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A universal approach to matching marginals and sums

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Abstract

For a given set of random variables $X_1,\ldots,X_d$ we seek as large a family as possible of random variables $Y_1,\ldots,Y_d$ such that the marginal laws and the laws of the sums match: $Y_i \overset{d}{=} X_i$ and $\sum_i Y_i \overset{d}{=} \sum_i X_i$. Under the assumption that $X_1,\ldots,X_d$ are identically distributed but not necessarily independent, using a symmetry-balancing approach we provide a universal construction with sufficient symmetry to satisfy the more stringent requirement that, for any symmetric function g, $g(Y) \overset{d}{=} g(X)$. The same ideas are shown to extend to the non-identically but "similarly" distributed case.

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1 Introduction

While the multivariate normal distribution is widely used for its tractability and general applicability, there are many situations of great significance where departure from normality is necessary. Such deviation may exist at the univariate level but may rest solely with the dependence structure at hand. The latter refers to situations where the marginal distributions are all normal without the multivariate distribution being normal. Modelling such multivariate distributions is often achieved with the use of copulas. The flexibility afforded by such a construction is limitless allowing, in the bivariate case, a dependence structure that covers the entire spectrum from co-monotonicity (perfect positive dependence) to counter-monotonicity (perfect negative dependence). See Sklar (1959).

Beyond the normal setting, modelling dependencies between stochastic variables is of interest to many areas of applications, notably insurance and finance. While the assumption of independence is technically convenient, in reality it usually does not hold, and one often resorts to copulas to generate more realistic dependence structures in a variety of fields of application.

In a recent paper, GH (2020), the authors produced a characterisation, by means of mean square expansions, of all multivariate distributions whose marginals and sums coincide with those of a set of independent random variables that belong to the same Meixner class. This characterisation is shown to enable specific constructions via finite (truncated) expansions and appropriate compensations. The Meixner class was identified in Meixner (1934) as the family of distributions for which the generating function of the

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associated orthogonal polynomials takes a specific form. It is made up of five types of distributions: normal, generalised gamma, generalised Poisson, generalised (negative) binomial and generalised hypergeometric. See Eagleson (1964); GH (2020) for more details.

In this paper, we propose a universal construction that goes beyond the Meixner class of distributions, beyond the independent setting and beyond the limitations of the finite expansion method.

Given random variables $X_1, \ldots, X_d$, we seek random variables $Y_1, \ldots, Y_d$, such that, for each i $Y_i \overset{d}{=} X_i$, $Y_1 + \ldots + Y_d \overset{d}{=} X_1 + \ldots + X_d$ and $(Y_1, \ldots, Y_d) \overset{d}{\neq} (X_1, \ldots, X_d)$. The construction we use is universal in the sense that it produces a large family of copulas that meet the above requirements irrespective of the marginals under consideration, albeit under the assumption that the random variables $X_1, \ldots, X_d$ are identically distributed. A construction in the non-identically distributed case is given in Section 4. Section 2 contains the main (copula) construction and Section 3 provides the sought answer in the case of identically distributed random variables.

To the best of our knowledge the only known construction, other than those given in our recent paper, GH (2020), is due to Stoyanov (2013).

Example 1.1. Stoyanov (2013) (see Counterexample 10.5) suggests the following construction of a pair (Y_1, Y_2) of dependent (but uncorrelated) standard normal random variables such that $Y_1 + Y_2$ is normal with mean 0 and variance equal to 2:

$$\varphi(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1^2 + x_2^2)\right) \left(1 + \kappa x_1 x_2 (x_1^2 - x_2^2) \exp\left(-\frac{1}{2}(x_1^2 + x_2^2)\right)\right). \quad (1.1)$$

Here κ is any positive constant that ensures that $\varphi(x_1, x_2) \geq 0$; e.g. $\kappa \leq e^2/8$.

While the above example provides one specific construction in the two-dimensional Gaussian and independent case, it does not shed any light on whether other solutions exist and how to construct them, nor does it fulfil the aspiration to extend the problem beyond the Gaussian and independent case.

In this paper, we propose to answer the question of matching marginals and sums in considerable generality. After listing a few basic facts that guide our discovery, we develop in Section 2 a symmetry-balancing approach that delivers sufficient symmetry to satisfy the more stringent requirement that, for any symmetric function g, $g(Y_1, \ldots, Y_d) \overset{d}{=} g(X_1, \ldots, X_d)$. The construction is universal in that it applies to any $(X_1, \ldots, X_d)$ as long as $X_1, \ldots, X_d$ admit a joint density f.

We shall assume throughout this paper that all multivariate random variables admit joint densities, denoted by f. We propose a generic construction under the assumption that the random variables $X_1, X_2, \ldots, X_d$ are identically distributed and admit a joint density but with no restriction on their dependence. An extension to non-identically distributed random variables will be discussed in Section 4. This leads to the introduction of the concept of similar distributions.

2 The "matching" copula

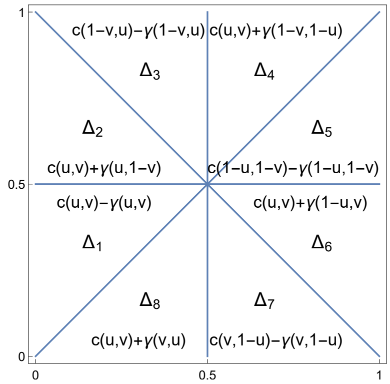
Let X_1 and X_2 be two identically distributed random variables with joint density f and marginal density ϕ. Then, for any (measurable) ℓ such that $0 \leq \ell \leq f$,

$$\varphi(x_1, x_2) = \begin{cases} f(x_1, x_2) - \ell(x_1, x_2) & (x_1, x_2) \in D_+ \\ f(x_1, x_2) + \ell(x_2, x_1) & (x_1, x_2) \in D_- \end{cases}$$

where $D_\pm = \{(x_1, x_2) : \pm x_2 < \pm x_1\}$, is the density of a pair (Y_1, Y_2) such that for any symmetric g, $g(Y_1, Y_2) \overset{d}{=} g(X_1, X_2)$.

While φ defines a pair that matches the law of $g(X_1, X_2)$, for all symmetric functions g, it does not necessarily preserve the marginal laws of f. To do so we need to also "compensate" in the x_1 and x_2 directions. We call this a *symmetry-balancing* approach. The proposed construction is universal in that it is described through the use of a "copula perturbation" that can then be applied to any distribution. It offers a continuum of settings that can be used to model a wide range of dependence structures, through perturbations of varying types and sizes.

Let c be the copula density of f. For any measurable $\gamma : [0,1] \times [0,1] \to [0,+\infty)$, the function θ_γ described in the figure below, and supposed to be non-negative, is a copula density we call the *octal copula*.



Furthermore, for any bounded and symmetric g, $\int g(u,v)\theta_\gamma(u,v)dudv = \int g(u,v)c(u,v)dudv$.

The proof is given in Theorem 2.3 in the more general setting of the d-dimensional case, $d \geq 2$.

We note that the Stoyanov example has a copula that is of the octal form.

We are now ready to extend the construction of θ_γ to the d-dimensional hypercube $[0,1]^d$. We shall retain from the two-dimensional case the idea that regions (i.e. $\Delta_2, \ldots, \Delta_8$) are mapped onto a reference region (i.e. Δ_1). Core to these mappings are the reflections $(u_1,u_2) \hookrightarrow (1-u_1,u_2)$, $(u_1,u_2) \hookrightarrow (u_1,1-u_2)$ and $(u_1,u_2) \hookrightarrow (1-u_1,1-u_2)$ as well as $(u_1,u_2) \hookrightarrow (u_2,u_1)$. Generalising these to the hypercube lead to the maps τ_α, for the first three, and the maps σ_β for the last one. These are introduced next.

- S_d denotes the space of permutations on $[d] = \{1,\ldots,d\}$, and for $\beta \in S_d$, σ_β denotes the function defined on $[0,1]^d$, $\sigma_\beta(u) = (u_{\beta(1)},\ldots,u_{\beta(d)})$.
- id denotes the identity function.
- $\mathcal{G}(\mathbb{R}^d)$ and $\mathcal{G}([0,1]^d)$, or simply $\mathcal{G}_d$, denote the sets of symmetric real-valued functions g on $\mathbb{R}^d$ and $[0,1]^d$, respectively; that is for any $\beta \in S_d$, $g \circ \sigma_\beta = g$.
- For $\alpha \in \{0,1\}^d$ and $u = (u_1,\ldots,u_d) \in [0,1]^d$, $\tau_\alpha(u) = \big(\alpha_i(1-u_i) + (1-\alpha_i)u_i\big)_{i=1}^d$.
- $\Delta(0) = (0,1/2)^d$ and for any other $\alpha \in \{0,1\}^d$, $\Delta(\alpha) = \{u \in [0,1]^d :\ \tau_\alpha(u) \in \Delta(0)\}$.
- $\Delta(0,id) = \{u \in [0,1]^d :\ 0 < u_1 < u_2 < \ldots < u_d < 1/2\}$ and, for any other pair $(\alpha,\beta) \in \{0,1\}^d \times S_d$,

$$\Delta(\alpha,\beta) = \{u \in [0,1]^d :\ \sigma_\beta(\tau_\alpha(u)) \in \Delta(0,id)\}.$$

  In the two-dimensional case, $\Delta(0,id)$ was referred to as Δ_1, $\Delta(0,(12))$ was referred to as Δ_8, $\Delta((0,1),id)$ was referred to as Δ_2 etc.
- $\Xi_d = [0,1]^d \setminus \bigcup_{(\alpha,\beta)} \Delta(\alpha,\beta)$.

The next lemma shows that we can essentially partition the hypercube into $2^d d!$ regions that all map onto the reference region $\Delta(0, id)$.

Lemma 2.1. *1. For $(\alpha,\beta) \neq (\alpha',\beta')$, $\Delta(\alpha,\beta) \cap \Delta(\alpha',\beta') = \emptyset$.*

2. $\forall\ u \in [0,1]^d \setminus \Xi_d$, $\exists!\ (\alpha,\beta) \in \{0,1\}^d \times S_d$ such that $q = \sigma_\beta(\tau_\alpha(u))$ satisfies the condition $0 < q_1 < \ldots < q_d < 1/2$.

3. For any $u \in [0,1]^d$, $\tau_{\alpha'}(\tau_\alpha(u)) = \tau_{\tau_{\alpha'}(\alpha)}(u)$ and $\sigma_\beta(\tau_\alpha(u)) = \tau_{\sigma_\beta(\alpha)}(\sigma_\beta(u))$.

4. $\tau_\alpha(u) \in \Delta(\alpha',\beta') \Leftrightarrow u \in \Delta(\tau_\alpha(\alpha'),\beta')$. $\sigma_\beta(u) \in \Delta(\alpha',\beta') \Leftrightarrow u \in \Delta(\sigma_{\beta^{-1}}(\alpha'), \beta' \circ \beta)$.

A necessary step in the construction is the embedding of the $(d-1)$-dimensional hypercube as a hyperplane in the d-dimensional hypercube and how the corresponding partitions carry across. To that end, we need the following notations and results.

- For $v \in [0,1]^{d-1}$, $r \in [0,1]$ and $k \in [d]$,

$$\omega_k(v,r) = (v_1, \ldots, v_{k-1}, r, v_k, \ldots, v_{d-1}),$$

  with the obvious adjustments for the cases $k = 1$ and $k = d$.
- For $b \in S_{d-1}$ and $j, k \in [d]$, $\beta = \chi_j(b,k)$ is the permutation in S_d

$$\beta(i) = \begin{cases} b(i) & \text{if } i \le j-1 \text{ and } b(i) \le k-1 \\ b(i)+1 & \text{if } i \le j-1 \text{ and } b(i) \ge k \\ k & \text{if } i = j \\ b(i-1) & \text{if } i \ge j+1 \text{ and } b(i-1) \le k-1 \\ b(i-1)+1 & \text{if } i \ge j+1 \text{ and } b(i-1) \ge k \end{cases}$$

  If permutation b is identified with the vector $(b(1), \ldots, b(d-1))$ and similarly for β, and the definition of ω_k is extended to $\mathbb{R}^d$, then β can be written as $\omega_j(b + 1_{b \ge k}, k)$.
- For $a \in \{0,1\}^{d-1}$, $b \in S_{d-1}$, $j \in [d]$ and $r \in [0,1]$, with a slight abuse of notation in the use of τ and σ, $\Delta_j(a,b,r) = \{v \in [0,1]^{d-1} :\ q_0 < \ldots < q_{j-1} < \bar{r} < q_j < \ldots < q_d\}$, where $\bar{r} = \min(r, 1-r)$ and, q stands for $\sigma_b(\tau_a(v))$ and has been augmented with the bounds $q_0 = 0$ and $q_d = 1/2$.

Lemma 2.2. *Let $a \in \{0,1\}^{d-1}$, $b \in S_{d-1}$, $j,k \in [d]$, $r \in [0,1]$, $\beta = \chi_j(b,k)$ and*

$$\alpha = \begin{cases} \omega_k(a,0) & r \in [0,1/2] \\ \omega_k(a,1) & r \in (1/2,1] \end{cases}$$

1. $\tau_\alpha(\omega_k(v,r)) = \omega_k(\tau_a(v),\bar{r})$ and $\sigma_\beta(\tau_\alpha(\omega_k(v,r))) = \omega_j(\sigma_b(\tau_a(v)),\bar{r})$.

2. $\omega_k(v,r) \in \Delta(\alpha,\beta) \quad \Leftrightarrow \quad v \in \Delta_j(a,b,r) \quad \Leftrightarrow \quad \sigma_b(\tau_a(v)) \in \Delta_j(0,id,r)$.

Since Ξ_d has Lebesgue measure zero, any integral over $[0,1]^d$ can be taken to mean an integral on $[0,1]^d \setminus \Xi_d$.

Theorem 2.3. *Let c be a density on $[0,1]^d$, U be a multivariate random variable with density c, γ be an integrable non-negative function on $\Delta(0,id)$ and $\varepsilon : \{0,1\}^d \times S_d \to [-1,1]$. We assume that $\gamma(u) \le \min_{\varepsilon(\alpha,\beta)>0} \dfrac{c(\tau_\alpha(\sigma_{\beta^{-1}}(u)))}{\varepsilon(\alpha,\beta)}$ and define on $[0,1]^d$ the function*

$$\theta_{\varepsilon,\gamma}(u) = c(u) - \sum_{\alpha,\beta} \varepsilon(\alpha,\beta)\gamma(\sigma_\beta(\tau_\alpha(u)))1_{\Delta(\alpha,\beta)}(u). \tag{2.1}$$

(D) $\theta_{\varepsilon,\gamma}$ is a density if and only if $\sum_{\alpha,\beta} \varepsilon(\alpha,\beta) = 0$.

Assume (D) and let V be a multivariate random variable with density $\theta_{\varepsilon,\gamma}$.

(G) $g(V) \overset{d}{=} g(U)$, *for any* $g \in \mathcal{G}_d$, *if and only if* $\forall\ \alpha \in \{0,1\}^d$, $\sum_\beta \varepsilon(\sigma_{\beta^{-1}}(\alpha), \beta) = 0$.

In particular, in this case, $V_1 + \ldots + V_d \overset{d}{=} U_1 + \ldots + U_d$.

(M_0) Fix $k \in [d]$. *If* $\forall\ j \in [d]$, $\forall\ a \in \{0,1\}^{d-1}$, $\forall\ b \in S_{d-1}$, $\sum_{\aleph=0,1} \varepsilon(\omega_k(a,\aleph), \chi_j(b,k)) = 0$, *then* $(V_1, \ldots, V_{k-1}, V_{k+1}, \ldots, V_d) \overset{d}{=} (U_1, \ldots, U_{k-1}, U_{k+1}, \ldots, U_d)$.

(C_0) Suppose c *is a copula density. If* $\forall\ j,k \in [d]$, $\sum_{a,b} \varepsilon(\omega_k(a,0), \chi_j(b,k)) = 0$ *and* $\sum_{a,b} \varepsilon(\omega_k(a,1), \chi_j(b,k)) = 0$, *then* $\theta_{\varepsilon,\gamma}$ *is a copula density.*

Proof. (D) Clearly, $\theta_{\varepsilon,\gamma}(u) \geq 0$. Let us show that it integrates to 1 or equivalently that $\sum_{\alpha,\beta} \varepsilon(\alpha,\beta)\gamma(\sigma_\beta(\tau_\alpha(u)))1_{\Delta(\alpha,\beta)}(u)$ integrates to 0. Applying (G) (below) to the case $g = 1$, we get that $\theta_{\varepsilon,\gamma}$ is a density if and only if

$$0 = \sum_\alpha \sum_\beta \varepsilon(\sigma_{\beta^{-1}}(\alpha), \beta) = \sum_\beta \sum_\alpha \varepsilon(\sigma_{\beta^{-1}}(\alpha), \beta) = \sum_{\alpha,\beta} \varepsilon(\alpha, \beta).$$

(G) Let $g \in \mathcal{G}_d$. $g(V) \overset{d}{=} g(U)$ if and only if for any bounded function G, $\mathbb{E}[G(g(V))] = \mathbb{E}[G(g(U))]$. Now $\overline{g} = G \circ g$ is bounded and symmetric, and

$$\begin{aligned} \int_{\Delta(\alpha,\beta)} \overline{g}(u)\gamma(\sigma_\beta(\tau_\alpha(u)))du &= \int_{\Delta(0,id)} \overline{g}(\tau_\alpha(\sigma_{\beta^{-1}}(u)))\gamma(u)du \\ = \int_{\Delta(0,id)} \overline{g}(\sigma_{\beta^{-1}}(\tau_{\sigma_\beta(\alpha)}(u)))\gamma(u)du &= \int_{\Delta(0,id)} \overline{g}(\tau_{\sigma_\beta(\alpha)}(u))\gamma(u)du. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{[0,1]^d} \overline{g}(u)\theta_{\varepsilon,\gamma}(u)du - \int_{[0,1]^d} \overline{g}(u)c(u)du &= -\sum_{\alpha,\beta} \varepsilon(\alpha,\beta) \int_{\Delta(\alpha,\beta)} \overline{g}(u)\gamma(\sigma_\beta(\tau_\alpha(u)))du \\ = -\sum_\alpha \int_{\Delta(0,id)} \overline{g}(\tau_\alpha(u))\gamma(u)du \sum_\beta \varepsilon(\sigma_{\beta^{-1}}(\alpha), \beta). \end{aligned}$$

Clearly if (G) holds, then $\int_{[0,1]^d} \overline{g}(u)\theta_{\varepsilon,\gamma}(u)du = \int_{[0,1]^d} \overline{g}(u)c(u)du$ and $g(V) \overset{d}{=} g(U)$.

Conversely, suppose that $g(V) \overset{d}{=} g(U)$, for any $g \in \mathcal{G}_d$. Then for any bounded $g \in \mathcal{G}_d$,

$$\sum_\alpha \int_{\Delta(\alpha,id)} \Gamma(\alpha)g(u)\gamma(\tau_\alpha(u))du = 0,$$

where $\Gamma(\alpha) = \sum_\beta \varepsilon(\sigma_{\beta^{-1}}(\alpha), \beta)$. Fix $\alpha_0 \in \{0,1\}^d$ and let e_0 be any bounded measurable function. Define e as $e(u) = e_0(u)1_{\Delta(\alpha_0,id)}(u)$ and define g by symmetrisation of e: $g(u) = \sum_\beta e(\sigma_\beta(u))$. Then ($g$ is symmetric and)

$$\begin{aligned} 0 &= \sum_\alpha \int_{\Delta(\alpha,id)} \Gamma(\alpha)\Big(\sum_\beta e(\sigma_\beta(u))\Big)\gamma(\tau_\alpha(u))du \\ &= \sum_\alpha \sum_\beta \int_{\Delta(\alpha,id)} \Gamma(\alpha)e(\sigma_\beta(u))\gamma(\tau_\alpha(u))du = \Gamma(\alpha_0) \sum_\beta \int_{\Delta(\alpha_0,id)} e(\sigma_\beta(u))\gamma(\tau_{\alpha_0}(u))du, \end{aligned}$$

where we use the fact that $\sigma_\beta(u) \in \Delta(\alpha_0, id)$ if and only if $u \in \Delta(\sigma_{\beta^{-1}}(\alpha_0), \beta)$. We conclude that $\Gamma(\alpha_0) = 0$. Repeating for all other α_0 in $\{0,1\}^d$, we prove the result.

(M_0) Fix $k \in [d]$, $v \in [0,1]^{d-1} \setminus \Xi_{d-1}$ and let $(a,b) \in \{0,1\}^{d-1} \times S_{d-1}$ such that $q = \sigma_b(\tau_a(v))$ satisfies $0 < q_1 < \ldots < q_{d-1} < 1/2$. Let $q_0 = 0$, $q_d = 1/2$.

Using Lemma 2.2, we get

$$\begin{aligned}
& \int_{[0,1]} \theta_{\varepsilon,\gamma}(\omega_k(v,r))dr - \int_{[0,1]} c(\omega_k(v,r))dr \\
= & -\sum_{j=0}^{d-1} \int_{[q_j,q_{j+1}]} \sum_{\alpha,\beta} \varepsilon(\alpha,\beta)\gamma(\sigma_\beta(\tau_\alpha(\omega_k(v,r))))1_{\Delta(\alpha,\beta)}(\omega_k(v,r))dr \\
& - \sum_{j=0}^{d-1} \int_{[1-q_{j+1},1-q_j]} \sum_{\alpha,\beta} \varepsilon(\alpha,\beta)\gamma(\sigma_\beta(\tau_\alpha(\omega_k(v,r))))1_{\Delta(\alpha,\beta)}(\omega_k(v,r))dr \\
= & -\sum_{j=0}^{d-1} \int_{[q_j,q_{j+1}]} \varepsilon(\omega_k(a,0),\chi_{j+1}(b,k))\gamma(\omega_{j+1}(\sigma_b(\tau_a(v)),r))dr \\
& - \sum_{j=0}^{d-1} \int_{[1-q_{j+1},1-q_j]} \varepsilon(\omega_k(a,1),\chi_{j+1}(b,k))\gamma(\omega_{j+1}(\sigma_b(\tau_a(v)),1-r))dr \\
= & -\sum_{j=0}^{d-1} \int_{[q_j,q_{j+1}]} \gamma(\omega_{j+1}(\sigma_b(\tau_a(v)),r))dr \sum_{\aleph=0,1} \varepsilon(\omega_k(a,\aleph),\chi_{j+1}(b,k))
\end{aligned}$$

If (M_0) holds, then for any $k \in [d]$, $\int_{[0,1]} \theta_{\varepsilon,\gamma}(\omega_k(v,r))dr = \int_{[0,1]} c(\omega_k(v,r))dr$ (a.e.); that is $(V_1,\ldots,V_{k-1},V_{k+1},\ldots,V_d) \stackrel{d}{=} (U_1,\ldots,U_{k-1},U_{k+1},\ldots,U_d)$.

(C_0) We fix $k \in [d]$ and proceed to show that under (C_0), $V_k \stackrel{d}{=} U_k$. Fix $r \in (0,1)$.

$$\begin{aligned}
& \int_{[0,1]^{d-1}} \theta_{\varepsilon,\gamma}(\omega_k(v,r))dv \\
= & \int_{[0,1]^{d-1}} c(\omega_k(v,r))dv - \int_{[0,1]^{d-1}} \sum_{\alpha,\beta} \varepsilon(\alpha,\beta)\gamma(\sigma_\beta(\tau_\alpha(\omega_k(v,r))))1_{\Delta(\alpha,\beta)}(\omega_k(v,r))dv \\
= & \ 1 - \int_{[0,1]^{d-1}} \sum_{j,a,b} \varepsilon(\alpha,\beta)\gamma(\sigma_\beta(\tau_\alpha(\omega_k(v,r))))1_{\Delta_j(a,b,r)}(v)dv,
\end{aligned}$$

where α and β are defined as in Lemma 2.2. It follows that

$$\begin{aligned}
\int_{[0,1]^{d-1}} \theta_{\varepsilon,\gamma}(\omega_k(v,r))dv - 1 \ &= \ -\sum_{j,a,b} \varepsilon(\alpha,\beta) \int_{\Delta_j(a,b,r)} \gamma(\sigma_\beta(\tau_\alpha(\omega_k(v,r))))dv \\
= \ -\sum_{j,a,b} \varepsilon(\alpha,\beta) \int_{\Delta_j(a,b,r)} \gamma(\omega_j(\sigma_b(\tau_a(v)),\bar{r}))dv \ &= \ -\sum_{j,a,b} \varepsilon(\alpha,\beta) \int_{\Delta_j(0,id,r)} \gamma(\omega_j(v,\bar{r}))dv,
\end{aligned}$$

which concludes the proof. □

Example 2.4. When $d = 2$, (C_0) produces 8 equations and (G) another 4. Solving these we see that ε must take the form $\varepsilon(\alpha,\beta) = \lambda(-1)^{|\alpha|}\text{sgn}(\beta)$, where $|\alpha| = \alpha_1 + \alpha_2$ and $\text{sgn}(\beta)$ is the signature of the permutation β, $\text{sgn}(id) = 1$ and $\text{sgn}((12)) = -1$. In other words, θ_γ defined at the beginning of Section 2 is, up to a multiplicative factor, the only copula such that, for any $g \in \mathcal{G}_2$, $g(V) \stackrel{d}{=} g(U)$.

On the other hand the equations in (M_0) yield solutions of the form $\varepsilon(\alpha, id) = (-1)^{|\alpha|}\lambda$ and $\varepsilon(\alpha,(12)) = (-1)^{|\alpha|}\mu$.

(M_0) and (C_0) are sufficient to guarantee that marginal distributions match and that $\theta_{\varepsilon,\gamma}$ is a copula density, respectively. The next result shows that they are necessary if we add the assumption that γ is bounded from above and away from 0.

Theorem 2.5. *Further to the setting of Theorem 2.3, we assume that γ is bounded from above and away from 0; i.e. we assume that*

$$\inf(\gamma) = \inf_{u\in\Delta(0,id)} \gamma(u) > 0 \text{ and } \sup(\gamma) = \sup_{u\in\Delta(0,id)} \gamma(u) < +\infty. \tag{2.2}$$

(M) Fix $k \in [d]$. $(V_1,\ldots,V_{k-1},V_{k+1},\ldots,V_d) \overset{d}{=} (U_1,\ldots,U_{k-1},U_{k+1},\ldots,U_d)$ if and only if

$$\forall\ j \in [d], \forall\ a \in \{0,1\}^{d-1}, \forall\ b \in S_{d-1},\ \sum_{\aleph=0,1} \varepsilon(\omega_k(a,\aleph),\chi_j(b,k)) = 0. \tag{2.3}$$

(C) Suppose c is a copula density. $\theta_{\varepsilon,\gamma}$ is a copula density if and only if

$$\forall\ j,k \in [d],\ \sum_{a,b} \varepsilon(\omega_k(a,0),\chi_j(b,k)) = \sum_{a,b} \varepsilon(\omega_k(a,1),\chi_j(b,k)) = 0. \tag{2.4}$$

Proof. The sufficiency of both statements was shown in Theorem 2.3.

(M) We know that $(V_1,\ldots,V_{k-1},V_{k+1},\ldots,V_d) \overset{d}{=} (U_1,\ldots,U_{k-1},U_{k+1},\ldots,U_d)$ if and only if, $\forall\ a \in \{0,1\}^{d-1}$, $\forall\ b \in S_{d-1}$,

$$\forall\ q \in (0,1/2)^{d-1} \text{ such that } q_1 < \ldots < q_{d-1},\ \sum_{j=1}^{d} \lambda_j \int_{[q_{j-1},q_j]} \gamma(\omega_j(q,r))dr = 0,$$

where $\lambda_j = \sum_{\aleph=0,1} \varepsilon(\omega_k(a,\aleph),\chi_j(b,k))$, $q_0 = 0$ and $q_d = 1/2$. We reason by contradiction and assume that $\{j \in [d] : \lambda_j \neq 0\} \neq \emptyset$. Then

$$\sum_{j=1}^{d} \lambda_j \int_{[q_{j-1},q_j]} \gamma(\omega_j(q,r))dr \geq \sum_{j:\lambda_j<0} \lambda_j \sup(\gamma)(q_j - q_{j-1}) + \sum_{j:\lambda_j>0} \lambda_j \inf(\gamma)(q_j - q_{j-1})$$

and shrinking $q_j - q_{j-1}$ whenever $\lambda_j < 0$ (and therefore expanding it whenever $\lambda_j > 0$) shows that the right hand side can be made strictly positive, thus contradicting the assumption that the left hand side is nil. Of course, if there is no j such that $\lambda_j > 0$, then the inequality can be reversed and the left hand side shown to be strictly negative, leading to a contradiction.

(C) We know that $\theta_{\varepsilon,\gamma}$ is a copula density if and only if $\forall\ k \in [d]$,

$$\forall\ r \in (0,1),\ \sum_{j=1}^{d} \lambda_j \int_{\Delta_j(0,id,r)} \gamma(\omega_j(v,\bar{r}))dv = 0, \tag{2.5}$$

where $\lambda_j = \sum_{a,b} \varepsilon(\alpha,\beta)$ and, α and β are as in Lemma 2.2. We also note that

$$\mathrm{Leb}(\Delta_j(0,id,r)) = \frac{\bar{r}^{j-1}(1-2\bar{r})^{d-j}}{2^{d-j}(j-1)!(d-j)!} \propto \bar{r}^{j-1}(1-2\bar{r})^{d-j}.$$

Again we reason by contradiction. First we assume that $\lambda_1 \neq 0$ and more specifically (wlog) $\lambda_1 > 0$. Then

$$\sum_{j=1}^{d} \lambda_j \int_{\Delta_j(0,id,r)} \gamma(\omega_j(v,\bar{r}))dv \geq \lambda_1 \inf(\gamma)\mathrm{Leb}(\Delta_1(0,id,r)) + \sum_{j=2}^{d} \lambda_j \int_{\Delta_j(0,id,r)} \gamma(\omega_j(v,\bar{r}))dv.$$

Since $\lim_{r\to 0} \text{Leb}(\Delta_j(0, id, r)) = 0$, for $j \in \{2, \ldots, d\}$, and γ is bounded, by making r approach 0, the second term in the right hand side can be made as small as we want, while the first term is strictly positive. It follows that the left hand side can be made strictly positive thus contradicting the fact that it must be nil for all r. We deduce that λ_1 must be nil and (2.5) becomes

$$\forall\, r \in (0,1),\ \sum_{j=2}^{d} \lambda_j \frac{1}{r} \int_{\Delta_j(0,id,r)} \gamma(\omega_j(v,\bar{r}))dv = 0.$$

Again, we assume (wlog) that $\lambda_2 > 0$. Then

$$\sum_{j=2}^{d} \frac{\lambda_j}{r} \int_{\Delta_j(0,id,r)} \gamma(\omega_j(v,\bar{r}))dv \geq \lambda_2 \inf(\gamma) \frac{\text{Leb}(\Delta_2(0,id,r))}{r} + \sum_{j=3}^{d} \frac{\lambda_j}{r} \int_{\Delta_j(0,id,r)} \gamma(\omega_j(v,\bar{r}))dv$$

and the second term in the right hand side can be made as small as we want, while the first term is strictly positive. It follows that the left hand side can be made strictly positive thus showing that λ_2 must be nil. We continue this way, adjusting (2.5) by increasing powers of r, to prove that $\lambda_1 = \ldots = \lambda_{d-1} = 0$ and finally that $\lambda_d = 0$ since

$$\forall\, r \in (0,1),\ \lambda_d \int_{\Delta_d(0,id,r)} \gamma(\omega_d(v,\bar{r}))dv = 0.$$ □

Corollary 2.6. *Suppose* (2.2) *holds and* ε *takes the form* $\varepsilon(\alpha,\beta) = \zeta(|\alpha|)\psi(\beta)$*, where* $|\alpha| = \alpha_1 + \ldots + \alpha_d$*.*

(D′) $\theta_{\varepsilon,\gamma}$ *is a density if and only if either* $\sum_{\alpha} \zeta(|\alpha|) = 0$ *or* $\sum_{\beta} \psi(\beta) = 0$*.*

(G′) $g(V) \overset{d}{=} g(U)$*, for any* g *symmetric, if and only if* $\sum_{\beta} \psi(\beta) = 0$*.*

(C′) Suppose c *is a copula density (i.e. the marginals are uniform).* $\theta_{\varepsilon,\gamma}$ *is a copula density if and only if*

$$\sum_{a\in\{0,1\}^{d-1}} \zeta(|a|) = \sum_{a\in\{0,1\}^{d-1}} \zeta(|a|+1) = 0 \text{ or } \forall\, j,k \in [d],\ \sum_{b\in S_{d-1}} \psi(\chi_j(b,k)) = 0.$$

Proposition 2.7. *A necessary and sufficient condition for*

$$\forall\, j,k \in [d], \forall\, a \in \{0,1\}^{d-1}, \forall\, b \in S_{d-1},\ \sum_{\aleph=0,1} \varepsilon(\omega_k(a,\aleph), \chi_j(b,k)) = 0$$

is that

$$\forall\, \alpha \in \{0,1\}^d, \forall\, \beta \in S_d\ \varepsilon(\alpha,\beta) = (-1)^{|\alpha|}\varepsilon(0,\beta).$$

Proof. Sufficiency is immediate. We prove necessity by induction on d. First we observe that for any $\beta \in S_d$, for any $k \in [d]$, there exists $j \in [d]$ and $b \in S_{d-1}$ such that $\beta = \chi_j(b,k)$. Indeed, letting $j = \beta^{-1}(k)$ and

$$b(i) = \begin{cases} \beta(i) & \text{if } i \leq j-1 \text{ and } \beta(i) \leq k-1 \\ \beta(i)-1 & \text{if } i \leq j-1 \text{ and } \beta(i) \geq k+1 \\ \beta(i+1) & \text{if } i \geq j \text{ and } \beta(i+1) \leq k-1 \\ \beta(i-1)+1 & \text{if } i \geq j \text{ and } \beta(i+1) \geq k+1 \end{cases}$$

we obtain the required identity. We shall therefore prove that, for any fixed $\beta \in S_d$, the condition

$$\forall\, k \in [d], \forall\, a \in \{0,1\}^{d-1},\ \varepsilon(\omega_k(a,0),\beta) + \varepsilon(\omega_k(a,1),\beta) = 0$$

implies the desired statement. As β is fixed throughout, we write $\zeta(\alpha)$ for $\varepsilon(\alpha,\beta)$.

The case $d=2$ can easily be checked. Suppose the necessity true for $d-1$. Then setting the first component in α and a to 0 reduces the dimensionality of the problem by 1 and leads to

$$\forall\ \alpha \in \{0\}\times\{0,1\}^{d-1},\ \zeta(\alpha) = (-1)^{|\alpha|}\zeta(0).$$

Similarly, setting the first component in α and a to 1 again reduces the dimensionality of the problem by 1 and leads to

$$\forall\ \alpha \in \{1\}\times\{0,1\}^{d-1},\ \zeta(\alpha) = (-1)^{|\alpha|-1}\zeta(\omega_1(0,1)).$$

Now taking $k=1$ and $a=0$ leads to $\zeta(\omega_1(0,1)) = -\zeta(0)$ and concludes the proof. □

Corollary 2.8. *Suppose $\varepsilon(\alpha,\beta) = (-1)^{|\alpha|}\psi(\beta)$. If $\sum_\beta \psi(\beta) = 0$ then all conditions of Theorem 2.3 are satisfied; that is, for any γ such that*

$$\gamma(u) \le \min_{(-1)^{|\alpha|}\psi(\beta)>0} \frac{c(\tau_\alpha(\sigma_{\beta^{-1}}(u)))}{|\psi(\beta)|},$$

$\theta_{\varepsilon,\gamma}$ is a copula density for which the $(d-1)$-dimensional marginals coincide with those of c and, for any g symmetric, $g(V)\overset{d}{=}g(U)$, where U and V have densities c and $\theta_{\varepsilon,\gamma}$, respectively.

In particular, this is true for $\varepsilon(\alpha,\beta) = (-1)^{|\alpha|}sgn(\beta)$, where $sgn(\beta)$ is the signature of the permutation β.

We end this section with a strong construction of the maximal perturbation of the independence copula (i.e. $c=1$).

Proposition 2.9. *Let $U_1,\dots,U_d$ be independent uniform random variables on $[0,1]$. Suppose $\varepsilon(\alpha,\beta) = (-1)^{|\alpha|}sgn(\beta)$ and define*

$$V = \sum_{\substack{\alpha,\beta\\ \varepsilon(\alpha,\beta)=-1}} U1_{U\in\Delta(\alpha,\beta)} + \sum_{\substack{\alpha,\beta\\ \varepsilon(\alpha,\beta)=+1}} R_{\alpha,\beta}(U)1_{U\in\Delta(\alpha,\beta)},$$

where $R_{\alpha,\beta} = \tau_\alpha\circ\sigma_{\beta^{-1}}\circ\sigma_{(12)}\circ\sigma_\beta\circ\tau_\alpha$. Then V has density $\theta_{\varepsilon,1}$. In particular, for any $k=1,\dots,d$, $(V_1,\dots,V_{k-1},V_{k+1},\dots,V_d)\overset{d}{=}(U_1,\dots,U_{k-1},U_{k+1},\dots,U_d)$ and $V_1+\dots+V_d\overset{d}{=}U_1+\dots+U_d$.

Proof. Let A be a measurable subset of $\Delta(\alpha',\beta')$. Then

$$\begin{aligned}\mathbb{P}(V\in A) &= \sum_{\varepsilon(\alpha,\beta)=-1}\mathbb{P}(U\in A, U\in\Delta(\alpha,\beta)) + \sum_{\varepsilon(\alpha,\beta)=+1}\mathbb{P}(R_{\alpha,\beta}(U)\in A, U\in\Delta(\alpha,\beta))\\ &= \mathbb{P}(U\in A)1_{\varepsilon(\alpha',\beta')=-1} + \mathbb{P}(U\in R_{\alpha',(12)\circ\beta'}(A))1_{\varepsilon(\alpha',(12)\circ\beta')=+1} = 2\mathbb{P}(U\in A)1_{\varepsilon(\alpha',\beta')=-1},\end{aligned}$$

which proves that V has density $\theta_{\varepsilon,1}$. □

3 The case of identically distributed random variables

We are now ready to deal with the case of d identically distributed arbitrary random variables. We stress here that we do not assume that the random variables are independent.

Proposition 3.1. *Suppose that $X_1,\dots,X_d$ are identically distributed, that $X = (X_1,\dots,X_d)$ has copula density c, marginal distribution function Φ and marginal density ϕ, so that its density is*

$$f(x) = c\big(\Phi(x_1),\dots,\Phi(x_d)\big)\prod_{k=1}^d \phi(x_k).$$

For any (ε, γ) satisfying conditions (G) and (C_0) of Theorem 2.3,

$$\varphi(x) = \theta_{\varepsilon,\gamma}\big(\Phi(x_1), \dots, \Phi(x_d)\big) \prod_{k=1}^{d} \phi(x_k)$$

generates a random variable $Y = (Y_1, \dots, Y_d)$ that satisfies the requirements that, for any $k \in [d]$, $Y_k \overset{d}{=} X_k$, and, for any $g \in \mathcal{G}_d$, $g(Y) \overset{d}{=} g(X)$.

Proof. Let $U_k = \Phi(X_k)$, $g \in \mathcal{G}(\mathbb{R}^d)$ and

$$h(u) = g(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)).$$

Then U has density c and $h \in \mathcal{G}([0,1]^d)$. Letting V be a random variable with density $\theta_{\varepsilon,\gamma}$, $Y_k = \Phi^{-1}(V_k)$, we get that

$$g(Y) = g(\Phi^{-1}(V_1), \dots, \Phi^{-1}(V_d)) \;=\; h(V) \overset{d}{=} h(U) = g(\Phi^{-1}(U_1), \dots, \Phi^{-1}(U_d)) = g(X). \quad \square$$

Example 3.2. Let Φ be the distribution function and ϕ be the density of the standard normal distribution. Then for any $\gamma \leq 1$,

$$\varphi(x) = \theta_{\gamma}\big(\Phi(x_1), \dots, \Phi(x_d)\big) \prod_{k=1}^{d} \phi(x_k),$$

where

$$\theta_\gamma(u) = 1 - \sum_{\alpha,\beta} (-1)^{|\alpha|} \mathrm{sgn}(\beta) \gamma(\sigma_\beta(\tau_\alpha(u))) 1_{\Delta(\alpha,\beta)}(u),$$

is the density of a d-dimensional random variable Y for which all $(d-1)$-dimensional marginals are independent and identically distributed standard normal random variables, $Y_1 + \ldots + Y_d$ is normal with mean 0 and variance d, and Y is non-Gaussian.

4 The case of non-identically distributed random variables

Can the above construction extend to the case of non-identically distributed (and non-independent) random variables? To answer this question, we return to the two-dimensional case. Let s_1, s_2 and s_{12} be the reflections

$$s_1(u_1, u_2) = (1 - u_1, u_2),\; s_2(u_1, u_2) = (u_1, 1 - u_2) \text{ and } s_{12}(u_1, u_2) = (u_2, u_1).$$

These three involutions are such that $s_1 s_2 = s_2 s_1$, $s_1 s_{12} = s_{12} s_2$ and $s_2 s_{12} = s_{12} s_1$. It follows that they generate a finite group $\mathcal{R} = \{id, s_1, s_2, s_{12}, s_1 s_2, s_1 s_{12}, s_2 s_{12}, s_1 s_2 s_{12}\}$, the dihedral group of order 8.

Each element of $\mathcal{R}$ corresponds to one of the eight regions $\Delta(\alpha, \beta)$, and $\varepsilon(\alpha, \beta)$ of Section 2 is simply $(-1)^{|s|}$, where $|s|$ is the word length of s, that is the number of generators in the decomposition of s (modulo 2).

In the case of non-identically distributed random variables, say with distribution functions Φ_1 and Φ_2, for the construction to hold for symmetric functions, and in particular for the sum, the generator s_{12} needs to be changed to

$$s_{12}(u_1, u_2) = (\Phi_1(\Phi_2^{-1}(u_2)), \Phi_2(\Phi_1^{-1}(u_1)).$$

One would then attempt a construction of the type

$$\theta(u) = c(u) - \sum_{s\in\mathcal{R}} \varepsilon(s)\gamma(s(u))1_\Delta(s(u)),$$

for an appropriate reference region Δ, one for which $\{s(\Delta), s \in \mathcal{R}\}$ forms a measurable partition of $[0,1]^2$. The next example illustrates the difficulty we face in general.

Example 4.1. Suppose $\Phi_2(x) = \Phi_1(x)^2$ (X_2 has the distribution of the maximum of two independent copies of X_1) so that $s_{12}(u_1,u_2) = (\sqrt{u_2}, u_1^2)$. Then $s_1s_{12}s_2s_{12}(u_1,u_2) = \left(1-\sqrt{1-u_1^2}, u_2\right)$ and $(s_1s_{12}s_2s_{12})^n$, obtained by iterating the map $1-\sqrt{1-r^2}$, yields an infinite sequence.

In the above example the identity $s_1s_{12} = s_{12}s_2$ fails resulting in $\mathcal{R}$ being infinite. This identity translates in the language of the previous sections to $\sigma_\beta(\tau_\alpha(u)) = \tau_{\sigma_\beta(\alpha)}(\sigma_\beta(u))$ which was crucial in our construction.

In order to retain the identity $s_1s_{12} = s_{12}s_2$ we introduce the following notion.

Definition 4.2. *Two random variables are said to be similarly distributed if their distribution functions Φ_1 and Φ_2, assumed to be continuous and strictly increasing (on some interval), satisfy the identity*

$$\Phi_1^{-1}(1-\Phi_1(x)) = \Phi_2^{-1}(1-\Phi_2(x)).$$

Note that if a random variable X has a strictly increasing and continuous (on some interval) distribution function Φ, then $\Psi(x) = \Phi^{-1}(1-\Phi(x))$ is the only strictly decreasing and continuous measure-preserving map of X: $\Psi(X) \stackrel{d}{=} X$.

Proposition 4.3. *Two identically distributed random variables are necessarily similarly distributed and two symmetrical distributions around the same median are similarly distributed.*

For two similarly distributed random variables with distribution functions Φ_1 and Φ_2, we let $\Psi(x) = \Phi_1^{-1}(1-\Phi_1(x)) = \Phi_2^{-1}(1-\Phi_2(x))$ and define

$$\sigma_1(x_1,x_2) = (\Psi(x_1),x_2),\ \sigma_2(x_1,x_2) = (x_1,\Psi(x_2)) \text{ and } \sigma_{12}(x_1,x_2) = (x_2,x_1). \tag{4.1}$$

Proposition 4.4. *The three involutions σ_1, σ_2 and σ_{12} are such that $\sigma_1\sigma_2 = \sigma_2\sigma_1$, $\sigma_1\sigma_{12} = \sigma_{12}\sigma_2$ and $\sigma_2\sigma_{12} = \sigma_{12}\sigma_1$. As such, they generate a finite group $\mathcal{R} = \{id, \sigma_1, \sigma_2, \sigma_{12}, \sigma_1\sigma_2, \sigma_1\sigma_{12}, \sigma_2\sigma_{12}, \sigma_1\sigma_2\sigma_{12}\}$.*

While it is possible to approach this situation via copulas, other than in the identically distributed case, the resulting θ turns out to depend on Φ_1 and Φ_2 making it not universal and therefore less desirable. Instead, we apply the symmetry-balancing approach directly to the density.

Theorem 4.5. *Let f be the joint density of two similarly distributed random variables, X_1 and X_2, m be the common median, $\Delta = \{x \in \mathbb{R}^2 : x_1 < x_2 < m\}$ and γ be such that*

$$\varphi(x_1,x_2) = f(x_1,x_2) - \sum_{\sigma\in\mathcal{R}} (-1)^{|\sigma|}\gamma(\sigma(x_1,x_2))|J_\sigma(x_1,x_2)|1_\Delta(\sigma(x_1,x_2)) \tag{4.2}$$

is non-negative, where J_σ denotes the Jacobian determinant of σ and $|\sigma|$ is the word length of σ, that is the number of generators in the decomposition of σ (modulo 2).

Then φ generates (Y_1,Y_2) such that $Y_1 \stackrel{d}{=} X_1$, $Y_2 \stackrel{d}{=} X_2$ and, for any $g \in \mathcal{G}_2$, $g(Y_1,Y_2) \stackrel{d}{=} g(X_1,X_2)$; in particular $Y_1+Y_2 \stackrel{d}{=} X_1+X_2$.

Proof. Let $\psi(x) = \Psi'(x)$. We start by checking that $\int \varphi(x_1,x_2)dx_1 = \int f(x_1,x_2)dx_1$. Suppose $x_2 < m$. Then $\Psi(x_2) > m$ and the sum in (4.2) only contains four non-zero expressions, those for $\sigma \in \{id, \sigma_1, \sigma_{12}, \sigma_{12}\sigma_1\}$. Now,

$$\begin{aligned}
&\sum_{\sigma\in\{id,\sigma_1\}} (-1)^{|\sigma|} \int \gamma(\sigma(x_1,x_2))|J_\sigma(x_1,x_2)|1_\Delta(\sigma(x_1,x_2))dx_1 \\
= &\int \gamma(x_1,x_2)1_\Delta(x_1,x_2)dx_1 - \int \gamma(\sigma_1(x_1,x_2))|J_{\sigma_1}(x_1,x_2)|1_\Delta(\sigma_1(x_1,x_2))dx_1 \\
= &\int_{(-\infty,x_2)} \gamma(x_1,x_2)dx_1 - \int_{(\Psi(x_2),+\infty)} \gamma(\Psi(x_1),x_2)|\psi(x_1)|dx_1 \\
= &\int_{(-\infty,x_2)} \gamma(x_1,x_2)dx_1 - \int_{(-\infty,x_2)} \gamma(z_1,x_2)dz_1 \; = \; 0
\end{aligned}$$

and similarly for $\sum_{\sigma\in\{\sigma_{12},\sigma_{12}\sigma_1\}} (-1)^{|\sigma|} \int \gamma(\sigma(x_1,x_2))|J_\sigma(x_1,x_2)|1_\Delta(\sigma(x_1,x_2))dx_1 = 0$. Swapping x_1 and x_2 leads to the conclusion that φ and f have the same marginal distributions.

Furthermore, if g is symmetric, G is bounded and $\overline{g} = G \circ g$, then

$$\begin{aligned}
&\sum_{\sigma\in\{id,\sigma_{12}\}} (-1)^{|\sigma|} \int \overline{g}(x)\gamma(\sigma(x))|J_\sigma(x)|1_\Delta(\sigma(x))dx \\
= &\int_\Delta \overline{g}(x)\gamma(x)dx - \int_\Delta \overline{g}(\sigma_{12}(x))\gamma(x)dx \; = \; \int_\Delta \overline{g}(x)\gamma(x)dx - \int_\Delta \overline{g}(x)\gamma(x)dx \; = \; 0,
\end{aligned}$$

$$\begin{aligned}
&\sum_{\sigma\in\{\sigma_2,\sigma_2\sigma_{12}\}} (-1)^{|\sigma|} \int \overline{g}(x)\gamma(\sigma(x))|J_\sigma(x)|1_\Delta(\sigma(x))dx \\
= &-\int \overline{g}(x)\gamma(\sigma_2(x))|\psi(x_2)|1_\Delta(\sigma_2(x))dx + \int \overline{g}(x)\gamma(\sigma_2\sigma_{12}(x))|\psi(x_1)|1_\Delta(\sigma_2\sigma_{12}(x))dx \\
= &-\int_\Delta \overline{g}(\sigma_2(z))\gamma(z)dz + \int \overline{g}(\sigma_{12}\sigma_2(z))\gamma(z)dz \; = \; 0,
\end{aligned}$$

and so on for the sums on $\{\sigma_1, \sigma_1\sigma_{12}\}$ and $\{\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_{12}\}$, which concludes the proof. □

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