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Generalized scale functions of standard processes with no positive jumps*

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Abstract

As a generalization of scale functions of spectrally negative Lévy processes, we define generalized scale functions of general standard processes with no positive jumps. For this purpose, we utilize the excursion theory. Using the generalized scale functions, we study Laplace transforms of hitting times, potential measures and duality.

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1 Introduction

We first recall the basic facts of scale functions of spectrally negative Lévy processes. Let $(X, \mathbb{P}_x^X)$ with $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process with $\mathbb{P}_x^X(X_0 = x) = 1$, i.e., a Lévy process which has neither positive jumps nor monotone paths. Then, there correspond to it the Laplace exponent Ψ and the q-scale function $W^{(q)}$ of X for all $q \geq 0$. The Laplace exponent Ψ is a function from $[0, \infty)$ to $\mathbb{R}$ defined by

$$\Psi(\lambda) = \log \mathbb{E}_0^X \left[e^{\lambda X_1} \right], \qquad \lambda \geq 0. \tag{1.1}$$

The q-scale function $W^{(q)}$ is a function which is equal to 0 on $(-\infty, 0)$, is continuous on $[0, \infty)$, and satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\Psi(\beta) - q}, \qquad \beta > \Phi(q), \tag{1.2}$$

where $\Phi(q) = \inf\{\lambda > 0 : \Psi(\lambda) > q\}$ (see, e.g., [12, Section 8] for the details). The scale function is useful since the Laplace transforms of hitting times and the q-potential measure can be characterized as follows: for $b < x < a$,

$$\mathbb{E}_x^X \left[e^{-qT_a^+}; T_a^+ < T_b^- \right] = \frac{W^{(q)}(x-b)}{W^{(q)}(a-b)}, \tag{1.3}$$

$$\mathbb{E}_x^X \left[\int_0^{T_a^+ \wedge T_b^-} e^{-qt} f(X_t) dt \right] = \int_b^a f(y) \left(\frac{W^{(q)}(x-b)}{W^{(q)}(a-b)} W^{(q)}(a-y) - W^{(q)}(x-y) \right) dy, \tag{1.4}$$

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where T_a^+ and T_b^- denote the first hitting times of $[a,\infty)$ and $(-\infty,b]$, respectively. Many other properties of the scale functions can be found in Kyprianou [12] and Kuznetsov–Kyprianou–Rivero [11].

The scale functions have a close connection to excursion measures. Bertoin [2, Proposition VII.5] connected $W^{(0)}$ with an excursion measure $\underline{n}$ of the reflected process of X at 0 as

$$W^{(0)}(x) \propto \frac{1}{\underline{n}\left[T_x^+ < \infty\right]}, \quad x > 0, \tag{1.5}$$

where by $\propto$ we mean that both sides coincide up to a multiplicative constant. The identity (1.5) has some applications (see, e.g., Doney [6]). Recently, Pardo–Pérez–Rivero [16] connected the two excursion measures of the original process and of the reflected process up to a multiplicative constant. This study was applied for insurance mathematics (see, e.g., Pardo–Pérez–Rivero [15] or Avram–Pérez–Yamazaki [1]). Avram–Pérez–Yamazaki [1] applied some results of [15] to prove the identity

$$W^{(q)}(x) \propto \frac{1}{n\left[e^{-qT_x^+}; T_x^+ < \infty\right]}, \qquad q \geq 0, \ \ x > 0, \tag{1.6}$$

where n is an excursion measure away from 0 of the original process.

In Noba–Yano [14, Section 3], we removed the ambiguity of the multiplicative constant in (1.6). When X has unbounded variation paths, we proved that the scale functions satisfy

$$W^{(q)}(x) = \frac{1}{n^X\left[e^{-qT_x^+}; T_x^+ < \infty\right]}, \qquad q \geq 0, \ \ x > 0, \tag{1.7}$$

where n^X is an excursion measure away from 0 subject to the normalization

$$n^X\left[1 - e^{-qT_0}\right] = \frac{1}{\Phi'(q)}, \qquad q > 0, \tag{1.8}$$

with T_0 denoting the first hitting time of 0.

In this paper, we generalize the scale functions for Lévy processes to those for standard processes (for the definition of the standard processes, see, e.g., [4, Definition 1.9.2]). Let $(X, \mathbb{P}_x^X)$ with $X = \{X_t : t \geq 0\}$ and $\mathbb{P}_x^X(X_0 = x) = 1$ be a standard process with no positive jumps. For $q \geq 0$, we define the *generalized q-scale function* of X as

$$W_X^{(q)}(x,y) = \begin{cases} \frac{1}{n_y^X\left[e^{-qT_x^+}; T_x^+ < \infty\right]}, & x \geq y, \\ 0, & x < y, \end{cases} \tag{1.9}$$

where T_x^+ denotes the upward hitting time of level x and n_y^X the excursion measure away from y under suitable normalization. In this paper, we investigate exit problems, potential densities, and duality of the generalized scale functions. This is in fact a generalization of the usual scale functions. When X is a spectrally negative Lévy process, we have

$$W_X^{(q)}(x,y) = W^{(q)}(x-y), \quad q \geq 0, \ x,y \in \mathbb{R}. \tag{1.10}$$

We prove (1.10) in Section A. (What we have to show is that n^X coincides with n_0^X, i.e., the coincidence of the two normalization requirements.)

We prove that the generalized scale functions characterize the two-sided exit problem and the killed potential densities of X as follows: for $q \geq 0$ and $x \in (b,a)$, we have

$$\mathbb{E}^X_x\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \frac{W^{(q)}_X(x,b)}{W^{(q)}_X(a,b)}, \tag{1.11}$$

$$\mathbb{E}^X_x\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL^{X,y}_t\right] = \frac{W^{(q)}_X(x,b)}{W^{(q)}_X(a,b)} W^{(q)}_X(a,y) - W^{(q)}_X(x,y), \tag{1.12}$$

where $L^{X,y}_t$ denotes the local time at $y \in (b,a)$ under the normalization corresponding to n^X_y.

As the standardness assumption is too weak to exclude processes that may not have a manageable normalization of local times, the duality is a useful condition which assures regularity of the resolvent. We give necessary and sufficient conditions of duality in terms of generalized scale functions. For two standard processes X and $\widehat{X}$ such that X and $-\widehat{X}$ have no positive jumps, we prove that X and $\widehat{X}$ are in duality if and only if for all x, y, we have

$$W^{(q)}_X(x,y) = W^{(q)}_{-\widehat{X}}(-y,-x). \tag{1.13}$$

This duality problem naturally arises with our generalization. When X is a spectrally negative Lévy process, we know that X and $-X$ are in duality relative to the Lebesgue measure and so the identity (1.13) automatically follows from (1.10).

There were two earlier studies on generalization of scale functions for some modified classes of Lévy processes. Kyprianou–Loeffen [13] introduced the refracted Lévy processes via a stochastic differential equation and the corresponding scale functions. Noba–Yano [14] generalized their results via the excursion theory.

The organization of this paper is as follows. In Section 2, we prepare some notations and recall preliminary facts about standard processes, local times, and excursion measures. In Section 3, we give the definition of the generalized scale functions and apply them to the exit problems and the potential measures. In Section 4, we study the duality problem. In Section A, we prove (1.10).

2 Preliminaries

Let $\mathbb{D}$ denote the set of functions $\omega : [0,\infty) \to \mathbb{R} \cup \{\partial\}$ which are càdlàg and satisfy $\omega(t) = \partial$, $t \geq \zeta$, where $\mathbb{R} \cup \{\partial\}$ is the one-point compactification of $\mathbb{R}$, $\zeta = \inf\{t > 0 : \omega(t) = \partial\}$ and $\inf \emptyset = \infty$. Let $\mathcal{B}(\mathbb{D})$ denote the class of Borel sets of $\mathbb{D}$ equipped with the Skorokhod topology. For $\omega \in \mathbb{D}$, we write

$$T^-_x(\omega) := \inf\{t > 0 : \omega(t) \leq x\}, \tag{2.1}$$

$$T^+_x(\omega) := \inf\{t > 0 : \omega(t) \geq x\}, \tag{2.2}$$

$$T_x(\omega) := \inf\{t > 0 : \omega(t) = x\}. \tag{2.3}$$

We sometimes write T^-_x, T^+_x, T_x simply for $T^-_x(X)$, $T^+_x(X)$, $T_x(X)$, respectively, when we consider these times for the process $(X, \mathbb{P}^X_x)$. Let $\mathbb{T}$ be an interval of $\mathbb{R}$ and set $a_0 = \sup \mathbb{T}$ and $b_0 = \inf \mathbb{T}$. We assume that the process $(X, \mathbb{P}^X_x)$ considered in this paper is a $\mathbb{T}$-valued standard process with no positive jumps with $\mathbb{P}^X_x(X_0 = x) = 1$, satisfying the following conditions:

(A1) $(x,y) \mapsto \mathbb{E}^X_x\left[e^{-T_y}\right] > 0$ is a $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T})$-measurable function.

(A2) X has a reference measure m on $\mathbb{T}$, i.e. for $q \geq 0$ and $x \in \mathbb{T}$, we have $R_X^{(q)} 1_{(\cdot)}(x) \ll m(\cdot)$, where

$$R_X^{(q)} f(x) := \mathbb{E}_x^X \left[\int_0^\infty e^{-qt} f(X_t) dt \right] \tag{2.4}$$

for non-negative measurable function f. Here and hereafter we use the notation $\int_b^a = \int_{(b,a]\cap\mathbb{R}}$. In particular, $\int_{b-}^a = \int_{[b,a]\cap\mathbb{R}}$.

By [7, Theorem 18.4], there exists a family of processes $\{L^{X,x}\}_{x\in\mathbb{T}}$ with $L^{X,x} = \left\{ L_t^{X,x} \right\}_{t\geq 0}$ for $x \in \mathbb{T}$, which we call *local times*, such that the following conditions hold: for all $q > 0$, $x \in \mathbb{T}$ and non-negative measurable function f

$$\int_0^t f(X_s) ds = \int_{\mathbb{T}} f(y) L_t^{X,y} m(dy), \quad \text{a.s.} \tag{2.5}$$

$$R_X^{(q)} f(x) = \int_{\mathbb{T}} f(y) \mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right] m(dy). \tag{2.6}$$

We have the following two cases:

- **Case 1.** If $x \in \mathbb{T}$ is regular for itself (for the definitions of “regular for itself” and “irregular for itself”, see, e.g., [2, pp. 65]), this $L^{X,x}$ is the continuous local time at x ([4, pp. 216]). Note that $L^{X,x}$ has no ambiguity of multiplicative constant because of (2.5) or (2.6).
- **Case 2.** If $x \in \mathbb{T}$ is irregular for itself, we have

$$L_t^{X,x} = l_x^X \#\{0 \leq s < t : X_s = x\}, \quad \text{a.s.} \tag{2.7}$$

for some constant $l_x^X \in (0, \infty)$.

In Case 1, let $\eta^{X,x}$ denote the inverse local time of $L^{X,x}$, i.e., $\eta^{X,x}(t) = \inf\{s > 0 : L_s^{X,x} > t\}$. Let n_x^X be the excursion measure away from x which is associated with $L^{X,x}$. For background on general excursion theory, see Itô [9], [2, Section IV], [3] and [10]. Then, for all $q > 0$, we have

$$-\log \mathbb{E}_x^X \left[e^{-q\eta^{X,x}(1)} \right] = \delta_x^X q + n_x^X \left[1 - e^{-qT_x} \right] \tag{2.8}$$

for a non-negative constant δ_x^X called the *stagnancy rate* (see, e.g., [2, Theorem IV.8]). We thus have

$$\mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,x} \right] = \mathbb{E}_x^X \left[\int_0^\infty e^{-q\eta^{X,x}(s)} ds \right] = \frac{1}{\delta_x^X q + n_x^X [1 - e^{-qT_x}]}. \tag{2.9}$$

In Case 2, we define $n_x^X = \frac{1}{l_x^X} \mathbb{P}_x^{X^x}$ where $\mathbb{P}_x^{X^x}$ denotes the law of X started from x and stopped at x. Then we have

$$\mathbb{E}_x^X \left[\int_{0-}^\infty e^{-qt} dL_t^{X,x} \right] = l_x^X \sum_{i=0}^\infty \left(\mathbb{E}_x^X \left[e^{-qT_x} \right] \right)^i = \frac{l_x^X}{\mathbb{E}_x^X [1 - e^{-qT_x}]} = \frac{1}{n_x^X [1 - e^{-qT_x}]}. \tag{2.10}$$

Remark 2.1. Any point $x \in \mathbb{T} \backslash \{a_0\}$ cannot be a holding point. To demonstrate, assume x is a holding point. Then X leaves x by jumps (see, e.g., [18, Theorem 1 (vi)]). However, X has no positive jumps, and thus X can not exceed x, which contradicts (A1).

3 Generalized scale functions

We define generalized scale functions for standard processes with no positive jumps using the excursion theory. In addition, we characterize the fluctuations of standard processes with no positive jumps using the generalized scale functions.

Definition 3.1. *For $q \geq 0$ and $x, y \in \mathbb{T}$, we define generalized q-scale function of X as*

$$W_X^{(q)}(x,y) = \begin{cases} \frac{1}{n_y^X\left[e^{-qT_x^+};T_x^+<\infty\right]}, & x \geq y, \\ 0, & x < y, \end{cases} \tag{1.9}$$

where $\frac{1}{\infty} = 0$.

Remark 3.2. Each $x \in \mathbb{T}\backslash\{b_0\}$ is regular for (x, ∞), i.e., $\mathbb{P}_x^X(\tau_x^+ = 0) = 1$ with $\tau_x^+ = \{t > 0 : X_t > x\}$, thanks to the assumptions of no positive jumps and (A1). Indeed, we have

$$\mathbb{E}_x\left[e^{-\tau_x^+}\right] = \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\tau_x^+}\right]\middle|\tau_x^+ < \infty\right] \tag{3.1}$$

$$= \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_x^+}}\left[e^{-\tau_x^+}\right]\middle|\tau_x^+ < \infty\right] = \mathbb{E}_x\left[\left(e^{-\tau_x^+}\right)\circ\theta_{\tau_x^+}\middle|\tau_x^+ < \infty\right] = 1. \tag{3.2}$$

When x is irregular for itself, we have $W_X^{(q)}(x,x) = l_x^X$ by the definition of n_x^X.

Remark 3.3. Let us characterize the generalized scale functions of diffusion processes in terms of their characteristics. Let m and s be two $\mathbb{R}$-valued strictly increasing continuous functions on the interval $[0, \infty)$ satisfying $s(0) = 0$. Let X be a $\frac{d}{dm}\frac{d}{ds}$-diffusion process with 0 being a reflecting boundary. Note that our n_0^X coincides with the excursion measure defined in [19, Definition 2.1] up to scale transformation. Let $\psi^{(q)}$ denote the increasing eigenfunction $\frac{d}{dm}\frac{d}{ds}\psi^{(q)} = q\psi^{(q)}$ such that $\frac{d}{ds}\psi^{(q)}(0) = 1$. In other words, $\psi^{(q)}$ is the unique solution of the integral equation

$$\psi^{(q)}(x) = s(x) + q\int_0^x (s(x) - s(y))\psi^{(q)}(y)dm(y), \qquad x \in [0, \infty). \tag{3.3}$$

Then, by [19, Corollary 2.4], for $q > 0$ and $x \in (0, \infty)$, we have

$$\psi^{(q)}(x) = \frac{1}{n_0^X\left[e^{-qT_x^+};T_x^+ < \infty\right]}, \tag{3.4}$$

which shows that $W_X^{(q)}(x, 0) = \psi^{(q)}(x)$. In particular, we have $W_X^{(0)}(x, 0) = s(x)$.

We fix $b, a \in \mathbb{T}$ with $b < a$. The exit from the upper barrier is characterized as follows.

Theorem 3.4. *For $q \geq 0$ and $x \in (b, a)$, we have*

$$\mathbb{E}_x^X\left[e^{-qT_a^+};T_a^+ < T_b^-\right] = \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)}. \tag{1.11}$$

Proof. Since $b < x < a$ and since X has no positive jumps, we have

$$n_b^X\left[e^{-qT_a^+};T_a^+ < \infty\right] = n_b^X\left[e^{-qT_x^+};T_x^+ < \infty\right]\mathbb{E}_x^X\left[e^{-qT_a^+};T_a^+ < T_b^-\right], \tag{3.5}$$

where we utilized the strong Markov property of n_b^X (see, e.g., [3]). □

In order to obtain killed potential density, we need the following lemma.

Lemma 3.5. *For $q \geq 0$ and $x \in (b, a)$, we have*

$$\mathbb{E}_x^X\left[e^{-qT_a^+};T_a^+ < T_b^-\right] = n_x^X\left[e^{-qT_a^+};T_a^+ < \infty\right]\mathbb{E}_x^X\left[\int_{0-}^{T_a^+\wedge T_b^-} e^{-qt}dL_t^{X,x}\right]. \tag{3.6}$$

Proof. The proof is almost the same as that of [14, Lemma 6.1], but a slight difference lies in presence of stagnancy.

$i)$ We assume that x is regular for itself.

We use the notations which were used the notations which was used in in [8, Section I.9]. Let $p : D(p) \to \mathbb{D}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ denote a Poisson point process with characteristic measure n_x^X. We write $\eta^{X,x}(s) = \delta_x^X s + \sum_{u \leq s} T_x(p(u))$. We set $A = \{T_a^+ < \infty\} \cup \{T_b^- < \infty\} \cup \{\zeta < \infty\}$ and $\kappa_A = \inf\{s > 0 : p(s) \in A\}$. By the same argument as in the proof of [14, Lemma 6.1], we have

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \mathbb{E}\left[e^{-q\eta^{X,x}(\kappa_A -)}\right] \frac{n_x^X\left[e^{-qT_a^+}; A\right]}{n_x^X[A]}. \tag{3.7}$$

We denote $p_A = p|_{D(p_A)}$ with $D(p_A) = \{s \in D(p) : p(s) \in A\}$. We write $\eta_{A^c}^{X,x}(s) = \eta^{X,x}(s) - \sum_{u \leq s} T_x(p_A(u))$ where $T_x(\partial) = 0$. Note that $\eta^{X,x}(t) = \eta_{A^c}^{X,x}(t)$ for $t < \kappa_A$ and that $\eta_{A^c}^{X,x}$ and κ_A are independent. We thus have

$$\mathbb{E}\left[e^{-q\eta^{X,x}(\kappa_A -)}\right] = n_x^X[A]\,\mathbb{E}\left[\int_0^{\kappa_A} \exp\bigl(-q\eta^{X,x}(t)\bigr)\,dt\right] \tag{3.8}$$

$$= n_x^X[A]\,\mathbb{E}_x^X\left[\int_{0-}^{T_a^+ \wedge T_b^-} e^{-qt} dL_t^{X,x}\right], \tag{3.9}$$

where we used the fact that $\mathbb{P}[\kappa_A > t] = e^{-t n_x^X[A]}$, and the identity

$$\mathbb{E}[f(\mathbf{e}_q)] = q\mathbb{E}\left[\int_0^{\mathbf{e}_q} f(t)dt\right] \tag{3.10}$$

for an exponential variable with $\mathbb{P}[\mathbf{e}_q > t] = e^{-tq}$ and a non-negative measurable function f. Therefore, we obtain (3.6).

$ii)$ We assume that x is irregular for itself.

Let $T_x^{(n)}$ denote the n-th hitting time to x and let $T_x^{(0)} = 0$. Then we have

$$\mathbb{E}_x^X\left[\int_{0-}^{T_a^+ \wedge T_b^-} e^{-qt} dL_t^{X,x}\right] = l_x^X \sum_{i=0}^{\infty} \mathbb{E}_x^X\left[e^{-qT_x^{(i)}}; T_x^{(i)} < T_a^+ \wedge T_b^-\right] \tag{3.11}$$

$$= l_x^X \sum_{i=0}^{\infty} \bigl(\mathbb{E}_x^X\left[e^{-qT_x}; T_x < T_a^+ \wedge T_b^-\right]\bigr)^i. \tag{3.12}$$

On the other hand, we have

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \sum_{i=0}^{\infty} \bigl(\mathbb{E}_x^X\left[e^{-qT_x}; T_x < T_a^+ \wedge T_b^-\right]\bigr)^i \mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_x \wedge T_b^-\right]. \tag{3.13}$$

Therefore, we obtain (3.6). □

By Theorem 3.4 and Lemma 3.5, for $q \geq 0$ and $x \in (b, a)$, we obtain

$$\mathbb{E}_x^X\left[\int_{0-}^{T_a^+ \wedge T_b^-} e^{-qt} dL_t^{X,x}\right] = \frac{W_X^{(q)}(x,b) W_X^{(q)}(a,x)}{W_X^{(q)}(a,b)}. \tag{3.14}$$

For $q \geq 0$, $x \in (b, a)$ and non-negative measurable function f, we define

$$\underline{\overline{R}}_X^{(q;b,a)} f(x) := \mathbb{E}_x^X\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} f(X_t) dt\right]. \tag{3.15}$$

Then, for $q \geq 0$, we have

$$\underline{\overline{R}}_X^{(q;b,a)} f(x) = \int_{(b,a)} f(y) \mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] m(dy). \tag{3.16}$$

The following theorem represents the potential density in terms of the generalized scale functions.

Theorem 3.6. *For* $q \geq 0$ *and* $x, y \in (b, a)$*, we have*

$$\mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] = \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y) - W_X^{(q)}(x,y). \tag{1.12}$$

Proof. $i)$ Let us consider the case where $x = y$.

When x is regular for itself, the continuity of the local time implies that

$$\mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] = \mathbb{E}_x^X \left[\int_{0-}^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right], \tag{3.17}$$

and the absence of positive jumps implies that

$$W_X^{(q)}(x,x) = \frac{1}{n_x^X \left[e^{-qT_x^+}; T_x^+ < \infty \right]} = \frac{1}{n_x^X \left[T_x^+ < \infty \right]} = 0. \tag{3.18}$$

Thus, (1.12) follows from (3.14).

When x is irregular for itself, we have

$$\mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] = \mathbb{E}_x^X \left[\int_{0-}^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] - l_x^X. \tag{3.19}$$

By (3.14) and Remark 3.2, we obtain (1.12).

$ii)$ Let us consider the case where $x \neq y$.

On one hand, we have

$$\mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] = \mathbb{E}_x^X \left[e^{-qT_y}; T_y < T_b^- \wedge T_a^+ \right] \mathbb{E}_y^X \left[\int_{0-}^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right]. \tag{3.20}$$

On the other hand, we can prove

$$\mathbb{E}_x^X \left[e^{-qT_y}; T_y < T_b^- \wedge T_a^+ \right] = \frac{W_X^{(q)}(a,b)}{W_X^{(q)}(y,b)} \left(\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} - \frac{W_X^{(q)}(x,y)}{W_X^{(q)}(a,y)} \right). \tag{3.21}$$

Indeed, for $x < y$, this is obvious, and, and for $x > y$ the left-hand-side of (3.21) equals

$$\frac{\mathbb{E}_x^X \left[e^{-qT_a^+}; T_y < T_a^+ < T_b^- \right]}{\mathbb{E}_y^X \left[e^{-qT_a^+}; T_a^+ < T_b^- \right]} = \frac{\mathbb{E}_x^X \left[e^{-qT_a^+}; T_a^+ < T_b^- \right] - \mathbb{E}_x^X \left[e^{-qT_a^+}; T_a^+ < T_y^- \right]}{\mathbb{E}_y^X \left[e^{-qT_a^+}; T_a^+ < T_b^- \right]}, \tag{3.22}$$

which leads to (3.21) by Theorem 3.4. Combining (3.20), (3.14) and (3.21), we obtain (1.12). □

The exit from the lower barrier is characterized as follows.

Corollary 3.7. *For $x, y \in (b_0, a_0)$, we define*

$$Z_X^{(q)}(x,y) = \begin{cases} 1 + q\int_{(y,x)} W_X^{(q)}(x,z)m(dz), & x > y, \\ 1, & x \le y. \end{cases} \tag{3.23}$$

Then we have

$$\mathbb{E}_x^X\left[e^{-qT_b^-}; T_b^- < T_a^+\right] = Z_X^{(q)}(x,b) - \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} Z_X^{(q)}(a,b). \tag{3.24}$$

Proof. We have

$$\mathbb{E}_x^X\left[e^{-qT_b^-}; T_b^- < T_a^+\right] = \mathbb{E}_x^X\left[e^{-q(T_b^- \wedge T_a^+)}; T_b^- \wedge T_a^+ < \infty\right] - \mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right]. \tag{3.25}$$

By Theorem 3.6 and by the identity $e^{-qs} = 1 - q\int_0^s e^{-qt}dt$, we have

$$\mathbb{E}_x^X\left[e^{-q(T_b^- \wedge T_a^+)}\right] = 1 - q\int_{(b,a)} \left(\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y) - W_X^{(q)}(x,y)\right) m(dy). \tag{3.26}$$

By (3.25), (3.26) and Theorem 3.4, we have

$$(3.25) = 1 + q\int_{(b,a)} W_X^{(q)}(x,y)m(dy) - \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)}\left(1 + q\int_{(b,a)} W_X^{(q)}(a,y)m(dy)\right), \tag{3.27}$$

and therefore, we obtain (3.24). □

4 Relation between duality and generalized scale functions

In this section, we give the necessary and sufficient conditions of duality in terms of generalized scale functions. To state our main result, we need several facts about duality.

Let X be a $\mathbb{T}$-valued standard process with no positive jumps satisfying (A1) and (A2). Let $(\widehat{X}, \mathbb{P}_x^{\widehat{X}})$ with $\widehat{X} = \left\{\widehat{X}_t : t \ge 0\right\}$ be a $\mathbb{T}$-valued standard process with no negative jumps satisfying the following conditions:

(B1) $(x,y) \mapsto \mathbb{E}_x^{\widehat{X}}\left[e^{-T_y}\right] > 0$ is a $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T})$-measurable function.

(B2) $\widehat{X}$ has a reference measure m on $\mathbb{T}$.

For $q \ge 0$ and non-negative measurable function f, we denote

$$R_{\widehat{X}}^{(q)} f(x) = \mathbb{E}_x^{\widehat{X}}\left[\int_0^\infty e^{-qt} f(\widehat{X}_t)dt\right]. \tag{4.1}$$

We define local times $\{L^{\widehat{X},x}\}_{x\in\mathbb{T}}$, excursion measures $\{n_x^{\widehat{X}}\}_{x\in\mathbb{T}}$, and generalized scale functions $\{W_{-\widehat{X}}^{(q)}\}_{q\ge 0}$ of $\widehat{X}$ in the same way as X's in Section 3.

Definition 4.1 (See, e.g., [5, Definition 13.1])**.** *Let m be a σ-finite Radon measure on $\mathbb{T}$. We say that X and $\widehat{X}$ are* in duality (relative to m) *if for $q > 0$ and for non-negative measurable functions f and g,*

$$\int_{\mathbb{T}} f(x) R_X^{(q)} g(x) m(dx) = \int_{\mathbb{T}} R_{\widehat{X}}^{(q)} f(x) g(x) m(dx). \tag{4.2}$$

Theorem 4.2 (See, e.g., [5, Theorem 13.2] or [17, pp. 517])**.** *Suppose X and $\widehat{X}$ are in duality relative to m. Then, for each $q > 0$, there exists a function $r_X^{(q)} : \mathbb{T} \times \mathbb{T} \to [0, \infty)$ such that*

(i) $r_X^{(q)}$ *is $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T})$-measurable,*
(ii) $x \mapsto r_X^{(q)}(x, y)$ *is q-excessive and finely continuous for each $y \in \mathbb{T}$,*
(iii) $y \mapsto r_X^{(q)}(x, y)$ *is q-coexcessive and cofinely continuous for each $x \in \mathbb{T}$, and*
(iv) *for all non-negative measurable function f,*

$$R_X^{(q)} f(x) = \int_{\mathbb{T}} f(y) r_X^{(q)}(x, y) m(dy), \qquad R_{\widehat{X}}^{(q)} f(y) = \int_{\mathbb{T}} f(x) r_X^{(q)}(x, y) m(dx). \tag{4.3}$$

By [17, Proposition of Section V.1], if X and $\widehat{X}$ are in duality relative to m, there exist local times $\{L^{X,x}\}_{x \in \mathbb{T}}$ of X and $\{L^{\widehat{X},x}\}_{x \in \mathbb{T}}$ of $\widehat{X}$ satisfying

$$\mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right] = r_X^{(q)}(x, y), \ \mathbb{E}_y^{\widehat{X}} \left[\int_0^\infty e^{-qt} dL_t^{\widehat{X},x} \right] = r_X^{(q)}(x, y) \tag{4.4}$$

for all $q > 0$. When X and $\widehat{X}$ are in duality, we always use the normalization of the local times above. In other cases, we use the normalization of the local times in Section 3.

Our main theorem is as follows.

Theorem 4.3. *If X and $\widehat{X}$ are in duality relative to m, then we have*

$$W_X^{(q)}(x, y) = W_{-\widehat{X}}^{(q)}(-y, -x), \qquad q > 0, \quad x, y \in (b_0, a_0). \tag{4.5}$$

The converse is also true, when $\mathbb{T}$ is open.

We postpone the proof of Theorem 4.3 until the end of this section. To prove Theorem 4.3, we need the following lemma, which gives us the relationship between the killed potential densities of X and $\widehat{X}$.

Lemma 4.4. *Let X and $\widehat{X}$ be in duality relative to m. Then, for all $b < a \in \mathbb{T}$ and $x, y \in (b, a)$, we have*

$$\mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] = \mathbb{E}_y^{\widehat{X}} \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{\widehat{X},x} \right]. \tag{4.6}$$

Proof. Let $X^{(b,a)}$ and $\widehat{X}^{(b,a)}$ denote the X and $\widehat{X}$ killed on exiting (b, a), respectively. We denote by $R_{X^{(b,a)}}^{(q)}$ and $R_{\widehat{X}^{(b,a)}}^{(q)}$ the q-resolvent operators of $X^{(b,a)}$ and $\widehat{X}^{(b,a)}$, respectively. For each $q > 0$, there exists a function $r_{X^{(b,a)}}^{(q)} : (b, a) \times (b, a) \to [0, \infty)$ such that all the conditions (i)–(iv) of Theorem 4.2 hold. By definition, we have

$$R_{X^{(b,a)}}^{(q)} f(y) = \overline{\underline{R}}_X^{(q;b,a)} f(y) = \int_{(b,a)} f(y) \mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] m(dy). \tag{4.7}$$

So, for all $x \in (b, a)$, we have

$$r_{X^{(b,a)}}^{(q)}(x, y) = \mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right], \quad m\text{-a.e. } y. \tag{4.8}$$

Let us remove the a.e. restriction. Since

$$\begin{aligned} \mathbb{E}_x^X \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] &= \mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right] \\ &\quad - \mathbb{E}_x^X \left[e^{-q(T_b^- \wedge T_a^+)} \mathbb{E}_{X_{T_b^- \wedge T_a^+}}^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right] \right] \end{aligned} \tag{4.9}$$

and from the dominated convergence theorem, the function $y \mapsto \mathbb{E}_x^X\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y}\right]$ is cofinely continuous. By the cofine continuity, we see that (4.8) holds for all $y \in (b,a)$.

In the same way, we have

$$r_{X^{(b,a)}}^{(q)}(x,y) = \mathbb{E}_y^{\widehat{X}}\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{\widehat{X},x}\right], \quad m\text{-a.e. } x \tag{4.10}$$

for all $x,y \in (b,a)$ via fine continuity. The proof is now completed. □

Remark 4.5. We suppose that X and $\widehat{X}$ are in duality. Then at each point $x \in \mathbb{T}$, fine continuity implies right continuity and cofine continuity implies left continuity, so by Theorem 4.2, $x \mapsto r_{X^{(b,a)}}^{(q)}(x,y)$ is right continuous and $y \mapsto r_{X^{(b,a)}}^{(q)}(x,y)$ is left continuous. By the proof of Lemma 4.4 and Theorem 3.6, the function $x \mapsto W_X^{(q)}(x,y)$ is finely continuous (and hence right continuous, since X has no positive jumps and we have Remark 3.2) and $y \mapsto W_X^{(q)}(x,y)$ is cofinely continuous (and hence left continuous). In the case of the spectrally negative Lévy process X, the scale functions are not continuous at 0 when X has bounded variation paths (and hence each $x \in \mathbb{R}$ is irregular for itself) and are continuous at 0 when X has bounded variation paths (and hence each $x \in \mathbb{R}$ is regular for itself) (see, e.g., [12, Lemma 8.6]). As in the case for the scale functions for Lévy processes, it holds that, for $x \in \mathbb{T}$, the function $y \mapsto W_X^{(q)}(y,x)$ is continuous at x or not according as x is regular for itself or not. More precisely, we have

$$W_X^{(q)}(x-,x) = 0, \quad W_X^{(q)}(x,x) = \frac{1}{n_x^X\left[T_x^+ < \infty\right]}. \tag{4.11}$$

Proof of Theorem 4.3. *i*) Let us assume that X and $\widehat{X}$ are in duality relative to m.

First, we fix $b,y,a \in \mathbb{T}$ with $b < y < a$. By Lemma 4.4 and Theorem 3.6, for all $q \geq 0$ and $x \in (b,y)$, we have

$$\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y) = \frac{W_{-\widehat{X}}^{(q)}(-y,-a)}{W_{-\widehat{X}}^{(q)}(-b,-a)} W_{-\widehat{X}}^{(q)}(-b,-x). \tag{4.12}$$

Hence, there exists a function $\gamma_1 : [0,\infty) \times \mathbb{T} \to (0,\infty)$ satisfying

$$W_X^{(q)}(x,b) = \gamma_1(q,b) W_{-\widehat{X}}^{(q)}(-b,-x) \qquad x \in (b,a_0). \tag{4.13}$$

Second, we fix $b,x,a \in \mathbb{T}$ with $b < x < a$. For $q \geq 0$ and $y \in (x,a)$, we have (4.12). Thus there exists a function $\gamma_2 : [0,\infty) \times \mathbb{T} \to (0,\infty)$ such that

$$W_X^{(q)}(a,y) = \gamma_2(q,a) W_{-\widehat{X}}^{(q)}(-y,-a) \qquad y \in (b_0,a). \tag{4.14}$$

For $q \geq 0$ and $a,b \in (b_0,a_0)$ with $b < a$, by (4.13) with $x = a$ and (4.14) with $y = b$, we have $\gamma_1(q,b) = \gamma_2(q,a)$, so γ_1 and γ_2 only depend on $q \geq 0$. We can rewrite $\gamma_1(q) = \gamma_1(q,\cdot)$.

By (4.12) and the definition of γ_1, for $q \geq 0$ and $a,b,x,y \in (b_0,a_0)$ with $b < x < y < a$, we have

$$\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y) = \frac{W_{-\widehat{X}}^{(q)}(-y,-a)}{W_{-\widehat{X}}^{(q)}(-b,-a)} W_{-\widehat{X}}^{(q)}(-b,-x) = \gamma_1(q) \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y), \tag{4.15}$$

so we have $\gamma_1 = \gamma_2 \equiv 1$. Thus, for $y,x \in (b_0,a_0)$ with $y < x$, we have $W_X^{(q)}(x,y) = W_{-\widehat{X}}^{(q)}(-y,-x)$. By the fine continuity of $W_X^{(q)}$ and the cofine continuity $W_{-\widehat{X}}^{(q)}$, for $x \in (b_0,a_0)$, we have $W_X^{(q)}(x,x) = W_{-\widehat{X}}^{(q)}(-x,-x)$.

ii) Let us assume that $\mathbb{T}$ is open and that (4.5) is satisfied. Then, for $b < a \in \mathbb{T}$ and $x, y \in (b, a)$, we have

$$\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y) - W_X^{(q)}(x,y) = \frac{W_{-\widehat{X}}^{(q)}(-y,-a)}{W_{-\widehat{X}}^{(q)}(-b,-a)} W_{-\widehat{X}}^{(q)}(-b,-x) - W_{-\widehat{X}}^{(q)}(-y,-x). \tag{4.16}$$

By Theorem 3.6, the first term and the second term of (4.16) are potential densities of X and $\widehat{X}$ killed on exiting (b, a), respectively. We therefore conclude the duality of the killed processes, which yields that of the original processes. □

A The proof of (1.10)

We prove (1.10). When X has unbounded variation paths, we have (1.7). Suppose X has bounded variation paths. In [14], we proved that

$$W^{(q)}(x) = \frac{1}{\delta_X \mathbb{E}_0^X \left[e^{-qT_x^+}; T_x^+ < T_0^- \right]}, \qquad q \geq 0, \ \ x \geq 0, \tag{A.1}$$

where δ_X denotes the drift parameter of X, and also proved that

$$\delta_X \mathbb{E}_0^X \left[1 - e^{-qT_0} \right] = \frac{1}{\Phi'(q)}, \quad q > 0. \tag{A.2}$$

So the only thing we have to show is that $n_0^X = n^X$ or $n_0^X = \delta_X \mathbb{E}_0^X$, that is, that the normalization of n_0^X coincides with (1.8) or (A.2).

By [12, Theorem 8.7], the spectrally negative Lévy process X has the q-potential density

$$r^{(q)}(x,y) = \Phi'(q) e^{-\Phi(q)(y-x)} - W^{(q)}(x-y), \quad x, y \in \mathbb{R} \tag{A.3}$$

with respect to the Lebesgue measure. Since $r^{(q)}(x, y)$ is right-continuous for x and left-continuous for y, we have

$$r^{(q)}(x,y) = \mathbb{E}_x \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right], \quad q \geq 0, \ x, y \in \mathbb{R}, \tag{A.4}$$

where $\{L^{X,x}\}_{x \in \mathbb{R}}$ is the local times defined by Theorem 4.2 with respect to the Lebesgue measure. Therefore, the generalized scale functions of X defined in Section 4 satisfy (1.9) for the excursion measures which satisfy

$$n_x^X \left[1 - e^{-qT_x} \right] = \frac{1}{\mathbb{E}_x \left[\int_{0-}^\infty e^{-qt} dL_t^{X,x} \right]} = \frac{1}{r^{(q)}(x, x+)} = \frac{1}{\Phi'(q)}, \quad q > 0, \ x \in \mathbb{R}. \tag{A.5}$$

By (1.9), (A.5) and (1.7), we obtain (1.10). The proof is now completed.

References

[1] Avram, F., Pérez, J. L. and Yamazaki, K.: Spectrally negative Lévy processes with Parisian reflection below and classical reflection above. *Stochastic Process. Appl.* **128**, (2018), no. 1, 255–290. MR-3729538

[2] Bertoin, J.: Lévy processes. *Cambridge Tracts in Mathematics*, **121**. Cambridge University Press, Cambridge, 1996. x+265 pp. MR-1406564

[3] Blumenthal, R. M.: Excursions of Markov processes. *Probability and its Applications*. Birkhäuser Boston, Inc., Boston, MA, 1992. xii+275 pp. MR-1138461

[4] Blumenthal, R. M. and Getoor, R. K.: Markov processes and potential theory. *Pure and Applied Mathematics*, **29**. Academic Press, New York-London, 1968. x+313 pp. MR-0264757

[5] Chung, K. L. and Walsh, J.: Markov processes, Brownian motion, and time symmetry. Second edition. *Grundlehren der Mathematischen Wissenschaften* [Fundamental Principles of Mathematical Sciences], **249**. Springer, New York, 2005. xii+431 pp. MR-2152573

[6] Doney, R. A.: Some excursion calculations for spectrally one-sided Lévy processes. *Séminaire de Probabilités* **XXXVIII**, 5–15, Lecture Notes in Math., 1857, Springer, Berlin, 2005. MR-2126963

[7] Geman, D. and Horowitz, J.: Occupation densities. *Ann. Probab.* **8** (1980), no. 1, 1–67. MR-0556414

[8] Ikeda, N. and Watanabe, S.: Stochastic differential equations and diffusion processes. *North-Holland Mathematical Library*, **24**. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981. xiv+464 pp. MR-0637061

[9] Itô, K.: Poisson point processes attached to Markov processes. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory, pp. 225–239. Univ. California Press, Berkeley, Calif., 1972. MR-0402949

[10] Itô, K.: Poisson point processes and their application to Markov processes. With a foreword by Shinzo Watanabe and Ichiro Shigekawa. *SpringerBriefs in Probability and Mathematical Statistics*. Springer, Singapore, 2015. xi+43 pp. MR-3467032

[11] Kuznetsov, A., Kyprianou, A. E. and Rivero, V.: The theory of scale functions for spectrally negative Lévy processes. *Lévy matters II*, 97–186, Lecture Notes in Math., **2061**, Lévy Matters. Springer, Heidelberg, 2012. MR-3014147

[12] Kyprianou, A. E.: Fluctuations of Lévy processes with applications. Introductory lectures. Second edition. *Universitext*. Springer, Heidelberg, 2014. xviii+455 pp. MR-3155252

[13] Kyprianou, A. E. and Loeffen, R. L.: Refracted Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010), no. 1, 24–44. MR-2641768

[14] Noba, K. and Yano, K.: Generalized refracted Lévy process and its application to exit problem. *Stochastic Process. Appl.* **129** (2019), no. 5, 1697–1725. MR-3944781

[15] Pardo, J. C., Pérez, J. L. and Rivero, V.: Lévy insurance risk processes with parisian type severity of debt, arXiv:1507.07255

[16] Pardo, J. C., Pérez, J. L. and Rivero, V.: The excursion measure away from zero for spectrally negative Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (2018), no. 1, 75–99. MR-3765881

[17] Revuz, D.: Mesures associées aux fonctionnelles additives de Markov. I. (French) *Trans. Amer. Math. Soc.* **148** (1970) 501–531. MR-0279890

[18] Salisbury, T. S.: On the Itô excursion process. *Probab. Theory Related Fields* **73** (1986), no. 3, 319–350. MR-0859838

[19] Yano, K.: Excursion measure away from an exit boundary of one-dimensional diffusion processes. *Publ. Res. Inst. Math. Sci.* **42** (2006), no. 3, 837–878. MR-2266999

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