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A switch convergence for a small perturbation of a linear recurrence equation

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Abstract. In this article we study a small random perturbation of a linear recurrence equation. If all the roots of its corresponding characteristic equation have modulus strictly less than one, the random linear recurrence goes exponentially fast to its limiting distribution in the total variation distance as time increases. By assuming that all the roots of its corresponding characteristic equation have modulus strictly less than one and rather mild conditions, we prove that this convergence happens as a switch-type, i.e., there is a sharp transition in the convergence to its limiting distribution. This fact is known as a cut-off phenomenon in the context of stochastic processes.

1 Introduction

Linear recurrence equations have been widely used in several areas of applied mathematics and computer science. In applied science, they can be used to model the future of a process that depends linearly on a finite string, for instance: in population dynamics to model population size and structure [Allen and Nowak (2015), Dubeau (1993), Smale and Williams (1976)]; in economics to model the interest rate, the amortization of a loan and price fluctuations [Ferguson (1960), Flannery and James (1984), Klee (1986)]; in computer science for analysis of algorithms [Cormen et al. (2009), Ouaknine and Worrell (2014)]; in statistics for the autoregressive linear model [Akaike (1969), Dahlhaus (1997)]. In theoretical mathematics, for instance: in differential equations to find the coefficients of series solutions [Chapters 4–5 in Coddington and Levinson (1987)]; in the proof of Hilbert's tenth problem over $\mathbb{Z}$ by Matiyasevich (1993); and in approximation theory to provide expansions of some second order operators by Spigler and Vianello (1992). For a complete understanding of applications of the linear recurrence equations, we recommend the Introduction of the monograph by Everest et al. (2003) and the references therein.

We consider a random dynamics that arises from a linear homogeneous recurrence equation with control term given by independent and identically distributed (i.i.d. for short) random variables with Gaussian distribution. To be precise, *given* $p \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_p \in \mathbb{R}$ with $\phi_p \neq 0$, we define the linear homogeneous recurrence of degree p as follows:

$$x_{t+p} = \phi_1 x_{t+p-1} + \phi_2 x_{t+p-2} + \cdots + \phi_p x_t \quad \text{for any } t \in \mathbb{N}_0, \tag{L}$$

where $\mathbb{N}_0$ denotes the set of non-negative integers. To single out a unique solution of (L) one should assign initial conditions $x_0, x_1, \ldots, x_{p-1} \in \mathbb{R}$. Recurrence (L) is called a recurrence with p-history since it only depends on a p-number of earlier values.

We consider a small perturbation of (L) by adding Gaussian noise as follows: given $\epsilon > 0$ fixed, consider the random dynamics

$$X^{(\epsilon)}_{t+p} = \phi_1 X^{(\epsilon)}_{t+p-1} + \phi_2 X^{(\epsilon)}_{t+p-2} + \cdots + \phi_p X^{(\epsilon)}_t + \epsilon \xi_{t+p} \quad \text{for any } t \in \mathbb{N}_0, \tag{SL}$$

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with initial conditions $X_0^{(\epsilon)} = x_0, \ldots, X_{p-1}^{(\epsilon)} = x_{p-1}$, and $(\xi_t : t \geq p)$ is a sequence of i.i.d. random variables with Gaussian distribution with zero mean and variance one. Since the random linear recurrence (SL) depends on its p-string past, it is not Markovian. However, it is straightforward to convert as a linear first-order random matrix recurrence, which is Markovian. Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space where the sequence $(\xi_t : t \geq p)$ is defined, then the random dynamics (SL) can be defined as a stochastic process in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Notice that $\epsilon > 0$ is a parameter that controls the magnitude of the noise. When $\epsilon = 0$ the deterministic model (L) is recovered from the stochastic model (SL). Since $(\xi_t : t \geq p)$ is a sequence of i.i.d. random variables with Gaussian distribution, the model (SL) could be understood as a regularization of (L).

Up to our knowledge, this type of model was originally used by Yule (1927) ($p = 2$) to model the presence of random disturbances of a harmonic oscillator for investigating hidden periodicities and their relation to the observations of sunspots.

In this article, we obtain a *nearly-complete characterization* of the convergence in the total variation distance between the distribution of $X_t^{(\epsilon)}$ and its limiting distribution as t increases. Under general conditions, that we state in Section 2, when the intensity of the control ϵ is fixed, as the time goes by, the random linear recurrence goes to a limiting distribution in the total variation distance. We show that this convergence is actually abrupt in the following sense: the total variation distance between the distribution of the random linear recurrence and its limiting distribution drops abruptly over a negligible time (time window) around a threshold time (cut-off time) from near one to near zero. It means that if we run the random linear recurrence before a time window around the cut-off time the process is not well mixed and after a time window around the cut-off time becomes well mixed. This fact is known as a cut-off phenomenon in the context of stochastic processes.

Suppose that we model a system by a random process $(X_t^{(\epsilon)} : t \geq 0)$, where the parameter ϵ denotes the intensity of the noise and assume that $X_\infty^{(\epsilon)}$ is its equilibrium. A natural question that arises is the following: *with a fixed ϵ and an error $\eta > 0$, how much time $\tau(\epsilon, \eta)$ do we need to run the model $(X_t^{(\epsilon)} : t \geq 0)$ in order to be close to its equilibrium $X_\infty^{(\epsilon)}$ with an error at most η in a suitable distance?* The latter is known as a *mixing time* in the context of random processes. In general, it is hard to compute and/or estimate $\tau(\epsilon, \eta)$. The cut-off phenomenon provides a strong answer in a small regime ϵ. Roughly speaking, as ϵ goes to zero, it means that after a deterministic time $\tau^*(\epsilon)$ the system is "almost" in its equilibrium within any error η. We provide a precise definition in Section 2.

The cut-off phenomenon was extensively studied in the eighties to describe the phenomenon of abrupt convergence which appears for example, in models of cards' shuffling, Ehrenfests' urn and random transpositions, see for instance Diaconis (1996). In general, it is a challenging problem to prove that a specific model exhibits a cut-off phenomenon. It requires a complete understanding of the dynamics of the specific random process. For an introduction to this concept, we recommend Chapter 18 in Levin and Peres (2017) for discrete Markov chains in a finite state, Martínez and Ycart (2001) for discrete Markov chains with infinite countable state space and [Barrera and Jara (2016, 2020), Barrera (2018)] for Stochastic Differential Equations in a continuous state space.

This article is organized as follows: In Section 2, we state the main result and its consequences. In Section 3, we give the proof of Theorem 2.1 which is the main result of this article. Also, we give conditions to verify the hypothesis of Theorem 2.1. In Section 4, we provide a complete understanding how to verify the conditions of Theorem 2.1 for a discretization of the celebrated Brownian oscillator. Lastly, some results about the distribution of the random linear recurrence and its limiting behavior present in Appendix A, Appendix B which summarizes some properties about the total variation distance between Gaussian distributions, and Appendix C which states some elementary limit behaviors.

2 Main theorem

One of the most important problems in dynamical systems is the study of the limit behavior of their evolution for forward times. To the linear recurrence (L), we can associate a characteristic polynomial

$$f(\lambda) = \lambda^p - \phi_1 \lambda^{p-1} - \cdots - \phi_p \quad \text{for any } \lambda \in \mathbb{C}. \tag{2.1}$$

From now on to the end of this article, we assume

$$\text{all the roots of (2.1) have modulus strictly less than one.} \tag{H}$$

From (H), we can prove that for any string of initial values $x_0, \ldots, x_{p-1} \in \mathbb{R}$, x_t goes exponentially fast to zero as t goes to infinity. For more details, see Theorem 1 in Lueker (1980). In the stochastic model (SL), (H) implies that the process $(X_t^{(\epsilon)}, t \in \mathbb{N}_0)$ is strongly ergodic, that is, for any initial data $x_0, \ldots, x_{p-1}$, the random recurrence $X_t^{(\epsilon)}$ converges in the so-called total variation distance as t goes to infinity to a random variable $X_\infty^{(\epsilon)}$. For further details see Lemma A.2 in Appendix A.

Given $m \in \mathbb{R}$ and $\sigma^2 \in (0, +\infty)$, denote by $\mathcal{N}(m, \sigma^2)$ the Gaussian distribution with mean m and variance σ^2. Later on, we see that for $t \geq p$ the random variable $X_t^{(\epsilon)}$ has distribution $\mathcal{N}(x_t, \epsilon^2 \sigma_t^2)$, where x_t is given by (L) and $\sigma_t^2 \in (0, +\infty)$. Moreover, the random variable $X_\infty^{(\epsilon)}$ has distribution $\mathcal{N}(0, \epsilon^2 \sigma_\infty^2)$ with $\sigma_\infty^2 \in (0, +\infty)$.

Since the random recurrence (SL) is linear in the inputs which are independent Gaussian random variables, the distribution of $X_t^{(\epsilon)}$ (for $t \geq p$) and its limiting distribution $X_\infty^{(\epsilon)}$ are also Gaussian. For details, see Lemma A.1 and Lemma A.2. Then a natural way to measure its discrepancy is by the total variation distance. Given two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on the measure space $(\Omega, \mathcal{F})$, the total variation distance between the probabilities $\mathbb{P}_1$ and $\mathbb{P}_2$ is given by

$$\mathbf{d}_{\mathrm{TV}}(\mathbb{P}_1, \mathbb{P}_2) = \sup_{F \in \mathcal{F}} \left| \mathbb{P}_1(F) - \mathbb{P}_2(F) \right|.$$

When X, Y are random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we write $\mathbf{d}_{\mathrm{TV}}(X, Y)$ instead of $\mathbf{d}_{\mathrm{TV}}(\mathbb{P}(X \in \cdot), \mathbb{P}(Y \in \cdot))$, where $\mathbb{P}(X \in \cdot)$ and $\mathbb{P}(Y \in \cdot)$ denote the distribution of X and Y under $\mathbb{P}$, respectively. Then we define

$$d^{(\epsilon)}(t) := \mathbf{d}_{\mathrm{TV}}\big(X_t^{(\epsilon)}, X_\infty^{(\epsilon)}\big) = \mathbf{d}_{\mathrm{TV}}\big(\mathcal{N}\big(x_t, \epsilon^2 \sigma_t^2\big), \mathcal{N}\big(0, \epsilon^2 \sigma_\infty^2\big)\big)$$

for any $t \geq p$. Notice that the above distance depends on the initial conditions $x_0, \ldots, x_{p-1} \in \mathbb{R}$. To make the notation more fluid, *we avoid its dependence from our notation.* For a complete understanding of the total variation distance between two arbitrary probabilities with densities, we recommend Section 3.3 in Reiss (2012) and Section 2.2 in DasGupta (2008). Nevertheless, for the sake of completeness, we provide an Appendix B that contains the properties and bounds for the total variation distance between Gaussian distributions that we used to prove Theorem 2.1, which is the main theorem of this article.

The goal is to study of the so-called *cut-off phenomenon* in the total variation distance when ϵ goes to zero for the family of the stochastic processes

$$\big(X^{(\epsilon)} := \big(X_t^{(\epsilon)} : t \in \mathbb{N}_0\big) : \epsilon > 0\big)$$

for *fixed* initial conditions $x_0, \ldots, x_{p-1}$.

In the sequel, we introduce the formal definition of cut-off phenomenon. Recall that for any $z \in \mathbb{R}$, $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. Consider the family of stochastic processes $(X^{(\epsilon)} : \epsilon > 0)$. According to Barrera and Ycart (2014), the cut-off phenomenon can be expressed in three increasingly sharp levels as follows.

Definition 2.1. The family $(X^{(\epsilon)} : \epsilon > 0)$ has

(i) *cut-off* at $(t^{(\epsilon)} : \epsilon > 0)$ with cut-off time $t^{(\epsilon)}$ if $t^{(\epsilon)}$ goes to infinity as ϵ goes to zero and

$$\lim_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor \delta t^{(\epsilon)} \rfloor) = \begin{cases} 1 & \text{if } 0 < \delta < 1, \\ 0 & \text{if } \delta > 1. \end{cases}$$

(ii) *window cut-off* at $((t^{(\epsilon)}, w^{(\epsilon)}) : \epsilon > 0)$ with cut-off time $t^{(\epsilon)}$ and time cut-off $w^{(\epsilon)}$ if $t^{(\epsilon)}$ goes to infinity as ϵ goes to zero, $w^{(\epsilon)} = o(t^{(\epsilon)})$ and

$$\lim_{b\to -\infty} \liminf_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)} + b w^{(\epsilon)} \rfloor) = 1$$

and

$$\lim_{b\to +\infty} \limsup_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)} + b w^{(\epsilon)} \rfloor) = 0.$$

(iii) *profile cut-off* at $((t^{(\epsilon)}, w^{(\epsilon)}) : \epsilon > 0)$ with cut-off time $t^{(\epsilon)}$, time cut-off $w^{(\epsilon)}$ and profile function $G : \mathbb{R} \to [0, 1]$ if $t^{(\epsilon)}$ goes to infinity as ϵ goes to zero, $w^{(\epsilon)} = o(t^{(\epsilon)})$,

$$\lim_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)} + b w^{(\epsilon)} \rfloor) =: G(b) \quad \text{exists for any } b \in \mathbb{R}$$

together with $\lim_{b\to -\infty} G(b) = 1$ and $\lim_{b\to +\infty} G(b) = 0$.

Bearing all this in mind, we can analyze how this convergence happens which is exactly the statement of the following theorem.

Theorem 2.1 (Main theorem). *Assume that* (H) *holds. For a given initial data* $x = (x_0, \dots, x_{p-1}) \in \mathbb{R}^p \setminus \{0_p\}$ *assume that there exist* $r = r(x) \in (0, 1)$, $l = l(x) \in \{1, \dots, p\}$ *and* $v_t = v(t, x) \in \mathbb{R}$ *such that*

(i)

$$\lim_{t\to +\infty} \left| \frac{x_t}{t^{l-1} r^t} - v_t \right| = 0,$$

(ii) $\sup_{t\geq 0} |v_t| < +\infty$,

(iii) $\liminf_{t\to +\infty} |v_t| > 0$.

Then the family of random linear recurrences

$$\big(X^{(\epsilon)} := \big(X^{(\epsilon)}(t) : t \in \mathbb{N}_0\big) : \epsilon > 0\big)$$

has window cut-off as ϵ *goes to zero with cut-off time*

$$t^{(\epsilon)} = \frac{\ln(\frac{1}{\epsilon})}{\ln(\frac{1}{r})} + (l-1)\frac{\ln(\frac{\ln(\frac{1}{\epsilon})}{\ln(\frac{1}{r})})}{\ln(\frac{1}{r})}$$

and time window

$$w^{(\epsilon)} = C + o_\epsilon(1),$$

where C *is any positive constant and* $\lim_{\epsilon\to 0^+} o_\epsilon(1) = 0$. *In other words*,

$$\lim_{b\to -\infty} \liminf_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)} + b w^{(\epsilon)} \rfloor) = 1$$

and

$$\lim_{b\to +\infty} \limsup_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)} + b w^{(\epsilon)} \rfloor) = 0,$$

where $d^{(\epsilon)}(t) = \mathbf{d}_{\mathrm{TV}}(X_t^{(\epsilon)}, X_\infty^{(\epsilon)})$ *for any* $t \geq p$.

Roughly speaking, the argument of the proof consists in fairly intricate calculations of the distributions of $X_t^{(\epsilon)}$, $t \geq p$ and its limiting distribution $X_\infty^{(\epsilon)}$ whose distributions are Gaussian. Then the cut-off phenomenon is proved from a refined analysis of their means and variances, and "explicit calculations and bounds" for the total variation distance between Gaussian distributions. This analysis also provides a delicate case in which the cut-off phenomenon does not occur.

Remark 2.1. Notice that $\sup_{t\geq 0} |v_t| < +\infty$ and $\limsup_{t\to+\infty} |v_t| < +\infty$ are actually equivalent. However, $\liminf_{t\to+\infty} |v_t| > 0$ does not always imply $\inf_{t\geq 0} |v_t| > 0$.

Remark 2.2. Roughly speaking, the number r corresponds to the absolute value of some roots of (2.1) and l is related to their multiplicities.

Remark 2.3. Under the conditions of Theorem 2.1, the total variation distance between the distribution of $X_t^{(\epsilon)}$ and its limiting distribution $X_\infty^{(\epsilon)}$ changes abruptly from one to zero in a time window $w^{(\epsilon)}$ of constant order around the cut-off time $t^{(\epsilon)}$ of logarithmic order in ϵ.

We introduce the definition of maximal set. We say that a set $\mathcal{A} \subset \mathbb{R}^p$ is a maximal set that satisfies the property **P** if and only if any set $\mathcal{B} \subset \mathbb{R}^d$ that satisfies the property **P** is a subset of $\mathcal{A}$.

In the case when all the roots of (2.1) are real numbers, we will see in Lemma 3.3 that there exists a maximal set $\mathcal{C} \subset \mathbb{R}^p$ such that any initial datum $x := (x_0, \ldots, x_{p-1}) \in \mathcal{C}$ fulfills Condition (i), Condition (ii) and Condition (iii) of Theorem 2.1. Moreover, $\mathcal{C}$ has full measure with respect to the Lebesgue measure on $\mathbb{R}^p$. If we only assume (H) and *no further assumptions*, we will see in Corollary 3.1 that Condition (iii) of Theorem 2.1 may not hold. We conjecture that cut-off phenomenon does not hold when condition (iii) fails.

Remark 2.4. If (H) does not hold, then it is not hard to prove that the variance of $X_t^{(\epsilon)}$ tends to $+\infty$ as $t \to +\infty$. As a consequence, the random linear recurrence (SL) does not converge in distribution. Therefore the cut-off phenomenon problem is not well-posed.

3 Proof

Observe that for any $t \geq p$, $X_t^{(\epsilon)}$ has Gaussian distribution with mean x_t and variance $\sigma^2(t, \epsilon, x_0, \ldots, x_{p-1}) \in (0, +\infty)$. By Lemma A.1 in Appendix A, under assumption (H), we obtain $\sigma^2(t, \epsilon, x_0, \ldots, x_{p-1}) = \epsilon^2\sigma_t^2$, where $\sigma_t^2 \in [1, +\infty)$ and it does not depend on the initial data $x_0, x_1, \ldots, x_{p-1}$.

The following lemma asserts that the random dynamics (SL) is strongly ergodic when (H) holds.

Lemma 3.1. *Assume that* (H) *holds. As t goes to infinity, $X_t^{(\epsilon)}$ converges in the total variation distance to a random variable $X_\infty^{(\epsilon)}$ that has Gaussian distribution with zero mean and variance $\epsilon^2\sigma_\infty^2 \in [\epsilon^2, +\infty)$.*

For the sake of brevity, the proof of the last lemma is given in Lemma A.2 in Appendix A. Recall that

$$d^{(\epsilon)}(t) = \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(x_t, \epsilon^2\sigma_t^2), \mathcal{N}(0, \epsilon^2\sigma_\infty^2)) \quad \text{for any } t \geq p.$$

In order to analyze the cut-off phenomenon for the distance $d^{(\epsilon)}(t)$, for the convenience of computations, we first study another distance as the following lemma states.

Lemma 3.2. *For any $t \geq p$ we have*

$$\left|d^{(\epsilon)}(t) - D^{(\epsilon)}(t)\right| \leq R(t)$$

where

$$D^{(\epsilon)}(t) = \mathbf{d}_{\mathrm{TV}}\left(\mathcal{N}\left(\frac{x_t}{\epsilon\sigma_\infty}, 1\right), \mathcal{N}(0,1)\right)$$

and

$$R(t) = \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(0, \sigma_t^2), \mathcal{N}(0, \sigma_\infty^2)).$$

Proof. Notice that the expressions $d^{(\epsilon)}(t)$ and $D^{(\epsilon)}(t)$ depend on the parameter ϵ and the initial data $x_0, x_1, \ldots, x_{p-1}$. Nevertheless, the term $R(t)$ does not depend on ϵ and on the initial data $x_0, x_1, \ldots, x_{p-1}$. Let $t \geq p$. By the triangle inequality, we obtain

$$d^{(\epsilon)}(t) \leq \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(x_t, \epsilon^2\sigma_t^2), \mathcal{N}(x_t, \epsilon^2\sigma_\infty^2)) + \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(x_t, \epsilon^2\sigma_\infty^2), \mathcal{N}(0, \epsilon^2\sigma_\infty^2)).$$

By item (i) and item (ii) of Lemma B.1 we have

$$d^{(\epsilon)}(t) \leq R(t) + D^{(\epsilon)}(t).$$

On the other hand, by item (ii) of Lemma B.1 we notice

$$D^{(\epsilon)}(t) = \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(x_t, \epsilon^2\sigma_\infty^2), \mathcal{N}(0, \epsilon^2\sigma_\infty^2)).$$

By the triangle inequality, we obtain

$$D^{(\epsilon)}(t) \leq \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(x_t, \epsilon^2\sigma_\infty^2), \mathcal{N}(x_t, \epsilon^2\sigma_t^2)) + \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(x_t, \epsilon^2\sigma_t^2), \mathcal{N}(0, \epsilon^2\sigma_\infty^2)).$$

Again, by item (i) and item (ii) of Lemma B.1 we have

$$D^{(\epsilon)}(t) \leq R(t) + d^{(\epsilon)}(t).$$

Gluing all pieces together we deduce

$$\left|d^{(\epsilon)}(t) - D^{(\epsilon)}(t)\right| \leq R(t) \quad \text{for any } t \geq p. \qquad \square$$

Now, we have all the tools to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 3.1 and Lemma B.4, we have

$$\lim_{t \to +\infty} R(t) = 0.$$

In order to analyze $D^{(\epsilon)}(t)$, we observe that

$$\frac{x_t}{\epsilon\sigma_\infty} = \frac{t^{l-1}r^t}{\epsilon\sigma_\infty}\left(\frac{x_t}{t^{l-1}r^t} - v_t\right) + \frac{t^{l-1}r^t}{\epsilon\sigma_\infty}v_t, \tag{3.1}$$

where $l \in \{1, \ldots, p\}$, $r \in (0,1)$, and v_t are given by Condition (i). By Lemma C.2 in Appendix C we have

$$\lim_{\epsilon \to 0^+} \frac{(t^{(\epsilon)})^{l-1} r^{t^{(\epsilon)}}}{\epsilon} = 1.$$

For any $t \geq 0$, define $p_t = \frac{t^{l-1}r^t}{\epsilon\sigma_\infty}(\frac{x_t}{t^{l-1}r^t} - v_t)$ and $q_t = \frac{t^{l-1}r^t}{\epsilon\sigma_\infty}v_t$. Then for any $b \in \mathbb{R}$ we have

$$|p_{\lfloor t^{(\epsilon)} + bw^{(\epsilon)}\rfloor}| \leq \left(\frac{t^{(\epsilon)} + bw^{(\epsilon)}}{t^{(\epsilon)}}\right)^{l-1} \frac{(t^{(\epsilon)})^{l-1} r^{t^{(\epsilon)} + bw^{(\epsilon)} - 1}}{\epsilon\sigma_\infty}$$
$$\times \left|\frac{x_{\lfloor t^{(\epsilon)} + bw^{(\epsilon)}\rfloor}}{(\lfloor t^{(\epsilon)} + bw^{(\epsilon)}\rfloor)^{l-1} r^{\lfloor t^{(\epsilon)} + bw^{(\epsilon)}\rfloor}} - v_{\lfloor t^{(\epsilon)} + bw^{(\epsilon)}\rfloor}\right|.$$

By Condition (i) we have

$$\lim_{\epsilon\to 0^+} p_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}=0 \quad \text{for any } b\in\mathbb{R}. \tag{3.2}$$

Now, we analyze an upper bound for $|q_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|$. Notice that

$$|q_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|\leq \left(\frac{t^{(\epsilon)}+bw^{(\epsilon)}}{t^{(\epsilon)}}\right)^{l-1}\frac{(t^{(\epsilon)})^{l-1}r^{t^{(\epsilon)}+bw^{(\epsilon)}-1}}{\epsilon\sigma_\infty}M,$$

where $M=\sup_{t\geq 0}|v_t|$. By Condition (ii) we know $M<+\infty$. Then

$$\limsup_{\epsilon\to 0^+}|q_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|\leq \frac{Mr^{bC-1}}{\sigma_\infty} \quad \text{for any } b\in\mathbb{R}. \tag{3.3}$$

From equality (3.1), relation (3.2), inequality (3.3) and item (ii) of Lemma C.1 we get

$$\limsup_{\epsilon\to 0^+}\frac{|x_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|}{\epsilon\sigma_\infty}\leq \frac{Mr^{bC-1}}{\sigma_\infty} \quad \text{for any } b\in\mathbb{R}.$$

Using item (i) of Lemma B.5, we have

$$\begin{aligned}\limsup_{\epsilon\to 0^+}\mathbf{d}_{\mathrm{TV}}\Big(\mathcal{N}\Big(\frac{|x_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|}{\epsilon\sigma_\infty},1\Big),\mathcal{N}(0,1)\Big)\\ \leq \mathbf{d}_{\mathrm{TV}}\Big(\mathcal{N}\Big(\frac{Mr^{bC-1}}{\sigma_\infty},1\Big),\mathcal{N}(0,1)\Big)\end{aligned}$$

for any $b\in\mathbb{R}$. Since $r\in(0,1)$, by Lemma B.4 we have

$$\lim_{b\to+\infty}\limsup_{\epsilon\to 0^+}\mathbf{d}_{\mathrm{TV}}\Big(\mathcal{N}\Big(\frac{|x_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|}{\epsilon\sigma_\infty},1\Big),\mathcal{N}(0,1)\Big)=0. \tag{3.4}$$

In order to analyze a lower bound for $|q_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|$, note

$$|q_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|\geq \left(\frac{t^{(\epsilon)}+bw^{(\epsilon)}-1}{t^{(\epsilon)}}\right)^{l-1}\frac{(t^{(\epsilon)})^{l-1}r^{t^{(\epsilon)}+bw^{(\epsilon)}}}{\epsilon\sigma_\infty}|v_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|$$

for any $b\in\mathbb{R}$. By Condition (iii) and item (iii) of Lemma C.1 we have

$$\liminf_{\epsilon\to 0^+}|q_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|\geq \frac{r^{bC}}{\sigma_\infty}\liminf_{\epsilon\to 0^+}|v_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|\geq \frac{mr^{bC}}{\sigma_\infty}, \tag{3.5}$$

where $m=\liminf_{t\to+\infty}|v_t|\in(0,+\infty)$. From equality (3.1), relation (3.2), inequality (3.5) and item (ii) of Lemma C.1 we get

$$\liminf_{\epsilon\to 0^+}\frac{|x_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|}{\epsilon\sigma_\infty}\geq \frac{mr^{bC}}{\sigma_\infty} \quad \text{for any } b\in\mathbb{R}.$$

From item (ii) of Lemma B.5 we have

$$\begin{aligned}\liminf_{\epsilon\to 0^+}\mathbf{d}_{\mathrm{TV}}\Big(\mathcal{N}\Big(\frac{|x_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|}{\epsilon\sigma_\infty},1\Big),\mathcal{N}(0,1)\Big)\\ \geq \mathbf{d}_{\mathrm{TV}}\Big(\mathcal{N}\Big(\frac{r^{bC}}{\sigma_\infty}m,1\Big),\mathcal{N}(0,1)\Big)\end{aligned}$$

for any $b\in\mathbb{R}$. Since $r\in(0,1)$, by item (iii) Lemma B.2 we have

$$\lim_{b\to-\infty}\liminf_{\epsilon\to 0^+}\mathbf{d}_{\mathrm{TV}}\Big(\mathcal{N}\Big(\frac{|x_{\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor}|}{\epsilon\sigma_\infty},1\Big),\mathcal{N}(0,1)\Big)=1. \tag{3.6}$$

From (3.4) and (3.6), we have

$$\lim_{b\to+\infty}\limsup_{\epsilon\to 0^+} D^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor)=0 \quad \text{and} \quad \lim_{b\to-\infty}\liminf_{\epsilon\to 0^+} D^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor)=1.$$

Recall that $\lim_{t\to+\infty} R(t)=0$. By Lemma 3.2 and item (i) of Lemma C.1 we obtain

$$\limsup_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor)\le \limsup_{\epsilon\to 0^+} D^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor).$$

Now, sending $b\to+\infty$ we get

$$\lim_{b\to+\infty}\limsup_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor)=0.$$

Similarly, by Lemma 3.2 and item (ii) of Lemma C.1 we obtain

$$\liminf_{\epsilon\to 0^+} D^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor)\le \liminf_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor).$$

Now, sending $b\to-\infty$ we get

$$\lim_{b\to-\infty}\liminf_{\epsilon\to 0^+} d^{(\epsilon)}(\lfloor t^{(\epsilon)}+bw^{(\epsilon)}\rfloor)=1.$$

□

3.1 Fulfilling the conditions of Theorem 2.1

Now, we provide a precise estimate of the rate of the convergence to zero of (L). Let us recall some well-known facts about p-linear recurrences. By the celebrated Fundamental Theorem of Algebra we have at most p roots in the complex numbers for (2.1). Denote by $\lambda_1,\ldots,\lambda_q\in\mathbb{C}$ the different roots of (2.1) with multiplicity $m_1,\ldots,m_q$ respectively, where $1\le q\le p$. Then

$$x_t=\sum_{j_1=1}^{m_1} c_{1,j_1}t^{j_1-1}\lambda_1^t+\sum_{j_2=1}^{m_2} c_{2,j_2}t^{j_2-1}\lambda_2^t+\cdots+\sum_{j_q=1}^{m_q} c_{q,j_q}t^{j_q-1}\lambda_q^t \tag{3.7}$$

for any $t\in\mathbb{N}_0$, where the coefficients $c_{1,1},\ldots,c_{1,m_1},\ldots,c_{q,1},\ldots,c_{q,m_q}$ are uniquely obtained from the initial data $x_0,\ldots,x_{p-1}$. For more details see Theorem 1 in Lueker (1980). Moreover, for any initial conditions $(x_0,\ldots,x_{p-1})\in\mathbb{R}^p\setminus\{0_p\}$ we have

$$(c_{1,1},\ldots,c_{1,m_1},\ldots,c_{q,1},\ldots,c_{q,m_q})\in\mathbb{C}^p\setminus\{0_p\}.$$

Notice that the right-hand side of (3.7) may have complex numbers. When all the roots of (2.1) are real numbers we can establish the precise exponential behavior of x_t as t goes by.

Lemma 3.3 (Real roots). *Assume that all the roots of* (2.1) *are real numbers. Then there exists a non-empty maximal set $\mathcal{C}\subset\mathbb{R}^p$ that satisfies the following: for any $x=(x_0,\ldots,x_{p-1})\in\mathcal{C}$ there exist $r:=r(x)>0$, $l:=l(x)\in\{1,\ldots,p\}$ and $v_t:=v(t,x)\in\mathbb{R}$ such that*

$$\lim_{t\to+\infty}\left|\frac{x_t}{t^{l-1}r^t}-v_t\right|=0.$$

Moreover, we have $\sup_{t\ge 0}|v_t|<+\infty$ *and* $\liminf_{t\to+\infty}|v_t|>0$.

Proof. Recall that the constants $c_{1,1},\ldots,c_{1,m_1},\ldots,c_{q,1},\ldots,c_{q,m_q}$ in representation (3.7) depend on the initial data $x_0,x_1,\ldots,x_{p-1}$. In order to avoid technicalities, without loss of generality we assume that for each $1\le j\le q$ there exists at least one $1\le k\le m_j$ such that $c_{j,j_k}\neq 0$. If the last assumption is not true for some $1\le j\le q$, then the root λ_j does not appear in representation (3.7) for an specific initial data $x_0,x_1,\ldots,x_{p-1}$, then we can remove it from representation (3.7) and apply the method described below.

Denote by $r=\max_{1\le j\le q}|\lambda_j|>0$. Since all the roots of (2.1) are real numbers, after multiplicity at most two roots of (2.1) have the same absolute value. The function $\operatorname{sign}(\cdot)$ is defined over the domain $\mathbb{R}\setminus\{0\}$ by $\operatorname{sign}(x)=\frac{x}{|x|}$. Only one of the following cases can occur.

(i) There exists a unique $1 \le j \le q$ such that $|\lambda_j| = r$. Let

$$l = \max\{1 \le s \le m_j : c_{j,s} \neq 0\}.$$

Then

$$\lim_{t\to+\infty}\left|\frac{x_t}{t^{l-1}r^t} - c_{j,l}\big(\mathrm{sign}(\lambda_j)\big)^t\right| = 0.$$

In this case $\mathcal{C} = \mathbb{R}^p \setminus \{0_p\}$.

(ii) There exist $1 \le j < k \le q$ such that $|\lambda_j| = |\lambda_k| = r$. Without loss of generality, we can assume $0 < \lambda_k = -\lambda_j$. Let

$$l_j = \max\{1 \le s \le m_j : c_{j,s} \neq 0\}$$

and

$$l_k = \max\{1 \le s \le m_k : c_{k,s} \neq 0\}.$$

If $l_j < l_k$ or $l_k < l_j$, then by taking $l = \max\{l_j, l_k\}$ we have

$$\lim_{t\to+\infty}\left|\frac{x_t}{t^{l-1}r^t} - c_{\star,l}\big(\mathrm{sign}(\lambda_\star)\big)^t\right| = 0,$$

where $\star = j$ if $l_j = l$ and $\star = k$ if $l_k = l$. In this case, $\mathcal{C} = \mathbb{R}^p \setminus \{0_p\}$. If $l_j = l_k$, then by taking $l = l_j$, $v_t = (-1)^t c_{j,l} + c_{k,l}$ we have

$$\lim_{t\to+\infty}\left|\frac{x_t}{t^{l-1}r^t} - v_t\right| = 0.$$

Notice that $\sup_{t\ge 0}|v_t| < +\infty$. By taking

$$\mathcal{C} = \big\{(x_0, \dots, x_{p-1}) \in \mathbb{R}^p : -c_{j,l} + c_{k,l} \neq 0 \text{ and } c_{j,l} + c_{k,l} \neq 0\big\}$$

we have $\liminf_{t\to+\infty}|v_t| > 0$. □

Remark 3.1. From the proof of Lemma 3.3, we can state precisely $\mathcal{C}$. Moreover, $\mathcal{C}$ has full measure with respect to the Lebesgue measure on $\mathbb{R}^p$.

Rather than the real roots case, the following lemma provides a fine estimate about the behavior of (L) as t increases in general setting.

Lemma 3.4 (General case). *For any $x = (x_0, \dots, x_{p-1}) \in \mathbb{R}^p \setminus \{0_p\}$ there exist $r := r(x) > 0$, $l := l(x) \in \{1, \dots, p\}$ and $v_t := v(t, x) \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty}\left|\frac{x_t}{t^{l-1}r^t} - v_t\right| = 0,$$

where

$$v_t = \sum_{j=1}^{m}\big(\alpha_j\cos(2\pi\theta_j t) + \beta_j\sin(2\pi\theta_j t)\big)$$

with $(\alpha_j, \beta_j) := (\alpha_j(x), \beta_j(x)) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $m := m(x) \in \{1, \dots, p\}$, and $\theta_j := \theta(x) \in [0, 1)$ for any $j \in \{1, \dots, m\}$. Moreover, $\sup_{t\ge 0}|v_t| < +\infty$.

Proof. From (3.7), we have for any $t \in \mathbb{N}_0$

$$x_t = \sum_{j_1=1}^{m_1} c_{1,j_1}t^{j_1-1}\lambda_1^t + \sum_{j_2=1}^{m_2} c_{2,j_2}t^{j_2-1}\lambda_2^t + \cdots + \sum_{j_q=1}^{m_q} c_{q,j_q}t^{j_q-1}\lambda_q^t.$$

Without loss of generality, we assume that for any $k \in \{1, \dots, q\}$ there exists $j \in \{1, \dots, m_k\}$ such that $c_{k,j} \neq 0$. Let $l_k := \max\{1 \leq j \leq m_k : c_{k,j} \neq 0\}$. Then x_t can be rewritten as

$$x_t = \sum_{j_1=1}^{l_1} c_{1,j_1} t^{j_1-1} \lambda_1^t + \sum_{j_2=1}^{l_2} c_{2,j_2} t^{j_2-1} \lambda_2^t + \cdots + \sum_{j_q=1}^{l_q} c_{q,j_q} t^{j_q-1} \lambda_q^t,$$

where $c_{k,l_k} \neq 0$ for each k. For each k let $r_k := \|\lambda_k\|$ be its complex modulus. Without loss of generality, we assume:

(i) $r_1 \leq \cdots \leq r_q$,
(ii) there exists an integer $\tilde{h}$ such that $r_{\tilde{h}} = \cdots = r_q$,
(iii) $l_{\tilde{h}} \leq \cdots \leq l_q$,
(iv) there exists an integer $h \geq \tilde{h}$ such that $l_h = \cdots = l_q$.

Let $r := r_q$ and $l := l_q$. By taking $v_t = r^{-t}(c_{h,l}\lambda_h^t + \cdots + c_{q,l}\lambda_q^t)$, we have

$$\lim_{t \to +\infty} \left| \frac{x_t}{t^{l-1} r^t} - v_t \right| = 0,$$

where $\lambda_h, \dots, \lambda_q$ have the same modulus r, but they have different arguments $\theta_j \in [0, 1)$. Then

$$v_t = \sum_{j=h}^{q} \big(\alpha_j \cos(2\pi \theta_j t) + \beta_j \sin(2\pi \theta_j t) \big).$$

Since $c_{k,l_k} \neq 0$ for each $h \leq k \leq q$, α_j and β_j are not both zero for any $h \leq j \leq q$. After relabelling, we have the desired result. □

Remark 3.2. Under no further conditions on Lemma 3.4, we cannot guarantee that $\liminf_{t \to +\infty} |v_t| > 0$. For instance, the following corollary provides sufficient conditions for which $\liminf_{t \to +\infty} |v_t| = 0$.

Following Viana and Oliveira (2016), the numbers $\vartheta_1, \dots, \vartheta_m$ are rationally independent if

$$k_1 \vartheta_1 + \cdots + k_m \vartheta_m \notin \mathbb{Z} \quad \text{for any } (k_1, \dots, k_m) \in \mathbb{Z}^m \setminus \{0_m\}.$$

Corollary 3.1. *Assume that $\theta_1, \dots, \theta_m$ are rationally independent. Then*

$$\liminf_{t \to +\infty} |v_t| = 0.$$

Proof. For any $j \in \{1, \dots, m\}$ notice that $d_j := \sqrt{\alpha_j^2 + \beta_j^2} > 0$, and let $\cos(\gamma_j) = \frac{\alpha_j}{d_j}$ and $\sin(\gamma_j) = \frac{\beta_j}{d_j}$. Then v_t can be rewritten as

$$v_t = \sum_{j=1}^{m} d_j \cos(2\pi \theta_j t - \gamma_j).$$

Let $\gamma = -(\frac{\gamma_1}{2\pi}, \dots, \frac{\gamma_m}{2\pi})$ be in the m-dimensional torus $(\mathbb{R}/\mathbb{Z})^m$. Then the set $\{(\gamma + (\theta_1 t, \dots, \theta_m t)) \in (\mathbb{R}/\mathbb{Z})^m, t \in \mathbb{N}\}$ is dense in $(\mathbb{R}/\mathbb{Z})^m$, for more details see Corollary 4.2.3 in Viana and Oliveira (2016). Consequently, $\liminf_{t \to +\infty} |v_t| = 0$. □

4 Example

In this section, we consider the celebrated Brownian oscillator

$$\ddot{x}_t + \gamma \dot{x}_t + \kappa x_t = \epsilon \dot{B}_t \quad \text{for any } t \geq 0, \tag{4.1}$$

where x_t denotes the position at time t of the holding mass m with respect to its equilibrium position, $\gamma > 0$ denotes the damping constant, $\kappa > 0$ denotes the restoration constant (Hooke's constant) and $(B_t : t \geq 0)$ is a Brownian motion. For each initial displacement from the equilibrium position $x_0 = u$ and initial velocity $\dot{x}_0 = v$, we have a unique solution of (4.1). For further details, see Chapter 8 in Mao (2008).

Without loss of generality, we can assume that the mass m is one. Using the classical *forward difference approximation* with the step size $h > 0$ (fixed), we obtain

$$\frac{1}{h^2}(x_{(n+2)h} - 2x_{(n+1)h} + x_{nh}) + \frac{\gamma}{h}(x_{(n+1)h} - x_{nh}) + \kappa x_{nh} = \frac{\epsilon}{h}(B_{(n+3)h} - B_{(n+2)h})$$

for any $n \in \mathbb{N}_0$ with the initial data $x_0 = u$ and $x_h = x_0 + \dot{x}_0 h = u + vh$. For consistency, let $X_t = x_{th}$ for any $t \in \mathbb{N}_0$. The latter can be rewritten as

$$X_{t+2} = (2 - \gamma h)X_{t+1} - \big(1 - \gamma h + \kappa h^2\big)X_t + \epsilon h(B_{(t+3)h} - B_{(t+2)h}) \tag{4.2}$$

for any $t \in \mathbb{N}_0$. Notice that the sequence $(B_{(t+3)h} - B_{(t+2)h} : t \in \mathbb{N}_0)$ are i.i.d. random variables with Gaussian distribution with zero mean and variance h. Therefore,

$$X_{t+2} = (2 - \gamma h)X_{t+1} - \big(1 - \gamma h + \kappa h^2\big)X_t + \epsilon h^{\frac{3}{2}} \xi_{t+2} \quad \text{for any } t \in \mathbb{N}_0,$$

where $(\xi_{t+2} : t \in \mathbb{N}_0)$ is a sequence of i.i.d. random variables with standard Gaussian distribution. This is exactly a linear recurrence of degree 2 with control sequence $(\epsilon h^{\frac{3}{2}} \xi_{t+2} : t \in \mathbb{N}_0)$, and its characteristic polynomial is given by

$$\lambda^2 + (\gamma h - 2)\lambda + \big(1 - \gamma h + \kappa h^2\big). \tag{4.3}$$

To fulfill assumption (H) we deduce the following conditions.

(i) If $\gamma^2 - 4k > 0$, then polynomial (4.3) has two distinct real roots. In this case a sufficient condition to verify (H) is $h \in (0, \frac{2}{\gamma})$.
(ii) If $\gamma^2 - 4k = 0$, then polynomial (4.3) has two repeated real roots. In this case (H) is equivalent to $h \in (0, \frac{\gamma}{\kappa})$.
(iii) If $\gamma^2 - 4k < 0$, then polynomial (4.3) has two complex conjugate roots. In this case (H) is equivalent to $h \in (0, \frac{\gamma}{\kappa})$.

In other words, there exists $h^* \in (0, 1)$ such that for each $h \in (0, h^*)$ the characteristic polynomial (4.3) satisfies assumption (H). From here to the end of this section, we assume that $h \in (0, h^*)$.

Now, we compute r, l, v_t and $\mathcal{C}$ which appear in Lemma 3.3. Let λ_1 and λ_2 be roots of (4.3). Denote $r_1 = \|\lambda_1\|$ and $r_2 = \|\lambda_2\|$. Recall the function $\operatorname{sign}(\cdot)$ is defined over the domain $\mathbb{R} \setminus \{0\}$ by $\operatorname{sign}(x) = \frac{x}{|x|}$. We assume that $(x_0, x_1) \neq (0, 0)$. We analyze as far as possible when the conditions of Theorem 2.1 are fulfilled for the model (4.2).

(i) *Real roots with different absolute values.* λ_1 and λ_2 are real and $r_1 \neq r_2$. In this case,

$$x_t = c_1 \lambda_1^t + c_2 \lambda_2^t \quad \text{for any } t \in \mathbb{N}_0,$$

where c_1 and c_2 are unique real constants given by initial data x_0, x_1. Since $(x_0, x_1) \neq (0, 0)$, we have $(c_1, c_2) \neq (0, 0)$. Without loss of generality, assume that $r_1 > r_2$.

(i.1) If $c_1 \neq 0$, then

$$\lim_{t\to+\infty}\left|\frac{x_t}{r_1^t} - c_1\big(\text{sign}(\lambda_1)\big)^t\right| = 0.$$

(i.2) If $c_1 = 0$, then $c_2 \neq 0$. Therefore,

$$\lim_{t\to+\infty}\left|\frac{x_t}{r_2^t} - c_2\big(\text{sign}(\lambda_2)\big)^t\right| = 0.$$

Consequently, $\mathcal{C} = \mathbb{R}^2 \setminus \{(0,0)\}$.

(ii) *Real roots with the same absolute value.* λ_1 and λ_2 are real and $r := r_1 = r_2$.

(ii.1) If $\lambda_1 = \lambda_2 = r\,\text{sign}(\lambda_1)$, then

$$x_t = c_1 r^t\big(\text{sign}(\lambda_1)\big)^t + c_2 t r^t\big(\text{sign}(\lambda_1)\big)^t \quad \text{for any } t \in \mathbb{N}_0,$$

where c_1 and c_2 are unique real constants given by initial data x_0, x_1. Since $(x_0, x_1) \neq (0,0)$, we have $(c_1, c_2) \neq (0,0)$. The following cases are analyzed.

(ii.1.1) If $c_2 \neq 0$, then

$$\lim_{t\to+\infty}\left|\frac{x_t}{t r^t} - c_2\big(\text{sign}(\lambda_1)\big)^t\right| = 0.$$

(ii.1.2) If $c_2 = 0$, then $c_1 \neq 0$. Therefore,

$$\lim_{t\to+\infty}\left|\frac{x_t}{r^t} - c_1\big(\text{sign}(\lambda_1)\big)^t\right| = 0.$$

Consequently, $\mathcal{C} = \mathbb{R}^2 \setminus \{(0,0)\}$.

(ii.2) If $\lambda_1 \neq \lambda_2$, then

$$x_t = c_1 r^t + c_2(-r)^t \quad \text{for any } t \in \mathbb{N}_0,$$

where c_1 and c_2 are unique real constants given by initial data x_0, x_1. Therefore,

$$\lim_{t\to+\infty}\left|\frac{x_t}{r^t} - \big(c_1 + c_2(-1)^t\big)\right| = 0.$$

Consequently,

$$\begin{aligned}\mathcal{C} &= \big\{(x_0, x_1) \in \mathbb{R}^2 : c_1 + c_2 \neq 0 \text{ and } c_1 - c_2 \neq 0\big\}\\ &= \big\{(x_0, x_1) \in \mathbb{R}^2 : x_0 \neq 0 \text{ and } x_1 \neq 0\big\}.\end{aligned}$$

(iii) *Complex conjugate roots.* Since the coefficients of the characteristic polynomial are real, if λ is a root of the polynomial, then conjugate $\overline{\lambda}$ is also a root. We can assume that $\lambda_1 = re^{i2\pi\theta}$ and $\lambda_2 = re^{-i2\pi\theta}$ with $r \in (0,1)$ and $\theta \in (0,1) \setminus \{\frac{1}{2}\}$. In this setting

$$x_t = c_1 r^t \cos(2\pi\theta t) + c_2 r^t \sin(2\pi\theta t) \quad \text{for any } t \in \mathbb{N}_0,$$

where c_1 and c_2 are unique real constants given by initial data x_0, x_1. Thus,

$$\lim_{t\to+\infty}\left|\frac{x_t}{r^t} - \big(c_1\cos(2\pi\theta t) + c_2\sin(2\pi\theta t)\big)\right| = 0.$$

Since $(x_0, x_1) \neq (0,0)$, we have $(c_1, c_2) \neq (0,0)$. Let $c = \sqrt{c_1^2 + c_2^2}$ and define γ satisfying $\cos(\gamma) = \frac{c_1}{c}$ and $\sin(\gamma) = \frac{c_2}{c}$. Consequently,

$$v_t := c_1\cos(2\pi\theta t) + c_2\sin(2\pi\theta t) = c\cos(2\pi\theta t - \gamma) \quad \text{for any } t \in \mathbb{N}_0.$$

Observe that γ depends on the initial data x_0 and x_1. Let us analyze under which conditions on x_0 and x_1 we have $\liminf_{t\to+\infty} |v_t| > 0$.

(iii.1) If θ is a rational number, then the sequence $(\cos(2\pi\theta t-\gamma), t\in\mathbb{N}_0)$ takes finite number of values. Notice that there exists $t_0\in\mathbb{N}_0$ such that $2\pi\theta t_0-\gamma=\frac{\pi}{2}+k\pi$ for some $k\in\mathbb{Z}$, if and only if $\cos(2\pi\theta t_0-\gamma)=0$. Therefore, $\liminf_{t\to+\infty}|v_t|>0$ if and only if

$$\mathcal{C}=\left\{(x_0,x_1)\in\mathbb{R}^2 : 2\pi\theta t-\gamma\neq\frac{\pi}{2}+k\pi \text{ for any } t\in\mathbb{N}_0, k\in\mathbb{Z}\right\}.$$

(iii.2) If θ is an irrational number, then by Corollary 4.2.3 in Viana and Oliveira (2016) the set $\{(\theta t-\frac{\gamma}{2\pi})\in\mathbb{R}/\mathbb{Z} : t\in\mathbb{N}_0\}$ is dense in the circle $\mathbb{R}/\mathbb{Z}$. Consequently, the set

$$\left\{\cos(2\pi\theta t-\gamma) : t\in\mathbb{N}_0\right\}\quad\text{is dense in } [-1,1].$$

Therefore, for any γ we have $\liminf_{t\to+\infty}|v_t|=0$, which implies $\mathcal{C}=\varnothing$.

Appendix A: Variance representation of $X_t^{(\epsilon)}$

Since $(\xi_t : t\geq 0)$ is a sequence of i.i.d. random variables with standard Gaussian distribution, it is not hard to see that for any $t\geq p$ the random variable $X_t^{(\epsilon)}$ has Gaussian distribution, whose expectation is x_t. The next lemma provides a representation of its variance under assumption (H).

Now, for the sake of intuitive reasoning and in a conscious abuse of notation we introduce the following notation. For each $s\in\mathbb{N}_0$ denote by $\sum k_j=s$ the set

$$\left\{(k_1,\ldots,k_p)\in\mathbb{N}_0^p : \sum_{j=1}^{p}k_j=s\right\}$$

and denote by $\sum_{\sum k_j=s}$ the sum of $\sum_{(k_1,\ldots,k_p)\in\sum k_j=s}$.

Lemma A.1. *Assume that* (H) *holds. For any $t\geq p$, $X_t^{(\epsilon)}$ has Gaussian distribution with mean x_t and variance $\epsilon^2\sigma_t^2$, where*

$$\sigma_t^2=1+\Bigg(\sum_{\sum k_j=1}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\Bigg)^2+\cdots+\Bigg(\sum_{\sum k_j=t-p}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\Bigg)^2$$

and $\lambda_1,\ldots,\lambda_p$ are the roots of (2.1).

Proof. By the superposition principle, the solution of the non-homogeneous linear recurrence (SL) can be written as the general solution of the homogeneous linear recurrence (L) plus a particular solution of the non-homogeneous linear recurrence (SL) as follows:

$$X_t^{(\epsilon)}=x_t^{\text{gen}}+X_t^{(\text{par},\epsilon)}\quad\text{for any } t\in\mathbb{N}_0,$$

where $X_t^{(\text{par},\epsilon)}$ solves the non-homogeneous linear recurrence (SL), x_t^{gen} solves the homogeneous linear recurrence (L) but possible both solutions do not fit the prescribed initial conditions. The initial conditions are fitting after adding themselves. For more details, see Section 2.4 in Elaydi (2005).

To find a particular solution, we introduce the *Lag operator* $\mathbb{L}$ which acts as follows: $x_{t-1}=\mathbb{L}\circ x_t$. Its inverse, $\mathbb{L}^{-1}$, is defined as $\mathbb{L}^{-1}\circ x_t=x_{t+1}$. For more details about the Lag operator, we recommend Chapter 2 in Hamilton (1994). Notice that the random linear recurrence (SL) can be rewritten as

$$(\mathbb{L}^{-p}-\phi_1\mathbb{L}^{-p+1}-\cdots-\phi_p)\circ X_t^{(\text{par},\epsilon)}=\epsilon\mathbb{L}^{-p}\circ\xi_t.$$

Then

$$(1-\lambda_1\mathbb{L})(1-\lambda_2\mathbb{L})\cdots(1-\lambda_p\mathbb{L})\circ X_t^{(\mathrm{par},\epsilon)}=\epsilon\xi_t,$$

where $\lambda_1,\dots,\lambda_p$ are the roots of (2.1). Since the modulus of the roots of (2.1) are strictly less than one, we have

$$X_t^{(\mathrm{par},\epsilon)}=\left(1+\lambda_1\mathbb{L}+\lambda_1^2\mathbb{L}^2+\cdots\right)\cdots\left(1+\lambda_p\mathbb{L}+\lambda_p^2\mathbb{L}^2+\cdots\right)\circ\epsilon\xi_t$$

for any $t\geq p$. Since ξ_t is only defined for $t\geq p$, we get

$$X_t^{(\mathrm{par},\epsilon)}=\left(1+\sum_{\sum k_i=1}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\mathbb{L}+\cdots+\sum_{\sum k_i=t-p}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\mathbb{L}^{t-p}\right)\circ\epsilon\xi_t.$$

Consequently,

$$X_t^{(\epsilon)}=x_t^{\mathrm{gen}}+\epsilon\left(\xi_t+\sum_{\sum k_i=1}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\xi_{t-1}+\cdots+\sum_{\sum k_i=t-p}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\xi_p\right)\tag{A.1}$$

for $t\geq p$, where x_t^{gen} satisfies (L). After fitting the initial conditions, we see that $(x_t^{\mathrm{gen}}:t\in\mathbb{N}_0)$ is the solution of (L) with initial data $x_0,\dots,x_{p-1}$. Therefore $x_t^{\mathrm{gen}}=x_t$ for any $t\in\mathbb{N}_0$. Since $(\xi_t:t\geq p)$ are i.i.d. Gaussian random variables with zero mean and unit variance, $X_t^{(\epsilon)}$ is a Gaussian distribution for any $t\geq p$. Therefore, it is characterized by its mean and variance. Since the expectation of $X_t^{(\epsilon)}$ is x_t, we only need to compute its variance. From (A.1) for any $t\geq p$, we get

$$\begin{aligned}&\mathrm{Var}\big(X_t^{(\epsilon)}\big)\\&=\epsilon^2\left(1+\Big(\sum_{\sum k_j=1}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\Big)^2+\cdots+\Big(\sum_{\sum k_j=t-p}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\Big)^2\right).\end{aligned}$$

□

Lemma A.2. *Assume that (H) holds. As t goes to infinity, $X_t^{(\epsilon)}$ converges in the total variation distance to a random variable $X_\infty^{(\epsilon)}$ that has Gaussian distribution with zero mean and variance $\epsilon^2\sigma_\infty^2\in[\epsilon^2,+\infty)$.*

Proof. From Lemma A.1, we have that for any $t\geq p$, $X_t^{(\epsilon)}$ has mean x_t which is the solution of (L) and variance $\epsilon^2\sigma_t^2$ where

$$\sigma_t^2=1+\Big(\sum_{\sum k_j=1}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\Big)^2+\cdots+\Big(\sum_{\sum k_j=t-p}\lambda_1^{k_1}\cdots\lambda_p^{k_p}\Big)^2.$$

Since all the roots of (2.1) have modulus strictly less than one, relation (3.7) yields that x_t converges to zero when t goes to infinity. By a simple counting argument, one can see that

$$\mathrm{Card}\Big(\sum k_j=s\Big)\leq(s+1)^p\quad\text{for any } s\in\mathbb{N}_0,$$

where Card denotes the cardinality of the given set. Then for any $t\geq p$

$$\begin{aligned}\sigma_t^2&=1+\Big(\sum_{\sum k_j=1}\lambda_1^{k_1}\dots\lambda_p^{k_p}\Big)^2+\cdots+\Big(\sum_{\sum k_j=t-p}\lambda_1^{k_1}\dots\lambda_p^{k_p}\Big)^2\\&\leq 1+\left(2^p\kappa\right)^2+\cdots+\left((t-p+1)^p\kappa^{t-p}\right)^2\\&=\sum_{j=0}^{t-p}(j+1)^{2p}\kappa^{2j}\leq\sum_{j=0}^{\infty}(j+1)^{2p}\kappa^{2j}<+\infty,\end{aligned}$$

where $\kappa = \max_{1\leq j\leq n} |\lambda_j| < 1$. Since $1 \leq \sigma_t^2 \leq \sigma_{t+1}^2 \leq \sum_{j=0}^{\infty}(j+1)^{2p}\kappa^{2j} < +\infty$ for any $t \geq p$, we deduce $\lim_{t\to+\infty}\sigma_t^2$ exists. Denote by σ_∞^2 its value. Observe that $\sigma_\infty^2 \in [1, +\infty)$. It follows from Lemma B.4 that $X_t^{(\epsilon)}$ converges in the total variation distance to $X_\infty^{(\epsilon)}$ as t goes to infinity, which has Gaussian distribution with zero mean and variance $\epsilon^2\sigma_\infty^2$. □

Appendix B: Total variation distance between Gaussian distributions

In this section, we provide some useful properties for the total variation distance between Gaussian distributions. Recall that $\mathcal{N}(m, \sigma^2)$ denotes the Gaussian distribution with mean $m \in \mathbb{R}$ and variance $\sigma^2 \in (0, +\infty)$. A straightforward computation leads

$$\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_1, \sigma_1^2), \mathcal{N}(m_2, \sigma_2^2)) = \frac{1}{2}\int_{\mathbb{R}}\left|\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(x-m_1)^2}{2\sigma_1^2}} - \frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(x-m_2)^2}{2\sigma_2^2}}\right| \mathrm{d}x$$

for any $m_1, m_2 \in \mathbb{R}$, $\sigma_1^2, \sigma_2^2 \in (0, +\infty)$. For details see Lemma 3.3.1 in Reiss (2012).

Lemma B.1. *Let $m_1, m_2 \in \mathbb{R}$ and $\sigma_1^2, \sigma_2^2 \in (0, +\infty)$. Then*

(i) $\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_1, \sigma_1^2), \mathcal{N}(m_2, \sigma_2^2)) = \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_1 - m_2, \sigma_1^2), \mathcal{N}(0, \sigma_2^2))$.
(ii) $\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(cm_1, c^2\sigma_1^2), \mathcal{N}(cm_2, c^2\sigma_2^2)) = \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_1, \sigma_1^2), \mathcal{N}(m_2, \sigma_2^2))$ *for any $c \neq 0$.*

Proof. The proofs of item (i) and item (ii) proceed from the Change of Variable theorem. □

Lemma B.2.

(i) *For any $m \in \mathbb{R}$ and $\sigma^2 \in (0, +\infty)$, we have*

$$\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m, \sigma^2), \mathcal{N}(0, \sigma^2)) = \frac{2}{\sqrt{2\pi}}\int_0^{\frac{|m|}{2\sigma}} e^{-\frac{x^2}{2}}\,\mathrm{d}x \leq \frac{|m|}{\sigma\sqrt{2\pi}}.$$

(ii) *For any $m_1, m_2 \in \mathbb{R}$ and $\sigma^2 \in (0, +\infty)$ such that $|m_1| \leq |m_2| < +\infty$ we have*

$$\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_1, \sigma^2), \mathcal{N}(0, \sigma^2)) \leq \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_2, \sigma^2), \mathcal{N}(0, \sigma^2)).$$

(iii) *If $\lim_{t\to+\infty}|m_t| = +\infty$ and $\sigma^2 \in (0, +\infty)$, then*

$$\lim_{t\to+\infty}\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_t, \sigma^2), \mathcal{N}(0, \sigma^2)) = 1.$$

Proof. Notice that item (ii) and item (iii) follow immediately from item (i). Therefore we only prove item (i). From item (ii) of Lemma B.1, we can assume that $m \geq 0$. Observe that

$$\begin{aligned}
&\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m, \sigma^2), \mathcal{N}(0, \sigma^2)) \\
&= \frac{1}{2\sqrt{2\pi}\sigma}\int_{-\infty}^{\frac{m}{2}}\left(e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{(x-m)^2}{2\sigma^2}}\right)\mathrm{d}x + \frac{1}{2\sqrt{2\pi}\sigma}\int_{\frac{m}{2}}^{+\infty}\left(e^{-\frac{(x-m)^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right)\mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}\sigma}\int_0^{\frac{m}{2}} e^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x.
\end{aligned}$$

The latter easily implies the result. □

Lemma B.3. *For any $\sigma^2 \in (0, 1) \cup (1, +\infty)$, we have*

$$\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(0, \sigma^2), \mathcal{N}(0, 1)) = \frac{2}{\sqrt{2\pi}}\int_{\min\{x(\sigma), \frac{x(\sigma)}{\sigma}\}}^{\max\{x(\sigma), \frac{x(\sigma)}{\sigma}\}} e^{-\frac{x^2}{2}}\,\mathrm{d}x \leq \frac{2}{\sqrt{2\pi}}x(\sigma)\left|\frac{1}{\sigma} - 1\right|,$$

where $x(\sigma) = \sigma\left(\frac{\ln(\sigma^2)}{\sigma^2-1}\right)^{\frac{1}{2}}$. Moreover, we have $\lim_{\sigma^2\to1} x(\sigma) = 1$.

Proof. In this case, a formula for $\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(0,\sigma^2),\mathcal{N}(0,1))$ can be computed explicitly as we did in the proof of item (i) of Lemma B.2. Indeed, if $\sigma^2 \in (0,1)$ observe that

$$\begin{aligned}
&\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(0,\sigma^2),\mathcal{N}(0,1))\\
&= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left|\frac{1}{\sigma}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2}}\right|\mathrm{d}x\\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}\left|\frac{1}{\sigma}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2}}\right|\mathrm{d}x\\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{x(\sigma)}\left(\frac{1}{\sigma}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2}}\right)\mathrm{d}x + \int_{x(\sigma)}^{+\infty}\left(e^{-\frac{x^2}{2}} - \frac{1}{\sigma}e^{-\frac{x^2}{2\sigma^2}}\right)\mathrm{d}x\right]\\
&= \frac{2}{\sqrt{2\pi}}\int_{0}^{x(\sigma)}\left(\frac{1}{\sigma}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2}}\right)\mathrm{d}x\\
&= \frac{2}{\sqrt{2\pi}}\int_{x(\sigma)}^{\frac{x(\sigma)}{\sigma}} e^{-\frac{x^2}{2}}\,\mathrm{d}x \le \frac{2}{\sqrt{2\pi}}x(\sigma)\left(\frac{1}{\sigma}-1\right).
\end{aligned}$$

On the other hand, if $\sigma^2 \in (1,+\infty)$ one can also deduce that

$$\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(0,\sigma^2),\mathcal{N}(0,1)) = \frac{2}{\sqrt{2\pi}}\int_{\frac{x(\sigma)}{\sigma}}^{x(\sigma)} e^{-\frac{x^2}{2}}\,\mathrm{d}x \le \frac{2}{\sqrt{2\pi}}x(\sigma)\left(1-\frac{1}{\sigma}\right).$$

The second part of the lemma is a direct computation. □

Lemma B.4. *If* $\lim_{t\to+\infty} m_t = m \in \mathbb{R}$ *and* $\lim_{t\to+\infty}\sigma_t^2 = \sigma^2 \in (0,+\infty)$, *then*

$$\lim_{t\to+\infty}\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_t,\sigma_t^2),\mathcal{N}(m,\sigma^2)) = 0.$$

Proof. The proof follows from the triangle inequality together with item (i) of Lemma B.1, item (i) of Lemma B.2 and Lemma B.3. □

Lemma B.5. *Let* $\sigma^2 \in (0,+\infty)$.

(i) *If* $\limsup_{t\to+\infty}|m_t| \le C_0 \in [0,+\infty)$, *then*

$$\limsup_{t\to+\infty}\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_t,\sigma^2),\mathcal{N}(0,\sigma^2)) \le \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(C_0,\sigma^2),\mathcal{N}(0,\sigma^2)).$$

(ii) *If* $\liminf_{t\to+\infty}|m_t| \ge C_1 \in [0,+\infty)$, *then*

$$\liminf_{t\to+\infty}\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_t,\sigma^2),\mathcal{N}(0,\sigma^2)) \ge \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(C_1,\sigma^2),\mathcal{N}(0,\sigma^2)).$$

Proof.

(i) Let $L := \limsup_{t\to+\infty}\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_t,\sigma^2),\mathcal{N}(0,\sigma^2))$. Then there exists a subsequence $(t_n : n \in \mathbb{N})$ such that $\lim_{n\to+\infty} t_n = +\infty$ and

$$\lim_{n\to+\infty}\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_{t_n},\sigma^2),\mathcal{N}(0,\sigma^2)) = L.$$

Since $\limsup_{t\to+\infty}|m_t| \le C_0$, we have $\limsup_{n\to+\infty}|m_{t_n}| \le C_0$. Then there exists a subsequence $(t_{n_k} : k \in \mathbb{N})$ of $(t_n : n \in \mathbb{N})$ such that $\lim_{k\to+\infty} t_{n_k} = +\infty$ and $\lim_{k\to+\infty}|m_{t_{n_k}}|$ exists. We define $C := \lim_{k\to+\infty}|m_{t_{n_k}}|$ and notice that $0 \le C \le C_0$. From Lemma B.4, we obtain

$$\lim_{k\to+\infty}\mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_{t_{n_k}},\sigma^2),\mathcal{N}(0,\sigma^2)) = \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(C,\sigma^2),\mathcal{N}(0,\sigma^2)).$$

Notice that $\lim_{k\to+\infty} \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(m_{t_{n_k}}, \sigma^2), \mathcal{N}(0,\sigma^2)) = L$, then by item (ii) of Lemma B.2 we deduce

$$L = \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(C,\sigma^2), \mathcal{N}(0,\sigma^2)) \le \mathbf{d}_{\mathrm{TV}}(\mathcal{N}(C_0,\sigma^2), \mathcal{N}(0,\sigma^2)).$$

(ii) The proof of item (ii) follows from similar arguments as we did in item (i). We left the details to the interested reader. □

Appendix C: Tools

In this section, we state some elementary tools that we used all along the article. We state them here for the sake of completeness.

Lemma C.1. *Let* $(a_\epsilon : \epsilon > 0)$ *and* $(b_\epsilon : \epsilon > 0)$ *be functions of real numbers. Assume that* $\lim_{\epsilon\to 0^+} b_\epsilon = b \in \mathbb{R}$. *Then*

(i) $\limsup_{\epsilon\to 0^+}(a_\epsilon + b_\epsilon) = \limsup_{\epsilon\to 0^+} a_\epsilon + b$.
(ii) $\liminf_{\epsilon\to 0^+}(a_\epsilon + b_\epsilon) = \liminf_{\epsilon\to 0^+} a_\epsilon + b$.
(iii) $\liminf_{\epsilon\to 0^+}(a_\epsilon b_\epsilon) = b \liminf_{\epsilon\to 0^+} a_\epsilon$ *when* $b > 0$.

Proof. The proofs proceed by definition of limit superior and limit inferior using subsequences. □

Lemma C.2. *For any* $\alpha \in \mathbb{R}$ *and* $r \in (0,1)$ *we have*

$$\lim_{\epsilon\to 0^+} \frac{(t^{(\epsilon)})^\alpha r^{t^{(\epsilon)}}}{\epsilon} = 1,$$

where $t^{(\epsilon)} = \frac{\ln(\frac{1}{\epsilon})}{\ln(\frac{1}{r})} + \alpha \frac{\ln\left(\frac{\ln(\frac{1}{\epsilon})}{\ln(\frac{1}{r})}\right)}{\ln(\frac{1}{r})}$.

Proof. Note that $t^{(\epsilon)} = \log_r(\epsilon) - \alpha \log_r(\log_r(\epsilon))$, where $\log_r(\cdot)$ denotes the base–r logarithm function. A straightforward computation shows

$$\lim_{\epsilon\to 0^+} \frac{(t^{(\epsilon)})^\alpha r^{t^{(\epsilon)}}}{\epsilon} = \lim_{\epsilon\to 0^+} \left(1 - \alpha \frac{\log_r(\log_r(\epsilon))}{\log_r(\epsilon)}\right)^\alpha = 1.$$ □

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