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A bivariate fatigue-life regression model and its application to fracture of metallic tools

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Abstract. The Birnbaum–Saunders distribution has been widely used to model reliability and fatigue data. In this paper, we propose a regression of generalized linear models type based on a new bivariate Birnbaum–Saunders distribution. This is parameterized in terms of its means and allows data to be described in their original scale. We estimate the model parameters and carry out inference with the maximum likelihood method. A case study with real-world reliability data is conducted for motivating our investigation, illustrating the potential applications of the proposed results. We obtain a predictive model which can be a useful addition to the tool-kit of diverse practitioners, reliability engineers, applied statisticians, and data scientists.

1 Introduction and bibliographical review

The fatigue-life distribution was proposed by Birnbaum and Saunders (1969) in a reliability setting motivated by problems of vibration in commercial aircrafts that caused fatigue in materials. The fatigue-life or Birnbaum–Saunders (BS) distribution is right-skewed, continuous and unimodal, with two parameters modifying its shape and scale. Therefore, it has been used quite effectively to model data with positive support which follow distributions skewed to the right. The BS distribution is particularly useful for describing fatigue, lifetime and reliability data, as well as crack growth data.

A number of authors have presented applications of the BS distribution to reliability problems; see Villegas, Paula and Leiva (2011), Barros et al. (2014), Marchant, Leiva and Cysneiros (2016), and Vila et al. (2020). The distribution has appealing features and properties, being one of them its relationship with the normal distribution. The study of methodological and theoretical aspects of the BS distribution has received increasing interest and a considerable amount of work is available; see Ferreira (2013), Leiva and Saunders (2015), Leiva (2016, 2019), Aykroyd, Leiva and Marchant (2018), Balakrishnan and Kundu (2019), Leiva, Aykroyd and Marchant (2019), Dasilva et al. (2020) and references therein, which summarize the works to the date.

Reparameterizations of statistical distributions are useful for several purposes; see for example Santos-Neto et al. (2012), where eleven parameterizations were proposed with different justifications. In particular, one of such parameterizations is indexed by two parameters related to the mean and precision of the data distribution. This reparameterized BS (RBS) distribution allows us to mimic the standard parameterization employed for the well-known normal or Gaussian distribution, but in an asymmetric framework, which is useful for modeling lifetime and reliability data, among others.

Key words and phrases. Bivariate Birnbaum–Saunders distribution, fatigue data, maximum likelihood method, predictive models, R software.

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Analyzing data for correlated variables separately with marginal distributions can be a problem because valuable information is not being considered. In addition, explanatory variables (covariates hereafter), which can improve the accuracy in estimation for the mean of the response variable (response hereafter) and its prediction, must be considered as well. Ignoring these two issues related to correlation between responses and incorporation of covariates may conduct to inaccurate decisions, which has been reported in a reliability setting by Marchant, Leiva and Cysneiros (2016). Therefore, bivariate versions of statistical distributions, as well as their associated regression models, are needed and useful for solving such issues. Moreover, regression modeling in its standard framework is often inadequate when describing lifetime and reliability data because of asymmetry and non-linearity. Then, wider classes of models considering asymmetry and non-linearity are of interest in reliability, for example, in the line of generalized linear models (GLM). Leiva et al. (2014a) and Santos-Neto et al. (2016) used the univariate RBS distribution for constructing GLM type RBS regressions.

A bivariate BS distribution based on the original parameterization proposed by Birnbaum and Saunders (1969) was studied by Kundu, Balakrishnan and Jamalizadeh (2010), where maximum likelihood (ML) and modified moment (MM) estimation of the model parameters was discussed; see Ng, Kundu and Balakrishnan (2003) for details about MM estimation in univariate BS models. Khosravi, Kundu and Jamalizadeh (2015) observed that the bivariate BS model proposed by Kundu, Balakrishnan and Jamalizadeh (2010) may be written as the weighted mixture of a bivariate inverse Gaussian distribution (Kocherlakota (1986)) and its reciprocal. They also introduced a mixture of two bivariate BS distributions and discussed its properties. Other bivariate distributions related to the BS model can be found in Vilca, Balakrishnan and Zeller (2014a, 2014b) and Kundu (2015a, 2015b). The readers are also referred to Vilca (2019) for a discussion of multivariate extensions of the BS model.

Saulo et al. (2020) proposed the bivariate RBS (BRBS) distribution. In this context, the primary objective of this paper is to introduce a regression model of GLM type based on the BRBS distribution. An important point of the proposed BRBS regression is that it does not require the logarithmic transformation of the responses, as it is the case of usual bivariate BS regression models; see, for example, Vilca, Romeiro and Balakrishnan (2016).

The rest of the paper proceeds as follows. In Section 2, we provide a motivating example for our study. Section 3 introduces the BRBS regression model as well as the ML estimators of the unknown parameters, their corresponding asymptotic results and model checking. In Section 4, we carry out two Monte Carlo simulation studies to evaluate the performance of the estimators and of a model checking tool. An illustrative example by using the real-world data presented in the case study is analyzed at the end of this section. In Section 5, we sketch some concluding remarks and also point out some problems worthy of further research. Some mathematical derivations are given in the Appendix.

2 Motivating example

In this section, we provide a motivating example for our investigation based on fatigue-life data with application to fracture of metallic tools.

2.1 Fatigue-life

Fatigue is concerning with the failure of materials, which occurs after a long time of service, caused by tension and stress; see Leiva and Vivanco (2017). Fatigue of metals is the increase of a crack, produced by stress, provoking their fracture. Then, fatigue is a process containing the initiation of a crack and its increase, until the material is finally fractured; see Leiva and Saunders (2015). Prediction of fatigue-life is relevant in determining the reliability of systems and components exposed to fatigue.

Improvement of productivity, manufacturing and quality of metallic tools may be considered as part of a process cycle. Then, fracture of these tools might be predicted by studying jointly random variables (RVs) associated with this process considering a random vector. Such a vector can include, for example, manufacturing force, stress, die lifetime and deformation. Due to the correlation often present in these RVs, fracture might be predicted by multivariate models. Therefore, engineers could make decisions based on these models after specifying technical priorities of such RVs assigning target values for each of them.

2.2 The data set

The motivation for our study came from a real-world reliability data set corresponding to die fracture proposed by Lepadatu et al. (2005). Die fracture is a type of metal fatigue produced by cyclic stress during the service life of dies (die lifetime). Although this fatigue may be mainly explained by die lifetime, other RVs correlated with it could be considered as responses to such a fatigue, which implies a multivariate problem. This case study is focused on a data analysis to model fatigue in a metal forming process. The responses to be considered are:

T_1: von Mises stress (in N/mm^2); and
T_2: die lifetime (in number of cycles).

The covariates which might explain T_1 and T_2 are:

X_1: friction coefficient (dimensionless);
X_2: angle of die (in °); and
X_3: work temperature (in °C).

As mentioned, the advantage of a multivariate regression over marginal models is that it considers the statistical correlation of the responses. Note that several responses about metal fatigue can be related during the process. Thus, this relationship must be studied using a correlation analysis, which should provide information with respect to whether the correlation must be incorporated in the modeling by a multivariate regression. Otherwise, marginal models, one for each response, should be used. Nevertheless, studying these variables separately, when correlations exist, can conduct to inaccurate prediction; see more details in Lepadatu et al. (2005). Recall that the data were collected by these authors and are detailed in Table 1.

2.3 An exploratory data analysis

A correlation analysis is conducted for each pair of variables from T_1, T_2, X_1, X_2 and X_3. We report some non-linear relationships between the responses T_1, T_2 and covariates X_1, X_2, X_3, as usual in reliability data. Figure 1 provides the scatterplots and correlation coefficients for all responses and covariates. From this figure, we have that: (i) X_1, X_2, X_3 are uncorrelated, discarding multicollinearity problems for the regression; (ii) T_1 and T_2 have a high negative correlation, being die lifetime a target variable; and (iii) high, moderate and low correlations between responses T_1, T_2 and covariates X_1, X_2, X_3 are detected. This exploratory data analysis allows us to support the use of a bivariate regression model of GLM type. The kind of distribution used for the parametric model depends on the characteristics of the marginal distributions of the responses T_1, T_2, as well as of the theoretical arguments for the fatigue data to be presented.

Figure 2 shows the theoretical probability versus empirical probability (PP) plot with acceptance bands for detecting if the the marginal BS distributions can be suitable in modeling the data related to T_1 and T_2. From this figure, note that both data sets seem to be adequately modeled by BS distributions. The total time on test (TTT) plot is often used for identifying

Table 1 *Fatigue data for the indicated variable*

X_1	X_2	X_3	T_1	T_2
0.07	23.00	581.08	1850	6420
0.07	23.00	818.92	470	33,700
0.07	31.96	581.08	1830	9430
0.07	31.96	818.92	523	36,600
0.13	23.00	581.08	2030	12,100
0.13	23.00	818.92	581	32,000
0.13	31.96	581.08	2230	13,200
0.13	31.96	818.92	632	32,100
0.05	27.50	700.00	889	19,900
0.15	27.50	700.00	1410	15,000
0.10	20.00	700.00	1060	20,900
0.10	35.00	700.00	1390	21,200
0.10	27.50	500.00	2430	9170
0.10	27.50	900.00	243	74,800
0.10	27.50	700.00	1130	19,900

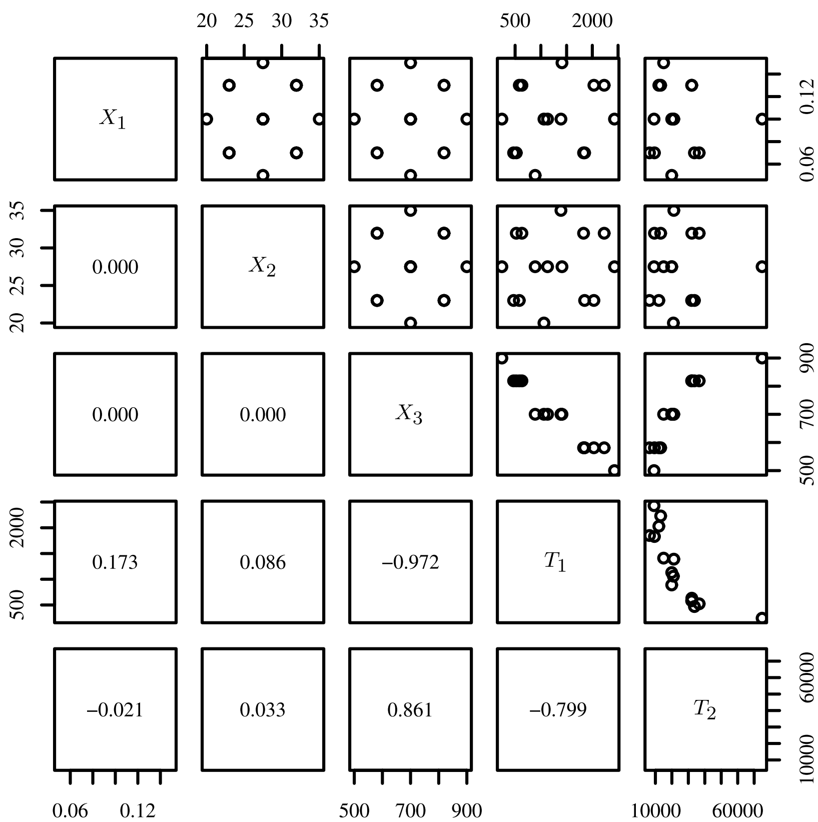

Figure 1 *Scatterplots with their correlations for the indicated variables.*

the shape of the failure rate (FR), and consequently, of the corresponding reliability function (RF). Based on the shape of the FR, we are able to propose lifetime distributions which may be suitable to model the associated reliability data; for details about theoretical TTT plot, see Figure 3 and Athayde (2017). We conduct a study for identifying the shape of the marginal FRs of T_1 and T_2. Figure 2 suggests some curvatures on the TTT plots that make

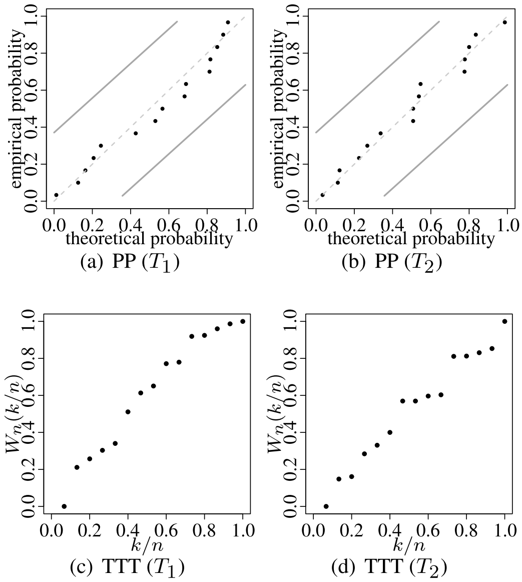

Figure 2 *PP plots with acceptance bands and TTT plots for the fatigue data.*

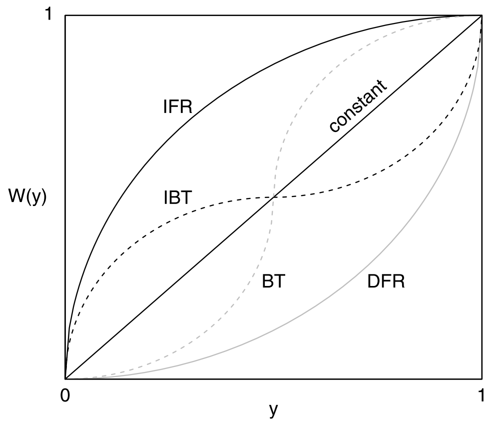

Figure 3 *Scaled TTT function W for distributions with the indicated FR shape*, *according to increasing/decreasing FR* (*IFR/DRF*), *inverse bathtub/bathtub* (*IBT/BT*) *and constant* (*exponential distribution*) *cases*.

us to suspect lightly increasing FRs for the data sets associated with T_1 and T_2. In order to confirm this identification, we carry out a small simulation generating BS data with a similar setting to that found in the real-world data. That is, estimating the univariate BS parameters with such data and then obtaining simulated BS data using these estimates as true values of the BS parameters. Thus, with the simulated BS data, once again we plot the marginal TTT curves, which show similar shapes to those identified with the TTT plots obtained from the real-world data related to T_1 and T_2.

In summary, based on the bivariate exploratory data analysis, the fatigue principles of the BS distribution detailed in Leiva (2016, pp. 5–11), as well as the PP and TTT plots displayed in Figure 2, we conjecture that a BRBS regression model can be suitable to describe these data. Therefore, this example serves as a motivation for the joint modeling of von Mises stress and die lifetime in terms of the friction coefficient, angle of die and work temperature, when predicting fatigue-life of metal tools.

3 BRBS regression model

In this section, the BRBS distribution and its associated regression model are presented. The interested reader on the BRBS distribution is referred to Saulo et al. (2020).

3.1 The BRBS distribution

If a bivariate random vector $\boldsymbol{T} = (T_1, T_2)^\top$ follows a BRBS distribution with parameters $(\mu_1, \mu_2, \delta_1, \delta_2, \rho)$, this is denoted by $\boldsymbol{T} \sim \text{BRBS}(\mu_1, \mu_2, \delta_1, \delta_2, \rho)$.

The joint cumulative distribution function (CDF) of (T_1, T_2) is given by

$$F(t_1, t_2) = \Phi_2\left[\sqrt{\frac{\delta_1}{2}}(a_1 - b_1), \sqrt{\frac{\delta_2}{2}}(a_2 - b_2); \rho\right], \quad t_1 > 0, t_2 > 0, \tag{3.1}$$

with

$$a_k = \sqrt{\frac{(\delta_k + 1)t_k}{\delta_k \mu_k}}, \qquad b_k = \sqrt{\frac{\delta_k \mu_k}{(\delta_k + 1)t_k}}, \quad k = 1, 2,$$

where $\mu_1 > 0$, $\delta_1 > 0$, $\mu_2 > 0$, $\delta_2 > 0$, $|\rho| < 1$, and Φ_2 is the bivariate standard normal CDF with correlation coefficient ρ. The joint probability density function (PDF) associated with (3.1) is expressed as

$$f(t_1, t_2) = \phi_2\left[\sqrt{\frac{\delta_1}{2}}(a_1 - b_1), \sqrt{\frac{\delta_2}{2}}(a_2 - b_2); \rho\right] \prod_{k=1}^{2} \frac{\sqrt{\delta_k}}{2\sqrt{2}t_k}(a_k + b_k), \tag{3.2}$$

where $t_1, t_2 > 0$ and ϕ_2 is the bivariate standard normal PDF defined as

$$\phi_2(u, v; \rho) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left[-\frac{1}{2(1 - \rho^2)}(u^2 - 2\rho uv + v^2)\right], \quad u, v \in \mathbb{R}.$$

3.2 Model formulation

Let $\boldsymbol{T} = (T_1, T_2)^\top \sim \text{BRBS}(\mu_1, \mu_2, \delta_1, \delta_2, \rho)$. Assume that there are p and q covariates, $\boldsymbol{X}^{(1)} = (X_1^{(1)}, \ldots, X_p^{(1)})^\top$ and $\boldsymbol{X}^{(2)} = (X_1^{(2)}, \ldots, X_q^{(2)})^\top$ namely, associated with the RVs T_1 and T_2, respectively. Then, we can establish the relationship among covariates and the mean of T_k, $\mu_k = \text{E}(T_k)$ namely, as

$$g(\mu_k) = \boldsymbol{\beta}_k^\top \boldsymbol{x}^{(k)} = \beta_{k0} + \beta_{k1} x_1^{(k)} + \cdots + \beta_{kl} x_l^{(k)}, \quad k = 1, 2,$$

where $\boldsymbol{\beta}_k = (\beta_{k0}, \beta_{k1}, \ldots, \beta_{kl})$ is a vector of l unknown regression coefficients to be estimated, with $l = p$ or q, $\boldsymbol{x}^{(k)}$ are the values of the covariate $\boldsymbol{X}^{(k)}$ and g is an invertible function with positive support and at least twice differentiable, such that $\mu_k = g^{-1}(\boldsymbol{\beta}_k^\top \boldsymbol{x}^{(k)})$, with g^{-1} being the inverse function of g. In this work, $g(\mu_k) = \log(\mu_k)$ based on the case study, but other link functions can also be assumed. Then, we have

$$\mu_k = \exp(\boldsymbol{\beta}_k^\top \boldsymbol{x}^{(k)}) = \exp\big(\beta_{k0} + \beta_{k1} x_1^{(k)} + \cdots + \beta_{kl} x_l^{(k)}\big). \tag{3.3}$$

Note that the precision parameter δ_k of the BRBS distribution is assumed to be independent of the covariates $\boldsymbol{X}^{(k)}$. In addition, since $\text{Var}[T_k] = \mu_k/\phi_k$, where $\phi_k = (\delta_k + 1)^2/(2\delta_k + 5)$, note that the BRBS variances are a function of μ_k, and consequently, of the covariates. Thus, we can analyze situations where a non-constant variance (but proportional to the mean) is present by using the structure defined in (3.3). This BRBS regression formulation allows us to describe the means of the bivariate data in their original scale. Hence, we avoid the power reduction when testing statistical hypotheses and the difficulties of interpretation from the information obtained using the BRBS regression model. These are consequences of the use of a logarithmic transformation of the data; see Leiva et al. (2014a).

3.3 Estimation

Let $\{(T_{1_i}, T_{2_i}), i = 1, \ldots, n\}$ be a bivariate sample of size n with observations (t_{1_i}, t_{2_i}), for $i = 1, \ldots, n$, from the BRBS distribution. In addition, consider values of covariates corresponding to t_{1_i} as $\boldsymbol{x}_i^{(1)} = (x_{1_i}^{(1)}, \ldots, x_{pi}^{(1)})$ and to t_{2_i} as $\boldsymbol{x}_i^{(2)} = (x_{1_i}^{(2)}, \ldots, x_{qi}^{(2)})$, for $i = 1, \ldots, n$. The problem of interest is to estimate the unknown parameter $\boldsymbol{\theta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \delta_1, \delta_2, \rho)^\top$. Let $L = L(\boldsymbol{\theta})$ be the likelihood function of these observations. From the BRBS PDF defined in (3.2), the log-likelihood function for $\boldsymbol{\theta}$ can be written as

$$
\begin{aligned}
\ell &= \log(L) \\
&= \text{constant} - \frac{n}{2}\log(1-\rho^2) + \frac{\rho}{2(1-\rho^2)} \sum_{i=1}^{n} \prod_{k=1}^{2} \sqrt{\delta_k}(a_{k_i} - b_{k_i}) \\
&\quad - \frac{1}{4(1-\rho^2)} \sum_{i=1}^{n} \sum_{k=1}^{2} \Big[\frac{\delta_k^2 \mu_{k_i}}{(\delta_k+1)t_{k_i}} - 2\delta_k a_{k_i} b_{k_i} + \frac{(\delta_k+1)t_{k_i}}{\mu_{k_i}} \Big] \\
&\quad + \sum_{k=1}^{2} \Big[\frac{n}{2}\log(\delta_k) + \sum_{i=1}^{n} \log(a_{k_i} + b_{k_i}) \Big],
\end{aligned} \tag{3.4}
$$

where

$$
a_{k_i} = \sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}}, \qquad b_{k_i} = \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}, \quad k = 1, 2, \tag{3.5}
$$

with $\mu_{k_i} = \exp(\beta_{k0} + \beta_{k1}x_{1_i}^{(k)} + \cdots + \beta_{kl}x_{l_i}^{(k)})$, for $k = 1, 2$, $i = 1, \ldots, n$ and $l = p, q$. If a maximum for $\boldsymbol{\theta}$, denoted by $\widehat{\boldsymbol{\theta}}$, exists, it must satisfy the likelihood equations given by

$$
\frac{\partial \log(L)}{\partial \beta_{kr}} = 0, \qquad \frac{\partial \log(L)}{\partial \delta_k} = 0, \qquad \frac{\partial \log(L)}{\partial \rho} = 0,
$$

for $k = 1, 2$, $r = 0, 1, \ldots, l$ and $l = p, q$. In the case that $\widehat{\boldsymbol{\theta}}$ provides the global maximum of $\log(L)$, it is called an ML estimate for $\boldsymbol{\theta}$. Under regularity conditions, the information matrix, denoted by $\boldsymbol{U}$, is stated as

$$
\boldsymbol{U} = -\Big\{ \mathrm{E}\Big[\frac{\partial^2 \log(L)}{\partial \vartheta \, \partial \vartheta'} \Big] \Big\}, \quad \vartheta, \vartheta' \in \{\beta_{kr}, \delta_k, \rho\},
$$

for $k = 1, 2$, $r = 0, 1, \ldots, l$ and $l = p, q$. Expressions for the second-order partial derivatives $\partial^2 \log(L)/\partial\vartheta\,\partial\vartheta'$ are given in the Appendix. Similarly to the case with no covariates, note that, for fixed values of $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, δ_1 and δ_2, the ML estimate of ρ can be obtained as

$$
\widehat{\rho}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \delta_1, \delta_2) = \frac{\sum_{i=1}^{n} \prod_{k=1}^{2} \sqrt{\frac{\delta_k}{2}}(\widehat{a}_{k_i} - \widehat{b}_{k_i})}{\prod_{k=1}^{2} \sqrt{\sum_{i=1}^{n} [\sqrt{\frac{\delta_k}{2}}(\widehat{a}_{k_i} - \widehat{b}_{k_i})]^2}},
$$

where, for $k = 1, 2$,

$$
\widehat{a}_{k_i} = \sqrt{\frac{(\widehat{\delta}_k + 1)t_{k_i}}{\widehat{\delta}_k \widehat{\mu}_{k_i}}}, \qquad \widehat{b}_{k_i} = \sqrt{\frac{\widehat{\delta}_k \widehat{\mu}_{k_i}}{(\widehat{\delta}_k + 1)t_{k_i}}}.
$$

Therefore, the ML estimate of $\boldsymbol{\theta}_1 = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \delta_1, \delta_2)^\top$ may be obtained by maximizing the profile log-likelihood function generated from (3.4), denoted by $\ell_p(\boldsymbol{\theta}_1)$, and given by

$$\begin{aligned}\ell_p(\boldsymbol{\theta}_1) = &\text{constant} - \frac{n}{2}\log[1-\widehat{\rho}(\boldsymbol{\theta}_1)^2]\\ &+ \frac{\widehat{\rho}(\boldsymbol{\theta}_1)}{2[1-\widehat{\rho}(\boldsymbol{\theta}_1)^2]}\sum_{i=1}^{n}\prod_{k=1}^{2}\sqrt{\delta_k}(a_{k_i}-b_{k_i})\\ &- \frac{1}{2[1-\widehat{\rho}(\boldsymbol{\theta}_1)^2]}\sum_{i=1}^{n}\sum_{k=1}^{2}\left[\frac{\delta_k^2\mu_{k_i}}{(\delta_k+1)t_{k_i}} - 2\delta_k a_{k_i}b_{k_i} + \frac{(\delta_k+1)t_{k_i}}{\mu_{k_i}}\right]\\ &+ \sum_{k=1}^{2}\left[\frac{n}{2}\log(\delta_k) + \sum_{i=1}^{n}\log(a_{k_i}+b_{k_i})\right]. \end{aligned} \tag{3.6}$$

The estimation of the parameters can be performed by numerical optimization of the profile log-likelihood function defined in (3.6). Initial values in the numerical procedure for estimation of δ_1 and δ_2 may be obtained by the MM method, that is, $\delta_k^{(0)} = (\sqrt{\bar{t}_k/\bar{t}_{kh}} - 1)^{-1}$, where $\bar{t}_k = (1/n)\sum_{i=1}^{n} t_{k_i}$ and $\bar{t}_{kh} = [(1/n)\sum_{i=1}^{n} 1/t_{k_i}]^{-1}$, with $k = 1, 2$; see Santos-Neto et al. (2014) for details. Moreover, the initial values for the regression coefficients might be generated from the least squares (LS) estimate of $\boldsymbol{\beta}_k$, that is, by minimizing the sum of squares defined as

$$S(\boldsymbol{\beta}_k) = \sum_{i=1}^{n}[\log(t_{k_i}) - \beta_{k0} - \beta_{k1}x_{1_i}^{(k)} - \cdots - \beta_{kl}x_{li}^{(k)}]^2,$$

for $k = 1, 2$, $i = 1, \ldots, n$ and $l = p, q$. Then, the initial value for the ML estimate of $\boldsymbol{\beta}_k$, based on the LS estimate, $\boldsymbol{\beta}_k^{(0)}$ namely, is established as

$$\boldsymbol{\beta}_k^{(0)} = [\boldsymbol{x}^{(k)\top}\boldsymbol{x}^{(k)}]^{-1}\boldsymbol{x}^{(k)\top}\log(\boldsymbol{t}_k),$$

where

$$\boldsymbol{x}^{(k)} = \begin{pmatrix} 1 & x_{11}^{(k)} & x_{21}^{(k)} & \cdots & x_{l1}^{(k)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{1n}^{(k)} & x_{2n}^{(k)} & \cdots & x_{ln}^{(k)} \end{pmatrix}$$

and

$$\log(\boldsymbol{t}_k) = \begin{pmatrix} \log(t_{k1}) \\ \vdots \\ \log(t_{kn}) \end{pmatrix}.$$

The asymptotic distribution of $\widehat{\boldsymbol{\theta}} = (\widehat{\delta}_1, \widehat{\delta}_2, \widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2, \widehat{\rho})$, as $n \to \infty$, under some regularity conditions (Cox and Hinkley (1974)), is stated as

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{\text{D}} \text{N}_{p+q+5}(\mathbf{0}, \boldsymbol{R}_{p+q+5}^{-1}),$$

where $\xrightarrow{\text{D}}$ denotes convergence in distribution, $\text{N}_{p+q+5}(\mathbf{0}_{p+q+5}, \boldsymbol{R}_{p+q+5}^{-1})$ stands for a $(p+q+5)$-variate normal distribution with $(p+q+5) \times 1$ vector of mean $\mathbf{0}_{p+q+5}$ and covariance matrix $\boldsymbol{R}_{p+q+5}^{-1}$. This covariance matrix may be obtained from the observed Fisher information, whose elements are presented in the Appendix.

3.4 Model checking

Model checking for the BRBS regression can be conducted by using the Mahalanobis distance (MD) expressed as

$$D_i(\boldsymbol{\theta}) = \boldsymbol{\xi}_i^\top \boldsymbol{\Psi}^{-1} \boldsymbol{\xi}_i, \quad i = 1, \dots, n, \tag{3.7}$$

where

$$\boldsymbol{\Psi} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

and

$$\boldsymbol{\xi}_i = \left[(a_{1_i} - b_{1_i}) \sqrt{\frac{\delta_1}{2}}, (a_{2_i} - b_{2_i}) \sqrt{\frac{\delta_2}{2}} \right]^\top,$$

with a_{1_i}, b_{1_i}, a_{2_i}, and b_{2_i} defined as in (3.5). Based on Marchant, Leiva and Cysneiros (2016), it follows that the MD established in (3.7), with $\boldsymbol{\theta}$ substituted by its ML estimator $\widehat{\boldsymbol{\theta}}$, has asymptotically a χ_4^2 distribution. We use the Wilson–Hilferty (WH) approximation for transforming to normality this distance; see Ibacache-Pulgar, Paula and Galea (2014) and references therein. Then, we check normality of the transformed distances with the WH approximation using theoretical quantile versus empirical quantile (QQ) plots.

4 Numerical studies

In this section, we report the results of two simulation studies carried out to assess (i) the statistical performance of the ML estimators in small and large samples, and (ii) the normality transformation of the MD, in both cases (i)–(ii) for the BRBS regression model. In addition, we illustrate the proposed methodology by applying it to a real-world reliability data set.

4.1 Simulation I: Evaluating performance of ML estimators

In order to assess the performance of the ML estimators for the BRBS model parameters, we assume that

$$\mu_{1_i} = \exp(\beta_{01} + \beta_{11} x_{11_i}), \qquad \mu_{2_i} = \exp(\beta_{02} + \beta_{12} x_{12_i}), \quad i = 1, \dots, n.$$

The sample sizes and true values of the parameters considered are $n = 50, 100, 300, 500$, $\delta_1 = 0.5, 2.0$, $\delta_2 = 0.5, 2.0$ and $\rho = -0.90, -0.50, -0.25, 0.25, 0.50, 0.90$. The regression coefficients to be considered are $\beta_{01} = 1.0$, $\beta_{11} = 0.5$, $\beta_{02} = 2.0$ and $\beta_{12} = 1.5$, with 5000 Monte Carlo replications for each sample size. The statistical indicators to evaluate the performance of the ML estimators are the empirical bias and mean square error (MSE), which are presented in Tables 2 ($\delta_k = 0.5$, for $k = 1, 2$) and 3 ($\delta_k = 2.0$, for $k = 1, 2$). From these tables, we observe that the bias and MSE become smaller as the sample size n increases, as expected. Furthermore, we note that, as the values of the correlation increase negatively or positively, the performance of the estimators of the intercepts β_{0k}, for $k = 1, 2$, improve. For example, when $n = 50$, $\rho = -0.25$ and $\delta_k = 0.50$, with $k = 1, 2$, the bias of $\widehat{\beta}_{01}$ and $\widehat{\beta}_{02}$ are -0.0477 and -0.0508, as well as -0.0308 and -0.0302, respectively, when $\rho = -0.95$. A similar behavior is detected when the correlation is positive. Nevertheless, the performance of the remaining estimators does not seem to depend on ρ.

Table 2 *Empirical bias and MSE (in parenthesis) of the ML estimators ($\delta_k = 0.50$, for $k = 1, 2$), for the BRBS regression*

ρ	n	ML estimator						
		$\widehat{\beta}_{01}$	$\widehat{\beta}_{11}$	$\widehat{\delta}_1$	$\widehat{\beta}_{02}$	$\widehat{\beta}_{22}$	$\widehat{\delta}_2$	$\widehat{\rho}$
−0.90	50	−0.0308 (0.1775)	0.0058 (0.1978)	0.0354 (0.0137)	−0.0302 (0.1885)	0.0086 (0.2165)	0.0365 (0.0151)	−0.0025 (0.0008)
	100	−0.0198 (0.1472)	0.0078 (0.1624)	0.0189 (0.0060)	−0.0117 (0.1556)	0.0058 (0.1778)	0.0184 (0.0061)	−0.0010 (0.0003)
	300	−0.0061 (0.1316)	0.0018 (0.1388)	0.0054 (0.0017)	−0.0027 (0.1349)	0.0030 (0.1395)	0.0051 (0.0017)	−0.0005 (0.0001)
	500	−0.0033 (0.1299)	0.0008 (0.1336)	0.0031 (0.0010)	−0.0021 (0.1311)	0.0003 (0.1312)	0.0030 (0.0010)	−0.0004 (0.0001)
−0.50	50	−0.0446 (0.2250)	0.0128 (0.3978)	0.0454 (0.0153)	−0.0460 (0.2541)	0.0144 (0.4701)	0.0467 (0.0166)	−0.0050 (0.0129)
	100	−0.0279 (0.1738)	0.0101 (0.2683)	0.0237 (0.0064)	−0.0148 (0.1858)	0.0103 (0.3058)	0.0220 (0.0062)	−0.0023 (0.0061)
	300	−0.0087 (0.1398)	0.0038 (0.1763)	0.0069 (0.0017)	−0.0050 (0.1433)	0.0003 (0.1777)	0.0068 (0.0017)	−0.0017 (0.0019)
	500	−0.0041 (0.1357)	0.0013 (0.1560)	0.0039 (0.0010)	−0.0049 (0.1362)	0.0041 (0.1525)	0.0044 (0.0011)	−0.0013 (0.0011)
−0.25	50	−0.0477 (0.2400)	0.0091 (0.4603)	0.0484 (0.0157)	−0.0508 (0.2714)	0.0146 (0.5479)	0.0499 (0.0169)	−0.0034 (0.0201)
	100	−0.0293 (0.1824)	0.0120 (0.3015)	0.0251 (0.0064)	−0.0160 (0.1952)	0.0118 (0.3449)	0.0233 (0.0062)	−0.0020 (0.0094)
	300	−0.0095 (0.1423)	0.0040 (0.1879)	0.0073 (0.0017)	−0.0059 (0.1457)	0.0010 (0.1891)	0.0076 (0.0018)	−0.0016 (0.0029)
	500	−0.0044 (0.1374)	0.0011 (0.1628)	0.0042 (0.0010)	−0.0062 (0.1374)	0.0046 (0.1589)	0.0050 (0.0011)	−0.0011 (0.0018)
0.25	50	−0.0455 (0.2421)	0.0045 (0.4652)	0.0484 (0.0156)	−0.0526 (0.2649)	0.0148 (0.5439)	0.0509 (0.0172)	−0.0025 (0.0195)
	100	−0.0267 (0.1834)	0.0064 (0.3034)	0.0251 (0.0064)	−0.0164 (0.1959)	0.0132 (0.3494)	0.0243 (0.0065)	−0.0005 (0.0093)
	300	−0.0091 (0.1428)	0.0028 (0.1886)	0.0073 (0.0017)	−0.0071 (0.1448)	0.0013 (0.1888)	0.0082 (0.0018)	−0.0004 (0.0029)
	500	−0.0045 (0.1374)	0.0006 (0.1628)	0.0042 (0.0010)	−0.0078 (0.1367)	0.0056 (0.1586)	0.0054 (0.0011)	−0.0001 (0.0017)
0.50	50	−0.0405 (0.2288)	0.0020 (0.4058)	0.0455 (0.0153)	−0.0484 (0.2433)	0.0137 (0.4677)	0.0478 (0.0166)	0.0048 (0.0122)
	100	−0.0229 (0.1757)	0.0024 (0.271)	0.0237 (0.0063)	−0.0157 (0.1860)	0.0123 (0.3118)	0.0236 (0.0065)	0.0010 (0.0059)
	300	−0.0082 (0.1406)	0.0018 (0.1775)	0.0069 (0.0017)	−0.0077 (0.1418)	0.0009 (0.1773)	0.0079 (0.0018)	0.0003 (0.0018)
	500	−0.0044 (0.1356)	0.0002 (0.1558)	0.0040 (0.0010)	−0.0072 (0.1349)	0.0055 (0.1521)	0.0051 (0.0011)	0.0002 (0.0011)
0.90	50	−0.0267 (0.1800)	0.0019 (0.2018)	0.0357 (0.0138)	−0.0321 (0.1792)	0.0071 (0.2185)	0.0361 (0.0137)	0.0026 (0.0008)
	100	−0.0139 (0.1491)	0.0034 (0.1669)	0.0189 (0.0060)	−0.0130 (0.1519)	0.0065 (0.1783)	0.0192 (0.0060)	0.0010 (0.0004)
	300	−0.0053 (0.1325)	0.0002 (0.1397)	0.0054 (0.0017)	−0.0058 (0.1324)	0.0003 (0.1395)	0.0060 (0.0017)	0.0004 (0.0002)
	500	−0.0036 (0.1298)	0.0002 (0.1331)	0.0031 (0.0010)	−0.0056 (0.1294)	0.0033 (0.1310)	0.0036 (0.0011)	0.0003 (0.0001)

Table 3 *Empirical bias and MSE (in parenthesis) of the ML estimators ($\delta_k = 2.0$, for $k = 1, 2$), for the BRBS regression*

		ML estimator						
ρ	n	$\widehat{\beta}_{01}$	$\widehat{\beta}_{11}$	$\widehat{\delta}_1$	$\widehat{\beta}_{02}$	$\widehat{\beta}_{22}$	$\widehat{\delta}_2$	$\widehat{\rho}$
−0.90	50	−0.0140 (0.1442)	0.0038 (0.1524)	0.1410 (0.2161)	−0.0130 (0.1495)	0.0054 (0.1599)	0.1453 (0.2361)	−0.0021 (0.0009)
	100	−0.0097 (0.1322)	0.0044 (0.1394)	0.0760 (0.0958)	−0.0039 (0.1384)	0.0034 (0.1467)	0.0738 (0.0974)	−0.0008 (0.0007)
	300	−0.0029 (0.1272)	0.0014 (0.1303)	0.0218 (0.0271)	−0.0020 (0.1294)	0.0022 (0.1307)	0.0205 (0.0270)	−0.0005 (0.0003)
	500	−0.0015 (0.1271)	0.0006 (0.1284)	0.0124 (0.0161)	−0.0012 (0.1274)	0.0011 (0.1268)	0.0121 (0.0163)	−0.0004 (0.0001)
−0.50	50	−0.0201 (0.1623)	0.0081 (0.2363)	0.1718 (0.2353)	−0.0209 (0.1736)	0.0104 (0.2648)	0.1774 (0.2551)	−0.0042 (0.0129)
	100	−0.0134 (0.1428)	0.0073 (0.1845)	0.0907 (0.1002)	−0.0147 (0.1515)	0.0065 (0.2041)	0.0842 (0.0986)	−0.0020 (0.0060)
	300	−0.0044 (0.1303)	0.0030 (0.1471)	0.0264 (0.0275)	−0.0031 (0.1327)	0.0042 (0.1476)	0.0262 (0.0284)	−0.0015 (0.0019)
	500	−0.0020 (0.1296)	0.0002 (0.1381)	0.0153 (0.0163)	−0.0049 (0.1294)	0.0009 (0.1355)	0.0172 (0.0172)	−0.0012 (0.0011)
−0.25	50	−0.0213 (0.1688)	0.0067 (0.2652)	0.1816 (0.2392)	−0.0233 (0.1809)	0.0174 (0.2996)	0.1878 (0.2572)	−0.0029 (0.0201)
	100	−0.0139 (0.1469)	0.0064 (0.2001)	0.0953 (0.1012)	−0.0053 (0.1556)	0.0112 (0.2227)	0.0881 (0.0982)	−0.0018 (0.0095)
	300	−0.0046 (0.1314)	0.0034 (0.1528)	0.0279 (0.0276)	−0.0024 (0.1338)	0.0009 (0.1532)	0.0290 (0.0287)	−0.0015 (0.0030)
	500	−0.0021 (0.1304)	0.0005 (0.1413)	0.0161 (0.0163)	−0.0038 (0.1299)	0.0004 (0.1385)	0.0194 (0.0174)	−0.0011 (0.0018)
0.25	50	−0.0172 (0.1705)	0.0016 (0.2683)	0.1720 (0.2389)	−0.0223 (0.1785)	0.0098 (0.2995)	0.1813 (0.2624)	−0.0041 (0.0194)
	100	−0.0101 (0.1480)	0.0014 (0.2016)	0.0907 (0.1012)	−0.0061 (0.1551)	0.0069 (0.2233)	0.0898 (0.1019)	−0.0006 (0.0093)
	300	−0.0037 (0.1316)	0.0013 (0.1533)	0.0264 (0.0276)	−0.0043 (0.1333)	0.0018 (0.1532)	0.0305 (0.0297)	−0.0008 (0.0029)
	500	−0.0021 (0.1304)	0.0004 (0.1413)	0.0153 (0.0163)	−0.0033 (0.1297)	0.0013 (0.1384)	0.0197 (0.0175)	−0.0003 (0.0017)
0.50	50	−0.0116 (0.1650)	0.0022 (0.2411)	0.1420 (0.2349)	−0.0152 (0.1694)	0.0049 (0.2650)	0.1436 (0.2546)	−0.0022 (0.0122)
	100	−0.0065 (0.1449)	0.0020 (0.1872)	0.0760 (0.1001)	−0.0061 (0.1502)	0.0035 (0.2048)	0.0769 (0.1029)	−0.0007 (0.0058)
	300	−0.0022 (0.1307)	0.0003 (0.1479)	0.0218 (0.0275)	−0.0028 (0.1319)	0.0023 (0.1477)	0.0242 (0.0298)	−0.0003 (0.0018)
	500	−0.0017 (0.1295)	0.0002 (0.1379)	0.0125 (0.0163)	−0.0024 (0.1288)	0.0003 (0.1353)	0.0145 (0.0175)	−0.0003 (0.0011)
0.90	50	−0.0195 (0.1459)	0.0027 (0.1552)	0.1815 (0.2165)	−0.0241 (0.1445)	0.0109 (0.16039)	0.1913 (0.2158)	−0.0022 (0.0074)
	100	−0.0120 (0.1339)	0.0029 (0.1419)	0.0953 (0.0955)	−0.0060 (0.1352)	0.0076 (0.1466)	0.0917 (0.0957)	−0.0003 (0.0036)
	300	−0.0043 (0.1276)	0.0022 (0.1312)	0.0279 (0.0271)	−0.0048 (0.1277)	0.0021 (0.1308)	0.0314 (0.0281)	−0.0005 (0.0012)
	500	−0.0022 (0.1270)	0.0003 (0.1281)	0.0231 (0.0161)	−0.0025 (0.1263)	0.0006 (0.1267)	0.0208 (0.0168)	−0.0002 (0.0007)

Table 4 *Summary statistics from simulation analysis of the transformed MD ($\delta_k = 0.5$, for $k = 1, 2$) for the BRBS regression*

ρ	n	Mean	SD	CS	CK
−0.90	50	0.0350	0.9503	0.0531	2.4654
	100	0.0232	0.9628	0.1111	2.5861
	300	0.0156	0.9707	0.1453	2.6761
	500	0.0142	0.9719	0.1548	2.6985
−0.50	50	0.0376	0.9469	0.0411	2.4588
	100	0.0245	0.9612	0.1042	2.5822
	300	0.0161	0.9701	0.1432	2.6734
	500	0.0145	0.9716	0.1534	2.6965
−0.25	50	0.0384	0.9458	0.0378	2.4557
	100	0.0249	0.9606	0.1024	2.5804
	300	0.0162	0.9698	0.1426	2.6730
	500	0.0146	0.9715	0.1529	2.6961
0.25	50	0.0383	0.9459	0.0379	2.4544
	100	0.0250	0.9605	0.1025	2.5790
	300	0.0162	0.9698	0.1428	2.6739
	500	0.0145	0.9715	0.1528	2.6964
0.50	50	0.0376	0.9468	0.0424	2.4566
	100	0.0246	0.9610	0.1046	2.5803
	300	0.0162	0.9700	0.1436	2.6750
	500	0.0145	0.9716	0.1532	2.6972
0.90	50	0.0353	0.9498	0.0551	2.4621
	100	0.0234	0.9626	0.1110	2.5844
	300	0.0156	0.9706	0.1456	2.6773
	500	0.0142	0.9719	0.1547	2.6990

4.2 Simulation II: Assessing normality of the transformed MD

In order to evaluate the normality transformation of the MD for the BRBS regression model, we consider the empirical distributions of this transformed MD under the same scenario of the simulation study I. The results are reported in Tables 4 ($\delta_k = 0.5$, for $k = 1, 2$) and 5 ($\delta_k = 2.0$, for $k = 1, 2$). These tables present the empirical mean, standard deviation (SD), coefficient of skewness (CS) and coefficient of kurtosis (CK), whose values are expected to be zero, one, zero, and three, respectively. The results reported into Tables 4 and 5 indicate that the MD transformation conforms well with the normal distribution, which is an important result since it allows us to use the MD transformation as a measure of model adjustment. In general, the results do not seem to depend on ρ.

4.3 Application to real data

We illustrate the proposed methodology with a data set related to die fracture presented in Lepadatu et al. (2005) and described in the case study of Section 2. We consider the regression structure defined in (3.3) for establishing our statistical model with link functions for the bivariate response, and $i = 1, \ldots, 15$, given by

$$\mu_{1_i} = \exp\big(\beta_{10} + \beta_{11} x_{1_i}^{(1)} + \beta_{12} x_{2_i}^{(1)} + \beta_{13} x_{3_i}^{(1)}\big), \tag{4.1}$$

$$\mu_{2_i} = \exp\big(\beta_{20} + \beta_{21} x_{1_i}^{(2)} + \beta_{22} x_{2_i}^{(2)} + \beta_{23} x_{3_i}^{(2)}\big), \tag{4.2}$$

recalling that $\mu_{1_i} = \mathrm{E}(T_{1_i})$, with T_{1_i} being the von Mises stress (in N/mm^2), $\mu_{2_i} = \mathrm{E}(T_{2_i})$, and T_{2_i} being the die lifetime (in number of cycles), where x_{1_i} is the value of the friction

Table 5 *Summary statistics from simulation analysis of the transformed MD ($\delta_k = 2.0$, for $k = 1, 2$) for the BRBS regression*

ρ	n	Mean	SD	CS	CK
−0.90	50	0.0295	0.9574	0.0835	2.4877
	100	0.0202	0.9665	0.1275	2.6050
	300	0.0146	0.9719	0.1509	2.6834
	500	0.0136	0.9726	0.1582	2.7032
−0.50	50	0.0306	0.9560	0.0765	2.4838
	100	0.0208	0.9658	0.1238	2.6026
	300	0.0148	0.9716	0.1500	2.6822
	500	0.0138	0.9725	0.1575	2.7021
−0.25	50	0.0311	0.9554	0.0749	2.4816
	100	0.0210	0.9656	0.1227	2.6018
	300	0.0149	0.9715	0.1497	2.6823
	500	0.0137	0.9725	0.1573	2.7020
0.25	50	0.0311	0.9554	0.0751	2.4804
	100	0.0211	0.9655	0.1233	2.6010
	300	0.0149	0.9714	0.1498	2.6832
	500	0.0137	0.9725	0.1572	2.7023
0.50	50	0.0308	0.9558	0.0775	2.4813
	100	0.0209	0.9656	0.1247	2.6012
	300	0.0149	0.9715	0.1503	2.6837
	500	0.0137	0.9725	0.1574	2.7028
0.90	50	0.0297	0.9570	0.0846	2.4842
	100	0.0203	0.9663	0.1286	2.6029
	300	0.0147	0.9718	0.1513	2.6847
	500	0.0137	0.9727	0.1582	2.7037

Table 6 *ML estimates, SEs and p-values for the indicated parameter with the fatigue data*

Parameter	Estimate	SE	p-value
δ_1	4.301	1.538	–
δ_2	4.763	1.882	–
β_{10}	10.138	1.826	<0.001
β_{11}	3.592	6.677	0.591
β_{12}	0.010	0.044	0.819
β_{13}	−0.005	0.001	0.002
β_{20}	5.914	1.705	<0.001
β_{21}	0.777	6.239	0.901
β_{22}	0.008	0.042	0.848
β_{23}	0.005	0.001	<0.001
ρ	−0.657	0.134	–

coefficient (dimensionless), x_{2_i} is the value of the die angle (in °) and x_{3_i} is the work temperature (in °C), for the specimen i. In addition, in the formulations given in (4.1) and (4.2), β_0, β_1, β_2 and β_3 are the regression coefficients to be estimated for obtaining the predictive model. Table 6 reports the ML estimates, standard errors (SEs) and p-values of the corresponding t-test for the BRBS regression parameters. The results of this table reveal that only

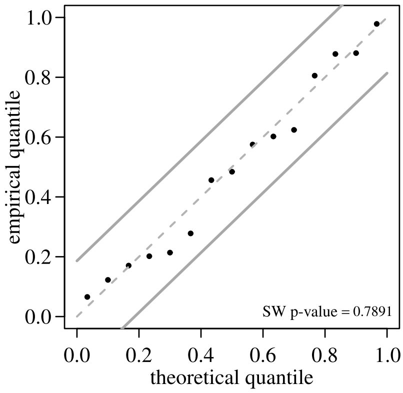

Figure 4 *QQ plots with acceptance bands at* 5% *and p-value of the SW test for the transformed MD with the fatigue data.*

the predictor X_3 is significant at 5%. Hence, the final predictive model is given by

$$\begin{aligned} \widehat{\mu}_{1_{\text{pred}}} &= \exp(10.823 - 0.005 x_{3_{\text{pred}}}), \\ \widehat{\mu}_{2_{\text{pred}}} &= \exp(6.255 + 0.006 x_{3_{\text{pred}}}). \end{aligned} \tag{4.3}$$

In order to check whether the BRBS regression model describes the fatigue bivariate data adequately, Figure 4 shows the QQ plot with acceptance bands of the transformed MD for the final BRBS regression model defined in (4.3). From this figure, note that the model provides a good agreement with the considered data. Such an agreement is corroborated by the associated p-value of 0.7891 corresponding to the Shapiro–Wilk (SW) test; see Yap and Sim (2011).

5 Concluding remarks and future research

This paper reported the following findings:

- (i) A new regression model of GLM type based on a bivariate Birnbaum–Saunders distribution has been proposed.
- (ii) The model parameters have been estimated with the maximum likelihood method and inference has been performed using this method to detect the significance of the regression coefficients.
- (iii) A numerical evaluation of the proposed methodology was considered by means of Monte Carlo simulations.
- (iv) By using a case study with real-world reliability data, we have motivated the development of the new bivariate regression model.

In summary, the new bivariate regression model was parameterized by its means permitting us to describe bivariate data in their original scale. The numerical evaluations of the proposed methodology with simulated and real data sets allowed us to show its good performance and its potential applications. We obtained a predictive model which can be a useful knowledge addition to the tool-kit of diverse practitioners, reliability engineers, applied statisticians, and data scientists.

Some open problems that arose from this study are the following:

- (i) The developing of likelihood inferential methods by considering censored data and random effects is of interest in this type of applications; see Villegas, Paula and Leiva (2011) and Desousa et al. (2020).

(ii) Extensions to the multivariate case is also of practical relevance; see Marchant, Leiva and Cysneiros (2016) and Sánchez et al. (2021).

(iii) Incorporation of time series, spatial and quantile regression structures in the modeling, as well as errors-in-variables, functional data analysis and PLS regression, based on the proposed bivariate distribution, are also of interest; see Leiva et al. (2014b, 2021), Vilca, Balakrishnan and Zeller (2014b), Garcia-Papani et al. (2017), Huerta et al. (2019), Martinez, Giraldo and Leiva (2019), Saulo et al. (2019), and Carrasco et al. (2020).

(iv) The derivation of influence diagnostic techniques to detect potential influential cases are needed, which are an important tool to be used in all statistical modeling; see Ibacache-Pulgar, Paula and Galea (2014), Garcia-Papani et al. (2018), Carrasco et al. (2020), Leiva et al. (2020), and Sánchez et al. (2021).

Therefore, the proposed methodology in this investigation promotes new challenges and offers an open door to explore other theoretical and numerical issues. Research on these and other issues are in progress and their findings will be reported in future articles.

Appendix

Consider observations (t_{1_i}, t_{2_i}), for $i = 1, \ldots, n$, from the BRBS distribution defined in (3.2). Note that the log-likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \delta_1, \delta_2, \rho)^\top$ stated in (3.4) based on these observations can be written as

$$\ell = \log(L) = \sum_{i=1}^{n} \log\big[\phi_2\big(a(t_{1_i}), a(t_{2_i}); \rho\big)\big] + \sum_{i=1}^{n}\sum_{k=1}^{2} \log\big[a'(t_{k_i})\big],$$

where

$$a(t_{k_i}) = \sqrt{\frac{\delta_k}{2}}\left[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} - \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\right],$$

$$a'(t_{k_i}) = \sqrt{\frac{\delta_k}{2}}\left[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} + \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\right]\frac{1}{2t_{k_i}},$$

with $\mu_{k_i} = \exp(\beta_{k0} + \beta_{k1}x_{1_i}^{(k)} + \cdots + \beta_{kl}x_{l_i}^{(k)})$, for $k = 1, 2$, $i = 1, \ldots, n$ and $l = p, q$. Next, we provide expressions for the first and second-order derivatives of $\log(L)$.

Let ϕ_2 be the bivariate standard normal joint PDF given in (3.2). Since

$$\frac{\partial \phi_2(u, v; \rho)}{\partial u} = 6 - \frac{\phi_2(u, v; \rho)}{1-\rho^2}(u - \rho v),$$

$$\frac{\partial \phi_2(u, v; \rho)}{\partial v} = -\frac{\phi_2(u, v; \rho)}{1-\rho^2}(v - \rho u),$$

by applying the chain rule with two variables, we have

$$\frac{\partial \log(L)}{\partial \vartheta} = \sum_{i=1}^{n} \frac{\frac{\partial \phi_2(a(t_{1_i}), a(t_{2_i}); \rho)}{\partial a(t_{1_i})}\frac{\partial a(t_{1_i})}{\partial \vartheta} + \frac{\partial \phi_2(a(t_{1_i}), a(t_{2_i}); \rho)}{\partial a(t_{2_i})}\frac{\partial a(t_{2_i})}{\partial \vartheta}}{\phi_2(a(t_{1_i}), a(t_{2_i}); \rho)}$$

$$+ \sum_{i=1}^{n}\sum_{k=1}^{2} \frac{1}{a'(t_{k_i})}\frac{\partial a'(t_{k_i})}{\partial \vartheta}(1 - \delta_{\vartheta,\rho})$$

$$= \frac{1}{1-\rho^2}\sum_{i=1}^{n}\sum_{k=1}^{2}\Big[\rho a(t_{k_i})\frac{\partial a(t_{k_i})}{\partial \vartheta} + \frac{(1-\delta_{\vartheta,\rho})}{a'(t_{k_i})}\frac{\partial a'(t_{k_i})}{\partial \vartheta}\Big]$$
$$- \frac{1}{1-\rho^2}\sum_{i=1}^{n}\Big[a(t_{2_i})\frac{\partial a(t_{1_i})}{\partial \vartheta} + a(t_{1_i})\frac{\partial a(t_{2_i})}{\partial \vartheta}\Big], \quad \vartheta \in \{\beta_{kr}, \delta_k, \rho\},$$

for $k = 1, 2$, $r = 0, 1, \ldots, l$ and $l = p, q$, where $\delta_{x,y}$ is the Kronecker delta function, that is, $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ in otherwise.

The second-order partial derivatives of $\log(L)$ may be obtained as

$$\frac{\partial^2 \log(L)}{\partial \vartheta' \partial \vartheta} = \frac{\rho}{1-\rho^2}\sum_{i=1}^{n}\sum_{k=1}^{2}\Big[\frac{\partial a(t_{k_i})}{\partial \vartheta'}\frac{\partial a(t_{k_i})}{\partial \vartheta} + a(t_{k_i})\frac{\partial^2 a(t_{k_i})}{\partial \vartheta' \partial \vartheta}\Big]$$
$$+ \frac{1}{1-\rho^2}\sum_{i=1}^{n}\sum_{k=1}^{2}\Big\{\frac{1}{[a'(t_{k_i})]^2}\frac{\partial a'(t_{k_i})}{\partial \vartheta'}\frac{\partial a'(t_{k_i})}{\partial \vartheta} + \frac{1}{a'(t_{k_i})}\frac{\partial^2 a'(t_{k_i})}{\partial \vartheta' \partial \vartheta}\Big\}$$
$$- \frac{1}{1-\rho^2}\sum_{i=1}^{n}\Big[\frac{\partial a(t_{2_i})}{\partial \vartheta'}\frac{\partial a(t_{1_i})}{\partial \vartheta} + a(t_{2_i})\frac{\partial^2 a(t_{1_i})}{\partial \vartheta' \partial \vartheta}\Big]$$
$$- \frac{1}{1-\rho^2}\sum_{i=1}^{n}\Big[\frac{\partial a(t_{1_i})}{\partial \vartheta'}\frac{\partial a(t_{2_i})}{\partial \vartheta} + a(t_{1_i})\frac{\partial^2 a(t_{2_i})}{\partial \vartheta' \partial \vartheta}\Big], \quad \vartheta' \neq \rho,$$

for each $\vartheta', \vartheta \in \{\beta_{kr}, \delta_k, \rho\}$, and

$$\frac{\partial^2 \log(L)}{\partial \rho \partial \vartheta} = \frac{1}{1-\rho^2}\Big[2\rho\frac{\partial \log(L)}{\partial \vartheta} - \sum_{i=1}^{n}\sum_{k=1}^{2} a(t_{k_i})\frac{\partial a(t_{k_i})}{\partial \vartheta}\Big]$$
$$= \frac{3\rho^2 - 1}{(1-\rho^2)^2}\sum_{i=1}^{n}\sum_{k=1}^{2} a(t_{k_i})\frac{\partial a(t_{k_i})}{\partial \vartheta}$$
$$- \frac{2\rho}{(1-\rho^2)^2}\sum_{i=1}^{n}\Big[a(t_{2_i})\frac{\partial a(t_{1_i})}{\partial \vartheta} + a(t_{1_i})\frac{\partial a(t_{2_i})}{\partial \vartheta}\Big], \quad \vartheta \in \{\beta_{kr}, \delta_k, \rho\}.$$

A straightforward computation shows that the above first-order partial derivatives of $a(t_{k_i})$ and $a'(t_{k_i})$ are given by

$$\frac{\partial a(t_{k_i})}{\partial \beta_{kr}} = -\frac{1}{2}\sqrt{\frac{\delta_k}{2}}\Big[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} + \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Big][x_{r_i}^{(k)}(1-\delta_{r,0}) + \delta_{r,0}],$$
$$\frac{\partial a(t_{k_i})}{\partial \delta_k} = -\frac{1}{4}\sqrt{\frac{2}{\delta_k}}\Big[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} - \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Big],$$
$$\frac{\partial a'(t_{k_i})}{\partial \beta_{kr}} = -\frac{1}{4}\sqrt{\frac{\delta_k}{2}}\Big[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} - \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Big]\frac{1}{t_{k_i}}[x_{r_i}^{(k)}(1-\delta_{r,0}) + \delta_{r,0}],$$
$$\frac{\partial a'(t_{k_i})}{\partial \delta_k} = -\frac{1}{8}\sqrt{\frac{2}{\delta_k}}\Big[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} + \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Big]\frac{1}{t_{k_i}},$$

where $\delta_{r,0}$ is the Kronecker delta function.

The second-order partial derivatives of $a(t_{k_i})$ can be written as

$$\frac{\partial^2 a(t_{k_i})}{\partial \beta_{kr}^2} = -\frac{(\delta_k+1)}{\delta_k \mu_{k_i}}\Big[\frac{\partial a(t_{k_i})}{\partial \beta_{kr}} + \frac{1}{4}\sqrt{\frac{2}{\delta_k}}\frac{(\delta_k+1)}{\delta_k \mu_{k_i}}\frac{\partial a(t_{k_i})}{\partial \delta_k}\Big][x_{r_i}^{(k)}(1-\delta_{r,0}) + \delta_{r,0}],$$

$$\frac{\partial^2 a(t_{k_i})}{\partial \delta_k^2} = \frac{\delta_k}{8}\sqrt{\frac{2}{\delta_k}}\Bigg[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} - \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Bigg],$$

$$\frac{\partial^2 a(t_{k_i})}{\partial \delta_k \partial \beta_{kr}} = \frac{\delta_k}{4}\Bigg[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} + \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Bigg]\big[x_{r_i}^{(k)}(1-\delta_{r,0})+\delta_{r,0}\big].$$

Finally, the second-order partial derivatives of $a'(t_{k_i})$ are expressed as

$$\frac{\partial^2 a'(t_{k_i})}{\partial \beta_{kr}^2} = -\frac{1}{4}\frac{(\delta_k+1)}{\delta_k \mu_{k_i}}\Bigg[\frac{\partial a'(t_{k_i})}{\partial \beta_{kr}} + \sqrt{\frac{2}{\delta_k}}\frac{(\delta_k+1)}{\delta_k \mu_{k_i}}\frac{\partial a'(t_{k_i})}{\partial \delta_k}\Bigg]\frac{1}{t_{k_i}}\big[x_{r_i}^{(k)}(1-\delta_{r,0})+\delta_{r,0}\big],$$

$$\frac{\partial^2 a'(t_{k_i})}{\partial \delta_k^2} = \frac{\delta_k}{32}\sqrt{\frac{2}{\delta_k}}\Bigg[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} + \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Bigg]\frac{1}{t_{k_i}^2},$$

$$\frac{\partial^2 a'(t_{k_i})}{\partial \delta_k \partial \beta_{kr}} = \frac{\delta_k}{16}\Bigg[\sqrt{\frac{(\delta_k+1)t_{k_i}}{\delta_k \mu_{k_i}}} - \sqrt{\frac{\delta_k \mu_{k_i}}{(\delta_k+1)t_{k_i}}}\Bigg]\frac{1}{t_{k_i}}\big[x_{r_i}^{(k)}(1-\delta_{r,0})+\delta_{r,0}\big].$$

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