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CORRECTION NOTE: "STATISTICAL INFERENCE FOR THE MEAN OUTCOME UNDER A POSSIBLY NONUNIQUE OPTIMAL TREATMENT RULE"

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The end of the proof of Lemma 6 in [2] contains an error. Specifically, it is not generally true that $n^{\beta-1}\sum_{j=1}^{n} R_j \overset{d}{=} n^{\beta-1}\sum_{j=1}^{n} R'_j(\omega')$. As is evidenced by the counterexample in [1], the claim of that lemma does not generally hold unless an additional condition is imposed. Below we state a correction to that lemma that adopts one such condition. Alternative sets of conditions that yield a similar conclusion can be found in [1].

LEMMA 1 (Revision of Lemma 6 in [2]). *Suppose that R_j is some sequence of real-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, \nu)$ and such that $j^{\beta}R_j \xrightarrow{a.s.} 0$ for some $\beta \in [0, 1)$. Then $n^{\beta-1}\sum_{j=1}^{n} R_j \xrightarrow{a.s.} 0$, and so $n^{-1}\sum_{j=1}^{n} R_j = o_p(n^{-\beta})$.*

PROOF. Let $\tilde{\Omega} \triangleq \{\omega \in \Omega : \lim_{j\to\infty} j^{\beta}R_j(\omega) = 0\}$. Fix $\epsilon > 0$ and let $\omega \in \tilde{\Omega}$. There exists some N that, for all $n \geq N$, $n^{\beta}|R_n(\omega)| < \frac{(1-\beta)\epsilon}{2}$. Also,

$$\frac{1}{n^{1-\beta}}\sum_{j=1}^{n} j^{-\beta} \leq \frac{1}{n^{1-\beta}}\int_{1}^{n}(j-1)^{-\beta}\,dj = \frac{1}{1-\beta}.$$

Hence, for all $n \geq N$,

$$\begin{aligned}\frac{1}{n^{1-\beta}}\sum_{j=1}^{n}|R_j(\omega)| &= \frac{1}{n^{1-\beta}}\sum_{j=1}^{N-1}|R_j(\omega)| + \frac{1}{n^{1-\beta}}\sum_{j=N}^{n}\frac{1}{j^{\beta}}j^{\beta}|R_j(\omega)| \\ &< \frac{1}{n^{1-\beta}}\sum_{j=1}^{N-1}|R_j(\omega)| + \frac{(1-\beta)\epsilon}{2n^{1-\beta}}\sum_{j=N}^{n}\frac{1}{j^{\beta}} \\ &\leq \frac{1}{n^{1-\beta}}\sum_{j=1}^{N-1}|R_j(\omega)| + \frac{\epsilon}{2}.\end{aligned}$$

It follows that $\frac{1}{n^{1-\beta}}\sum_{j=1}^{n}|R_j(\omega)| < \epsilon$ for all n large enough, and thus that $n^{\beta-1}\sum_{j=1}^{n} R_j(\omega)$ converges to zero as $n \to \infty$. As $\omega \in \tilde{\Omega}$ was arbitrary and $\nu(\tilde{\Omega}) = 1$, $n^{\beta-1}\sum_{j=1}^{n} R_j$ converges to zero almost surely and, therefore, also in probability. □

The following statements in [2] built on Lemma 6 and, therefore, must be modified to account for the above revision of this lemma:

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- page 726, lines 5–6 from the bottom, should read: As will become apparent after reading Section 7, the almost-sure strengthening of (C6), namely that $n^{1/2}\Gamma_n[\Psi_{d_n}(P_0) - \psi_1] \xrightarrow{a.s.} 0$, will typically imply (C7) (see Lemma 6).
- page 729, bottom: The condition in Theorem 7 that $\widetilde{\mathcal{D}}_1 \triangleq \{\tilde{D}(d, \bar{Q}, g) : d, \bar{Q}, g\}$ be a P_0 Glivenko–Cantelli class should be replaced by the stronger condition that $\widetilde{\mathcal{D}}_1$ be a strong P_0 Glivenko–Cantelli class, that is, that $\|P_n - P_0\|_{\widetilde{\mathcal{D}}_1} \to 0$ almost surely.
- page 730, lines 19–20, should read: More generally, Lemma 6 shows that condition (C4) holds if $\tilde{\sigma}_{0,j}^2/\tilde{\sigma}_j^2 \xrightarrow{a.s.} 1$, $\Pr_0(0 < g_j(1|W) < 1) = 1$ with probability 1 for all j, and

$$j^{1/2} E_0\left[\left(1 - \frac{g_0(d_j(W)|W)}{g_j(d_j(W)|W)}\right)\left(\bar{Q}_j(d_j(W), W) - \bar{Q}_0(d_j(W), W)\right)\right] \xrightarrow{a.s.} 0.$$

- page 731, lines 9–12 should read: The above theorem thus shows that $j^{1/2}[\Psi_{d_j}(P_0) - \Psi_{d_0^*}(P_0)] \xrightarrow{a.s.} 0$ if the distribution of $|\bar{Q}_{b,0}(W)|$ and our estimates of $\bar{Q}_{b,0}$ satisfy reasonable conditions. If, additionally, $\tilde{\sigma}_{0,j}$ is estimated well in the sense that $\tilde{\sigma}_{0,j}^2/\tilde{\sigma}_j^2 \xrightarrow{a.s.} 1$, then an application of Lemma 6 shows that (C5) is satisfied.

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