$ a.s. on $\{\Delta_k(s) > 0\}$. To estimate the last term $\mathcal{R}^7_{k,m}(s)$ we use property (ii) so that $\phi_m(\Delta_k(s) - z_k \mathbb{1}_{\{r\leq X_j(s)\}}) - \phi_m(\Delta_k(s)) \leq 0$ and hence $\mathcal{R}^7_{k,m}(s) \leq 0$ since $\mu_j(dz) \geq \widetilde{\mu}_j(dz)$. Combining all estimates proves (2.5) and hence the assertion. □

The next theorem shows that the additional restriction $\mu_k \circ \mathrm{pr}_k^{-1} = \widetilde{\mu}_k \circ \mathrm{pr}_k^{-1}$, $k = 1, \dots, d$, can be omitted if these jump measures have finite first moment also for the small jumps.

Theorem 2.3. *Let (c, β, B, ν, μ) and $(c, \widetilde{\beta}, \widetilde{B}, \widetilde{\nu}, \widetilde{\mu})$ be admissible parameters satisfying (A1), $\nu(dz) \geq \widetilde{\nu}(dz)$, and $\mu_k(dz) \geq \widetilde{\mu}_k(dz)$ for each $k = 1, \dots, d$. Let X and Y be constructed as above. Suppose that*

$$\int_{|z|\leq 1} |z| \mu_k(dz) < \infty, \qquad k = 1, \dots, d, \tag{2.6}$$

and that $x_k \geq y_k$ for all $k = 1, \dots, d$. Then $\mathbb{P}[X_k(t) \geq Y_k(t), \quad \forall t \geq 0] = 1$ for each $k = 1, \dots, d$.

Proof. Due to condition (2.6) we can absorb the compensation in the stochastic integrals against $N_{\widetilde{\mu}_k}$ and $N_{\mu_k - \widetilde{\mu}_k}$ which readily gives

$$\begin{aligned}
\Delta_k(t) = y_k - x_k &+ \int_0^t \left(\widetilde{\beta}_k - \beta_k + \sum_{j=1}^d (\widetilde{g}_{kj} Y_j(s) - g_{kj} X_j(s)) \right) ds \\
&+ \sqrt{2c_k} \int_0^t \left(\sqrt{Y_k(s)} - \sqrt{X_k(s)} \right) dW_k(s) - \int_0^t \int_{\mathbb{R}^d_+} z_k dN_{\nu - \widetilde{\nu}} \\
&+ \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d_+} \int_{\mathbb{R}_+} z_k \delta_k(r, s-) dN_{\widetilde{\mu}_j} - \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d_+} \int_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \leq X_j(s-)\}} dN_{\mu_j - \widetilde{\mu}_j},
\end{aligned}$$

where $g_{kj} = b_{kj}$, $\widetilde{g}_{kj} = \widetilde{b}_{kj}$ for $k \neq j$, and $g_{kk} = b_{kk} - \int_{\mathbb{R}^d_+} z_k \mu_k(dz) = \widetilde{g}_{kk}$. The assertion can now be literarly shown in the same way as in Theorem 2.2. □

3 Application to multi-type CBI processes

Here and below we denote by X a multi-type CBI process with admissible parameters (c, β, B, ν, μ). We start with the simple case where one component of the multi-type CBI process has bounded variation.

Proposition 3.1. *Suppose that there exists* $k \in \{1, \dots, d\}$ *such that*

$$c_k = 0 \qquad \textit{and} \qquad \int_{|z|\le 1} z_k \mu_k(dz) < \infty. \tag{3.1}$$

Then X_k *has bounded variation and*

$$X_k(t) \ge \begin{cases} e^{\theta_k t} x_k + \beta_k \frac{e^{\theta_k t}-1}{\theta_k}, & \textit{if } \theta_k \ne 0 \\ x_k + \beta_k t, & \textit{if } \theta_k = 0 \end{cases}, \qquad t \ge 0, \tag{3.2}$$

where $\theta_k = b_{kk} - \int_{\mathbb{R}_+^d} z_k \mu_k(dz) \in \mathbb{R}$.

Proof. Let $(c, \widetilde{\beta}, \widetilde{B}, \widetilde{\nu}, \widetilde{\mu})$ be admissible parameters given by $\widetilde{\beta} = \beta$, $\widetilde{b}_{kj} = 0$ for $k \ne j$ and $\widetilde{b}_{kk} = b_{kk}$, $\widetilde{\nu} = 0$, and $\widetilde{\mu}_1 = \dots = \widetilde{\mu}_d = 0$. In view of condition (3.1) the process X_k obtained from (2.1) also satisfies

$$\begin{aligned} X_k(t) &= x_k + \int_0^t \left(\beta_k + \sum_{j=1}^d g_{kj} X_j(s) \right) ds + \int_0^t \int_{\mathbb{R}_+^d} z_k N_\nu(ds, dz) \\ &+ \sum_{j=1}^d \int_0^t \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \le X_j(s-)\}} N_{\mu_j}(ds, dz, dr), \end{aligned}$$

where $g_{kj} = b_{kj}$ for $k \ne j$ and $g_{kk} = \theta_k$. Hence X_k has finite variation. The process Y given by (2.2) satisfies $Y_k(t) = x_k + \int_0^t (\beta_k + \theta_k Y(s))\, ds$, i.e.,

$$Y_k(t) = \begin{cases} x_k e^{\theta_k t} + \beta_k \frac{e^{\theta_k t}-1}{\theta_k}, & \text{if } \theta_k \ne 0 \\ x_k + \beta_k t, & \text{if } \theta_k = 0 \end{cases}, \qquad t \ge 0.$$

Using Theorem 2.3 yields $\mathbb{P}[X_k(t) \ge Y_k(t)] = 1$ for all $t \ge 0$ and this fixed choice of k. This proves the assertion. □

From this we easily obtain the following corollary.

Corollary 3.2. *Let* $k \in \{1, \dots, d\}$ *and suppose that* (3.1) *holds. If either* $x_k > 0$ *or* $\beta_k > 0$, *then* $\mathbb{P}[X_k(t) > 0, \ \ t \ge 0] = 1$.

The next proposition gives a multi-dimensional analogue of this result. For $x, y \in \mathbb{R}^d$ we will write $x \le y$ to mean that $x_i \le y_i$ for all $i = 1, \dots, d$.

Proposition 3.3. *Suppose that* (3.1) *holds for all* $k \in \{1, \dots, d\}$. *Then* X *has bounded variation and it holds that*

$$X(t) \ge e^{tG} x + \int_0^t e^{sG} \beta ds, \tag{3.3}$$

where $G = (g_{kj})_{k,j \in \{1,\dots,d\}}$ *is given by*

$$g_{kj} = \begin{cases} b_{kj}, & k \ne j \\ b_{kk} - \int_{\mathbb{R}_+^d} z_k \mu_k(dz), & k = j. \end{cases} \tag{3.4}$$

Proof. Let $(c, \widetilde{\beta}, \widetilde{B}, \widetilde{\nu}, \widetilde{\mu})$ be admissible parameters given by

$$\widetilde{\beta} = \beta, \widetilde{B} = B, \widetilde{\nu} = 0, \text{ and } \widetilde{\mu}_1 = \dots = \widetilde{\mu}_d = 0. \tag{3.5}$$

Observe that under (3.1) the process X obtained from (2.1) also satisfies

$$X_k(t) = x_k + \int_0^t \left(\beta_k + \sum_{j=1}^d g_{kj} X_j(s)\right) ds + \int_0^t \int_{\mathbb{R}_+^d} z_k N_\nu(ds, dz)$$
$$+ \sum_{j=1}^d \int_0^t \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+} z_k \mathbb{1}_{\{r \le X_j(s-)\}} N_{\mu_j}(ds, dz, dr)$$

for each $k = 1, \dots, d$. Let $Y(t)$ be the unique solution to (2.2) with parameters given as in (3.5), i.e., $Y(t) = x + \int_0^t (\beta + GY(s))\, ds$, which is given by $Y(t) = e^{tG}x + \int_0^t e^{sG}\beta ds$. Theorem 2.3 yields $\mathbb{P}[X_k(t) \ge Y_k(t)] = 1$ for all $t \ge 0$ and $k \in \{1, \dots, d\}$. This proves the assertion. □

In view of this estimate we restrict our further analysis to the case where (3.1) does not hold, i.e., the process has unbounded variation. In this case we define, for $k \in \{1, \dots, d\}$, the projected immigration and branching mechanisms by

$$F^{(k)}(\xi) = \beta_k \xi + \int_{\mathbb{R}_+^d} \left(1 - e^{-\xi z_k}\right) \nu(dz), \tag{3.6}$$

$$R^{(k)}(\xi) = -b_{kk}\xi + c_k \xi^2 + \int_{\mathbb{R}_+^d} \left(e^{-\xi z_k} - 1 + \xi z_k\right) \mu_k(dz). \tag{3.7}$$

Each of these immigration and branching mechanisms describes a one-dimensional CBI process which is obtained from a multi-type CBI process with admissible parameters (c, β, B, ν, μ) by ignoring all possibilities that a particle of type k may create another particle of type $j \neq k$.

Theorem 3.4. *Suppose that there exists $k \in \{1, \dots, d\}$ and $\kappa > 0$ such that $R^{(k)}(\xi) > 0$ for $\xi \ge \kappa$. If $c_k > 0$ or $\int_{|z| \le 1} z_k \mu_k(dz) = \infty$, and it holds that*

$$\int_\kappa^\infty \exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du\right) \frac{1}{R^{(k)}(\xi)} d\xi = \infty, \tag{3.8}$$

then $\mathbb{P}[X_k(t) > 0, \quad t \ge 0] = 1$, provided $x_k > 0$.

Proof. Let $(c, \widetilde{\beta}, \widetilde{B}, \widetilde{\nu}, \widetilde{\mu})$ be admissible parameters with $\widetilde{\beta} = \beta$, $\widetilde{B} = \operatorname{diag}(b_{11}, \dots, b_{dd})$, $\widetilde{\nu} = \nu$, $\widetilde{\mu}_k = \mu_k$, and $\widetilde{\mu}_j = 0$ for $j \neq k$. Applying Theorem 2.2 gives

$$\mathbb{P}[Y_k(t) > 0, \quad t \ge 0] = 1 \;\Rightarrow\; \mathbb{P}[X_k(t) > 0, \quad t \ge 0] = 1, \tag{3.9}$$

where X and Y are the unique solutions to (2.1) and (2.2), respectively. It is easy to see that Y_k is a CBI process with immigration and branching mechanisms given by (3.6) and (3.7), respectively. In view of (3.8) Y_k satisfies the conditions of [12, Corollary 6] which proves the assertion. □

From this we directly deduce the following corollary.

Corollary 3.5. *If for each $k \in \{1, \dots, d\}$ the conditions of Theorem 3.4 are satisfied, then $\mathbb{P}[X(t) \in \Gamma, \quad t \ge 0] = 1$, provided $x \in \Gamma = \{x \in \mathbb{R}_+^d \mid x_1, \dots, x_d > 0\}$.*

The following remark provides a sufficient condition for (3.8).

Lemma 3.6. *Suppose that for some $k \in \{1, \dots, d\}$ the following conditions are satisfied:*

(i) There exists $M_0 > 0$ such that $R^{(k)}(\xi) > 0$ for $\xi \ge M_0$.
(ii) There exists $\gamma_k \in (0, 1]$ and $M_1, C_1 > 0$ such that $F^{(k)}(\xi) \ge C_1 \xi^{\gamma_k}$ for $\xi \ge M_1$.
(iii) There exists $\alpha_k \in (1, 2]$ and $M_2, C_2 > 0$ such that $R^{(k)}(\xi) \le C_2 \xi^{\alpha_k}$ for $\xi \ge M_2$.

Then (3.8) *is satisfied, provided one of the following conditions holds:*

(a) $\alpha_k \in (0, 1+\gamma_k)$.
(b) $\alpha_k = 1+\gamma_k$ *and* $\gamma_k \le \frac{C_1}{C_2}$.

Note that, if $\beta_k > 0$, then $F^{(k)}(\xi) \ge \beta_k \xi$ and hence $\gamma_k = 1$. However, Corollary 3.5 also applies in the particular case where $\beta_1 = \dots = \beta_d = 0$.

Proof of Remark 3.6. Set $\kappa = \max\{M_0, M_1, M_2\}$. If $\alpha_k < 1+\gamma_k$, then $\frac{F^{(k)}(u)}{R^{(k)}(u)} \ge \frac{C_1}{C_2} u^{\gamma_k - \alpha_k}$, for $u \in [\kappa, \xi]$, and hence

$$\begin{aligned}\exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du\right) &\ge \exp\left(\frac{C_1}{C_2}\int_\kappa^\xi u^{\gamma_k-\alpha_k} du\right)\\ &= \exp\left(-\frac{C_1}{C_2}\frac{\kappa^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k}\right)\exp\left(\frac{C_1}{C_2}\frac{\xi^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k}\right)\end{aligned}$$

and

$$\begin{aligned}&\int_\kappa^\infty \exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du\right)\frac{d\xi}{R^{(k)}(\xi)}\\ &\quad \ge \frac{\exp\left(-\frac{C_1}{C_2}\frac{\kappa^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k}\right)}{C_2}\int_\kappa^\infty \exp\left(\frac{C_1}{C_2}\frac{\xi^{1+\gamma_k-\alpha_k}}{1+\gamma_k-\alpha_k}\right)\frac{d\xi}{\xi^{\alpha_k}} = \infty.\end{aligned}$$

This proves (3.8) under (a). If $\alpha_k = 1+\gamma_k$, then we obtain for $\xi \ge \kappa$ and $u \in [\kappa, \xi]$,

$$\exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du\right) \ge \exp\left(\frac{C_1}{C_2}\int_\kappa^\xi u^{\gamma_k-\alpha_k} du\right) = \kappa^{-\frac{C_1}{C_2}}\xi^{\frac{C_1}{C_2}}.$$

Using $\alpha_k \le 1 + \frac{C_1}{C_2}$ gives

$$\int_\kappa^\infty \exp\left(\int_\kappa^\xi \frac{F^{(k)}(u)}{R^{(k)}(u)} du\right)\frac{d\xi}{R^{(k)}(\xi)} \ge \frac{\kappa^{-\frac{C_1}{C_2}}}{C_2}\int_\kappa^\xi \frac{\xi^{\frac{C_1}{C_2}}}{\xi^{\alpha_k}} d\xi = \infty,$$

and hence proves (3.8) under (b). □

Our next statement provides a sufficient condition for one component of a multi-type CBI process to converge to infinity.

Theorem 3.7. *Let* $k \in \{1, \dots, d\}$ *and suppose that* $R^{(k)}(\xi) > 0$ *holds for all* $\xi > 0$. *Then* $\mathbb{P}[\lim_{t\to\infty} X_k(t) = \infty] = 1$, *provided one of the following conditions is satisfied:*

(a) $b_{kk} > 0$.
(b) $b_{kk} \le 0$ *and*

$$\int_0^1 \exp\left(-\int_\xi^1 \frac{F^{(k)}(u)}{R^{(k)}(u)} du\right)\frac{d\xi}{R^{(k)}(\xi)} < \infty. \tag{3.10}$$

Proof. Let $(c, \widetilde{\beta}, \widetilde{B}, \widetilde{\nu}, \widetilde{\mu})$ and Y_k be the same as in the proof of Theorem 3.4. Applying Theorem 2.2 gives

$$\mathbb{P}[\lim_{t\to\infty} Y_k(t) = \infty] = 1 \Longrightarrow \mathbb{P}[\lim_{t\to\infty} X_k(t) = \infty] = 1.$$

In view of Proposition 2.1, Y_k satisfies the conditions of [8, Theorem 3] which proves the assertion. □

Let us close this section with the example of an anisotropic stable JCIR process, i.e., the multi-type CBI process X with admissible parameters $(c = 0, \beta, B, \nu, \mu)$, where $\mu = (\mu_1, \dots, \mu_d)$ are, for $\alpha_1, \dots, \alpha_d \in (1, 2)$, given by

$$\mu_j(dz) = \mathbb{1}_{\mathbb{R}_+}(z_j) \frac{dz_j}{z_j^{1+\alpha_j}} \otimes \prod_{k \neq j} \delta_0(dz_k). \tag{3.11}$$

Example 3.8. Let X be the anisotropic stable JCIR process starting from $x \in \mathbb{R}_+^d$. Fix $k \in \{1, \dots, d\}$.

(a) Suppose that there exist $C, M > 0$ and $\gamma_k \in (0, 1]$ such that

$$\beta_k \xi + \int_{\mathbb{R}_+^d} \left(1 - e^{-\xi z_k}\right) \nu(dz) \geq C \xi^{\gamma_k}, \qquad \xi \geq M. \tag{3.12}$$

If $x_k > 0$ and $\alpha_k \in (1, 1 + \gamma_k)$, then $\mathbb{P}[X_k(t) > 0, \quad t \geq 0] = 1$.

(b) If $b_{kk} > 0$, then $\mathbb{P}[\lim_{t \to \infty} X_k(t) = \infty] = 1$.

Proof. Assertion (b) follows immediately from Theorem 3.7 (a). Let us prove assertion (a). Since $\alpha_1, \dots, \alpha_d \in (1, 2)$, it follows that X has unbounded variation. Hence it suffices to show that Theorem 3.4 is applicable. First observe that

$$F^{(k)}(\xi) = \beta_k \xi + \int_{\mathbb{R}_+^d} \left(1 - e^{-\xi z_k}\right) \nu(dz),$$
$$R^{(k)}(\xi) = -b_{kk} \xi + \int_0^\infty \left(e^{-\xi z} - 1 + \xi z\right) \frac{dz}{z^{1+\alpha_k}} = -b_{kk} \xi + K \xi^{\alpha_k},$$

where $K = \int_0^\infty \left(e^{-w} - 1 + w\right) \frac{dw}{w^{1+\alpha_k}} > 0$. Next it is easily seen that

$$R^{(k)}(\xi) > 0, \quad \text{whenever } \xi > \left(\frac{\max\{0, b_{kk}\}}{K}\right)^{\frac{1}{\alpha_k - 1}}.$$

Moreover, one finds $R^{(k)}(\xi) \leq (|b_{kk}| + K)\, \xi^{\alpha_k}$ for $\xi \geq 1$, and hence the assertion follows from Remark 3.6 since $\alpha_k \in (1, 1 + \gamma_k)$. $\square$

In Remark 3.6, if $\beta_k > 0$, then we may take $\gamma_k = 1$ so that (3.12) is satisfied. However, if $\beta_k = 0$, then (3.12) may be still satisfied as it is shown in the following example.

Example 3.9. Let $\gamma \in (0, 1)$ and set $\nu(dz) = \mathbb{1}_{\mathbb{R}_+^d}(z) \frac{dz}{|z|^{d+\gamma}}$. Then $\int_{\mathbb{R}_+^d} (1 \wedge |z|) \nu(dz) < \infty$ and

$$\int_{\mathbb{R}_+^d} \left(1 - e^{-\xi z_k}\right) \frac{dz}{|z|^{d+\gamma}} = \xi^\gamma \int_{\mathbb{R}_+^d} \left(1 - e^{-w_k}\right) \frac{dw}{|w|^{d+\gamma}}.$$

So (3.12) holds for $\gamma_k = \gamma$. Hence the assumptions of Example 3.8 (a) are satisfied, if $\alpha_k \in (1, 1 + \gamma)$.

It is worthwhile to mention that there exists a large class of measures which satisfy (3.12) but are not of the form $\nu(dz) = \mathbb{1}_{\mathbb{R}_+^d}(z) \frac{dz}{|z|^{d+\gamma}}$, see, e.g., [25], [14] and [15].

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Acknowledgments. The authors would like to thank the anonymous referees for many useful comments and remarks on this work, including the suggestion to study a more general comparison principle, as well as for pointing out several references on this topic.