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On the completion of Skorokhod space

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Abstract

We consider the classical Skorokhod space $\mathbb{D}[0,1]$ and the space of continuous functions $\mathbb{C}[0,1]$ equipped with the standard Skorokhod distance ρ.

It is well known that neither $(\mathbb{D}[0,1],\rho)$ nor $(\mathbb{C}[0,1],\rho)$ is complete. We provide an explicit description of the corresponding completions. The elements of these completions can be regarded as usual functions on $[0,1]$ except for a countable number of instants where their values vary "instantly".

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1 Introduction

We consider the classical Skorokhod space $\mathbb{D} := \mathbb{D}[0,1]$ (the space of all *càdlàg* functions[1] on $[0,1]$) and the space of continuous functions $\mathbb{C} := \mathbb{C}[0,1]$, both equipped with the standard simple Skorokhod distance

$$\rho(f,g) := \inf_{\gamma} \left[\sup_{s\in[0,1]} |f(s) - g(\gamma(s))| + \sup_{s\in[0,1]} |s-\gamma(s)| \right]$$

where the infimum is taken over all continuous strictly increasing maps γ acting from $[0,1]$ onto itself.

Recall that $\mathbb{C}$ is a closed linear subspace in $(\mathbb{D},\rho)$. Indeed, if $g_n \in \mathbb{C}$, $f \in \mathbb{D}$, and $\rho(f,g_n) \to 0$, as $n \to \infty$, then there exist continuous functions γ_n such that

$$\lim_{n\to\infty} \sup_{s\in[0,1]} |f(s) - g_n(\gamma_n(s))| \le \lim_{n\to\infty} \left[\rho(f,g_n) + n^{-1}\right] = 0.$$

Therefore, f is a uniform limit of continuous functions $g_n \circ \gamma_n$, hence, $f \in \mathbb{C}$.

The space $\mathbb{D}$ equipped with the Skorokhod topology generated by ρ is a de facto standard framework in the theory of stochastic processes with jumps, such as Lévy processes, random walks, etc. (see Skorokhod [7, 8]).

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[1] A function is càdlàg if it is right-continuous and has left limit at every point.

It is well known that neither $(\mathbb{C}, \rho)$ nor $(\mathbb{D}, \rho)$ is complete (see Example 1.1 below). Usually this issue is solved by introducing the other, more delicate, distance generating the same topology (see e.g. Billingsley [1, Section 14]). The other option is to enlarge the spaces $\mathbb{C}$ and $\mathbb{D}$ completing them with respect to ρ. The goal of this paper is to describe the completions explicitly.

We mention a related study by Whitt [9, Section 15], aimed to enlarge the space $\mathbb{D}$ in order to let more sequences converge (in the Skorokhod topologies M_1 and M_2). Whitt's motivation was in queueing theory.

The completions of the spaces $(\mathbb{C}, \rho)$ and $(\mathbb{D}, \rho)$ appeared, respectively, in the important functional large deviations principles of Mogulskii for random walks [5] and Lévy processes [6]. Both papers used the completed spaces without caring about their nature. However, it is not quite clear how to work in such abstract setting, e.g. how to calculate the large deviations rate function on the entire completed space.

We finish this introduction with a simple but representative and instructive example.

Example 1.1. Consider a triangular function $g(x) := (1 - |x|)_+$ and introduce a family of functions $\{g_\theta\}_{\theta>2}$ in $\mathbb{C}$ given by

$$g_\theta(s) := g\left(\theta(s - \tfrac{1}{2})\right), \qquad 0 \le s \le 1.$$

By using a piecewise linear variable change it is straightforward to show that

$$\rho\left(g_{\theta_1}, g_{\theta_2}\right) \le \left|\frac{1}{\theta_1} - \frac{1}{\theta_2}\right| \to 0, \qquad \text{as } \theta_1, \theta_2 \to \infty.$$

Hence, $\{g_\theta\}_\theta$ has the Cauchy property, as $\theta \to \infty$. However, it is clear that there is no limit $\lim_{\theta\to\infty} g_\theta$ in $(\mathbb{D}, \rho)$ because

$$\lim_{\theta\to\infty} g_\theta(s) = \begin{cases} 1, & s = \frac{1}{2}, \\ 0, & s \ne \frac{1}{2}. \end{cases}$$

Notice that every function g_θ takes all values between zero and one going first up, then down. Informally, one may say that the limiting "function" should take all these values in the same order (up and down) at the single time instant $s_* = \frac{1}{2}$. We call this paradoxical behaviour an "*instanton*" and formalize it in the following sections.

2 Notation and construction

2.1 Turbofunctions

Let us denote $I := [0, 1]$ and $I^{(\mathrm{T})} := [0, 1]$, stressing the following subtle but important difference. We consider I as a usual time interval equipped with the standard distance while $I^{(\mathrm{T})}$ is considered as a *topological* space, without any distance on it.

Distinguishing between I and $I^{(\mathrm{T})}$ is an important part of our notation system. These objects coincide as sets but they are equipped with different structures and have different meaning. The set I denotes the usual time interval (clock) and one may measure the distance between the time instants, see e.g. the second term in the definition of Skorokhod distance ρ. On the contrary, $I^{(\mathrm{T})}$ is not considered as a true time interval, and one never measures the distance between its elements. We use $I^{(\mathrm{T})}$ just as a technical tool for parametrisation, as one does in the definition of a path in general topology. We will be rather consequent in distinguishing notation for the elements of two sets, using $t \in I^{(\mathrm{T})}$ and $s \in I$.

Use the standard notation $\mathbb{C}[I^{(\mathrm{T})}]$ and $\mathbb{D}[I^{(\mathrm{T})}]$ for the spaces of continuous and càdlàg functions on $I^{(\mathrm{T})}$, respectively.

We denote by Γ the class of all increasing homeomorphisms of I, i.e. strictly increasing continuous functions acting from I onto I. Similarly, $\Gamma^{(\mathrm{T})}$ denotes the class of all increasing homeomorphisms of $I^{(\mathrm{T})}$. Denote by Σ the class of all continuous *non-decreasing* mappings acting from $I^{(\mathrm{T})}$ onto I.

The *extended Skorokhod space* is defined as a Cartesian product $\mathbb{D}^+ := \mathbb{D}[I^{(\mathrm{T})}] \times \Sigma$. Its elements, the pairs

$$F^\sigma := (F, \sigma), \qquad F \in \mathbb{D}[I^{(\mathrm{T})}],\ \sigma \in \Sigma,$$

will be called *turbofunctions*. Similarly, put $\mathbb{C}^+ := \mathbb{C}[I^{(\mathrm{T})}] \times \Sigma$.

If $\sigma \in \Sigma$ is *strictly* monotone, hence invertible, then every turbofunction F^σ may be *visualized* as a usual function $\widehat{F^\sigma} \in \mathbb{D}[I]$ defined by

$$\widehat{F^\sigma}(s) := F(\sigma^{-1}(s)), \qquad s \in I. \tag{2.1}$$

If σ is not strictly monotone, such simple visualization is not possible. In this case, there exists a finite or countable number of $s \in I$ such that $\sigma^{-1}(s)$ is a non-degenerate interval in $I^{(\mathrm{T})}$. This corresponds to an *instanton* at time s, meaning that F^σ takes all values $\{F(t) : t \in \sigma^{-1}(s)\}$ “instantly” at time s.

In the opposite direction, every function $f \in \mathbb{D}[I]$ can be interpreted as a turbofunction

$$f^+ := (f \circ \varsigma)^\varsigma, \tag{2.2}$$

where $\varsigma \in \Sigma$ is defined by $\varsigma(t) := t$. Here and elsewhere the symbol $\circ$ stands for superposition of mappings.

In the following we will equip $\mathbb{D}^+$ with a relevant Skorokhod-type semi-distance and show that, after the natural factorization, $\mathbb{D}^+$ can be interpreted as the completion of $(\mathbb{D}, \rho)$.

For continuous functions the situation is similar. If $f \in \mathbb{C}$, then $f^+ \in \mathbb{C}^+$ and the natural factorization of $\mathbb{C}^+$ can be interpreted as the completion of $(\mathbb{C}, \rho)$.

2.2 Distance and factorizations

Let us define the Skorokhod semi-distance[2] ρ^+ on turbofunctions in $\mathbb{D}^+$ by

$$\rho^+\left(F_1^{\sigma_1}, F_2^{\sigma_2}\right) := \inf_{\gamma \in \Gamma^{(\mathrm{T})}} \left[\sup_{t \in I^{(\mathrm{T})}} |(F_1 \circ \gamma)(t) - F_2(t)| + \sup_{t \in I^{(\mathrm{T})}} |(\sigma_1 \circ \gamma)(t) - \sigma_2(t)| \right].$$

For the restriction of ρ^+ on $\mathbb{C}^+$, we can replace the suprema in the above definition by the maxima; however, this will play no role in the following.

It is easy to show that all properties of semi-distance are verified. Indeed, the symmetry follows from

$$\begin{aligned} & \sup_{t \in I^{(\mathrm{T})}} |(F_1 \circ \gamma)(t) - F_2(t)| + \sup_{t \in I^{(\mathrm{T})}} |(\sigma_1 \circ \gamma)(t) - \sigma_2(t)| \\ = \ & \sup_{\tau \in I^{(\mathrm{T})}} |F_1(\tau) - F_2(\gamma^{-1}(\tau))| + \sup_{\tau \in I^{(\mathrm{T})}} |\sigma_1(\tau) - \sigma_2(\gamma^{-1}(\tau))|, \end{aligned}$$

where we used the variable change $t = \gamma^{-1}(\tau)$. To prove the triangle inequality

$$\rho^+\left(F_1^{\sigma_1}, F_3^{\sigma_3}\right) \le \rho^+\left(F_1^{\sigma_1}, F_2^{\sigma_2}\right) + \rho^+\left(F_2^{\sigma_2}, F_3^{\sigma_3}\right), \tag{2.3}$$

note that for all $t \in I^{(\mathrm{T})}$ and all $\gamma_{12}, \gamma_{23} \in \Gamma^{(\mathrm{T})}$ we have

$$|(F_1 \circ \gamma_{12} \circ \gamma_{23})(t) - F_3(t)| \le |(F_1 \circ \gamma_{12} \circ \gamma_{23})(t) - (F_2 \circ \gamma_{23})(t)| + |(F_2 \circ \gamma_{23})(t) - F_3(t)|.$$

[2] Semi-distance $\rho(\cdot, \cdot)$ is a non-negative symmetric function satisfying triangle inequality and $\rho(x, x) = 0$ but allowing $\rho(x, y) = 0$ for $x \neq y$.

Taking supremum over $t \in I^{(\mathrm{T})}$ and using the variable change $\tau = \gamma_{23}(t)$ in the first supremum on the right-hand side, we get

$$\begin{aligned}&\sup_{t\in I^{(\mathrm{T})}} |(F_1 \circ \gamma_{12} \circ \gamma_{23})(t) - F_3(t)| \\ \leq\ & \sup_{\tau\in I^{(\mathrm{T})}} |(F_1 \circ \gamma_{12})(\tau) - F_2(\tau)| + \sup_{t\in I^{(\mathrm{T})}} |(F_2 \circ \gamma_{23})(t) - F_3(t)|.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sup_{t\in I^{(\mathrm{T})}} |(\sigma_1 \circ \gamma_{12} \circ \gamma_{23})(t) - \sigma_3(t)| \\ \leq\ & \sup_{\tau\in I^{(\mathrm{T})}} |(\sigma_1 \circ \gamma_{12})(\tau) - \sigma_2(\tau)| + \sup_{t\in I^{(\mathrm{T})}} |(\sigma_2 \circ \gamma_{23})(t) - \sigma_3(t)|.\end{aligned}$$

Adding the two inequalities and subsequently optimizing over γ_{12}, γ_{23} yields the triangle inequality (2.3).

Remark 1. The space $\mathbb{C}^+$ is a closed subset of $(\mathbb{D}^+, \rho^+)$. The proof is almost the same as the one, recalled above, for the classical case $\mathbb{C} \subset \mathbb{D}$. Indeed, if $F_n^{\sigma_n} \in \mathbb{C}^+$, $F^\sigma \in \mathbb{D}^+$, and $\rho^+(F_n^{\sigma_n}, F^\sigma) \to 0$, as $n \to \infty$, then there exists a sequence of functions $\gamma_n \in \Gamma^{(\mathrm{T})}$ such that

$$\lim_{n\to\infty} \sup_{t\in I^{(\mathrm{T})}} |(F_n \circ \gamma_n)(t) - F(t)| \leq \lim_{n\to\infty} \left[\rho^+(F^\sigma, F_n^{\sigma_n}) + n^{-1}\right] = 0.$$

Therefore, F is a uniform limit of continuous functions $F_n \circ \gamma_n$, hence, $F \in \mathbb{C}$ and $F^\sigma \in \mathbb{C}^+$.

The natural equivalence generated by the semi-distance ρ^+ is as follows:

$$F_1^{\sigma_1} \equiv F_2^{\sigma_2} \quad \text{iff} \quad \rho^+(F_1^{\sigma_1}, F_2^{\sigma_2}) = 0.$$

It is worthwhile to compare this equivalence with the other one, which is more natural and very close to $\equiv$ but still different from it. Namely, let $F_1^{\sigma_1} \overset{(\mathrm{t})}{\equiv} F_2^{\sigma_2}$ iff there exists a homeomorphism $\gamma \in \Gamma^{(\mathrm{T})}$ such that $F_2 = F_1 \circ \gamma$ and $\sigma_2 = \sigma_1 \circ \gamma$. Within a slightly different context, the equivalence $\overset{(\mathrm{t})}{\equiv}$ appeared in [9, Section 15.7].

Proposition 2. Let $F_1^{\sigma_1}, F_2^{\sigma_2} \in \mathbb{D}^+$. If $F_1^{\sigma_1} \overset{(\mathrm{t})}{\equiv} F_2^{\sigma_2}$, then $F_1^{\sigma_1} \equiv F_2^{\sigma_2}$. Conversely, if $F_1^{\sigma_1} \equiv F_2^{\sigma_2}$ and σ_1, σ_2 are strictly increasing, then $F_1^{\sigma_1} \overset{(\mathrm{t})}{\equiv} F_2^{\sigma_2}$.

Proof. The first claim follows from the definitions of the semi-distance ρ^+ and the equivalences. For the second claim, let us choose a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ in $\Gamma^{(\mathrm{T})}$ such that

$$\lim_{n\to\infty} \left[\sup_{t\in I^{(\mathrm{T})}} |(F_1 \circ \gamma_n)(t) - F_2(t)| + \sup_{t\in I^{(\mathrm{T})}} |(\sigma_1 \circ \gamma_n)(t) - \sigma_2(t)|\right] = 0.$$

Then for an arbitrary $t' \in I^{(\mathrm{T})}$ it is true that $(\sigma_1 \circ \gamma_n)(t') \to \sigma_2(t')$. Hence $(F_1 \circ \gamma_n)(t') \to (F_1 \circ \sigma_1^{-1} \circ \sigma_2)(t')$ provided that the function $F_1 \circ \sigma_1^{-1}$ is continuous at t'. The latter assumption may cease to hold only for finitely or countably many $t' \in I^{(\mathrm{T})} \setminus \{0, 1\}$ since F_1 is a càdlàg function and σ_1^{-1} is a continuous one. On the other hand, we know that $(F_1 \circ \gamma_n)(t') \to F_2(t')$. Thus, $F_2(t') = F_1 \circ \sigma_1^{-1} \circ \sigma_2(t')$, and it follows that $F_2 = F_1 \circ \sigma_1^{-1} \circ \sigma_2$ since the functions on both sides of this equality are càdlàg. Since $\sigma_1^{-1} \circ \sigma_2 \in \Gamma^{(\mathrm{T})}$, we have $F_1^{\sigma_1} \overset{(\mathrm{t})}{\equiv} F_2^{\sigma_2}$, as required. □

Corollary 3. If $F_1^{\sigma_1} \overset{(\mathrm{t})}{\equiv} F_2^{\sigma_2}$, then for every $F_3^{\sigma_3} \in \mathbb{D}^+$ we have

$$\rho^+\left(F_1^{\sigma_1}, F_3^{\sigma_3}\right) = \rho^+\left(F_2^{\sigma_1}, F_3^{\sigma_3}\right). \tag{2.4}$$

This claim immediately follows from Proposition 2 and triangle inequality.

Somewhat surprisingly, the full converse in Proposition 2 is not true, i.e. there exist $F_1^{\sigma_1}, F_2^{\sigma_2}$ such that $F_1^{\sigma_1} \equiv F_2^{\sigma_2}$ but $F_1^{\sigma_1} \overset{(\mathrm{t})}{\equiv} F_2^{\sigma_2}$ does not hold. Here is a counter-example.

Example 2.1. Let $F_{\mathbf{1}}(\cdot) := 1$ and recall that $\varsigma \in \Sigma$ is defined as $\varsigma(t) := t$. Then for every strictly increasing $\sigma \in \Sigma$ it is true that $(F_{\mathbf{1}})^\sigma \overset{(\mathrm{t})}{\equiv} (F_{\mathbf{1}})^\varsigma$. However, let us consider some σ that is not strictly increasing and approximate it uniformly with strictly increasing $\sigma_\delta \in \Sigma$, $0 < \delta < 1$, so that

$$\sup_{t \in I^{(\mathrm{T})}} |\sigma_\delta(t) - \sigma(t)| \leq \delta.$$

This can be done by letting

$$\sigma_\delta(t) := (1-\delta)\sigma(t) + \delta t, \qquad t \in I^{(\mathrm{T})}. \tag{2.5}$$

We have $\rho^+\left((F_{\mathbf{1}})^{\sigma_\delta}, (F_{\mathbf{1}})^\sigma\right) \leq \delta$, which can be seen if we let $\gamma \in \Gamma^{(\mathrm{T})}$ be the identical homeomorphism of $I^{(\mathrm{T})}$ in the definition of ρ^+. Furthermore,

$$\rho^+\left((F_{\mathbf{1}})^\varsigma, (F_{\mathbf{1}})^\sigma\right) \leq \rho^+\left((F_{\mathbf{1}})^\varsigma, (F_{\mathbf{1}})^{\sigma_\delta}\right) + \rho^+\left((F_{\mathbf{1}})^{\sigma_\delta}, (F_{\mathbf{1}})^\sigma\right) \leq 0 + \delta = \delta.$$

Hence $\rho^+\left((F_{\mathbf{1}})^\varsigma, (F_{\mathbf{1}})^\sigma\right) = 0$ by taking $\delta \searrow 0$. In other words, $(F_{\mathbf{1}})^\varsigma \equiv (F_{\mathbf{1}})^\sigma$.

On the other hand, $(F_{\mathbf{1}})^\varsigma \overset{(\mathrm{t})}{\equiv} (F_{\mathbf{1}})^\sigma$ does not hold. Indeed, assuming otherwise would imply that there is a representation $\sigma = \varsigma \circ \gamma$ with some $\gamma \in \Gamma^{(\mathrm{T})}$. The right-hand side of this equality is strictly increasing, while the left-hand side is not, which is a contradiction.

The next proposition shows that both equivalences $\equiv$ and $\overset{(\mathrm{t})}{\equiv}$ respect the visualization mapping.

Proposition 4. Let $F_1^{\sigma_1}, F_2^{\sigma_2} \in \mathbb{D}^+$. Assume that σ_1, σ_2 are strictly increasing and $F_1^{\sigma_1} \equiv F_2^{\sigma_2}$. Then $\widehat{F_1^{\sigma_1}} = \widehat{F_2^{\sigma_2}}$.

Proof. By Proposition 2 we have $F_1^{\sigma_1} \overset{(\mathrm{t})}{\equiv} F_2^{\sigma_2}$, i.e. there exists a $\gamma \in \Gamma^{(\mathrm{T})}$ such that $F_2 = F_1 \circ \gamma$ and $\sigma_2 = \sigma_1 \circ \gamma$. Then by visualization's definition we have

$$\widehat{F_2^{\sigma_2}} = F_2 \circ \sigma_2^{-1} = (F_1 \circ \gamma) \circ (\gamma^{-1} \circ \sigma_1^{-1}) = F_1 \circ \sigma_1^{-1} = \widehat{F_1^{\sigma_1}}. \qquad \square$$

3 Embedding of $\mathbb{D}$ into $\mathbb{D}^+$

Proposition 5. The mapping $f \mapsto f^+$ defined by equation (2.2) provides a dense isometric embedding of $(\mathbb{D}, \rho)$ into $(\mathbb{D}^+, \rho^+)$. In particular, it isometrically embeds $(\mathbb{C}, \rho)$ into $(\mathbb{C}^+, \rho^+)$.

Proof. Let $f_1, f_2 \in \mathbb{D}$. Then

$$\begin{aligned} \rho^+(f_1^+, f_2^+) &= \inf_{\gamma \in \Gamma^{(\mathrm{T})}} \left[\sup_{t \in I^{(\mathrm{T})}} |(f_1 \circ \varsigma \circ \gamma)(t) - (f_2 \circ \varsigma)(t)| + \sup_{t \in I^{(\mathrm{T})}} |(\varsigma \circ \gamma)(t) - \varsigma(t)| \right] \\ &= \inf_{\gamma' \in \Gamma} \left[\sup_{s \in I} |(f_1 \circ \gamma')(s) - f_2(s)| + \sup_{s \in I} |\gamma'(s) - s| \right] \\ &= \rho(f_1, f_2), \end{aligned}$$

with $\gamma' := \varsigma \circ \gamma \circ \varsigma^{-1}$. This proves the isometry property.

Now we prove that the image $\{f^+ : f \in \mathbb{D}\}$ is dense in $\mathbb{D}^+$. To see this, let us consider some arbitrary $F^\sigma \in \mathbb{D}^+$ with strictly increasing σ. Then we have

$$\begin{aligned} \left[\widehat{F^\sigma}\right]^+ &= \left[F \circ \sigma^{-1}\right]^+ = (F \circ \sigma^{-1} \circ \varsigma)^\varsigma \\ &= (F \circ \sigma^{-1} \circ \varsigma)^{\sigma \circ (\sigma^{-1} \circ \varsigma)} \overset{(\mathrm{t})}{\equiv} F^\sigma. \end{aligned} \tag{3.1}$$

We see that F^σ is $\overset{(\mathrm{t})}{\equiv}$-equivalent to an element of the image.

Finally, let F^σ be an arbitrary element of $\mathbb{D}^+$. Consider its approximations F^{σ_δ} with strictly increasing σ_δ introduced in (2.5). Then we have

$$\rho^+(F^\sigma, F^{\sigma_\delta}) \le \sup_{t \in I^{(\mathrm{T})}} |\sigma(t) - \sigma_\delta(t)| \le \delta \to 0, \qquad \text{as } \delta \to 0.$$

By Corollary 3 we have

$$\limsup_{\delta \to 0} \rho^+\left(F^\sigma, \left[\widehat{F^{\sigma_\delta}}\right]^+\right) = \limsup_{\delta \to 0} \rho^+(F^\sigma, F^{\sigma_\delta}) = 0,$$

which proves that $\{f^+ : f \in \mathbb{D}\}$ is dense in $(\mathbb{D}^+, \rho^+)$. $\square$

In Example 1.1, we constructed a family of functions $\{g_\theta\}_{\theta > 2}$ in $\mathbb{D}$ with the Cauchy property but not having a limit in $(\mathbb{D}, \rho)$, as $\theta \to \infty$. We show now that the isometrically embedded family $\{g_\theta^+\}_{\theta > 2}$ *has a limit* in $(\mathbb{D}^+, \rho^+)$. This is a good hint at completeness of $(\mathbb{D}^+, \rho^+)$.

Example 3.1. (Example 1.1 continued).

We use the notation from the mentioned example. Let

$$F(t) := g_4(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{4}, \\ 1 - 4|t - \frac{1}{2}|, & \text{if } \frac{1}{4} \le t \le \frac{3}{4}, \\ 0, & \text{if } \frac{3}{4} \le t \le 1. \end{cases}$$

Introduce the time changes $\sigma_\theta \in \Sigma$ as piecewise linear functions with nodes

$$\sigma_\theta(0) := 0; \quad \sigma_\theta(1/4) := \frac{1}{2} - \frac{1}{\theta}; \quad \sigma_\theta(3/4) := \frac{1}{2} + \frac{1}{\theta}; \quad \sigma_\theta(1) := 1.$$

Then we have

$$g_\theta(s) = F(\sigma_\theta^{-1})(s) = \widehat{F^{\sigma_\theta}}(s), \qquad s \in I,$$

and the sequence $g_\theta^+ = [\widehat{F^{\sigma_\theta}}]^+ \overset{(\mathrm{t})}{\equiv} F^{\sigma_\theta}$ (where we used (3.1) at the last step) has a limit F^σ in $\mathbb{C}^+$ (as $\theta \to \infty$) with $\sigma \in \Sigma$ being the piecewise linear function with nodes

$$\sigma(0) := 0; \quad \sigma(1/4) := \frac{1}{2}; \quad \sigma(3/4) := \frac{1}{2}; \quad \sigma(1) := 1,$$

because

$$\rho^+\left(g_\theta^+, F^\sigma\right) = \rho^+\left(F^{\sigma_\theta}, F^\sigma\right) \le \sup_{t \in I^{(\mathrm{T})}} |\sigma_\theta(t) - \sigma(t)| = \frac{1}{\theta}.$$

Since the limiting variable change σ is not strictly increasing, the limiting turbofunction F^σ has an instanton and can not be interpreted as a usual function.

4 Completeness of $(\mathbb{D}^+, \rho^+)$

In the following proposition we essentially reach our goal by showing that $(\mathbb{D}^+, \rho^+)$ is complete.

Proposition 6. Let $\{F_n^{\sigma_n}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbb{D}^+, \rho^+)$. Then there exists a limit $F^\sigma \in \mathbb{D}^+$ such that $\lim_{n\to\infty} \rho^+\big(F_n^{\sigma_n}, F^\sigma\big) = 0$. Moreover, if $F_n^{\sigma_n} \in \mathbb{C}^+$ for all $n \in \mathbb{N}$, then $F^\sigma \in \mathbb{C}^+$.

Proof. It suffices to show that $\{F_n^{\sigma_n}\}_{n\in\mathbb{N}}$ contains a converging subsequence. Choose a subsequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\rho^+(F_{n_k}^{\sigma_{n_k}}, F_{n_{k+1}}^{\sigma_{n_{k+1}}}) < 1/2^k$. By definition of the semi-metric ρ^+, for every $k \in \mathbb{N}$ there exists a $\gamma_k \in \Gamma^{(\mathrm{T})}$ such that

$$\sup_{t\in I^{(\mathrm{T})}} |(F_{n_{k+1}} \circ \gamma_k)(t) - F_{n_k}(t)| + \sup_{t\in I^{(\mathrm{T})}} |(\sigma_{n_{k+1}} \circ \gamma_k)(t) - \sigma_{n_k}(t)| < \frac{1}{2^k}.$$

Define $\lambda_k := \gamma_{k-1} \circ \ldots \circ \gamma_1$ for integer $k \geq 2$ and let λ_1 be the identical homeomorphism of $I^{(\mathrm{T})}$. Since $\lambda_k \in \Gamma^{(\mathrm{T})}$, the change of variables $t = \lambda_k(\tau)$ yields

$$\sup_{\tau\in I^{(\mathrm{T})}} |(F_{n_{k+1}} \circ \lambda_{k+1})(\tau) - (F_{n_k} \circ \lambda_k)(\tau)| \\ + \sup_{\tau\in I^{(\mathrm{T})}} |(\sigma_{n_{k+1}} \circ \lambda_{k+1})(\tau) - (\sigma_{n_k} \circ \lambda_k)(\tau)| < \frac{1}{2^k}.$$

Hence the sequences $\{F_{n_k} \circ \lambda_k\}_{k\in\mathbb{N}}$ and $\{\sigma_{n_k} \circ \lambda_k\}_{k\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{D}[I^{(\mathrm{T})}]$ (resp. $\mathbb{C}[I^{(\mathrm{T})}]$) equipped with the uniform distance. Since these metric spaces are complete (see [1, Section 18]), we can find $F \in \mathbb{D}[I^{(\mathrm{T})}]$ and $\sigma \in \mathbb{C}[I^{(\mathrm{T})}]$ such that

$$\lim_{k\to\infty} \left[\sup_{\tau\in I^{(\mathrm{T})}} |(F_{n_k} \circ \lambda_k)(\tau) - F(\tau)| + \sup_{\tau\in I^{(\mathrm{T})}} |(\sigma_{n_k} \circ \lambda_k)(\tau) - \sigma(\tau)| \right] = 0.$$

We also have $\sigma \in \Sigma$, since σ is non-decreasing as a pointwise limit of non-decreasing functions $\sigma_{n_k} \circ \lambda_k$. Finally, the above equality implies that $\rho^+(F_{n_k}^{\sigma_{n_k}}, F^\sigma) \to 0$, as $k \to \infty$, as required.

Under the additional assumption $\{F_n^{\sigma_n}\}_{n\in\mathbb{N}} \subset \mathbb{C}^+$, one may observe that $\{F_{n_k} \circ \lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}[I^{(\mathrm{T})}]$, thus $F \in \mathbb{C}[I^{(\mathrm{T})}]$ and $F^\sigma \in \mathbb{C}^+$. ☐

5 Concluding formalities

Consider the quotient space $\mathbb{D}^\pm_\equiv := \mathbb{D}^+/\equiv$. Let $\pi : \mathbb{D}^+ \mapsto \mathbb{D}^\pm_\equiv$ denote the natural projection. The subspace $\mathbb{C}^+$ is factorized correctly because from $F_1^{\sigma_1} \equiv F_2^{\sigma_2}$ and $F_1^{\sigma_1} \in \mathbb{C}^+$ it follows that $F_2^{\sigma_2} \in \mathbb{C}^+$. We denote $\mathbb{C}^\pm_\equiv := \mathbb{C}^+/\equiv$, so $\mathbb{C}^\pm_\equiv = \pi(\mathbb{C}^+)$.

The *semi-distance* ρ^+ on $\mathbb{D}^+$ generates the *distance* on $\mathbb{D}^\pm_\equiv$. We will also denote this distance by ρ^+, which should not cause any confusion.

Theorem 7. The spaces $(\mathbb{D}^\pm_\equiv, \rho^+)$ and $(\mathbb{C}^\pm_\equiv, \rho^+)$ are isometrically isomorphic to the completions of the respective spaces $(\mathbb{D}, \rho)$ and $(\mathbb{C}, \rho)$.

Proof. From Proposition 6 it follows that $(\mathbb{D}^\pm_\equiv, \rho^+)$ is a complete metric space. From Proposition 5 it follows that the mapping

$$f \mapsto \pi\big(f^+\big)$$

is an injective isometric embedding of $(\mathbb{D}, \rho)$ into $(\mathbb{D}^\pm_\equiv, \rho^+)$ and its image $\big\{\pi\big(f^+\big) : f \in \mathbb{D}\big\}$ is dense in $(\mathbb{D}^\pm_\equiv, \rho^+)$.

Therefore, we can identify $(\mathbb{D}^\pm_\equiv, \rho^+)$ with the completion of $(\mathbb{D}, \rho)$.

By the same arguments, we can identify $(\mathbb{C}^\pm_\equiv, \rho^+)$ with the completion of $(\mathbb{C}, \rho)$. ☐

Remark 8. An equivalence class, i.e. an element of $\mathbb{D}^+_{\equiv}$, may be viewed as an element of the completion, while the turbofunctions from this class may be viewed as its different parametrizations. This is quite similar to a curve on a manifold having a multitude of parametrizations.

6 Pointwise convergence

Finally, we relate convergence in $(\mathbb{D}^+, \rho^+)$ to pointwise convergence.

Recall that every function $\sigma \in \Sigma$ is non-decreasing. From this point on, let us agree to understand σ^{-1} as the *right-continuous inverse*, namely

$$\sigma^{-1}(s) := \max\{t \in I^{(\mathrm{T})} : \sigma(t) \le s\}, \quad s \in [0,1]. \tag{6.1}$$

By doing so, we extend definition (2.1) of the visualization mapping to the whole $\mathbb{D}$. Then $\widehat{F^\sigma} \in \mathbb{D}$ for any $F^\sigma \in \mathbb{D}^+$.

Theorem 9. Suppose that $\lim_{n\to\infty} \rho^+(F_n^{\sigma_n}, F^\sigma) = 0$ for some turbofunctions $F^\sigma, F_1^{\sigma_1}, F_2^{\sigma_2}$, $\dots$ in $\mathbb{D}^+$. Then $\lim_{n\to\infty} \widehat{F_n^{\sigma_n}}(s) = \widehat{F^\sigma}(s)$ for $s = 1$ and for every continuity point s of σ^{-1} such that F is continuous at $\sigma^{-1}(s)$.

We first give two corollaries.

Corollary 10. Any Cauchy sequence in $(\mathbb{D}, \rho)$ converges pointwise to an element of $\mathbb{D}$ except, possibly, for at most countable set of points in $[0,1)$.

This follows from Theorem 9 combined with Propositions 5 and 6, and using that the functions σ^{-1} and F have at most countably many discontinuities.

Remark 11. We stress that the limiting element in $\mathbb{D}$ does not characterize completely the limiting turbofunction F^σ because it skips the instantons. In Example 3.1 the limiting function is zero but F^σ is not degenerate.

The next corollary asserts that Proposition 4 still holds for the extended notion of visualization.

Corollary 12. Let $F_1^{\sigma_1}, F_2^{\sigma_2} \in \mathbb{D}^+$ and $F_1^{\sigma_1} \equiv F_2^{\sigma_2}$. Then $\widehat{F_1^{\sigma_1}} = \widehat{F_2^{\sigma_2}}$.

This follows from Theorem 9 once we take $F_2^{\sigma_2} = F_3^{\sigma_3} = \dots$ and $F^\sigma = F_1^{\sigma_1}$ and use the fact that a càdlàg function on $[0,1]$ is identified by its values on any dense set that includes 1.

Proof of Theorem 9. By definition of the semi-metric ρ^+, there exist homeomorphisms $\gamma_1, \gamma_2, \dots$ in $\Gamma^{(\mathrm{T})}$ such that

$$\lim_{n\to\infty} \left[\sup_{t\in I^{(\mathrm{T})}} |(F_n \circ \gamma_n)(t) - F(t)| + \sup_{t\in I^{(\mathrm{T})}} |(\sigma_n \circ \gamma_n)(t) - \sigma(t)| \right] = 0. \tag{6.2}$$

We first consider the case $s = 1$. Since $\gamma_n(1) = 1$ for all n, equality (6.2) implies $F_n(1) \to F(1)$. This yields the claim for $s = 1$, since $\sigma^{-1}(1) = \sigma_n^{-1}(1) = 1$ for all $n \in \mathbb{N}$ by definition (6.1) of right-continuous inverse[3] and we obtain

$$\lim_{n\to\infty} \widehat{F_n^{\sigma_n}}(1) = \lim_{n\to\infty} F_n\big(\sigma_n^{-1}(1)\big) = \lim_{n\to\infty} F_n(1) = F(1) = F\big(\sigma^{-1}(1)\big) = \widehat{F^\sigma}(1).$$

It remains to consider the case when σ^{-1} is continuous at s and F is continuous at $\sigma^{-1}(s)$.

[3] Notice that this argument does not apply to the opposite endpoint $s = 0$ of the time interval because we do not have $\sigma^{-1}(0) = 0$ in general.

We have

$$\begin{aligned}\big|\widehat{F_n^{\sigma_n}}(s)-\widehat{F^\sigma}(s)\big| &= \big|(F_n\circ\gamma_n)\circ(\gamma_n^{-1}\circ\sigma_n^{-1})(s)-(F\circ\sigma^{-1})(s)\big|\\ &\le \big|(F_n\circ\gamma_n)\big((\gamma_n^{-1}\circ\sigma_n^{-1})(s)\big)-F\big((\gamma_n^{-1}\circ\sigma_n^{-1})(s)\big)\big|\\ &\quad+\big|F\big((\gamma_n^{-1}\circ\sigma_n^{-1})(s)\big)-F(\sigma^{-1}(s))\big|,\end{aligned}$$

hence

$$\big|\widehat{F_n^{\sigma_n}}(s)-\widehat{F^\sigma}(s)\big| \le \sup_{t\in I^{(\mathrm{T})}}|(F_n\circ\gamma_n)(t)-F(t)|+\big|F\big((\gamma_n^{-1}\circ\sigma_n^{-1})(s)\big)-F(\sigma^{-1}(s))\big|.$$

By equality (6.2), the first term on the right-hand side vanishes as $n\to\infty$, and since F is continuous at $\sigma^{-1}(s)$, it remains to prove that

$$\lim_{n\to\infty}(\gamma_n^{-1}\circ\sigma_n^{-1})(s)=\sigma^{-1}(s).$$

The function σ^{-1} is continuous at s by theorem's assumption, hence there is a unique $t\in I^{(\mathrm{T})}$ such that $\sigma(t)=s$. Assume first that $0<t<1$. Let $t_1,t_2\in I^{(\mathrm{T})}$ be such that $t_1<t<t_2$. Then $\sigma(t_1)<\sigma(t)=s<\sigma(t_2)$. By equality (6.2), we have

$$\lim_{n\to\infty}(\sigma_n\circ\gamma_n)(t_j)=\sigma(t_j),\quad j=1,2.$$

Hence, for all n large enough we have

$$(\sigma_n\circ\gamma_n)(t_1)<s<(\sigma_n\circ\gamma_n)(t_2).$$

Since the function $\gamma_n^{-1}\circ\sigma_n^{-1}$ is non-decreasing, these inequalities yield

$$t_1\le(\gamma_n^{-1}\circ\sigma_n^{-1})(s)\le t_2. \tag{6.3}$$

By taking the limits, we obtain

$$t_1\le\liminf_{n\to\infty}(\gamma_n^{-1}\circ\sigma_n^{-1})(s)\le\limsup_{n\to\infty}(\gamma_n^{-1}\circ\sigma_n^{-1})(s)\le t_2.$$

Finally, letting $t_1\nearrow t$ and $t_2\searrow t$, we get

$$\lim_{n\to\infty}(\gamma_n^{-1}\circ\sigma_n^{-1})(s)=t=\sigma^{-1}(s),$$

as required.

If $t=0$ (resp. $t=1$), we can not choose t_1 (resp. t_2) as before. However, inequality (6.3) obviously holds with $t_1=t=0$ (resp. $t=t_2=1$), and our conclusion for the limit follows as explained above. □

7 Concluding remarks

1. For using the suggested explicit space construction in concrete questions, one should first of all study the related compactness and tightness criteria. This might be a line of subsequent research.

2. All results of the paper remain valid for càdlàg functions taking values in a complete separable metric space, e.g. in $\mathbb{R}^d$ with $d>1$. The proofs are carried over without any change.

3. Looking beyond probabilistic applications, one may mention *regulated functions* used in solving differential and integral problems involving discontinuous solutions, see e.g. [2, 3, 4]. A regulated function is a function having finite one-sided limits at every point, like a càdlàg function, but one-sided continuity is not assumed. Therefore, Skorokhod space is a subspace of the space of regulated functions. Introducing the regulated turbofunctions along the lines of the present article seems perfectly possible, although we are not aware of any immediate application fields for them.

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