Brazilian Journal of Probability and Statistics
2021, Vol. 35, No. 1, 138–157
https://doi.org/10.1214/20-BJPS469

Confidence intervals of the index C_{pk} for normally distributed quality characteristics using classical and Bayesian methods of estimation

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Abstract. One of the indicators for evaluating the capability of a process potential and performance in an effective way is the process capability index (PCI). It is of great significance to quality control engineers as it quantifies the relation between the actual performance of the process and the pre-set specifications of the product. Most of the traditional PCIs performed well when process follows the normal behaviour. In this article, we consider a process capability index, C_{pk}, suggested by Kane (*Journal of Quality Technology* **18** (1986) 41–52) which can be used for normal random variables. The objective of this article is three fold: First, we address different methods of estimation of the process capability index C_{pk} from frequentist approaches for the normal distribution. We briefly describe different frequentist approaches, namely, maximum likelihood estimators, least squares and weighted least squares estimators, maximum product of spacings estimators, Cramèr–von-Mises estimators, Anderson–Darling estimators and Right-Tail Anderson–Darling estimators and compare them in terms of their mean squared errors using extensive numerical simulations. Second, we compare three parametric bootstrap confidence intervals (BCIs) namely, standard bootstrap, percentile bootstrap and bias-corrected percentile bootstrap. Third, we consider Bayesian estimation under squared error loss function using normal prior for location parameter and inverse gamma for scale parameter for the considered model. Monte Carlo simulation study has been carried out to compare the performances of the classical BCIs and highest posterior density (HPD) credible intervals of C_{pk} in terms of average widths and coverage probabilities. Finally, two real data sets have been analyzed for illustrative purposes.

1 Introduction

Process capability indices (PCIs) have received much interest in the statistical literature and in quality assurance work in recent years. As far as the concept is concerned, there is a strong agreement that process capability refers to the ability to produce output according to specified requirements. They are also regarded as convenient indicators of the 'capability' of a process to produce items with a specified measurable characteristic between lower (L) and upper (U) specification limits. For thorough discussions of different capability indices see, for instance, Kane (1986), Chan et al. (1988), Pearn et al. (1992, 1998), Kotz and Lovelace (1998), and Kotz and Johnson (2002).

The first process capability index (PCI) C_p was developed by Juran (1974) which does not depend on process mean and cannot reflect the tendency of process centering and thus gives

Key words and phrases. Process capability index, normal distribution, different classical methods of estimation, bootstrap confidence intervals, sensitivity analysis.

Received August 2019; accepted March 2020.

no indication of the actual process performance. Later on, many PCIs have been studied by numerous authors and one of the modified capability indices introduced by Kane (1986) to reflect the impact of μ (process mean) on the PCIs, defined as

$$C_{pk} = \min\left\{\frac{U-\mu}{3\sigma}, \frac{\mu-L}{3\sigma}\right\}. \tag{1.1}$$

The index C_{pk} takes the process variation and process centring into account, but not considering the process targeting to the preset target. The relationship between the indices C_p and C_{pk} are discussed by Gensidy (1985), Barnett (1988), Kotz and Johnson (1999). If the process is centered at process mean, then the index C_{pk} coincides with the index C_p.

While assessing PCIs, statistician and quality control engineers often focus their efforts on point and interval estimation. No doubt, the point estimator of the PCI is useful for measuring the process performance but there is variability associated with such an estimate. In such a situation a confidence interval (CI) provides better results and information regarding the variability of the estimator. For more details, see Chan et al. (1988), Smithson (2001), Thompson (2002) and Steiger (2004). Hsiang and Taguchi (1985) initiated the construction of CIs for the PCI. Since then, several researchers have developed numerous techniques for constructing confidence intervals. In this regard, readers may refer to Peng (2010a, 2010b); Leiva et al. (2014); Pearn et al. (2014, 2016); Kashif et al. (2016, 2017); Weber et al. (2016); Pina-Monarrez et al. (2016); Rao et al. (2016); Dey et al. (2017); Saha et al. (2018); Dey and Saha (2019) and the references cited therein.

In the recent past, besides classical point and interval estimation of PCIs, several authors considered Bayesian point and interval estimation for PCI. To name a few, Saxena and Singh (2006) considered the Bayesian estimation of the PCI C_p when the underlying distribution is normal. Ouyang et al. (2002) derived credible intervals for some PCIs. Lin et al. (2011) considered Bayesian approach to assess process capability for asymmetric tolerances based on PCI C_{pmk}. Huiming et al. (2007) considered Bayesian approach for the problem of estimation and testing PCI based on sub-samples collected over time from an in-control process. They used non-informative priors with squared error loss function (SELF) for inference purposes. Miao et al. (2011) studied Bayesian approach under SELF for calculating process capability indices. Wu and Lin (2009) considered one-sided lower Bayesian estimation of C_{pmk}. They also obtained the credible intervals of C_{pmk}. Recently, Kargar et al. (2014) used the Bayesian approach with normal prior based on sub-samples to check process capability using capability index C_{pk}. Maiti and Saha (2012) studied the Bayesian estimation of the index C_{py} for normal, exponential and Poisson process distributions based on SELF. Ali and Riaz (2014) studied the generalized capability indices from the Bayesian view point under symmetric and asymmetric loss functions for the simple and mixture of generalized lifetime models.

The objective of this paper is three fold: First, we obtain the estimates of C_{pk} based on seven different classical methods of estimation and Bayesian method of estimation. For estimating the parameter(s) of a distribution, one often uses traditional classical methods of estimation, viz., method of maximum likelihood (ML), method of least squares (LS), method of weighted least squares (WLS). Each has its own advantages and limitations but among these methods the most popular method of estimation is the ML estimation method. Besides, the above cited methods, we consider four additional methods to estimate the parameters of normal distribution and subsequently we estimate the PCI C_{pk}. These additional methods of estimation are: method of maximum product of spacing (MPS), method of Cramèr–von-Mises (CM), method of Anderson–Darling (AD) and method of Right-tail Anderson–Darling (RAD). Inspite of not having good theoretical properties, these methods are used for estimating the parameters of the model as they sometimes provide better estimates of the unknown parameter(s) than the ML estimator. In this regard, several authors have discussed different methods of estimation for estimating parameters of different distributions [see, Kundu and Raqab (2005); Alkasasbeh and Raqab (2009); Teimouri et al. (2013) and Dey et al. (2014, 2015, 2017a, 2017b, 2017c, 2018)]. The performance of the methods of estimation are demonstrated in terms of their mean squared errors (MSEs) based on simulated samples and for different sample sizes through simulation study. Second objective is to obtain three bootstrap confidence intervals (BCIs) of C_{pk} based on above cited classical methods of estimation. The performance of the BCIs are demonstrated in terms of estimated coverage probabilities and average widths. Third objective is to obtain Bayes estimates of the PCI C_{pk} under squared error loss function using normal prior for location parameter and inverse gamma for scale parameter of the considered model. We further obtain Bayes credible intervals and compare them with BCIs. Further, we have considered the net sensitivity (NS) analysis for the given PCI C_{pk}. To the best of our knowledge thus far, no work was carried out to study the PCI C_{pk} using three BCIs based on aforementioned classical methods of estimation and Bayesian method of estimation for the normal distribution. The study aims to develop a guideline for choosing the best method of estimation of the index C_{pk}, which we think would be of deep interest to applied statisticians and quality control engineers, where the item/subgroup quality characteristic follows normal distribution.

This article unfolds as follows: In Section 2, we describe different classical methods of estimation (MLE, LSE, WLSE, MPSE, CME, ADE, RADE) of the index C_{pk}. In Section 3, bootstrap confidence intervals, viz., standard bootstrap ($\mathcal{S}$-boot), percentile bootstrap ($\mathcal{P}$-boot) and bias-corrected percentile bootstrap ($\mathcal{BC}_p$-boot) based on aforementioned methods of estimation of the PCI C_{pk} have been discussed. In Section 4, we have derived the Bayes estimators of the index C_{pk} under squared error loss function using normal prior for location parameter and inverse gamma for scale parameter of the model. In Section 5, Monte Carlo simulation study is carried out to see the performance of the aforementioned classical estimators and Bayes estimators of C_{pk} in terms of their MSEs. Also we asses the performance of different bootstrap confidence intervals ($\mathcal{S}$-boot, $\mathcal{P}$-boot, $\mathcal{BC}_p$-boot) under the aforementioned methods of estimation and Bayes Credible intervals in terms of coverage probabilities and average widths of the intervals. In Section 6, net sensitivity analysis is carried out. For illustrative purposes, two real data sets are analyzed in Section 7. Finally, concluding remarks are given in Section 8.

2 Different classical methods of estimation of C_{pk}

Here, we briefly describe different classical estimators, namely, maximum likelihood estimators (MLE), ordinary and weighted least square estimators (LSE and WLSE), Cramèr–von-Mises estimators (CME), maximum product spacing estimators (MPSE), Anderson–Darling estimators (ADE) and Right-tail Anderson–Darling estimators (RADE) of the parameters μ and σ as well as the corresponding estimator of C_{pk}.

A random variable X is said to follow normal distribution with parameter $\Theta = (\mu, \sigma)$ if its probability density function (PDF) and cumulative distribution function (CDF) are given as;

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}; \quad x, \mu \in \Re, \sigma > 0, \tag{2.1}$$

where, μ is the mean and σ is the standard deviation of the normal distribution.

$$F(x; \mu, \sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right), \tag{2.2}$$

where, Φ is the distribution function of standard normal variate.

2.1 Maximum likelihood estimator

Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n observed from normal distribution, defined in Equation (2.1). Then, the likelihood function of the parameters μ and σ are given by

$$L(\mu, \sigma \mid x) = \prod_{i=1}^{n} f(x_i, \mu, \sigma) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(\frac{x-\mu}{\sigma})^2}. \tag{2.3}$$

The corresponding log-likelihood function is

$$\ln L(\mu, \sigma \mid x) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2. \tag{2.4}$$

The MLEs of the parameters μ and σ can be obtained by solving the following normal equations:

$$\frac{\partial \ln L}{\partial \mu} = 0, \qquad \frac{\partial \ln L}{\partial \sigma} = 0,$$

which yields

$$\hat{\mu}_{\mathrm{MLE}} = \bar{x}.$$

Recalling that $\hat{\mu}_{\mathrm{MLE}} = \bar{x}$, we obtain

$$\hat{\sigma}_{\mathrm{MLE}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

Substituting the MLEs, we can get the estimator of C_{pk} as

$$\hat{C}_{pk}^{\mathrm{MLE}} = \min\left\{\frac{U - \hat{\mu}_{\mathrm{MLE}}}{3\hat{\sigma}_{\mathrm{MLE}}}, \frac{\hat{\mu}_{\mathrm{MLE}} - L}{3\hat{\sigma}_{\mathrm{MLE}}}\right\}. \tag{2.5}$$

2.2 Ordinary and weighted least square estimators

The LSE and the WLSE were proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. Suppose $F(X_{(i:n)})$ denotes the distribution function of the ordered random variables $X_{(1:n)} < X_{(2:n)} < \cdots < X_{(n:n)}$ of size n from a distribution function $F(\cdot)$ from Equation (2.2). Then, the LSEs of the parameters μ and σ are obtained by minimizing

$$\mathcal{S}(\mu, \sigma) = \sum_{i=1}^{n}\left[\Phi\left(\frac{x_{(i:n)} - \mu}{\sigma}\right) - \frac{i}{n+1}\right]^2. \tag{2.6}$$

The least square estimators $\hat{\mu}_{\mathrm{LSE}}$ and $\hat{\sigma}_{\mathrm{LSE}}$ of the parameters μ and σ can be obtained by solving the following non-linear equations:

$$\sum_{i=1}^{n}\left[\Phi\left(\frac{x_{(i:n)} - \mu}{\sigma}\right) - \frac{i}{n+1}\right]\phi_1\left(\frac{x_{(i:n)} - \mu}{\sigma}\right) = 0 \tag{2.7}$$

and

$$\sum_{i=1}^{n}\left[\Phi\left(\frac{x_{(i:n)} - \mu}{\sigma}\right) - \frac{i}{n+1}\right](x_{(i:n)} - \mu)\phi_2\left(\frac{x_{(i:n)} - \mu}{\sigma}\right) = 0. \tag{2.8}$$

The above normal equations cannot be solved analytically, therefore, we use non-linear minimization (nlm) function [see, Dannis and Schnabel (1983)] to obtained the solutions. Substituting the LSEs, we can get the estimator of C_{pk} as

$$\hat{C}_{pk}^{\text{LSE}} = \min\left\{\frac{U-\hat{\mu}_{\text{LSE}}}{3\hat{\sigma}_{\text{LSE}}}, \frac{\hat{\mu}_{\text{LSE}}-L}{3\hat{\sigma}_{\text{LSE}}}\right\}. \tag{2.9}$$

The WLSEs of the parameters μ and σ can be obtained by minimising

$$\mathcal{W}(\mu,\sigma) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\Phi\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) - \frac{i}{n+1}\right]^2. \tag{2.10}$$

These estimators can also be obtained by solving the following equations:

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\Phi\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) - \frac{i}{n+1}\right]\phi_1\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) = 0, \tag{2.11}$$

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\Phi\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) - \frac{i}{n+1}\right](x_{(i:n)}-\mu)\phi_2\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) = 0, \tag{2.12}$$

where, $\phi_1(x_{(i:n)},\mu,\sigma)$ and $\phi_2(x_{(i:n)},\mu,\sigma)$ are the first derivatives of $\Phi(\frac{x_{(i:n)}-\mu}{\sigma})$ with respect to μ and σ respectively. Substituting the WLSEs, we can get the estimator of C_{pk} as

$$\hat{C}_{pk}^{\text{WLSE}} = \min\left\{\frac{U-\hat{\mu}_{\text{WLSE}}}{3\hat{\sigma}_{\text{WLSE}}}, \frac{\hat{\mu}_{\text{WLSE}}-L}{3\hat{\sigma}_{\text{WLSE}}}\right\}. \tag{2.13}$$

2.3 Maximum product of spacings estimator

This method was introduced by Cheng and Amin (1979) as an alternative to the method of MLE. The method is briefly described as follows. The CDF of the normal distribution is given in the Equation (2.2), using the same notations in Subsection 2.2, define the uniform spacings of a random sample from the normal distribution as:

$$\begin{aligned}\mathcal{D}_i(\mu,\sigma) &= F(x_{i:n} \mid \mu,\sigma) - F(x_{i-1:n} \mid \mu,\sigma)\\ &= \left[\Phi\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) - \Phi\left(\frac{x_{(i-1:n)}-\mu}{\sigma}\right)\right]; \quad i = 1,2,\ldots,n+1,\end{aligned} \tag{2.14}$$

where, $F(x_{(0:n)} \mid \mu,\sigma) = 0$ and $F(x_{(n+1:n)} \mid \mu,\sigma) = 1$. Clearly $\sum_{i=1}^{n+1}\mathcal{D}_i(\mu,\sigma) = 1$. The MPSEs $\hat{\mu}_{\text{MPSE}}$ and $\hat{\sigma}_{\text{MPSE}}$, of the parameters μ and σ are obtained by maximizing with respect to μ and σ, the geometric mean of the spacings:

$$\mathcal{G} = \sqrt[n+1]{\left(\prod_{i=1}^{n+1}\mathcal{D}_i(\mu,\sigma)\right)}. \tag{2.15}$$

Taking logarithm on both sides of Equation (2.15), we get,

$$\ln\mathcal{G} = \frac{1}{(n+1)}\sum_{i=1}^{n+1}\ln\mathcal{D}_i(\mu,\sigma). \tag{2.16}$$

The MPSEs are obtained by solving the following non-linear equations:

$$\sum_{i=1}^{n+1}\frac{1}{\mathcal{D}_i(\mu,\sigma)}\left[\eta_1(x_{(i:n)} \mid \mu,\sigma) - \eta_1(x_{(i-1:n)} \mid \mu,\sigma)\right] = 0 \tag{2.17}$$

and

$$\sum_{i=1}^{n+1} \frac{1}{D_i(\mu,\sigma)}\left[\eta_2(x_{(i:n)} \mid \mu,\sigma) - \eta_2(x_{(i-1:n)} \mid \mu,\sigma)\right] = 0, \tag{2.18}$$

where, $\eta_1(x_{(i:n)}, \mu, \sigma)$ and $\eta_2(x_{(i:n)}, \mu, \sigma)$ are the first derivatives of $\mathcal{D}_i(\mu,\sigma)$ with respect to μ and σ respectively. Substituting the MPSEs, we can get the estimator of C_{pk} as

$$\hat{C}_{pk}^{\text{MPSE}} = \min\left\{\frac{U - \hat{\mu}_{\text{MPSE}}}{3\hat{\sigma}_{\text{MPSE}}}, \frac{\hat{\mu}_{\text{MPSE}} - L}{3\hat{\sigma}_{\text{MPSE}}}\right\}. \tag{2.19}$$

2.4 Cramèr–von-Mises estimator

To motivate our choice of Cramèr-von Mises type minimum distance estimators, MacDonald (1971) provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, the Cramèr-von Mises estimators $\hat{\mu}_{\text{CME}}$ and $\hat{\sigma}_{\text{CME}}$ of the parameter μ, σ are obtained by minimizing the following expression:

$$\mathcal{C}(\mu,\sigma) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{(i:n)} \mid \mu,\sigma) - \frac{2i-1}{2n}\right]^2 \tag{2.20}$$

the minimization of the above equation yields

$$\sum_{i=1}^{n}\left[\Phi\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) - \frac{2i-1}{2n}\right]\phi_1(x_{(i:n)} \mid \mu,\sigma) = 0 \tag{2.21}$$

and

$$\sum_{i=1}^{n}\left[\Phi\left(\frac{x_{(i:n)}-\mu}{\sigma}\right) - \frac{2i-1}{2n}\right](x_i-\mu)\phi_2(x_{(i:n)} \mid \mu,\sigma) = 0, \tag{2.22}$$

where, $\phi_1(x_{(i:n)}, \mu, \sigma)$ and $\phi_2(x_{(i:n)}, \mu, \sigma)$ are the first derivatives of $\Phi(\frac{x_{(i:n)}-\mu}{\sigma})$ with respect to μ and σ, respectively. Substituting the CMEs, we can get the estimator of C_{pk} as

$$\hat{C}_{pk}^{\text{CME}} = \min\left\{\frac{U - \hat{\mu}_{\text{CME}}}{3\hat{\sigma}_{\text{CME}}}, \frac{\hat{\mu}_{\text{CME}} - L}{3\hat{\sigma}_{\text{CME}}}\right\}. \tag{2.23}$$

2.5 Anderson–Darling and right-tail Anderson–Darling estimators

Another method of estimation based on the minimum distance was suggested by Anderson and Darling (1952) as an alternative to other statistical tests for detecting sample distributions departure from normality. The ADEs $(\hat{\mu}_{\text{ADE}}, \hat{\sigma}_{\text{ADE}})$ of the parameters (μ, σ) are obtained by minimizing the following function, with respect to μ and σ;

$$\mathcal{AD}(x_i;\mu,\sigma) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F(x_{i:n} \mid \mu,\sigma) + \ln \bar{F}(x_{n+1-i:n} \mid \mu,\sigma)\right]. \tag{2.24}$$

Using, the CDF of normal distribution, we get

$$\begin{aligned}&\mathcal{AD}(x_i;\mu,\sigma)\\ &\quad = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln\Phi\left(\frac{x_{i:n}-\mu}{\sigma}\right) + \ln\left(1-\Phi\left(\frac{x_{(n+1-i):n}-\mu}{\sigma}\right)\right)\right].\end{aligned} \tag{2.25}$$

The respective estimators of μ and σ are obtained by solving the following non-linear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\zeta_1(x_{i:n} \mid \mu,\sigma)}{F(x_{i:n} \mid \mu,\sigma)} - \frac{\zeta_1(x_{(n+1-i):n} \mid \mu,\sigma)}{\bar{F}(x_{(n+1-i):n} \mid \mu,\sigma)}\right] = 0 \tag{2.26}$$

and

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\zeta_2(x_{i:n} \mid \mu,\sigma)}{F(x_{i:n} \mid \mu,\sigma)} - \frac{\zeta_2(x_{(n+1-i):n} \mid \mu,\sigma)}{\bar{F}(x_{(n+1-i):n} \mid \mu,\sigma)}\right] = 0. \tag{2.27}$$

Substituting the ADEs, we can get the estimator of C_{pk} as

$$\hat{C}_{pk}^{\mathrm{ADE}} = \min\left\{\frac{U-\hat{\mu}_{\mathrm{ADE}}}{3\hat{\sigma}_{\mathrm{ADE}}}, \frac{\hat{\mu}_{\mathrm{ADE}}-L}{3\hat{\sigma}_{\mathrm{ADE}}}\right\}. \tag{2.28}$$

The RADEs $\hat{\mu}_{\mathrm{RADE}}$ and $\hat{\sigma}_{\mathrm{RADE}}$ of the parameters μ and σ are obtained by minimizing the function:

$$\mathcal{RAD}(\mu,\sigma) = \frac{n}{2} - 2\sum_{i=1}^{n}\ln F(x_{i:n} \mid \mu,\sigma) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\bar{F}(x_{n+1-i:n} \mid \mu,\sigma) \tag{2.29}$$

or equivalently

$$\begin{aligned}\mathcal{RAD}(\mu,\sigma) = {} & \frac{n}{2} - 2\sum_{i=1}^{n}\ln\Phi\left(\frac{x_{i:n}-\mu}{\sigma}\right) \\ & - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(1-\Phi\left(\frac{x_{(n+1-i):n}-\mu}{\sigma}\right)\right).\end{aligned} \tag{2.30}$$

The estimates can also be obtained by solving the non-linear equations

$$-2\sum_{i=1}^{n}\frac{\zeta_1(x_{i:n} \mid \mu,\sigma)}{F(x_{i:n},\mu,\sigma)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\zeta_1(x_{(n+1-i):n} \mid \mu,\sigma)}{\bar{F}(x_{(n+1-i):n},\mu,\sigma)} = 0 \tag{2.31}$$

and

$$-2\sum_{i=1}^{n}\frac{\zeta_2(x_{i:n} \mid \mu,\sigma)}{F(x_{i:n} \mid \mu,\sigma)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\zeta_2(x_{(n+1-i):n} \mid \mu,\sigma)}{\bar{F}(x_{(n+1-i):n} \mid \mu,\sigma)} = 0. \tag{2.32}$$

Substituting the RADEs, we can get the estimator of C_{pk} as

$$\hat{C}_{pk}^{\mathrm{RADE}} = \min\left\{\frac{U-\hat{\mu}_{\mathrm{RADE}}}{3\hat{\sigma}_{\mathrm{RADE}}}, \frac{\hat{\mu}_{\mathrm{RADE}}-L}{3\hat{\sigma}_{\mathrm{RADE}}}\right\}. \tag{2.33}$$

3 Bootstrap confidence intervals

In this section, we propose three confidence intervals based on bootstrap methods: (i) standard bootstrap ($\mathcal{S}$-boot); (ii) percentile bootstrap ($\mathcal{P}$-boot) based on the idea of Efron (1982), and (ii) bias-corrected percentile bootstrap ($\mathcal{BC}_p$-boot). Below, we are discussing the algorithm for all the three methods. The algorithm of bootstrap method is displayed below graphically [see, Figure 1].

The arrangement of the entire collection of the bootstrap estimates are arranged from smallest to largest which would constitute an empirical bootstrap distribution of the index C_{pk} will be denoted as $\hat{C}_{pk}^{*(1)} \le \hat{C}_{pk}^{*(2)} \le \cdots \le \hat{C}_{pk}^{*(\mathcal{B})}$.

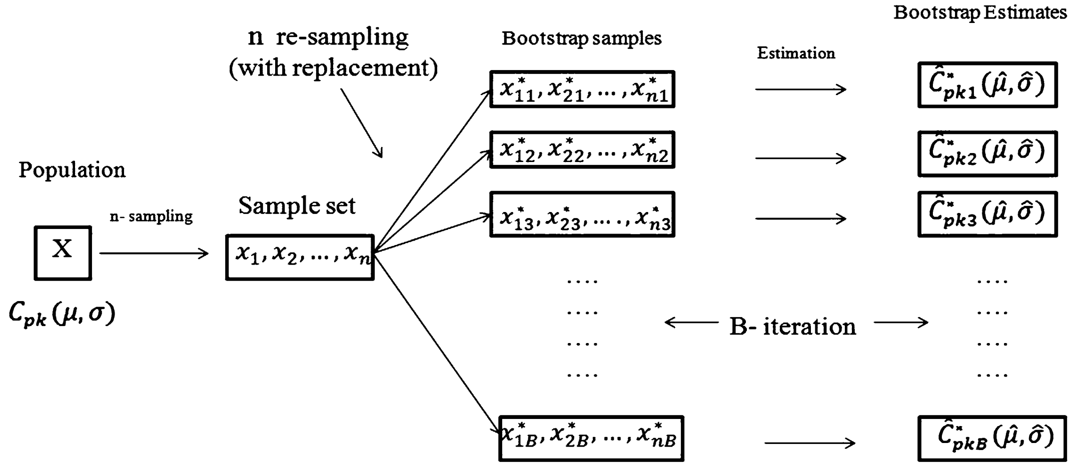

Figure 1 *Bootstrap Procedure.*

3.1 $\mathcal{S}$-boot

Let $\bar{\hat{C}}^*_{pk}$ and Se^* be the sample mean and sample standard deviation of $\{\hat{C}^{*(\mathcal{J})}_{pmk}; \mathcal{J} = 1, 2, \dots, \mathcal{B}\}$, i.e.,

$$\bar{\hat{C}}^*_{pmk} = \frac{1}{\mathcal{B}} \sum_{j=1}^{\mathcal{B}} \hat{C}^{*(\mathcal{J})}_{pk}$$

and

$$Se^* = \sqrt{\frac{1}{(\mathcal{B}-1)} \sum_{\mathcal{J}=1}^{\mathcal{B}} (\hat{C}^{*(\mathcal{J})}_{pk} - \bar{\hat{C}}^*_{pk})^2},$$

respectively. A $100(1-\alpha)\%$ $\mathcal{S}$-boot confidence interval of C_{pk} is given by

$$\{\bar{\hat{C}}^*_{pk} - z_{(\alpha/2)}.Se^*, \bar{\hat{C}}^*_{pk} + z_{(\alpha/2)}.Se^*\}. \tag{3.1}$$

Here, $z_{(\alpha/2)}$ is obtained by using upper $(\alpha/2)$th point of the standard normal deviate.

3.2 $\mathcal{P}$-boot

Let $\hat{C}^{*(\eta)}_{pk}$ be the η percentile of $\{\hat{C}^{*(\mathcal{J})}_{pTk}; \mathcal{J} = 1, 2, \dots, \mathcal{B}\}$, i.e., $\hat{C}^{*(\eta)}_{pk}$ is such that

$$\frac{1}{\mathcal{B}} \sum_{j=1}^{\mathcal{B}} I(\hat{C}^{*(\mathcal{J})}_{pk} \le \hat{C}^{*(\eta)}_{pk}) = \eta; \quad 0 < \eta < 1,$$

where, I is indicator function. Then, $100(1-\zeta)\%$ $\mathcal{P}$ CI of C_{pk} is

$$\{\hat{C}^{*(\mathcal{B}.(\zeta/2))}_{pk}, \hat{C}^{*(\mathcal{B}.(1-\zeta/2))}_{pk}\}. \tag{3.2}$$

3.3 $\mathcal{B}C_p$-boot

The idea of this method lies to correct for the potential bias. At first, locate the observed $\hat{C}_{pk}$ in the order statistics $\hat{C}^{*(1)}_{pk} \le \hat{C}^{*(2)}_{pk} \le \cdots \le \hat{C}^{*(\mathcal{B})}_{pk}$. Then, compute the probability

$$\Upsilon_0 = \frac{1}{\mathcal{B}} \sum_{\mathcal{J}=1}^{\mathcal{B}} I(\hat{C}^{*(\mathcal{J})}_{pk} \le \hat{C}_{pk}).$$

Calculate $\Psi_0 = \Phi^{-1}(\Upsilon_0)$, where, $\Phi(\cdot)$ is the standard normal CDF and ψ_l and ψ_u are defined as

$$\psi_l = \Phi(2\Psi_0 - \xi_{(1-\zeta/2)}) \quad \text{and} \quad \psi_u = \Phi(2\Psi_0 + \xi_{(1-\zeta/2)}).$$

Then, $100(1-\zeta)\%$ $\mathcal{B}C_p$-boot confidence interval of C_{pk} is

$$\{\hat{C}_{pk}^{*(\mathcal{B}.\psi_l)}, \hat{C}_{pk}^{*(\mathcal{B}.\psi_u)}\}. \tag{3.3}$$

To study the different confidence intervals, we consider their estimated average widths and coverage probabilities. For each of the methods considered, the probability that the true value of C_{pk} is covered by the $100(1-\alpha)\%$ BCI, which is called the "coverage probability". In addition, the average width of the BCIs is calculated based on the $\mathcal{K} = 5000$ different trials. The average width and estimated coverage probability are given by

$$\text{Average width} = \frac{\sum_{\mathcal{I}=1}^{\mathcal{K}}(U_{\mathcal{I}} - L_{\mathcal{I}})}{\mathcal{K}},$$

and

$$\text{Coverage probability} = \frac{(L_{\mathcal{W}} \leq C_{pk} \leq U_{\mathcal{P}})}{K},$$

where, $(L_{\mathcal{W}}, U_{\mathcal{P}})$ denote the $100(1-\alpha)\%$ confidence intervals based on $\mathcal{K}$ replicates.

4 Bayesian estimation

In this section, we present Bayesian estimation of the index C_{pk}. Bayesian analysis is a natural way to combine the observed information with the prior information. Here, we have considered two independent conjugate priors for the parameters μ and σ^2. The considered prior distributions are given by

$$g(\mu) \propto \text{Normal}(a,b), \qquad g(\sigma^2) \propto \text{Inverse-gamma}(c,d),$$

where, a, b, c and d are the hyper-parameters which are assumed to be known. Therefore, the joint prior distribution of μ and σ^2 is:

$$\pi(\mu,\sigma^2) \propto g(\mu) \times g(\sigma^2). \tag{4.1}$$

Now, using the Equations (2.3) and (4.1), the joint posterior distribution can be written as:

$$\begin{aligned} P(\mu,\sigma \mid \mathbf{x}) &\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(\frac{x-\mu}{\sigma})^2} e^{-\frac{1}{2b}(\mu-a)^2} \frac{1}{\sigma^2}^{c+1} e^{-\frac{d}{\sigma^2}} \\ &\propto \left(\frac{1}{\sigma^2}\right)^{n/2+c+1} e^{-\frac{1}{\sigma^2}(d+ns^2)} e^{-\frac{nb+\sigma^2}{2b\sigma^2}(\mu-\frac{a\sigma^2+b\bar{x}n}{\sigma^2+nb})}. \end{aligned} \tag{4.2}$$

Since, the considered priors are conjugate, therefore, the marginal posterior distribution of μ and σ^2 belongs to the same familiarity of distribution, given as

$$P_\sigma(\sigma^2|\mathbf{x}) \sim \text{Inverse-Gamma}(n/2+c, d+ns^2) \tag{4.3}$$

and

$$P_\mu(\mu|\sigma^2,\mathbf{x}) \sim N\left(\frac{a\sigma^2+b\bar{x}n}{\sigma^2+nb}, \frac{b\sigma^2}{nb+\sigma^2}\right) \tag{4.4}$$

respectively, where, $s^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2/n$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

In the case of Bayes point estimation theory, one of the most important element is the consideration of loss function. Here, we have considered squared error loss function (SELF). Note that, estimators using other loss functions can be obtained similarly. Under SELF, the Bayes estimate is the posterior mean of the corresponding posterior density functions, mentioned in Equations (4.3) and (4.4), respectively.

The Bayes estimate of C_{pk} under SELF is obtained as;

$$\hat{C}_{pk}^{\text{BAYES}} = E_P(C_{pk}|X) = K \int_{\mu} \int_{\sigma} C_{pk} P(\mu, \sigma \mid \mathbf{x})\, d\mu\, d\sigma, \tag{4.5}$$

where, K is the normalizing constant.

From the Equation (4.5), it is clear that the Bayes estimator of C_{pk} cannot be explicitly obtained due to involvement of ratio of two integrals. Therefore, for any Bayes computational techniques, viz., Lindley's method, Markov Chain Monte Carlo (MCMC) method, may be employed. Here, we have used MCMC method. The following steps have been used to obtain the estimate of C_{pk}.

1. Generate the sequence of σ^2, i.e., $\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2$ from $P_\sigma(\sigma^2|\mathbf{x})$.
2. Using sequence of σ^2 from Step 1, generate the sequence of μ, i.e., $\mu_1, \mu_2, \ldots, \mu_N$ from $P_\mu(\mu|\sigma^2, \mathbf{x})$.
3. Using Steps 1 and 2 for desired values U and L, generate the sequence of posterior sample: $\{C_{pk}^i; i = 1, 2, \ldots, N\}$.
4. Then Bayes estimate of C_{pk} is given as $\hat{C}_{pk}^{\text{BAYES}} = \frac{1}{N}\sum_{i=1}^{N} C_{pk}^i$
5. After simulating the posterior samples of C_{pk}, the $100(1-\alpha)\%$ HPD credible intervals for C_{pk} can be obtained by applying the algorithm suggested by Chen and Shao (1999).

5 Comparison among the estimators of C_{pk}

Here, we have carried out Monte Carlo simulation study to compare the performances of the classical methods of estimation and the Bayesian method of estimation of the PCI C_{pk}. The performances of the estimates (classical as well as Bayes) are compared in terms of their MSEs. Also, we have obtained three bootstrap confidence intervals (BCIs), viz., standard bootstrap ($\mathcal{S}$-boot); percentile bootstrap ($\mathcal{P}$-boot) and bias-corrected percentile bootstrap ($\mathcal{BC}_p$-boot) and HPD credible intervals. The performances of the CIs are compared in terms of their average widths (AWs) and coverage probabilities (CPs). Here, for the simulation study, we have considered the sample sizes $n = 10, 20, 30, 50$ and set the lower specification limit, the upper specification limit as 0.0, 8.0 and $(\mu, \sigma) = (1.0, 2.0), (1.0, 3.0), (2.0, 3.0), (2.0, 4.0)$, respectively.

For each of the design, $B = 1000$ bootstrap samples with each of size n are drawn from the original sample and replicated $M = 1000$ times. To obtain the Bayes estimate of the index C_{pk}, we chose the hyper parameters $a = 0.98; b = 0.95; c = 1.5; d = 5$ respectively. The results are reported in Table 1. The results of the 95% BCIs, viz., $\mathcal{S}$-boot, $\mathcal{P}$-boot and $\mathcal{BC}_p$-boot constructed by each of the classical methods estimation for C_{pk} and also 95% highest posterior density (HPD) credible interval are reported in Tables 2–9, respectively.

From Table 1, it is observed that, for all the considered parameters values, the Bayes estimate gives least MSEs as compared to other classical methods of estimation. Also, in almost all the cases, MSEs decreases as the sample sizes increases. Form Tables 2–8 it is observed that, MPSE gives the least AWs in all the cases ($\mathcal{S}$-boot and $\mathcal{BC}_p$-boot) compared to other classical methods of estimation (MLE, LSE, WLSE, CME, ADE, RADE) and the order of best method of estimation is MPSE < MLE < WLSE < LSE < CME < ADE < RADE. For all the parameter values, AWs decreases and CPs increases as we increase the sample sizes. From Table 9, it observed that AWs of HPD credible intervals decreases and CPs increases as we increase the sample size for almost all the set up of parameter values.

Table 1 *True value of C_{pk} along with average estimates of different estimators (MLE, LSE, WLSE, CME, MPSE, ADE, RADE, Bayes), their corresponding absolute biases (in each second row) and MSEs (in each third row)*

				Estimates of C_{pk} and corresponding biases, MSEs							
n	μ	σ	C_{pk}	MLE	LSE	WLSE	CME	MPSE	ADE	RADE	Bayes
10	1.00	2.00	0.16667	0.19214	0.17306	0.17214	0.17144	0.15293	0.17555	0.20592	0.18724
				0.20142	0.20213	0.20212	0.20163	0.02013	0.02108	0.02111	0.01935
				0.01252	0.02231	0.02089	0.02076	0.01142	0.01511	0.03339	0.00568
20	1.00	2.00	0.16667	0.17853	0.16662	0.16877	0.16671	0.15553	0.17089	0.18258	0.17249
				0.00476	0.00481	0.00481	0.00483	0.00473	0.00488	0.00489	0.00469
				0.00694	0.00741	0.00697	0.00695	0.00531	0.00671	0.00971	0.00438
30	1.00	2.00	0.16667	0.17784	0.16413	0.16621	0.16566	0.15615	0.16762	0.17276	0.16988
				0.00379	0.00379	0.00379	0.00382	0.00378	0.00384	0.00384	0.00362
				0.00450	0.00466	0.00433	0.00431	0.00389	0.00429	0.00542	0.00376
50	1.00	2.00	0.16667	0.17774	0.16661	0.16761	0.16756	0.15931	0.16812	0.17231	0.16682
				0.00018	0.00026	0.00023	0.00026	0.00018	0.00027	0.00028	0.00012
				0.00263	0.00294	0.00278	0.00253	0.00237	0.00277	0.00352	0.00224
10	1.00	3.00	0.11111	0.10148	0.11024	0.10753	0.10746	0.10079	0.11324	0.13148	0.13653
				0.02133	0.02139	0.02137	0.02142	0.02132	0.02139	0.02145	0.02127
				0.01354	0.01343	0.01514	0.01513	0.01210	0.01577	0.02255	0.00647
20	1.00	3.00	0.11111	0.12404	0.11497	0.11682	0.11679	0.10824	0.11855	0.12889	0.12543
				0.01041	0.01052	0.01048	0.01053	0.0144	0.01057	0.01058	0.01038
				0.00638	0.00683	0.00644	0.00667	0.00488	0.00686	0.00859	0.00459
30	1.00	3.00	0.11111	0.12756	0.10755	0.10977	0.10739	0.10355	0.11096	0.11516	0.11484
				0.00343	0.00344	0.00344	0.00349	0.00341	0.00353	0.00356	0.00329
				0.00453	0.00458	0.00454	0.00437	0.00412	0.00436	0.00488	0.00404
50	1.00	3.00	0.11111	0.12762	0.10806	0.10889	0.10884	0.10373	0.10929	0.11217	0.11232
				0.00151	0.00154	0.00154	0.00157	0.00148	0.00166	0.00169	0.00137
				0.00252	0.00253	0.00245	0.00242	0.00236	0.00244	0.00293	0.00233
10	2.00	3.00	0.22222	0.23291	0.21303	0.21662	0.21661	0.19761	0.22438	0.25466	0.23922
				0.02262	0.02276	0.02271	0.02279	0.02252	0.02291	0.02293	0.02247
				0.02792	0.02911	0.02809	0.02904	0.02144	0.03517	0.03519	0.02094
20	2.00	3.00	0.22222	0.23761	0.21839	0.22347	0.21904	0.20714	0.22752	0.23905	0.24483
				0.01803	0.01813	0.01809	0.01818	0.01793	0.01823	0.01826	0.01788
				0.00844	0.00926	0.00806	0.00931	0.00642	0.00996	0.01088	0.00432
30	2.00	3.00	0.22222	0.20483	0.21731	0.22252	0.22154	0.20888	0.22473	0.23079	0.22473
				0.00247	0.00258	0.00253	0.00263	0.00242	0.00266	0.00271	0.00238
				0.00784	0.00862	0.00828	0.00926	0.00731	0.00954	0.00979	0.00670
50	2.00	3.00	0.22222	0.19982	0.21954	0.22268	0.22133	0.21224	0.22313	0.22634	0.22278
				0.00163	0.00174	0.00171	0.00179	0.00154	0.00186	0.00192	0.00147
				0.00745	0.00842	0.00803	0.00894	0.00657	0.00899	0.00914	0.00630
10	4.00	3.00	0.44444	0.39978	0.45513	0.45554	0.42584	0.32679	0.37155	0.39186	0.44343
				0.00194	0.00223	0.00217	0.00229	0.00179	0.00234	0.00242	0.00133
				0.01191	0.01871	0.01810	0.01879	0.01098	0.01895	0.02040	0.00970
20	4.00	3.00	0.44444	0.41051	0.45623	0.38783	0.47923	0.36044	0.39439	0.40067	0.44517
				0.00112	0.00125	0.00117	0.00138	0.00078	0.00173	0.00184	0.00045
				0.00852	0.01050	0.00879	0.01177	0.00582	0.01377	0.01408	0.00370
30	4.00	3.00	0.44444	0.43242	0.45388	0.39832	0.41426	0.37556	0.40219	0.40567	0.44484
				0.00068	0.00065	0.00064	0.00073	0.00053	0.00083	0.00092	0.00034
				0.00508	0.00509	0.00506	0.00671	0.00386	0.00781	0.00823	0.00290
50	4.00	3.00	0.44444	0.43243	0.50117	0.40874	0.41739	0.38967	0.41062	0.41279	0.44453
				0.00036	0.00038	0.00035	0.00044	0.00028	0.00046	0.00064	0.00026
				0.00411	0.00414	0.00412	0.00432	0.00234	0.00488	0.00554	0.00210

Table 2 *True value of C_{pk} along with estimated average widths and coverage probabilities of BCIs based on MLEs*

				Average Widths			Coverage Probabilities		
n	μ	σ	C_{pk}	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot
10	1.00	2.00	0.16667	2.54621	2.36348	2.24156	0.915	0.915	0.919
20	1.00	2.00	0.16667	1.43781	1.00591	1.00081	0.916	0.919	0.921
30	1.00	2.00	0.16667	0.44435	0.38335	0.33276	0.921	0.923	0.923
50	1.00	2.00	0.16667	0.15284	0.10062	0.10041	0.924	0.926	0.926
10	1.00	3.00	0.11111	1.24968	1.14254	1.13561	0.918	0.919	0.917
20	1.00	3.00	0.11111	1.09224	1.05493	1.04483	0.919	0.921	0.921
30	1.00	3.00	0.11111	0.98293	0.95195	0.94337	0.920	0.921	0.923
50	1.00	3.00	0.11111	0.34651	0.33416	0.32872	0.927	0.929	0.930
10	2.00	3.00	0.22222	1.07502	0.93245	0.92765	0.918	0.919	0.919
20	2.00	3.00	0.22222	1.00007	0.92975	0.91897	0.921	0.922	0.922
30	2.00	3.00	0.22222	0.69373	0.67920	0.66873	0.924	0.926	0.927
50	2.00	3.00	0.22222	0.44550	0.42138	0.41773	0.930	0.931	0.933
10	4.00	3.00	0.44444	1.03256	0.92001	0.91254	0.917	0.919	0.920
20	4.00	3.00	0.44444	0.94359	0.93745	0.92113	0.919	0.920	0.921
30	4.00	3.00	0.44444	0.74589	0.72916	0.71771	0.921	0.921	0.922
50	4.00	3.00	0.44444	0.72014	0.68543	0.66558	0.929	0.930	0.932

Table 3 *True value of C_{pk} along with estimated average widths and coverage probabilities of BCIs based on LSEs*

				Average Widths			Coverage Probabilities		
n	μ	σ	C_{pk}	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot
10	1.00	2.00	0.16667	2.52663	2.33876	2.23115	0.916	0.917	0.919
20	1.00	2.00	0.16667	1.42116	1.00378	1.00063	0.918	0.919	0.920
30	1.00	2.00	0.16667	0.42786	0.37234	0.32116	0.923	0.924	0.923
50	1.00	2.00	0.16667	0.15003	0.10033	0.10017	0.926	0.927	0.926
10	1.00	3.00	0.11111	1.23264	1.12453	1.13114	0.917	0.919	0.917
20	1.00	3.00	0.11111	1.07454	1.04713	1.02784	0.919	0.920	0.920
30	1.00	3.00	0.11111	0.96773	0.93654	0.93672	0.921	0.921	0.922
50	1.00	3.00	0.11111	0.33245	0.32241	0.31782	0.926	0.927	0.927
10	2.00	3.00	0.22222	1.05352	0.91786	0.91135	0.917	0.918	0.919
20	2.00	3.00	0.22222	0.99766	0.91776	0.90565	0.920	0.921	0.922
30	2.00	3.00	0.22222	0.68343	0.65767	0.65435	0.923	0.924	0.925
50	2.00	3.00	0.22222	0.43264	0.41426	0.40896	0.926	0.928	0.928
10	4.00	3.00	0.44444	1.01236	0.91652	0.90373	0.917	0.918	0.919
20	4.00	3.00	0.44444	0.92447	0.91786	0.90115	0.919	0.920	0.921
30	4.00	3.00	0.44444	0.73523	0.70452	0.69884	0.920	0.921	0.922
50	4.00	3.00	0.44444	0.71143	0.67112	0.64786	0.927	0.928	0.928

6 Sensitivity analysis

The net sensitivity (NS) analysis using a distribution function for a given PCI is defined as [see, Flaig (1999), Maiti et al. (2010)]

$$NS = \frac{1}{p_0} \lim_{\delta \to 0} \left[\frac{\{F(U) - F(L)\} - \{F(U-\delta) - F(L-\delta)\}}{\delta} \right]$$

Table 4 *True value of C_{pk} along with estimated average widths and coverage probabilities of BCIs based on WLSEs*

				Average Widths			Coverage Probabilities		
n	μ	σ	C_{pk}	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot
10	1.00	2.00	0.16667	2.51456	2.33124	2.22544	0.915	0.917	0.918
20	1.00	2.00	0.16667	1.41674	1.00287	1.00044	0.917	0.918	0.919
30	1.00	2.00	0.16667	0.42113	0.36784	0.31543	0.922	0.923	0.923
50	1.00	2.00	0.16667	0.15001	0.10013	0.10015	0.926	0.927	0.927
10	1.00	3.00	0.11111	1.22784	1.12113	1.12768	0.916	0.917	0.917
20	1.00	3.00	0.11111	1.07244	1.04242	1.02443	0.918	0.919	0.920
30	1.00	3.00	0.11111	0.96245	0.93112	0.93324	0.921	0.921	0.921
50	1.00	3.00	0.11111	0.32785	0.31342	0.31114	0.925	0.926	0.926
10	2.00	3.00	0.22222	1.03655	0.91332	0.90896	0.916	0.918	0.919
20	2.00	3.00	0.22222	0.99223	0.91443	0.90114	0.919	0.920	0.920
30	2.00	3.00	0.22222	0.67894	0.65337	0.65111	0.922	0.922	0.923
50	2.00	3.00	0.22222	0.42775	0.41068	0.40089	0.925	0.926	0.926
10	4.00	3.00	0.44444	1.01114	0.91222	0.89773	0.916	0.917	0.918
20	4.00	3.00	0.44444	0.92046	0.91137	0.88364	0.919	0.920	0.921
30	4.00	3.00	0.44444	0.73114	0.70342	0.69117	0.921	0.921	0.921
50	4.00	3.00	0.44444	0.70677	0.66891	0.63885	0.926	0.927	0.927

Table 5 *True value of C_{pk} along with estimated average widths and coverage probabilities of BCIs based on CMEs*

				Average Widths			Coverage Probabilities		
n	μ	σ	C_{pk}	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot
10	1.00	2.00	0.16667	2.52334	2.33767	2.22112	0.916	0.917	0.919
20	1.00	2.00	0.16667	1.40897	1.00131	1.00023	0.918	0.918	0.920
30	1.00	2.00	0.16667	0.42036	0.36223	0.31221	0.922	0.923	0.923
50	1.00	2.00	0.16667	0.14894	0.99846	0.10061	0.926	0.927	0.927
10	1.00	3.00	0.11111	1.22844	1.12889	1.12843	0.917	0.918	0.919
20	1.00	3.00	0.11111	1.07899	1.05637	1.03644	0.918	0.919	0.920
30	1.00	3.00	0.11111	0.96766	0.94362	0.94377	0.922	0.922	0.923
50	1.00	3.00	0.11111	0.33367	0.32331	0.32111	0.926	0.926	0.928
10	2.00	3.00	0.22222	1.04367	0.92337	0.91227	0.916	0.918	0.919
20	2.00	3.00	0.22222	0.99887	0.92351	0.91226	0.918	0.920	0.920
30	2.00	3.00	0.22222	0.68447	0.65892	0.65776	0.921	0.922	0.923
50	2.00	3.00	0.22222	0.42993	0.41674	0.41037	0.926	0.926	0.927
10	4.00	3.00	0.44444	1.03674	0.91778	0.90133	0.917	0.918	0.918
20	4.00	3.00	0.44444	0.93224	0.91784	0.89614	0.918	0.920	0.920
30	4.00	3.00	0.44444	0.74773	0.71373	0.70568	0.920	0.921	0.922
50	4.00	3.00	0.44444	0.71336	0.67431	0.64114	0.926	0.926	0.927

$$= \frac{f(U) - f(L)}{p_0},$$

where, p_0 is the desirable yield. The positive NS values imply that the distribution is more sensitive (or less robust) at upper specification than at lower specification, and for negative NS values, it is opposite. Lower the value of NS (in absolute sense) implies less sensitiveness/more robustness with respect to the PCI.

Table 6 *True value of C_{pk} along with estimated average widths, coverage probabilities and relative coverages of BCIs based on MPSEs*

				Average Widths			Coverage Probabilities		
n	μ	σ	C_{pk}	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot
10	1.00	2.00	0.16667	2.50764	2.31896	2.21657	0.915	0.916	0.917
20	1.00	2.00	0.16667	1.40284	1.00114	1.00016	0.916	0.917	0.917
30	1.00	2.00	0.16667	0.41673	0.35724	0.30784	0.920	0.921	0.922
50	1.00	2.00	0.16667	0.14896	0.10011	0.10009	0.925	0.926	0.926
10	1.00	3.00	0.11111	1.21374	1.12037	1.12117	0.915	0.916	0.916
20	1.00	3.00	0.11111	1.06343	1.04111	1.02226	0.916	0.917	0.918
30	1.00	3.00	0.11111	0.95474	0.92225	0.92785	0.920	0.921	0.921
50	1.00	3.00	0.11111	0.32224	0.30472	0.30761	0.924	0.925	0.925
10	2.00	3.00	0.22222	1.03423	0.90762	0.90225	0.915	0.917	0.919
20	2.00	3.00	0.22222	0.98336	0.91112	0.89974	0.916	0.918	0.920
30	2.00	3.00	0.22222	0.67224	0.65016	0.64661	0.920	0.920	0.922
50	2.00	3.00	0.22222	0.42237	0.40896	0.40015	0.924	0.925	0.926
10	4.00	3.00	0.44444	0.99786	0.90764	0.89114	0.916	0.916	0.918
20	4.00	3.00	0.44444	0.91889	0.91058	0.87446	0.917	0.919	0.920
30	4.00	3.00	0.44444	0.72667	0.70111	0.68335	0.919	0.920	0.921
50	4.00	3.00	0.44444	0.69742	0.66332	0.63114	0.924	0.925	0.925

Table 7 *True value of C_{pk} along with estimated average widths and coverage probabilities of BCIs based on ADEs*

				Average Widths			Coverage Probabilities		
n	μ	σ	C_{pk}	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot
10	1.00	2.00	0.16667	2.52432	2.33476	2.22441	0.916	0.917	0.919
20	1.00	2.00	0.16667	1.41447	1.00221	1.00064	0.917	0.918	0.921
30	1.00	2.00	0.16667	0.43227	0.36467	0.31772	0.919	0.922	0.922
50	1.00	2.00	0.16667	0.14773	0.10044	0.10037	0.925	0.926	0.928
10	1.00	3.00	0.11111	1.22667	1.13114	1.12078	0.916	0.918	0.919
20	1.00	3.00	0.11111	1.07224	1.04221	1.02783	0.918	0.920	0.921
30	1.00	3.00	0.11111	0.96453	0.93667	0.92884	0.920	0.922	0.923
50	1.00	3.00	0.11111	0.32778	0.31558	0.31637	0.926	0.927	0.927
10	2.00	3.00	0.22222	1.05223	0.92781	0.90896	0.916	0.917	0.919
20	2.00	3.00	0.22222	0.99782	0.92112	0.91078	0.919	0.921	0.922
30	2.00	3.00	0.22222	0.68771	0.65482	0.64991	0.921	0.922	0.923
50	2.00	3.00	0.22222	0.43294	0.41783	0.40891	0.924	0.926	0.927
10	4.00	3.00	0.44444	1.01452	0.91463	0.90224	0.917	0.918	0.919
20	4.00	3.00	0.44444	0.93116	0.92362	0.90887	0.918	0.920	0.921
30	4.00	3.00	0.44444	0.73112	0.72022	0.70792	0.921	0.922	0.922
50	4.00	3.00	0.44444	0.71762	0.67883	0.66114	0.925	0.926	0.927

In Table 10, we have reported the net sensitivity values for the normal, gamma and exponential distributions. The net sensitivity values are expressed as defective per million (dpm). We observe that the gamma and exponential distributions are more sensitive (less robust) than normal distribution for $(L, U) = (0.00, 10.00)$ and $p_0 = 0.95$, respectively.

Table 8 *True value of C_{pk} along with estimated average widths and coverage probabilities of BCIs based on RADEs*

				Average Widths			Coverage Probabilities		
n	μ	σ	C_{pk}	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{BC}_p$-boot
10	1.00	2.00	0.16667	2.55224	2.36887	2.25114	0.917	0.918	0.919
20	1.00	2.00	0.16667	1.44556	1.04671	1.03624	0.919	0.920	0.921
30	1.00	2.00	0.16667	0.45381	0.39972	0.34878	0.922	0.923	0.924
50	1.00	2.00	0.16667	0.17226	0.10267	0.10167	0.926	0.928	0.929
10	1.00	3.00	0.11111	1.25243	1.15772	1.14764	0.919	0.919	0.920
20	1.00	3.00	0.11111	1.10375	1.07226	1.06781	0.920	0.921	0.922
30	1.00	3.00	0.11111	0.99452	0.95889	0.94773	0.922	0.923	0.923
50	1.00	3.00	0.11111	0.35625	0.34664	0.33714	0.926	0.927	0.928
10	2.00	3.00	0.22222	1.09363	0.94772	0.93463	0.919	0.920	0.920
20	2.00	3.00	0.22222	1.04613	0.93771	0.92117	0.920	0.922	0.922
30	2.00	3.00	0.22222	0.70382	0.68615	0.67881	0.924	0.925	0.926
50	2.00	3.00	0.22222	0.45293	0.43738	0.42553	0.927	0.928	0.928
10	4.00	3.00	0.44444	1.05283	0.93773	0.92392	0.918	0.919	0.920
20	4.00	3.00	0.44444	0.95671	0.94884	0.93682	0.920	0.921	0.921
30	4.00	3.00	0.44444	0.75653	0.73899	0.72369	0.921	0.922	0.924
50	4.00	3.00	0.44444	0.74112	0.70783	0.69221	0.927	0.928	0.928

Table 9 *True value of C_{pk} along with estimated average widths and coverage probabilities of credible intervals based on Bayesian estimation method*

n	μ	σ	C_{pk}	Average width	Coverage probability
10	1.00	2.00	0.16667	1.33904	0.929
20	1.00	2.00	0.16667	0.92226	0.933
30	1.00	2.00	0.16667	0.29071	0.937
50	1.00	2.00	0.16667	0.09474	0.944
10	1.00	3.00	0.11111	1.12019	0.931
20	1.00	3.00	0.11111	1.01959	0.936
30	1.00	3.00	0.11111	0.87077	0.939
50	1.00	3.00	0.11111	0.24870	0.946
10	2.00	3.00	0.22222	0.85483	0.932
20	2.00	3.00	0.22222	0.84215	0.935
30	2.00	3.00	0.22222	0.58489	0.938
50	2.00	3.00	0.22222	0.35528	0.943
10	4.00	3.00	0.44444	0.87984	0.929
20	4.00	3.00	0.44444	0.84458	0.932
30	4.00	3.00	0.44444	0.67982	0.936
50	4.00	3.00	0.44444	0.62961	0.939

Table 10 *Sensitivity analysis*

Distribution	$f(U)$	$f(L)$	NS (dpm)
Normal($\mu = 4, \sigma = 1$)	6.075883×10^{-09}	0.0001338302	−140.8675
Gamma($\alpha = 4, \lambda = 1$)	0.007566655	0.000	7964.900
Exponential($\lambda = 1$)	4.539993×10^{-5}	1.000	−1052584

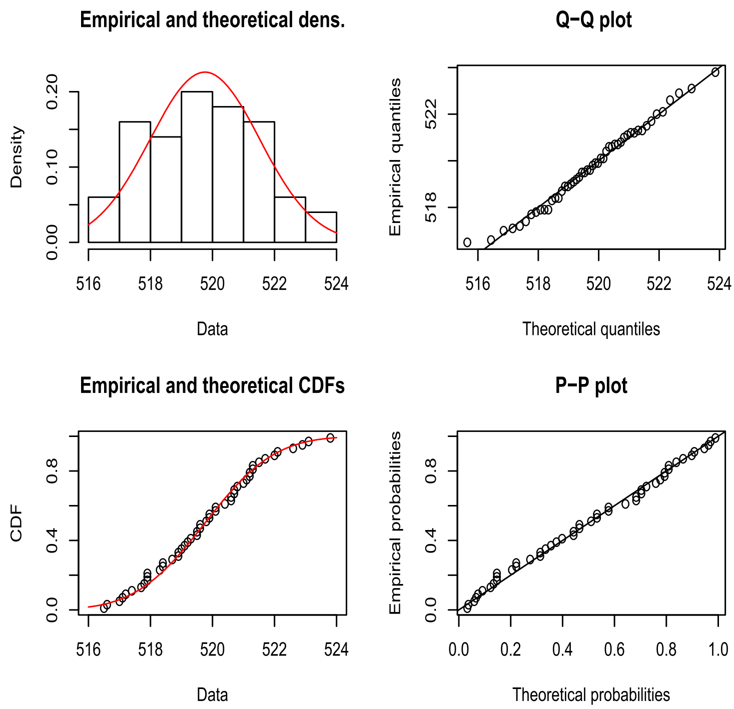

Figure 2 *Histogram, density, CDFs, P–P plot and Q–Q plot.*

7 Applications

In this section, two real data sets are considered to illustrate the different methods of estimation and BCIs ($\mathcal{S}$-boot, $\mathcal{P}$-boot and $\mathcal{BC}_p$-boot) of the PCI C_{pk} for the normal distribution. At first, we check whether the considered data sets come from the normal distribution by goodness of fit test. We have used histogram, density, theoretical and empirical CDFs, P–P plot and Q–Q plot to test the goodness of fit test for the normal distribution, displayed in the Figures 2 and 3, respectively. Also, we provide the descriptive statistics, AIC, BIC, the Kolmogorov–Smirnov (KS) statistic and the corresponding p-values of the considered data sets in Tables 11 and 12, respectively, which have been done using fitdistrplus package of R software [see, Ikha and Gentleman (1996)]

- *Data Set I: Data of voltages for aluminium foils.*
  This study cites data from the suppliers, who provided aluminium foil materials to an electronics company in Taiwan, to demonstrate the proposed procedure, given in Chen and Tong (2003). Aluminium foil is a key component that governs the quality of capacitors and the voltage is an important quality characteristic of aluminium foil: the production specifications (USL, T, LSL) of the voltage are (530, 520, 510). If the voltage falls outside this interval, the aluminium foil will break, and thus be rejected.
- *Data Set II: Application of the novel procedure to a color STN display process.*
  The data set involving a colour STN (Super Twist Nematic) displays product was taken from a manufacturing industry in Taiwan and was originally discussed by Chen and Chen (2004). Colour STN displays are created by adding colour filters to traditional monochrome STN displays. The specification limits are $12{,}000 \pm 500A^0$ (where, $1A^0 = 10^{-7}$ mm), that is, the upper and the lower specification limits are set to $USL = 12{,}500$, $LSL = 11{,}500$ and the target value is set to $T = 12{,}000$. If the thickness of membrane does not fall within the tolerance (LSL, USL), color STN displays will suffer chromatic aberration.

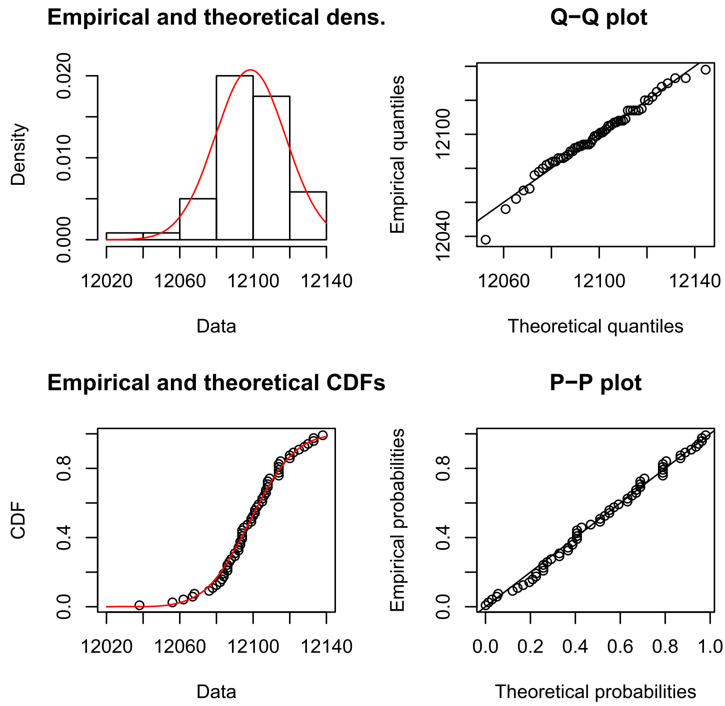

Figure 3 *Histogram, density, CDFs, P–P plot and Q–Q plot.*

Table 11 *Descriptive statistics for the considered data sets*

Data	n	Min.	Q_1	Median	Q_3	Max.	CS	CK
I	50	516.5	518.4	519.7	521.1	523.8	0.1244902	2.343682
II	60	12040	12090	12100	12110	12140	−0.41697	3.636393

Table 12 *MLEs of μ and σ, KS and p-values for the considered data sets*

Data	MLEs	Log-likelihood	AIC	BIC	KS Statistic	KS p-value
I	$\hat{\mu} = 519.756$ $\hat{\sigma} = 1.783731$	−99.37723	202.7545	206.5785	0.073389	0.9505
II	$\hat{\mu} = 12098.51667$ $\hat{\sigma} = 19.23061$	−262.5265	529.053	533.2417	0.061872	0.9757

For the considered data sets, we have calculated the point estimates of C_{pk} using different aforementioned classical methods of estimation and the Bayesian estimation method, reported in Table 13. Further, the widths of BCIs ($\mathcal{S}$-boot, $\mathcal{P}$-boot and $\mathcal{BC}_p$-boot) using different classical methods of estimation as well as widths of HPD credible intervals are reported in Table 14 and it is observed that the width of the HPD interval is minimum among the widths of BCIs ($\mathcal{S}$-boot, $\mathcal{P}$-boot and $\mathcal{BC}_p$-boot), which is also echoed our simulation results.

Table 13 *Estimates of the index C_{pk} using different methods of estimation*

Data Set	MLE	LSE	WLSE	CME	MPSE	ADE	RADE	BAYES
I	0.88617	0.85338	0.85112	0.85223	0.82894	0.89734	0.90067	0.88353
II	6.95910	6.55118	6.54723	6.54223	6.45613	6.95372	6.97335	6.64917

Table 14 *Widths of BCIs and HPD credible intervals of C_{pk} for the data sets*

		Width of BCIs		
Data Set	Width of HPD Credible intervals	$\mathcal{S}$-boot	$\mathcal{P}$-boot	$\mathcal{B}C_p$-boot
I	0.24074	0.27683	0.26779	0.24011
II	1.75805	1.83772	1.81684	1.77083

8 Conclusions

In this paper, we have considered three BCIs of the PCI C_{pk} based on seven different classical methods of estimation as well as by Bayesian method of estimation using squared error loss function for the normal distribution. We have considered the MLEs, LSEs, WLSEs, CMEs, MPSEs, ADEs and RADEs of the parameters μ and σ to obtain the estimates and BCIs for the PCI, C_{pk}. As it is not feasible to compare these methods theoretically, we have performed extensive simulation study to compare these methods with different sample sizes and different combinations of the unknown parameters. Next, we have considered Bayesian estimation of the unknown parameters and the index C_{pk}. Besides, point estimation, we have considered three BCIs ($\mathcal{S}$-boot, $\mathcal{P}$-boot, $\mathcal{B}C_p$-boot) and HPD credible intervals for the index C_{pk}. Simulation results suggest that Bayes estimators perform better than the considered classical methods of estimation. It is worth mentioning that the choice of hyper-parameters of the prior distributions need to be carefully chosen. Among classical methods of estimation, MPSE gives the best results in terms of MSE for almost all the choices of sample sizes among the other classical methods of estimation. In real data analysis for both the data sets, the Bayesian method of estimation gives smallest widths from the other methods of estimation.

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