Diffusions on a space of interval partitions: construction from Bertoin's $\mathrm{BES}_0(d)$, $d \in (0,1)$

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Abstract

In 1990, Bertoin constructed a measure-valued Markov process in the framework of a Bessel process of dimension between 0 and 1. In the present paper, we represent this process in a space of interval partitions. We show that this is a member of a class of interval partition diffusions introduced recently and independently by Forman, Pal, Rizzolo and Winkel using a completely different construction from spectrally positive stable Lévy processes with index between 1 and 2 and with jumps marked by squared Bessel excursions of a corresponding dimension between -2 and 0.

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1 Introduction

We define interval partitions, following Aldous [3, Section 17] and Pitman [12, Chapter 4].

Definition 1.1. *An* interval partition *is a set β of disjoint, open subintervals of some finite real interval $[0, M]$, that cover $[0, M]$ up to a Lebesgue-null set. We write $\|\beta\|$ to denote M. We refer to the elements of an interval partition as its* blocks. *The Lebesgue measure of a block is called its* mass *or* size.

In this paper we construct diffusion processes in a space of interval partitions in Bertoin's [4, 5] framework of a Bessel process of dimension $d \in (0,1)$. Bertoin studied the excursions of such a Bessel process. Specifically, he first decomposed the Bessel process

$$\mathbf{R} = \mathbf{B} - (1-d)\mathbf{H} \tag{1.1}$$

into a Brownian motion $\mathbf{B}$ and a path-continuous process $\mathbf{H}$ with zero quadratic variation. He constructed excursions of the Markov process $(\mathbf{R}, \mathbf{H})$ away from $(0,0)$, each consisting of infinitely many excursions of $\mathbf{R}$ away from 0. By extracting suitable statistics, namely the set $\{\mathbf{R}(t): t \geq 0,\ \mathbf{H}(t) = y\}$, he showed [5, Theorems II.2–II.3] that the measure-valued process

$$y \mapsto \mu^y_{[0,T]} := \sum_{0 \leq t \leq T\,:\ \mathbf{H}(t)=y,\mathbf{R}(t)\neq 0} \delta_{\mathbf{R}(t)} \tag{1.2}$$

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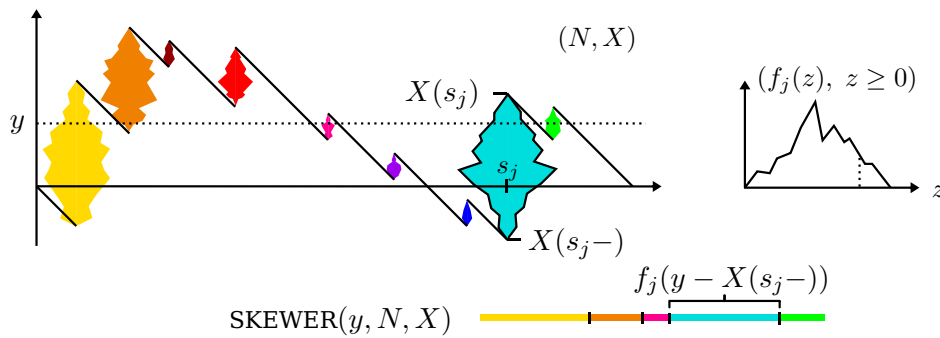Interval partitions from Bertoin's study of $\mathrm{BES}_0(d)$

Figure 1: Left: The sloping black lines show the scaffolding X. Shaded blobs decorating jumps show the corresponding spindles: points (s_j, f_j) of N. Right: Graph of one spindle. Bottom: A skewer, with blocks shaded to correspond to spindles; not to scale.

is path-continuous (in the vague topology for sigma-finite point measures on $(0, \infty)$) and Markovian when T is chosen suitably such as an inverse local time $\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)$, $u \geq 0$, of $(\mathbf{R}, \mathbf{H})$ at $(0,0)$. He further showed in [4, Theorem 4.2] and [5, Corollary II.4] that

$$y \mapsto \lambda^y(T) := 2 \int_{(0,\infty)} x \mu_{[0,T]}^y(dx) = 2 \sum_{0 \leq t \leq T \,:\, \mathbf{H}(t)=y} \mathbf{R}(t) \qquad (1.3)$$

is $\mathrm{BESQ}(0)$, a zero-dimensional squared Bessel process. We provide a more comprehensive review of Bertoin's results in Section 2. In this paper, we represent his measure-valued process (1.2) as a diffusion in a space of interval partitions.

Theorem 1.2. *In the setting of* (1.1)–(1.3), *with* $T = \tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)$, *the interval partitions*

$$\beta^y := \left\{ \left(\lambda^y(t-), \lambda^y(t) \right) \colon t \in [0,T], \ \mathbf{R}(t) \neq 0, \ \mathbf{H}(t) = y \right\}, \quad y \geq 0,$$

form a diffusion process in a suitable space interval partitions.

While the interval lengths $\lambda^y(t) - \lambda^y(t-)$ of β^y are (twice) the locations $\mathbf{R}(t)$ of atoms of $\mu_{[0,T]}^y$, the order of the intervals is not captured by $\mu_{[0,T]}^y$. Hence, this theorem is not an immediate consequence of Bertoin's corresponding results for $(\mu_{[0,T]}^y, y \geq 0)$.

Indeed, we prove this theorem by identifying this diffusion process as an instance of a class of diffusion processes introduced in [8], where we gave a general construction of processes in a space of interval partitions based on spectrally positive Lévy processes (*scaffolding*) whose point process of jump heights (interpreted as lifetimes of individuals) is marked by excursions (*spindles*, giving "sizes" varying during the lifetime, one for each level crossed). Informally, the interval partition evolution, indexed by level, considers for each level $y \geq 0$ the jumps crossing that level and records for each such jump an interval whose length is the "size" of the individual (width of the spindle) when crossing that level, ordered from left to right without leaving gaps. This construction and terminology is illustrated in Figure 1.

Specifically, if $N = \sum_{i \in I} \delta_{(s_i, f_i)}$ is a point process of times $s_i \in [0, S]$ and excursions f_i of excursion lengths ζ_i (spindle heights), and X is a real-valued process with jumps $\Delta X(s_i) := X(s_i) - X(s_i-) = \zeta_i$ at times s_i, $i \in I$, we define the interval partition $\mathrm{SKEWER}(y, N, X)$ at level y, as follows.

Definition 1.3. *For* $y \in \mathbb{R}$, $s \in [0, S]$, *the* aggregate mass *in* (N, X) *at level* y, *up to time* s *is*

$$M_{N,X}^y(s) := \sum_{i \in I \,:\, s_i \leq s} f_i(y - X(s_i-)). \qquad (1.4)$$

The skewer *of* (N, X) *at level* y*, denoted by* $\mathrm{SKEWER}(y, N, X)$*, is defined as*

$$\left\{ \left(M_{N,X}^y(s-), M_{N,X}^y(s) \right) \colon s \in [0, S], \, M_{N,X}^y(s-) < M_{N,X}^y(s) \right\} \qquad (1.5)$$

and the skewer process *as* $\overline{\mathrm{SKEWER}}(N, X) := \left(\mathrm{SKEWER}(y, N, X), y \geq 0 \right)$.

This definition is meaningful when X has finitely many jumps as in Figure 1, and also when X has a dense set of jump times and the f_i are such that $M_{N,X}$ is finite. In [8], we established criteria under which $\overline{\mathrm{SKEWER}}(\mathbf{N}, \mathbf{X})$ is a diffusion. Specifically, $\mathbf{N}$ is a Poisson random measure (PRM) with intensity measure $\mathrm{Leb} \otimes \nu$, where ν is the Pitman–Yor excursion law [14] associated with a suitable (self-similar) $[0, \infty)$-valued diffusion, and $\mathbf{X}$ is an associated Lévy process, suitably stopped at a time $\mathbf{S}$ when $\mathbf{X}$ is zero. In this interval partition evolution, each interval length (block) evolves independently according to the $[0, \infty)$-valued diffusion, which we call *block diffusion*, while between (the infinitely many) blocks, new blocks appear at the pre-jump levels of $\mathbf{X}$. The PRM of jumps is obtained by mapping the PRM of spindles onto the spindle heights. Conversely, we may view the PRM of spindles as marking the PRM of jumps by *block excursions.* See Section 3 for more details.

Theorem 1.4. *When the block diffusion is* $\mathrm{BESQ}(-2(1-d))$*, a squared Bessel process of dimension* $-2(1-d) \in (-2, 0)$*, and the scaffolding Lévy process is* $\mathrm{Stable}\,(2-d)$ *stopped at an inverse local time* $\tau_{\mathbf{X}}^0(v)$ *of* $\mathbf{X}$ *at 0, the interval partition evolution associated via* $\overline{\mathrm{SKEWER}}$ *is distributed as the diffusion in Theorem 1.2, for* $u = 2^{d-1}v$.

The remainder of this paper is organised, as follows. In Sections 2 and 3, we state the main results of [4, 5] and [8, 9], in order to exhibit the parallels. A few of these are used in Section 4 to make precise the connections between the two frameworks and to deduce the theorems we have stated. In Section 5, we discuss some further observations. We explore the one-to-one correspondence between frameworks to reprove results and to obtain new results. Furthermore, we discuss the prospects and the challenges to use representations in terms of Brownian motion such as the developments in this paper, or indeed in [13], to simplify the approach of [8, 9], which we have used to solve two long-standing conjectures by constructing a continuum-tree-valued process [1, 2, 7] and a two-parameter family of Fleming–Viot processes [11, 6, 10]. But more work is needed here, and we conclude by posing a problem.

2 Bertoin's study of Bessel processes [4, 5]

Consider a Bessel process $\mathbf{R} \sim \mathrm{BES}_x(d)$ of dimension $d \in (0, 1)$ starting from $x > 0$. Let $T_{\mathbf{R}}(0) = \inf\{t \geq 0 \colon \mathbf{R}(t) = 0\}$. On $[0, T_{\mathbf{R}}(0))$, the Bessel process $\mathbf{R}$ satisfies an SDE that yields

$$\mathbf{R}(t) = x + \mathbf{B}(t) - \frac{1-d}{2} \int_0^t \frac{du}{\mathbf{R}(u)}. \qquad (2.1)$$

Furthermore, this singular integral is finite as $t \uparrow T_{\mathbf{R}}(0)$. By time reversal, this means that this integral is also well-defined under the excursion measure of the Bessel process. While the excursions can be stitched together to form a Bessel process that has 0 as a reflecting boundary, the positive values of these integrals are not summable so that the representation (2.1) fails beyond $T_{\mathbf{R}}(0)$. However, (2.1) can be extended beyond $T_{\mathbf{R}}(0)$ if some compensation is introduced, as follows. It is well-known that the Bessel process $\mathbf{R}$ has jointly continuous space-time local times on $(0, \infty)^2$. To obtain a family of local times that extends continuously to $[0, \infty)^2$, it is convenient to choose the level-a local time $(L^a(t), t \geq 0)$, $a > 0$, of $\mathbf{R}$ such that the occupation density of $\mathbf{R}$ is $(a^{d-1}L^a(t), a > 0, t \geq 0)$.

By the occupation density formula and since $L^0(t) = 0$ for $t < T_{\mathbf{R}}(0)$, we can write

$$\frac{1}{2}\int_0^t \frac{du}{\mathbf{R}(u)} = \frac{1}{2}\int_0^\infty a^{d-1}L^a(t)\frac{da}{a} = \frac{1}{2}\int_0^\infty a^{d-2}(L^a(t) - L^0(t))da =: \mathbf{H}(t) \qquad (2.2)$$

for $t < T_{\mathbf{R}}(0)$. Bertoin showed that defining $\mathbf{H}(t)$ by the right-most integral in (2.2) also for $t \geq T_{\mathbf{R}}(0)$ yields a path-continuous process $\mathbf{H}$ with unbounded variation, but zero quadratic variation (the finiteness of $\mathbf{H}$ follows from the Hölder continuity of $L^a(t)$ in a). Clearly, this process $\mathbf{H}$ is increasing on all excursion intervals of $\mathbf{R}$ away from zero, but the effect of the compensating local time at zero is that $\mathbf{H}$ does not increase across the zero-set of $\mathbf{R}$. With this notation, we have

$$\mathbf{R}(t) = x + \mathbf{B}(t) - (1-d)\mathbf{H}(t), \qquad \text{for all } t \geq 0. \qquad (2.3)$$

Bertoin noted that $(\mathbf{R}, \mathbf{H})$ is a Markov process and that $(0,0)$ is recurrent for this Markov process. It is instructive to consider the excursions of $(\mathbf{R}, \mathbf{H})$ away from $(0,0)$ by plotting $\mathbf{R}(t)$ against "time" $\mathbf{H}(t)$. Since $\mathbf{H}$ increases during each excursion of $\mathbf{R}$ away from 0, on (ℓ, r), say, such a plot shows a time-changed excursion of $\mathbf{R}$ starting from 0 at "time" $\mathbf{H}(\ell)$ and returning to 0 at "time" $\mathbf{H}(r) > \mathbf{H}(\ell)$. As $\mathbf{H}$ does not increase across the zero-set of $\mathbf{R}$, the excursions for different (ℓ, r) overlap, in general, when included in the same plot.

Since $\mathbf{H}$ is increasing when $\mathbf{R}$ is away from 0, and can only decrease across the zero-set of $\mathbf{R}$, the excursions of $(\mathbf{R}, \mathbf{H})$ away from $(0,0)$ typically consist of many excursions of $\mathbf{R}$. Specifically, each excursion of $(\mathbf{R}, \mathbf{H})$ can be decomposed into three parts: first, at the "beginning", there is an escape from $(0,0)$ towards the left by an accumulation of short $\mathbf{R}$-excursions until, in the "middle", one $\mathbf{R}$-excursion takes $(\mathbf{R}, \mathbf{H})$ across to positive $\mathbf{H}$-values and, at the "end", there is a final approach back to $(0,0)$ from the right by an accumulation of short $\mathbf{R}$-excursions.

The main objects of interest in Bertoin's work [4, 5] are

- the excursions away from $(0,0)$ of $(\mathbf{R}, \mathbf{H})$, and associated quantities,
- the excursions away from 0 of $\widetilde{\mathbf{R}} := 2\mathbf{R} \circ T_{\mathbf{H}}^+$, where $T_{\mathbf{H}}^+(y) = \inf\{t \geq 0 : \mathbf{H}(t) > y\}$,
- local time processes $(\lambda^y(t), y \geq 0)$ and $(\lambda^{-y}(t), y \geq 0)$ of $\mathbf{H}$, up to time $t \geq 0$,
- measure-valued processes $y \mapsto \mu^y_{[0,T]} = \sum_{0 \leq t \leq T:\ \mathbf{H}(t)=y, \mathbf{R}(t)\neq 0} \delta_{\mathbf{R}(t)}$ for some $T > 0$.

Specifically, some of the main results of [4] are the following. We use Bertoin's numbering for ease of reference.

2.4 The inverse local time $\tau^{(0,0)}_{\mathbf{R},\mathbf{H}}$ of $(\mathbf{R}, \mathbf{H})$ at $(0,0)$ is stable with index $(1-d)/2$.

3.1 The Itô excursion process of $(\mathbf{R}, \mathbf{H})$ is a PRM.

3.2 Under the excursion measure, excursions of $(\mathbf{R}, \mathbf{H})$ are time-reversible.

3.3   (i) A.s., all excursions of $(\mathbf{R}, \mathbf{H})$ away from $(0,0)$ start into $[0,\infty) \times (-\infty, 0)$, cross $(0,\infty) \times \{0\}$ at a unique time U and finish from $[0,\infty) \times (0,\infty)$.
  (ii) The PRM has points at excursions whose value of $\mathbf{R}$ when the excursion is crossing the line $\mathbf{H} = 0$ has (sigma-finite) law $((1-d)/\Gamma(d))x^{d-2}dx$, and
  (iii) points at excursions with an $\mathbf{H}$-infimum below $-y$ occur at rate y^{d-1}.

3.4 Mid-excursion Markov property: conditionally given an $\mathbf{R}$-value of $\mathbf{R}(U) = x$ at the crossing time U of the line $\mathbf{H} = 0$, the post-U part of the excursion and the time-reversed pre-U part are independent and distributed as the process $(\mathbf{R}, \mathbf{H})$ starting from $(x, 0)$ and stopped when hitting $(0, 0)$.

4.1 In the setting of result 3.4, conditionally given $\mathbf{R}(U) = x$, the above-0 and below-0 local time processes $(\lambda^y(t), y \geq 0)$ and $(\lambda^{-y}(t), y \geq 0)$ of $\mathbf{H}$ during an excursion of $(\mathbf{R}, \mathbf{H})$ are two independent $\mathrm{BESQ}_{2x}(0)$.

4.2 The level-0 local time λ^0 of $\mathbf{H}$ time-changed by the inverse local time $\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}$ is a stable subordinator of index $1-d$. Given $\lambda^0(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u))=x$, the processes $(\lambda^y(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)), y\geq 0)$ and $(\lambda^{-y}(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)), y\geq 0)$ are two independent $\text{BESQ}_x(0)$.

The main additional results of [5] are the following.

I.5 Under the Itô excursion measure of $\widetilde{\mathbf{R}}$, excursions

    (i) start positive with initial values at rate $(2^{1-d}(1-d)/\Gamma(d))x^{d-2}dx$, and

    (ii) when starting from x evolve as $\text{BESQ}_x(-2(1-d))$.

I.6 The semi-group of $\widetilde{\mathbf{R}}$ is characterised by its Laplace transforms, for $\gamma \geq 0$, $\mathbb{E}_x(\exp(-\gamma\widetilde{\mathbf{R}}(y)))=\exp(-x/2y)\left((1+2\gamma y)^{1-d}\exp(x/(2+4\gamma y))-(2\gamma y)^{1-d}\right)$.

II.1 The measure-valued process $y\mapsto \mu_{[0,T_{\mathbf{H}}(-1)]}^{y}$ for $T_{\mathbf{H}}(-1):=\inf\{t\geq 0\colon \mathbf{H}(t)=-1\}$ admits a continuous version in the space $\mathcal{N}((0,\infty))$ of point measures that are finite on (ε,∞) for all $\varepsilon>0$, equipped with the topology of vague convergence.

II.2   (i) The process $(\mu_{[0,T_{\mathbf{H}}(-1)]}^{y}, y\geq 0)$ is Markovian.

    (ii) Its semi-group $\kappa_y^{\mathcal{N}}$, $y\geq 0$, acts on functions $f_\varphi(\sum_{i\in I}n_i\delta_{x_i})=\prod_{i\in I}(\varphi(x_i))^{n_i}$ for continuous $\varphi\colon (0,\infty)\to[0,1]$ as $\kappa_y^{\mathcal{N}}f_\varphi=f_{\varphi_y}$, where $\varphi_y(x)$ is given by $e^{-x/y}+\int_0^\infty \varphi(a)p_y(x,da)/\left(1+y^{1-d}\int_0^\infty((1-d)/\Gamma(d))s^{d-2}(1-\varphi(s))e^{-s/y}ds\right)$, where $\int_0^\infty e^{-\gamma a}p_y(x,da)=(1+\gamma y)^{1-d}\left(e^{-\gamma x/(1+\gamma y)}-e^{-x/y}\right)$, for all $\gamma\geq 0$.

    (iii) The process $(\mu_{[0,T_{\mathbf{H}}(-1)]}^{-1+y}, 0\leq y\leq 1)$ is Markovian with semi-group $\widetilde{\kappa}_y^{\mathcal{N}}$, $y\geq 0$, given by $\widetilde{\kappa}_y^{\mathcal{N}}f_\varphi=f_{\varphi_y}/\left(1+y^{1-d}\int_0^\infty((1-d)/\Gamma(d))s^{d-2}(1-\varphi(s))e^{-s/y}ds\right)$.

II.3 Given $\mu_{[0,\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(1)]}^{0}$, the processes $(\mu_{[0,\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(1)]}^{y}, y\geq 0)$ and $(\mu_{[0,\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(1)]}^{-y}, y\geq 0)$ are conditionally independent and have the semi-group $(\kappa_y^{\mathcal{N}}, y\geq 0)$, of II.2(ii).

II.4   (i) The process $(\lambda^{-1+y}(T_{\mathbf{H}}(-1)), 0\leq y\leq 1)$ is a $\text{BESQ}_0(2-2d)$.

    (ii) Given $\lambda^0(T_{\mathbf{H}}(-1))=x$, the process $(\lambda^y(T_{\mathbf{H}}(-1)), y\geq 0)$ is a $\text{BESQ}_x(0)$.

    (iii) Given $\lambda^0(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(1))=x$, the process $(\lambda^y(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(1)), y\geq 0)$ is a $\text{BESQ}_x(0)$.

3 Skewer processes of marked Lévy processes [8, 9]

Let $\alpha\in(0,1)$ and $\mathbf{X}$ a spectrally positive $\text{Stable}\,(1+\alpha)$-process with Laplace exponent $\psi(c)=c^{1+\alpha}/2^\alpha\Gamma(1+\alpha)$. We call $\mathbf{X}$ *scaffolding* and proceed to decorate it. Specifically, consider the $\text{PRM}\ \sum_{i\in\mathbf{I}}\delta_{(\mathbf{s}_i,\Delta\mathbf{X}(\mathbf{s}_i))}$ of its jumps. For each jump $\Delta\mathbf{X}(\mathbf{s}_i)$, consider an independent $\text{BESQ}(-2\alpha)$ excursion (*spindle*) $\mathbf{f}_i$ of length $\zeta(\mathbf{f}_i)=\Delta\mathbf{X}(\mathbf{s}_i)$. These excursions were studied by Pitman and Yor [14], who also noted, in their Remark (5.8) on pp. 453f., that when conditioned on their length, they are $\text{BESQ}(4+2\alpha)$ bridges from 0 to 0. By standard marking of PRMs, $\mathbf{N}:=\sum_{i\in\mathbf{I}}\delta_{(\mathbf{s}_i,\mathbf{f}_i)}$ is itself a PRM on the space $[0,\infty)\times\mathcal{E}$, where $\mathcal{E}$ is the space of (continuous) excursion paths. This is illustrated in a simplified way in Figure 1. The intensity measure $\text{Leb}\otimes\nu$ of $\mathbf{N}$ is the Pitman–Yor excursion measure of [14], which can be described by entrance laws and a further evolution as unconditioned $\text{BESQ}(-2\alpha)$ processes.

Recall that the skewer of Definition 1.3 extracts from $N=\sum_{i\in I}\delta_{(s_i,f_i)}$ all level-y spindle masses $f_i(y-X(s_i))$, where $y\in(X(s_i-),X(s_i))$, $i\in I$, and builds the interval partition that has these as interval lengths in the order given by the s_i, $i\in I$. The set $\mathcal{I}_H$ of all interval partitions can be equipped with a distance d_H that applies the Hausdorff metric to the set of points not covered by the intervals.

Since $\mathbf{X}$ is spectrally positive, its (càdlàg) excursions away from 0 (or any other level y) start negative, jump across zero and end positive. Applied to $(\mathbf{N},\mathbf{X})$, the skewer at level y extracts one block from each excursion of $\mathbf{X}$ away from y. In [8], we denote

the PRM of excursions of $\mathbf{X}$ away from y by $\mathbf{G}^y$ and enhance the excursion theory of $\mathbf{X}$ to include $\mathbf{N}$: each excursion $e_{[\ell,r]} := (-y + \mathbf{X}|_{[\ell,r]}(\ell + s), s \in [0, r - \ell])$ of $\mathbf{X}$ has its jumps marked by spindles. We denote by $\mathbf{F}^y$ the associated random measure whose points are pairs of $e_{[\ell,r]}$ and the restriction $\mathbf{N}|_{[\ell,r] \times \mathcal{E}}$ shifted to $[0, r - \ell] \times \mathcal{E}$. In each excursion with spindle marks, the *central spindle* crossing 0 can be viewed as the "middle" of three parts, separating the spindles of the "beginning" where $\mathbf{X}$ is negative from the spindles of the "end" where $\mathbf{X}$ is positive.

We refer to excursions of $(\mathbf{X}, \mathbf{N})$ as *bi-clades*, to the negative part of such an excursion including the central spindle up to level 0 as an *anti-clade*, and to the remainder as a *clade*. To start an interval-partition-valued process from any interval partition β, we consider *clades starting from* $\mathrm{Leb}(V)$, $V \in \beta$, as follows. For each interval $V \in \beta$ independently, consider $\mathbf{f}_V \sim \mathrm{BESQ}_{\mathrm{Leb}(V)}(-2\alpha)$ and an independent $(\mathbf{X}, \mathbf{N})$ stopped at $S_{\mathbf{X}}(-\zeta(\mathbf{f}_V)) := \inf\{s \geq 0 \colon \mathbf{X}(s) = -\zeta(\mathbf{f}_V)\}$, for the length $\zeta(\mathbf{f}_V)$ of $\mathbf{f}_V$, then form the clade $(\mathbf{X}_V, \mathbf{N}_V) := (\zeta(\mathbf{f}_V) + \mathbf{X}|_{[0, S_{\mathbf{X}}(-\zeta(\mathbf{f}_V))]}, \delta_{(0, \mathbf{f}_V)} + \mathbf{N}|_{[0, S_{\mathbf{X}}(-\zeta(\mathbf{f}_V))] \times \mathcal{E}})$. We stitch together all excursions $\mathbf{X}_V$ in the left-to-right order of $V \in \beta$ to form a scaffolding $\mathbf{X}_\beta$, similarly build $\mathbf{N}_\beta$ from $\mathbf{N}_V$, $V \in \beta$, and consider $\overline{\mathrm{SKEWER}}(\mathbf{N}_\beta, \mathbf{X}_\beta)$.

Some of the main objects of interest are

- the pair $(\mathbf{X}, \mathbf{N})$ of the $\mathtt{Stable}\,(1 + \alpha)$ scaffolding $\mathbf{X}$ and the PRM $\mathbf{N}$ of spindles,
- the random point measures $\mathbf{F}^y$, $y \geq 0$, of bi-clades of $(\mathbf{X}, \mathbf{N})$,
- the *type-1 evolution* $(\beta^y, y \geq 0) := \overline{\mathrm{SKEWER}}(\mathbf{N}_\beta, \mathbf{X}_\beta)$, extracting intervals from the spindles in jumps of $\mathbf{X}_\beta$ crossing level y, for any initial interval partition β.
- the total mass process $(\|\beta^y\|,\, y \geq 0)$.

Some of the main results of [8] are the following, in the numbering of [8].

1.3 The interval-partition-valued process $y \mapsto \beta^y$ admits a continuous version.

1.4 Type-1 evolutions $y \mapsto \beta^y$ are path-continuous Hunt processes: they can be started from any interval partition in a Lusin state space $(\mathcal{I}_H, d_H)$, are continuous in the initial condition and satisfy the strong Markov property.

3.2 The level-y aggregate mass process $s \mapsto M_{\mathbf{N},\mathbf{X}}^y(s)$ of (1.4), time-changed by the inverse local time $\tau_{\mathbf{X}}^y$ of $\mathbf{X}$ at level y is a stable subordinator of index α.

4.9 $\mathbf{F}^y$ is a PRM, whose intensity measure we call *bi-clade excursion measure*.

4.11 Bi-clades are space/time-reversible in the sense that reversing scaffolding time and block diffusion time in spindles yields the same bi-clade excursion measure.

4.15 Mid-bi-clade Markov property: conditionally given a spindle mass $f_R(-\mathbf{X}_{R-}) = x$ of the spindle (R, f_R) in $\mathbf{N}$ at the time R when the scaffolding $\mathbf{X}$ crosses the line $\mathbf{X} = 0$, the clade part (post-R) and the time-reversed anti-clade part (pre-R) are independent and distributed as clades starting from x.

5.6 The skewer processes of $(\mathbf{N}, \mathbf{X})$ stopped at stopping times including $\tau_{\mathbf{X}}^0(u)$ and $S_{\mathbf{X}}(-u)$ for $u \geq 0$ are type-1 evolutions.

In [9], we further prove the following.

1.2 The type-1 semi-group $\kappa_y^{\mathcal{I}}$, $y \geq 0$, is as follows. Independently for each block $V \in \beta^0$ of size $b = \mathrm{Leb}(V)$, there is a contribution to level y with probability $1 - e^{-b/2y}$. Such a contribution consists of a left-most interval with Laplace transform $(1 + \gamma/r)^\alpha (e^{br^2/(r+\gamma)} - 1)/(e^{br} - 1)$, where $r = 1/2y$, concatenated with a scaled $\mathrm{PDIP}(\alpha, \alpha)$, the interval partition formed by the excursion intervals of a $\mathrm{BES}(2 - 2\alpha)$-bridge, scaled by an independent $\mathrm{Gamma}(\alpha, 1/2y)$-distributed random factor. The contributions are concatenated in the order of $V \in \beta^0$ to give the distribution under the time-y transition kernel starting from β^0.

1.3 The kernels $\widetilde{\kappa}_y^{\mathcal{I}}$ obtained by concatenating a $\mathtt{PDIP}(\alpha, \alpha)$ scaled by an independent $\mathtt{Gamma}(\alpha, 1/2y)$ with the interval partition from $\kappa_y^{\mathcal{I}}$, $y \geq 0$, also form the semi-group of a path-continuous Hunt process, called *type-0 evolution*.

1.4 (i) The total mass process $(\|\beta^y\|, y \geq 0)$ of a type-1 evolution is $\mathtt{BESQ}_{\|\beta^0\|}(0)$, for all initial $\beta^0 \in \mathcal{I}_H$, hence including the case of a single clade.

(ii) The total mass processes of type-0 evolutions are $\mathtt{BESQ}_{\|\beta^0\|}(2\alpha)$.

3.2 (i) The $\mathtt{PRM}$ $\mathbf{F}^y$ has points at bi-clades whose value of the central spindle mass when crossing $\mathbf{X} = 0$ has (sigma-finite) law $(\alpha/\Gamma(1-\alpha))x^{-\alpha-1}dx$, and

(ii) points at bi-clades with $\mathbf{X}$-supremum above y at rate $2^{-\alpha}y^{-\alpha}$.

3.10 $(\mathrm{SKEWER}(y, \mathbf{N}|_{[0, S_{\mathbf{X}}(-u)] \times \mathcal{E}}, u + \mathbf{X}|_{[0, S_{\mathbf{X}}(-u)]}), 0 \leq y \leq u)$ is a type-0 evolution.

4 Construction of $(\mathbf{X}, \mathbf{N})$ from $(\mathbf{R}, \mathbf{H})$ and vice versa

Let $\tau_{\mathbf{R}}^0(s) = \inf\{L^0(t) > s\}$, $s \geq 0$, be the inverse local time of $\mathbf{R}$ at 0, and $\mathbf{K}$ the $\mathtt{PRM}$ of excursions of $\mathbf{R}$ away from 0. For each excursion interval $(\tau_{\mathbf{R}}^0(s-), \tau_{\mathbf{R}}^0(s)) = (\ell, r)$ of $\mathbf{R}$, we decompose the Bessel excursion $(\mathbf{R}(\ell+t), 0 \leq t \leq r-\ell)$ in $\mathbf{K}$ as in $\mathbf{R} = \mathbf{B} - (1-d)\mathbf{H}$ in (2.3), and we define the associated occupation density local time process $\lambda_s := (\lambda_s^y, y \geq 0)$ of $(\mathbf{H}(\ell + t) - \mathbf{H}(\ell), 0 \leq t \leq r - \ell)$.

Proposition 4.1. *The random measure $\sum_{s \geq 0 : \tau_{\mathbf{R}}^0(s-) < \tau_{\mathbf{R}}^0(s)} \delta_{\lambda_s}$ is a $\mathtt{PRM}(\mathrm{Leb} \otimes \overline{\nu})$, where $\overline{\nu}$ is a Pitman–Yor excursion measure associated with $\mathtt{BESQ}(-2(1-d))$, the process $\mathbf{H} \circ \tau_{\mathbf{R}}^0$ is a spectrally positive stable process of index $2-d$. The pair has the same distribution as $(\mathbf{N}, \mathbf{X})$ in Section 3, with $\alpha = 1 - d$, up to a linear time-change.*

Proof. Since $t \mapsto \mathbf{H}(t)$ is differentiable almost everywhere, with derivative $\mathbf{H}'(t) = 1/2\mathbf{R}(t)$, its local time at level $\mathbf{H}(t)$ increases by a jump of $2\mathbf{R}(t)$ at time t. Specifically, during each excursion interval $(\tau_{\mathbf{R}}^0(s-), \tau_{\mathbf{R}}^0(s)) = (\ell, r)$ of $\mathbf{R}$ away from 0, we get

$$\lambda_s^{\mathbf{H}(\ell+t)-\mathbf{H}(\ell)} = 2\mathbf{R}(\ell+t), \quad 0 \leq t \leq r - \ell, \tag{4.1}$$

i.e. the local times of $\mathbf{H}$ during excursions are continuous time-changes of the excursions of $2\mathbf{R}$. Hence, the jump sizes of $\mathbf{H} \circ \tau_{\mathbf{R}}^0$ are

$$\mathbf{H}(\tau_{\mathbf{R}}^0(s)) - \mathbf{H}(\tau_{\mathbf{R}}^0(s-)) = \inf\{y > 0 : \lambda_s^y = 0\} = \sup\{y \geq 0 : \lambda_s^y > 0\}, \ s \geq 0.$$

Bertoin [4, Proof of Lemma 3.2] showed that $\mathbf{H} \circ \tau_{\mathbf{R}}^0$ is a spectrally positive stable process of index $2-d$. Furthermore, by standard mapping of PRMs, $\sum_{s \geq 0 : \tau_{\mathbf{R}}^0(s-) < \tau_{\mathbf{R}}^0(s)} \delta_{\lambda_s}$ is a PRM. We will identify its intensity measure as a $\mathtt{BESQ}(-2(1 - d))$ excursion measure by [14, (3.1) First description].

Specifically, this description requires us to check three points. (i) Neither excursion measure charges the zero excursion. (ii) Whether the hitting time T_x of level x by the excursion of $\mathbf{R}$ is finite or not is not affected by the time-change, and the associated rate under either excursion measure is proportional to $1/s(x)$ where $s(x) = x^{2-d}$ is the common scale function of $\mathtt{BES}(d)$ and $\mathtt{BESQ}(-2(1 - d))$, see e.g. [14, (3.5) Examples]. (iii) We will show that the pre-T_x and post-T_x processes are, as required. The pre-T_x part of the excursion of $\mathbf{R}$ is a $\mathtt{BES}_0(d)$ conditioned to stay positive, i.e. a $\mathtt{BES}_0(4 - d)$ (see again [14, (3.5) Examples]). The time-change relation (4.1) transforms this into a $\mathtt{BESQ}_0(4 + 2(1-d))$, by [16, Proposition XI.(1.11)], which is a $\mathtt{BESQ}_0(-2(1-d))$ conditioned to stay positive, as required. Bertoin [5, bottom of p. 117] noted the corresponding time-change relation for $\mathtt{BES}(d)$ and $\mathtt{BESQ}(-2(1 - d))$ starting from y stopped when hitting 0. This identifies the post-T_x parts of the excursions and completes the proof. $\square$

Proof of Theorems 1.2 and 1.4. This will follow from Proposition 4.1 because applying $\mathrm{SKEWER}(y, \cdot)$ to the scaffolding-and-spindles pair of the proposition yields an interval partition with blocks

$$\lambda_s^{y-\mathbf{H}(\tau_\mathbf{R}^0(s-))} = 2\mathbf{R}(t) = \lambda^y(t) - \lambda^y(t-) \quad \text{if } y = \mathbf{H}(t) \text{ and } t \in (\tau_\mathbf{R}^0(s-), \tau_\mathbf{R}^0(s)), \qquad (4.2)$$

by (4.1) and (1.3). Since $\overline{\mathrm{SKEWER}}(y, \mathbf{N}, \mathbf{X})$ is unaffected by (linear) changes of scaffolding time of $\mathbf{X}$ and $\mathbf{N}$, the process of Theorem 1.2 can be constructed as claimed in Theorem 1.4, when stopped at a time that corresponds to an inverse local time $\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)$ of $(\mathbf{R}, \mathbf{H})$ at $(0,0)$ and that after time change by $\tau_\mathbf{R}^0$, is an inverse local time of the $\mathtt{Stable}(2-d)$ process $\mathbf{H} \circ \tau_\mathbf{R}^0$.

Let us work out the constant c for which stopping $(\mathbf{X}, \mathbf{N})$ at $\tau_\mathbf{X}^0(cu)$ yields the same initial distribution for the skewer process as the stopped scaffolding-and-spindles pair constructed from $(\mathbf{R}, \mathbf{H})$ stopped at $\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)$. We do this using the parts of [4, Lemma 3.3] and [9, Proposition 3.2] that we recalled in Sections 2 and 3 here. Specifically, the statistics of excursions of $(\mathbf{R}, \mathbf{H})$ of $\mathbf{H}$-infima directly transfer to $\mathbf{H} \circ \tau_\mathbf{R}^0$-infima that correspond to $\mathbf{X}$-infima, which, by bi-clade reversibility (see 4.11 above) or the mid-bi-clade Markov property (see 4.15 above) have the same rates as $\mathbf{X}$-suprema in a bi-clade. But the rates of $\mathbf{H}$-infima and $\mathbf{X}$-suprema differ by the constant $c = 2^\alpha = 2^{1-d}$, hence $\mathbf{X}$ needs to run longer than $\mathbf{H} \circ \tau_\mathbf{R}^0$, by a factor of c, to achieve the same number of excursions exceeding any given level y.

Finally, we note that the skewer process associated with $(\mathbf{N}|_{[0,\tau_\mathbf{X}^0(v)] \times \mathcal{E}}, \mathbf{X}|_{[0,\tau_\mathbf{X}^0(v)]})$ is a diffusion by [8, Theorem 1.4], again as recalled in Section 3 here. $\qquad\square$

A similar argument to work out c can be based on the values of $\mathbf{R}$ when crossing $\mathbf{H} = 0$ and the mass of the central spindle of $\mathbf{N}$ when crossing $\mathbf{X} = 0$. Note, however, that these also differ by a factor of 2, by (4.1).

Corollary 4.2. *In Bertoin's setting, under the Itô excursion measure of $\mathbf{R}$, the local time process of $\mathbf{H}$ has as its law a Pitman–Yor excursion measure of $\mathtt{BESQ}(-2(1-d))$.*

Proposition 4.1 makes precise the sense in which the framework of a single Bessel process $\mathbf{R} \sim \mathtt{BES}_0(d)$ of [4, 5], via $(\mathbf{R}, \mathbf{H})$, yields the scaffolding-and-spindles framework $(\mathbf{N}, \mathbf{X})$ of [8, 9]. The main step in the proof is time-changing the excursions of $\mathbf{R}$ away from 0 to form $\mathtt{BESQ}(-2(1-d))$ spindles. Let us invert this time-change and construct $\mathbf{R}$ from the spindles of $\mathbf{N}$. To this end, recall our notation ν for the Pitman–Yor excursion measure of $\mathtt{BESQ}(-2\alpha)$ of Section 3.

Proposition 4.3. *For $\mathbf{N} = \sum_{i \in \mathbf{I}} \delta_{(\mathbf{s}_i, \mathbf{f}_i)} \sim \mathrm{PRM}(\mathrm{Leb} \otimes \nu)$, set $\zeta_i := \int_0^{\zeta(\mathbf{f}_i)} \mathbf{f}_i(y) dy$ and*

$$\mathbf{e}_i(t) := \frac{1}{2}\mathbf{f}_i\left(\inf\left\{z \geq 0 \colon \int_0^z \mathbf{f}_i(y)dy > t\right\}\right), \quad t \in [0, \zeta_i), \quad \text{and} \quad \mathbf{e}_i(\zeta_i) = 0.$$

Then $\sum_{i \in \mathbf{I}} \delta_{(\mathbf{s}_i, \mathbf{e}_i)}$ has the same distribution as the Itô excursion process $\mathbf{K}$ of $\mathbf{R} \sim \mathtt{BES}_0(d)$, up to a linear time change, with $d = 1 - \alpha$. In particular, the $\mathbf{e}_i$ can be stitched together in the order of the $\mathbf{s}_i$, $i \in \mathbf{I}$, to yield a process $\overline{\mathbf{R}} \sim \mathtt{BES}_0(1-\alpha)$.

Proof. This follows from Proposition 4.1. Specifically, mapping $\mathbf{f}_i$ to $\mathbf{e}_i$ is elementary since all $\mathbf{f}_i$ are continuous with compact support a.s.. In present notation, we can write (4.1) as

$$\mathbf{f}_i\left(\int_0^t \frac{du}{\mathbf{e}_i(u)}\right) = 2\mathbf{e}_i(t), \quad t \in [0, \zeta_i].$$

This is a.s. well-defined for all $\mathbf{e}_i$, $i \in \mathbf{I}$, so the time-changes relating $\mathbf{f}_i$ and $\mathbf{e}_i$ are bijective, and hence the associated PRMs are bijectively related by standard mapping of PRMs. In particular, we deduce the claimed distributional identities up to a linear time change. The

construction of Markov processes from excursions has been well-studied [17]. Note that a linear time change of the PRM has no effect on the $\mathrm{BES}_0(1-\alpha)$-excursions themselves. Specifically, we define $\overline{\tau}(s) = \sum_{i \in \mathbf{I}: \, \mathbf{s}_i \leq s} \zeta_i$, $s \geq 0$, and $\overline{\mathbf{R}}(\overline{\tau}(\mathbf{s}_i-) + t) = \mathbf{e}_i(t)$, $0 \leq t \leq \zeta_i$, also setting $\overline{\mathbf{R}}(t) = 0$ for $t \notin \bigcup_{i \in \mathbf{I}} [\overline{\tau}(\mathbf{s}_i-), \overline{\tau}(\mathbf{s}_i)]$ and obtain $\overline{\mathbf{R}} \sim \mathrm{BES}_0(1-\alpha)$, and this is the same process as if we replace $\mathbf{s}_i$ by $a\mathbf{s}_i$, $i \in \mathbf{I}$, throughout, $a > 0$. The process $\overline{\tau}$ is an inverse local time of $\overline{\mathbf{R}}$ at 0, and replacing $\mathbf{s}_i$ by $a\mathbf{s}_i$ corresponds to a different choice of local time. $\qquad\square$

Corollary 4.4. *For* $(\mathbf{X}, \mathbf{N})$ *as in Section 3 and notation* $\overline{\mathbf{R}}$ *as in Proposition 4.3, with* $\overline{\tau}(s) = \sum_{i \in \mathbf{I}: \, \mathbf{s}_i \leq s} \zeta_i$, $s \geq 0$, *define* $\overline{\mathbf{H}}$ *on the range of* $\overline{\tau}$ *as* $\overline{\mathbf{H}}(\overline{\tau}(s)) := \mathbf{X}(s)$, $s \geq 0$, *and outside the range of* $\overline{\tau}$ *as*

$$\overline{\mathbf{H}}\left(\overline{\tau}(\mathbf{s}_i-) + \frac{1}{2}\int_0^z \mathbf{f}_i(y)dy\right) := \mathbf{X}(\mathbf{s}_i-) + z, \quad 0 \leq z < \Delta\mathbf{X}(\mathbf{s}_i) = \zeta(\mathbf{f}_i).$$

Then the pair $(\overline{\mathbf{R}}, \overline{\mathbf{H}})$ *has the same distribution as* $(\mathbf{R}, \mathbf{H})$ *of Section 2.*

Proof. Since $\mathbf{H}$ is determined by $\mathbf{R}$ via (2.2) and $\overline{\mathbf{R}} \stackrel{d}{=} \mathbf{R}$, it suffices to show that $\overline{\mathbf{H}}$ relates to $\overline{\mathbf{R}}$ in the same way. Indeed, we have $\overline{\mathbf{H}} \circ \overline{\tau} = \mathbf{X}$, by construction, and $\mathbf{X}(s)$ is the compensated limit of its jumps $\Delta\mathbf{X}(\mathbf{s}_i) = \zeta(\mathbf{f}_i)$ for $i \in \mathbf{I}$ with $\mathbf{s}_i \leq s$. But

$$\zeta(\mathbf{f}_i) = \int_0^{\zeta(\mathbf{e}_i)} \frac{du}{\mathbf{e}_i(u)} = \int_0^\infty a^{d-2} L_i^a(\infty) da,$$

where $(a^{d-1} L_i^a(\infty), a \geq 0)$ is the continuous version of the total occupation density local time of $\mathbf{e}_i$ at level a. This entails that the right-most equality of (2.2) holds for $t = \overline{\tau}(s)$, when $(\mathbf{R}, \mathbf{H})$ is replaced by $(\overline{\mathbf{R}}, \overline{\mathbf{H}})$. Since these limits exist almost surely uniformly for s in compact intervals, they also hold at $t = \overline{\tau}(\mathbf{s}_i-)$, $i \in \mathbf{I}$. Beyond the range of $\overline{\tau}$, we have, for each $i \in \mathbf{I}$,

$$\overline{\mathbf{H}}\left(\overline{\tau}(\mathbf{s}_i-) + \frac{1}{2}\int_0^z \mathbf{f}_i(y)dy\right) = \overline{\mathbf{H}}(\overline{\tau}(\mathbf{s}_i-)) + z, \quad 0 \leq z \leq \zeta(\mathbf{f}_i).$$

But according to the bijective time change relationships transforming $\mathbf{f}_i$ into $\mathbf{e}_i$ noted in the proof of Proposition 4.3, we have

$$z = \int_0^t \frac{du}{\mathbf{e}_i(u)} \quad \text{if and only if} \quad t = \frac{1}{2}\int_0^z \mathbf{f}_i(y)dy.$$

Hence, we obtain

$$\overline{\mathbf{H}}(\overline{\tau}(\mathbf{s}_i-) + t) = \overline{\mathbf{H}}(\overline{\tau}(\mathbf{s}_i-)) + \int_0^t \frac{du}{\mathbf{e}_i(u)}, \quad 0 \leq t \leq \zeta(\mathbf{e}_i),$$

and this completes the proof. $\qquad\square$

5 Further consequences of the connection between [4, 5] and [8, 9]

In the light of the results of Section 4, the results of [4, 5] and [8, 9] are closely related. Indeed, many results of [4, 5] can now be deduced from [8, 9], and the approach of [4, 5] could be refined to handle the additional order structure needed for the interval partitions of [8, 9]. Table 1 pairs the analogous results, which will mostly have been evident already from the formulations in Sections 2 and 3.

One may note, however, that these results differ in detail, not just because $(\beta^y, y \geq 0)$ and $(\mu_{[0,T]}^y, y \geq 0)$ have different state spaces. Specifically, [8, Theorem 1.4] and [9,

Table 1: Each column lists pairs of results (or groups of results) from [8] or [9], and from [4, 5] that are analogues of each other.

[8]‖[9]	1.3	1.4	4.9	4.11	4.15	5.5	1.2	1.2, 1.3, 3.10	1.4	3.2
[4, 5]	II.1	II.2	3.1	3.2	3.4	II.2	II.3	II.2	4.1, 4.2, II.4	3.3

Theorem 1.3] establish interval-partition-valued processes as path-continuous Hunt processes that are continuous in the initial condition, while [5, Theorem II.2] does not push beyond the simple Markov property. On the other hand, [4, Proposition 2.4] and [8, Proposition 3.2] find stable inverse local times of different indices, but fundamentally play the same role, since they provide the time parameterisations for the PRMs of excursions of $(\mathbf{R}, \mathbf{H})$ and of bi-clades, respectively.

The observation of Corollary 4.2, that $\mathbf{H}$-local time processes in $\mathbf{R}$-excursions are $\mathrm{BESQ}(-2(1-d))$, is related to [5, Theorem I.5 or (0.3)], which notes $\mathrm{BESQ}(-2(1-d))$ evolution of time-changed $\mathbf{R}$-excursions after they exceed previous $\mathbf{H}$-suprema. In the context of [8, 9], the corresponding result is a consequence of the construction from $\mathrm{BESQ}(-2\alpha)$ spindles (and the Markov property). But [5, Theorem I.5] goes further and yields the following result when translated into the framework of [8, 9].

Corollary 5.1. *For each $y \geq 0$, let $T_{\mathbf{X}}^+(y) = \inf\{s \geq 0 \colon \mathbf{X}(s) > y\}$ and denote by $\mathbf{L}(y) := \mathbf{f}_{T_{\mathbf{X}}^+(y)}(y - \mathbf{X}(T_{\mathbf{X}}^+(y)-))$ the value of the left-most spindle $\mathbf{f}_{T_{\mathbf{X}}^+(y)}$ that crosses level y. Then $(\mathbf{L}(y), y \geq 0)$ is a Markov process whose excursions away from 0 start with a jump of intensity $(2^\alpha \alpha / \Gamma(1-\alpha)) x^{-1-\alpha} dx$ and then evolve as $\mathrm{BESQ}_x(-2\alpha)$.*

Similarly, [5, Theorem I.6] then yields the semi-group of $\mathbf{L}$.

Less immediate are the consequences of some further results of [9], which we have not stated in Section 3, about what we call pseudo-stationarity of type-0 and type-1 evolutions, and the passage to normalised interval-partition evolutions on the subspace $\mathcal{I}_{H,1}$ of interval partitions of $[0,1]$ via suitable time-change. We observe in the context of Bertoin [5, Theorem II.2] that the marginal distributions of $\mu_{[0,T_{\mathbf{H}}(-1)]}^{-1+y}$, $y \in [0,1]$, a process starting from the zero measure $0 \in \mathcal{N}((0,\infty))$, can be read from

$$\widetilde{\kappa}_y^{\mathcal{N}} f_\varphi(0) = \frac{y^{d-1}}{y^{d-1} + \int_0^\infty (1-\varphi(s))\Pi_y(ds)} = \mathbb{E}\left(f_\varphi\left(\sum_{j \in \mathbf{J} \colon 0 \leq \mathbf{r}_j \leq E_y} \delta_{\Delta\sigma_y(\mathbf{r}_j)}\right)\right), \qquad (5.1)$$

where $\Pi_y(ds) = ((1-d)/\Gamma(d)) s^{d-2} e^{-s/y} ds$ is the Lévy measure of a subordinator $(\sigma_y(r), r \geq 0)$ with PRM $\sum_{j \in \mathbf{J}} \delta_{(\mathbf{r}_j, \Delta\sigma_y(\mathbf{r}_j))}$ of its jumps, and $E_y \sim \mathrm{Exp}(y^{d-1})$ is an independent random variable. Now [15, Proposition 21] showed, for $y = 1$, that the decreasing rearrangement $(\sigma_y(E_y))^{-1}(\Delta\sigma_y(t), 0 \leq t \leq E_y)^{\downarrow}$ of normalised jump sizes of $\sigma_y|_{[0,E_y]}$ has Poisson–Dirichlet distribution $\mathrm{PD}(1-d, 1-d)$, and is independent of $\sigma_y(E_y) \sim \mathrm{Gamma}(1-d, 1/y)$, and a simple change of variables extends this to all $y > 0$. As $\mathrm{PD}(1-d, 1-d)$ is preserved at all times $y > 0$ (while $\mathrm{Gamma}(1-d, 1/y)$ depends on $y > 0$), we call this behaviour *pseudo-stationarity*, cf. [9, Theorem 1.5].

Furthermore, it is well-known that adding an independent $\mathbf{R}(0) \sim \mathrm{Gamma}(d, 1/y)$ variable to the jumps of $\sigma_y|_{[0,E_y]}$, we obtain $\mathbf{R}(0) + \sigma_y(E_y) \sim \mathrm{Exp}(1/y)$ independent of $(\mathbf{R}(0) + \sigma_y(E_y))^{-1}(\mathbf{R}(0); \Delta\sigma_y(t), 0 \leq t \leq E_y)^{\downarrow} \sim \mathrm{PD}(1-d, 0)$, and there is pseudo-stationarity in the following sense.

Theorem 5.2. *Let $\mathbf{R}$ be a $\mathrm{BES}(d)$ starting from $\mathbf{R}(0) \sim \mathrm{Gamma}(d, \rho)$ and let $\mathbf{Q}$ be a $\mathrm{BESQ}(0)$ starting from $\mathbf{Q}(0) \sim \mathrm{Exp}(\rho/2)$ independent of $(\mathbf{x}_i, i \geq 1) \sim \mathrm{PD}(1-d, 0)$. Then $\mu_{[0,T_{\mathbf{H}}(-1/\rho)]}^y$ has the same distribution as $\sum_{i \geq 1} \delta_{\mathbf{x}_i \mathbf{Q}(y)/2}$, for each fixed $y > 0$.*

Proof. Let $\mathbf{f}_0 := \lambda_0 = (\lambda_0^y, y \geq 0)$ be the local time process of $(\mathbf{H}(t), 0 \leq t \leq T_{\mathbf{R}}(0))$. Then $\mathbf{f}_0(y) = \widetilde{\mathbf{R}}(y)$ for all $0 \leq y < \mathbf{H}(T_{\mathbf{R}}(0)) = \zeta(\mathbf{f}_0)$, and by [5, Theorem I.5], $\mathbf{f}_0$ is a $\mathrm{BESQ}(-2(1-d))$, starting from $2\mathbf{R}(0) \sim \mathrm{Gamma}(d, \rho/2)$. Proceeding as in Proposition 4.1, where $\mathbf{R}(0) = 0$, we obtain here, after the linear time-change noted in that proposition, a point measure $\mathbf{N}_0 := \delta_{(0,\mathbf{f}_0)} + \mathbf{N}$, in which $\mathbf{N} \sim \mathrm{PRM}(\mathrm{Leb} \otimes \nu)$ is independent of $\mathbf{f}_0 \sim \mathrm{BESQ}_{2\mathbf{R}(0)}(-2(1-d))$ and scaffolding $\mathbf{X}_0 := \zeta(\mathbf{f}_0) + \mathbf{X}$.

By the strong Markov property of $(\mathbf{R}, \mathbf{H})$ at $T_{\mathbf{H}}(0)$, and by (5.1) with $y = 1/\rho$, we find that $\mu^0_{[0,T_{\mathbf{H}}(-1/\rho)]}$ has the claimed initial distribution. Similarly, but now based on the strong Markov property of $(\mathbf{N}_0, \mathbf{X}_0)$ at $S_{\mathbf{X}_0}(0)$ and on [9, Theorem 1.3], $(\mathbf{N}_0, \mathbf{X}_0)$ correspondingly stopped at $S_{\mathbf{X}_0}(-1/\rho)$, has a skewer process $(\beta^y, y \geq 0)$ starting from a $\mathrm{PDIP}(\alpha, 0)$ scaled by the independent $\mathrm{Exp}(\rho/2)$. By [9, Theorem 1.5], $(\beta^y, y \geq 0)$ is pseudo-stationary with β^y distributed as a $\mathrm{PDIP}(\alpha, 0)$ scaled by an independent $\mathbf{Q}(y)$, where $\mathbf{Q}$ is a $\mathrm{BESQ}(0)$ starting from $\mathbf{Q}(0) \sim \mathrm{Exp}(\rho/2)$.

But as $(\mathbf{N}_0, \mathbf{X}_0)$ has been constructed from $(\mathbf{R}, \mathbf{H})$ as Proposition 4.1 did for the proof of Theorems 1.2 and 1.4, we read from (4.2) the coupling

$$\mu^y_{[0,T_{\mathbf{H}}(-1/\rho)]} = \phi(\beta^y), \qquad \text{where } \phi(\beta) = \sum_{V \in \beta} \delta_{\mathrm{Leb}(V)/2}, \tag{5.2}$$

so the distribution of $\mu^y_{[0,T_{\mathbf{H}}(-1/\rho)]}$ follows from the distribution of the ranked sequence of interval lengths of the pseudo-stationary β^y, which are $\mathrm{PD}(\alpha, 0)$ scaled by independent $\mathbf{Q}(y)$, as required. $\qquad\square$

In the light of this coupling (5.2), [9, Theorem 1.6] has the following corollary. Let $\mathcal{N}_1((0,\infty)) := \left\{ \sum_{i \in I} \delta_{x_i} \in \mathcal{N}((0,\infty)) \colon \sum_{i \in I} x_i = 1 \right\}$ and consider the map $\mu = \sum_{i \in I} \delta_{x_i} \mapsto \overline{\mu} := \sum_{i \in I} \delta_{x_i / \sum_{j \in I} x_j}$ from $\mathcal{N}((0,\infty)) \setminus \{0\}$ to $\mathcal{N}_1((0,\infty))$.

Corollary 5.3. *Let $\mathbf{R}$ be as in Theorem 5.2, set $T := T_{\mathbf{H}}(-1)$ and consider the time-change*

$$\varrho(u) := \inf \left\{ y \geq 0 \colon \int_0^y \frac{dz}{\lambda^z(T)} > u \right\}, \qquad u \geq 0.$$

Then the process $\left(\overline{\mu}^{\varrho(u)}_{[0,T]}, u \geq 0 \right)$, obtained from $\left(\mu^y_{[0,T]}, y \geq 0 \right)$ by first time-changing by ϱ and then mapping under $\mu \mapsto \overline{\mu}$, is a stationary Markov process whose invariant distribution is the distribution of $\sum_{i \geq 1} \delta_{\mathbf{x}_i}$ for $\left(\mathbf{x}_i, i \geq 1 \right) \sim \mathrm{PD}(\alpha, 0)$.

Proof. By [9, Theorem 1.6], the corresponding interval-partition-valued process is a stationary Markov process, whose invariant distribution is $\mathrm{PDIP}(\alpha, 0)$. Specifically, for $\beta \in \mathcal{I}_H \setminus \{\emptyset\}$, let $\overline{\beta} := \{(a/\|\beta\|, b/\|\beta\|) \colon (a,b) \in \beta\} \in \mathcal{I}_{H,1} := \{\gamma \in \mathcal{I}_H \colon \|\gamma\| = 1\}$. Recall that with the coupling (5.2), for $\rho = 1$, we have $\lambda^y(T) = \|\beta^y\|$ for all $y \geq 0$. Then

$$\overline{\mu}^{\varrho(u)}_{[0,T]} = \sum_{V \in \beta^{\varrho(u)}} \delta_{\mathrm{Leb}(V)/\|\beta^{\varrho(u)}\|} = \sum_{V \in \overline{\beta}^{\varrho(u)}} \delta_{\mathrm{Leb}(V)}.$$

This yields that each $\overline{\mu}^{\varrho(u)}_{[0,T]}$ has the claimed distribution. The Markov property will follow from Dynkin's criterion for when a function of a Markov process is a Markov process. Specifically, any two $\beta, \gamma \in \mathcal{I}_H$ with $\phi(\beta) = \phi(\gamma)$ have intervals of the same lengths, so there is a bijection $\eta \colon \gamma \to \beta$ such that $\mathrm{Leb}(\eta(V)) = \mathrm{Leb}(V)$ for all $V \in \gamma$. We construct coupled type-1 evolutions $(\beta^y, y \geq 0)$ starting from β and $(\gamma^y, y \geq 0)$ starting from γ by building $(\mathbf{X}_\beta, \mathbf{N}_\beta)$ from $(\mathbf{X}_V, \mathbf{N}_V)$, $V \in \beta$, as in Section 3, and then building $(\mathbf{X}_\gamma, \mathbf{N}_\gamma)$ by stitching together the same $(\mathbf{X}_{\eta(V)}, \mathbf{N}_{\eta(V)})$, $V \in \gamma$, in the order given by γ. Then $\phi(\beta^y) = \phi(\gamma^y)$ and $\phi(\overline{\beta}^{\varrho(u)}) = \phi(\overline{\gamma}^{\varrho(u)})$. In particular, the distributions of $\phi(\overline{\beta}^{\varrho(u)})$ and $\phi(\overline{\gamma}^{\varrho(u)})$ coincide, as required for Dynkin's criterion. $\qquad\square$

Finally, we rewrite $\mathbf{R}(t) = \mathbf{B}(t) - (1-d)\mathbf{H}(t)$ as a decomposition of Brownian motion $\mathbf{B}(t) = \mathbf{R}(t) + (1-d)\mathbf{H}(t)$. In Proposition 4.1, we time-changed $\mathbf{H}$ by the inverse local time $\tau_{\mathbf{R}}^0$. But then $(1-d)\mathbf{H}(\tau_{\mathbf{R}}^0(s)) = \mathbf{B}(\tau_{\mathbf{R}}^0(s))$ is a time-changed Brownian motion. Also, during each jump $\mathbf{H}(\tau_{\mathbf{R}}^0(s)) - \mathbf{H}(\tau_{\mathbf{R}}^0(s-))$, we can write

$$\mathbf{B}(\tau_{\mathbf{R}}^0(s-)+t) - \mathbf{B}(\tau_{\mathbf{R}}^0(s-)) = \mathbf{R}(\tau_{\mathbf{R}}^0(s-)+t) + (1-d)\big(\mathbf{H}(\tau_{\mathbf{R}}^0(s-)+t) - \mathbf{H}(\tau_{\mathbf{R}}^0(s-))\big),$$

which is the part of the Brownian motion from which the corresponding excursion of $\mathbf{R}$ away from 0 is built. From that excursion, we built the corresponding (increasing) stretch of $(1-d)\mathbf{H}$, whose local time is a $\mathrm{BESQ}(-2(1-d))$ excursion of length $\mathbf{H}(\tau_{\mathbf{R}}^0(s)) - \mathbf{H}(\tau_{\mathbf{R}}^0(s-))$, by Corollary 4.2. Note that this part of the Brownian motion is positive (relative to its starting level) and indeed it stays above the increasing stretch of $(1-d)\mathbf{H}$ since $\mathbf{R}(t) > 0$ during $(\tau_{\mathbf{R}}^0(s-), \tau_{\mathbf{R}}^0(s))$.

At $\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)$, when $\mathbf{R}$ and $\mathbf{H}$ both vanish, we also have $\mathbf{B}(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)) = 0$. Bertoin [4, Lemma 3.2] noted that $\widetilde{\mathbf{R}}(t) = \mathbf{R}(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u) - t)$ and $\widetilde{\mathbf{H}}(t) = -\mathbf{H}(\tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u) - t)$ give $\widetilde{\mathbf{R}} \overset{d}{=} \mathbf{R}$ and $\widetilde{\mathbf{H}} \overset{d}{=} \mathbf{H}$ on $[0, \tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)]$. This yields

$$-\widetilde{\mathbf{R}}(t) = \widetilde{\mathbf{B}}(t) - (1-d)\widetilde{\mathbf{H}}(t), \qquad t \in [0, \tau_{\mathbf{R},\mathbf{H}}^{(0,0)}(u)],$$

with a minus sign on the left-hand side, so the reversibility is rather subtle. It would be interesting to understand more fully the behaviour of $\mathbf{B}$ on intervals $[\tau_{\mathbf{R}}^0(s-), \tau_{\mathbf{R}}^0(s)]$. E.g., what are the local times of $\mathbf{B}$ on $[\tau_{\mathbf{R}}^0(s-), \tau_{\mathbf{R}}^0(s)]$, $s \geq 0$?

The wider context for this problem includes the natural question of whether the construction of interval partition evolutions in Theorem 1.2, or related evolutions, from a single Brownian motion, provides a simpler approach than [8, 9]. The one-to-one correspondence established here is a promising step, but results such as path properties in the framework of [4, 5] pose a challenge beyond the scope of this short paper, particularly when stitching together an initial condition from infinitely many intervals.

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