A lower bound for point-to-point connection probabilities in critical percolation

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Abstract

Consider critical site percolation on $\mathbb{Z}^d$ with $d \geq 2$. We prove a lower bound of order n^{-d^2} for point-to-point connection probabilities, where n is the distance between the points.

Most of the work in our proof concerns a 'construction' which finally reduces the problem to a topological one. This is then solved by applying a topological fact, Lemma 2.12 below, which follows from Brouwer's fixed point theorem.

Our bound improves the lower bound with exponent $2d(d-1)$, used by Cerf in 2015 [1] to obtain an *upper* bound for the so-called two-arm probabilities. Apart from being of interest in itself, our result gives a small improvement of the bound on the two-arm exponent found by Cerf.

1 Introduction and statement of the main results

Consider site percolation on $\mathbb{Z}^d$ with $d \geq 2$. Let $p_c = p_c(d)$ denote the critical probability. We use the notation $\Lambda(n)$ for the box $[-n, n]^d$. Our main result is the following, Theorem 1.1 below. Although the result is not relevant for dimension 2 and dimensions ≥ 11, we state it for all $d \geq 2$ because the proof works for all these values of d 'simultaneously'.

Theorem 1.1. *There is a constant $c > 0$, which depends only on the dimension d, such that for all $x \in \mathbb{Z}^d, x \neq 0$,*

$$\mathbb{P}_{p_c}(0 \leftrightarrow x \text{ in } \Lambda(4\|x\|)) \geq \frac{c}{\|x\|^{d^2}}. \tag{1.1}$$

Here $\|x\| := \max_{1 \leq i \leq d} |x_i|$, and $\{0 \leftrightarrow x \text{ in } \Lambda(4\|x\|)\}$ is the event that there is an open path from 0 to x which lies in $\Lambda(4\|x\|)$.

Let us first explain why this result is not relevant for dimension 2 and for 'high' dimensions: For 'nice' two-dimensional lattices (including the square lattice) a much better exponent ($2/3$ or even smaller) than the exponent 4 from (1.1) is known. For these

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lattices it is believed (and known for site percolation on the triangular lattice from [9]) that for every $C > 1$, $\mathbb{P}_{p_c}(0 \leftrightarrow x \text{ in } \Lambda(C\|x\|)) \approx \|x\|^{-5/24}$.

Further, it has been proved for dimensions $d \geq 11$ and is strongly believed for $d \geq 7$, that the 'unrestricted' event in (1.1) (i.e. without the restriction "in $\Lambda(4\|x\|)$") has probability of order $\|x\|^{-(d-2)}$, see [3], [6] and [7]. Moreover, it seems to be believed that for such d a suitably restricted form of this event (with a suitable constant instead of the factor 4 in the l.h.s. of (1.1)) also has probability of order $\|x\|^{-(d-2)}$.

The methods used for the above mentioned results for $d = 2$ don't work for $d \in \{3, \ldots, 6\}$, and presumably this also holds for the methods used for $d \geq 11$. To our knowledge, for dimension $d \in \{3, \ldots, 6\}$ no version of Theorem 1.1 with exponent $\leq d^2$ exists in the literature, even if we drop the restriction "in $\Lambda(4\|x\|)$". In fact, we don't know if the probabilities for the events with and without the restriction are really of different order. We stated our result for the case with restriction because it is stronger, and because of its applicability for other purposes, see Remark 1.4 below.

Non-rigorous numerical estimates in the literature indicate that the exponent d^2 in our theorem is still far from the 'true' one. To illustrate, for $d = 3$ these estimates suggest that the exponent is roughly 0.95, see for instance [8].

Theorem 1.1 has the following implication:

Corollary 1.2. *There is a constant $c' > 0$, which depends only on the dimension d, such that for all $n \geq 1$, and all $x, y \in \Lambda(n)$,*

$$\mathbb{P}_{p_c}(x \leftrightarrow y \text{ in } \Lambda(9n)) \geq \frac{c'}{n^{d^2}}. \tag{1.2}$$

Indeed, this follows readily from Theorem 1.1 by observing that, for all $x, y \in \Lambda(n)$, we have that $\|y - x\| \leq 2n$ and that $\mathbb{P}_{p_c}(x \leftrightarrow y \text{ in } \Lambda(9n)) \geq \mathbb{P}_{p_c}(x \leftrightarrow y \text{ in } x + \Lambda(8n))$, which by translation invariance is equal to $\mathbb{P}_{p_c}(0 \leftrightarrow y - x \text{ in } \Lambda(8n))$.

Corollary 1.2 clearly improves (apart from the factor 9 in the l.h.s. of (1.2), see Remark 1.4(ii) below) the following result by Cerf, which is one of the motivations for our work:

Lemma 1.3 (Cerf, Lemma 6.1 in [1])**.** *There is a constant $\tilde{c} > 0$, which depends only on the dimension d, such that for all $n \geq 1$, and all $x, y \in \Lambda(n)$,*

$$\mathbb{P}_{p_c}(x \leftrightarrow y \text{ in } \Lambda(2n)) \geq \frac{\tilde{c}}{n^{2d(d-1)}}. \tag{1.3}$$

Remark 1.4. We make the following remarks on this lemma.

(i) Lemma 1.3 above was used in a clever way by Cerf in [1] to obtain new *upper* bounds for two-arm probabililities. Cerf's work on two-arm probabililities is, in turn, mainly motivated by the famous conjecture that, for any dimension ≥ 2, there is no infinite open cluster at p_c. This conjecture, one of the main open problems in percolation theory, has only been proved for dimension two and for high dimensions.

(ii) The precise factor 2 in the expression $\Lambda(2n)$ in (1.3) is not essential for Cerf's applications mentioned in Remark 1.4(ii): a bigger constant, which may even depend on the dimension, would work as well (with tiny, straightforward modifications of Cerf's arguments). This is why we have not seriously tried to reduce the factor 9 in the expression $\Lambda(9n)$ in Corollary 1.2.

(iii) What *does* matter for Cerf's applications is an improvement of the power $2d(d-1)$ in the r.h.s. of (1.3). By a simple adaptation of a step in Cerf's proof of Lemma 1.3, the exponent $2d(d-1)$ can be replaced by $(2d-1)(d-1)$, see Section 1.1 below. Our

Corollary 1.2 states that it can even be replaced by d^2. So, for instance, for $d = 3$ the exponents given by Cerf, by the small adaptation of Cerf's argument mentioned above, and by our Corollary 1.2, are 12, 10 and 9 respectively; and for $d = 4$ they are 24, 21 and 16 respectively. Following step by step the computations in Sections 8 and 9 in Cerf's paper, shows that our improvement of Lemma 1.3 also provides a small improvement of Cerf's lower bound for the two-arm exponent in Theorem 1.1 in his paper.

1.1 Main ideas in the proof of Theorem 1.1

We first present, somewhat informally, Cerf's proof of Lemma 1.3 (and the proof of the small adaptation mentioned in Remark 1.4(iii) above).

To focus on the main idea we ignore here the restriction "in $\Lambda(2n)$" in Lemma 1.3. The key ingredient is a result, Corollary 2.3 below, which goes back to work by Hammersley. Applied to the special case where Γ is a box centered at 0, and using symmetry, this result gives that there is a $C = C(d)$ such that for every n we can select a site v^n on the boundary of $\Lambda(n)$ with $v_1^n = n$ and such that

$$\mathbb{P}_{p_c}(0 \leftrightarrow v^n) \geq \frac{C}{n^{d-1}}. \tag{1.4}$$

In the remainder of this section the sites v^n, $n = 1, 2, \ldots$ are called *selected* sites.

Now let x be a vertex. By symmetry we may assume that its coordinates x_i, $i = 1, \ldots, d$, are non-negative. For simplicity we also assume that they are even. Let $x(i)$ be the vertex of which the ith coordinate is equal to x_i and all other coordinates are 0.

We have by translation invariance and FKG that $\mathbb{P}_{p_c}(0 \leftrightarrow x)$ is larger than or equal to

$$\prod_{i=1}^{d} \mathbb{P}_{p_c}(0 \leftrightarrow x(i)). \tag{1.5}$$

Using symmetry, the ith factor in this product is equal to $\mathbb{P}_{p_c}(0 \leftrightarrow (x_i, 0, 0, \cdots, 0))$, and hence, by FKG (and again by symmetry), at least

$$(\mathbb{P}_{p_c}(0 \leftrightarrow v^{x_i/2}))^2, \tag{1.6}$$

which as we saw is $\geq C^2 \cdot \|x\|^{-2(d-1)}$. Hence the product of the d factors is at least of order $\|x\|^{-2d(d-1)}$. This gives essentially (apart from some relatively straightforward issues) Lemma 1.3.

The small adaptation mentioned in Remark 1.4(iii) comes from an observation concerning the first step of the argument above: Note that $\mathbb{P}_{p_c}(0 \leftrightarrow x)$ is also larger than or equal to

$$\mathbb{P}_{p_c}(0 \leftrightarrow v^{x_1}) \, \mathbb{P}_{p_c}(v^{x_1} \leftrightarrow x). \tag{1.7}$$

The first factor is at least $C \cdot \|x\|^{-(d-1)}$. And the second factor can be written as $\mathbb{P}_{p_c}(0 \leftrightarrow v^{x_1} - x)$ to which we can apply Cerf's argument mentioned above. However, since $v^{x_1} - x$ has first coordinate 0, and hence has at most $d - 1$ non-zero coordinates, that argument gives a lower bound of order $\|x\|^{-2(d-1)(d-1)}$ for that factor. Hence, the product of the two factors is at least of order $\|x\|^{-(2d-1)(d-1)}$ in agreement with Remark 1.4(iii).

So, roughly speaking, the explanation of Lemma 1.3 is that x can be written as the sum of $2d$ (and in fact, as the above adaptation showed, $2d - 1$) selected points, where each selected point 'costs' a factor of order $\|x\|^{-(d-1)}$.

Our proof of Theorem 1.1 (and thus of Corollary 1.2) also uses the idea of certain 'selected' points, which will be called 'good' points. Our notion of good points is weaker

than that of the selected points mentioned above, in the sense that they 'cost' a factor of order $\|x\|^{-d}$. However, we will prove that, roughly speaking, each point is the sum of 'just' d good points, which together with the previous statement gives the exponent d^2 in Theorem 1.1. To prove this we first show, again using Corollary 2.3 (but now with general Γ, not only boxes) the existence of suitable paths of good points and then turn the problem into a topological issue. This is then finally solved by applying the 'topological fact' Lemma 2.12.

2 Proof of Theorem 1.1

2.1 Reformulation of Theorem 1.1

Before we start, we reformulate Theorem 1.1 as follows:

Proposition 2.1. *There is a constant $c > 0$, which depends only on the dimension d, such that for all $n \geq 1$, and all $x \in \Lambda(n)$,*

$$\mathbb{P}_{p_c}(0 \leftrightarrow x \text{ in } \Lambda(4n)) \geq \frac{c}{n^{d^2}}. \tag{2.1}$$

This proposition is trivially equivalent to Theorem 1.1. This reformulation is less 'compact' than that of Theorem 1.1 but has the advantage that it is more natural with respect to the approach in our proof (where, as we will see, we first fix an n and then distinguish between 'good' and other points in $\Lambda(n)$).

2.2 Preliminaries

First we introduce some notation and precise definitions: For two vertices v, w, the notation $v \sim w$ means that v and w are neighbours, i.e. that $\|v - w\|_1 := \sum_{i=1}^d |v_i - w_i| = 1$. If a and b are vertices, a *path* from a to b is a sequence of vertices (a $= v(1), v(2), \ldots, v(k) = b$) with $\|v(i+1) - v(i)\|_1 \leq 1$ for all $i = 1, \ldots, k-1$. (Note that we allow consecutive vertices to be equal; this is done for convenience later in this paper.)

We write $\{a \rightarrow b\}$ for the event that there is a path from a to b of which all vertices $\neq a$ are open. (The addition "$\neq a$" is made for convenience later; note that it effects the probability of the event simply by a factor $\frac{1}{p}$).

If W is a set of vertices, we define

$$\partial^{\text{out}} W = \{v \in W^c : \exists w \in W \text{ s.t. } w \sim v\}, \quad \text{and} \tag{2.2}$$

$$\partial^{\text{in}} W = \{w \in W : \exists v \in W^c \text{ s.t. } w \sim v\}. \tag{2.3}$$

We will use a result, Lemma 2.2 below, which goes back to work by Hammersley in the late fifties. Hammersley only proved the special case (used by Cerf) where Γ is of the form $\Lambda(k)$. The proof of the general case (which we use) is very similar, see Section 2 of [2].

Lemma 2.2 (Hammersley [5], Duminil-Copin and Tassion [2].)**.** *Let Γ be a finite, connected set of vertices containing 0. Then*

$$\sum_{x \in \partial^{out} \Gamma} \mathbb{P}_{p_c}(0 \leftrightarrow x \text{ 'in' } \Gamma) \geq 1, \tag{2.4}$$

where $\{0 \leftrightarrow x \text{ 'in' } \Gamma\}$ is the event that there is an open path from 0 to x of which all vertices, except x itself, are in Γ.

Clearly, this lemma has the following consequence.

Corollary 2.3. *For every finite, connected set of vertices Γ containing 0, there is an $x \in \partial^{out} \Gamma$ such that*

$$\mathbb{P}_{p_c}(0 \leftrightarrow x \text{ 'in' } \Gamma) \geq \frac{1}{|\partial^{out} \Gamma|}. \tag{2.5}$$

2.3 The set of 'good' vertices and its properties

Since, throughout the proof, the percolation parameter p is equal to p_c, we will simply write $\mathbb{P}$ instead of $\mathbb{P}_{p_c}$.

Let $n \geq 1$ be fixed, and consider the box $\Lambda(n)$.

Definition 2.4. *A vertex $v \in \Lambda(n)$ is called* good *if $v = 0$ or*

$$\mathbb{P}(0 \leftrightarrow v \text{ in } \Lambda(n)) \geq \frac{1}{|\Lambda(n)|}. \tag{2.6}$$

The set of all good vertices is denoted by G.

Lemma 2.5. *The set G has the following properties:*

(i) *G is invariant under the automorphisms of $\Lambda(n)$.*

(ii) *For each $i = 1, \ldots, d$, there is a path $\pi^{(i)} \subset G \cap [0, n]^d$ which starts in 0 and ends in $[0, n]^{i-1} \times \{n\} \times [0, n]^{d-i}$.*

Proof. Part (i) is obvious.

As to part (ii), we define, inductively, a finite sequence $\Gamma_1 \subset \Gamma_2 \subset \cdots$ of finite connected subsets of $\Lambda(n)$ as follows: $\Gamma_1 := \{0\}$. If $j \geq 1$ and Γ_j is given, then, if $\Gamma_j \cap \partial^{\text{in}} \Lambda(n)$ is nonempty we stop the procedure. If it is empty, we can find, by Corollary 2.3, a vertex $x \in \partial^{\text{out}} \Gamma_j \cap \Lambda(n)$ such that

$$\mathbb{P}(0 \leftrightarrow x \text{ in } \Lambda(n)) \geq \mathbb{P}(0 \leftrightarrow x \text{ 'in' } \Gamma_j) \geq \frac{1}{|\partial^{\text{out}} \Gamma_j|} \geq \frac{1}{|\Lambda(n)|}. \tag{2.7}$$

In that case we take $\Gamma_{j+1} := \Gamma_j \cup \{x\}$.

Let Γ denote the final set obtained at the end of the procedure (i.e. Γ is the union of the Γ_j's obtained by the procedure). It is immediate from the procedure that all the vertices of Γ are good, and that there is a path $\pi = (x(1) = 0, x(2), \ldots, x(m))$ in Γ from 0 to $\partial^{\text{in}} \Lambda(n)$. Clearly, (where, for $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we use the notation $|y|$ for $(|y_1|, \ldots, |y_d|)$), $(|x(1)|, |x(2)|, \ldots, |x(m)|)$ is a path in $[0, n]^d$, which starts in 0 and of which the endpoint is in $[0, n]^{i-1} \times \{n\} \times [0, n]^{d-i}$ for some $i \in \{1, \ldots, d\}$. Moreover, by part (i) of the lemma, each vertex of this path belongs to G. Hence, the property in part (ii) of the lemma holds for some $i \in \{1, \ldots, d\}$. Using again part (i), the property follows for *each* $i \in \{1, \ldots, d\}$. $\qquad \square$

Lemma 2.6. *For each $i = 1, \ldots, d$ there is a path $\tilde{\pi}^{(i)}$ of good points in the box $[0, n]^{i-1} \times [-n, n] \times [0, n]^{d-i}$, starting at a point of which the ith coordinate is $-n$ and ending at a point of which the ith coordinate is n. (So $\tilde{\pi}^{(i)}$ traverses the box 'in the long direction'.)*

Proof. Fix an $i \in \{1, \ldots, d\}$. For each vertex $v = (v_1, \ldots, v_d)$, let the vertex $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_d)$ be defined by:

$$\tilde{v}_j = \begin{cases} v_j, & j \neq i \\ -v_j, & j = i \end{cases} \tag{2.8}$$

Note that, by Lemma 2.5(i), $v \in G$ iff $\tilde{v} \in G$.

Let $\pi^{(i)}$ be the path in the statement of Lemma 2.5(ii), and write it as $(v(1), v(2), \ldots, v(k))$. Then, clearly,

$$\tilde{\pi}^{(i)} := (\tilde{v}(k), \tilde{v}(k-1), \ldots, \tilde{v}(2), \tilde{v}(1), v(1), v(2), \ldots, v(k)) \tag{2.9}$$

is a path with the desired property. $\qquad \square$

Now, for each $i = 1, 2, \ldots, d$, we associate the path $\tilde{\pi}^{(i)}$ of Lemma 2.6 with a continuous curve $c^{(i)}$, simply by following the path along the edges of the graph at constant speed, chosen such that the total travel time along the path is 1. By Lemma 2.6 we immediately have the following:

Lemma 2.7. *For each $i = 1, \ldots, d$, the function $c^{(i)} : [0,1] \to \mathbb{R}^d$ is continuous and has the following properties:*

(i) $\forall s \in [0,1]$: $\exists t \in [0,1]$ *s.t.* $c^{(i)}(t) \in G$ *and* $\|c^{(i)}(t) - c^{(i)}(s)\|_1 \leq 1/2$.

(ii) $\forall s \in [0,1]$: $-n = (c^{(i)}(0))_i \leq (c^{(i)}(s))_i \leq (c^{(i)}(1))_i = n$.

(iii) $\forall s \in [0,1]$ *and* $j \neq i$: $(c^{(i)}(s))_j \in [0,n]$.

2.4 Preparations for a topological approach

We use the curves $c^{(i)}, i = 1, \ldots, d$, introduced in the previous subsection, to define a continuous function $g : [0,1]^d \to \mathbb{R}^d$. We use this function to approximate the set of sums of good points. In the next section we will apply a topological lemma to this function to complete the proof of our main result.

First we define for each $t = (t_1, \ldots, t_d) \in [0,1]^d$ the matrix $C(t) \in \mathbb{R}^{d \times d}$ by

$$C(t) = \begin{bmatrix} c^{(1)}(t_1) & c^{(2)}(t_2) & \cdots & c^{(d)}(t_d) \end{bmatrix}. \tag{2.10}$$

One might hope that the set $\{\sum_{j=1}^d c^{(j)}(t_j) : (t_1, \ldots, t_d) \in [0,1]^d\}$ covers (almost) the entire box $[0,n]^d$. However, to obtain and prove a coverage of such flavour, we need more than the simple sums mentioned above, and will exploit the reflection symmetry in a more subtle way. To do this we next introduce functions $h_m : \mathbb{R}^m \to \mathbb{R}$ defined recursively for $m \geq 1$ by

$$\begin{aligned} h_1(x) &= |x| & \text{for } x \in \mathbb{R}, \\ h_m(x) &= |h_{m-1}(x_1, \ldots, x_{m-1}) - |x_m|| & \text{for } x \in \mathbb{R}^m, \, m \geq 2. \end{aligned} \tag{2.11}$$

Obviously, $h_m(x) \geq 0$ for each $m \geq 1$ and all $x \in \mathbb{R}^m$. The following lemma gives other useful properties of h_m.

Lemma 2.8. *For all $m \geq 1$ and all $x \in \mathbb{R}^m$ there exist weights $a_j \in \{-1, 1\}$ for $j = 1, \ldots, m$ such that the following holds:*

$$0 \leq h_m(x) = \sum_{j=1}^m a_j x_j \leq \|x\| \quad \text{and} \quad \forall k \leq m-1 : \left| \sum_{j=1}^k a_j x_j \right| \leq \|x\|. \tag{2.12}$$

Proof. The proof is by induction. The case $m = 1$ is obvious. Assume that the lemma holds for a certain m, and let $y \in \mathbb{R}^{m+1}$. Denote $x = (y_1, \ldots, y_m) \in \mathbb{R}^m$. By the definition we have $h_{m+1}(y) = |h_m(x) - |y_{m+1}||$. By the induction hypothesis there exist $a_1, \ldots, a_m \in \{-1, 1\}$ such that (2.12) holds.

We now distinguish two cases: *Case 1* is the case where $0 \leq h_m(x) \leq |y_{m+1}|$. In this case we set $\hat{a}_j = -a_j$ for $j = 1, \ldots, m$, and we set $\hat{a}_{m+1} = \text{sgn}(y_{m+1})$. So we have

$$h_{m+1}(y) = |y_{m+1}| - h_m(x) = \sum_{j=1}^{m+1} \hat{a}_j y_j, \tag{2.13}$$

as desired. Moreover, we clearly have $0 \le h_{m+1}(y) \le \max(|y_{m+1}|, h_m(x))$, which by the induction hypothesis is at most $\max(|y_{m+1}|, \|x\|)$, which is equal to $\|y\|$. Finally, for all $k = 1, \ldots, m$ we have

$$\left| \sum_{j=1}^{k} \hat{a}_j y_j \right| = \left| \sum_{j=1}^{k} -a_j y_j \right| = \left| \sum_{j=1}^{k} a_j y_j \right|, \tag{2.14}$$

which by the induction hypothesis is at most $\|x\|$ and hence at most $\|y\|$. This settles Case 1.

In *Case 2*, where $h_m(x) \ge |y_{m+1}|$, we set $\hat{a}_j = a_j$ for $j = 1, \ldots, m$, and $\hat{a}_{m+1} = -\mathrm{sgn}(y_{m+1})$, after which the argument is a straighforward adaptation of that for the first case. $\qquad \square$

Now we define $g : [0,1]^d \to \mathbb{R}^d$ by describing the d coordinates of $g(t)$.

Definition 2.9. *For $t \in [0,1]^d$, let $C = C(t)$. For all $1 \le i \le d$, we define*

$$(g(t))_i = C_{i,i} + h_{d-1}(C_{i,1}, \ldots, C_{i,i-1}, C_{i,i+1}, \ldots, C_{i,d}). \tag{2.15}$$

Lemma 2.10. *The function $g : [0,1]^d \to \mathbb{R}^d$ has the following properties:*

(i) *g is continuous;*

(ii) *$(g(t))_i \le 0$ if $t_i = 0$ and $(g(t))_i \ge n$ if $t_i = 1$;*

(iii) *For each $x \in g([0,1]^d)$, there exist $z^{(1)}, \ldots, z^{(d)} \in G$ such that*

 (a) *$\sum_{j=1}^{k} z^{(j)} \in \Lambda(2n + \lceil \frac{d}{2} \rceil)$ for each $k = 1, \ldots, d$;*

 (b) *$\left\| x - \sum_{j=1}^{d} z^{(j)} \right\|_1 \le \frac{d}{2}$.*

Proof. Part (i) follows because g is a composition of continuous functions.

As to part (ii), note that $C_{i,j}(t) = (c^{(j)}(t_j))_i$ so that $|C_{i,j}(t)| \le n$ for all i, j by Lemma 2.7(ii) and (iii). By (2.12), this implies that

$$0 \le h_{d-1}(C_{i,1}, \ldots, C_{i,i-1}, C_{i,i+1}, \ldots, C_{i,d}) \le n. \tag{2.16}$$

Together with Lemma 2.7(ii), this proves part (ii).

As to part (iii): By definition of g and by Lemma 2.8, for each $t \in [0,1]^d$ there exist weights $a_{ij}(t) \in \{-1, 1\}$ such that

$$g(t) = \tilde{C}(t)\mathbf{1} = \sum_{j=1}^{d} \tilde{c}^{(j)}(t_j), \tag{2.17}$$

where $\mathbf{1}$ denotes the 'all 1's vector', $\tilde{C}(t)$ is the matrix with elements $\tilde{C}_{i,j}(t) = a_{ij}(t)C_{i,j}(t)$, and $\tilde{c}^{(j)}(t_j)$ denotes the jth column of $\tilde{C}(t)$. For all i, the partial sums of (2.17) satisfy

$$\left(\sum_{j=1}^{k} \tilde{c}^{(j)}(t_j) \right)_i = \begin{cases} \sum_{j=1}^{k} a_{ij} C_{i,j} & \text{if } k < i, \\ C_{ii} + \sum_{j=1, j \neq i}^{k} a_{ij} C_{i,j} & \text{if } k \ge i. \end{cases} \tag{2.18}$$

Applying Lemma 2.8 gives that all partial sums are in $\Lambda(2n)$ for all t.

Since Lemma 2.7(i) applies to the columns of $C(t)$, it also applies to the columns of $\tilde{C}(t)$ by Lemma 2.5(i). Consequently, there exist $s = (s_1, \ldots, s_d) \in [0,1]^d$ such that $z^{(j)} := \tilde{c}^{(j)}(s_j) \in G$ for all j and such that for all $k \le d$

$$\left\| \sum_{j=1}^{k} \tilde{c}^{(j)}(t_j) - \sum_{j=1}^{k} z^{(j)} \right\|_1 \le \sum_{j=1}^{k} \left\| \tilde{c}^{(j)}(t_j) - z^{(j)} \right\|_1 \le \frac{k}{2} \le \frac{d}{2}, \tag{2.19}$$

completing the proof of part (iii). $\qquad \square$

Lemma 2.11. *There exists a constant $c > 0$ (which depends on d only) such that inequality (2.1) in Proposition 2.1 holds for all lattice points $x \in g([0,1]^d) \cap \Lambda(n)$.*

Proof. Clearly, we can assume that $3n + d/2 < 4n$. Let x be a lattice point in $g([0,1]^d) \cap \Lambda(n)$. Choose $z^{(1)}, \dots, z^{(d)}$ as in Lemma 2.10(iii). Applying this lemma, we get that the event $\{0 \leftrightarrow x \text{ in } \Lambda(3n + \frac{d}{2})\}$, and hence the event $\{0 \leftrightarrow x \text{ in } \Lambda(4n)\}$, occurs if each of the events

$$\left\{ 0 \overset{\text{in } \Lambda(n)}{\longleftrightarrow} z^{(1)} \right\},$$

$$\bigcap_{k=1}^{d-1} \left\{ \sum_{i=1}^{k} z^{(i)} \longleftrightarrow \sum_{i=1}^{k+1} z^{(i)} \text{ in } \sum_{i=1}^{k} z^{(i)} + \Lambda(n) \right\}, \text{ and}$$

$$\left\{ \sum_{i=1}^{d} z^{(i)} \leftrightarrow x \text{ in } \sum_{i=1}^{d} z^{(i)} + \Lambda(d) \right\}$$

occurs. Hence, by FKG and using that each $z^{(i)}$ is good,

$$\mathbb{P}(0 \leftrightarrow x \text{ in } \Lambda(4n)) \geq \frac{1}{|\Lambda(n)|^d} \, p_c^{d/2}, \tag{2.20}$$

which completes the proof of Lemma 2.11. $\qquad\square$

2.5 Completion of the proof of Theorem 1.1

By Lemma 2.11 it is clear that the result of Proposition 2.1, and hence Theorem 1.1, follows if

$$g([0,1]^d) \supset [0,n]^d. \tag{2.21}$$

Indeed, if $g([0,1]^d) \supset [0,n]^d$, then, by Lemma 2.11, the bound (2.1) holds for all vertices in $[0,n]^d$, and hence, by symmetry (part (i) of Lemma 2.5), also for all vertices in $\Lambda(n)$.

Finally, (2.21) follows by applying the following 'topological fact', Lemma 2.12, to the function $\frac{g(\cdot)}{n}$, and by noting that Lemma 2.10 (i) and (ii) guarantees that this function satisfies the conditions of Lemma 2.12. We don't know if this lemma is explicitly in the literature. It can be obtained from a theorem by Miranda [10], which in turn follows from Brouwer's fixed-point theorem. See also remarks (C2) and (C3) on page 100 in [4]. To keep our paper as self-contained as possible, we present a proof, directly from Brouwer's theorem, for which we thank Lex Schrijver.

Lemma 2.12. *Let $f : [0,1]^d \to \mathbb{R}^d$ be a continuous function with the following properties.*

  (i) For all $i \in \{1, \dots, d\}$ and all $x \in [0,1]^d$ with $x_i = 0$, $(f(x))_i \leq 0$, and

  (ii) for all $i \in \{1, \dots, d\}$ and all $x \in [0,1]^d$ with $x_i = 1$, $(f(x))_i \geq 1$.

Then $f([0,1]^d) \supset [0,1]^d$.

Proof. Since $f([0,1]^d)$ is compact, it is sufficient to prove that $f([0,1]^d) \supset (0,1)^d$. Further, w.l.o.g. we may assume that $f([0,1]^d) \subset [0,1]^d$ and that the inequalities at the end of (i) and (ii) in Lemma 2.12 are *equalities*. To see this, replace f by the function $\tilde{f}$ defined by

$$(\tilde{f}(x))_i = \begin{cases} 1, & (f(x))_i \geq 1 \\ 0, & (f(x))_i \leq 0 \\ (f(x))_i, & \text{otherwise} \end{cases} \tag{2.22}$$

and note that $\tilde{f}([0,1]^d) \cap (0,1)^d = f([0,1]^d) \cap (0,1)^d$.

So, from now on, we assume that $f : [0,1]^d \to [0,1]^d$ is continuous and has the property that:

(i) For all $i \in \{1,\ldots,d\}$ and all $x \in [0,1]^d$ with $x_i = 0$, $(f(x))_i = 0$, and

(ii) for all $i \in \{1,\ldots,d\}$ and all $x \in [0,1]^d$ with $x_i = 1$, $(f(x))_i = 1$.

Note that this implies in particular that, for all $x \in \partial[0,1]^d$, $f(x) \in \partial[0,1]^d$ and

$$f(x) \neq \mathbf{1} - x. \tag{2.23}$$

Now suppose there is an $a \in (0,1)^d$ which is not in $f([0,1]^d)$. We will see that this leads to a contradiction. Fix such an a and let π be the projection from a on $\partial[0,1]^d$. (More precisely, for every $x \neq a$, $\pi(x)$ is the unique intersection of the half-line $\{a + \lambda(x-a) : \lambda > 0\}$ with $\partial[0,1]^d$). Finally, let $\psi(x) := \mathbf{1} - x$.

Now consider the function $\psi \circ \pi \circ f : [0,1]^d \to \partial[0,1]^d$. Note that it is well-defined (by the special property of a) and continuous. Hence, by Brouwer's fixed-point theorem, it has a fixed point, which we denote by b. It is clear that $b \in \partial[0,1]^d$. So we have

$$b = (\psi \circ \pi \circ f)(b) = \psi(f(b)), \tag{2.24}$$

and hence $\psi(b) = f(b)$. However, this last equality violates (2.23), so we have indeed obtained a contradiction. □

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