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Strong Feller property and continuous dependence on initial data for one-dimensional stochastic differential equations with Hölder continuous coefficients*

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Abstract

In this paper, under the assumption of Hölder continuous coefficients, we prove the strong Feller property and continuous dependence on initial data for the solution to one-dimensional stochastic differential equations whose proof are based on the technique of local time, coupling method and Girsanov's transform.

Keywords: strong Feller property; continuous dependence of initial data; stochastic differential equations; Hölder continuous coefficients; local time; coupling method; Girsanov's transform.
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1 Introduction

In the present paper, we are concerned with the problem of the strong Feller property and continuous dependence on initial data for the following one-dimensional stochastic differential equations (SDEs in short) with Hölder continuous coefficients:

$$X(t,x) = x + \int_0^t b(X(s,x))ds + \int_0^t \sigma(X(s,x))dW(s), \tag{1.1}$$

where $W(t)$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathfrak{F}, \mathbf{P}; (\mathfrak{F}_t)_{t\geq 0})$, $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are two continuous functions.

As far as we know, there are many ways to establish the strong Feller property on $P_t\varphi(x) := \mathbf{E}[\varphi(X(t,x))]$. First of all, Bismut-Elworthy-Li's formula gives an explicit formula for $\nabla P_t\varphi(x)$ (cf. [1]), and hence the strong Feller property can be obtained. Moreover, the Hanarck inequality which is established by Wang (cf. [5]) gives some quantitative estimate for $P_t\varphi(x)$ for finite and infinite dimensional systems, which can also be used to derive the strong Feller property. For the case of non-Lipschitz continuous coefficient, Zhang (cf. [8]) first used the coupling method combined with Girsanov's transformation to establish the strong Feller property for the solution to Eq. (1.1) with the Log-Lipschitz continuous coefficients. Based on Krylov's estimate, Zvonkin's transform and Bismut-Elworthy-Li's formula, the same result was also obtained by Zhang (cf. [9])

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under the assumption of singular time dependent drifts and non-degenerate Sobolev diffusion coefficients. Recently, a Harnack inequality for the solution to Eq. (1.1) with non-degenerate Sobolev or Lipschitz continuous diffusion coefficient and non-regular time-dependent drift coefficient was established by Li, Luo and Wang (cf. [4]), which contains the cases where the drift coefficients are uniformly Hölder continuous with respect to the spatial variable. However, for the case of SDEs that does not satisfy the above conditions, such as SDEs with Hölder continuous diffusion coefficients, due to the ineffectiveness of the existing tools, the previous methods are no longer applicable or some difficulties need to be overcome.

In this article, we prove the strong Feller property for SDEs with Hölder continuous coefficients whose proof is based on the technique of local time, coupling method and Girsanov's transform. Indeed, under the non-Lipschitz continuous coefficients, especially including the cases of Hölder continuous coefficients, the quantitative continuity estimate for $|P_t\varphi(x) - P_t\varphi(y)|$ in terms of x and y is also obtained. In addition, using a precise estimate on local times, the continuous dependence on initial data for SDEs with Hölder continuous coefficients is also established, which was not obtained before. To the best of our knowledge, so far little is known about these topics, and the aim of this paper is to close this gap.

We now give an outline of the paper. In Section 2, we recall the well-known result on the existence and uniqueness for SDEs with Hölder continuous coefficients. The strong Feller property for SDEs with Hölder continuous coefficients is stated in Section 3. In Section 4, using a precise estimate on local time, we also give the continuous dependence on initial data for SDEs with Hölder continuous coefficients.

Let $C(\lambda)$ be a quantity depending only on some parameter λ and $0 < C(\lambda) < \infty$, whose value may change from line to line. When we do not want to emphasis this dependence we just use C instead.

2 Preliminaries

In the present article, we will make use of the following assumptions.

Assumption 2.1 ($\mathbf{C}_\sigma$). *1. $\sigma : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the linear growth condition.*

2. The solution to Eq. (1.1) is pathwise unique.

Assumption 2.2 ($\mathbf{H}_\sigma$). *$\sigma : \mathbb{R} \to \mathbb{R}$ is measurable, and there exist some constants $C > 0$ and $\alpha \in [0, 1]$ such that*

$$|\sigma(x) - \sigma(y)|^2 \leq C|x - y|^{1+\alpha}.$$

Assumption 2.3 ($\mathbf{H_b}$). *$b = b_1 + b_2$, where $b_1 : \mathbb{R} \to \mathbb{R}$ is monotone decreasing, $b_2 : \mathbb{R} \to \mathbb{R}$ is measurable, and there exist some constants $C > 0$ and $\beta \in (0, 1]$ such that*

$$|b_1(x) - b_1(y)| \leq C|x - y|^\beta, \quad |b_2(x) - b_2(y)| \leq C|x - y|.$$

Remark 2.4. Using the martingale problem of Stroock and Varadhan, and the principle of Yamada-Watanabe, it is well known that under the assumptions ($\mathbf{C}_\sigma$) and ($\mathbf{H_b}$), there exists a unique strong solution $X(\cdot, x)$ to Eq. (1.1).

Remark 2.5. Under the assumptions ($\mathbf{H}_\sigma$) and ($\mathbf{H_b}$), the solution $X(\cdot, x)$ to Eq. (1.1) is pathwise unique (see, e.g., [2, 3]), which implies that the assumption ($\mathbf{C}_\sigma$) is satisfied.

Remark 2.6. There are many authors who discussed the pathwise uniqueness for the solution to one-dimensional SDEs with rough coefficients. For more details, the readers is referred to [3, Chapter 5.5] and the related references therein.

Remark 2.7. As can be seen from [6], once the coefficients b, σ are continuous, then the solution $X(\cdot, x)$ to Eq. (1.1) is a Markov process, which yields that P_t is a semigroup.

3 Strong Feller property

In this section, the strong Feller property for the solution to Eq. (1.1) is established. In order to achieve this goal, an additional hypothesis needs to be introduced as follows.

Assumption 3.1. **(E)** *There exists a constant* $\lambda > 0$ *such that* $\sigma(x) > \lambda$ *for any* $x \in \mathbb{R}$.

Now we can state the result on the strong Feller property for one-dimensional SDEs with Hölder continuous coefficients.

Theorem 3.2. *Under the assumptions* $(\mathbf{C}_\sigma)$, $(\mathbf{H_b})$ *and* **(E)**, *the semigroup* P_t *is strong Feller. In particular, let* $T > 0$ *and* $0 < \gamma < 1$ *be given numbers, then for any* $\varphi \in \mathcal{B}_b(\mathbb{R})$ *and* $x, y \in \mathbb{R}$,

$$\begin{aligned} |P_T\varphi(x) - P_T\varphi(y)| \quad \leq \quad & \|\varphi\|_0 (\lambda^{-1}\sqrt{T}\exp\{\frac{1}{2}\lambda^{-2}T(x-y)^{2\gamma}\}(x-y)^\gamma \\ & + \frac{\sqrt{3+\exp\{\lambda^{-2}T(x-y)^{2\gamma}\}}}{\sqrt{T}} \exp\{CT\}(x-y)^{(1-\gamma)/2}), \end{aligned}$$

where C *and* λ *are the constants appeared in the assumptions* $(\mathbf{H_b})$ *and* **(E)** *respectively, and* $\|\varphi\|_0 := \sup_{z\in\mathbb{R}} |\varphi(z)|$.

Remark 3.3. It is worth noting that this estimate does not depend on the Hölder exponents of the coefficients, which is an interesting fact.

Proof. In order to make the details of the proof clearer, we will split our proof into four steps.

Step 1: Without loss of generality, we may assume that $x > y$ from now on. Let $Y(\cdot, y)$ be the solution of the following SDEs:

$$\begin{aligned} Y(t,y) \quad = \quad & y + \int_0^t b(Y(s,y))ds + \int_0^t \sigma(Y(s,y))dW(s) \\ & + \int_0^t (x-y)^\gamma \frac{X(s,x) - Y(s,y)}{|X(s,x) - Y(s,y)|} I_{\{s<\tau\}} ds, \end{aligned} \tag{3.1}$$

where $\gamma \in (0,1)$ and τ is the coupling time given by

$$\tau := \inf\{s \geq 0; |X(s,x) - Y(s,y)| = 0\}.$$

It is worth noting that for any $0 \leq s \leq t < \tau$,

$$\frac{X(s,x) - Y(s,y)}{|X(s,x) - Y(s,y)|} = 1.$$

Then the last term in (3.1) can be rewritten as follows:

$$(x-y)^\gamma (t \wedge \tau).$$

In order to show that the solution to Eq. (3.1) is well defined, we deal with as follows. First of all, it is easy to see that there exists a unique solution $\overline{Y}(\cdot, y)$ to the following SDEs:

$$\begin{aligned} \overline{Y}(t,y) \quad = \quad & y + \int_0^t b(\overline{Y}(s,y))ds + \int_0^t \sigma(\overline{Y}(s,y))dW(s) \\ & + (x-y)^\gamma t. \end{aligned}$$

Now introduce the stopping time $\bar{\tau}$ by:

$$\bar{\tau} := \inf\{s \geq 0; |X(s,x) - \overline{Y}(s,y)| = 0\}.$$

Then define the process $Y(\cdot,y)$ as follows:

$$Y(t,y)=\begin{cases}\overline{Y}(t,y), & t<\bar{\tau},\\ X(t,x), & t\geq\bar{\tau}.\end{cases}$$

It is easy to verify that $Y(\cdot,y)$ solves Eq. (3.1), hence the desired result is obtained.

Step 2: First of all, it is obvious that

$$\begin{aligned}X(t,x)-Y(t,y) &= (x-y)+\int_0^t(b(X(s,x))-b(Y(s,y)))ds\\ &\quad+\int_0^t(\sigma(X(s,x))-\sigma(Y(s,y)))dW(s)\\ &\quad-\int_0^t(x-y)^\gamma\frac{X(s,x)-Y(s,y)}{|X(s,x)-Y(s,y)|}I_{\{s<\tau\}}ds.\end{aligned}$$

Then by the Meyer-Itô formula, we obtain

$$\begin{aligned}&|X(t\wedge\tau,x)-Y(t\wedge\tau,y)|\\ =\ &(x-y)+\int_0^{t\wedge\tau}sign(X(s,x)-Y(s,y))(b(X(s,x))-b(Y(s,y)))ds\\ &+\int_0^{t\wedge\tau}sign(X(s,x)-Y(s,y))(\sigma(X(s,x))-\sigma(Y(s,y)))dW(s)\\ &-\int_0^{t\wedge\tau}sign(X(s,x)-Y(s,y))(x-y)^\gamma\frac{X(s,x)-Y(s,y)}{|X(s,x)-Y(s,y)|}ds\\ &+L^0_{X(\cdot,x)-Y(\cdot,y)}(t\wedge\tau).\end{aligned}$$

It is worth noting that b_1 is monotone decreasing, and hence $sign(X(s,x)-Y(s,y))(b_1(X(s,x))-b_1(Y(s,y)))\leq 0$. Now since $X(\cdot,x)-Y(\cdot,y)$ is being stopped before the coupling time τ and $dL^0_{X(\cdot,x)-Y(\cdot,y)}(t)$ is almost surely carried by the set $\{t;X(t,x)-Y(t,y)=0\}$, the local time $L^0_{X(\cdot,x)-Y(\cdot,y)}(t\wedge\tau)$ vanishes, and hence using the assumption $(\mathbf{H_b})$, we have

$$\begin{aligned}&\mathbf{E}[|X(t\wedge\tau,x)-Y(t\wedge\tau,y)|]\\ =\ &(x-y)+\mathbf{E}[\int_0^{t\wedge\tau}sign(X(s,x)-Y(s,y))(b(X(s,x))-b(Y(s,y)))ds]\\ &-\mathbf{E}[\int_0^{t\wedge\tau}sign(X(s,x)-Y(s,y))(x-y)^\gamma\frac{X(s,x)-Y(s,y)}{|X(s,x)-Y(s,y)|}ds]\\ \leq\ &(x-y)+C\mathbf{E}[\int_0^{t\wedge\tau}|X(s,x)-Y(s,y)|ds]-(x-y)^\gamma\mathbf{E}[t\wedge\tau]\\ =\ &(x-y)+C\mathbf{E}[\int_0^{t}|X(s\wedge\tau,x)-Y(s\wedge\tau,y)|ds]-(x-y)^\gamma\mathbf{E}[t\wedge\tau].\end{aligned}\tag{3.2}$$

Therefore it is obvious by Gronwall's inequality that

$$\mathbf{E}[|X(t\wedge\tau,x)-Y(t\wedge\tau,y)|]\leq\exp\{Ct\}(x-y).$$

On the other hand, it can be seen from (3.2) that

$$(x-y)^\gamma\mathbf{E}[t\wedge\tau]\leq(x-y)+C\mathbf{E}[\int_0^t|X(s\wedge\tau,x)-Y(s\wedge\tau,y)|ds].$$

This implies that

$$\mathbf{E}[t\wedge\tau]\leq\exp\{Ct\}(x-y)^{1-\gamma}.\tag{3.3}$$

Step 3: Now fix a $T > 0$ and define

$$Z(t) := \exp[-\int_0^t H(X(s,x), Y(s,y))dW(s) - \frac{1}{2}\int_0^t |H(X(s,x), Y(s,y))|^2 ds]$$

and

$$\widetilde{W}(t) := W(t) + \int_0^t H(X(s,x), Y(s,y))ds,$$

where

$$H(m,n) := (x-y)^\gamma \sigma^{-1}(n)\frac{m-n}{|m-n|}I_{\{s<\tau\}}.$$

According to the assumption **(E)**, $\sigma^{-1}(n) \le \lambda^{-1}$, and hence we have

$$|H(m,n)|^2 \le \lambda^{-2}(x-y)^{2\gamma}.$$

This implies that

$$\mathbf{E}[\exp\{\frac{1}{2}\int_0^T |H(X(s,x), Y(s,y))|^2 ds\}] \le \exp\{\frac{1}{2}\lambda^{-2}(x-y)^{2\gamma}T\},$$

and hence the Novikov condition is satisfied. Therefore

$$\mathbf{E}[Z(T)] = 1 \tag{3.4}$$

and

$$\mathbf{E}[|Z(T)|^2] \le \exp\{\lambda^{-2}T(x-y)^{2\gamma}\}. \tag{3.5}$$

In view of Girsanov's theorem, $(\widetilde{W}(t), t \in [0,T])$ is a Brownian motion under the new probability measure $\mathbf{Q} := Z(T)\mathbf{P}$. Then due to the fact that $Y(\cdot, y)$ also solves

$$Y(t,y) = y + \int_0^t b(Y(s,y))ds + \int_0^t \sigma(Y(s,y))d\widetilde{W}(s),$$

it is easy to show that the law of $X(T,y)$ under $\mathbf{P}$ is the same as that of $Y(T,y)$ under $\mathbf{Q}$. Now on the one hand, according to the elementary inequality $e^r - 1 \le re^r$ for $r \ge 0$, Hölder's inequality, the estimates (3.4) and (3.5), we have

$$\begin{aligned}(\mathbf{E}[|1-Z(T)|])^2 &\le \mathbf{E}[|1-Z(T)|^2]\\ &= \mathbf{E}[|Z(T)|^2] - 1\\ &\le \exp\{\lambda^{-2}T(x-y)^{2\gamma}\} - 1\\ &\le \lambda^{-2}T\exp\{\lambda^{-2}T(x-y)^{2\gamma}\}(x-y)^{2\gamma}\end{aligned} \tag{3.6}$$

On the other hand, since

$$\mathbf{P}[\tau \ge T] = \mathbf{P}[(2T)\wedge\tau \ge T],$$

it can also be seen from the Hölder's inequality, the estimates (3.4) and (3.5) that

$$\begin{aligned}(\mathbf{E}[(1+Z(T))I_{\{\tau\ge T\}}])^2 &\le \mathbf{E}[(1+Z(T))^2]\mathbf{P}[\tau \ge T]\\ &= (3+\mathbf{E}[|Z(T)|^2])\mathbf{P}[\tau \ge T]\\ &\le (3+\exp\{\lambda^{-2}T(x-y)^{2\gamma}\})\mathbf{P}[(2T)\wedge\tau \ge T]\\ &\le \frac{3+\exp\{\lambda^{-2}T(x-y)^{2\gamma}\}}{T}\mathbf{E}[(2T)\wedge\tau].\end{aligned} \tag{3.7}$$

Step 4: Consequently, combining all the above estimates (3.3), (3.6) and (3.7), for any $\varphi \in \mathcal{B}_b(\mathbb{R})$, $T > 0$ and $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
& |P_T\varphi(x) - P_T\varphi(y)| \\
= \; & |\mathbf{E}[\varphi(X(T,x)) - Z(T)\varphi(Y(T,y))]| \\
\leq \; & \mathbf{E}[|(1-Z(T))\varphi(X(T,x))I_{\{\tau<T\}}|] \\
& +\mathbf{E}[|(\varphi(X(T,x)) - Z(T)\varphi(Y(T,y)))I_{\{\tau\geq T\}}|] \\
\leq \; & \|\varphi\|_0\mathbf{E}[|1-Z(T)|] + \|\varphi\|_0\mathbf{E}[(1+Z(T))I_{\{\tau\geq T\}}] \\
\leq \; & \|\varphi\|_0(\lambda^{-1}\sqrt{T}\exp\{\frac{1}{2}\lambda^{-2}T(x-y)^{2\gamma}\}(x-y)^{\gamma} \\
& +\frac{\sqrt{3+\exp\{\lambda^{-2}T(x-y)^{2\gamma}\}}}{\sqrt{T}}\sqrt{\mathbf{E}[(2T)\wedge\tau]}) \\
\leq \; & \|\varphi\|_0(\lambda^{-1}\sqrt{T}\exp\{\frac{1}{2}\lambda^{-2}T(x-y)^{2\gamma}\}(x-y)^{\gamma} \\
& +\frac{\sqrt{3+\exp\{\lambda^{-2}T(x-y)^{2\gamma}\}}}{\sqrt{T}}\exp\{CT\}(x-y)^{(1-\gamma)/2}).
\end{aligned}$$

Therefore the proof is complete. □

4 Continuous dependence on initial data

In this section, using the technique introduced by Yan (cf. [7]), we can establish the following precise estimate on local time which will play an important role in the proof of the continuous dependence on initial data for the solution to Eq. (1.1).

Theorem 4.1. *Let Z be a continuous semimartingale with $Z(0) = z \geq 0$. For any $0 \leq \eta \leq z$, we define a double sequence of stopping times (α_n, β_n) by*

$$\alpha_1 = 0, \quad \beta_1 = \inf\{t > 0 : Z(t) = \eta\},$$
$$\cdots$$
$$\alpha_n = \inf\{t > \beta_{n-1} : Z(t) = 0\}, \quad \beta_n = \inf\{t > \alpha_n : Z(t) = \eta\}.$$

Then we have

$$\begin{aligned}
L_Z^0(t) \quad \leq \quad & \frac{2\int_0^t(2Z^+(s)-\eta)\zeta_Z(s)dZ(s) + 2\int_0^t\zeta_Z(s)d[Z,Z](s)}{\eta} \\
& +\frac{2z(z-\eta)}{\eta} \\
& +2\eta,
\end{aligned}$$

where

$$\zeta_Z(s) = \sum_{n=1}^{\infty} I_{\{\alpha_n<s\leq\beta_n, 0<Z(s)\leq\eta\}}.$$

Proof. By the Meyer-Itô formula,

$$Z^+(\beta_n\wedge t) - Z^+(\alpha_n\wedge t) = \int_{\alpha_n\wedge t}^{\beta_n\wedge t} I_{\{Z(s)>0\}}dZ(s) + \frac{1}{2}(L_Z^0(\beta_n\wedge t) - L_Z^0(\alpha_n\wedge t)).$$

Since the measure $dL_Z^0(t)$ is almost surely carried by the set $\{t; Z(t) = 0\}$,

$$L_Z^0(\alpha_{n+1}\wedge t) = L_Z^0(\beta_n\wedge t).$$

Therefore we get

$$\begin{aligned} & \sum_{n=1}^{\infty}(Z^+(\beta_n \wedge t) - Z^+(\alpha_n \wedge t)) \\ = & \int_0^t \zeta_Z(s)dZ(s) + \frac{1}{2}L_Z^0(t). \end{aligned}$$

Let

$$\gamma(t) = \sup\{n \in \mathbb{N}; \beta_n < t\},$$

and denote

$$\delta(t) = t \wedge \alpha_{\gamma(t)+1}.$$

It is worth noting that using the definition of the double sequence of stopping times (α_n, β_n), we also have

$$\sum_{n=1}^{\infty}(Z^+(\beta_n \wedge t) - Z^+(\alpha_n \wedge t)) = \eta(\gamma(t)-1) + \eta - z + Z^+(t) - Z^+(\delta(t)).$$

These imply that

$$\int_0^t \zeta_Z(s)dZ(s) + \frac{1}{2}L_Z^0(t) = \eta(\gamma(t)-1) + \eta - z + Z^+(t) - Z^+(\delta(t)),$$

which can be rewritten as the following expression:

$$\eta(\gamma(t)-1) = \int_0^t \zeta_Z(s)dZ(s) + \frac{1}{2}L_Z^0(t) - \eta + z - Z^+(t) + Z^+(\delta(t)). \tag{4.1}$$

On the other hand, by the Meyer-Itô formula, we obtain

$$\begin{aligned} (Z^+(\beta_n \wedge t))^2 - (Z^+(\alpha_n \wedge t))^2 &= 2\int_{\alpha_n \wedge t}^{\beta_n \wedge t} Z^+(s) I_{\{Z(s)>0\}} dZ(s) \\ &\quad + \int_{\alpha_n \wedge t}^{\beta_n \wedge t} Z^+(s) dL_Z^0(s) \\ &\quad + \int_{\alpha_n \wedge t}^{\beta_n \wedge t} I_{\{Z(s)>0\}} d[Z,Z](s), \end{aligned}$$

which yields by summing up the above formula for all n that

$$\begin{aligned} \sum_{n=1}^{\infty}((Z^+(\beta_n \wedge t))^2 - (Z^+(\alpha_n \wedge t))^2) &= 2\int_0^t Z^+(s)\zeta_Z(s)dZ(s) \\ &\quad + \int_0^t Z^+(s)dL_Z^0(s) \\ &\quad + \int_0^t \zeta_Z(s)d[Z,Z](s) \\ &= 2\int_0^t Z^+(s)\zeta_Z(s)dZ(s) + \int_0^t \zeta_Z(s)d[Z,Z](s). \end{aligned}$$

Similarly as above, we have

$$\sum_{n=1}^{\infty}((Z^+(\beta_n \wedge t))^2 - (Z^+(\alpha_n \wedge t))^2) = \eta^2(\gamma(t)-1) + \eta^2 - z^2 + (Z^+(t))^2 - (Z^+(\delta(t)))^2.$$

Therefore

$$\begin{aligned} & 2\int_0^t Z^+(s)\zeta_Z(s)dZ(s) + \int_0^t \zeta_Z(s)d[Z,Z](s) \\ = \ & \eta^2(\gamma(t)-1) + \eta^2 - z^2 + (Z^+(t))^2 - (Z^+(\delta(t)))^2, \end{aligned}$$

which implies that

$$\begin{aligned} \eta^2(\gamma(t)-1) &= 2\int_0^t Z^+(s)\zeta_Z(s)dZ(s) + \int_0^t \zeta_Z(s)d[Z,Z](s) \\ &\quad -\eta^2 + z^2 - (Z^+(t))^2 + (Z^+(\delta(t)))^2. \end{aligned} \tag{4.2}$$

Consequently, combining with the identities (4.1) and (4.2), we obtain

$$\begin{aligned} & \eta\Big(\int_0^t \zeta_Z(s)dZ(s) + \frac{1}{2}L_Z^0(t) - \eta + z - Z^+(t) + Z^+(\delta(t))\Big) \\ = \ & 2\int_0^t Z^+(s)\zeta_Z(s)dZ(s) + \int_0^t \zeta_Z(s)d[Z,Z](s) - \eta^2 + z^2 - (Z^+(t))^2 + (Z^+(\delta(t)))^2, \end{aligned}$$

and hence

$$\begin{aligned} L_Z^0(t) &= \frac{2\int_0^t (2Z^+(s)-\eta)\zeta_Z(s)dZ(s) + 2\int_0^t \zeta_Z(s)d[Z,Z](s)}{\eta} \\ &\quad + \frac{2z(z-\eta)}{\eta} \\ &\quad + \frac{2((Z^+(\delta(t)))^2 - (Z^+(t))^2) + 2\eta(Z^+(t) - Z^+(\delta(t)))}{\eta}. \end{aligned} \tag{4.3}$$

Noting that

$$\beta_{\gamma(t)} \le t < \alpha_{\gamma(t)+1} \Rightarrow \delta(t) = t$$

and

$$\alpha_{\gamma(t)+1} \le t < \beta_{\gamma(t)+1} \Rightarrow \delta(t) = \alpha_{\gamma(t)+1}, \quad Z^+(\delta(t)) = 0, \quad Z^+(t) \le \eta,$$

we have

$$0 \le Z^+(t) - Z^+(\delta(t)) \le \eta, \quad (Z^+(\delta(t)))^2 - (Z^+(t))^2 \le 0.$$

Therefore the proof is completed by the estimate (4.3). □

Our main result in this section is stated as follows.

Theorem 4.2. *Under the assumption* $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H_b})$, *there exists a constant* $C > 0$ *such that for any* $t \in [0,T]$ *and* $x, y \in \mathbb{R}$,

$$\mathbf{E}[|X(t,x) - X(t,y)|] \le C|x-y|^{\alpha\wedge\beta}.$$

Proof. Similarly as in the proof of Theorem 3.2, we may also assume that $x \ge y$ from now on. Since

$$\begin{aligned} & X(t,x) - X(t,y) \\ = \ & (x-y) + \int_0^t (b(X(s,x)) - b(X(s,y)))ds \\ & + \int_0^t (\sigma(X(s,x)) - \sigma(X(s,y)))dW(s), \end{aligned}$$

by the Meyer-Itô formula, we obtain

$$\begin{aligned}
& |X(t,x)-X(t,y)| \\
= & (x-y)+\int_0^t sign(X(s,x)-X(s,y))(b(X(s,x))-b(X(s,y)))ds \\
& +\int_0^t sign(X(s,x)-X(s,y))(\sigma(X(s,x))-\sigma(X(s,y)))dW(s) \\
& +L^0_{X(\cdot,x)-X(\cdot,y)}(t).
\end{aligned}$$

Therefore using the fact that $sign(X(s,x)-Y(s,y))(b_1(X(s,x))-b_1(Y(s,y)))\leq 0$, and applying Theorem 4.1 with $Z(t)=X(t,x)-X(t,y)$, $z=x-y$ and $\eta=x-y$, we have

$$\begin{aligned}
& \mathbf{E}[|X(t,x)-X(t,y)|] \\
= & (x-y)+\mathbf{E}[\int_0^t sign(X(s,x)-X(s,y))(b(X(s,x))-b(X(s,y)))ds] \\
& +\mathbf{E}[L^0_{X(\cdot,x)-X(\cdot,y)}(t)] \\
\leq & 3(x-y)+C\mathbf{E}[\int_0^t |X(s,x)-X(s,y)|ds] \\
& +\frac{2\mathbf{E}[\int_0^t(2(X(s,x)-X(s,y))^+-(x-y))\zeta_{X(\cdot,x)-X(\cdot,y)}(s)d(X(s,x)-X(s,y))]}{x-y} \\
& +\frac{2\mathbf{E}[\int_0^t \zeta_{X(\cdot,x)-X(\cdot,y)}(s)d[X(\cdot,x)-X(\cdot,y)](s)]}{x-y} \\
= & 3(x-y)+C\mathbf{E}[\int_0^t |X(s,x)-X(s,y)|ds] \\
& +\frac{2\mathbf{E}[\int_0^t(2(X(s,x)-X(s,y))^+-(x-y))\zeta_{X(\cdot,x)-X(\cdot,y)}(s)(b(X(s,x))-b(X(s,y)))ds]}{x-y} \\
& +\frac{2\mathbf{E}[\int_0^t \zeta_{X(\cdot,x)-X(\cdot,y)}(s)(\sigma(X(s,x))-\sigma(X(s,y)))^2ds]}{x-y} \\
\leq & 3(x-y)+C\mathbf{E}[\int_0^t |X(s,x)-X(s,y)|ds] \\
& +\frac{2\mathbf{E}[\int_0^t(2(X(s,x)-X(s,y))^+-(x-y))\zeta_{X(\cdot,x)-X(\cdot,y)}(s)|X(s,x)-X(s,y)|^\beta ds]}{x-y} \\
& +\frac{2\mathbf{E}[\int_0^t(2(X(s,x)-X(s,y))^+-(x-y))\zeta_{X(\cdot,x)-X(\cdot,y)}(s)|X(s,x)-X(s,y)|ds]}{x-y} \\
& +\frac{2\mathbf{E}[\int_0^t \zeta_{X(\cdot,x)-X(\cdot,y)}(s)|X(s,x)-X(s,y)|^{1+\alpha}ds]}{x-y}.
\end{aligned}$$

Since

$$\zeta_{X(\cdot,x)-X(\cdot,y)}(s)=\sum_{n=1}^\infty I_{\{\alpha_n<s\leq\beta_n,0<X(s,x)-X(s,y)\leq(x-y)\}},$$

for any $0<s<T$,

$$\begin{aligned}
(2(X(s,x)-X(s,y))^+-(x-y))|X(s,x)-X(s,y)|^\beta\zeta_{X(\cdot,x)-X(\cdot,y)}(s) &\leq (x-y)^{1+\beta}, \\
(2(X(s,x)-X(s,y))^+-(x-y))|X(s,x)-X(s,y)|\zeta_{X(\cdot,x)-X(\cdot,y)}(s) &\leq (x-y)^2, \\
|X(s,x)-X(s,y)|^{1+\alpha}\zeta_{X(\cdot,x)-X(\cdot,y)}(s) &\leq (x-y)^{1+\alpha}.
\end{aligned}$$

This implies that

$$\begin{aligned} & \mathbf{E}[|X(t,x)-X(t,y)|] \\ \leq \quad & C(x-y)+C(x-y)^{\alpha}+C(x-y)^{\beta}+C\mathbf{E}[\int_0^t |X(s,x)-X(s,y)|ds], \end{aligned}$$

and hence by the Gronwall's inequality, we have

$$\mathbf{E}[|X(t,x)-X(t,y)|] \leq C(x-y)+C(x-y)^{\alpha}+C(x-y)^{\beta}.$$

Therefore we have completed the proof. □

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