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# Large deviations related to the law of the iterated logarithm for Itô diffusions*

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**Abstract**

When a Brownian motion is scaled according to the law of the iterated logarithm, its supremum converges to one as time tends to zero. Upper large deviations of the supremum process can be quantified by writing the problem in terms of hitting times and applying a result of Strassen (1967) on hitting time densities. We extend this to a small-time large deviations principle for the supremum of scaled Itô diffusions, using as our main tool a refinement of Strassen's result due to Lerche (1986).

**Keywords:** large deviations principle; law of the iterated logarithm; boundary crossings; Itô diffusion.
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## 1 Introduction and main results

For a standard Brownian motion $W$ and

$$h(u) := \sqrt{2u \log\log \frac{1}{u}},$$

we have

$$\limsup_{t \searrow 0} \frac{W_t}{h(t)} = \lim_{t \searrow 0} \sup_{0<u<t} \frac{W_u}{h(u)} = 1 \quad \text{a.s.},$$

by Khinchin's law of the iterated logarithm, and there are extensions to the diffusion case; see the proof of Proposition 1.2 below for some references. In this note we are not interested in a.s. convergence, but rather in small-time large deviations of the process $\sup_{0<u<t} X_u/h(u)$ for an Itô diffusion $X$. For Brownian motion, a large deviations estimate follows from a result of Strassen [8], which gives precise tail asymptotics for the last (or, by time inversion, first) time at which a Brownian motion hits a smooth curve. For fixed $\varepsilon > 0$, it yields

$$P\left( \sup_{0<u<t} \frac{W_u}{h(u)} \geq \sqrt{1+\varepsilon} \right) = e^{-\varepsilon(\log\log \frac{1}{t})(1+\mathrm{o}(1))}, \quad t \searrow 0. \tag{1.1}$$

See Section 2 for details. In Theorem 2.1 below, we cite an extension of Strassen's result due to Lerche [5], which we will use when extending the estimate (1.1) to Itô diffusions. We make the following assumptions on our diffusion process. Simple sufficient conditions, just concerning smoothness and growth of $b$ and $\sigma$, are given in Proposition 1.2.

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**Assumption 1.1.** *(i) The continuous one-dimensional stochastic process $X = (X_t)_{t\geq 0}$ satisfies the SDE*

$$X_t = \int_0^t b(X_u, u)\, du + \int_0^t \sigma(X_u, u)\, dW_u, \quad t > 0, \\ X_0 = 0. \tag{1.2}$$

*(ii) The coefficients $b$ and $\sigma$ are continuous functions from $\mathbb{R} \times [0, \infty)$ to $\mathbb{R}$ with*

$$\sigma_0 := \sigma(X_0, 0) = \sigma(0, 0) > 0.$$

*(iii) The process $X$ satisfies a small-time sample path moderate deviations principle in Hölder space. More explicitly, for $1 \leq \lambda(\varepsilon) = \mathrm{o}(\varepsilon^{-1/2})$ and $\alpha \in [0, \frac{1}{2})$, the family of processes $(\sqrt{\varepsilon}\lambda(\varepsilon))^{-1}(X_{\varepsilon t})_{t\in[0,1]}$ satisfies the LDP (large deviations principle) in $C_0^\alpha([0,1], \mathbb{R})$ as $\varepsilon \searrow 0$ with speed $\lambda^2(\varepsilon)$ and good rate function*

$$\psi \mapsto \begin{cases} \|\psi\|^2_{\mathcal{H}}/(2\sigma_0^2) & \psi \in \mathcal{H}, \\ \infty & \psi \notin \mathcal{H}, \end{cases}$$

*where $\mathcal{H}$ is the one-dimensional Cameron–Martin space (see p. 260f in [3] for definitions of $C_0^\alpha$ and $\mathcal{H}$).*

*(iv) The process $X$ satisfies the small-time law of the iterated logarithm, i.e.,*

$$\limsup_{t \searrow 0} \frac{X_t}{h(t)} = \lim_{t\searrow 0} \sup_{0<u<t} \frac{X_u}{h(u)} = \sigma_0, \quad \textit{a.s.}$$

By inspecting our proofs (see Lemma 3.2 and (3.9)), it is not hard to see that the continuity assumption (ii) can be slightly weakened. We do not make this explicit, since the available sufficient conditions implying the moderate deviations principle (iii) require much smoother coefficients. In part (iv), the second equality could be replaced by $\geq$. The following proposition gives sufficient conditions for Assumption 1.1.

**Proposition 1.2.** *Suppose that the coefficients of the SDE* (1.2) *satisfy*

*(i) $b : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is continuous, continuously differentiable on the interior of its domain, and has at most linear growth, i.e. there is some $M > 0$ such that*

$$b^2(x, t) \leq M(1 + x^2 + t^2), \quad \textit{for all } (x, t) \in \mathbb{R} \times [0, \infty),$$

*(ii) $\sigma : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is locally Lipschitz continuous and of at most linear growth. Furthermore, $\sigma_0 := \sigma(0, 0) > 0$.*

*Then, the diffusion equation* (1.2) *admits a unique strong solution, and all parts of Assumption 1.1 are satisfied.*

*Proof.* It is well-known that Lipschitz and linear growth conditions (w.r.t. the space variable) imply strong existence and uniqueness, see e.g. Section 5.2 in [4]. The coefficients $b$ and $\sigma$ satisfy (A.1)–(A.3) from [3], and so (iii) follows from Corollary 4.1 in [3]. Part (iv) is a special case of the functional law of the iterated logarithm in Theorem 4.3 in [3]. See also p. 57 in [6] and p. 11 in [1]. □

**Theorem 1.3.** *Under Assumption 1.1, the process $\sup_{0<u<t} X_u/h(u)$ satisfies a small-time large deviations principle with speed $\log\log(1/t)$ and rate function*

$$J(x) := \begin{cases} (x/\sigma_0)^2 - 1 & x \geq \sigma_0, \\ \infty & x < \sigma_0. \end{cases}$$

*This means that*

$$\liminf_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in O\right) \geq -J(O) \tag{1.3}$$

*for any open set* $O$ *and*

$$\limsup_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in C\right) \leq -J(C) \tag{1.4}$$

*for any closed set* $C$*, where* $J(M) := \inf_{x\in M} J(x)$*.*

Obviously, $J$ is a *good rate function* in the sense of [2], i.e. the level sets $\{J \leq c\}$, $c \in \mathbb{R}$, are compact. The main estimate needed to prove Theorem 1.3 is contained in the following result.

**Theorem 1.4.** *Under parts* (i)–(iii) *of Assumption 1.1, for* $\varepsilon > 0$ *we have*

$$\begin{aligned} P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon}\right) &= e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))} \\ &= \left(\log\frac{1}{t}\right)^{-\varepsilon+\mathrm{o}(1)}, \quad t\searrow 0. \end{aligned}$$

After some preparations, the proofs of Theorems 1.3 and 1.4 are given at the end of Section 3. We note that part (iv) of Assumption 1.1 is not needed to prove the lower bound (1.3). Moreover, note that our approach does not easily extend to the case of a *multi-dimensional* diffusion, and so we leave this for future research. Even the case of two correlated Brownian motions is not trivial. Let $B$, $W$ be independent standard Brownian motions and $\rho \in (0,1)$. While a joint LDP for the independent processes $\sup B/h$ and $\sup W/h$ clearly holds, it is not obvious how to treat

$$\Big(\sup_{u\leq t}\frac{B_u}{h(u)},\ \sup_{u\leq t}\frac{\rho B_u + \sqrt{1-\rho^2}W_u}{h(u)}\Big).$$

## 2 Brownian motion

We can quickly see that there are positive constants $\gamma_1, \gamma_2$ (depending on $\varepsilon$) such that

$$e^{-\gamma_1(\log\log\frac{1}{t})(1+\mathrm{o}(1))} \leq P\left(\sup_{0<u<t}\frac{W_u}{h(u)} \geq \sqrt{1+\varepsilon}\right) \leq e^{-\gamma_2(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \quad t\searrow 0. \tag{2.1}$$

As for the lower estimate, note that $h(u)$ increases for small $u > 0$, and thus

$$P\left(\sup_{0<u<t}\frac{|W_u|}{h(u)} \geq \sqrt{1+\varepsilon}\right) \geq P\left(\sup_{0<u<t}|W_u| \geq \sqrt{1+\varepsilon}\,h(t)\right), \quad t \text{ small}.$$

From this and the reflection principle, it is very easy to see that we can take $\gamma_1 = \varepsilon + 1$ in (2.1). The upper estimate in (2.1) follows from applying the Borell inequality (Theorem D.1 in [7]) to the centered Gaussian process $(W_u/h(u))_{0<u<t}$, but neither of these estimates is sharp. To get the optimal constants $\gamma_1 = \gamma_2 = \varepsilon$, we use a result of Strassen [8] on boundary crossings (which is *not* directly related to Strassen's well-known functional law of the iterated logarithm). By time inversion, we have

$$\begin{aligned} P\left(\sup_{0<u<t}\frac{W_u}{h(u)} \geq \sqrt{1+\varepsilon}\right) &= P\big(\inf\{u : W_u \geq \sqrt{1+\varepsilon}\,h(u)\} \leq t\big) \\ &= P\left(\sup\{v : W_v \geq \sqrt{1+\varepsilon}\,vh(1/v)\} \geq \frac{1}{t}\right). \end{aligned}$$

Define $\varphi(v) = \sqrt{1+\varepsilon}\, vh(1/v)$. Then, by Theorem 1.2 of [8], the random variable $\sup\{v : W_v \geq \varphi(v)\}$ has a density $D_\varphi(s)$ (except possibly for some mass at zero, which is irrelevant for our asymptotic estimates), which satisfies

$$D_\varphi(s) \sim \varphi'(s)(2\pi s)^{-1/2} \exp\left(-\varphi(s)^2/2s\right), \quad s \nearrow \infty.$$

From this, the estimate (1.1) easily follows, very similarly as in the proof of Theorem 2.2 below. That theorem strengthens (1.1), replacing $\varepsilon$ by some quantity that converges to $\varepsilon$. To prove it, we apply the following theorem due to Lerche:

**Theorem 2.1** (Theorem 4.1 in [5], p. 60)**.** *Let $T_a := \inf\{u > 0 : W_u \geq \psi_a(u)\}$ for some positive, increasing, continuously differentiable function $u \mapsto \psi_a(u)$, which depends on a positive parameter $a$. Assume that there are $0 < t_1 \leq \infty$ and $0 < \alpha < 1$ such that*

*(i)* $P(T_a < t_1) \to 0$ *as* $a \nearrow \infty$,

*(ii)* $\psi_a(u)/u^\alpha$ *is monotone decreasing in* $u$ *for each* $a$,

*(iii) for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $a$

$$\left|\frac{\psi_a'(s)}{\psi_a'(u)} - 1\right| < \varepsilon \quad \text{if} \quad \left|\frac{s}{u} - 1\right| < \delta,$$

*for* $s, u \in (0, t_1)$.

*Then the density of* $T_a$ *satisfies*

$$p_a(u) = \frac{\Lambda_a(u)}{u^{3/2}} n\Big(\frac{\psi_a(u)}{\sqrt{u}}\Big)\,(1 + \mathrm{o}(1)) \tag{2.2}$$

*uniformly on* $(0, t_1)$ *as* $a \nearrow \infty$. *Here,* $n$ *is the Gaussian density*

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

*and* $\Lambda_a$ *is defined by*

$$\Lambda_a(u) := \psi_a(u) - u\psi_a'(u).$$

We can now prove the following variant of Theorem 1.4, where $X$ is specialized to Brownian motion, but $\varepsilon$ is generalized to $\varepsilon + \mathrm{o}(1)$.

**Theorem 2.2.** *Let $d(t)$ be a deterministic function with $d(t) = \mathrm{o}(1)$ as $t \searrow 0$. Then, for $\varepsilon > 0$,*

$$P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \geq \sqrt{1+\varepsilon+d(t)}\right) = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \quad t \searrow 0. \tag{2.3}$$

*Proof.* We put

$$q(t) := \sqrt{1+\varepsilon+d(t)} \tag{2.4}$$

and $a = 1/t$, to make the notation similar to [5]. We can write the probability in (2.3) as a boundary crossing probability,

$$\begin{aligned} P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \geq q(t)\right) &= P\left(\inf\left\{u > 0 : W_u \geq q(1/a)h(u)\right\} < \frac{1}{a}\right) \\ &= P\left(\inf\left\{au > 0 : W_u \geq q(1/a)h(u)\right\} < 1\right) \\ &= P\left(\inf\left\{s > 0 : W_{s/a} \geq q(1/a)h(s/a)\right\} < 1\right) \\ &= P\left(\inf\left\{s > 0 : \sqrt{a}W_{s/a} \geq q(1/a)\sqrt{a}h(s/a)\right\} < 1\right) \\ &= P\left(\inf\left\{s > 0 : W_s' \geq q(1/a)\sqrt{a}h(s/a)\right\} < 1\right), \end{aligned} \tag{2.5}$$

where $W'$ is again a Brownian motion, using the scaling property. We will verify in Lemma 2.3 below that the function

$$\psi_a(u) := q(1/a)\sqrt{a}h(u/a) \tag{2.6}$$

satisfies the assumptions of Theorem 2.1. By (2.5) and the uniform estimate (2.2), we thus obtain

$$P\left(\sup_{0<u<t}\frac{W_u}{h(u)} \geq q(t)\right) \sim \int_0^1 \frac{\Lambda_a(u)}{u^{3/2}} n\Big(\frac{\psi_a(u)}{\sqrt{u}}\Big) du, \quad a = \frac{1}{t} \nearrow \infty.$$

An easy calculation shows that

$$\Lambda_a(u) \sim const \cdot \sqrt{u \log\log\frac{a}{u}},$$

uniformly in $u \in (0,1)$, and so

$$\begin{aligned}\int_0^1 \frac{\Lambda_a(u)}{u^{3/2}} n\Big(\frac{\psi_a(u)}{\sqrt{u}}\Big) du &\sim const \cdot \int_0^1 \frac{1}{u}\sqrt{\log\log\frac{a}{u}}\Big(\log\frac{a}{u}\Big)^{-(1+\varepsilon+d(t))} du \\ &= const \cdot \int_a^\infty \frac{1}{x}\sqrt{\log\log x}\,(\log x)^{-(1+\varepsilon+d(t))} dx \\ &= const \cdot \int_a^\infty \frac{1}{x}(\log x)^{-(1+\varepsilon+\mathrm{o}(1))} dx \\ &= const \cdot (\log a)^{-\varepsilon+\mathrm{o}(1)} = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}.\end{aligned}$$

As for the third line, note that

$$\log\log x = (\log x)^{\frac{\log\log\log x}{\log\log x}},$$

and that the exponent is $\mathrm{o}(1)$ for $x \geq a$ and $a \nearrow \infty$. □

**Lemma 2.3.** *The function $\psi_a$ defined in (2.6), with $q$ defined in (2.4), satisfies the assumptions of Theorem 2.1.*

*Proof.* To verify condition (ii) of Theorem 2.1, it suffices to note that $h(u)/u^\alpha$ decreases for small $u$ and $\alpha \in (\frac{1}{2}, 1)$. The continuity condition (iii) easily follows from

$$\log(t) \sim \log(T), \quad t/T \nearrow 1, \quad t, T \nearrow \infty.$$

It remains to show condition (i), i.e., that

$$\begin{aligned}P(T_a < 1) &= P\left(\inf\left\{s > 0 : W'_s \geq q(1/a)\sqrt{a}h(s/a)\right\} < 1\right) \\ &= P\left(\sup_{0<s\leq 1}\frac{W'_s}{\sqrt{2s\log\log\frac{a}{s}}} \geq q(1/a)\right)\end{aligned} \tag{2.7}$$

converges to zero as $a \nearrow \infty$. Choose $a_0 > 0$ such that

$$q(1/a) \geq \sqrt{1 + \frac{2}{3}\varepsilon}, \quad a \geq a_0. \tag{2.8}$$

By the law of the iterated logarithm for Brownian motion, we have

$$\lim_{s_0 \searrow 0}\sup_{0<s\leq s_0}\frac{|W'_s|}{\sqrt{2s\log\log\frac{a_0}{s}}} = 1 \quad \text{a.s.}$$

From this we get that there exists an $s_0 > 0$ such that

$$\sup_{0<s\le s_0} \frac{|W'_s|}{\sqrt{2s\log\log\frac{a_0}{s}}} \le \sqrt{1+\frac{1}{2}\varepsilon} \quad \text{a.s.}$$

By monotonicity w.r.t. $a$, we obtain

$$\frac{|W'_s|}{\sqrt{2s\log\log\frac{a}{s}}} \le \frac{|W'_s|}{\sqrt{2s\log\log\frac{a_0}{s}}} \le \sqrt{1+\frac{1}{2}\varepsilon}, \quad a \ge a_0,\ s\in(0,s_0] \quad \text{a.s.} \tag{2.9}$$

For $s \in [s_0, 1]$, note that the first factor of

$$\frac{W'_s}{\sqrt{2s}} \cdot \frac{1}{\sqrt{\log\log\frac{a}{s}}}$$

is bounded pathwise, and that the second factor satisfies

$$\frac{1}{\sqrt{\log\log\frac{a}{s}}} = \frac{1}{\sqrt{\log\log a + \mathrm{o}(1)}} \to 0, \quad a \nearrow \infty,$$

uniformly on $[s_0, 1]$. From this and (2.9), we get

$$\limsup_{a\nearrow\infty} \sup_{0<s\le 1} \frac{W'_s}{\sqrt{2s\log\log\frac{a}{s}}} \le \sqrt{1+\frac{1}{2}\varepsilon},$$

and together with (2.8) this implies that (2.7) converges to zero. □

## 3 Itô diffusions

We now show that our results about Itô diffusions can be reduced to the case of Brownian motion, which was handled in the preceding section. The following easy consequence of the sample path moderate deviations principle will be used repeatedly.

**Lemma 3.1.** *Suppose that parts* (i) *and* (iii) *of Assumption 1.1 hold. Define*

$$\mathcal{A}_t := \{|X_u| \le u^{1/4},\ u \le t\}.$$

*Then there is $c > 0$ such that for $t > 0$ sufficiently small*

$$P(\mathcal{A}_t^{\mathrm{c}}) \le \exp(-c/\sqrt{t}).$$

*Proof.* For $\frac{1}{4} < \alpha < \frac{1}{2}$, it is easy to see that the map $\Phi : C_0^\alpha \to [0,\infty)$ defined by

$$\Phi(f) := \sup_{0<u\le 1} |f(u)| u^{-1/4}$$

is continuous. Using part (iii) of Assumption 1.1 with $\lambda(\varepsilon) = \varepsilon^{-1/4}$ and the contraction principle, we get that the family of random variables

$$\sup_{0<u\le 1} \frac{|X_{\varepsilon u}|}{(\varepsilon u)^{1/4}}, \quad \varepsilon > 0,$$

satisfies an LDP with speed $\varepsilon^{-1/2}$. The assertion now follows from

$$P(\mathcal{A}_t^{\mathrm{c}}) \le P\left(\sup_{u\le t} \frac{|X_u|}{u^{1/4}} \ge 1\right) = P\left(\sup_{u\le 1} \frac{|X_{\varepsilon u}|}{(\varepsilon u)^{1/4}} \ge 1\right)\Bigg|_{\varepsilon=t}. \qquad \square$$

The drift of $X$ can be easily controlled by continuity and the preceding lemma. Define

$$D_t := \sup_{0<u<t} \frac{|\int_0^u b(X_v, v)\,dv|}{h(u)}. \tag{3.1}$$

**Lemma 3.2.** *Under parts* (i)–(iii) *of Assumption 1.1, there is $c > 0$ such that for $t > 0$ sufficiently small*

$$P\left(D_t > \sqrt{t}\right) \leq \exp(-c/\sqrt{t}). \tag{3.2}$$

*Proof.* By the continuity of $b$,

$$c_1 := \sup\left\{|b(x,v)| : |x| \leq 1,\ v \leq 1\right\} < \infty.$$

Therefore, for small $t$ we have

$$\left|\int_0^u b(X_v, v)\,dv\right| \leq c_1 u, \quad u \leq t, \text{ on } \mathcal{A}_t.$$

This implies

$$D_t \leq \sup_{0<u<t} \frac{c_1 u}{h(u)} = c_1 \sqrt{\frac{t}{2\log\log\frac{1}{t}}} \quad \text{on } \mathcal{A}_t,$$

and thus

$$P\left(D_t > \sqrt{t},\ \mathcal{A}_t\right) = 0, \quad t \text{ small}.$$

Then Lemma 3.1 implies the result. □

Note that the decay rate in (3.2) is clearly negligible in comparison to (1.1). The next step in the proof of Theorem 1.4 is contained in Lemma 3.4, which allows us to deal with the local martingale part, after expressing it as a time-changed Brownian motion. We will require the following well-known result.

**Theorem 3.3** (Lévy modulus of continuity, Theorem 2.9.25 in [4])**.** *For $f(\delta) := \sqrt{2\delta\log(1/\delta)}$, we have*

$$\limsup_{\delta\searrow 0} \frac{1}{f(\delta)} \max_{\substack{0\leq s<t\leq 1\\ |t-s|\leq\delta}} |W_t - W_s| = 1 \quad \text{a.s.}$$

**Lemma 3.4.** *Suppose that parts* (i)–(iii) *of Assumption 1.1 hold. Let $\widehat{W}$ be a standard Brownian motion, and $d(t)$ a deterministic function satisfying $d(t) = \mathrm{o}(1)$ as $t \searrow 0$. Then*

$$P\left(\sup_{0<u<t} \frac{|\widehat{W}_{\langle X\rangle_u}|}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + d(t)\right) = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \tag{3.3}$$

$$P\left(\sup_{0<u<t} \frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + d(t)\right) = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \quad t \searrow 0. \tag{3.4}$$

*Proof.* Define

$$g(u) := \sup_{\substack{|x|\leq u^{1/4}\\ s<u}} \left|\sigma^2(x,s) - \sigma_0^2\right| = \mathrm{o}(1), \quad u \searrow 0.$$

Since

$$\langle X\rangle_u = \int_0^u \sigma^2(X_v, v)\,dv,$$

we conclude from the mean value theorem that

$$\big|\langle X\rangle_u - \sigma_0^2 u\big| = \hat{u}|\sigma^2(X_{\hat{u}},\hat{u}) - \sigma_0^2| \le u g(u), \quad u \le t, \tag{3.5}$$

on the event $\mathcal{A}_t$ from Lemma 3.1. The mean value theorem also implies

$$\langle X\rangle_u = \tilde{u}\sigma^2(X_{\tilde{u}},\tilde{u}) \le 2u\sigma_0^2. \tag{3.6}$$

This estimate, and (3.7)–(3.9) below, hold for $t$ sufficiently small and $u \le t$ on the event $\mathcal{A}_t$. Putting $(s,t) = (x,y)/(2\sigma_0^2 u)$ in Theorem 3.3, and using Brownian scaling, we obtain

$$\max_{\substack{0\le x<y\le 2\sigma_0^2 u \\ |y-x|\le 2\sigma_0^2 u\delta}} |\widehat{W}_y - \widehat{W}_x| \le \sigma_0\sqrt{2u}\sqrt{3\delta\log\left(\frac{1}{\delta}\right)} \tag{3.7}$$

for $\delta > 0$ sufficiently small. In particular, with $\delta := \frac{g(u)}{2\sigma_0^2}$ we get

$$\max_{\substack{0\le x<y\le 2\sigma_0^2 u \\ |y-x|\le u g(u)}} |\widehat{W}_y - \widehat{W}_x| \le \sqrt{3u g(u)\left(\log\frac{1}{g(u)} + \log(2\sigma_0^2)\right)}. \tag{3.8}$$

Together with (3.5) and (3.6), this estimate implies

$$\begin{aligned}\sup_{0<u<t} \frac{\big|\widehat{W}_{\langle X\rangle_u} - \widehat{W}_{\sigma_0^2 u}\big|}{h(u)} \\ \le \sup_{0<u<t} \frac{\sqrt{3u g(u)\left(\log\left(\frac{1}{g(u)}\right) + \log(2\sigma_0^2)\right)}}{h(u)} =: r(t) = \mathrm{o}(1), \quad t \searrow 0, \end{aligned}\tag{3.9}$$

on the event $\mathcal{A}_t$. We conclude that, for small $t$,

$$\begin{aligned} P&\left(\sup_{0<u<t} \frac{\big|\widehat{W}_{\langle X\rangle_u}\big|}{h(u)} \ge \sigma_0\sqrt{1+\varepsilon} + d(t),\ \mathcal{A}_t\right) \\ &\le P\left(\sup_{0<u<t} \frac{\big|\widehat{W}_{\sigma_0^2 u}\big|}{h(u)} \ge \sigma_0\sqrt{1+\varepsilon} + d(t) - r(t)\right) \\ &= P\left(\sup_{0<u<t} \frac{\big|\widetilde{W}_u\big|}{h(u)} \ge \sqrt{1+\varepsilon} + \frac{d(t)-r(t)}{\sigma_0}\right) \\ &\le 2P\left(\sup_{0<u<t} \frac{\widetilde{W}_u}{h(u)} \ge \sqrt{1+\varepsilon} + \frac{d(t)-r(t)}{\sigma_0}\right), \end{aligned}$$

where $\widetilde{W}$ is again a Brownian motion. Now the upper estimate in (3.3) follows from Theorem 2.2 and Lemma 3.1. To complete the proof of the lemma, a lower estimate for the left-hand side of (3.4) is needed. We have

$$\sup_{0<u<t} \frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \ge \sup_{0<u<t} \frac{\widehat{W}_{\sigma_0^2 u}}{h(u)} - \sup_{0<u<t} \frac{\big|\widehat{W}_{\langle X\rangle_u} - \widehat{W}_{\sigma_0^2 u}\big|}{h(u)},$$

and thus, by (3.9),

$$\begin{aligned} P\left(\sup_{0<u<t}\frac{\widehat{W}_{\langle X\rangle_u}}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}+d(t),\ \mathcal{A}_t\right) &\geq P\left(\sup_{0<u<t}\frac{\widehat{W}_{\sigma_0^2 u}}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}+d(t)+r(t),\ \mathcal{A}_t\right)\\ &\geq P\left(\sup_{0<u<t}\frac{\widetilde{W}_u}{h(u)}\geq \sqrt{1+\varepsilon}+\frac{d(t)+r(t)}{\sigma_0}\right)-P\left(\mathcal{A}_t^{\mathrm{c}}\right), \end{aligned} \tag{3.10}$$

using $P(A\cap B)\geq P(A)-P(B^{\mathrm{c}})$. The first probability in (3.10) can be estimated by Theorem 2.2, and the second probability in (3.10) is asymptotically smaller by Lemma 3.1. □

We now conclude the paper by proving our main results, Theorem 1.4 and its consequence, Theorem 1.3.

*Proof of Theorem 1.4.* Recalling the definition of $D_t$ in (3.1), we have

$$P\left(\sup_{0<u<t}\frac{X_u}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}\right)\leq P\left(\sup_{0<u<t}\frac{\left|\int_0^u \sigma(X_v,v)\,dW_v\right|}{h(u)}+D_t\geq \sigma_0\sqrt{1+\varepsilon}\right). \tag{3.11}$$

By the Dambis–Dubins–Schwarz theorem (Theorem 3.4.6 and Problem 3.4.7 in [4]), the local martingale can be written as

$$\int_0^u \sigma(X_v,v)\,dW_v=\widehat{W}_{\langle X\rangle_u} \tag{3.12}$$

with a Brownian motion $\widehat{W}$. The upper estimate thus follows from applying Lemma 3.2 and (3.3) to (3.11). We proceed with the lower estimate in Theorem 1.4. From

$$\sup_{0<u<t}\frac{X_u}{h(u)}\geq \sup_{0<u<t}\frac{\int_0^u \sigma(X_v,v)\,dW_v}{h(u)}-\sup_{0<u<t}\frac{\left|\int_0^u b(X_v,v)\,dv\right|}{h(u)}$$

and (3.12), we get

$$P\left(\sup_{0<u<t}\frac{X_u}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}\right)\geq P\left(\sup_{0<u<t}\frac{\widehat{W}_{\langle X\rangle_u}}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}+D_t\right).$$

Since we need a lower bound, we can intersect with the event $D_t\leq\sqrt{t}$. Using $P(A\cap B)\geq P(A)-P(B^{\mathrm{c}})$, we obtain

$$\begin{aligned} P\left(\sup_{0<u<t}\frac{\widehat{W}_{\langle X\rangle_u}}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}+D_t\right) &\geq P\left(\sup_{0<u<t}\frac{\widehat{W}_{\langle X\rangle_u}}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}+\sqrt{t},\ D_t\leq\sqrt{t}\right)\\ &\geq P\left(\sup_{0<u<t}\frac{\widehat{W}_{\langle X\rangle_u}}{h(u)}\geq \sigma_0\sqrt{1+\varepsilon}+\sqrt{t}\right)-P(D_t>\sqrt{t}). \end{aligned}$$

The lower estimate now follows from Lemma 3.2 and (3.4). □

*Proof of Theorem 1.3.* First, let $C\subseteq\mathbb{R}$ be a closed set. Then, the increasing process $\sup_{0<u<t}X_u/h(u)$ converges to $\sigma_0$ as $t\searrow 0$ by part (iv) of Assumption 1.1, and hence its values are $\geq\sigma_0$ a.s.; note that this is the only place where part (iv) is used. Moreover, the rate function satisfies $J(C)=J(C\cap[\sigma_0,\infty))$. We may thus assume $C\subseteq[\sigma_0,\infty)$.

If $\inf C = \sigma_0$, then $J(C) = 0$, and it suffices to estimate the probability in (1.4) by $1$. Otherwise, let $\sigma_0\sqrt{1+\kappa} := \inf C$ with $\kappa > 0$. Then, by Theorem 1.4,

$$\begin{aligned}\limsup_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in C\right) \\ &\le \limsup_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \ge \sigma_0\sqrt{1+\kappa}\right) \\ &= -\kappa = -J(C).\end{aligned}$$

Now consider an open set $O \neq \emptyset$, and define $\tilde{O} := O \cap [\sigma_0, \infty)$. It is clear that $J(O) = J(\tilde{O})$. If $\tilde{O} = \emptyset$, then $J(O) = J(\tilde{O}) = \infty$, and so the lower bound is trivial. Hence we may suppose that $\tilde{O} \neq \emptyset$. For arbitrary $\lambda > 0$, we can pick $x > 1$ and $\delta > 0$ such that

$$\inf \tilde{O} < \sigma_0\sqrt{x-\delta} < \sigma_0\sqrt{x+\delta} < \inf \tilde{O} + \lambda$$

and

$$\big(\sigma_0\sqrt{x-\delta}, \sigma_0\sqrt{x+\delta}\big) \subseteq \tilde{O}.$$

Then,

$$\begin{aligned}P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in O\right) &\ge P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in \big(\sigma_0\sqrt{x-\delta}, \sigma_0\sqrt{x+\delta}\big)\right) \\ &= P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \ge \sigma_0\sqrt{x-\delta}\right) - P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \ge \sigma_0\sqrt{x+\delta}\right) \\ &= e^{-(x-\delta-1)(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \quad t \searrow 0,\end{aligned}$$

by Theorem 1.4. Therefore,

$$\begin{aligned}\liminf_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in O\right) &\ge -(x-\delta-1) \\ &\ge -\Big(\frac{\inf \tilde{O} + \lambda}{\sigma_0}\Big)^2 + 1 = -J(\tilde{O}) + \mathrm{O}(\lambda),\ \lambda \searrow 0.\end{aligned}$$

As $J(O) = J(\tilde{O})$, this yields (1.3). □

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