Electron. Commun. Probab. **25** (2020), no. 24, 1–12.
https://doi.org/10.1214/20-ECP305
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The contact process on periodic trees*

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Abstract

A little over 25 years ago Pemantle [6] pioneered the study of the contact process on trees, and showed that on homogeneous trees the critical values λ_1 and λ_2 for global and local survival were different. He also considered trees with periodic degree sequences, and Galton-Watson trees. Here, we will consider periodic trees in which the number of children in successive generations is $(n, a_1, \dots, a_k)$ with $\max_i a_i \le Cn^{1-\delta}$ and $\log(a_1 \cdots a_k)/\log n \to b$ as $n \to \infty$. We show that the critical value for local survival is asymptotically $\sqrt{c(\log n)/n}$ where $c = (k-b)/2$. This supports Pemantle's claim that the critical value is largely determined by the maximum degree, but it also shows that the smaller degrees can make a significant contribution to the answer.

Keywords: local survival; block construction; sharp asymptotics.
AMS MSC 2010: 60K35.
Submitted to ECP on September 22, 2019, final version accepted on March 8, 2020.

1 Introduction

The contact process can be defined on any graph as follows: occupied sites become vacant at rate 1, while vacant sites become occupied at rate λ times the number of occupied neighbors. In 1974 Harris introduced the contact process on $\mathbb{Z}^d$. In 1992 Pemantle [6] began the study of contact processes on trees. Let ξ_t be the set of occupied sites at time t and use ξ_t^0 to denote the process with $\xi_0^0 = \{0\}$ where 0 is the root of the tree. His main new result was that the process had two phase transitions:

$$\lambda_1 = \inf\{\lambda : P(\xi_t^0 \neq \emptyset \text{ for all } t) > 0\}$$
$$\lambda_2 = \inf\{\lambda : \liminf_{t\to\infty} P(0 \in \xi_t^0) > 0\}.$$

Let $\mathbb{T}_d$ be the tree in which each vertex has $d+1$ neighbors. When $d=1$, $\mathbb{T}_1 = \mathbb{Z}$, so we restrict our attention to $d \ge 2$. Pemantle showed that $\lambda_1 < \lambda_2$ when $d \ge 3$ by getting upper bounds on λ_1 and lower bounds on λ_2.1n 1996 Liggett proved that in $d=2$ $\lambda_1 < 0.605 < 0.609 < \lambda_2$ to settle the last case. At about the same time, Stacey gave an elegant proof that on $\mathbb{T}_d$ and a number of other graphs we have $\lambda_1 < \lambda_2$. For an account of the history and the theory of the contact process see Liggett's 1999 book, [5].

*Partially supported by DMS grants DMS 1505215 and 1809967 from the Probability Program. This original version of this paper [4] was produced in a DOMath project at Duke University in the summer of 2018 with undergraduates Yufeng Jiang, Remy Kassem, Grayson York, and Brandon Zhao.

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Pemantle also considered periodic trees and Galton-Watson trees. In the special case that the number of children alternates between a and b he showed $1/(\sqrt{a}+\sqrt{b}) \le \lambda_2$. He did not give the details of the proof, but this can easily be proved using Lemma 3.1 in Pemantle and Stacey [7], which gives a formula for the critical value for local survival for branching random walk on a general graph. Pemantle also showed, see the first sentence after (7) on page 2103, that for a general period k tree with degree sequence $(a_1, a_2, \dots, a_k)$ the critical value $\lambda_2 \le C/(a_1 \cdots a_k)^{1/2k}$. When $k = 2$ the bound is $C/(ab)^{1/4}$. When $a = b$, the upper and lower bounds differ by a factor of 2. However when $a = 1$ and $b = n$

$$1/(\sqrt{a}+\sqrt{b}) = 1/(1+\sqrt{n}) \qquad C/(ab)^{1/4} = C/n^{1/4}. \tag{1.1}$$

As Pemantle notes on page 2103, the upper and lower bounds are different orders of magnitude. He continues with "Which of these asymptotics for λ_2 is sharp if either? The somewhat surprising answer is that the lower bound is sharp even though the geometric mean $(a_1 \cdots a_k)^{1/k}$ is clearly a better representative for the growth rate of the tree." The next result shows that the lower bound is more accurate than the upper bound, but it is not quite sharp.

Theorem 1.1. *On the $(1, n)$ tree, as $n \to \infty$ the critical value*

$$\lambda_2 \sim \sqrt{c_2 (\log n)/n} \quad \textit{where } c_2 = 1/2.$$

On page 2103, Pemantle says that "for reasonably regular non-homogeneous trees the critical value is determined by M the maximum number of children and is at most $rM^{-1/2}$ where r is a logarithmic measure of how far apart vertices with M children are from each other." The next result confirms his intuition about the importance of the maximum degree but also shows that the lower degree vertices can have a significant influence on λ_2.

Theorem 1.2. *Consider the $(n, a_1, a_2, \dots, a_k)$ tree where k is a fixed integer and a_i can depend on n. Suppose $\max_i a_i \le C n^{1-\delta}$ for some positive C, δ and*

$$b = \lim_{n\to\infty} \frac{\log(a_1 a_2 \cdots a_k)}{\log n}.$$

As $n \to \infty$ the critical value $\lambda_2 \sim \sqrt{c_k \log n/n}$ where $c_k = (k-b)/2$.

Theorem 1.1 is the special case $k = 1$, $a_1 = 1$, so it suffices to prove Theorem 1.2.

Using the methods developed to study λ_2 we can derive asymptotics for λ_1

Theorem 1.3. *Under the assumptions of Theorem 1.2. (i) If $c'_k = (k+1)/2 - (b+1) > 0$ then as $n \to \infty$ the critical value $\lambda_1 \sim \sqrt{c'_k \log n/n}$. (ii) If $(k+1)/2 - (b+1) < 0$ then $(\log \lambda_1)/\log n \to -(b+1)/(k+1)$.*

When $b = k$ the critical value is of order $1/n$, which is the correct order of magnitude value for the homogeneous tree in which all vertices have degree n. The difference between (i) and (ii) is that in the second case $\lambda_1 \ll n^{-1/2}$ so the contact process does not survive very long on stars.

1.1 The survival time of the contact process on finite trees

Let ρ be the root of the periodic tree $(n, a_1, \dots, a_k)$. Truncating the periodic tree at height k gives a subgraph $S_k = \{x : d(\rho, x) \le k\}$, where d is the distance on the tree. A vertex $x \in S_k$ at distance i from the center ρ is said to be in the set L_i ("level i"). In S_k, vertices on level $1 \le i < k$ have a_i children while vertices on level k are leaves, i.e., they have no children. When the context is clear we also write $S_k = (n, a_1, \dots, a_k)$, where the sequence corresponds to the offspring number on each level. When we delete the root from S_k we end up with n subtrees $\{T_{k,i}\}_{i=1}^n$ with $T_{k,i} = (a_1, \dots, a_k)$.

The first step in the proof of Theorem 1.2 is to prove in Section 2 an upper bound on the survival time of contact process on S_k.

Theorem 1.4. *Suppose $\lambda = \sqrt{c(\log n)/n}$ where $c > 0$. Let τ_k be the survival time of contact process on $S_k = (n, a_1, \dots, a_k)$ starting from all sites occupied where $\max_i a_i \le Cn^{1-\delta}$ for some positive C and δ. For any $\epsilon > 0$, when n is sufficiently large*

$$E\tau_k \le C_0(\log n)e^{(1+\epsilon)\lambda^2 n} = C_0(\log n)n^{c(1+\epsilon)},$$

where C_0 is some positive constant depending on k but not on C, δ.

When $k = 1$, S_k reduces to the star graph. In this case the result holds for all λ (see [4] for details). Theorem 1.4 gives the only upper bound we know of for the survival time for the contact process on the star. There are many lower bounds for the survival time on stars. See Theorem 4.1 in [6], Lemma 5.3 in [1], and Lemma 1.1 in [2]. These bounds can be used to show that the critical value for prolonged survival of the contact process on some random graphs is 0, but to identify the asymptotics for the critical value on the $(n, a_1, \dots, a_k)$ tree, we need a more precise result on the survival time on the star graph with n leaves. We denote the state of the star by (i, j) where i is the number of occupied leaves and $j = 1, 0$ when the center is occupied, vacant. Let $P_{i,j}$ be the law of the process starting at state (i, j) and let $T_{0,0}$ be the time to hit $(0, 0)$.

Theorem 1.5. *Let $L = (1 - 4\delta)\lambda n$ with $\delta > 0$. If $\eta > 0$ is small then*

$$P_{L,1}\left(T_{0,0} \ge e^{(1-\eta)\lambda^2 n}\right) \to 1 \qquad \text{as } n \to \infty.$$

Combining this with Theorem 1.4 shows that if $\lambda^2 n \to \infty$ the survival time on the star is $\exp((1 + o(1))\lambda^2 n)$.

2 Upper bound on survival times on S_k

2.1 Equilibrium on T_k^*

To prepare for the proof of Theorem 1.4, we first consider the contact process on S_k with the center ρ permanently occupied. Let $T_k = (a_1, \dots, a_k)$ denote a generic subtree of ρ and let T_k^* be T_k with ρ attached to the root and with ρ permanently occupied. Since the center ρ is always occupied, the contact process on S_k can be simply treated as n independent contact processes on the $T_{k,i}^*$'s. As ρ is always occupied, the contact process ξ_t on T_k^* has a stationary distribution ξ_∞.

We use x_i to denote a generic vertex in L_i. To compute the occupancy probability for a vertex $x_i \in L_i$ we need the notion of the dual process ζ_t of ξ_t. To construct ξ_t by graphical representation, we assign a Poisson process N_x of rate 1 to each vertex $x \in T_k^* \backslash \{\rho\}$ and a Poisson process $N_{(x,y)}$ of rate λ to each ordered pair of vertices that are joined by an edge of T_k^*. The dual process ζ_t is constructed by looking at the dual path on the same graphical representation we used to construct ξ_t. (See Liggett [5] for an account of graphical representation and duality.) It follows from duality that

$$P(x_i \in \xi_t) = P(\rho \in \zeta_s^{x_i} \text{ for some } s \le t) \le P(\rho \in \zeta_s^{x_i} \text{ for some } s \ge 0).$$

Letting $t \to \infty$ gives $P(x_i \in \xi_\infty) \le P(\rho \in \zeta_s^{x_i}$ for some $s \ge 0)$, that is, if $x_i \in \xi_\infty$ then the dual contact process $\zeta_t^{x_i}$ starting from x_i has to reach ρ at some time. If we have a dual path of length $i + 2m$ from x_i to ρ then $i + m$ steps will be toward ρ and m steps away. Let $(y_0, y_1, \dots, y_{i+2m})$ denote a path from x_i to ρ with $y_0 = x_i$ and $y_{i+2m} = \rho$. To produce a particle at ρ, we need a birth from y_j to y_{j+1} to occur before the particle at y_j dies for all $j = 0, \dots i + 2m - 1$. So the expected number of particles produced at ρ by this path is $(\lambda/(1+\lambda))^{i+2m} \le \lambda^{i+2m}$. If we let $d = Cn^{1-\delta}$ so that $a_i \le d$ for all $i = 1, \dots, k$, then the expected number of particles $N_{x_i,\rho}$ that reach ρ has

$$
\begin{aligned}
EN_{x_i,\rho} &\le \sum_{m=0}^{\infty} \binom{i+2m}{m} \lambda^{i+2m} d^m \le \lambda^i (1 + \sum_{m=1}^{\infty} 2^{i+2m} \lambda^{2m} d^m) \\
&= \lambda^i (1 + 2^i \sum_{m=1}^{\infty} (4\lambda^2 d)^m) \le (1+\eta)\lambda^i. \qquad (2.1)
\end{aligned}
$$

Since $\lambda^2 d \to 0$ as $n \to \infty$, $\eta > 0$ can be arbitrarily small if n is large enough. It follows that

$$
P(x_i \in \xi_\infty) \le P(N_{x_i,\rho} \ge 1) \le E(N_{x_i,\rho}) \le (1+\eta)\lambda^i. \qquad (2.2)
$$

2.2 Proof of Theorem 1.4

Proof. Starting with all sites occupied on S_k, we will set the center ρ to be occupied for an amount of time M chosen so that the distribution of the contact process on each $T^*_{k,i}$ becomes close to the equilibrium ξ_∞. After time M when the center ρ first becomes vacant we run a trial to see if the following event occurs:

$$
G = \{\rho \text{ is not recolonized before the contact process dies out on } S_k\}.
$$

If G occurs we terminate the process and obtain an upper bound on the survival time; if G does not occur then we set all of the sites in S_k to be 1, make ρ occupied for the next M units of time, and begin the next try.

Our first step is to choose M. The contact process ξ_t on T^*_k is additive so we can write $\xi_t = \xi^\rho_t \cup \hat{\xi}^1_t$ where $\hat{\xi}^1_t$ is the contact process on T_k with initially all 1's, and ξ^ρ_t the contact process on T^*_k with ρ initially and permanently occupied. By the time $\hat{\xi}^1_t$ dies out, we have $\xi_t = \xi^\rho_t$, whose distribution is stochastically dominated by the stationary distribution ξ_∞.

For our purpose M should be chosen to be roughly the extinction time of $\hat{\xi}^1_t$. To simplify notation we let $d = Cn^{1-\delta}$, the upper bound on $\max_i a_i$ and consider the contact process on the regular tree $\mathbb{T}_d$. Let A_t be the contact process on $\mathbb{T}_d$ with birth rate γ and death rate 1. Since $T_k \subset \mathbb{T}_d$, if we take $\gamma > \lambda$ the contact process ξ_t on T_k is stochastically dominated by A_t. Following the proof of Theorem 4.1 in part I of Liggett [5], we define $w_\theta(A_t) = \sum_{y \in A_t} \theta^{\ell(y)}$. where $\ell(y)$ is a function from $\mathbb{T}^d$ to $\mathbb{Z}$ so that for each $y \in \mathbb{T}^d$, $\ell(z) = \ell(y) - 1$ for exactly one neighbor z of y and $\ell(z) = \ell(y) + 1$ for the other d neighbors z of y. Liggett shows that if $\theta = 1/\sqrt{d}$

$$
\frac{d}{dt} E_A w_\theta(A_t) \bigg|_{t=0} \le [2\sqrt{d}\gamma - 1] w_\theta(A_0)
$$

When n is sufficiently large and $\gamma = 1/(\log n)\sqrt{d} \approx (\log n)^{-1} n^{-(1-\delta)/2} \gg \lambda$, for any small $\eta > 0$ we have

$$
E_{T_k} w_\theta(A_t) \le w_\theta(T_k) e^{-(1-\eta)t} \le (d\theta)^k e^{-(1-\eta)t}. \qquad (2.3)
$$

Markov's inequality implies that

$$
d^{-k/2} P_{T_k}(\ell(x) \le k \text{ for some } x \in A_t) \le d^{k/2} e^{-(1-\eta)t} \le d^{k/2} e^{-t/2}.
$$

It follows that

$$
P(\hat{\xi}^1_t \ne \emptyset) \le P_{T_k}(\ell(x) \le k \text{ for some } x \in A_t) \le d^k e^{-t/2}, \qquad (2.4)
$$

so the process dies out with high probability when $t \geq 4k \log d$. Since $d = Cn^{1-\delta}$ we can choose $M = 4k \log n$.

With the choice of M fixed, next we will estimate the probability of the event G and obtain an upper bound on the survival time. Now we start the contact process on T_k^* with all sites occupied and set ρ to be occupied for the first M units of time. After time M we allow ρ to become vacant at rate 1. For any $\eta > 0$ and $t \geq M$, when n is large enough

$$\begin{aligned} P(\xi_t(x_i) = 1) &\leq P(\xi_t^\rho(x_i) = 1) + P(\hat{\xi}_t^1(x_i) = 1) \leq P(x_i \in \xi_\infty) + P(\hat{\xi}_t^1 \neq \varnothing) \\ &\leq (1+\eta)\lambda^i + n^{-(1-\delta)k} \leq (1+2\eta)\lambda^i \qquad \text{by (2.2).} \end{aligned}$$

Call the subgraph consisting of ρ and all of its neighbors the *central star*. When the center ρ first becomes vacant (after time M), we use D_1 to denote the set of occupied sites on the central star and D_2 the set of occupied sites outside the central star. By the additivity of the process we observe that G is the intersection of the following two events:

$$G_1 = \{\rho \notin \xi_t^{D_1} \text{ for any } t\} \quad \text{ and } \quad G_2 = \{\rho \notin \xi_t^{D_2} \text{ for any } t\},$$

where ξ_t^A is the contact process on S_k with an initial set A of occupied sites. Since G_1 and G_2 are both decreasing events, i.e., having more births or fewer deaths is bad for them, by the Harris-FKG inequality

$$P(G) = P(G_1 \cap G_2) \geq P(G_1)P(G_2).$$

We begin by estimating $P(G_2)$. Note that the expected number of particles outside the central star at time $t \geq M$ is

$$\leq (1+2\eta)\left(na_1\lambda^2 + na_1a_2\lambda^3 + \ldots na_1a_2\cdots a_{k-1}\lambda^k\right).$$

Since $\lambda = \sqrt{(c\log n)/n}$, if $a_i = Cn^{1-\delta}$ for all i with $\delta < 1/2$ this grows rapidly as $n \to \infty$.

Fortunately, if we start a contact process on S_k from a site on level i and freeze any particle that reaches the center ρ, then the expected number of such particles is $\leq (1+\eta)\lambda^i$ by the same dual path argument as in (2.1). Therefore the expected number N_ρ of particles reaching the center ρ is

$$\leq (1+\eta)(1+2\eta)\left(na_1\lambda^4 + na_1a_2\lambda^6 + \ldots na_1a_2\cdots a_{k-1}\lambda^{2k+2}\right) \leq n^{-\delta/2}.$$

Hence when n is large

$$P(G_2^c) \leq P(N_\rho \geq 1) \leq E(N_\rho) \leq n^{-\delta/2} \leq \eta.$$

Now we turn to event G_1. Let $\zeta_t^{x_1}$ and $\tilde{\zeta}_t^{x_1}$ denote the contact process on T_k and T_k^*, respectively, with the site $x_1 \in L_1$ initially occupied. We need to estimate the probability of event $B = \{\tilde{\zeta}_t^{x_1}$ dies out before giving birth to $\rho\}$. Let σ be the amount of time 0 is occupied in $\zeta_t^{x_1}$. Using the definition of w_θ and a calculation similar to (2.3)

$$E\sigma \leq \int_0^\infty E_{x_1} w_\theta(A_t)dt \leq \int_0^\infty e^{-(1-\eta)t}dt \leq 1+2\eta.$$

when η is small. Note that on event B, $\zeta_t^{x_1}$ and $\tilde{\zeta}_t^{x_1}$ can be coupled together exactly. Hence event B occurs if there is no arrival of infection arrows pointing from x_1 to ρ during the survival time σ of $\zeta_t^{x_1}$, i.e.,

$$P(B) \geq P(Poisson(\lambda\sigma) = 0) = Ee^{-\lambda\sigma} \geq e^{-\lambda E\sigma} \geq e^{-(1+2\eta)\lambda}.$$

If there are m occupied neighbors of ρ when it becomes vacant then we need event B to occur on m subtrees to ensure the occurrence of G_1. Again by the Harris-FKG inequality

$P(G_1) \geq P(B)^m \geq e^{-(1+2\eta)m\lambda}$. Since the occupancy of sites adjacent to the root are independent events, (2.2) and the law of large numbers implies that for sufficiently large n we have $P(m \leq (1+3\eta)\lambda n) \geq 1-\eta$. From this it follows that for small η and large n

$$P(G_1) \geq (1-\eta)e^{-(1+2\eta)(1+3\eta)\lambda^2 n} \geq (1-\eta)e^{-(1+6\eta)\lambda^2 n}. \tag{2.5}$$

Therefore, $P(G) \geq P(G_1)P(G_2) \geq (1-\eta)^2 e^{-(1+6\eta)\lambda^2 n}$. That is, in expectation we need to try $(1-\eta)^{-2}e^{(1+6\eta)\lambda^2 n}$ times to have a success. In each trial, it takes an expected amount of time $M+1$ for ρ to become vacant. Let $\{X_i\}_{i=1}^n$ be a set of i.i.d. random variables representing the survival time of the contact process $\hat{\xi}_t^1$ on $\{T_{k,i}\}_{i=1}^n$, respectively. We observe that it takes time $\leq E(\max_{1\leq i\leq n} X_i)$ to determine if G occurs. Since

$$\begin{aligned} E(\max_{1\leq i\leq n} X_i) &\leq 2(k+1)\log n + \int_{2(k+1)\log n}^{\infty} P(\max_{1\leq i\leq n} X_i > t)\, dt \\ &\leq 2(k+1)\log n + \int_{2(k+1)\log n}^{\infty} n\cdot (Cn^{1-\delta})^k e^{-t/2}\, dt \leq 4k\log n \qquad \text{by (2.4)}, \end{aligned}$$

each round takes at most $M+1+4k\log n \leq 9k\log n$ units of time in expectation. It follows that

$$E\tau_k \leq (9k\log n)(1-\eta)^{-2}e^{(1+6\eta)\lambda^2 n} \leq C_0(\log n)e^{(1+6\eta)\lambda^2 n}$$

for some $C_0 > 0$ and sufficiently large n. □

3 Lower bound on λ_2

Lemma 3.1. *Let $c_k = (k-b)/2$ and $\epsilon > 0$. When n is sufficiently large, the critical value λ_2 of the contact process on the $(n, a_1, a_2, \dots, a_k)$ tree in Theorem 1.2 satisfies*

$$\lambda_2 \geq \sqrt{\frac{c_k \log n}{(1+\epsilon)n}}.$$

Proof. Let $S(\rho) \equiv S_k \cup L_{k+1}$, where we recall that $L_{k+1} = \{y : d(y,\rho) = k+1\}$. We start the contact process on $S(\rho)$. When a site in S_k gives birth onto a site on L_{k+1} we freeze the particle at y_{k+1} We begin with only ρ occupied and run the process until there are no particles on S_k. These particles will be the descendants of the initial particle ρ in a branching random walk that we use to dominate the contact process on the periodic tree. When the contact process on S_k dies out we are left with frozen particles in L_{k+1}. Each frozen particle at y_{k+1} starts a new contact process on a subgraph $S(y_{k+1}) \equiv \{z : |z - y_{k+1}| \leq k+1\}$ which is isomorphic to $S(\rho)$ and has center y_{k+1}. If there are several frozen particles at the same site they start independent contact processes. Then we freeze every particle that escapes from $S(y_{k+1})$, and so on.

Let $B(S(\rho), y_{k+1})$ be the total number of particles frozen at $y_{k+1} \in L_{k+1}$ in the contact process on $S(\rho)$. Let y_1 denote the neighbor of ρ that is at distance k to y_{k+1}. When the center ρ is occupied, it gives birth to a particle at y_1 at rate λ. By the same reasoning as in (2.1), starting from a particle at y_1, to produce a particle at y_{k+1} we need a path from y_1 to y_{k+1} where in each step a birth occurs before death. If we set the center ρ to be always occupied, then we can ignore the paths from y_1 to y_{k+1} that go through ρ. Therefore, starting from a particle at y_1 the expected number of particles reaching y_{k+1} is $\leq (1+\eta)\lambda^k$ by the same computation as (2.1).

By Theorem 1.4 the expected survival time on S_k is $\leq C(\log n)e^{(1+\eta)\lambda^2 n}$. If during this whole time the center ρ is occupied and pushing particles to y_1 at rate λ, then there are an expected number of $\leq \lambda \cdot C(\log n)e^{(1+\eta)\lambda^2 n}$ times we start a process from a particle at y_1 to produce particles at y_{k+1}. Hence

$$EB(S(\rho), y_{k+1}) \leq \lambda \cdot C(\log n)e^{(1+\eta)\lambda^2 n} \cdot (1+\eta)\lambda^k \leq \lambda^{k+1} n^{(1+2\eta)c}. \tag{3.1}$$

To bound the number of particles on the periodic tree that reach the root ρ, we consider a tree consisting of the vertices of degree n in which each vertex is connected to the others vertices of degree n at distance $k+1$. This is a N-regular tree with $N = n(a_1 a_2 \cdots a_k)$. Starting from the root, there are

$$\leq \binom{2m}{m} N^m \cdot 1^m \leq 2^{2m} N^m$$

paths of length $2m$ that returns to it. So the expected number of particles returning to the root is

$$\leq 2^{2m} N^m (\lambda^{k+1} n^{(1+2\eta)c})^{2m} = (2N^{1/2} \lambda^{k+1} n^{(1+2\eta)c})^{2m} \tag{3.2}$$

We have $\log(\lambda)/\log n \to -1/2$ and $(\log N)/\log n \to 1+b$ so

$$\lim_{n\to\infty} \frac{\log(2\lambda^{k+1} N^{1/2} n^{(1+\epsilon)c})}{\log n} = -\frac{k+1}{2} + \frac{1+b}{2} + (1+2\eta)c < 0$$

if $c < (k-b)/2(1+2\eta)$. In this case, the expected number of particles that return to the origin is finite, which means the process does not survive locally. Taking $\epsilon = 2\eta$ completes the proof. $\square$

4 Lower bound on survival time on stars

Here, following the approach of Chatterjee and Durrett [2], we will reduce the contact process on a star to a one dimensional chain. We denote the state of the star by (j,k) where j is the number of occupied leaves and $k = 1, 0$ when the center is occupied, vacant. We will only look at times when the center is occupied. When the center is vacant and there are j occupied leaves, the next event will occur after exponential time with mean $1/(j\lambda + j)$. The probability that it will be a birth at the center is $\lambda/(\lambda+1)$. The probability it will be the death of a leaf particle is $1/(\lambda+1)$. Thus, the number of leaf particles Z that will be lost while the center is vacant has a shifted geometric distribution with success probability $\lambda/(\lambda+1)$, i.e.,

$$P(Z = j) = \left(\frac{1}{\lambda+1}\right)^j \cdot \frac{\lambda}{\lambda+1} \quad \text{for } j \geq 0. \tag{4.1}$$

Note that $EZ = 1/\lambda$. Since we are interested in a lower bound on the survival time, we can simply ignore the time spent when the center is vacant. Here we will construct a process X_t that gives a lower bound on the number of occupied leaves in the contact process.

Let $\delta > 0$ and $L = (1-4\delta)\lambda n$. When there are $k \leq L$ occupied leaves and the center is occupied, new leaves become occupied at rate $\lambda(n-k) \geq \lambda(n - \lambda n) \geq \lambda(1-\delta)n$ for sufficiently large n since $\lambda = \sqrt{c \log n / n} \to 0$ as $n \to \infty$. Let X_t have the following transition rates:

jump	at rate
$X_t \to X_t - 1$	L
$X_t \to \min\{X_t + 1, L\}$	$(1-\delta)\lambda n$
$X_t \to X_t - Z$	1

Here Z is independent of X_t and has the distribution given in (4.1).

Lemma 4.1. *Let $\delta > 0$. Suppose $\lambda = \sqrt{c(\log n)/n}$ and let*

$$\theta = \frac{1}{\lambda+1}\left(\lambda - \frac{1}{\delta\lambda n}\right)$$

If n is large then $h(X_t) \equiv (1-\theta)^{X_t}$ is a supermartingale when $X_t < L$.

Proof. Suppose the current value is $V=(1-\theta)^{X_t}$ where $X_t \le L=(1-4\delta)\lambda n$. We have: $V \to V/(1-\theta)$ t rate $\le L$, $V \to V(1-\theta)$ at rate $\ge (1-\delta)\lambda n$, and $V \to V(1-\theta)^{-Z}$ at rate 1. The changes in value due to the first two transitions are, if θ is small,

$$V\left(\frac{1}{1-\theta}-1\right) \le (1-\delta)^{-1}\theta V \quad \text{at rate} \le L$$
$$V[(1-\theta)-1] = -\theta V \quad \text{at rate} \ge (1-\delta)\lambda n$$

We have $L=(1-4\delta)\lambda n < (1-\delta)(1-3\delta)\lambda n$, so the first two types of jumps have a net drift

$$\left((1-\delta)^{-1}L-(1-\delta)\lambda n\right)\theta V \le -(2\delta\lambda n)\theta V. \tag{4.2}$$

In the third case, ignoring the fact that the number of occupied leaves cannot drop below 0, we have

$$\begin{aligned} E(1-\theta)^{-Z} &\le \sum_{k=0}^{\infty}\left(\frac{1}{1+\lambda}\right)^k \frac{\lambda}{1+\lambda}\cdot(1-\theta)^{-k} = \frac{\lambda}{1+\lambda}\sum_{k=0}^{\infty}\left(\frac{1}{(1+\lambda)(1-\theta)}\right)^k \\ &= \frac{\lambda}{1+\lambda}\cdot\frac{1}{1-1/(1+\lambda)(1-\theta)} = \frac{\lambda(1-\theta)}{\lambda-\theta-\theta\lambda} \end{aligned}$$

so we have $V(E(1-\theta)^{-Z}-1) = \frac{\theta V}{\lambda-\theta(1+\lambda)} = (\delta\lambda n)\theta V$ for the chosen value of θ. Combining this with (4.2) gives that for any $\delta>0$, $h(X_t)$ is a supermartingale for large n. □

We use P_i to denote the law of the process X_t starting with $X_0=i$. Since X_t omits some time intervals from the contact process on the star, the next result implies Theorem 1.5.

Lemma 4.2. *Let* $L=(1-4\delta)\lambda n$ *and* $T_a^- = \inf\{t : X_t < a\}$. *If* $\eta>0$ *is small then*

$$P_{L-1}\left(T_{\eta L}^- \ge \frac{1}{\lambda^2 n}e^{(1-4\eta)\lambda^2 n}\right) \to 1 \quad \textit{as } n\to\infty.$$

Proof. Suppose $a<x<b$ are integers. Let $T_b=\inf\{t : X_t=b\}$ and note that $X(T_b)=b$ while $X(T_a^-)\le a-1$. Since $h(X_t)$ is a supermartingale and h is decreasing

$$h(x) \ge h(a-1)P_x(T_a^- < T_b) + h(b)[1-P_x(T_a^- < T_b)]$$

Rearranging we have

$$P_x(T_a^- < T_b) \le \frac{h(x)-h(b)}{h(a-1)-h(b)}.$$

When $x=b-1$ this implies

$$P_x(T_a^- < T_b) \le \frac{h(b-1)-(1-\theta)h(b-1)}{h(a-1)-h(b-1)} = \frac{\theta h(b-1)/h(a-1)}{1-h(b-1)/h(a-1)}$$

Let $\eta>0$. We will apply this result with $b=L=(1-4\delta)\lambda n$ and $a=\eta b$. If δ is small $b\ge(1-\eta)\lambda n$. If λ is small then $1-\theta<1-(1-\eta)\lambda$. With these choices

$$h(b-1)/h(a-1) = (1-\theta)^{b-a} < (1-(1-\eta)\lambda)^{(1-2\eta)\lambda n} \le \exp\left(-(1-3\eta)\lambda^2 n\right).$$

If n is large,

$$P_{b-1}(T_a^- < T_b) \le 2\lambda\exp(-(1-3\eta)\lambda^2 n). \tag{4.3}$$

Let $G_L=\{X_t \text{ returns } (1/2\lambda)e^{(1-4\eta)\lambda^2 n} \text{ times to } L \text{ before going } <\eta L\}$. It follows from (4.3) that $P(G_L)\ge 1-e^{-\eta\lambda^2 n}$. In order to return to L we have to jump from $L-1$ to L, a time that dominates an exponential random variable with parameter $\lambda n/2$ so the law of large numbers tells us that the total amount of time before $X_t<\eta L$ is $\ge \frac{1}{\lambda^2 n}e^{(1-4\eta)\lambda^2 n}$ on G_L which completes the proof. □

5 Ignition on a star graph

In this section we will describe a mechanism for the contact process on the period tree $(n, a_1, \dots, a_k)$ to survive, which basically relies on the dynamics on the star graphs of degree n embedded in tree. Vertices with degree n will be called *hubs*.

There are three ingredients in the proof of the upper bounds on λ_2

1. survival of the process on the star graph containing a hub for a long time,
2. pushing particles from one hub to other hubs at distance $k+1$,
3. "ignition", which refers to increasing the number of occupied leaves at the new hub to L.

The first point was taken care of in the previous section. The third is covered in this one. Starting from only the central vertex of a degree n hub occupied, we need to increase the number of occupied leaves to $L = (1-4\delta)\lambda n$ by time $n^c/4$, which is referred to as the *ignition* of a hub. We treat L and K in the following lemma as integers for simplicity.

Lemma 5.1. *Suppose $\lambda = \sqrt{c_0(\log n)/n}$. Let $T_{0,0}$ be the first time the star is vacant and T_i be the first time the star has i occupied leaves. For any small $\delta > 0$ if $K = \lambda n/\sqrt{\log n}$ and $L = (1-4\delta)\lambda n$, then for large n*

$$\begin{aligned}
(i) &\quad P_{0,1}(T_K > T_{0,0}) \le 3/\sqrt{\log n}, \\
(ii) &\quad P_{K,1}(T_{0,0} < T_L) \le 2\exp(-(c_0/3)\sqrt{\log n}) \\
(iii) &\quad E_{0,1}\min\{T_{0,0}, T_L\} \le (1+\log n)/2\delta
\end{aligned}$$

Proof. Let $p_0(t)$ be the probability a leaf is occupied at time t when there are no occupied leaves at time 0 and the central vertex has been occupied for all $s \le t$. $p_0(0) = 0$ and

$$\frac{dp_0(t)}{dt} = -p_0(t) + \lambda(1-p_0(t)) = \lambda - (\lambda+1)p_0(t)$$

Solving gives $p_0(t) = \lambda(1-e^{-(\lambda+1)t})/(\lambda+1)$. As $t \to 0$, $p_0(t) \sim \lambda t$ so if t is small $p_0(t) \ge \lambda t/2$ Taking $t = 2/\sqrt{\log n}$ it follows that if $B = \text{Binomial}(n, \lambda/\sqrt{\log n})$

$$P_{0,1}(T_K < T_{0,0}) \ge P(B > K)\exp(-2/\sqrt{\log n})$$

The second factor is the probability that the center stays occupied until time $2/\sqrt{\log n}$, and $\exp(-2/\sqrt{\log n}) \ge 1 - 2/\sqrt{\log n}$. B has mean $\lambda n/\sqrt{\log n}$ and variance $\le \lambda n/\sqrt{\log n}$ so Chebyshev's inequality implies

$$P(B < \lambda n/(2\sqrt{\log n})) \le \frac{\lambda n/\sqrt{\log n}}{(\lambda n/(2\sqrt{\log n}))^2} \le \frac{4\sqrt{\log n}}{\lambda n} \le \frac{1}{\sqrt{\log n}}.$$

For (ii) we use the supermartingale $h(X_t)$ from Lemma 4.1.

$$\begin{aligned}
P_{K,1}(T_{0,0} < T_L) &\le 2(1-\lambda/3)^{\lambda n/\sqrt{\log n}} \\
&\le 2\exp(-\lambda^2 n/3\sqrt{\log n}) = 2\exp(-(c_0/3)\sqrt{\log n}).
\end{aligned}$$

For (iii) we compare with the process X_t in which we ignore the time spent when the center is vacant. To bound the time for the process X_t to reach L or die out we note that $EZ = (\lambda+1)/\lambda - 1 = 1/\lambda$ so when n is large

$$\mu = (1-\delta)\lambda n - (1-4\delta)\lambda n - 1/\lambda = 3\delta\lambda n - 1/\lambda \ge 2\delta\lambda n$$

gives a lower bound on the drift. Let $\hat{T}_{0,0}$ be the first time X_t hits 0 and $\hat{T}_L$ be the first time X_t hits L. $X_t - \mu t$ is a submartingale before time $V_L = \hat{T}_{0,} \wedge \hat{T}_L$. Stopping the submartingale $X_t - \mu t$ at the bounded stopping time $V_L \wedge s$

$$EX(V_L \wedge s) - \mu E(V_L \wedge s) \geq EX_0 = 0.$$

Since $EX(V_L \wedge s) \leq L$, it follows that $E(V_L \wedge s) \leq L/\mu$.

Letting $s \to \infty$ we have $EV_L \leq L/\mu \leq 1/2\delta$ since $L = (1-4\delta)\lambda n$ and $\mu \geq 2\delta\lambda n$. Note that the above calculation is for X_t which ignores the time when the center is vacant. To bound the time when the center is vacant, we note that the most extreme excursion that starts at n and goes to 0 takes a time with mean $(\log n)/(1+\lambda)$. During time $[0, V_L]$ the excursions occur at rate 1, that is, $E_{0,1} \min\{T_{0,0}, T_L\} \leq (1+\log n)EV_L \leq (1+\log n)/2\delta$. □

6 Upper bound on λ_2

We will prove the result for $(n, a_1, \dots, a_k)$ tree. Suppose that $\lambda = \sqrt{c(\log n)/n}$ with $c = c_k + \epsilon$ where $c_k \equiv (k-b)/2$. We select one hub to call the root.

Step 1. Pushing the particles out to distance $(k+1)m$. Lemma 4.2 implies that if there at least L occupied leaves before time $n^c/4$ then with high probability we have

$$G_0 = \{\text{there will be at least } \eta L \text{ occupied leaves during } [n^c/4, 3n^c/4]\}.$$

During this time interval the hub will try to push particles to hubs at distance $k+1$. The first step is to show that the center is never vacant for very long.

Lemma 6.1. *Suppose that the number of occupied leaves is always $\geq \eta L$ on $I = [n^c/4, 3n^c/4]$. Let $t_0 = 2/(1-4\delta)\eta$ and $G_1 = \{$there is no interval of length $\geq t_0$ in I during which the center is always vacant$\}$. As $n \to \infty$, $P(G_1) \to 1$.*

Proof. The center becomes vacant at times of a Poisson process with rate 1. Using large deviations results for the Poisson process, the probability there are more than n^c arrivals in an interval of length $n^c/2$ is $\leq \exp(-\gamma n^c)$. Suppose the center is vacant and let R be the time needed until it becomes occupied.

$$P(R > t_0) \leq \exp(-t_0 \lambda \eta L) = \exp(-2c \log n) = n^{-2c}$$

Hence $P(G_1^c) \leq \exp(-\gamma n^c) + n^c n^{-2c} \to 0$. □

When the center is occupied there is probability $e^{-1}(1-e^{-\lambda})$ that it will stay occupied for time 1 and give birth onto a given leaf within time 1. With probability e^{-1} that leaf will stay occupied until time 1. Doing this for $k+1$ times the probability of passing a particle to a given hub at distance $k+1$ is

$$\geq \left[e^{-1}(1-e^{-\lambda})e^{-1}\right]^{k+1} \geq \lambda^{k+1}/(2e^2)^{k+1} \geq C_1 \left(\frac{\log n}{n}\right)^{(k+1)/2} \tag{6.1}$$

where $C_1 = (\sqrt{c}/2e^2)^{k+1}$. Since our cycle takes time $t_1 \equiv t_0 + 2$, we have n^c/t_1 chances to do this during $[n^c/4, 3n^c/4]$. The probability that all attempts fail is

$$\leq \left(1 - C_1 \left(\frac{\log n}{n}\right)^{(k+1)/2}\right)^{n^c/t_1} \leq \exp\left(-C_2 n^{c-(k+1)/2} (\log n)^{(k+1)/2}\right) \tag{6.2}$$

where $C_2 = C_1/t_1$.

It follows from (6.2) that the probability of a successful push in $[n^c/4, 3n^c/4]$ is

$$\geq 1 - \exp\left(-C_2 n^{c-(k+1)/2} (\log n)^{(k+1)/2}\right) \geq n^{c-(k+1)/2} \quad \text{when } n \text{ is large.} \tag{6.3}$$

We say that a hub at distance $(k+1)m$ that is a descendant of the root is *wet* if it has $\geq \eta L$ occupied leaves at time mn^c. Starting with a wet hub at time 0, the center of the hub will become occupied at rate at least ηL and then Lemma 5.1 implies that the hub will be ignited within time $n^c/4$ with high probability. When the hub successfully pushes a particle to an adjacent hub during time $[n^c/4, 3n^c/4]$, that adjacent hub can ignite within the next $n^c/4$ units of time with high probability and hence be wet at time n^c. Therefore a wet hub can make an adjacent hub wet with probability $\geq n^{c-(k+1)/2}$ when n is large. The pushing events for different neighbors are not independent, but we can estimate the expected number Z_m of hubs that become wet at distance $(k+1)m$ at time mn^c. Let $N \equiv n(a_1 \cdots a_k)$. Then $EZ_m = (Nn^{c-(k+1)/2})^m$.

Step 2. Bringing a particle back to the root. To simplify notation we will write $N \equiv n(a_1 \cdots a_k)$. Let $\mathbb{T}_N$ denote an N-regular tree. We will compare with an oriented percolation in $\mathbb{T}_N \times \mathbb{Z}_+$, where the probability for successfully pushing a particle to a neighbor is $p \equiv n^{c-(k+1)/2}$ so that it is dominated by the contact process (see (6.3)). The mean number of paths N_m that go out a distance $(k+1)m$ and lead back to the origin is $EN_m = N^m p^{2m}$.

Note that the pushing events to different neighbors in the oriented percolation are not independent due to the underlying contact process. To estimate the second moment EN_m^2 we need to control the correlation between different paths back to the root. For simplicity in notation we write $a_0 = n$. For some $0 \leq l < m$ and $0 \leq r < k+1$, the number of pairs of paths from distance $(k+1)m$ back to the root that agree in the last $l(k+1)+r$ steps is

$$N^l \left(\prod_{i=0}^{r-1} a_i\right) \left(N^{m-l-1} \prod_{i=r}^{k} a_i\right)^2 \leq N^l N^{2(m-l-1)} N^2 = N^{2m-l}.$$

Since the two paths merge at distance $l(k+1)+r$ from the root, the corresponding step in the oriented percolation involves two dependent pushing events from two vertices to their common neighbor. The probability of this event is trivially upper bounded by p. Hence the probability that all the edges in the combined path are successful pushes is $\leq p^l (p^{m-l-1})^2 p = p^{2m-l-1}$. Thus the second moment of the number of successful paths out and back is

$$EN_m^2 \leq \sum_{l=0}^{m} N^{2m-l} (p^{2m-l-1})^2 \leq (Np^2)^{2m} p^{-2} \left(\sum_{l=0}^{m} (Np^2)^{-l}\right).$$

Since $c = c_k + \epsilon$, $Np^2 = n^{2(c_k+\epsilon)-(k-b)} > 1$ when n is sufficiently large, which then gives $EN_m^2 \leq Cp^{-2}(Np^2)^{2m}$ for some constant $C > 0$.

The Cauchy-Schwarz inequality implies that $E(N_m 1_{\{N_m>0\}})^2 \leq E(N_m^2) P(N_m > 0)$. Rearranging we conclude that

$$P(N_m > 0) \geq \frac{(EN_m)^2}{E(N_m^2)} \geq \frac{(Np^2)^{2m}}{Cp^{-2}(Np^2)^{2m}} = \frac{p^2}{C} = n^{2c-(k+1)}/C.$$

Since m is arbitrary we have that a particle returns to the root at arbitrarily large times in the oriented percolation. This implies that the dominating contact process survives strongly with probability $\geq n^{2c-(k+1)}/C > 0$ for any sufficiently large n. That is, $\lambda_2 \leq \sqrt{c(\log n)/n}$ when n is large enough.

7 Proof of Theorem 1.3

Case 1: $c'_k = (k+1)/2 - (b+1) > 0$**.**

We begin with the lower bound, which is proved using results from Section 3. Let $\alpha = c'_k - \epsilon$ and $\lambda = \sqrt{\alpha(\log n)/n}$. Using (3.1) it follows that the total number of frozen particles on neighboring hubs is $\leq C_0(\log n)n^{(1+\eta)\alpha}n^{1+b}\lambda^{k+1}$ for some constant C_0. From this we see that if η is small then the expected number of particles that escape from S_k is < 1 and comparing with a branching process implies that the process dies out.

Turning to the upper bound, let $\beta = c'_k + \epsilon$ and $\lambda = \sqrt{\beta(\log n)/n}$. Let $L = (1-4\delta)\lambda n$ with $\delta > 0$. Theorem 1.5 implies that starting from a "wet hub" (that has L occupied neighbors) then with high probability (i) the infection on the associated star survives for at least $\exp((1-\eta)\lambda^2 n) = n^c$ where $c = (1-\eta)\beta$, and (ii) has ηL occupied neighbors during this time. Computations in Step 3 of Section 6 imply that the probability of successfully pushing the infection to a neighboring hub during $[n^c/4, 3n^c/4]$ is $\geq n^{c-(k+1)/2}$ when n is large. The ignition result, Lemma 5.1, implies that at time n^c the new hub will have at least L occupied neighbors with high probability. The expected number of new wet neighboring hubs is $\geq n^{c+b+1-(k+1)/2}$. If η is small then under our choice of β the number of wet hubs dominates a supercritical branching process.

Case 2: $(k+1)/2 - (b+1) < 0$**.**

Again we begin with the lower bound. Suppose $\lambda = n^{-\alpha}$ with $\alpha > 1/2$. In this case, Theorem 1.4 implies that the contact process survives for $O(\log n)$ on the graph S_k. Using (3.1) again, the expected number of particles that escaped from S_k is $\leq C(\log n)n^{b+1}\lambda^{k+1} = C(\log n)n^{b+1-\alpha(k+1)}$ for some positive constant C. If $\alpha > \frac{b+1}{k+1}$ the above is < 1 when n is large. Comparing with a branching process implies the the process dies out.

Starting with the center of S_k infected, the probability that it successfully pushes the infection to a neighboring hub is $\geq C_1\lambda^{k+1}$ by (6.1). If we only use neighboring hubs that are further from the root then we can compare with a branching process whose expected number of offspring is $\geq C_1\lambda^{k+1}n^{1+b} = C_1 n^{-\alpha(k+1)+1+b}$. Hence when $\alpha < \frac{b+1}{k+1}$ the contact process dominates a supercritical branching process, which implies $\lambda > \lambda_1$.

References

[1] Berger, N., Borgs, C., Chayes, J.T. and Saberi, A. (2005) On the spread o viruses on the internet. *Proceedings of the 1CM-SIAM Symposium on Discrete Algorithms*, 301–310. MR-2298278

[2] Chatterjee, Shirshendu, and Durrett, R. (2009) Contact processes on random graphs with power law degree distributions have critical value 0. *Ann. Probab.* 37, 2332–2356. MR-2573560

[3] Huang, X. and Durret, R. (2020) The contact process on random graphs and Galton-Watson trees. *ALEA*, to appear.

[4] Jiang, Y., Kasssem, R., York G., Zhao, B., Huang, X., Junge, M., and Durrett, R. (2018) The contact process on periodic trees. arXiv:1808.01863.

[5] Liggett, T.M. (1999) *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, New York. MR-1717346

[6] Pemantle, R. (1992) The contact process on trees. *Ann. Probab.* 20, 2089–2116. MR-1188054

[7] Pemantle, R., and Stacey, A.M. (2001) The branching random walk and contact process on Galton-Watson and nonhomogeneous trees. *Ann. Probab.* 29, 1563–1590. MR-1880232