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ASYMPTOTICS OF THE EIGENVALUES OF THE ANDERSON HAMILTONIAN WITH WHITE NOISE POTENTIAL IN TWO DIMENSIONS

BY KHALIL CHOUK[1] AND WILLEM VAN ZUIJLEN[2]

[1]*School of Mathematics, University of Edinburgh, khalil.chouk@gmail.com*
[2]*Weierstrass Institute for Applied Analysis and Stochastics, Berlin, vanzuijlen@wias-berlin.de*

In this paper we consider the Anderson Hamiltonian with white noise potential on the box $[0, L]^2$ with Dirichlet boundary conditions. We show that all of the eigenvalues divided by $\log L$, converge as $L \to \infty$, almost surely to the same deterministic constant which is given by a variational formula.

1. Introduction. We consider the Anderson Hamiltonian (also called random Schrödinger operator), formally defined by $\mathscr{H} = \Delta + \xi$, under Dirichlet boundary conditions on the two-dimensional box $[0, L]^2$, where ξ is considered to be white noise. We are interested in the behaviour of this operator, as the size of the box, L, tends to infinity. In this paper we prove the following asymptotics of the eigenvalues. Let $\lambda(L) = \lambda_1(L) > \lambda_2(L) \geq \lambda_3(L) \cdots$ be the eigenvalues of the Anderson Hamiltonian on $[0, L]^2$. For all $n \in \mathbb{N}$, almost surely

$$\lim_{\substack{L\in\mathbb{Q} \\ L\to\infty}} \frac{\lambda_n(L)}{\log L} = 4 \sup_{\substack{\psi\in C_c^\infty(\mathbb{R}^2) \\ \|\psi\|_{L^2}^2=1}} \|\psi\|_{L^4}^2 - \int_{\mathbb{R}^2} |\nabla\psi|^2 = \chi,$$

where χ is the smallest $C > 0$ such that $\|f\|_{L^4}^4 \leq C\|\nabla f\|_{L^2}^2 \|f\|_{L^2}^2$ for all $f \in H^1(\mathbb{R}^2)$ (this is Ladyzhenskaya's inequality).

1.1. *Main challenge and literature.* In the one-dimensional setting, that is, on the box $[0, L]$, the Anderson Hamiltonian can be defined using the associated Dirichlet form, as the white noise is sufficiently regular; see Fukushima and Nakao [14] (see [35] for the regularity of white noise). In dimension two the regularity of white noise is too small to allow for the same approach. A naive way to tackle the problem of the construction is to take a smooth approximation of the white noise ξ_ε so that the operator $\mathscr{H}_\varepsilon = \Delta + \xi_\varepsilon$ is well defined as an unbounded selfadjoint operator, and then take the limit $\varepsilon \downarrow 0$. However, $\mathscr{H}_\varepsilon$ does not converge, but $\mathscr{H}_\varepsilon - c_\varepsilon$ does converge to an operator $\mathscr{H}$ for certain renormalisation constants $c_\varepsilon \nearrow_{\varepsilon\downarrow 0} \infty$. This has been shown by Allez and Chouk [1] for periodic boundary conditions, using the techniques of paracontrolled distributions introduced by Gubinelli, Imkeller and Perkowski [16] in order to study singular stochastic partial differential equations. In this paper we extend this to Dirichlet boundary conditions.

Recently, also Labbé [21] constructed the Anderson Hamiltonian with both periodic and Dirichlet boundary conditions, using the tools of regularity structures. Gubinelli, Ugurcan and Zachhuber [17] extend the work of Allez and Chouk to define the Anderson Hamiltonian with periodic boundary conditions also for dimension 3.

One of the main interests in the study of this operator is due to its universal property, more precisely, it was proved by Chouk, Gairing and Perkowski [8], Theorem 6.1, that, under

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periodic boundary conditions, the operator $\mathscr{H}$ is the limit under a suitable renormalisation of the discrete Anderson Hamiltonian $\mathscr{H}_N = \Delta_N + \frac{1}{N}\eta_N$, defined on the periodic lattice $(\frac{1}{N}\mathbb{Z}/N\mathbb{Z})^2$, where Δ_N is discrete Laplacian and $(\eta_N(i), i \in \mathbb{Z}^2)$ are centred I.I.D. random variables with normalised variance and finite pth moment, for some $p > 6$.

Recently, Dumaz and Labbé [13] proved the Anderson localization for the one-dimensional case for the largest eigenvalues, and they obtain the exact fluctuation of the eigenvalue and the exact behaviour of the eigenfunctions near their maxima. Unfortunately, their approach used to tackle the Anderson localization in the one-dimensional setting is strongly attached to the SDE, obtained by the so-called Riccati transform, and cannot be adapted to the two-dimensional setting. Also, Chen [7] not only the one-dimensional setting for the white noise (and shows $\lambda(L) \approx (\log L)^{\frac{2}{3}}$) but also a higher-dimensional setting for the more regular fractional white noise (where $\lambda(L) \approx (\log L)^{\beta}$ for some $\beta \in (\frac{1}{2}, 1)$ (and $\beta \in (\frac{1}{2}, \frac{2}{3})$ for $d = 1$), where β is a function of the degree of singularity of the covariance at zero). The techniques in his work do not allow for an extension to a higher-dimensional setting with a white noise potential.

The asymptotics of the principal eigenvalue is of particular interest for the asymptotics of the total mass of the solution to the parabolic Anderson model: $\partial_t u = \Delta u + \xi u = \mathscr{H} u$. Chen [7] shows that with $U(t)$, the total mass of $u(t, \cdot)$, one has $\log U(t) \approx t\lambda(L_t)$ for some almost linear L_t, so that the asymptotics of $\lambda(L)$ lead to asymptotics of $\log U(t)$: In $d = 1$ with ξ white noise, $\log U(t) \approx t(\log t)^{\frac{2}{3}}$; for $d \geq 1$ with ξ a fractional white noise $\log U(t) \approx t(\log t)^{\beta}$, with β as above. For smooth Gaussian fields ξ, Carmona and Molchanov [5] show $\log U(t) \sim t(\log t)^{\frac{1}{2}}$. In a future work by König, Perkowski and van Zuijlen, the following asymptotics of the total mass of the solution to the parabolic Anderson model with white noise potential in two dimensions will be shown: $\log U(t) \approx t \log t$.

For a general overview about the parabolic Anderson model and the Anderson Hamiltonian, we refer to the book by König [20].

Let us mention that our main result is already applied in [27] to prove that the super Brownian motion in static random environment is almost surely super-exponentially persistent.

1.2. *Outline*. In Section 2 we state the main results of this paper. In Section 3 we give a proof of the tail bounds of the eigenvalues, using the other ingredients presented in Section 2, and use this to prove the main theorem. The definitions of our Dirichlet and Neumann (Besov) spaces and para- and resonance products between those spaces are given in Section 4. With the definitions given we can properly define the Anderson Hamiltonian on its Dirichlet domain and state the spectral properties in Section 5. In Section 6 we prove the convergence to enhanced white noise that will be used to extend properties for smooth potentials to analogue properties where enhanced white noise is taken. In Section 7 we prove scaling and translation properties. In Section 8 we compare eigenvalues on boxes of different size. In Section 9 we prove the large deviation principle of the enhanced white noise. This leads to the large deviation principle for the eigenvalues. In Section 10 we study infima over the large deviation rate function which are used to express the limit of the eigenvalues. The more cumbersome calculations needed to prove convergence to enhanced white noise are postponed to Section 11 and Section 12.

1.3. *Notation*. $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N}_{-1} = \{-1\} \cup \mathbb{N}_0$. $\delta_{k,l}$ is the Kronecker delta, that is, $\delta_{k,k} = 1$ and $\delta_{k,l} = 0$ for $k \neq l$. $\mathrm{i} = \sqrt{-1}$. For $f, g \in L^2(D)$, for some domain $D \subset \mathbb{R}^d$ we write $\langle f, g\rangle_{L^2(D)} = \int_D f\overline{g}$. We write $\mathbb{T}_L^d$ for the d-dimensional torus of length $L > 0$, that is, $\mathbb{R}^d/L\mathbb{Z}^d$. $(\Omega, \mathbb{P})$ will be our underlying complete probability space. In order to avoid cumbersome administration of constants, for families $(a_i)_{i\in\mathbb{I}}$ and $(b_i)_{i\in\mathbb{I}}$ in $\mathbb{R}$, we also write $a_i \lesssim b_i$ to denote that there exists a $C > 0$ such that $a_i \le Cb_i$ for all $i \in \mathbb{I}$ and $a_i \approx b_i$ to denote that both $a_i \lesssim b_i$ and $a_i \gtrsim b_i$ (i.e., $b_i \lesssim a_i$). We write $C_c^\infty(A)$ for those functions in $C^\infty(A)$ that have compact support in A°.

2. Main results. In this section we give the main results of this paper without the technical details and definitions; the main theorem is Theorem 2.8.

We build on the methods on the construction of the Anderson Hamiltonian in [1]. In that paper the operator is considered on the torus or, differently said, on a box with periodic boundary conditions. In order to consider Dirichlet boundary conditions, we will consider the domain to be a subset of H_0^1. The construction in [1] relies on Bony estimates for para- and resonance products. We therefore have to find the right space in which we take ξ in order to be able to take para- and resonance products of ξ with elements in the domain. For this reason we construct the framework of Dirichlet, $B_{p,q}^{\mathfrak{d},\alpha}$, and Neumann Besov spaces, $B_{p,q}^{\mathfrak{n},\alpha}$ in Section 4. We will show that H_0^γ agrees with $B_{2,2}^{\mathfrak{d},\gamma}$ and show that the Bony estimates extend to products between elements of Dirichlet and Neumann spaces. Basically, the idea is as follows, for $d = 1$ and $L = 1$. Instead of the basis for the periodic Besov space L^2, given by $x \mapsto e^{2\pi i k x}$, we build the Dirichlet Besov space by the basis of L^2 given by $x \mapsto \sin(\pi k x)$ and the Neumann Besov space by $x \mapsto \cos(\pi k x)$. The elements of the Dirichlet/Neumann Besov space on $[0, L]$ then extend oddly/evenly to elements of the periodic Besov space on $\mathbb{T}_{2L}$. We show that the extension of a product is the same as the product of the respective extensions which allows us to obtain the Bony estimates from the periodic spaces. Moreover, this also allows us to extend the main theorem in [1] to Dirichlet boundary conditions on $Q_L = [0, L]^2$, as we present in the following theorem. We will consider ξ in $\mathcal{C}_\mathfrak{n}^\alpha$ and its enhancement in $\mathfrak{X}_\mathfrak{n}^\alpha$ which are the Neumann analogues of $\mathcal{C}^\alpha$ and $\mathfrak{X}^\alpha$.

THEOREM 2.1 (Summary of Theorem 5.4). *Let* $\alpha \in (-\frac{4}{3}, -1)$. *Let* $y \in \mathbb{R}^2$, $L > 0$ *and* $\Gamma = y + Q_L$. *For an enhanced Neumann distribution* $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}_\mathfrak{n}^\alpha(\Gamma)$, *we construct a stongly paracontrolled Dirichlet domain* $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}(\Gamma)$ *such that the Anderson Hamiltonian on* $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}(\Gamma)$ *maps in* $L^2(\Gamma)$ *and is selfadjoint as an operator on* $L^2(\Gamma)$ *with a countable spectrum given by eigenvalues* $\lambda(\Gamma, \boldsymbol{\xi}) = \lambda_1(\Gamma, \boldsymbol{\xi}) > \lambda_2(\Gamma, \boldsymbol{\xi}) \ge \cdots$ (*counting multiplicities*). *For all* $n \in \mathbb{N}$, *the map* $\boldsymbol{\xi} \mapsto \lambda_n(\Gamma, \boldsymbol{\xi})$ *is locally Lipschitz. Moreover, a Courant–Fischer formula is given for* λ_n (*see* (44)).

In Section 6 we show that there exists a canonical enhanced white noise in $\mathfrak{X}_\mathfrak{n}^\alpha$:

THEOREM 2.2 (See Theorem 6.4 and 6.5). *Let* $\alpha \in (-\frac{4}{3}, -1)$. *For all* $y \in \mathbb{R}^2$ *and* $L > 0$, *there exists a canonical* $\boldsymbol{\xi}_L^y = (\xi_L^y, \Xi_L^y) \in \mathfrak{X}_\mathfrak{n}^\alpha(y + Q_L)$ *such that* ξ_L^y *is a white noise* (*in the sense that is described in that theorem*).

We will write $\boldsymbol{\xi}_L = \boldsymbol{\xi}_L^0$, $\xi_L = \xi_L^0$, $\Xi_L = \Xi_L^0$ and, for $\beta > 0$,

$$\boldsymbol{\lambda}_n(y + Q_L, \beta) = \lambda_n\big(y + Q_L, (\beta \xi_L^y, \beta^2 \Xi_L^y)\big), \qquad \boldsymbol{\lambda}_n(y + Q_L) = \boldsymbol{\lambda}_n(y + Q_L, 1).$$

Now, we have the framework set and can get to the key ingredients, of which two are given in Section 7, the scaling and translation properties:

2.3.

(a) (Lemma 7.3) For $L, \beta, \varepsilon > 0$, $\boldsymbol{\lambda}_n(Q_L, \beta) \overset{d}{=} \frac{1}{\varepsilon^2} \boldsymbol{\lambda}_n(Q_{\frac{L}{\varepsilon}}, \varepsilon\beta) + \frac{1}{2\pi} \log \varepsilon$.

(b) (Lemma 7.4) For $y \in \mathbb{R}^2$ and $L, \beta > 0$, $\boldsymbol{\lambda}_n(Q_L, \beta) \overset{d}{=} \boldsymbol{\lambda}_n(y + Q_L, \beta)$. Moreover, if $y + Q_L^\circ \cap Q_L^\circ = \varnothing$, then $\boldsymbol{\lambda}_n(Q_L, \beta)$ and $\boldsymbol{\lambda}_n(y + Q_L, \beta)$ are independent.

In [15], Proposition 1, and [3], Lemma 4.6, the principal eigenvalue on a large box is bounded by maxima of principal eigenvalues on smaller boxes. We extend these results from smooth potentials to enhanced potentials.

THEOREM 2.4 (Consequence of Theorem 8.6[1]). *There exists a $K > 0$ such that, for all $\varepsilon > 0$ and $L > r \geq 1$, the following inequalities hold almost surely*:

$$\max_{k \in \mathbb{N}_0^2, |k|_\infty < \frac{L}{r} - 1} \boldsymbol{\lambda}(rk + Q_r, \varepsilon) \leq \boldsymbol{\lambda}(Q_L, \varepsilon) \leq \max_{k \in \mathbb{N}_0^2, |k|_\infty < \frac{L}{r} + 1} \boldsymbol{\lambda}(rk + Q_{\frac{3}{2}r}, \varepsilon) + \frac{4K}{r^2}.$$

Moreover, for $n \in \mathbb{N}$ and $L > r \geq 1$, if $x, y \in \mathbb{R}^2$ and $x + Q_r \subset y + Q_L$, then $\boldsymbol{\lambda}_n(x + Q_r, \varepsilon) \leq \boldsymbol{\lambda}_n(y + Q_L, \varepsilon)$; if $y, y_1, \dots, y_n \in \mathbb{R}^2$ are such that $(y_i + Q_r)_{i=1}^n$ are pairwise disjoint subsets of $y + Q_L$, then almost surely $\boldsymbol{\lambda}_n(y + Q_L, \varepsilon) \geq \min_{i \in \{1,\dots,n\}} \boldsymbol{\lambda}(y_i + Q_r, \varepsilon)$.

Another important tool that we prove is the large deviations of the eigenvalues, which—by the contraction principle and continuity of the eigenvalues in terms of its enhanced distribution—is a consequence of the large deviations of $(\sqrt{\varepsilon}\xi_L, \varepsilon \Xi_L)$, proven in Section 9.

THEOREM 2.5 (See Corollary 9.3). $\boldsymbol{\lambda}_n(Q_L, \sqrt{\varepsilon}) = \lambda_n(Q_L, (\sqrt{\varepsilon}\xi_L, \varepsilon \Xi_L))$ *satisfies the large deviation principle with rate ε and rate function $I_{L,n} : \mathbb{R} \to [0, \infty]$ given by*

$$I_{L,n}(x) = \inf_{\substack{V \in L^2(Q_L) \\ \lambda_n(Q_L, V) = x}} \frac{1}{2} \|V\|_{L^2}^2.$$

In Section 10 we study infima over the large deviation rate function over half-lines, in terms of which the almost sure limit of the eigenvalues will be described.

THEOREM 2.6. *There exists a $C > 0$ such that, for all $n \in \mathbb{N}$, $\varrho_n = \inf_{L>0} \inf I_{L,n}[1, \infty) = \lim_{L \to \infty} \inf I_{L,n}[1, \infty) > C$ and*

$$\frac{2}{\varrho_n} = 4 \sup_{\substack{V \in C_c^\infty(\mathbb{R}^2) \\ \|V\|_{L^2}^2 \leq 1}} \sup_{\substack{F \sqsubset C_c^\infty(\mathbb{R}^2) \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2}^2 = 1}} \int_{\mathbb{R}^2} -|\nabla \psi|^2 + V \psi^2. \tag{1}$$

Moreover,

$$\frac{2}{\varrho_1} = 4 \sup_{\substack{\psi \in C_c^\infty(\mathbb{R}^2) \\ \|\psi\|_{L^2}^2 = 1}} \|\psi\|_{L^4}^2 - \int_{\mathbb{R}^2} |\nabla \psi|^2 = \chi, \tag{2}$$

where χ is the smallest $C > 0$ such that $\|f\|_{L^4}^4 \leq C \|\nabla f\|_{L^2}^2 \|f\|_{L^2}^2$ for all $f \in H^1(\mathbb{R}^2)$ (this is Ladyzhenskaya's inequality).

Using the scaling and translation properties of 2.3, the comparison of the eigenvalue with maxima of eigenvalues of smaller boxes in Theorem 2.4 and the large deviations in Theorem 2.5, we obtain the following tail bounds in Section 3.

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[1] In this statement we have choosen $a = \frac{1}{2}r$.

THEOREM 2.7. *Let $K > 0$ be as in Theorem 8.6. Let $r, \beta > 0$. We will abbreviate $I_{r,1}$ by I_r. For all $\mu > \inf I_r(1, \infty)$ and $\kappa < \inf I_{\frac{3}{2}r}[1 - \frac{16K}{r^2})$, there exists an $M > 0$ such that, for all $L, x > 0$ with $L\sqrt{x} > M$,*

$$\mathbb{P}\big(\boldsymbol{\lambda}(Q_L, \beta) \le x\big) \le \exp\biggl(-\frac{e^{2\log L - \frac{\mu}{\beta^2}x}x}{2r^2}\biggr), \tag{3}$$

$$\mathbb{P}\big(\boldsymbol{\lambda}(Q_L, \beta) \ge x\big) \le \frac{2}{r^2}xe^{2\log L - \frac{\kappa}{\beta^2}x}. \tag{4}$$

Using the tail bounds and the limit in Theorem 2.6 we obtain our main result by a Borel–Cantelli argument and the "moreover" part of Theorem 2.4. For the details, see Section 3.

THEOREM 2.8. *Let $\mathbb{I} \subset (1, \infty)$ be an unbounded countable set, and let $\beta > 0$. For $L \in \mathbb{I}$ let $y_L \in \mathbb{R}^2$ be such that $y_r + Q_r \subset y_L + Q_L$ for $r, L \in \mathbb{I}$ with $L > r$. Then, for $n \in \mathbb{N}$,*

$$\lim_{\substack{L\in\mathbb{I}\\ L\to\infty}} \frac{\boldsymbol{\lambda}_n(y_L + Q_L, \beta)}{\log L} = \frac{2\beta^2}{\varrho_1} = \beta^2\chi \quad a.s.$$

3. Proofs of Theorem 2.7 and Theorem 2.8. In this section we prove Theorem 2.7 and Theorem 2.8 by using 2.1–2.6.

3.1. Let $K > 0$ be as in Theorem 2.4. To simplify notation, we take $\beta = 1$. By consecutively applying the scaling in 2.3(a), the bounds in Theorem 2.4 and then the independence and translation properties in 2.3(b), we get, for $L, r, \varepsilon > 0$ with $\frac{L}{\varepsilon} > r \ge 1$,

$$\begin{aligned}\mathbb{P}\big(\varepsilon^2\boldsymbol{\lambda}(Q_L) \le 1\big) &= \mathbb{P}\biggl(\boldsymbol{\lambda}(Q_{\frac{L}{\varepsilon}}, \varepsilon) + \frac{\varepsilon^2}{2\pi}\log\varepsilon \le 1\biggr)\\ &\le \mathbb{P}\biggl(\max_{k\in\mathbb{N}_0^2, |k|_\infty < \frac{L}{\varepsilon r}-1} \boldsymbol{\lambda}(rk + Q_r, \varepsilon) \le 1 - \frac{\varepsilon^2}{2\pi}\log\varepsilon\biggr)\\ &= \mathbb{P}\biggl(\boldsymbol{\lambda}(Q_r, \varepsilon) \le 1 - \frac{\varepsilon^2}{2\pi}\log\varepsilon\biggr)^{\#\{k\in\mathbb{N}_0^2 : |k|_\infty < \frac{L}{\varepsilon r}-1\}},\end{aligned} \tag{5}$$

and, similarly,

$$\begin{aligned}\mathbb{P}\big(\varepsilon^2\boldsymbol{\lambda}(Q_L) \ge 1\big) &= \mathbb{P}\biggl(\boldsymbol{\lambda}(Q_{\frac{L}{\varepsilon}}, \varepsilon) + \frac{\varepsilon^2}{2\pi}\log\varepsilon \ge 1\biggr)\\ &\le \mathbb{P}\biggl(\max_{k\in\mathbb{N}_0^2, |k|_\infty < \frac{L}{\varepsilon r}+1} \boldsymbol{\lambda}(rk + Q_{\frac{3}{2}r}, \varepsilon) + \frac{4K}{r^2} + \frac{\varepsilon^2}{2\pi}\log\varepsilon \ge 1\biggr)\\ &\le \#\biggl\{k \in \mathbb{N}_0^2 : |k|_\infty < \frac{L}{\varepsilon r} + 1\biggr\}\mathbb{P}\biggl(\boldsymbol{\lambda}(Q_{\frac{3}{2}r}, \varepsilon) \ge 1 - \frac{4K}{r^2} - \frac{\varepsilon^2}{2\pi}\log\varepsilon\biggr).\end{aligned} \tag{6}$$

As $\#\{k \in \mathbb{N}_0^2 : |k|_\infty \le n\} = (n+1)^2$ for $n \in \mathbb{N}$, we have

$$\lim_{M\to\infty} \frac{\#\{k \in \mathbb{N}_0^2 : |k|_\infty < M \pm 1\}}{M^2} = 1.$$

Observe that there exists an $M > 0$ such that, for all $L, r, \varepsilon > 0$ with $\frac{L}{\varepsilon r} > M$,

$$\frac{1}{2}\biggl(\frac{L}{\varepsilon r}\biggr)^2 \le \#\biggl\{k \in \mathbb{N}_0^2 : |k|_\infty < \frac{L}{\varepsilon r} \pm 1\biggr\} \le 2\biggl(\frac{L}{\varepsilon r}\biggr)^2.$$

By combining the above observations we have obtained the following.

LEMMA 3.2. *Let $K > 0$ be as in Theorem 8.6. Let $\beta > 0$. There exists an $M > 1$ such that, for all $L, r, \varepsilon > 0$ with $\frac{L}{\varepsilon} > Mr > r \geq 1$,*

$$\mathbb{P}(\varepsilon^2 \boldsymbol{\lambda}(Q_L, \beta) \leq 1) \leq \mathbb{P}\Big(\boldsymbol{\lambda}(Q_r, \varepsilon\beta) \leq 1 - \frac{\varepsilon^2}{2\pi}\log\varepsilon\Big)^{\frac{1}{2}(\frac{L}{\varepsilon r})^2}, \tag{7}$$

$$\mathbb{P}(\varepsilon^2 \boldsymbol{\lambda}(Q_L, \beta) \geq 1) \leq 2\Big(\frac{L}{\varepsilon r}\Big)^2 \mathbb{P}\Big(\boldsymbol{\lambda}(Q_{\frac{3}{2}r}, \varepsilon\beta) \geq 1 - \frac{4K}{r^2} - \frac{\varepsilon^2}{2\pi}\log\varepsilon\Big). \tag{8}$$

3.3. Let $r > 0$. Let us now use the large deviation principle in Corollary 9.3. First, observe that as $\lim_{\varepsilon\downarrow 0} \frac{\varepsilon^2}{2\pi}\log\varepsilon = 0$, also $\boldsymbol{\lambda}(Q_r, \varepsilon\beta) + \frac{\varepsilon^2}{2\pi}\log\varepsilon$ satisfies the large deviation principle with the rate function $\beta^{-2} I_{r,n}$ (by exponential equivalence; see [10], Theorem 4.2.13). Hence, for all $\mu > \inf I_{r,n}(1,\infty)$ and $\kappa < \inf I_{\frac{3}{2}r,n}[1 - \frac{4K}{r^2}, \infty)$, there exists a ε_0 such that, for $\varepsilon \in (0, \varepsilon_0)$, we have the following bound on the probability appearing in (7) (using that $1 - x \leq e^{-x}$ for $x \geq 0$):

$$\mathbb{P}\Big(\boldsymbol{\lambda}(Q_r, \varepsilon\beta) \leq 1 - \frac{\varepsilon^2}{2\pi}\log\varepsilon\Big) \leq 1 - e^{-\frac{\mu}{\varepsilon^2\beta^2}} \leq e^{-e^{-\frac{\mu}{\varepsilon^2\beta^2}}}, \tag{9}$$

$$\mathbb{P}\Big(\boldsymbol{\lambda}(Q_{\frac{3}{2}r}, \varepsilon\beta) \geq 1 - \frac{4K}{r^2} - \frac{\varepsilon^2}{2\pi}\log\varepsilon\Big) \leq e^{-\frac{\kappa}{\varepsilon^2\beta^2}}. \tag{10}$$

PROOF OF THEOREM 2.7. This now follows by Lemma 3.2 and the bounds (9) and (10). □

First, we prove the convergence of the eigenvalues along the set $\{2^m : m \in \mathbb{N}\}$ before proving Theorem 2.8. Observe that in Theorem 3.4, contrary to Theorem 2.8, we do not impose a condition on the sequence $(y_m)_{m\in\mathbb{N}}$.

THEOREM 3.4. *Let $n \in \mathbb{N}$ and $\beta > 0$. For any sequence $(y_m)_{m\in\mathbb{N}}$ in $\mathbb{R}^2$.*

$$\lim_{m\in\mathbb{N}, m\to\infty} \frac{\boldsymbol{\lambda}_n(y_m + Q_{2^m}, \beta)}{\log 2^m} = \frac{2\beta^2}{\varrho_1} = 4\beta^2 \sup_{\substack{V\in C_c^\infty(\mathbb{R}^2)\\ \|V\|_{L^2}^2\leq 1}} \sup_{\substack{\psi\in C_c^\infty(\mathbb{R}^2)\\ \|\psi\|_{L^2}^2 = 1}} \int_{\mathbb{R}^2} -|\nabla\psi|^2 + V\psi^2 \quad a.s.$$

PROOF. Without loss of generality, we may assume $y_m = 0$ for all $m \in \mathbb{N}$ and take $\beta = 1$:

• First, we prove the convergence of the principal eigenvalue, that is, we consider $n = 1$. Let $p, q \in \mathbb{R}$ be such that $p < \frac{2}{\varrho_1} < q$. We show that

$$\liminf_{m\to\infty} \frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} > p \quad \text{a.s.}, \qquad \limsup_{m\to\infty} \frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} < q \quad \text{a.s.}$$

By the lemma of Borel–Cantelli, it is sufficient to show that

$$\sum_{m=1}^{\infty} \mathbb{P}\Big[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} < p\Big] < \infty, \qquad \sum_{m=1}^{\infty} \mathbb{P}\Big[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} > q\Big] < \infty.$$

By Lemma 10.1,

$$\lim_{r\to\infty} \inf I_r(1,\infty) = \lim_{r\to\infty} \inf I_{\frac{3}{2}r}\Big[1 - \frac{16K}{r^2}, \infty\Big) = \varrho_1.$$

Let $r > 0$ be large enough such that

$$p \inf I_r(1,\infty) < 2 < q \inf I_{\frac{3}{2}r}\left[1 - \frac{16K}{r^2}, \infty\right).$$

Let $\mu > \inf I_r(1,\infty)$ be such that $p\mu < 2$ and $\kappa < \inf I_{\frac{3}{2}r}[1-\frac{16K}{r^2},\infty)$ be such that $q\kappa > 2$. By Theorem 2.7 for $M \in \mathbb{N}$ large enough,

$$\sum_{m=M}^{\infty} \mathbb{P}\left[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} < p\right] \le \sum_{m=M}^{\infty} 2^{-m\frac{p2^{(2-p\mu)m}}{2r^2}} < \infty,$$

which is finite because $\frac{p2^{(2-p\mu)m}}{8r^2} > 1$ for large m, as $2 - p\mu > 0$. Also

$$\sum_{m=M}^{\infty} \mathbb{P}\left[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} > q\right] \le \sum_{m=M}^{\infty} \frac{2m\log 2}{r^2} 2^{(2-\kappa q)m},$$

which is finite, as $2 - \kappa q < 0$ (and because $2^{-\alpha m} m \to 0$ for $\alpha > 0$).

- Let $n \in \mathbb{N}$. Let us first observe that, as $\boldsymbol{\lambda}_n(Q_{2^m}) \le \boldsymbol{\lambda}(Q_{2^m})$, we have $\limsup_{m\to\infty} \frac{\boldsymbol{\lambda}_n(Q_{2^m})}{\log 2^m} \le \frac{2}{\varrho_1}$. Let $x_1, \dots, x_n \in Q_{2^n}$ be such that $(x_i + Q_1)_{i=1}^n$ are disjoint. By Theorem 2.4 we obtain almost surely

$$\liminf_{m\to\infty} \frac{\boldsymbol{\lambda}_n(Q_{2^{n+m}})}{\log 2^{n+m}} \ge \min_{i\in\{1,\dots,n\}} \lim_{m\to\infty} \frac{\boldsymbol{\lambda}(2^m x_i + Q_{2^m})}{\log 2^n + \log 2^m} = \frac{2}{\varrho_1}. \qquad \square$$

PROOF OF THEOREM 2.8. The condition on y_L is assumed in order to have the monotonicity of $L \mapsto \boldsymbol{\lambda}_n(y_L)$ on $\mathbb{I}$. Therefore and for convenience, we assume $y_L = 0$ for all $L \in \mathbb{I}$. Also, we take $\beta = 1$. Write $s = \frac{2}{\varrho_1}$. Let $\varepsilon \in (0, s)$. By Theorem 3.4 there exists an M such that, for all $m \ge M$,

$$(\log 2^m)(s - \varepsilon) \le \boldsymbol{\lambda}_n(Q_{2^m}) \le (\log 2^m)(s+\varepsilon) \quad \text{a.s.}$$

Let $a \in [1, 2]$, then almost surely, as $L \mapsto \boldsymbol{\lambda}_n(Q_L)$ is an increasing function

$$(\log a2^{m-1})(s-\varepsilon) \le \boldsymbol{\lambda}_n(Q_{2^m}) \le \boldsymbol{\lambda}_n(Q_{a2^m}) \le \boldsymbol{\lambda}_n(Q_{2^{m+1}}) \le (\log a2^{m+1})(s+\varepsilon),$$

and

$$\begin{aligned}\left(1 - \frac{\log 2}{\log(2^m)}\right)(s-\varepsilon) &\le \left(1 - \frac{\log 2}{\log(a2^m)}\right)(s-\varepsilon)\\ &\le \frac{\boldsymbol{\lambda}_n(Q_{a2^m})}{\log(a2^m)} \le \left(1 + \frac{\log 2}{\log(2^m)}\right)(s+\varepsilon).\end{aligned}$$

From this it follows that almost surely $\lim_{L\in\mathbb{I}, L\to\infty} \frac{\boldsymbol{\lambda}_n(Q_L)}{\log(L)} = s$. $\square$

4. Dirichlet and Neumann Besov spaces, para- and resonance products. Let $d \in \mathbb{N}$. Let $L > 0$. We will first introduce Dirichlet and Neumann spaces on $Q_L = [0, L]^d$. In order to do this, we use three different bases of $L^2([0, L]^d)$, one standard (the e_k's), one as an underlying basis for Dirichlet spaces (the $\mathfrak{d}_k$'s) and one as an underlying basis for Neumann spaces (the $\mathfrak{n}_k$'s). After defining these spaces (in Definition 4.9), we prove a few results that compare Besov and Sobolev spaces. Later, in Definition 4.19 we show how to generalize this to spaces on general boxes of the form $\prod_{i=1}^d [a_i, b_i]$. Then, we present bounds on Fourier multipliers (Theorem 4.20) and define para- and resonance products (Definition 4.24) and state their Bony estimates (Theorem 4.26).

In the following we will introduce some notation. For $\mathfrak{q} \in \{-1,1\}^d$ and $x \in \mathbb{R}^d$, we use the following shorthand notation ($\mathfrak{q} \circ x$ is known as the Hadamard product):

$$\Big(\prod \mathfrak{q}\Big) = \prod_{i=1}^d \mathfrak{q}_i, \qquad \mathfrak{q} \circ x = (\mathfrak{q}_1 x_1, \dots, \mathfrak{q}_d x_d).$$

We call a function $f : [-L, L]^d \to \mathbb{C}$ *odd* if $f(x) = (\prod \mathfrak{q}) f(\mathfrak{q} \circ x)$ for all $\mathfrak{q} \in \{-1,1\}^d$, and, similarly, we call f *even* if $f(x) = f(\mathfrak{q} \circ x)$ for all $\mathfrak{q} \in \{-1,1\}^d$. For any $f : [0, L]^d \to \mathbb{C}$, we write $\tilde{f} : [-L, L]^d \to \mathbb{C}$ for its odd extension (the $\sim$ notation is taken as it looks like the graph of an odd function) and $\overline{f} : [-L, L]^d \to \mathbb{C}$ for its even extension (similarly, the notation—is taken, as it looks like the graph of an even function), that is, for the functions that satisfy

$$\tilde{f}(\mathfrak{q} \circ x) = \Big(\prod \mathfrak{q}\Big) f(x), \qquad \overline{f}(\mathfrak{q} \circ x) = f(x) \quad \text{for all } x \in [0, L]^d, \mathfrak{q} \in \{-1,1\}^d.$$

If a function $f : [-L, L]^d \to \mathbb{C}$ is *periodic*, which means that $f(y, L) = f(y, -L)$ and $f(L, y) = f(-L, y)$ for all $y \in [-L, L]$, then it can be extended periodically on $\mathbb{R}^d$ (with period $2L$); we will also consider it to be a function on the domain $\mathbb{T}^d_{2L}$. Note that if f is periodic and odd, then $f = 0$ on $\partial[0, L]^d$.

For $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, let $\nu_k = 2^{-\frac{1}{2}\#\{i : k_i = 0\}}$, and write $\mathfrak{d}_{k,L}$ and $\mathfrak{n}_{k,L}$ or, simply, $\mathfrak{d}_k$ and $\mathfrak{n}_k$ for the functions $[0, L]^d \to \mathbb{C}$ and $e_{k,2L}$ or, simply, e_k for the function $[-L, L]^d \to \mathbb{C}$ given by:

$$\mathfrak{d}_{k,L}(x) = \mathfrak{d}_k(x) = \Big(\frac{2}{L}\Big)^{\frac{d}{2}} \prod_{i=1}^d \sin\Big(\frac{\pi}{L} k_i x_i\Big), \tag{11}$$

$$\mathfrak{n}_{k,L}(x) = \mathfrak{n}_k(x) = \nu_k \Big(\frac{2}{L}\Big)^{\frac{d}{2}} \prod_{i=1}^d \cos\Big(\frac{\pi}{L} k_i x_i\Big), \tag{12}$$

$$e_{k,2L}(x) = e_k(x) = \Big(\frac{1}{2L}\Big)^{\frac{d}{2}} e^{\frac{\pi \mathrm{i}}{L}\langle k, x\rangle}. \tag{13}$$

Note that $\tilde{\mathfrak{d}}_k(x)$ equals the right-hand side of (11) and $\overline{\mathfrak{n}}_k(x)$ equals the right-hand side of (12) for $x \in [-L, L]^d$ so that $\tilde{\mathfrak{d}}_k$ and $\overline{\mathfrak{n}}_k$ are elements of $C^\infty(\mathbb{T}^d_{2L})$. We can also write $\tilde{\mathfrak{d}}_k$ and $\overline{\mathfrak{n}}_k$ as follows:

$$\tilde{\mathfrak{d}}_k(x) = \Big(\frac{2}{L}\Big)^{\frac{d}{2}} \prod_{i=1}^d \frac{e^{\frac{\pi \mathrm{i}}{L} k_i x_i} - e^{-\frac{\pi \mathrm{i}}{L} k_i x_i}}{2\mathrm{i}} = (-\mathrm{i})^d \sum_{\mathfrak{q} \in \{-1,1\}^d} \Big(\prod \mathfrak{q}\Big) e_{\mathfrak{q} \circ k}(x), \tag{14}$$

$$\overline{\mathfrak{n}}_k(x) = \nu_k \Big(\frac{2}{L}\Big)^{\frac{d}{2}} \prod_{i=1}^d \frac{e^{\frac{\pi \mathrm{i}}{L} k_i x_i} + e^{-\frac{\pi \mathrm{i}}{L} k_i x_i}}{2} = \nu_k \sum_{\mathfrak{q} \in \{-1,1\}^d} e_{\mathfrak{q} \circ k}(x). \tag{15}$$

For an integrable function $f : \mathbb{T}^d_{2L} \to \mathbb{C}$, its kth Fourier coefficient is defined by

$$\mathcal{F} f(k) = \langle f, e_k\rangle = \frac{1}{(2L)^{\frac{d}{2}}} \int_{\mathbb{T}^d_{2L}} f(x) e^{-\frac{\pi \mathrm{i}}{L}\langle k, x\rangle} \, \mathrm{d}x \quad \big(k \in \mathbb{Z}^d\big).$$

4.1. It is not difficult to see that for $\varphi, \psi \in L^2([0, L]^d)$, the following equalities hold:

$$\mathcal{F}(\tilde{\varphi})(k) = \Big(\prod \mathfrak{q}\Big) \mathcal{F}(\tilde{\varphi})(\mathfrak{q} \circ k) \quad \text{for all } k \in \mathbb{Z}^d, \mathfrak{q} \in \{-1,1\}^d, \tag{16}$$

$$\mathcal{F}(\tilde{\varphi})(k) = 0 \quad \text{for all } k \in \mathbb{Z}^d \text{ with } k_i = 0 \text{ for some } i, \tag{17}$$

$$\mathcal{F}(\overline{\varphi})(k) = \mathcal{F}(\overline{\varphi})(\mathfrak{q} \circ k) \quad \text{for all } k \in \mathbb{Z}^d, \mathfrak{q} \in \{-1, 1\}^d, \tag{18}$$

$$\langle \tilde{\varphi}, \tilde{\psi} \rangle_{L^2[-L,L]^d} = 2^d \langle \varphi, \psi \rangle_{L^2[0,L]^d} = \langle \overline{\varphi}, \overline{\psi} \rangle_{L^2[-L,L]^d}, \tag{19}$$

$$\langle \varphi, \mathfrak{d}_k \rangle = \mathrm{i}^d \mathcal{F}(\tilde{\varphi})(k) \quad \text{for all } k \in \mathbb{N}^d, \tag{20}$$

$$\langle \varphi, \mathfrak{n}_k \rangle = \mathcal{F}(\overline{\varphi})(k) \quad \text{for all } k \in \mathbb{N}_0^d. \tag{21}$$

4.2. By partial integration one obtains that $\mathcal{F}(\partial^\alpha f)(k) = (\frac{\pi \mathrm{i}}{L} k)^\alpha \mathcal{F}(f)(k)$. So that $\mathcal{F}(\Delta f)(k) = -|\frac{\pi}{L} k|^2 \mathcal{F}(f)(k)$. Consequently, $\langle \Delta f, \mathfrak{d}_k \rangle = -|\frac{\pi}{L} k|^2 \langle f, \mathfrak{d}_k \rangle$ and $\langle \Delta f, \mathfrak{n}_k \rangle = -|\frac{\pi}{L} k|^2 \langle f, \mathfrak{n}_k \rangle$. This will be used later to define $(a - \Delta)^{-1}$ for $a \in \mathbb{R} \setminus \{0\}$.

Moreover, from this one obtains that the spectrum of $-\Delta$ is given by $\{\frac{\pi^2}{L^2} |k|^2 : k \in \mathbb{Z}^d\}$ and that every e_k is an eigenvector.

LEMMA 4.3. $\{\mathfrak{d}_k : k \in \mathbb{N}^d\}$ *and* $\{\mathfrak{n}_k : k \in \mathbb{N}_0^d\}$ *form orthonormal bases for* $L^2([0, L]^d)$.

PROOF. We leave it to the reader to check that those sets are orthonormal. Let $\varphi \in L^2([0, L]^d)$. By expressing $\tilde{\varphi}$ and $\overline{\varphi}$ in terms of the basis $\{e_k : k \in \mathbb{Z}^d\}$ and using 4.1, one obtains $\tilde{\varphi} = \sum_{k \in \mathbb{N}^d} \langle \varphi, \mathfrak{d}_k \rangle_{L^2[0,L]^2} \tilde{\mathfrak{d}}_k$ and $\overline{\varphi} = \sum_{k \in \mathbb{N}_0^d} \langle \varphi, \mathfrak{n}_k \rangle_{L^2[0,L]^2} \overline{\mathfrak{n}}_k$. □

DEFINITION 4.4. We define the set of test functions on $[0, L]^d$ that oddly and evenly extend to smooth functions on $\mathbb{T}_{2L}^d$ (here $\mathcal{S}(\mathbb{T}_{2L}^d) = C^\infty(\mathbb{T}_{2L}^d)$),

$$\mathcal{S}_0([0, L]^d) := \{\varphi \in C^\infty([0, L]^d) : \tilde{\varphi} \in \mathcal{S}(\mathbb{T}_{2L}^d)\},$$
$$\mathcal{S}_{\mathfrak{n}}([0, L]^d) := \{\varphi \in C^\infty([0, L]^d) : \overline{\varphi} \in \mathcal{S}(\mathbb{T}_{2L}^d)\}.$$

We equip $\mathcal{S}_0([0, L]^d)$, $\mathcal{S}_{\mathfrak{n}}([0, L]^d)$ and $\mathcal{S}(\mathbb{T}_{2L}^d)$ with the Schwarz-seminorms. Note that[2] $C_c^\infty([0, L]^d)$ is a subset of both $\mathcal{S}_0([0, L]^d)$ and $\mathcal{S}_{\mathfrak{n}}([0, L]^d)$.

In the following theorem we state how one can represent elements of $\mathcal{S}$, $\mathcal{S}_0$ and $\mathcal{S}_{\mathfrak{n}}$ and of $\mathcal{S}'$, $\mathcal{S}_0'$ and $\mathcal{S}_{\mathfrak{n}}'$ in terms of series in terms of e_k, $\mathfrak{d}_k$ and $\mathfrak{n}_k$.

THEOREM 4.5. *(a) Every* $\omega \in \mathcal{S}(\mathbb{T}_{2L}^d)$, $\varphi \in \mathcal{S}_0([0, L]^d)$ *and* $\psi \in \mathcal{S}_{\mathfrak{n}}([0, L]^d)$ *can be represented by*

$$\omega = \sum_{k \in \mathbb{Z}^d} a_k e_k, \qquad \varphi = \sum_{k \in \mathbb{N}^d} b_k \mathfrak{d}_k, \qquad \psi = \sum_{k \in \mathbb{N}_0^d} c_k \mathfrak{n}_k, \tag{22}$$

where $(a_k)_{k \in \mathbb{Z}^d}$, $(b_k)_{k \in \mathbb{N}^d}$ *and* $(c_k)_{k \in \mathbb{N}_0^d}$ *in* $\mathbb{C}$ *are such that*

$$\begin{aligned} \forall n \in \mathbb{N} : \quad & \sup_{k \in \mathbb{Z}^d} (1 + |k|)^n |a_k| < \infty, \qquad \sup_{k \in \mathbb{N}^d} (1 + |k|)^n |b_k| < \infty, \\ & \sup_{k \in \mathbb{N}_0^d} (1 + |k|)^n |c_k| < \infty, \end{aligned} \tag{23}$$

and $a_k = \langle \omega, e_k \rangle$, $b_k = \langle \varphi, \mathfrak{d}_k \rangle$ *and* $c_k = \langle \psi, \mathfrak{n}_k \rangle$.

Conversely, if $(a_k)_{k \in \mathbb{Z}^d}$, $(b_k)_{k \in \mathbb{N}^d}$ *and* $(c_k)_{k \in \mathbb{N}_0^d}$ *satisfy* (23), *then* $\sum_{k \in \mathbb{Z}^d} a_k e_k$, $\sum_{k \in \mathbb{N}^d} b_k \mathfrak{d}_k$ *and* $\sum_{k \in \mathbb{N}_0^d} c_k \mathfrak{n}_k$ *converge in* $\mathcal{S}(\mathbb{T}_{2L}^d)$, $\mathcal{S}_0([0, L]^d)$ *and* $\mathcal{S}_{\mathfrak{n}}([0, L]^d)$, *respectively.*

[2]For the notation, see Section 1.3.

(*b*) *Every* $w \in \mathcal{S}'(\mathbb{T}_{2L}^d)$, $u \in \mathcal{S}_0'([0,L]^d)$ *and* $v \in \mathcal{S}_{\mathfrak{n}}'([0,L]^d)$ *can be represented by*

$$w = \sum_{k\in\mathbb{Z}^d} a_k e_k, \qquad u = \sum_{k\in\mathbb{N}^d} b_k \mathfrak{d}_k, \qquad v = \sum_{k\in\mathbb{N}_0^d} c_k \mathfrak{n}_k, \tag{24}$$

where $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ *and* $(c_k)_{k\in\mathbb{N}_0^d}$ *in* $\mathbb{C}$ *are such that*

$$\begin{aligned} \exists n \in \mathbb{N}: \quad & \sup_{k\in\mathbb{Z}^d} \frac{|a_k|}{(1+|k|)^n} < \infty, \qquad \sup_{k\in\mathbb{N}^d} \frac{|b_k|}{(1+|k|)^n} < \infty, \\ & \sup_{k\in\mathbb{N}_0^d} \frac{|c_k|}{(1+|k|)^n} < \infty \end{aligned} \tag{25}$$

and $a_k = \langle w, e_k\rangle$, $b_k = \langle u, \mathfrak{d}_k\rangle$ *and* $c_k = \langle v, \mathfrak{n}_k\rangle$.

Conversely, if $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ *and* $(c_k)_{k\in\mathbb{N}_0^d}$ *satisfy* (25), *then* $\sum_{k\in\mathbb{Z}^d} a_k e_k$, $\sum_{k\in\mathbb{N}^d} b_k \mathfrak{d}_k$ *and* $\sum_{k\in\mathbb{N}_0^d} c_k \mathfrak{n}_k$ *converge in* $\mathcal{S}'(\mathbb{T}_{2L}^d)$, $\mathcal{S}_0'([0,L]^d)$ *and* $\mathcal{S}_{\mathfrak{n}}'([0,L]^d)$, *respectively.*

PROOF. Let $\omega \in \mathcal{S}(\mathbb{T}_{2L}^d)$. As one has the relation $\mathcal{F}(\Delta^n\omega)(k) = (-\frac{\pi^2}{L^2}|k|^2)^n \mathcal{F}(\omega)(k)$ for all $n \in \mathbb{N}_0$, we have (23) and $\sum_{k\in\mathbb{Z}^d:|k|\le N} \mathcal{F}(\omega)(k) e_k \xrightarrow{N\to\infty} \omega$ in $\mathcal{S}(\mathbb{T}_{2L}^d)$; see also [33], Corollary 2.2.4.

Let $\varphi \in \mathcal{S}_0([0,L]^d)$. Using the shown convergence above for $\omega = \tilde{\varphi}$, by (14), (16), (17) and (20)

$$\sum_{\substack{k\in\mathbb{Z}^d\\|k|\le N}} \mathcal{F}(\tilde{\varphi})(k) e_k = \sum_{\substack{k\in\mathbb{N}^d\\|k|\le N}} \sum_{\mathfrak{q}\in\{-1,1\}^d} \mathcal{F}(\tilde{\varphi})(\mathfrak{q}\circ k) e_{\mathfrak{q}\circ k} = \sum_{\substack{k\in\mathbb{N}^d\\|k|\le N}} \langle \varphi, \mathfrak{d}_k\rangle \tilde{\mathfrak{d}}_k.$$

Hence, $\sum_{k\in\mathbb{N}^d:|k|\le N} \langle \varphi, \mathfrak{d}_k\rangle \mathfrak{d}_k$ converges to φ in $\mathcal{S}_0([0,L]^d)$.

Let $\psi \in \mathcal{S}_{\mathfrak{n}}([0,L]^d)$. Using the shown convergence above for $\overline{\psi}$, by (15), (18) and (21),

$$\sum_{\substack{k\in\mathbb{Z}^d\\|k|\le N}} \mathcal{F}(\overline{\psi})(k) e_k = \sum_{\substack{k\in\mathbb{N}_0^d\\|k|\le N}} 2^{-\#\{i:k_i=0\}} \sum_{\mathfrak{q}\in\{-1,1\}^d} \mathcal{F}(\overline{\psi})(\mathfrak{q}\circ k) e_{\mathfrak{q}\circ k} = \sum_{\substack{k\in\mathbb{N}_0^d\\|k|\le N}} c_k \overline{\mathfrak{n}}_k.$$

Hence, $\sum_{k\in\mathbb{N}^d:|k|\le N} \langle \psi, \mathfrak{n}_k\rangle \mathfrak{n}_k$ converges to ψ in $\mathcal{S}_{\mathfrak{n}}([0,L]^d)$.

(b) follows from (a). □

For $\varphi \in \mathcal{S}_0([0,L]^d)$, note that $\tilde{\varphi} = \sum_{k\in\mathbb{N}^d} \langle \varphi, \mathfrak{d}_k\rangle \tilde{\mathfrak{d}}_k$. Moreover, note that $\omega \in \mathcal{S}(\mathbb{T}_{2L}^d)$ is odd if and only if $\langle \omega, e_{\mathfrak{q}\circ k}\rangle = (\prod \mathfrak{q})\langle \omega, e_k\rangle$ for all $k \in \mathbb{Z}^d$ and $\mathfrak{q} \in \{-1,1\}^d$. This motivates the following definition.

DEFINITION 4.6. For $u \in \mathcal{S}_0'([0,L]^d)$, we write $\tilde{u}$ for the distribution in $\mathcal{S}'(\mathbb{T}_{2L}^d)$ given by $\tilde{u} = \sum_{k\in\mathbb{N}^d} \langle u, \mathfrak{d}_k\rangle \tilde{\mathfrak{d}}_k$. For $v \in \mathcal{S}_{\mathfrak{n}}'([0,L]^d)$, we write $\overline{v}$ for the distribution in $\mathcal{S}'(\mathbb{T}_{2L}^d)$ given by $\overline{v} = \sum_{k\in\mathbb{N}_0^d} \langle u, \mathfrak{n}_k\rangle \overline{\mathfrak{n}}_k$. A $w \in \mathcal{S}'(\mathbb{T}_{2L}^d)$ is called *odd* if $\langle w, e_{\mathfrak{q}\circ k}\rangle = (\prod \mathfrak{q})\langle w, e_k\rangle$ for all $k \in \mathbb{Z}^d$ and $\mathfrak{q} \in \{-1,1\}^d$. If instead $\langle w, e_{\mathfrak{q}\circ k}\rangle = \langle w, e_k\rangle$ for all $k \in \mathbb{Z}^d$ and $\mathfrak{q} \in \{-1,1\}^d$, then w is called *even*.

Note that $\tilde{u}$ is odd and $\overline{v}$ is even.

By (19) and Theorem 4.5, for $u \in \mathcal{S}_0'([0,L]^d)$, $\varphi \in \mathcal{S}_0([0,L]^d)$ and $v \in \mathcal{S}_{\mathfrak{n}}'([0,L]^d)$, $\psi \in \mathcal{S}_{\mathfrak{n}}([0,L]^d)$,

$$\langle u, \varphi\rangle = 2^{-d}\langle \tilde{u}, \tilde{\varphi}\rangle, \qquad \langle v, \psi\rangle = 2^{-d}\langle \overline{v}, \overline{\psi}\rangle. \tag{26}$$

THEOREM 4.7.

(a) *We have*

$$\tilde{\mathcal{S}}_0(\mathbb{T}^d_{2L}) := \{\tilde{\varphi} : \varphi \in \mathcal{S}_0([0, L]^d)\} = \{\psi \in \mathcal{S}(\mathbb{T}^d_{2L}) : \psi \text{ is odd}\},$$
$$\overline{\mathcal{S}}_{\mathfrak{n}}(\mathbb{T}^d_{2L}) := \{\overline{\varphi} : \varphi \in \mathcal{S}_{\mathfrak{n}}([0, L]^d)\} = \{\psi \in \mathcal{S}(\mathbb{T}^d_{2L}) : \psi \text{ is even}\},$$

and $\tilde{\mathcal{S}}_0(\mathbb{T}^d_{2L})$ *and* $\overline{\mathcal{S}}_{\mathfrak{n}}(\mathbb{T}^d_{2L})$ *are closed in* $\mathcal{S}(\mathbb{T}_{2L})$.

(b) $\mathcal{S}(\mathbb{T}^d_{2L})$, $\mathcal{S}_0([0, L]^d)$ *and* $\mathcal{S}_{\mathfrak{n}}([0, L]^d)$ *are complete.*

(c) *We have*

$$\tilde{\mathcal{S}}'_0(\mathbb{T}^d_{2L}) := \{\tilde{u} : u \in \mathcal{S}'_0([0, L]^d)\} = \{w \in \mathcal{S}'(\mathbb{T}^d_{2L}) : w \text{ is odd}\},$$
$$\overline{\mathcal{S}}'_{\mathfrak{n}}(\mathbb{T}^d_{2L}) := \{\overline{v} : v \in \mathcal{S}'_{\mathfrak{n}}([0, L]^d)\} = \{w \in \mathcal{S}'(\mathbb{T}^d_{2L}) : w \text{ is even}\},$$

and $\tilde{\mathcal{S}}'_0(\mathbb{T}^d_{2L})$ *and* $\overline{\mathcal{S}}'_{\mathfrak{n}}(\mathbb{T}^d_{2L})$ *are closed in* $\mathcal{S}'(\mathbb{T}^d_{2L})$.

(d) $\mathcal{S}'(\mathbb{T}^d_{2L})$, $\mathcal{S}'_0([0, L]^d)$ *and* $\mathcal{S}'_{\mathfrak{n}}([0, L]^d)$ *are* (*weak**) *sequentially complete.*

PROOF. (a) follows as convergence in $\mathcal{S}$ implies pointwise convergence and, therefore, the limit of odd and even functions is again odd and even, respectively. (b) follows from (a), as $\mathcal{S}(\mathbb{T}^d_{2L})$ is complete (see [12], Page 134). (c) If a net $(w_\iota)_{\iota\in\mathbb{I}}$ in $\tilde{\mathcal{S}}'_0$ converges in $\mathcal{S}'$ to some w, then $\langle w_\iota, e_k\rangle \to \langle w, e_k\rangle$ for all k so that w is odd. (d) follows from (c), as $\mathcal{S}'(\mathbb{T}^d_{2L})$ is weak* sequentially complete (see [12], page 137). □

As we index the basis e_k, $\mathfrak{d}_k$ and $\mathfrak{n}_k$ by elements k in $\mathbb{Z}^d$ and not in $\frac{1}{L}\mathbb{Z}^d$, in the next definition of a Fourier multiplier we have an additional $\frac{1}{L}$ factor in the argument of the functions τ and σ.

DEFINITION 4.8. Let $\tau : \mathbb{R}^d \to \mathbb{R}$, $\sigma : [0, \infty)^d \to \mathbb{R}$, $w \in \mathcal{S}'(\mathbb{T}^d_{2L})$, $u \in \mathcal{S}'_0([0, L]^d)$ and $v \in \mathcal{S}'_{\mathfrak{n}}([0, L]^d)$. We define (at least formally) the so-called *Fourier multipliers* by

$$\tau(\mathrm{D})w = \sum_{k\in\mathbb{Z}^d} \tau\left(\frac{k}{L}\right)\langle w, e_k\rangle e_k,$$
$$\sigma(\mathrm{D})u = \sum_{k\in\mathbb{N}^d} \sigma\left(\frac{k}{L}\right)\langle u, \mathfrak{d}_k\rangle \mathfrak{d}_k, \tag{27}$$
$$\sigma(\mathrm{D})v = \sum_{k\in\mathbb{N}_0^d} \sigma\left(\frac{k}{L}\right)\langle v, \mathfrak{n}_k\rangle \mathfrak{n}_k.$$

Let $(\rho_j)_{j\in\mathbb{N}_{-1}}$ form a *dyadic partition of unity*, that is, ρ_{-1} and ρ_0 are C^∞ radial functions on $\mathbb{R}^d$, where ρ_{-1} is supported in a ball and ρ_0 is supported in an annulus, $\rho_j = \rho(2^{-j}\cdot)$ for $j \in \mathbb{N}_0$, and

$$\sum_{j\in\mathbb{N}_{-1}} \rho_j(y) = 1, \qquad \frac{1}{2} \le \sum_{j\in\mathbb{N}_{-1}} \rho_j(y)^2 \le 1 \quad (y \in \mathbb{R}^d), \tag{28}$$

$$|j - k| \ge 2 \implies \operatorname{supp}\rho_j \cap \operatorname{supp}\rho_k = \varnothing \quad (j, k \in \mathbb{N}_0). \tag{29}$$

Let $w \in \mathcal{S}'(\mathbb{T}^d_{2L})$, $u \in \mathcal{S}'_0([0, L]^d)$ and $v \in \mathcal{S}'_{\mathfrak{n}}([0, L]^d)$. We define the Littlewood–Paley blocks $\Delta_j w$, $\Delta_j u$ and $\Delta_j v$ for $j \in \mathbb{N}_{-1}$ by $\Delta_j w = \rho_j(\mathrm{D})w$, $\Delta_j u = \rho_j(\mathrm{D})u$, $\Delta_j v = \rho_j(\mathrm{D})v$, that is,

$$\Delta_j w = \sum_{k\in\mathbb{Z}^d} \langle w, e_k\rangle \rho_j\left(\frac{k}{L}\right) e_k,$$

$$\Delta_j u = \sum_{k\in\mathbb{N}^d} \langle u, \mathfrak{d}_k\rangle \rho_j\left(\frac{k}{L}\right) \mathfrak{d}_k,$$

$$\Delta_j v = \sum_{k\in\mathbb{N}_0^d} \langle v, \mathfrak{n}_k\rangle \rho_j\left(\frac{k}{L}\right) \mathfrak{n}_k.$$

Let $\overline{\sigma}:\mathbb{R}^d\to\mathbb{R}$ be the even extension of σ, that is, $\overline{\sigma}(\mathfrak{q}\circ x)=\sigma(x)$ for all $x\in[0,\infty)^d$ and $\mathfrak{q}\in\{-1,1\}^d$. As $\sigma(\mathrm{D})\mathfrak{d}_k=\sigma(\frac{k}{L})\mathfrak{d}_k$ and $\overline{\sigma}(\mathrm{D})\tilde{\mathfrak{d}}_k=\sigma(\frac{k}{L})\tilde{\mathfrak{d}}_k$, by Theorem 4.5 we obtain that, for all $u\in\mathcal{S}_0'([0,L]^d)$ and $v\in\mathcal{S}_\mathfrak{n}'([0,L]^d)$,

$$\widetilde{\sigma(\mathrm{D})u}=\overline{\sigma}(\mathrm{D})\tilde{u}, \qquad \overline{\sigma(\mathrm{D})v}=\overline{\sigma}(\mathrm{D})\overline{v}. \tag{30}$$

Moreover, with $a_{d,p}=2^{-\frac{d}{p}}$ for $p<\infty$ and $a_{d,\infty}=1$ we have, for all $p\in[1,\infty]$,

$$\|\sigma(\mathrm{D})u\|_{L^p([0,L]^d)} = a_{d,p}\|\widetilde{\sigma(\mathrm{D})u}\|_{L^p(\mathbb{T}_{2L}^d)} = a_{d,p}\|\overline{\sigma}(\mathrm{D})\tilde{u}\|_{L^p(\mathbb{T}_{2L}^d)},$$

$$\|\sigma(\mathrm{D})v\|_{L^p([0,L]^d)} = a_{d,p}\|\overline{\sigma(\mathrm{D})v}\|_{L^p(\mathbb{T}_{2L}^d)} = a_{d,p}\|\overline{\sigma}(\mathrm{D})\overline{v}\|_{L^p(\mathbb{T}_{2L}^d)}.$$

Therefore, by applying the above to $\sigma=\rho_j$, with $\|\cdot\|_{B^\alpha_{p,q}}$ the standard Besov norm,

$$a_{d,p}\|\tilde{u}\|_{B^\alpha_{p,q}} = \|(2^{i\alpha}\|\Delta_i u\|_{L^p})_{i\in\mathbb{N}_{-1}}\|_{\ell^q}, \qquad a_{d,p}\|\overline{v}\|_{B^\alpha_{p,q}} = \|(2^{i\alpha}\|\Delta_i v\|_{L^p})_{i\in\mathbb{N}_{-1}}\|_{\ell^q}.$$

This motivates the following definition.

DEFINITION 4.9. Let $\alpha\in\mathbb{R}$, $p,q\in[1,\infty]$. We define the *Dirichlet Besov space* $B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)$ to be the space of $u\in\mathcal{S}_0'([0,L]^d)$ for which $\|u\|_{B^{\mathfrak{d},\alpha}_{p,q}}:=a_{d,p}\|\tilde{u}\|_{B^\alpha_{p,q}}<\infty$. Similarly, we define the *Neumann Besov space* $B^{\mathfrak{n},\alpha}_{p,q}([0,L]^d)$ as the space of $v\in\mathcal{S}_\mathfrak{n}'([0,L]^d)$ for which $\|v\|_{B^{\mathfrak{n},\alpha}_{p,q}}:=a_{d,p}\|\overline{v}\|_{B^\alpha_{p,q}}<\infty$.

We will abbreviate $\mathcal{C}^\alpha_\mathfrak{n}=B^{\mathfrak{n},\alpha}_{\infty,\infty}$, $H^\alpha_\mathfrak{n}=B^{\mathfrak{n},\alpha}_{2,2}$. In Theorem 4.14 we show $H^\alpha_0=B^{\mathfrak{d},\alpha}_{2,2}$.

As $B^\alpha_{p,q}(\mathbb{T}^d_{2L})$ is a Banach space, $\|\cdot\|_{B^{\mathfrak{d},\alpha}_{p,q}}$ is a norm on $B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)$ under which it is a Banach space. Similarly, $\|\cdot\|_{B^{\mathfrak{n},\alpha}_{p,q}}$ is a norm on $B^{\mathfrak{n},\alpha}_{p,q}([0,L]^d)$ under which it is a Banach space.

THEOREM 4.10. *$C_c^\infty([0,L]^d)$ is dense in $B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)$ for all $\alpha\in\mathbb{R}$, $p,q\in[1,\infty)$.*

PROOF. The proof follows the same strategy as the proof of [2], Proposition 2.74. □

THEOREM 4.11. *For $\alpha>0$, $H^\alpha(\mathbb{R}^d)=B^\alpha_{2,2}(\mathbb{R}^d)=\Lambda^\alpha_{2,2}(\mathbb{R}^d)$ and their norms are equivalent* (*for the definitions*, *see* [34], *p*. 36).

PROOF. For $H^\alpha(\mathbb{R}^d)=F^\alpha_{2,2}(\mathbb{R}^d)$, see [34], p. 88; for $F^\alpha_{2,2}(\mathbb{R}^d)=B^\alpha_{2,2}(\mathbb{R}^d)$, see [34], p. 47, and for $B^\alpha_{2,2}(\mathbb{R}^d)=\Lambda^\alpha_{2,2}(\mathbb{R}^d)$, see [34], p. 90. □

LEMMA 4.12. *For $\alpha \in \mathbb{R}$, the spaces $B^{\alpha}_{2,2}(\mathbb{T}^d_{2L})$ and $H^{\alpha}(\mathbb{T}^d_{2L})$ (see [31], p. 168) are equal with equivalent norms. Here, $H^{\alpha}(\mathbb{T}^d_{2L})$ is the space of distributions in $\mathcal{S}'(\mathbb{T}^d_{2L})$ for which $\|u\|_{H^\alpha} < \infty$, where*

$$\|u\|_{H^\alpha} = \sqrt{\sum_{k \in \mathbb{N}_0^2} \left(1 + \left|\frac{k}{L}\right|^2\right)^{\alpha} \langle u, e_k\rangle^2}.$$

PROOF. Observe that by the properties of the dyadic partition, for all $\alpha \in \mathbb{R}$, there exist $c_\alpha, C_\alpha > 0$ such that

$$c_\alpha \left(1 + \left|\frac{k}{L}\right|^2\right)^{\alpha} \le \sum_{j \in \mathbb{N}_{-1}} 2^{2\alpha j} \rho_j \left(\frac{k}{L}\right)^2 \le C_\alpha \left(1 + \left|\frac{k}{L}\right|^2\right)^{\alpha}. \tag{31}$$

Therefore, the equivalence of the norms follows by Plancherel's formula. □

The following is a consequence of the fact that the norms of $H^{\alpha}(\mathbb{T}^d_{2L})$ (see [31], p. 168) and $B^{\alpha}_{2,2}(\mathbb{T}^d_{2L})$ are equivalent.

THEOREM 4.13. *For all $\alpha \in \mathbb{R}$, we have, for $u \in \mathcal{S}'_{\mathfrak{n}}([0, L]^d)$ and $v \in \mathcal{S}'_0([0, L]^d)$,*

$$\|u\|_{B^{\mathfrak{n},\alpha}_{2,2}} \eqsim \sqrt{\sum_{k \in \mathbb{N}_0^2} \left(1 + \left|\frac{k}{L}\right|^2\right)^{\alpha} \langle u, \mathfrak{n}_k\rangle^2}, \qquad \|v\|_{B^{\mathfrak{d},\alpha}_{2,2}} \eqsim \sqrt{\sum_{k \in \mathbb{N}_0^2} \left(1 + \left|\frac{k}{L}\right|^2\right)^{\alpha} \langle v, \mathfrak{d}_k\rangle^2}.$$

THEOREM 4.14. *For $\alpha > 0$, the spaces $B^{\mathfrak{d},\alpha}_{2,2}([0, L]^d)$ and $H^{\alpha}_0([0, L]^d)$ are equal with equivalent norms, where $H^{\alpha}_0([0, L]^d)$ is the closure of $C^\infty_c([0, L]^d)$ in $H^{\alpha}(\mathbb{R}^d)$.*

PROOF. As $C^\infty_c([0, L]^d)$ is dense in $B^{\mathfrak{d},\alpha}_{2,2}([0, L]^d)$ (Theorem 4.10), it is sufficient to prove the equivalence of the norms on $C^\infty_c([0, L]^d)$. Let $f \in C^\infty_c([0, L]^d)$. By definition of the $\Lambda^{\alpha}_{2,2}$ norm, $\|f\|_{\Lambda^{\alpha}_{2,2}(\mathbb{T}^d_L)} = \|f\|_{\Lambda^{\alpha}_{2,2}(\mathbb{R}^d)}$. As $D^\beta \tilde{f} = \widetilde{D^\beta f}$, we have $\|\tilde{f}\|_{\Lambda^{\alpha}_{2,2}(\mathbb{T}^d_{2L})} = 2^{\frac{d}{2}} \|f\|_{\Lambda^{\alpha}_{2,2}(\mathbb{T}^d_L)}$. Because $\|\tilde{f}\|_{B^{\alpha}_{2,2}(\mathbb{T}^d_{2L})} = 2^{\frac{d}{2}} \|f\|_{B^{\mathfrak{d},\alpha}_{2,2}([0,L]^d)}$ (by definition), the proof follows by Theorem 4.11. □

THEOREM 4.15. *Let $p, q \in [1, \infty]$ and $\beta, \gamma \in \mathbb{R}$, $\gamma < \beta$. Then, $B^{\beta}_{p,q}(\mathbb{T}^d_{2L})$ is compactly embedded in $B^{\gamma}_{p,q}(\mathbb{T}^d_{2L})$, that is, every bounded set in $B^{\beta}_{p,q}(\mathbb{T}^d_{2L})$ is compact in $B^{\gamma}_{p,q}(\mathbb{T}^d_{2L})$. The analogous statement holds for $B^{\mathfrak{d},\beta}_{p,q}([0, L]^d)$ and $B^{\mathfrak{n},\beta}_{p,q}([0, L]^d)$ spaces. In particular, the injection $j : H^{\beta}_0([0, L]^d) \to H^{\gamma}_0([0, L]^d)$ is a compact operator.*

PROOF. We consider the underlying space to be $\mathbb{T}^d_{2L}$, that is, periodic boundary conditions; the other cases follow by Theorem 4.7. Suppose that $u_n \in B^{\beta}_{p,q}$ and $\|u_n\|_{B^{\beta}_{p,q}} \le 1$ for all $n \in \mathbb{N}$. We prove that there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ that converges in $B^{\gamma}_{p,q}$. By [2], Theorem 2.72, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, which we assume to be the sequence itself, such that $u_n \to u$ in $\mathcal{S}'$ and $\|u\|_{B^{\beta}_{p,q}} \le 1$. As $\langle u_n, e_k\rangle \to \langle u, e_k\rangle$ for all $k \in \mathbb{Z}^d$, we have $\|\Delta_j(u_n - u)\|_{L^p} \to 0$ for all $j \in \mathbb{N}_{-1}$. Let $\varepsilon > 0$. Choose $J \in \mathbb{N}$ large enough such that $2^{(\gamma-\beta)J} < \varepsilon$ so that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \left(2^{\gamma j} \|\Delta_j(u_n - u)\|_{L^p}\right)_{j=J+1}^{\infty} \right\|_{\ell^q} &\le 2^{(\gamma-\beta)J} \left\| \left(2^{\beta j} \|\Delta_j(u_n - u)\|_{L^p}\right)_{j=J+1}^{\infty} \right\|_{\ell^q} \\ &\le 2^{(\gamma-\beta)J} \left(\|u_n\|_{B^{\beta}_{p,q}} + \|u\|_{B^{\beta}_{p,q}}\right) < 2\varepsilon. \end{aligned}$$

Then, by choosing $N \in \mathbb{N}$ large enough such that $\|(2^{\gamma j}\|\Delta_j(u_n - u)\|_{L^p})_{j=-1}^J\|_{\ell^q} < \varepsilon$ for all $n \geq N$, one has with the above bound that $\|u_n - u\|_{B^\gamma_{p,q}} < 3\varepsilon$ for all $n \geq N$. □

4.16. Observe that by Lemma 4.3 $H^0_0([0,L]^d) = H^0_{\mathfrak{n}}([0,L]^d) = L^2([0,L]^d)$ and $\|\cdot\|_{H^0_0} \eqsim \|\cdot\|_{H^0_{\mathfrak{n}}} \eqsim \|\cdot\|_{L^2}$.

4.17. By 4.2 we have $(a-\Delta)^{-1} f = \sigma(\mathrm{D}) f$ for $\sigma(x) = (a + \pi^2|x|^2)^{-1}$.

4.18. For any function φ and $\lambda \in \mathbb{R}$, we write $l_\lambda \varphi$ for the function $x \mapsto \varphi(\lambda x)$. For a distribution u we write $l_\lambda u$ for the distribution given by $\langle l_\lambda u, \varphi\rangle = \lambda^{-d}\langle u, l_{\frac{1}{\lambda}}\varphi\rangle$. As $l_\lambda e_{k,2L} = \lambda^{-\frac{d}{2}} e_{k,\frac{2L}{\lambda}}$ and $\langle l_\lambda u, e_{k,\frac{2L}{\lambda}}\rangle = \lambda^{-\frac{d}{2}}\langle u, e_{k,2L}\rangle$, we have, for $u \in \mathcal{S}'(\mathbb{T}^d_{2L})$,

$$l_\lambda\big[\sigma(\lambda \mathrm{D})u\big] = \sigma(\mathrm{D})[l_\lambda u]. \tag{32}$$

Similarly, (32) holds for $u \in \mathcal{S}'_0([0,L]^d)$ and $u \in \mathcal{S}'_{\mathfrak{n}}([0,L]^d)$ (use, e.g., 4.1).

DEFINITION 4.19. Let $y \in \mathbb{R}^d$, $s \in (0,\infty)^d$ and $\Gamma = y + \prod_{i=1}^d [0, s_i]$. Let $l : \prod_{i=1}^d [0,s_i] \to [0,1]^d$ be given by $l(x) = (\frac{x_1}{s_1}, \dots, \frac{x_d}{s_d})$. For a function φ we define new functions $l\varphi$ and $\mathcal{T}_y\varphi$ by $l\varphi(x) = \varphi \circ l(x)$ and $\mathcal{T}_y\varphi(x) = \varphi(x-y)$, and for a distribution u we define the distributions lu and $\mathcal{T}_y u$ by $\langle lu, \varphi\rangle = |\det l|^{-1}\langle u, l^{-1}\varphi\rangle$ and $\langle \mathcal{T}_y u, \varphi\rangle = \langle u, \mathcal{T}_y^{-1}\varphi\rangle$. We define

$$\begin{aligned} &\mathcal{S}_0(\Gamma) := \mathcal{T}_y l\big[\mathcal{S}_0([0,1]^d)\big], \qquad \mathcal{S}'_0(\Gamma) := \mathcal{T}_y l\big(\mathcal{S}'_0([0,1]^d)\big), \\ &\sigma(\mathrm{D})u := \mathcal{T}_y l\big[(l\sigma)(\mathrm{D})\big((\mathcal{T}_y l)^{-1}u\big)\big] \quad \text{for } u \in \mathcal{S}'_0(\Gamma). \end{aligned} \tag{33}$$

Note that the definition of $\sigma(\mathrm{D})u$ is consistent with (27) by 4.18. Moreover, we define $\Delta_i = \rho_i(\mathrm{D})$ (as in (33)) and

$$\|u\|_{B^{\mathfrak{d},\alpha}_{p,q}(\Gamma)} := \big\|\big(2^{i\alpha}\|\Delta_i u\|_{L^p}\big)_{i\in\mathbb{N}_{-1}}\big\|_{\ell^q}.$$

Similarly, we define $\mathcal{S}_{\mathfrak{n}}(\Gamma)$, $\mathcal{S}'_{\mathfrak{n}}(\Gamma)$, $B^{\mathfrak{n},\alpha}_{p,q}(\Gamma)$ and $\|\cdot\|_{B^{\mathfrak{n},\alpha}_{p,q}(\Gamma)}$.

The following theorem gives a bound on Fourier multipliers, similar as in [2], Theorem 2.78. However, considering the particular choice $H^\gamma(\mathbb{T}^d_{2L}) = B^\gamma_{2,2}(\mathbb{T}^d_{2L})$ allows us to reduce condition to control all derivatives of σ to a condition that only controls the growth of σ itself.

THEOREM 4.20. *Let $\gamma, m \in \mathbb{R}$ and $M > 0$. There exists a $C > 0$ such that the following statements hold*:

(a) *For all bounded $\sigma : \mathbb{R}^d \to \mathbb{R}$ such that $|\sigma(x)| \leq M(1+|x|)^{-m}$ for all $x \in \mathbb{R}^d$*,

$$\big\|\sigma(\mathrm{D})w\big\|_{H^{\gamma+m}} \leq C\|w\|_{H^\gamma} \quad \big(w \in \mathcal{S}'(\mathbb{T}^d_{2L})\big). \tag{34}$$

By (30), *one may replace "H" and "$\mathcal{S}'(\mathbb{T}^d_{2L})$" by "$H_0$" and "$\mathcal{S}'_0([0,L]^d)$" or "$H_{\mathfrak{n}}$" and "$\mathcal{S}'_{\mathfrak{n}}([0,L]^d)$" in* (34).

(b) *For all $\sigma : \mathbb{R}^d \to \mathbb{R}$, which are C^∞ on $\mathbb{R}^d \setminus \{0\}$, such that $|\partial^\alpha\sigma(x)| \leq M|x|^{-m-|\alpha|}$ for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 2\lfloor 1 + \frac{d}{2}\rfloor$*,

$$\big\|\sigma(\mathrm{D})w\big\|_{\mathcal{C}^{\gamma+m}} \leq C\|w\|_{\mathcal{C}^\gamma} \quad \big(w \in \mathcal{S}'(\mathbb{T}^d_{2L})\big). \tag{35}$$

By (30), *one may replace "$\mathcal{C}$" and "$\mathcal{S}'(\mathbb{T}^d_{2L})$" by "$\mathcal{C}_{\mathfrak{n}}$" and "$\mathcal{C}'_{\mathfrak{n}}([0,L]^d)$" in* (35).

PROOF. Let $a > 0$ be such that $\rho(k) = 0$ if $|k| < a$. Then, for $j \geq 0$ one has $|\rho_j(k)\sigma(k)| \leq M(1 + \frac{a2^j}{L})^{-m}\rho_j(k) \leq ML^m a^{-m}2^{-jm}\rho_j(k)$ for all $k \in \mathbb{Z}^d$. As σ is bounded on the support of ρ_{-1}, there exists a $C > 0$ such that, for all $j \in \mathbb{N}_{-1}$,

$$\|\sigma(\mathrm{D})\Delta_j w\|_{L^2} = \sqrt{\sum_{k \in \mathbb{Z}^d} |w(e_k)|^2 \left|\sigma\left(\frac{k}{L}\right)\right|^2 |\rho_j(k)|^2} \leq C2^{-jm}\|\Delta_j w\|_{L^2}.$$

(35) follows from [2], Lemma 2.2. □

4.21. Using the multivariate chain rule (Faà di Bruno's formula), one can prove that $\sigma(x) = (1 + \pi^2|x|^2)^{-1}$ satisfies the conditions in Theorem 4.20 (those needed for (35)).

One other bound that we will refer to is a special case of [2], Proposition 2.71.

THEOREM 4.22. *For all $\alpha \in \mathbb{R}$, there exists a $C > 0$ such that $\|w\|_{C^\alpha_{\mathfrak{n}}} \leq C\|w\|_{H_{\mathfrak{n}}^{\alpha+\frac{d}{2}}}$ for all $w \in \mathcal{S}'_{\mathfrak{n}}([0, L]^d)$.*

Now, we consider (para- and resonance-) products between elements of $\mathcal{S}'_{\mathfrak{d}}([0, L]^d)$ and $\mathcal{S}'_{\mathfrak{n}}([0, L]^d)$ and between elements of $\mathcal{S}'_{\mathfrak{n}}([0, L]^d)$.

4.23. Let $w_1, w_2 \in \mathcal{S}'(\mathbb{T}^d_{2L})$ be represented by $w_1 = \sum_{k\in\mathbb{Z}^d} a_k e_k$ and $w_2 = \sum_{l\in\mathbb{Z}^d} b_l e_l$. Then, formally, $w_1 w_2 = \sum_{m\in\mathbb{Z}^d} c_m e_m$, with $c_m = \sum_{k,l\in\mathbb{Z}^d, k+l=m} a_k b_l$.

Of course this series is not always convergent (e.g., take $a_k = b_k = |k|^n$ for some $n \in \mathbb{N}$ and see (25)). But if it does, then due to the identities

$$(2L)^{\frac{d}{2}}\tilde{\mathfrak{d}}_k\overline{\mathfrak{n}}_l = \nu_l \sum_{\mathfrak{p}\in\{-1,1\}^d} \tilde{\mathfrak{d}}_{k+\mathfrak{p}\circ l}, \tag{36}$$

$$(2L)^{\frac{d}{2}}\tilde{\mathfrak{d}}_k\tilde{\mathfrak{d}}_l = (-1)^d \sum_{\mathfrak{p}\in\{-1,1\}^d} \nu_{k+\mathfrak{p}\circ l}^{-1}\Big(\prod \mathfrak{p}\Big)\overline{\mathfrak{n}}_{k+\mathfrak{p}\circ l}, \tag{37}$$

$$(2L)^{\frac{d}{2}}\overline{\mathfrak{n}}_k\overline{\mathfrak{n}}_l = \sum_{\mathfrak{p}\in\{-1,1\}^d} \frac{\nu_k\nu_l}{\nu_{k+\mathfrak{p}\circ l}}\overline{\mathfrak{n}}_{k+\mathfrak{p}\circ l}, \tag{38}$$

the product obeys the following rules:

even × even = even, odd × even = odd, odd × odd = even.

For example, if $u \in \mathcal{S}'_{\mathfrak{d}}$ and $v \in \mathcal{S}'_{\mathfrak{n}}$ and uv exists in a proper sense, then $uv \in \mathcal{S}'_{\mathfrak{d}}$.

DEFINITION 4.24. For $u \in \mathcal{S}'_{\mathfrak{d}}([0, L]^d) \cup \mathcal{S}'_{\mathfrak{n}}([0, L]^d)$ and $v \in \mathcal{S}'_{\mathfrak{n}}([0, L]^d)$, we write (at least formally)

$$u \otimes v = v \ominus u = \sum_{\substack{i,j\in\mathbb{N}_{-1}\\ i\leq j-1}} \Delta_i u \Delta_j v, \qquad u \odot v = \sum_{\substack{i,j\in\mathbb{N}_{-1}\\ |i-j|\leq 1}} \Delta_i u \Delta_j v. \tag{39}$$

4.25. As $\widetilde{\mathfrak{d}_k\mathfrak{n}_m} = \tilde{\mathfrak{d}}_k\overline{\mathfrak{n}}_m$ and $\overline{\mathfrak{n}_k\mathfrak{n}_m} = \overline{\mathfrak{n}}_k\overline{\mathfrak{n}}_m$, we have (at least formally)

$$\widetilde{u \otimes v} = \tilde{u} \otimes \overline{v}, \qquad \widetilde{u \ominus v} = \tilde{u} \ominus \overline{v}, \qquad \widetilde{u \odot v} = \tilde{u} \odot \overline{v}, \tag{40}$$

$$\overline{u \otimes v} = \overline{u} \otimes \overline{v}, \qquad \overline{u \ominus v} = \overline{u} \ominus \overline{v}, \qquad \overline{u \odot v} = \overline{u} \odot \overline{v}. \tag{41}$$

With this one can extend the Bony estimates on the (para-/resonance) products on the torus to Bony estimates between elements of $B^{\mathfrak{d},\alpha}_{p,q}([0, L]^d)$ and $B^{\mathfrak{n},\beta}_{p,q}([0, L]^d)$ and between elements of $B^{\mathfrak{n},\beta}_{p,q}([0, L]^d)$. We list some Bony estimates in Theorem 4.26.

THEOREM 4.26 (Bony estimates).

(a) *For all* $\alpha < 0$, $\gamma \in \mathbb{R}$, *there exists a* $C > 0$ *such that*, *for all* $L > 0$,

$$\|f \circledcirc \xi\|_{H_0^{\alpha+\gamma}} \le C\|f\|_{H_0^\gamma}\|\xi\|_{\mathcal{C}_\mathfrak{n}^\alpha} \quad (f \in \mathcal{S}_0'([0,L]^d), \xi \in \mathcal{S}_\mathfrak{n}'([0,L]^d)).$$

(b) *For all* $\delta > 0$, $\gamma \ge -\delta$ *and* $\beta \in \mathbb{R}$, *there exists a* $C > 0$ *such that*, *for all* $L > 0$,

$$\|f \circledcirc \xi\|_{H_0^{\beta-\delta}} \le C\|f\|_{H_0^\gamma}\|\xi\|_{\mathcal{C}_\mathfrak{n}^\beta} \quad (f \in \mathcal{S}_0'([0,L]^d), \xi \in \mathcal{S}_\mathfrak{n}'([0,L]^d)).$$

(c) *For all* $\alpha, \gamma \in \mathbb{R}$ *with* $\alpha + \gamma > 0$, *there exists a* $C > 0$ *such that*, *for all* $L > 0$,

$$\|f \odot \xi\|_{H_0^{\alpha+\gamma}} \le C\|f\|_{H_0^\gamma}\|\xi\|_{\mathcal{C}_\mathfrak{n}^\alpha} \quad (f \in \mathcal{S}_0'([0,L]^d), \xi \in \mathcal{S}_\mathfrak{n}'([0,L]^d)),$$

$$\|f \odot \xi\|_{\mathcal{C}_\mathfrak{n}^{\alpha+\gamma}} \le C\|f\|_{\mathcal{C}_\mathfrak{n}^\gamma}\|\xi\|_{\mathcal{C}_\mathfrak{n}^\alpha} \quad (f, \xi \in \mathcal{S}_\mathfrak{n}'([0,L]^d)).$$

(d) *For all* $\alpha, \gamma \in \mathbb{R}$ *with* $\alpha + \gamma > 0$ *and* $\delta > 0$, *there exists a* $C > 0$ *such that*, *for all* $L > 0$,

$$\|f\xi\|_{H_0^{\alpha\wedge\gamma-\delta}} \le C\|f\|_{H_0^\gamma}\|\xi\|_{\mathcal{C}_\mathfrak{n}^\alpha} \quad (f \in \mathcal{S}_0'([0,L]^d), \xi \in \mathcal{S}_\mathfrak{n}'([0,L]^d)).$$

The above statements also hold by simultaneously replacing "H_0" and "$\mathcal{S}_0'$" with "$H_\mathfrak{n}$" and "$\mathcal{S}_\mathfrak{n}'$."

PROOF. By 4.25 it is sufficient to consider the analogue statements with periodic boundary conditions, that is, considering the underlying space $\mathbb{T}_{2L}^d$. For (a) and (b), see [28], Lemma 2.1, and [2], Proposition 2.82, where the underlying space is $\mathbb{R}^d$ rather than the torus. For (c), see [2], Proposition 2.85. (d) follows from the rest. □

The last observation we make is that one can also define Besov spaces with mixed boundary conditions, to which we refer in Definition 5.2.

4.27 (Besov spaces with mixed boundary conditions). Beside the Dirichlet and Neumann Besov spaces one can define Besov spaces with mixed boundary conditions as follows. First, observe that, for $k \in \mathbb{N}_0^d$, the function $\mathfrak{d}_{k,L}$ is the product of the one-dimensional functions $\mathfrak{d}_{k_i,L}$, in the sense that $\mathfrak{d}_{k,L}(x) = \prod_{i=1}^d \mathfrak{d}_{k_i,L}(x_i)$. Similarly, $\mathfrak{n}_{k,L}(x) = \prod_{i=1}^d \mathfrak{n}_{k_i,L}(x_i)$. One could interpret this as taking Dirichlet (or Neumann) boundary conditions in every direction. Instead, one could, for example, for $d = 2$, take the function $f_{k,L}(x) = \mathfrak{d}_{k_1,L}(x_1)\mathfrak{n}_{k_2,L}(x_2)$ and, analogously to Definition 4.9, define a Besov space with mixed boundary conditions. Moreover, analogous to Definition 4.24 one can define the para- and resonance products as in (39) and obtain the Bony estimates as in Theorem 4.26 for elements with "opposite boundary conditions."

5. The operator $\Delta + \xi$ with Dirichlet boundary conditions.

We define the Anderson Hamiltonian with Dirichlet boundary conditions and study its spectral properties that will be used in the rest of the paper. *In this section we assume* $d = 2$, $y \in \mathbb{R}^2$ *and* $s \in (0,\infty)^2$ *and write* $\Gamma = y + \prod_{i=1}^2 [0, s_i]$. *Moreover, we let* $\alpha \in (-\frac{4}{3}, -1)$ *and* $\xi \in \mathcal{C}_\mathfrak{n}^\alpha(\Gamma)$. *We abbreviate* $\mathcal{C}_\mathfrak{n}^\alpha(\Gamma)$ *by* $\mathcal{C}_\mathfrak{n}^\alpha$, $H_0^\gamma(\Gamma)$ *by* H_0^γ, *etc. We write* $\sigma : \mathbb{R}^2 \to (0,\infty)$ *for the function given by*

$$\sigma(x) = \frac{1}{1 + \pi^2|x|^2}.$$

Additional assumptions are given in 5.10. Remember, see 4.17 that $\sigma(\mathrm{D}) = (1-\Delta)^{-1}$.

DEFINITION 5.1. For $\beta \in \mathbb{R}$, we define the *space of enhanced Neumann distributions*, written $\mathfrak{X}_{\mathfrak{n}}^{\beta}$, to be the closure in $\mathcal{C}_{\mathfrak{n}}^{\beta} \times \mathcal{C}_{\mathfrak{n}}^{2\beta+2}$ of the set

$$\{(\zeta, \zeta \odot \sigma(\mathrm{D})\zeta - c) : \zeta \in \mathcal{S}_{\mathfrak{n}}, c \in \mathbb{R}\}.$$

We equip $\mathfrak{X}_{\mathfrak{n}}^{\beta}$ with the relative topology with respect to $\mathcal{C}_{\mathfrak{n}}^{\beta} \times \mathcal{C}_{\mathfrak{n}}^{2\beta+2}$.

We will now define the Dirichlet domain of the Anderson Hamiltonian analogously to [1] did on the torus.

DEFINITION 5.2. Let $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}_{\mathfrak{n}}^{\alpha}$. For $\gamma \in (0, \alpha+2)$, we define $\mathcal{D}_{\xi}^{\mathfrak{d},\gamma} = \{f \in H_0^{\gamma} : f^{\sharp\xi} \in H_0^{2\gamma}\}$, where $f^{\sharp\xi} := f - f \otimes \sigma(\mathrm{D})\xi$. Moreover, we define an inner product on $\mathcal{D}_{\xi}^{\mathfrak{d},\gamma}$, written $\langle \cdot, \cdot \rangle_{\mathcal{D}_{\xi}^{\mathfrak{d},\gamma}}$, by $\langle f, g \rangle_{\mathcal{D}_{\xi}^{\mathfrak{d},\gamma}} = \langle f, g \rangle_{H_0^{\gamma}} + \langle f^{\sharp\xi}, g^{\sharp\xi} \rangle_{H_0^{2\gamma}}$.

For $\gamma \in (-\frac{\alpha}{2}, \alpha+2)$, we define the space of *strongly paracontrolled distributions* by $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma} = \{f \in H_0^{\gamma} : f^{\flat\boldsymbol{\xi}} \in H_0^2\}$, where $f^{\flat\boldsymbol{\xi}} := f^{\sharp\xi} - B(f, \boldsymbol{\xi})$ and $B(f, \boldsymbol{\xi}) = \sigma(\mathrm{D})(f\Xi + f \otimes \xi - ((\Delta - 1)f) \otimes \sigma(\mathrm{D})\xi - 2\sum_{i=1}^{d} \partial_{x_i} f \otimes \partial_{x_i}\sigma(\mathrm{D})\xi)$ (for the paraproducts under the sum, see 4.27). We define an inner product on $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}$, written $\langle \cdot, \cdot \rangle_{\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}}$, by $\langle f, g \rangle_{\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}} = \langle f, g \rangle_{H_0^{\gamma}} + \langle f^{\flat\boldsymbol{\xi}}, g^{\flat\boldsymbol{\xi}} \rangle_{H_0^2}$. As in the periodic setting, one has $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma} \subset H_0^{\alpha+2-}$ for all $\gamma \in (-\frac{\alpha}{2}, \alpha+2)$. We write $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} = \{f \in H_0^{\alpha+2-} : f^{\flat\boldsymbol{\xi}} \in H_0^2\}$.

We will define the Anderson Hamiltonian on the Dirichlet domain in a similar sense, as is done on the periodic domain; however, we choose to change the sign in front of the Laplacian, as this is more common in literature on the parabolic Anderson model.

DEFINITION 5.3. Let $\gamma \in (-\frac{\alpha}{2}, \alpha+2)$, $\boldsymbol{\xi} \in \mathfrak{X}_{\mathfrak{n}}^{\alpha}$. We define[3] the operator $\mathscr{H}_{\boldsymbol{\xi}} : \mathcal{D}_{\xi}^{\mathfrak{d},\gamma} \to H_0^{\gamma-2}$ by

$$\mathscr{H}_{\boldsymbol{\xi}} f = \Delta f + f \diamond \boldsymbol{\xi},$$

where $f \diamond \boldsymbol{\xi} = f \otimes \xi + f^{\sharp\xi} \odot \xi + \mathcal{R}(f, \sigma(\mathrm{D})\xi, \xi) + f\Xi + f \circledcirc \xi$ and $\mathcal{R}(f, g, h) := (f \otimes g) \odot h - f(g \odot h)$.

We state the main results about the spectrum of the Anderson Hamiltonian on its Dirichlet domain. These results are analogous to the Anderson Hamiltonian on the torus [1] (one can just read the theorem below without the Dirichlet and Neumann notations, i.e., the sub- or superscripts "0, $\mathfrak{d}$, $\mathfrak{n}$" and with the spaces interpreted to be defined on a torus). Moreover, they are similar to the results of [21] which proof is based on the theory of regularity structures.

THEOREM 5.4. *For $\gamma \in (-\frac{\alpha}{2}, \alpha+2)$, there exists a $C > 0$ such that*

$$\|\mathscr{H}_{\boldsymbol{\xi}} f\|_{H_0^{\gamma-2}} \le C \|f\|_{\mathcal{D}_{\xi}^{\mathfrak{d},\gamma}} (1 + \|\boldsymbol{\xi}\|_{\mathfrak{X}_{\mathfrak{n}}^{\alpha}})^2 \quad (f \in \mathcal{D}_{\xi}^{\mathfrak{d},\gamma}, \boldsymbol{\xi} \in \mathfrak{X}_{\mathfrak{n}}^{\alpha}). \tag{42}$$

$\mathscr{H}_{\boldsymbol{\xi}}(\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}) \subset L^2$ and $\mathscr{H}_{\boldsymbol{\xi}} : \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} \to L^2$ is closed and selfadjoint as an operator on L^2, and $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}$ is dense in L^2. There exist $\lambda_1(\Gamma, \boldsymbol{\xi}) > \lambda_2(\Gamma, \boldsymbol{\xi}) \ge \lambda_3(\Gamma, \boldsymbol{\xi}) \ge \cdots$ such that $\lim_{n\to\infty} \lambda_n(\Gamma, \boldsymbol{\xi}) =$

[3] The definition needs, of course, justification to show $H_0^{\gamma-2}$ is really the codomain; this is shown in Theorem 5.4.

$-\infty$, $\sigma(\mathscr{H}_{\boldsymbol{\xi}}) = \sigma_p(\mathscr{H}_{\boldsymbol{\xi}}) = \{\lambda_n(\Gamma, \boldsymbol{\xi}) : n \in \mathbb{N}\}$ *and* $\#\{n \in \mathbb{N} : \lambda_n(\Gamma, \boldsymbol{\xi}) = \lambda\} = \dim \ker(\lambda - \mathscr{H}_{\boldsymbol{\xi}}) < \infty$ *for all* $\lambda \in \sigma(\mathscr{H}_{\boldsymbol{\xi}})$. *One has*

$$\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} = \bigoplus_{\lambda \in \sigma(\mathscr{H}_{\boldsymbol{\xi}})} \ker(\lambda - \mathscr{H}_{\boldsymbol{\xi}}).$$

There exists an $M > 0$ *such that, for all* $n \in \mathbb{N}$ *and* $\boldsymbol{\xi}, \boldsymbol{\theta} \in \mathfrak{X}_{\mathfrak{n}}^{\alpha}$,

$$\left|\lambda_n(\Gamma, \boldsymbol{\xi}) - \lambda_n(\Gamma, \boldsymbol{\theta})\right| \le M \|\boldsymbol{\xi} - \boldsymbol{\theta}\|_{\mathfrak{X}_{\mathfrak{n}}^{\alpha}} \left(1 + \|\boldsymbol{\xi}\|_{\mathfrak{X}_{\mathfrak{n}}^{\alpha}} + \|\boldsymbol{\theta}\|_{\mathfrak{X}_{\mathfrak{n}}^{\alpha}}\right)^M. \tag{43}$$

With the notation $\sqsubset$ *for "is a linear subspace of,"*

$$\lambda_n(\Gamma, \boldsymbol{\xi}) = \sup_{\substack{F \sqsubset \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2} = 1}} \langle \mathscr{H}_{\boldsymbol{\xi}} \psi, \psi \rangle_{L^2} \tag{44}$$

In particular, $\lambda_1(\Gamma, \boldsymbol{\xi}) = \sup_{\psi \in \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} : \|\psi\|_{L^2} = 1} \langle \mathscr{H}_{\boldsymbol{\xi}} \psi, \psi \rangle_{L^2}$.

REMARK 5.5. Let us mention that, in an analogous way, one can state (and prove) the same statement for the operator with Neumann boundary conditions by replacing "$\mathfrak{d}$" by "$\mathfrak{n}$" and "H_0" by "$H_{\mathfrak{n}}$."

REMARK 5.6. In [1] it is pointed out that in (44) one may replace $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}$ by $\mathcal{D}_{\xi}^{\gamma}$ for $\gamma \in (\frac{2}{3}, \alpha + 2)$ and $\langle \mathscr{H}_{\boldsymbol{\xi}} \psi, \psi \rangle_{L^2}$ by ${}_{H_0^{-\gamma}}\langle \mathscr{H}_{\boldsymbol{\xi}} \psi, \psi \rangle_{H_0^{\gamma}}$, where ${}_{H_0^{-\gamma}}\langle \cdot, \cdot \rangle_{H_0^{\gamma}} : H_0^{-\gamma} \times H_0^{\gamma} \to \mathbb{R}$ is the continuous bilinear map (see [2], Theorem 2.76) given by

$${}_{H_0^{-\gamma}}\langle f, g \rangle_{H_0^{\gamma}} = \sum_{\substack{i,j \in \mathbb{N}_{-1} \\ |i-j| \le 1}} \langle \Delta_i f, \Delta_j g \rangle_{L^2}.$$

This is done for the periodic setting, but the arguments can easily be adapted to our setting.

5.7. Let $\eta \in L^2$ (which equals $H_{\mathfrak{n}}^0$, see 4.16). By Theorem 4.20 $\sigma(\mathrm{D})\eta \in H_{\mathfrak{n}}^2$ which is included in $\mathcal{C}_{\mathfrak{n}}^1$ by Theorem 4.22. Then, by Theorem 4.26, $\eta \odot \sigma(\mathrm{D})\eta \in H_{\mathfrak{n}}^1$. Moreover, if $\eta_\varepsilon \to \eta$ in L^2, then $\eta_\varepsilon \odot \sigma(\mathrm{D})\eta_\varepsilon \to \eta \odot \sigma(\mathrm{D})\eta$ in $H_{\mathfrak{n}}^1$ (by the same theorems). Hence, by Theorem 4.22 we obtain the following convergence in $\mathfrak{X}_{\mathfrak{n}}^{\alpha}$ for all $\alpha \le -1$:

$$\left(\eta_\varepsilon, \eta_\varepsilon \odot \sigma(\mathrm{D})\eta_\varepsilon\right) \to \left(\eta, \eta \odot \sigma(\mathrm{D})\eta\right).$$

We write $\lambda_n(\Gamma, \eta) = \lambda_n(\Gamma, (\eta, \eta \odot \sigma(\mathrm{D})\eta))$.

By 5.7 and the continuity of $\boldsymbol{\xi} \mapsto \lambda_n(\Gamma, \boldsymbol{\xi})$ (see (43) in Theorem 5.4), we obtain the following lemma.

LEMMA 5.8. *The map* $L^2(\Gamma) \to \mathbb{R}, \eta \mapsto \lambda_n(\Gamma, \eta)$ *is continuous.*

5.9. Let $\zeta \in \mathcal{S}_{\mathfrak{n}}^{\infty}$. Then, $\boldsymbol{\zeta} := (\zeta, \zeta \odot \sigma(\mathrm{D})\zeta) \in \mathfrak{X}_{\mathfrak{n}}^{\beta}$, $f \otimes \sigma(\mathrm{D})\zeta \in H_0^{\beta}$ for all $\beta \in \mathbb{R}$ and $B(f, \boldsymbol{\zeta}) \in H_0^2$ and $f \in H_0^{\gamma}$ with $\gamma \in (0,1)$ (use Theorems 4.20, 4.21 and 4.26). Therefore, for all $\gamma \in (0,1)$, $\mathcal{D}_{\zeta}^{\mathfrak{d},\gamma} = H_0^{2\gamma}$ and $\mathfrak{D}_{\boldsymbol{\zeta}}^{\mathfrak{d},\gamma} = H_0^2$ and for $f \in H_0^{\gamma}$, $f \odot \zeta = f^{\sharp\zeta} \odot \zeta + \mathcal{R}(f, \sigma(\mathrm{D})\zeta, \zeta) + f(\zeta \odot \sigma(\mathrm{D})\zeta)$ so that

$$\mathscr{H}_{\zeta} f := \Delta f + f\zeta = \mathscr{H}_{\boldsymbol{\zeta}} f. \tag{45}$$

Now, suppose $\zeta \in L^\infty \subset \mathcal{C}_{\mathfrak{n}}^{\infty}$. Then, $\boldsymbol{\zeta} := (\zeta, \zeta \odot \sigma(\mathrm{D})\zeta) \in \mathfrak{X}_{\mathfrak{n}}^0$, but the Bony estimates give $f \otimes \sigma(\mathrm{D})\zeta \in H_0^{2-}$ (and not $\in H_0^2$). Nevertheless, by the Kato–Rellich theorem [29], Theorem

X.12, on the domain H_0^2, the operator $\mathscr{H}_\zeta$, defined as in (45), is selfadjoint. As the injection map, $H_0^2 \to L^2$ is compact (see Theorem 4.15); every resolvent is compact. Hence, by the Riesz–Schauder theorem [29], Theorem VI.15, and the Hilbert–Schmidt theorem [29], Theorem VI.16, there exist $\lambda_1(\Gamma,\zeta) \geq \lambda_2(\Gamma,\zeta) \geq \cdots$ such that $\sigma(\mathscr{H}_\zeta) = \sigma_p(\mathscr{H}_\zeta) = \{\lambda_n(\Gamma,\zeta) : n \in \mathbb{N}\}$ and $\#\{n \in \mathbb{N} : \lambda_n(\Gamma,\zeta) = \lambda\} = \dim\ker(\lambda - \mathscr{H}_\zeta) < \infty$ for all $\lambda \in \sigma(\mathscr{H}_\zeta)$. Moreover, by Fischer's principle [23], Section 28, Theorem 4, p. 318,[4] and Lemma A.2,

$$
\begin{aligned}
\lambda_n(\Gamma,\zeta) &= \sup_{\substack{F \sqsubset H_0^2 \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2}=1}} \langle \mathscr{H}_\zeta \psi, \psi \rangle_{L^2} \\
&= \sup_{\substack{F \sqsubset C_c^\infty \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2}=1}} \int -|\nabla \psi|^2 + \zeta \psi^2.
\end{aligned} \tag{46}
$$

The proof of Theorem 5.4 follows from the results of the Anderson Hamiltonian on the torus with the help of Lemma 5.12. The proof is written below Lemma 5.12. We may restrict ourselves to the case $\Gamma = Q_L$.

5.10. *For the rest of this section $y = 0$ and $b_i = L$ for all i, that is, $\Gamma = Q_L = [0, L]^2$.*

5.11. For $\mathfrak{q} \in \{-1,1\}^d$ and $w \in \mathcal{S}'$, we write $l_\mathfrak{q} w$ for the element in $\mathcal{S}'$ given by $\langle l_\mathfrak{q} w, \varphi\rangle = \langle w, \varphi(\mathfrak{q} \circ \cdot)\rangle$ for $\varphi \in \mathcal{S}$. Then, w is odd if and only if $w = (\prod \mathfrak{q}) l_\mathfrak{q} w$ for all $\mathfrak{q} \in \{-1,1\}^d$, and w is even if and only if $w = l_\mathfrak{q} w$ for all $\mathfrak{q} \in \{-1,1\}^d$.

LEMMA 5.12. *Let $\boldsymbol{\xi} \in \mathfrak{X}_\mathfrak{n}^\alpha$. Let $\frac{2}{3} < \gamma < \alpha + 2$. Write $\overline{\boldsymbol{\xi}} = (\overline{\xi}, \overline{\Xi})$, $\mathcal{D}_{\overline{\xi}}^\gamma = \mathcal{D}_{\overline{\xi}}^\gamma(\mathbb{T}_{2L}^d)$, $\mathfrak{D}_{\overline{\boldsymbol{\xi}}}^\gamma = \mathfrak{D}_{\overline{\boldsymbol{\xi}}}^\gamma(\mathbb{T}_{2L}^d)$:*

(a) $\widetilde{\mathcal{D}_\xi^{\mathfrak{d},\gamma}} = \{w \in \mathcal{D}_{\overline{\xi}}^\gamma : w \text{ is odd}\}$, $\widetilde{\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}} = \{w \in \mathfrak{D}_{\overline{\boldsymbol{\xi}}}^\gamma : w \text{ is odd}\}$, $\widetilde{\mathscr{H}_{\boldsymbol{\xi}} f} = \mathscr{H}_{\overline{\boldsymbol{\xi}}} \tilde{f}$ *and* $\|f\|_{\mathcal{D}_\xi^{\mathfrak{d},\gamma}} \eqsim \|\tilde{f}\|_{\mathcal{D}_{\overline{\xi}}^\gamma}$ *uniformly for all* $f \in \mathcal{D}_\xi^{\mathfrak{d},\gamma}$ *and* $\|f\|_{\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}} \eqsim \|\tilde{f}\|_{\mathfrak{D}_{\overline{\boldsymbol{\xi}}}^\gamma}$ *uniformly for all* $f \in \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}$.

(b) $\mathscr{H}_{\boldsymbol{\xi}}(\mathcal{D}_\xi^{\mathfrak{d},\gamma}) \subset H_0^{\gamma-2}$, $\mathscr{H}_{\boldsymbol{\xi}}(\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}) \subset L^2$.

(c) $\mathscr{H}_{\overline{\boldsymbol{\xi}}}(l_\mathfrak{q} f) = l_\mathfrak{q} \mathscr{H}_{\overline{\boldsymbol{\xi}}} f$ *for all* $f \in \mathcal{D}_{\overline{\xi}}^\gamma$ *and* $\mathfrak{q} \in \{-1,1\}^2$.

(d) $\sigma(\mathscr{H}_{\boldsymbol{\xi}}) \subset \sigma(\mathscr{H}_{\overline{\boldsymbol{\xi}}})$ *(for the operators either on the $\mathcal{D}$ or $\mathfrak{D}$ domains) and for all $a \in \mathbb{C} \setminus \sigma(\mathscr{H}_{\overline{\boldsymbol{\xi}}})$ the inverse of $a - \mathscr{H}_{\boldsymbol{\xi}} : \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} \to L^2$ is selfadjoint and compact.*

(e) $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}$ *is dense in* $\mathcal{D}_\xi^{\mathfrak{d},\gamma}$, *and* $\mathcal{D}_\xi^{\mathfrak{d},\gamma}$ *is dense in* L^2.

PROOF. (a) follows from the identities (40), $\widetilde{f^{\sharp\xi}} = \tilde{f}^{\sharp\overline{\xi}}$, $\widetilde{B(f,\boldsymbol{\xi})} = B(\tilde{f}, \overline{\boldsymbol{\xi}})$, $\widetilde{f^{\flat\boldsymbol{\xi}}} = \tilde{f}^{\flat\overline{\boldsymbol{\xi}}}$ and because $\|\tilde{g}\|_{H^\gamma} \eqsim \|g\|_{H_0^\gamma}$ for all $\gamma \in \mathbb{R}$ and $g \in H_0^\gamma([0,L]^d)$ (indeed, $\|g\|_{B_{2,2}^{\mathfrak{d},\gamma}} = \|\tilde{g}\|_{B_{2,2}^\gamma}$ by definition and $\|\cdot\|_{H_0^\gamma} \eqsim \|\cdot\|_{B_{2,2}^{\mathfrak{d},\gamma}}$ and $\|\cdot\|_{B_{2,2}^\gamma} \eqsim \|\cdot\|_{H^\gamma}$ by Theorems 4.11 and 4.14).

(b) follows from (a) as $\mathscr{H}_{\overline{\boldsymbol{\xi}}}(\mathcal{D}_{\overline{\xi}}^\gamma) \subset H^{\gamma-2}$ and $\mathscr{H}_{\overline{\boldsymbol{\xi}}}(\mathfrak{D}_{\overline{\boldsymbol{\xi}}}) \subset H^0$ (see [1]).

(c) follows by a straightforward calculation; use that $\mathcal{F}(l_\mathfrak{q} f) = l_\mathfrak{q}\mathcal{F}(f)$, $l_\mathfrak{q}\rho_i = \rho_i$, $l_\mathfrak{q}\overline{\xi} = \overline{\xi}$ and $l_\mathfrak{q}\overline{\Xi} = \overline{\Xi}$ for $\mathfrak{q} \in \{-1,1\}^2$.

[4]In this reference the operator is actually assumed to be compact and symmetric, whereas we apply it to $\mathscr{H}_{\boldsymbol{\xi}}$. But the compactness is only assumed to guarantee that the spectrum is countable and ordered so that the arguments still hold.

(d) Let $a \in \mathbb{C}$ be such that $a - \mathscr{H}_{\tilde{\xi}}$ has a bounded inverse $\mathcal{R}_a$. By (c) $(a - \mathscr{H}_{\tilde{\xi}})f$ is odd if and only if f is odd; indeed, if $(a - \mathscr{H}_{\tilde{\xi}})f$ is odd, then $(a - \mathscr{H}_{\tilde{\xi}})[f - (\prod \mathfrak{q})l_{\mathfrak{q}} f] = 0$ (see 5.11), and, thus, $f = (\prod \mathfrak{q})l_{\mathfrak{q}} f$. Hence, $a - \mathscr{H}_{\boldsymbol{\xi}}$ has a bounded inverse $\mathcal{R}_a^{\mathfrak{d}}$ such that $\widetilde{\mathcal{R}_a^{\mathfrak{d}} h} = \mathcal{R}_a \tilde{h}$. From the fact that $\mathcal{R}_a$ is selfadjoint and compact, it follows that $\mathcal{R}_a^{\mathfrak{d}}$ is too.

$\mathcal{S}_0$ and thus L^2 is dense in $H_0^{\gamma-2}$ (see [2], Theorem 2.74, and Theorem 4.14), therefore, for $a \notin \sigma(\mathscr{H}_{\boldsymbol{\xi}})$ and $\mathcal{G}_a = (a - \mathscr{H}_{\boldsymbol{\xi}})^{-1}$, $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} = \mathcal{G}_a L^2$ is dense in $\mathcal{D}_{\xi}^{\mathfrak{d},\gamma} = \mathcal{G}_a H_0^{\gamma-2}$. That $\mathcal{D}_{\xi}^{\mathfrak{d},\gamma}$ is dense in L^2 follows from the periodic counterpart which is proven in [1], Lemma 4.12. This proves (e). □

PROOF OF THEOREM 5.4. By Lemma 5.12 it follows that $\mathscr{H}_{\boldsymbol{\xi}}$ is a closed densely defined symmetric operator and that $\sigma(\mathscr{H}_{\boldsymbol{\xi}}) \subset \sigma(\mathscr{H}_{\tilde{\xi}})$ so that $\mathscr{H}_{\boldsymbol{\xi}}$ is indeed selfadjoint (see [9], Theorem X.2.9). As the resolvents are compact, the statements in Theorem 5.4 up to (43) follow by the Riesz–Schauder theorem [29], Theorem VI.15, and the Hilbert–Schmidt theorem [29], Theorem VI.16, because of the following identity, where $\mathcal{R}_\mu = (\mu - \mathscr{H}_{\boldsymbol{\xi}})^{-1}$:

$$\sigma(\mathscr{H}_{\boldsymbol{\xi}}) = \sigma_p(\mathscr{H}_{\boldsymbol{\xi}}) = \left\{\mu - \frac{1}{\lambda} : \lambda \in \sigma_p(\mathcal{R}_\mu) \setminus \{0\}\right\};$$

this means that $\lambda - \mathcal{R}_\mu$ is boundedly invertible (or injective) if and only if $\mu - \frac{1}{\lambda} - \mathscr{H}_{\boldsymbol{\xi}}$ is and, in turn, follows from the identity

$$\begin{aligned}\lambda\left(\mu - \frac{1}{\lambda} - \mathscr{H}_{\boldsymbol{\xi}}\right) &= \lambda(\mu - \mathscr{H}_{\boldsymbol{\xi}}) - 1 = (\lambda - \mathcal{R}_\mu)(\mu - \mathscr{H}_{\boldsymbol{\xi}}) \\ &= (\mu - \mathscr{H}_{\boldsymbol{\xi}})\lambda - 1 = (\mu - \mathscr{H}_{\boldsymbol{\xi}})(\lambda - \mathcal{R}_\mu).\end{aligned}$$

As every eigenvalue of $\mathscr{H}_{\boldsymbol{\xi}}$ is an eigenvalue of $\mathscr{H}_{\tilde{\xi}}$, which is locally lipschitz in the analogues sense of (43), also (43) holds by the equivalences of norms in Lemma 5.12(a). (44) follows from Fischer's principle [23], Section 28, Theorem 4, p. 318. That $\lambda_1 > \lambda_2$ or, in other words that the first eigenvalue is simple, follows from [30], Theorem XIII.44. The only condition to prove for that theorem is that the semigroup $e^{t\mathscr{H}_{\boldsymbol{\xi}}}$ is positivity improving or, differently, called the strong maximum principle for $e^{t\mathscr{H}_{\boldsymbol{\xi}}}$. The strategy to obtain this we borrow from [4], Theorem 5.1. With $u_t := e^{t\mathscr{H}_{\boldsymbol{\xi}}} u_0$, the map $(t,x) \mapsto u_t(x)$ is the solution to the parabolic Anderson model $\partial_t u = \Delta u + u \diamond \boldsymbol{\xi}$, hence satisfies $\sup_{s\in[0,t]} \|u_s\|_{B_{\infty,\infty}^{\mathfrak{d},1-\varepsilon}} < \infty$ for all $\varepsilon > 0$ (see [16]; the extension to Dirichlet boundary conditions follows similar as the extension of the operator) and $u_t = P_t u_0 + \int_0^t P_{t-s}(u_s \diamond \boldsymbol{\xi})\,\mathrm{d}s$, where $P_t u_0(x) = p_t * u_0(x)$ and p_t the standard heat kernel $p_t(x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}$. The next step is to prove that $P_t u_0$ is larger than the supremum norm of $\int_0^t P_{t-s}(u_s \diamond \boldsymbol{\xi})\,\mathrm{d}s$. In [4] it is shown that, for all $\rho > 0$, there exists a t_ρ such that $P_t \mathbb{1}_{B(x,\delta)} \geq \frac{1}{4}\mathbb{1}_{B(x,\delta+\rho t)}$ for $t \in (0, t_\rho]$. On the other hand, one can prove that, for $\varepsilon \in (0,1)$, there exists a $C > 0$ such that $\|\int_0^t P_{t-s}(u_s \diamond \boldsymbol{\xi})\,\mathrm{d}s\|_{B_{\infty,\infty}^{\mathfrak{d},\varepsilon}} \leq Ct^{1-\varepsilon}$. Hence, we can choose $t_0 \in (0, t_\rho)$ such that $Ct_0^{1-\varepsilon} \leq \frac{1}{8}$. This implies that $u_t \geq \frac{1}{8}$ on $B(x, \delta + \rho t_0)$. Let $T, \rho > 0$, by choosing n such that $\frac{T}{n} \leq t_0$, by repeating the argument we have $u_T \geq (\frac{1}{8})^n$ on $B(x, \delta + \rho T)$. As this holds for arbitrary $\rho > 0$, this implies that U_T is strictly positive everywhere. □

6. Enhanced white noise. In this section we prove Theorem 6.4; we first recall a definition and introduce notation.

DEFINITION 6.1. A *white noise* on $\mathbb{R}^d$ is a random variable $W : \Omega \to \mathcal{S}'(\mathbb{R}^d, \mathbb{R})$ such that, for all $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, the random variable $\langle W, f\rangle$ is a centered Gaussian random variable.

6.2. Because $\|\langle W, f\rangle\|_{L^2(\Omega,\mathbb{P})} = \|f\|_{L^2(\mathbb{R}^d)}$, the function $f \mapsto \langle W, f\rangle$ extends to a bounded linear operator $\mathcal{W} : L^2(\mathbb{R}^d) \to L^2(\Omega, \mathbb{P})$ such that, for all $f \in L^2(\mathbb{R}^d)$, $\mathcal{W}f$ is a complex Gaussian random variable, $\mathcal{W}\overline{f} = \overline{\mathcal{W}f}$ and $\mathbb{E}[\mathcal{W}f\overline{\mathcal{W}g}] = \langle f, g\rangle_{L^2}$ for all $f, g \in L^2(\mathbb{R}^d)$.

6.3. Let W be a white noise on $\mathbb{R}^2$ and $\mathcal{W}$ be as in 6.2. For the rest of this section, we fix $L > 0$. Unless mentioned otherwise, $\tau \in C_c^\infty(\mathbb{R}^d, [0, 1])$ is an even function that is equal to 1 on a neighbourhood of 0. Define $\xi_{L,\varepsilon} \in \mathcal{S}_\mathfrak{n}([0, L]^d)$ by (for $\langle \mathcal{W}, \mathfrak{n}_{k,L}\rangle$, we interpret $\mathfrak{n}_{k,L}$ to be the function in $L^2(\mathbb{R}^d)$ being equal to $\mathfrak{n}_{k,L}$ on $[0, L]^d$ and equal to 0 elsewhere)

$$\xi_{L,\varepsilon} = \sum_{k\in\mathbb{N}_0^d} \tau\left(\frac{\varepsilon}{L}k\right)\langle \mathcal{W}, \mathfrak{n}_{k,L}\rangle \mathfrak{n}_{k,L}. \tag{47}$$

For $k \in \mathbb{N}_0^d$, define $Z_k := \langle \mathcal{W}, \mathfrak{n}_{k,L}\rangle$. Then, Z_k is a (real) normal random variable with

$$\mathbb{E}[Z_k] = 0, \qquad \mathbb{E}[Z_k Z_l] = \delta_{k,l}. \tag{48}$$

Before we state the convergence to the enhanced white noise, let us discuss our choice of regularization (47). We use the regularisation by means of a Fourier multiplier, as in [1]. This basically means we "project" the white noise on the Neumann space on the box and then take the regularisation corresponding to a Fourier multiplier. Another option is to consider mollified white noise on the full space by convolution and then project the white noise on the Neumann space. In a future work by König, Perkowski and van Zuijlen, it will be shown that both choices lead to the same limiting object (up to a constant, by using techniques from Section 11). This also confirms that our construction of the Anderson Hamiltonian with enhanced white noise agrees with the construction of the Anderson Hamiltonian in [21], where the Anderson Hamiltonian is considered a limit of the operators with mollified white noise as potentials.

THEOREM 6.4. *Let $d = 2$. For all $\alpha < -1$, there exists a $\boldsymbol{\xi}_L \in \mathfrak{X}_\mathfrak{n}^\alpha$ such that the following convergence holds almost surely in $\mathfrak{X}_\mathfrak{n}^\alpha$, that is, on a measurable set Ω_L with $\mathbb{P}(\Omega_L) = 1$:*

$$\lim_{\varepsilon\downarrow 0,\varepsilon\in\mathbb{Q}\cap(0,\infty)} (\xi_{L,\varepsilon}, \xi_{L,\varepsilon} \odot \sigma(\mathrm{D})\xi_{L,\varepsilon} - c_\varepsilon) = \boldsymbol{\xi}_L, \tag{49}$$

where $c_\varepsilon = \frac{1}{2\pi}\log(\frac{1}{\varepsilon}) + c_\tau \in \mathbb{R}$ and c_τ only depends on τ. $\boldsymbol{\xi}_L$ does not depend on the choice of τ. ξ_L is a white noise in the sense that for $\varphi, \psi \in \mathcal{S}_\mathfrak{n}(Q_L)$, $\xi_L(\varphi)$ and $\xi_L(\psi)$ are Gaussian random variables with

$$\mathbb{E}[\xi_L(\varphi)] = 0, \qquad \mathbb{E}[\xi_L(\varphi)\xi_L(\psi)] = \langle \varphi, \psi\rangle_{L^2([0,L]^d)}. \tag{50}$$

Moreover, for $\varphi \in C_c^\infty(Q_L)$ one has almost surely (i.e., on Ω_L)

$$\langle \xi_L, \varphi\rangle = \lim_{\varepsilon\downarrow 0}\langle \xi_{L,\varepsilon}, \varphi\rangle = \sum_{k\in\mathbb{N}_0^d} \langle \mathcal{W}, \mathfrak{n}_{k,L}\rangle\langle \mathfrak{n}_{k,L}, \varphi\rangle = \langle W, \varphi\rangle.$$

Hence, for every $L > 0$, the W viewed as an element of $\mathcal{D}'(Q_L)$ extends almost surely uniquely to a ξ_L in $\mathcal{C}_\mathfrak{n}^\alpha$.

Instead of taking Q_L as an underlying space, we can also take a shift of the box, that is, $y + Q_L$.

6.5. For $y \in \mathbb{R}^d$, we define

$$\xi_{L,\varepsilon}^y = \mathcal{T}_y\bigg[\sum_{k\in\mathbb{N}_0^d} \tau\bigg(\frac{\varepsilon}{L}k\bigg)\langle \mathcal{T}_y^{-1}\mathcal{W}, \mathfrak{n}_{k,L}\rangle \mathfrak{n}_{k,L}\bigg].$$

If $d=2$, by Theorem 6.4 there exists a $\boldsymbol{\xi}_L^y = (\xi_L^y, \Xi_L^y) \in \mathfrak{X}_\mathfrak{n}^\alpha(y+Q_L)$ such that almost surely

$$\lim_{\varepsilon\downarrow 0, \varepsilon\in\mathbb{Q}\cap(0,\infty)} \bigg(\xi_{L,\varepsilon}^y, \xi_{L,\varepsilon}^y \odot \sigma(\mathrm{D})\xi_{L,\varepsilon}^y - \frac{1}{2\pi}\log\bigg(\frac{1}{\varepsilon}\bigg)\bigg) = \boldsymbol{\xi}_L^y \tag{51}$$

and such that ξ_L^y is a white noise in the sense described in Theorem 6.4 (i.e., $\mathcal{T}_{-y}\xi_L^y$ satisfies (50)).

For the rest of this section, we fix $L>0$ and drop the subindex L; we write $\xi_\varepsilon = \xi_{L,\varepsilon}$ and $\mathfrak{n}_k = \mathfrak{n}_{k,L}$.

DEFINITION 6.6. Define $\Xi_\varepsilon \in \mathcal{S}_\mathfrak{n}(Q_L)$ by

$$\Xi_\varepsilon(x) = \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x) - \mathbb{E}\big[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)\big]. \tag{52}$$

The strategy of the proof of the following theorem is rather similar to the proof on the torus in [1], but, due to the differences of the Dirichlet setting and for the sake of selfcontainedness, we provide the proof.

THEOREM 6.7. *For all $\alpha < -\frac{d}{2}$, ξ_ε converges almost surely as $\varepsilon \downarrow 0$ in $\mathcal{C}_\mathfrak{n}^\alpha$ to the white noise ξ_L (as in Theorem 6.4). Moreover, for $d=2$ and all $\alpha<-1$, Ξ_ε converges almost surely as $\varepsilon\downarrow 0$ in $\mathcal{C}_\mathfrak{n}^{2\alpha+2}$; the limit is independent of the choice of τ.*

PROOF. The proof relies on the Kolmogorov–Chentsov theorem (Theorem 6.8). Lemma 6.10(a) shows that the required bound for this theorem can be reduced to bounds on the second moments of $\Delta_i(\xi_\varepsilon - \xi_\delta)(x)$ and $\Delta_i(\Xi_\varepsilon - \Xi_\delta)(x)$, given in 6.11 (the proofs of these bounds are lengthy and, therefore, postponed to Section 11). (50) follows from

$$\mathbb{E}\big[\langle \xi_\varepsilon, \varphi\rangle\langle\xi_\varepsilon,\psi\rangle\big] = \sum_{k\in\mathbb{N}_0^d}\tau(\varepsilon k)^2\langle\varphi,\mathfrak{n}_k\rangle\langle\psi,\mathfrak{n}_k\rangle \xrightarrow{\varepsilon\downarrow 0} \sum_{k\in\mathbb{N}_0^d}\langle\varphi,\mathfrak{n}_k\rangle\langle\psi,\mathfrak{n}_k\rangle = \langle\varphi,\psi\rangle.$$

That the limit of Ξ_ε is independent of the choice of τ, follows from Theorem 11.2 (a). □

THEOREM 6.8 (Kolmogorov–Chentsov theorem). *Let ζ_ε be a random variable with values in a Banach space $\mathfrak{X}$ for all $\varepsilon>0$. Suppose there exist $a,b,C>0$ such that, for all $\varepsilon,\delta>0$,*

$$\mathbb{E}\big[\|\zeta_\varepsilon - \zeta_\delta\|_\mathfrak{X}^a\big] \le C|\varepsilon-\delta|^{1+b}.$$

Then, there exists a random variable ζ with values in $\mathfrak{X}$ such that in $L^a(\Omega,\mathfrak{X})$ and almost surely $\lim_{\varepsilon\downarrow 0,\varepsilon\in\mathbb{Q}\cap(0,\infty)}\zeta_\varepsilon = \zeta$.

PROOF. This follows from the proof of [19], Theorem 2.23. □

In Lemma 6.10(a) we show how we obtain L^p bounds on the $\mathcal{C}_\mathfrak{n}$ norm from bounds on squares of the Littlewood–Paley blocks. Lemma 6.10(b) follows from (a) and will be used in Section 8 to prove Theorem 8.7.

To prove Lemma 6.10, we use the following auxiliary lemma. It is generally known that the pth moment of a centered Gaussian random variable Z can be bounded by its second moment, as $\mathbb{E}[|Z|^p] = (p-1)!!\mathbb{E}[|Z|^2]^{\frac{p}{2}}$ (see [26], p. 110). We will use the generalisation of this bound which is a consequence of the so-called hypercontractivity.

LEMMA 6.9 ([25], Theorem 1.4.1 and equation (1.71)). *Suppose that Z_n for $n \in \mathbb{N}$ are independent standard Gaussian random variables. If Z is a random variable in the first or second Wiener chaos, which means it is of the form $\sum_{n\in\mathbb{N}} a_n Z_n$ or $\sum_{n,m\in\mathbb{N}} a_{n,m}(Z_n Z_m - \mathbb{E}[Z_n Z_m])$ with $a_n, a_{n,m} \in \mathbb{C}$, then, for $p > 1$,*

$$\mathbb{E}\big[|Z|^p\big] \le p^p \mathbb{E}\big[|Z|^2\big]^{\frac{p}{2}}.$$

LEMMA 6.10. *Let $A > 0$ and $a \in \mathbb{R}$:*

(a) *Suppose ζ is a random variable with values in $\mathcal{S}'_{\mathfrak{n}}([0,L]^d)$ such that $\Delta_i\zeta(x)$ is a random variable of the form as Z is, as in Lemma 6.9 for all $i \in \mathbb{N}_{-1}$ and $x \in [0,L]^d$. Suppose that, for all $i \in \mathbb{N}_{-1}$, $x \in [0,L]^d$*

$$\mathbb{E}\big[\big|\Delta_i\zeta(x)\big|^2\big] \le A 2^{ai}. \tag{53}$$

Then, for all $\kappa > 0$, there exists a $C > 0$ independent of ζ such that, for all $p \ge 1$,

$$\mathbb{E}\big[\|\zeta\|^p_{\mathcal{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}}\big] \le C p^p L^d A^{\frac{p}{2}}. \tag{54}$$

(b) *Suppose that $(\zeta_\varepsilon)_{\varepsilon>0}$ is a family of such random variables for which (53) holds for all $i \in \mathbb{N}_{-1}$ and $x \in [0,L]^d$, and that, for all $k \in \mathbb{N}_0^d$,*

$$\mathbb{E}\big[\big|\langle\zeta_\varepsilon, \mathfrak{n}_{k,L}\rangle\big|^2\big] \to 0. \tag{55}$$

Then, for all $\kappa > 0$ and $p > 1$,

$$\mathbb{E}\big[\|\zeta_\varepsilon\|^p_{\mathcal{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}}\big] \to 0.$$

Consequently, we have $\zeta_\varepsilon \xrightarrow{\mathbb{P}} 0$ (convergence in probability) in $\mathcal{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}([0,L]^d)$.

PROOF. (a) For $\kappa > 0$, by Lemma 6.9 with $C_\kappa = \sum_{i=-1}^\infty 2^{-\kappa i}$,

$$\mathbb{E}\big[\|\zeta\|^p_{B_{p,p}^{\mathfrak{n},-\frac{a}{2}-\kappa}}\big] = \sum_{i=-1}^{\infty} 2^{(-\frac{a}{2}-\kappa)pi}\mathbb{E}\big[\|\Delta_i\zeta\|^p_{L^p}\big] \le p^p L^d \Bigg(\sum_{i=-1}^{\infty} 2^{-p\kappa i}\Bigg) A^{\frac{p}{2}} \le C_\kappa p^p L^d A^{\frac{p}{2}}.$$

Using the embedding property of Besov spaces [2], Proposition 2.71, which implies the existence of a $C > 0$ such that $\|\cdot\|_{\mathcal{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}} \le C\|\cdot\|_{B_{p,p}^{\mathfrak{n},-\frac{a}{2}-\kappa}}$, one obtains (54).

(b) By Lemma 6.9 (and Fubini)

$$\mathbb{E}\big[\|\Delta_i\zeta_\varepsilon\|^p_{L^p}\big] \le p^p \int \mathbb{E}\big[\big|\Delta_i\zeta_\varepsilon(x)\big|^2\big]^{\frac{p}{2}}\,\mathrm{d}x \lesssim p^p L^d \Bigg(\sum_{k\in\mathbb{N}_0^d} \rho_i\bigg(\frac{k}{L}\bigg)^2 \mathbb{E}\big[\big|\langle\zeta_\varepsilon, \mathfrak{n}_k\rangle\big|^2\big]\Bigg)^{\frac{p}{2}},$$

and so

$$\mathbb{E}\big[\|\zeta_\varepsilon\|^p_{B_{p,p}^{\mathfrak{n},-\frac{a}{2}-\kappa}}\big] \le p^p L^d \Bigg(\sum_{i=-1}^{I} 2^{(-\frac{a}{2}-\kappa)pi}\Bigg(\sum_{k\in\mathbb{N}_0^d} \rho_i\bigg(\frac{k}{L}\bigg)^2 \mathbb{E}\big[\big|\langle\zeta_\varepsilon, \mathfrak{n}_k\rangle\big|^2\big]\Bigg)^{\frac{p}{2}} + A^{\frac{p}{2}} \sum_{i\ge I+1} 2^{-\kappa i}\Bigg).$$

The latter becomes arbitrarily small by choosing I large and subsequently ε small. □

6.11. The following two statements are proved in Section 11:

(a) (Lemma 11.4) For all $\gamma \in (0,1)$, there exists a $C > 0$ such that, for all $i \in \mathbb{N}_{-1}$, $\varepsilon, \delta > 0$, $x \in [0,L]^d$,

$$\mathbb{E}\big[\big|\Delta_i(\xi_\varepsilon - \xi_\delta)(x)\big|^2\big] \le C 2^{(d+2\gamma)i}|\varepsilon - \delta|^\gamma.$$

(b) (Lemma 11.11) Let $d = 2$. For all $\gamma \in (0,1)$, there exists a $C > 0$ such that, for all $i \in \mathbb{N}_{-1}$, $\varepsilon, \delta > 0$, $x \in Q_L$,

$$\mathbb{E}\big[\big|\Delta_i(\Xi_\varepsilon - \Xi_\delta)(x)\big|^2\big] \le C 2^{2\gamma i}|\varepsilon - \delta|^\gamma.$$

DEFINITION 6.12. Define $c_{\varepsilon,L} \in \mathbb{R}$ by

$$c_{\varepsilon,L} = \frac{1}{4L^2}\sum_{k\in\mathbb{Z}^2}\frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}. \tag{56}$$

In the periodic setting, one has that with ξ_ε defined as in [1], $\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] = c_{\varepsilon,L}$. Observe that it is independent of x. In our setting, the Dirichlet setting, we have (remember (48) and use that $\sum_{i,j\in\mathbb{N}_{-1},|i-j|\le 1}\rho_i(\frac{k}{L})\rho_j(\frac{k}{L}) = 1$)

$$\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] = \sum_{k\in\mathbb{N}_0^2}\frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}\mathfrak{n}_k(x)^2. \tag{57}$$

By (38), as $\mathfrak{n}_0(x) = \frac{2}{L}\nu_0 = \frac{1}{L}$ and $\nu_{2k} = \nu_k$,

$$\mathfrak{n}_k(x)^2 = \frac{1}{2L}\nu_k\mathfrak{n}_{2k}(x) + \frac{1}{2L}\frac{\nu_k^2}{\nu_{(k_1,0)}}\mathfrak{n}_{(2k_1,0)}(x) + \frac{1}{2L}\frac{\nu_k^2}{\nu_{(0,k_2)}}\mathfrak{n}_{(0,2k_2)}(x) + \frac{\nu_k^2}{L^2}. \tag{58}$$

Note that

$$c_{\varepsilon,L} = \sum_{k\in\mathbb{N}_0^2}\frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}\frac{\nu_k^2}{L^2} = \frac{1}{4}\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(0)]. \tag{59}$$

Lemma 6.15 deals with this x dependence of $\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)]$.

The following observations will be used multiple times.

6.13. As $0 \le \rho_i \le 1$ and there is a $b \ge 1$ such that ρ_i is supported in a ball of radius $2^i b$ for all $i \in \mathbb{N}_{-1}$, one has, for all $i \in \mathbb{N}_{-1}$, $x \in \mathbb{R}^d$ and $\gamma > 0$

$$\rho_i(x) \le \left(2b\frac{2^i}{1+|x|}\right)^\gamma. \tag{60}$$

THEOREM 6.14. *Let $\tau : \mathbb{R}^2 \to [0,1]$ be a compactly supported even function that equals 1 on a neighbourhood of 0. There exists a $C > 0$ such that, for all $\gamma \in \mathbb{R}$, $L > 0$ and $h \in H_\mathfrak{n}^\gamma(Q_L)$, we have $\|h - \tau(\varepsilon\mathrm{D})h\|_{H_\mathfrak{n}^\gamma} \to 0$ and, for $\beta < \gamma$,*

$$\|h - \tau(\varepsilon\mathrm{D})h\|_{H_\mathfrak{n}^\beta} \le C\varepsilon^{\gamma-\beta}\|h\|_{H_\mathfrak{n}^\gamma}.$$

PROOF. By assumption on τ, there exists an $a > 0$ such that $\tau = 1$ on $B(0,a)$. Then,

$$\begin{cases} 1 - \tau\left(\frac{\varepsilon}{L}k\right) = 0, & |k| < \frac{La}{\varepsilon}, \\ \left(1 + \left|\frac{k}{L}\right|^2\right)^{\beta-\gamma} \lesssim \varepsilon^{2(\gamma-\beta)}, & |k| \ge \frac{La}{\varepsilon}. \end{cases}$$

By the following bounds the theorem is proved; by Theorem 4.13

$$\|h - \tau(\varepsilon\mathrm{D})h\|_{H_\mathfrak{n}^\beta} \lesssim \sqrt{\sum_{k\in\mathbb{N}_0^d}\left(1 + \left|\frac{k}{L}\right|^2\right)^\beta\left(1 - \tau\left(\frac{\varepsilon}{L}k\right)\right)^2\langle h,\mathfrak{n}_k\rangle^2} \lesssim \varepsilon^{\gamma-\beta}\|h\|_{H_\mathfrak{n}^\gamma}.$$

□

LEMMA 6.15. *Let $\tau : \mathbb{R}^2 \to [0,1]$ be a compactly supported even function that equals* 1 *on a neighbourhood of* 0. *Then, $x \mapsto \mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] - c_{\varepsilon,L}$ converges in $\mathcal{C}_{\mathfrak{n}}^{-\gamma}$ to a limit that is independent of τ, as $\varepsilon \downarrow 0$ for all $\gamma > 0$.*

PROOF. Let $\gamma > 0$. As there are only finitely many $k \in \mathbb{N}_0^2$ for which $\tau(\frac{\varepsilon}{L}k) \neq 0$, $x \mapsto \mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] - c_{\varepsilon,L}$ is smooth. We can rewrite (58) and find uniformly bounded a_k, b_k such that $\mathfrak{n}_k(x)^2 - \frac{\nu_k^2}{L^2} = \frac{1}{2L}[\mathfrak{n}_k + a_k \mathfrak{n}_{(k_1,0)} + b_k \mathfrak{n}_{(0,k_2)}](2x)$. By (59) this means that $\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)]$ (see (57)) can be decomposed into three sums.

For the first sum (by taking the part with "$\mathfrak{n}_k$"), as $\delta_0 \in H_{\mathfrak{n}}^{-1}$ and $\langle \delta_0, \mathfrak{n}_k \rangle = \frac{2}{L}$ for all $k \in \mathbb{N}_0^2$,

$$\frac{1}{2L} \sum_{k \in \mathbb{N}_0^2} \frac{\tau(\frac{\varepsilon}{L}k)^2}{1 + \frac{\pi^2}{L^2}|k|^2} \mathfrak{n}_k(2x) = \frac{1}{4}\big[\tau(\varepsilon \mathrm{D})^2 \sigma(\mathrm{D})\delta_0\big](2x).$$

By Theorem 4.20 $\sigma(\mathrm{D})\delta_0 \in H_{\mathfrak{n}}^1$, so that by Theorem 6.14 $\tau(\varepsilon\mathrm{D})^2\sigma(\mathrm{D})\delta_0 \to \sigma(\mathrm{D})\delta_0$ in $H_{\mathfrak{n}}^{1-\gamma}$ and thus in $\mathcal{C}_{\mathfrak{n}}^{-\gamma}$ (by [2], Theorem 2.71). This convergence is "stable" under "multiplying the argument by 2" (see also 4.18).

Now, let us show the convergence of the other sums. We only consider the sum with "$a_k \mathfrak{n}_{(k_1,0)}$" in it, as the sum with "$b_k \mathfrak{n}_{(0,k_2)}$" follows similarly. Let us write h_ε for

$$h_\varepsilon(x) = \sum_{l,m \in \mathbb{N}_0} \frac{\tau(\frac{\varepsilon}{L}(l,m))^2}{1 + \frac{\pi^2}{L^2}(l^2 + m^2)} a_{(l,m)} \mathfrak{n}_{(l,0)}(x).$$

With (60) $\|\Delta_i \mathfrak{n}_{(l,0)}\|_{L^\infty} \lesssim |\rho_i(\frac{l}{L}, 0)| \lesssim 2^{\gamma i}(1 + \frac{l^2}{L^2})^{-\gamma}$. Hence,

$$\sup_{i \in \mathbb{N}_{-1}} 2^{-\gamma i} \|\Delta_i(h_\varepsilon - h_0)\|_{L^\infty} \lesssim \sum_{l,m \in \mathbb{N}_0} \left(1 + \frac{l^2}{L^2}\right)^{-\gamma} \frac{|\tau(\frac{\varepsilon}{L}(l,m))^2 - 1|}{1 + \frac{\pi^2}{L^2}(l^2+m^2)}.$$

By Lebesgue's dominated convergence theorem and the next bound, it follows that $h_0 \in \mathcal{C}_{\mathfrak{n}}^{-\gamma}$ and $h_\varepsilon \to h_0$ in $\mathcal{C}_{\mathfrak{n}}^{-\gamma}$. By using that $1 + l^2 + m^2 \geq (1+l)^{1-\frac{\gamma}{2}}(1+m)^{1+\frac{\gamma}{2}}$,

$$\sum_{l,m \in \mathbb{N}_0} \frac{(1 + \frac{l^2}{L^2})^{-\gamma}}{1 + \frac{\pi^2}{L^2}(l^2+m^2)} \lesssim \sum_{l,m \in \mathbb{N}_0} \frac{1}{(1+l)^{1+\frac{\gamma}{2}}(1+m)^{1+\frac{\gamma}{2}}} < \infty.$$

By these convergences and by plugging in the factor 2 also here, the convergence is proved. □

Before we give the proof of Theorem 6.4, we study the behaviour of $c_{\varepsilon,L}$.

LEMMA 6.16. *Let $\tau : \mathbb{R}^2 \to [0,1]$ be almost everywhere continuous, be equal to* 1 *on $B(0,a)$ and zero outside $B(0,b)$ for some a, b with $0 < a < b$. There exist a $c_\tau \in \mathbb{R}$ that only depends on τ, and $(C_L)_{L \geq 1}$ in $\mathbb{R}$ that do not depend on τ with $C_L \xrightarrow{L \to \infty} 0$ such that $c_{\varepsilon,L} - \frac{1}{2\pi}\log\frac{1}{\varepsilon} - c_\tau \xrightarrow{\varepsilon \downarrow 0} C_L$ for all $L \geq 1$.*

PROOF. We define $\lfloor y \rfloor = (\lfloor y_1 \rfloor, \lfloor y_2 \rfloor)$ and $h_L(y) = (L^2 + \pi^2|y|^2)^{-1}$ for $y \in \mathbb{R}^2$. Then, $4c_{\varepsilon,L} = \int_{\mathbb{R}^2} \tau(\frac{\varepsilon}{L}\lfloor y \rfloor)^2 h_L(\lfloor y \rfloor)\,\mathrm{d}y$. We first show that $4c_{\varepsilon,L} - \int_{\mathbb{R}^2} \tau(\frac{\varepsilon}{L}y)^2 h_L(y)\,\mathrm{d}y \to 0$. Write

$A(s,t)$ for the annulus $\{y \in \mathbb{R}^2 : s \le |y| \le t\}$. To shorten notation, we write $\delta = \frac{\varepsilon}{L}$. As $|\lfloor y \rfloor - y| \le \sqrt{2}$,

$$
\begin{aligned}
&4c_{\varepsilon,L} - \int_{\mathbb{R}^2} \tau\Big(\frac{\varepsilon}{L} y\Big)^2 h_L(y)\,\mathrm{d}y \\
&\quad = \int_{B(0,\frac{a}{\delta}-\sqrt{2})} h_L(\lfloor y \rfloor) - h_L(y)\,\mathrm{d}y \\
&\qquad + \int_{A(\frac{a}{\delta}-\sqrt{2},\frac{b}{\delta}+\sqrt{2})} \tau(\delta\lfloor y \rfloor)^2 h_L(\lfloor y \rfloor) - \tau(\delta y)^2 h_L(y)\,\mathrm{d}y.
\end{aligned}
$$

As $h_L(\lfloor y \rfloor) - h_L(y) = h_L(\lfloor y \rfloor) h_L(y)(|y|^2 - |\lfloor y \rfloor|^2)$, $h_L(\lfloor y \rfloor) \lesssim h_L(y)$ and $(|y|^2 - |\lfloor y \rfloor|^2) \lesssim 1 + |y|$, we have $h_L(\lfloor y \rfloor) - h_L(y) \lesssim (1+|y|) h_L(y)^2$. As the latter function is integrable over $\mathbb{R}^2$, it follows by Lebesgue's dominated convergence theorem that $\int_{B(0,\frac{a}{\delta}-\sqrt{2})} h_L(\lfloor y \rfloor) - h_L(y)\,\mathrm{d}y$ converges in $\mathbb{R}$ to a C_L for which $C_L \xrightarrow{L\to\infty} 0$. On the other hand, the integral over the annulus can be written as

$$
\int_{A(a-\sqrt{2}\delta,b+\sqrt{2}\delta)} \frac{\tau(\delta\lfloor \frac{x}{\delta} \rfloor)^2}{\delta^2 L^2 + \pi^2 \delta^2 |\lfloor \frac{x}{\delta} \rfloor|^2} - \frac{\tau(x)^2}{\delta^2 L^2 + \pi^2 |x|^2}\,\mathrm{d}x. \tag{61}
$$

Again, by a domination argument (note that $\frac{1}{|x|^2}$ is integrable over annuli), using that $|\frac{x}{\delta}|^2 \le 4 + 2|\lfloor \frac{x}{\delta} \rfloor|^2 \le 4(L^2 + |\lfloor \frac{x}{\delta} \rfloor|^2)$, we conclude that (61) converges to 0. Observe that

$$
\int_{A(a-\sqrt{2}\delta,b+\sqrt{2}\delta)} \frac{\tau(x)^2}{\delta^2 L^2 + \pi^2 |x|^2}\,\mathrm{d}x \xrightarrow{\delta\downarrow 0} \int_{A(a,b)} \frac{\tau(x)^2}{\pi^2 |x|^2}\,\mathrm{d}x.
$$

By some substitutions (remember $\delta = \frac{\varepsilon}{L}$), for $\varepsilon < a$

$$
\frac{1}{2\pi} \int_{B(0,\frac{a}{\delta}-\sqrt{2})} h_L(y)\,\mathrm{d}y = \int_0^1 \frac{s}{1+\pi^2 s^2}\,\mathrm{d}s + \int_1^{\frac{a}{\varepsilon}} \frac{s}{1+\pi^2 s^2}\,\mathrm{d}s - \int_{a-\frac{\sqrt{2}\varepsilon}{L}}^{a} \frac{s}{\varepsilon^2 + \pi^2 s^2}\,\mathrm{d}s.
$$

The last integral converges as $\varepsilon \downarrow 0$ to zero. For the second integral we consider

$$
\int_1^{\frac{a}{\varepsilon}} \frac{s}{1+\pi^2 s^2} - \frac{1}{\pi^2 s}\,\mathrm{d}s = \int_1^{\frac{a}{\varepsilon}} \frac{-1}{\pi^2 s(1+\pi^2 s^2)}\,\mathrm{d}s, \qquad \int_1^{\frac{a}{\varepsilon}} \frac{1}{\pi^2 s}\,\mathrm{d}s = \frac{1}{\pi^2} \log\Big(\frac{a}{\varepsilon}\Big).
$$

Observe that if $a \le 1$, then $\int_{A(a,1)} \frac{1}{\pi^2 |x|^2}\,\mathrm{d}x = -\frac{2}{\pi} \log a$, and if $a \ge 1$, then $\int_{A(1,a)} \frac{1}{\pi^2|x|^2}\,\mathrm{d}x = \frac{2}{\pi} \log a$. Therefore, with

$$
c_\tau = \int_{A(a\wedge 1,b)} \frac{\tau(x)^2}{\pi^2 |x|^2}\,\mathrm{d}x - \int_{A(a\wedge 1,1)} \frac{1}{\pi^2 |x|^2}\,\mathrm{d}x + \int_0^1 \frac{2\pi s}{1+\pi^2 s^2}\,\mathrm{d}s - \int_1^\infty \frac{2}{\pi s(1+\pi^2 s^2)}\,\mathrm{d}s,
$$

we obtain that $c_{\varepsilon,L} - \frac{1}{2\pi} \log \frac{1}{\varepsilon} - C_L - c_\tau \xrightarrow{\varepsilon\downarrow 0} 0$. Observe that c_τ does not depend on the choice of a, b (such that $\tau = 1$ on $B(0,a)$ and $\tau = 0$ outside $B(0,b)$). □

PROOF OF THEOREM 6.4. This is a consequence of Theorem 6.7 and Lemmas 6.15 and 6.16. □

7. Scaling and translation.

In this section we prove the scaling properties of the eigenvalues by scaling the size of the box and the noise. *In this section we fix $L > 0$ and $n \in \mathbb{N}$.*

LEMMA 7.1. *Suppose that $V \in L^\infty([0,L]^d)$. For all $\beta > 0$,*

$$
\lambda_n([0,L]^d, V) = \frac{1}{\beta^2} \lambda_n\Big(\Big[0, \frac{L}{\beta}\Big]^d, \beta^2 V(\beta\cdot)\Big).
$$

PROOF. Fix $n \in \mathbb{N}$, and write $\lambda = \lambda_n([0,L]^d, V)$. Suppose that $g \in H_0^2$ (see 5.9) is an eigenfunction for λ of $\Delta + V$. With $g_\beta(x) := g(\beta x)$, we have, for almost all x,

$$\Delta g_\beta(x) + \beta^2 V(\beta x) = \beta^2(\Delta g)(\beta x) + \beta^2 V(\beta x) = \beta^2 \lambda g_\beta(x).$$

So that $\beta^2\lambda$ is an eigenvalue of $\Delta + \beta^2 V(\beta\cdot)$ on $[0, \frac{L}{\beta}]^d$. As the multiplicities of the eigenvalues on $[0,L]^d$ and $[0, \frac{L}{\beta}]^d$ are the same, $\beta^2\lambda = \lambda_n([0, \frac{L}{\beta}]^d, \beta^2 V(\beta\cdot))$. □

7.2. For $y \in \mathbb{R}^2$, $L > 0$ and $\beta \in \mathbb{R}$, we write

$$\boldsymbol{\lambda}_n(y + Q_L, \beta) = \lambda_n\big(y + Q_L, \big(\beta\xi_L^y, \beta^2 \Xi_L^y\big)\big), \qquad \boldsymbol{\lambda}_n(y + Q_L) = \boldsymbol{\lambda}_n(y + Q_L, 1),$$

where $\boldsymbol{\xi}_L^y = (\xi_L^y, \Xi_L^y)$ is as in 6.5.

LEMMA 7.3. *For $\alpha, \beta > 0$,*

$$\boldsymbol{\lambda}_n(Q_L, \beta) \overset{d}{=} \frac{1}{\alpha^2}\boldsymbol{\lambda}_n(Q_{\frac{L}{\alpha}}, \alpha\beta) + \frac{1}{2\pi}\log\alpha.$$

PROOF. For simplicity, we take $\beta = 1$. $\alpha l_\alpha \xi_L$ is a white noise on $Q_{\frac{L}{\alpha}}$ so that $\langle \alpha l_\alpha \xi_L, \mathfrak{n}_k\rangle \overset{d}{=} \langle \xi_{\frac{L}{\alpha}}, \mathfrak{n}_k\rangle$ for all $k \in \mathbb{N}_0^2$, and, thus, $\frac{1}{\alpha}\xi_{\frac{L}{\alpha}} \overset{d}{=} l_\alpha \xi_L$. By 4.18, $l_\alpha \xi_{L,\varepsilon} = \tau(\frac{\varepsilon}{\alpha}\mathrm{D})[l_\alpha \xi_L] \overset{d}{=} \frac{1}{\alpha}\xi_{\frac{L}{\alpha},\frac{\varepsilon}{\alpha}}$. So that by Lemma 7.1,

$$\begin{aligned}
&\lambda_n\bigg(Q_L, \bigg(\xi_{L,\varepsilon}, \xi_{L,\varepsilon} \odot \sigma(\mathrm{D})\xi_{L,\varepsilon} - \frac{1}{2\pi}\log\Big(\frac{1}{\varepsilon}\Big)\bigg)\bigg)\\
&\quad = \lambda_n(Q_L, \xi_{L,\varepsilon}) - \frac{1}{2\pi}\log\Big(\frac{1}{\varepsilon}\Big)\\
&\quad \overset{d}{=} \frac{1}{\alpha^2}\lambda_n(Q_{\frac{L}{\alpha}}, \alpha\xi_{\frac{L}{\alpha},\frac{\varepsilon}{\alpha}}) - \frac{1}{2\pi}\log\Big(\frac{1}{\varepsilon}\Big)\\
&\quad \overset{d}{=} \frac{1}{\alpha^2}\lambda_n\bigg(Q_{\frac{L}{\alpha}}, \bigg(\alpha\xi_{\frac{L}{\alpha},\frac{\varepsilon}{\alpha}}, \alpha^2\bigg[\xi_{\frac{L}{\alpha},\frac{\varepsilon}{\alpha}} \odot \sigma(\mathrm{D})\xi_{\frac{L}{\alpha},\frac{\varepsilon}{\alpha}} - \frac{1}{2\pi}\log\Big(\frac{\alpha}{\varepsilon}\Big)\bigg]\bigg)\bigg) + \frac{1}{2\pi}\log\alpha.
\end{aligned}$$

Now, we can subtract c_τ from both sides and take the limit $\varepsilon \downarrow 0$. □

LEMMA 7.4. *For $y \in \mathbb{R}^2$ and $\beta > 0$,*

$$\boldsymbol{\lambda}_n(Q_L, \beta) \overset{d}{=} \boldsymbol{\lambda}_n(y + Q_L, \beta).$$

Moreover, if $y + Q_L^\circ \cap Q_L^\circ = \varnothing$, then $\boldsymbol{\lambda}_n(Q_L, \beta)$ and $\boldsymbol{\lambda}_n(y + Q_L, \beta)$ are independent.

PROOF. As (see also Definition 4.19, in particular, (33)) $\mathscr{H}_{\boldsymbol{\xi}_L^y} f = \mathcal{T}_y(\mathscr{H}_{\mathcal{T}_{-y}\boldsymbol{\xi}_L^y}(\mathcal{T}_{-y} f))$, it is sufficient to show $\boldsymbol{\xi}_L \overset{d}{=} \mathcal{T}_{-y}\boldsymbol{\xi}_L^y$. As $\mathcal{T}_{-y}\mathcal{W} \overset{d}{=} \mathcal{W}$, we have $\mathcal{T}_{-y}\xi_{L,\varepsilon}^y \overset{d}{=} \xi_{L,\varepsilon}$ and hence obtain $\boldsymbol{\xi}_L \overset{d}{=} \mathcal{T}_{-y}\boldsymbol{\xi}_L^y$ by (49) and (51).

For the "moreover," note that $(\langle \mathcal{T}_y^{-1}\mathcal{W}, \mathfrak{n}_{k,L}\rangle)_{k\in\mathbb{N}_0^2}$ and $(\langle \mathcal{W}, \mathfrak{n}_{k,L}\rangle)_{k\in\mathbb{N}_0^2}$ are independent when $y + Q_L^\circ \cap Q_L^\circ = \varnothing$ (as $\mathbb{E}[\langle \mathcal{T}_y^{-1}\mathcal{W}, \mathfrak{n}_{k,L}\rangle\langle \mathcal{W}, \mathfrak{n}_{m,L}\rangle] = \langle \mathcal{T}_y \mathfrak{n}_{k,L}, \mathfrak{n}_{k,L}\rangle = 0$). □

8. Comparing eigenvalues on boxes of different size.

8.1. *Bounded potentials.* In this section we prove the bounds comparing eigenvalues on large boxes with eigenvalues on smaller boxes for bounded potentials; see Lemma 8.1, Theorem 8.4 and Theorem 8.5. In Section 8.2, Theorem 8.6, we extend this for white noise potentials. We fix $d \in \mathbb{N}$ and use the notation $|k|_\infty = \max_{i\in\{1,\dots,d\}} |k_i|$.

LEMMA 8.1. *Let $L > r > 0$ and $\zeta \in L^\infty([0, L]^d)$. For all $y \in \mathbb{R}^2$ such that $y + [0, r]^d \subset [0, L]^d$, we have*

$$\lambda_n(y + [0, r]^d, \zeta) \le \lambda_n([0, L]^d, \zeta).$$

PROOF. This follows from (46), as one can identify a finite-dimensional $F \sqsubset H_0^2(y + [0, r]^d)$ with a linear subspace of $H_0^2([0, L]^d)$ with the same dimension. □

We will now prove an upper bound for $\lambda_n(Q_L, \zeta)$ in terms of a maximum over smaller boxes. For this we cover Q_L by smaller boxes that overlap and correct the potential with a function that takes into account the overlaps. We use the following lemma.

LEMMA 8.2. *Let $r > a > 0$. There exists a smooth function $\eta : \mathbb{R}^d \to [0, 1]$ with $\eta = 1$ on $[0, r - a]^d$ and $\operatorname{supp} \eta \subset [-a, r]^d$ such that $\|\nabla\eta\|_\infty \le \frac{K}{a}$ for some $K > 0$ that does not depend on r and a, and*

$$\sum_{k\in\mathbb{Z}^d} \eta(x - rk)^2 = 1 \quad (x \in \mathbb{R}^d). \tag{62}$$

PROOF. We adapt the proof of [15], Proposition 1, and [3], Lemma 4.6. Let $\varphi : \mathbb{R} \to [0, 1]$ be smooth, $\varphi = 0$ on $(-\infty, -1]$ and $\varphi = 1$ on $[1, \infty)$ for all $x \in \mathbb{R}$. Let

$$\zeta(x) = \sqrt{\varphi\left(\frac{2x}{a} + 1\right)\left(1 - \varphi\left(\frac{2(x - r)}{a} + 1\right)\right)}.$$

Then, $\zeta = 0$ outside $[-a, r]$, $\zeta = 1$ on $[0, r - a]$, and $\sum_{k\in\mathbb{Z}} \zeta(x - rk)^2 = 1$. Moreover, $\|\zeta'\|_\infty \le \frac{2}{a}[\|\sqrt{\varphi}'\|_\infty + \|\sqrt{1 - \varphi}'\|_\infty]$. Hence, with $\eta : \mathbb{R}^d \to [0, 1]$, defined by $\eta(x) = \prod_{i=1}^d \zeta(x_i)$, we have (62) and $\|\nabla\eta\|_\infty \le \frac{C}{a}$ for some $C > 0$. □

8.3 (IMS formula). Write $\eta_k(x) = \eta(x - rk)$. Then,

$$\eta_k^2 \Delta\psi + \Delta(\eta_k^2\psi) - 2\eta_k \Delta(\eta_k\psi) = \psi|\nabla\eta_k|^2.$$

Consequently, with $\mathscr{H}_k\psi = \eta_k \mathscr{H}(\eta_k\psi)$ (where $\mathscr{H} = \mathscr{H}_\zeta$) and $\Phi = \sum_{k\in\mathbb{Z}^d} |\nabla\eta_k|^2$

$$\mathscr{H} - \Phi = \sum_{k\in\mathbb{Z}^d} \mathscr{H}_k. \tag{63}$$

(63) is also called the IMS-formula; see also [32], Lemma 3.1, with references to first works in which it appears. The technique to prove [15], Proposition 1, which we slightly generalize, is basically the IMS-formula.

THEOREM 8.4. *For all $r > a > 0$, there is a smooth function $\Phi_{a,r} : \mathbb{R}^d \to [0, \infty)$ whose support is contained in the a-neighbourhood of the grid $r\mathbb{Z}^d + \partial[0, r]^d$, is periodic in each coordinate with period r, with $\|\Phi_{a,r}\|_\infty \le \frac{K}{a}$ for some $K > 0$ that does not depend on a and r such that $\zeta \in L^\infty(\mathbb{R}^d)$ and $L > r$,*

$$\lambda([0, L]^d, \zeta) - \frac{K}{a} \le \lambda([0, L]^d, \zeta - \Phi_{a,r}) \le \max_{k\in\mathbb{N}_0^d, |k|_\infty < \frac{L}{r}+1} \lambda(rk + [-a, r]^d, \zeta). \tag{64}$$

PROOF. Let η be as in Lemma 8.2, $\eta_k(x) = \eta(x - rk)$ and $\Phi_{a,r} = \Phi = \sum_{k\in\mathbb{Z}^d} |\nabla \eta_k|^2$. By Lemma 8.2 it follows that $\|\Phi\|_\infty \le \frac{K}{a}$ for some $K > 0$ that does not depend on a and r. Observe that $\sum_{k\in\mathbb{N}_0^d:|k|_\infty<\frac{L}{r}+1} \eta_k^2$ equals 1 on $[0, L]^d$. With $\mathscr{H}_k$ as in 8.3, $\mathscr{H}_k$ is selfadjoint, and $\mathscr{H}_k \le \lambda(rk + [-a, r]^d)\eta_k^2$ for all $k \in \mathbb{Z}^d$. Hence, we have by the IMS-formula (63) on $H_0^2([0, L]^d)$,

$$\mathscr{H} - \Phi \le \sum_{k\in\mathbb{N}_0^d,|k|_\infty<\frac{L}{r}+1} \lambda\big(rk + [-a, r]^d\big)\eta_k^2 \le \max_{k\in\mathbb{N}_0^d,|k|_\infty<\frac{L}{r}+1} \lambda\big(rk + [-a, r]^d\big).$$

□

THEOREM 8.5. *Let $\zeta \in L^\infty(\mathbb{R}^d)$. Let $x, y_1, \dots, y_n \in \mathbb{R}^d$, $L > r > 0$ be such that $(y_i + [0, r]^d)_{i=1}^n$ are pairwise disjoint subsets of $x + [0, L]^d$. Then,*

$$\lambda_n\big(x + [0, L]^d, \zeta\big) \ge \min_{i\in\{1,\dots,n\}} \lambda\big(y_i + [0, r]^d, \zeta\big). \tag{65}$$

PROOF. By (46) (see also (110)),

$$\lambda_n\big(x + [0, L]^d, \zeta\big) \ge \sup_{\substack{f_1,\dots,f_n,\\ f_i\in C_c^\infty(y_i+[0,r]^d),\|f_i\|_{L^2}=1}} \min_{i\in\{1,\dots,n\}} \int -|\nabla f_i|^2 + \zeta f_i^2,$$

which proves (65) by (46) with $n = 1$. □

8.2. *White noise as potential.* In this section we prove analogous bounds to those in Lemma 8.1, Theorem 8.4 and Theorem 8.5 by replacing the bounded potential ζ by white noise, that is, we prove Theorem 8.6.

THEOREM 8.6. *Let $L \ge r \ge 1$.*
For all $\kappa > 0$ and $x, y \in \mathbb{R}^2$ such that $y + Q_r \subset x + Q_L$,

$$\boldsymbol{\lambda}_n(y + Q_r, \kappa) \le \boldsymbol{\lambda}_n(x + Q_L, \kappa) \quad a.s. \tag{66}$$

There exists a $K > 0$ such that, for all $\kappa > 0$, $x \in \mathbb{R}^2$ and $a \in (0, r)$,

$$\boldsymbol{\lambda}(x + Q_L, \kappa) \le \max_{k\in\mathbb{N}_0^2,|k|_\infty<\frac{L}{r}+1} \boldsymbol{\lambda}(x + rk + Q_{r+a}, \kappa) + \frac{K}{a^2} \quad a.s. \tag{67}$$

For $\kappa > 0$ and $x, y_1, \dots, y_n \in \mathbb{R}^2$ such that $(y_i + Q_r)_{i=1}^n$ are pairwise disjoint subsets of $x + Q_L$,

$$\boldsymbol{\lambda}_n(x + Q_L, \kappa) \ge \min_{i\in\{1,\dots,n\}} \boldsymbol{\lambda}(y_i + Q_r, \kappa) \quad a.s. \tag{68}$$

Let us describe how the proof of Theorem 8.6 follows from the following theorem. Let $L \ge r \ge 1, \kappa > 0$. By performing a translation over x, we may assume $x = 0$.

It is sufficient to show that, for all $y \in \mathbb{R}^2$ and $r > 0$ such that $y + Q_r \subset Q_L$, one has the following convergences in probability (and thus almost surely along a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, 1)$ that converges to 0),

$$\lambda_n\big(y + Q_r, \kappa\big(\xi'_{L,\varepsilon} - c'_\varepsilon\big)\big) \xrightarrow{\mathbb{P}} \boldsymbol{\lambda}_n(y + Q_r, \kappa) \tag{69}$$

for the right choices $\xi'_{L,\varepsilon}$ and c'_ε. Indeed, for (66) and (68) this is clearly sufficient. For (67) this is sufficient by "replacing L" in (69) with "$3L$" and "replacing r" with either "L" or "$r + a$."

In this case we choose $\xi'_{L,\varepsilon}$ like $\xi_{L,\varepsilon}$ in (47) but with $\tau' = \mathbb{1}_{(-1,1)^2}$ instead of τ and $c'_\varepsilon = \frac{1}{2\pi}\log\frac{1}{\varepsilon} + c_{\tau'}$ (the choice of $\tau' = \mathbb{1}_{(-1,1)^2}$ is convenient for calculations in Section 12). Observe that

$$\lambda_n(y+Q_r, \kappa\xi'_{L,\varepsilon}) = \lambda_n(y+Q_r, \kappa\theta^y_\varepsilon) = \lambda_n(y+Q_r, (\kappa\theta^y_\varepsilon, \kappa^2\theta^y_\varepsilon \odot \sigma(\mathrm{D})\theta^y_\varepsilon))$$

for θ^y_ε (which equals $\xi_{L,\varepsilon}|_{y+Q_r}$ in $L^2(y+Q_r)$), given by

$$\begin{aligned}\theta^y_\varepsilon &= \sum_{k\in\mathbb{N}_0^2} \langle \xi_{L,\varepsilon}, \mathcal{T}_y \mathfrak{n}_{k,r}\rangle_{L^2(y+Q_r)} \mathcal{T}_y\mathfrak{n}_{k,r} \\ &= \sum_{k\in\mathbb{N}_0^2}\sum_{m\in\mathbb{N}_0^2} \mathbb{1}_{(-1,1)^2}\left(\frac{\varepsilon}{L}m\right)\langle \mathcal{W}, \mathfrak{n}_{m,L}\rangle\langle \mathfrak{n}_{m,L}, \mathcal{T}_y\mathfrak{n}_{k,r}\rangle_{L^2(y+Q_r)}\mathcal{T}_y\mathfrak{n}_{k,r}. \end{aligned} \tag{70}$$

Therefore, the following theorem resembles the missing part of the proof. Observe that $\theta^y_\varepsilon \odot \sigma(\mathrm{D})\theta^y_\varepsilon \in H^1_\mathfrak{n} \subset \mathcal{C}^0_\mathfrak{n}$ as $\theta^y_\varepsilon \in L^2 = H^0_\mathfrak{n}$ (see also 5.7).

THEOREM 8.7. *Let $L > r \ge 1$ and $x, y \in \mathbb{R}^2$ be such that $y + Q_r \subset x + Q_L$. Let θ^y_ε be as in (70). Then, $(\xi'_{L,\varepsilon}, \xi'_{L,\varepsilon} \odot \sigma(\mathrm{D})\xi'_{L,\varepsilon} - c'_\varepsilon) \xrightarrow{\mathbb{P}} \boldsymbol{\xi}_L$ in $\mathfrak{X}^\alpha_\mathfrak{n}(Q_L)$, and $(\theta^y_\varepsilon, \theta^y_\varepsilon \odot \sigma(\mathrm{D})\theta^y_\varepsilon - c'_\varepsilon) \xrightarrow{\mathbb{P}} \boldsymbol{\xi}^y_r$ in $\mathfrak{X}^\alpha_\mathfrak{n}(y+Q_r)$.*

We prove Theorem 8.7 in Section 11; it follows from Theorem 11.3.

9. Large deviation principle of the enhancement of white noise.

In this section we assume $L > 0$ and write $\boldsymbol{\xi} = (\xi, \Xi)$ for the limit $\boldsymbol{\xi}_L$, as in Theorem 6.4. We prove the following theorem.

THEOREM 9.1. *$(\sqrt{\varepsilon}\xi, \varepsilon\Xi)$ satisfies the large deviation principle with rate ε and rate function $\mathfrak{X}^\alpha_\mathfrak{n} \to [0,\infty]$, $(\psi_1,\psi_2) \mapsto \frac{1}{2}\|\psi_1\|^2_{L^2}$.*

REMARK 9.2. Analogously, by some lines of the proof in a straightforward way, the statement in Theorem 9.1 holds with underlying space the torus and (ξ, Ξ) being the analogue limit, as in Theorem 6.4 as is considered in [1].

As a direct consequence of this large deviation principle and the continuity of the eigenvalues in the (enhanced) noise (see (43)), we obtain the following by an application of the contraction principle (see [10], Theorem 4.2.1).

COROLLARY 9.3. *$\boldsymbol{\lambda}_n(Q_L, \varepsilon) = \lambda_n(Q_L, (\varepsilon\xi_L, \varepsilon^2\Xi_L))$ satisfies the large deviation principle with rate ε^2 and rate function $I_{L,n} : \mathbb{R} \to [0,\infty]$ given by*

$$I_{L,n}(x) = \inf_{\substack{V\in L^2(Q_L)\\ \lambda_n(Q_L,V)=x}} \frac{1}{2}\|V\|^2_{L^2}. \tag{71}$$

Theorem 9.1 is an extension of the following theorem. A proof can be given by using [11], Theorem 3.4.5, but as our proof is rather simple and, to our knowledge, different from proofs in literature, we include it.

THEOREM 9.4. *$\sqrt{\varepsilon}\xi$ satisfies the large deviation principle with rate function $\mathcal{C}^\alpha_\mathfrak{n}([0,L]^d) \to [0,\infty]$ given by $\psi \mapsto \frac{1}{2}\|\psi\|^2_{L^2}$.*

PROOF. We use the Dawson–Gärtner projective limit theorem [10], Theorem 4.6.1, and the inverse contraction principle [10], Theorem 4.2.4. Let $J = \mathbb{N}$ with its natural ordering. Let $\mathcal{Y}_i = \mathbb{R}^i$ for all $i \in J$. Let p_{ij} be the projection $\mathcal{Y}_j \to \mathcal{Y}_i$ on the first i-coordinates. Let $\mathcal{Y}$ be the projective limit $\lim_{\leftarrow} \mathcal{Y}_j$ (see [10], above Theorem 4.6.1, it is a subset of $\prod_{j\in J} \mathcal{Y}_j$). Let $p_j : \mathcal{Y} \to \mathcal{Y}_i$ be the canonical projection.

Let $\mathfrak{s} : \mathbb{N} \to \mathbb{N}_0^d$ be a bijection. Write $\mathfrak{d}'_n = \mathfrak{d}_{\mathfrak{s}(n)}$. Let $\Phi : \mathcal{C}^\alpha_\mathfrak{n}([0, L]^d) \to \mathcal{Y}$ be given by $\Phi(u) = (\langle u, \mathfrak{d}'_1\rangle, \dots, \langle u, \mathfrak{d}'_n\rangle)_{n\in\mathbb{N}}$. This Φ is continuous and injective. We first prove that $\Phi \circ \xi$ satisfies the large deviation principle.

For every $n \in \mathbb{N}$, the vector $(\langle \xi, \mathfrak{d}'_1\rangle, \dots, \langle \xi, \mathfrak{d}'_n\rangle)$ is an n-dimensional standard normal variable, whence $\sqrt{\varepsilon}(\langle \xi, \mathfrak{d}'_1\rangle, \dots, \langle \xi, \mathfrak{d}'_n\rangle) = (\langle \sqrt{\varepsilon}\xi, \mathfrak{d}'_1\rangle, \dots, \langle \sqrt{\varepsilon}\xi, \mathfrak{d}'_n\rangle)$ satisfies a large deviation principle on $\mathbb{R}^n$ with rate function given by $I_n(y) := \frac{1}{2}|y|^2 = \frac{1}{2}\sum_{i=1}^n y_i^2$. By the Dawson–Gärtner projective limit theorem, the sequence $\sqrt{\varepsilon}(\langle \xi, \mathfrak{d}'_1\rangle, \dots, \langle \xi, \mathfrak{d}'_n\rangle)_{n\in\mathbb{N}}$ satisfies the large deviation principle on $\mathcal{Y}$ with rate function

$$I\big((y_1, \dots, y_n)_{n\in\mathbb{N}}\big) = \sup_{n\in\mathbb{N}} I_n(y_1, \dots, y_n) = \sup_{n\in\mathbb{N}} \frac{1}{2}\sum_{i=1}^n y_i^2.$$

The image of $\mathcal{C}^\alpha_\mathfrak{n}$ under Φ is measurable, which follows from the following identity:

$$\Phi(\mathcal{C}^\alpha_\mathfrak{n}) = \left\{(a_1, \dots, a_n)_{n\in\mathbb{N}} : \sup_{i\in\mathbb{N}_{-1}} \left\| \sum_{n\in\mathbb{N}} \rho_i\left(\frac{\mathfrak{s}(n)}{L}\right) a_n \mathfrak{d}'_n \right\|_\infty < \infty\right\}.$$

As $\mathbb{P}(\Phi(\sqrt{\varepsilon}\xi) \in \Phi(\mathcal{C}^\alpha_\mathfrak{n})) = 1$ and the domain on which I is finite is contained in $\Phi(\mathcal{C}^\alpha_\mathfrak{n})$, that is, $\{y \in \mathcal{Y} : I(y) < \infty\} \subset \Phi(\mathcal{C}^\alpha_\mathfrak{n})$, by [10], Theorem 4.1.5, $\Phi(\sqrt{\varepsilon}\xi)$ satisfies the large deviation principle on $\Phi(\mathcal{C}^\alpha_\mathfrak{n})$ with rate function I (restricted to $\Phi(\mathcal{C}^\alpha_\mathfrak{n})$).

Now, we apply the inverse contraction principle. $\Phi : \mathcal{C}^\alpha_\mathfrak{n} \to \Phi(\mathcal{C}^\alpha_\mathfrak{n})$ is a continuous bijection. Also, $I \circ \Phi(\psi) = \frac{1}{2}\|\psi\|^2_{L^2}$ (by Parseval's identity). Hence, the proof is finished by showing that $\sqrt{\varepsilon}\xi$ is exponentially tight in $\mathcal{C}^\alpha_\mathfrak{n}$. Let $m > 0$ and $K_m := \{\psi \in \mathcal{C}^\alpha_\mathfrak{n} : I \circ \Phi(\psi) \le m\}$. As L^2 is compactly embedded in $H^{\alpha+1}_\mathfrak{n}$ by Theorem 4.15, which is continuously embedded in $\mathcal{C}^\alpha_\mathfrak{n}$ (by [2], Theorem 2.71, K_m is relatively compact in $\mathcal{C}^\alpha_\mathfrak{n}$. By the large deviation principle of $\Phi(\sqrt{\varepsilon}\xi)$ on $\Phi(\mathcal{C}^\alpha_\mathfrak{n})$ and because $\overline{K_m}^c \subset K_m^c$, it follows that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}\big(\sqrt{\varepsilon}\xi \in \overline{K_m}^c\big) = \limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}\big(\Phi(\sqrt{\varepsilon}\xi) \in \{y \in \mathcal{Y} : I \le m\}^c\big) \le -m.$$

This proves the exponential tightness of $\sqrt{\varepsilon}\xi$ in $\mathcal{C}^\alpha_\mathfrak{n}$ which finishes the proof. □

To prove Theorem 9.1, we use Theorem 9.4 and the extension of the contraction principle:

THEOREM 9.5 ([10], Theorem 4.2.23). *Let $\mathcal{X}$ be a Hausdorff space and $(\mathcal{Y}, d)$ be a metric space. Suppose that $(\eta_\varepsilon)_{\varepsilon>0}$ are random variables with values in $\mathcal{X}$ that satisfy the large deviation principle with (rate ε and) rate function $I : \mathcal{X} \to [0, \infty]$. Furthermore, suppose that $F_\delta : \mathcal{X} \to \mathcal{Y}$ is a continuous map for all $\delta > 0$, $F : \mathcal{X} \to \mathcal{Y}$ is measurable and that, for all $q \in [0, \infty)$,*

$$\lim_{\delta\downarrow 0} \sup_{x\in\mathcal{X}: I(x)\le q} d\big(F_\delta(x), F(x)\big) = 0 \tag{72}$$

and that $F_\delta(\eta_\varepsilon)$ are exponential good approximations for $F(\eta_\varepsilon)$, that is, if for all $\kappa > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}\big(d\big(F_\delta(\eta_\varepsilon), F(\eta_\varepsilon)\big) > \kappa\big) = -\infty. \tag{73}$$

Then, $F(\eta_\varepsilon)$ satisfies the large deviation principle with rate function $\mathcal{Y} \to [0, \infty]$ given by

$$y \mapsto \inf_{x\in\mathcal{X}: F(x)=y} I(x).$$

LEMMA 9.6. *Let $\alpha \in (-\frac{4}{3}, -1)$. Let $\tau : \mathbb{R}^2 \to [0, 1]$ be a compactly supported function that equals* 1 *on a neighbourhood of* 0. *Write $h_\delta = \tau(\delta \mathrm{D})h$. There exists a $C > 0$ such that, for all $\delta > 0$ and $h \in L^2$,*

$$\|h_\delta \odot \sigma(\mathrm{D})h_\delta - h \odot \sigma(\mathrm{D})h\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}} \le C\delta^{-\alpha-1}\|h\|_{L^2}^2. \tag{74}$$

PROOF. This follows by Theorem 4.26 (note $2\alpha + 4 > 0$), Theorem 4.22 (also using $\|h_\delta\|_{H_\mathfrak{n}^{\alpha+1}} \lesssim \|h\|_{H_\mathfrak{n}^{\alpha+1}} \lesssim \|h\|_{L^2}$, see also 4.16) and Theorem 6.14,

$$\begin{aligned}
&\|h_\delta \odot \sigma(\mathrm{D})h_\delta - h \odot \sigma(\mathrm{D})h\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}} \\
&\quad \le \|(h - h_\delta) \odot \sigma(\mathrm{D})h_\delta\|_{H_\mathfrak{n}^{2\alpha+4}} + \|h \odot \sigma(\mathrm{D})(h_\delta - h)\|_{H_\mathfrak{n}^{2\alpha+4}} \\
&\quad \lesssim \|h - h_\delta\|_{H_\mathfrak{n}^{\alpha+1}}\|h\|_{H_\mathfrak{n}^{\alpha+1}} \lesssim \delta^{-\alpha-1}\|h\|_{L^2}^2.
\end{aligned}$$

□

PROOF OF THEOREM 9.1. For $\delta > 0$, we write $h_\delta = \tau(\delta \mathrm{D})h$ for τ as in 6.3 and define $F_\delta : \mathcal{C}_\mathfrak{n}^\alpha(Q_L) \to \mathfrak{X}_\mathfrak{n}^\alpha(Q_L)$ by

$$F_\delta(h) = (h, h_\delta \odot \sigma(\mathrm{D})h_\delta).$$

We define $F : \mathcal{C}_\mathfrak{n}^\alpha(Q_L) \to \mathfrak{X}_\mathfrak{n}^\alpha(Q_L)$ as follows. If for $h \in \mathcal{C}_\mathfrak{n}^\alpha(Q_L)$ the function $h_\delta \odot \sigma(\mathrm{D})h_\delta$ converges in $\mathcal{C}_\mathfrak{n}^{2\alpha+2}$, then $F(h) = \lim_{\delta\downarrow 0}(h, h_\delta \odot \sigma(\mathrm{D})h_\delta)$; if $h_\delta \odot \sigma(\mathrm{D})h_\delta$ does not converge, but $h_\delta \odot \sigma(\mathrm{D})h_\delta - c_\delta$ does (where $c_\delta = \frac{1}{2\pi}\log(\frac{1}{\delta}) + c_\tau$), then define $F(h) = \lim_{\delta\downarrow 0}(h, h_\delta \odot \sigma(\mathrm{D})h_\delta - c_\delta)$; whereas if $h_\delta \odot \sigma(\mathrm{D})h_\delta - c_\delta$ also does not converge, then $F(h) = 0$.

With $\mathcal{X} = \mathcal{C}_\mathfrak{n}^\alpha(Q_L)$ and $\mathcal{Y} = \mathfrak{X}_\mathfrak{n}^\alpha(Q_L)$ and $\eta_\varepsilon = \sqrt{\varepsilon}\xi$, by Theorem 9.4 and Theorem 9.5 it is sufficient to prove that (72) and (73) hold, because when $F(\phi) = (\psi_1, \psi_2) \neq 0$, then $\phi = \psi_1$:

- First, we check (72). By Lemma 9.6 we have ($F(h) = (h, h \odot \sigma(\mathrm{D})h)$ and)

$$\sup_{h \in \mathcal{C}_\mathfrak{n}^\alpha(Q_L):\|h\|_{L^2} \le q} \|F_\delta(h) - F(h)\|_{\mathfrak{X}_\mathfrak{n}^\alpha} \lesssim \delta^{-\alpha-1}q^2,$$

for all $q \ge 0$, that is, (72) holds.

- Now, we check (73). Let $\kappa > 0$. We have that $\Xi := \lim_{\delta\downarrow 0} \xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - c_\delta$ exists almost surely by Theorem 6.4. Hence, for $p > 1$,

$$\begin{aligned}
\mathbb{P}(\|F_\delta(\sqrt{\varepsilon}\xi) - F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}_\mathfrak{n}^\alpha} > \kappa) &\le \frac{\varepsilon^p}{\kappa^p}\mathbb{E}[\|\xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - \Xi\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}}^p] \\
&\le \frac{\varepsilon^p 2^p}{\kappa^p}(c_\delta^p + \mathbb{E}[\|\xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - c_\delta - \Xi\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}}^p]).
\end{aligned}$$

Let $\eta = -(2\alpha + 2)$. By Lemmas 6.10, 6.15, 6.16 and 11.11, there exists a $C > 0$ such that, for all $p > 1$,

$$\mathbb{E}[\|\xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - c_\delta - \Xi\|_{\mathcal{C}_\mathfrak{n}^{2\alpha+2}}^p] \le C^p p^p \delta^{\eta p}.$$

Therefore (using that $a^p + b^p \le (a + b)^p$),

$$\mathbb{P}(\|F_\delta(\sqrt{\varepsilon}\xi) - F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}_\mathfrak{n}^\alpha} > \kappa) \le \left[\frac{2\varepsilon}{\kappa}(c_\delta + Cp\delta^\eta)\right]^p.$$

Hence, with $p = \frac{1}{\varepsilon}$ we obtain

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}(\|F_\delta(\sqrt{\varepsilon}\xi) - F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}_\mathfrak{n}^\alpha} > \kappa) &\le \limsup_{\varepsilon\downarrow 0} \log\left[\frac{2}{\kappa}(\varepsilon c_\delta + C\delta^\eta)\right] \\
&\le \log\left(\frac{2C}{\kappa}\delta^\eta\right).
\end{aligned}$$

So that

$$\lim_{\delta\downarrow 0}\limsup_{\varepsilon\downarrow 0}\varepsilon\log\mathbb{P}\big(\|F_\delta(\sqrt{\varepsilon}\xi)-F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}_\mathfrak{n}^\alpha}>\kappa\big)=-\infty,$$

that is, (73) holds. □

10. Infima over the large deviation rate function. In this section we consider infima over sets of the rate function $I_{L,n}$, as in (71). We prove the results summarized in Theorem 2.6.

LEMMA 10.1. *For $a,b\in\mathbb{R}$ and all $\delta>0$,*

$$(1-\delta)\inf I_{L,n}[b,\infty)+\frac{1}{2}\Big(1-\frac{1}{\delta}\Big)L^2a^2\le\inf I_{L,n}[b+a,\infty)$$
$$\le(1+\delta)\inf I_{L,n}[b,\infty)+\frac{1}{2}\Big(1+\frac{1}{\delta}\Big)L^2a^2.$$

Consequently, for $(a_L)_{L>0}$ in $\mathbb{R}$ with $\lim_{L\to\infty}La_L=0$,

$$\lim_{L\to\infty}\inf I_{L,n}[b,\infty)=\lim_{L\to\infty}\inf I_{L,n}[b+a_L,\infty)$$
$$=\lim_{L\to\infty}\inf I_{L,n}(b+a_L,\infty)=\lim_{L\to\infty}\inf I_{L,n}(b,\infty).$$

PROOF. As $\lambda_n(Q_L,V)+a=\lambda_n(Q_L,V+a\mathbb{1}_{Q_L})$, $\|a\mathbb{1}_{Q_L}\|_{L^2}=aL$, and $2\langle V,a\mathbb{1}_{Q_L}\rangle\le\delta\|V\|_{L^2}^2+\frac{1}{\delta}a^2L^2$ for all $\delta>0$;

$$\inf I_{L,n}[b+a,\infty)=\inf_{\substack{V\in L^2(Q_L)\\ \lambda_n(Q_L,V)\ge b}}\frac{1}{2}\|V+a\mathbb{1}_{Q_L}\|_{L^2(Q_L)}^2$$
$$\le(1+\delta)\inf_{\substack{V\in L^2(Q_L)\\ \lambda_n(Q_L,V)\ge b}}\frac{1}{2}\|V\|_{L^2(Q_L)}^2+\frac{1}{2}\Big(1+\frac{1}{\delta}\Big)a^2L^2.$$

The lower bound can be proven similarly. □

We define

$$\mu_{L,n}:=\inf I_{L,n}[1,\infty),\qquad \varrho_n:=\inf_{L>0}\mu_{L,n}. \tag{75}$$

We prove that ϱ_n is bounded away from 0 uniformly in n (Lemma 10.4) and give an alternative variational formula for ϱ_n (Lemma 10.5) from which we conclude Theorem 2.6.

LEMMA 10.2. $\mu_{L,n}=\inf I_{L,n}(1,\infty)=\inf_{\substack{V\in C_c^\infty(Q_L)\\ \lambda_n(Q_L,V)\ge 1}}\frac{1}{2}\|V\|_{L^2}^2.$

PROOF. The first equality follows by Lemma 10.1. The second follows by Lemma 5.8. □

We will use Ladyzhenskaya's inequality [22] which is a special case of the Gagliardo–Nirenberg interpolation inequality [24].

LEMMA 10.3 (Ladyzhenskaya's inequality). *There exists a $C>0$ such that, for $f\in H^1(\mathbb{R}^2)$,*

$$\|f\|_{L^4}^4\le C\|\nabla f\|_{L^2}^2\|f\|_{L^2}^2. \tag{76}$$

LEMMA 10.4. *Let $C > 0$ be as in Lemma 10.3. Then, $\varrho_n \geq \frac{2}{C}$ for all $n \in \mathbb{N}$.*

PROOF. Let $n \in \mathbb{N}$. Let $L > 0$ and $\varepsilon > 0$. Let $V \in C_c^\infty(Q_L)$ be such that $\lambda_n(Q_L, V) \geq 1$ and $\frac{1}{2}\|V\|_{L^2}^2 \leq \mu_{L,n} + \varepsilon$. By (46) there is a $\psi \in C_c^\infty(Q_L)$ with $\|\psi\|_{L^2} = 1$ such that (by integration by parts)

$$1 - \varepsilon \leq -\|\nabla\psi\|_{L^2}^2 + \int V\psi^2 \leq -\|\nabla\psi\|_{L^2}^2 + \|V\|_{L^2}\|\psi\|_{L^4}^2.$$

Hence, by using Ladyzhenskaya's inequality (76), which implies $\|\nabla\psi\|_{L^2}^2 \geq \frac{1}{C}\|\psi\|_{L^4}^4$,

$$\|V\|_{L^2} \geq \frac{1 - \varepsilon + \|\nabla\psi\|_{L^2}^2}{\|\psi\|_{L^4}^2} \geq \frac{1-\varepsilon}{\|\psi\|_{L^4}^2} + \frac{1}{C}\|\psi\|_{L^4}^2$$

As $a^2 + b^2 \geq 2ab$, we have $\mu_{L,n} + \varepsilon \geq \frac{1}{2}\|V\|_{L^2}^2 \geq 2\frac{1-\varepsilon}{C}$. As this holds for all $\varepsilon > 0$, we conclude that $\mu_{L,n} \geq \frac{2}{C}$ for all $L > 0$. Hence, $\varrho_n \geq \frac{2}{C}$. $\square$

LEMMA 10.5. *For all $n \in \mathbb{N}$, $a > 0$,*

$$\inf_{L>0} \inf_{\substack{V \in C_c^\infty(Q_L) \\ \lambda_n(Q_L,V) \geq a}} \frac{1}{2}\|V\|_{L^2(Q_L)}^2 = \inf_{L>0} \inf_{\substack{V \in C_c^\infty(Q_L) \\ \|V\|_{L^2}^2 \leq \frac{1}{a}}} \frac{1}{2\lambda_n(Q_L, V)}. \tag{77}$$

Moreover, $\mu_{L,n}$ is decreasing in L, and one could replace "$\inf_{L>0}$" in (77) by "$\lim_{L\to\infty}$." In particular, $\varrho_n = \lim_{L\to\infty}\mu_{L,n}$.

PROOF. With $W = L^2V(L\cdot)$, we have $W \in C_c^\infty(Q_1)$, $\|W\|_{L^2(Q_1)}^2 = L^2\|V\|_{L^2(Q_L)}^2$ and by Theorem 7.1 $\lambda_n(Q_L, V) = \lambda_n(Q_L, \frac{1}{L^2}W(\frac{1}{L}\cdot)) = \frac{1}{L^2}\lambda_n(Q_1, W)$. Therefore,

$$\inf_{\substack{V \in C_c^\infty(Q_L) \\ \lambda_n(Q_L,V) \geq a}} \frac{1}{2}\|V\|_{L^2(Q_L)}^2 = \inf_{\substack{W \in C_c^\infty(Q_1) \\ \lambda_n(Q_1,W) \geq aL^2}} \frac{1}{2}\frac{1}{L^2}\|W\|_{L^2(Q_1)}^2, \tag{78}$$

$$\inf_{\substack{V \in C_c^\infty(Q_L) \\ \|V\|_{L^2}^2 \leq \frac{1}{a}}} \frac{1}{2\lambda_n(Q_L, V)} = \inf_{\substack{W \in C_c^\infty(Q_1) \\ \|W\|_{L^2}^2 \leq \frac{L^2}{a}}} \frac{L^2}{2\lambda_n(Q_1, W)}. \tag{79}$$

With this, (77) follows directly from Lemma 10.6. That $\mu_{L,n}$ and the left-hand side of (79) are decreasing in L follows from Lemma 8.1. $\square$

LEMMA 10.6. *Let $\mathcal{Y}$ be a topological space and $f, g : \mathcal{Y} \to \mathbb{R}$ be continuous functions. Let $a > 0$, and suppose that $\varrho := \inf_{L>0}\inf_{w\in\mathcal{Y}: f(w)\geq aL} \frac{g(w)}{L} > 0$. Then,*

$$\inf_{L>0} \inf_{\substack{w\in\mathcal{Y} \\ f(w)\geq aL}} \frac{g(w)}{L} = \inf_{L>0} \inf_{\substack{w\in\mathcal{Y} \\ g(w)\leq \frac{L}{a}}} \frac{L}{f(w)}.$$

PROOF. By definition, we have $\forall L > 0\ \forall w \in \mathcal{Y} : \frac{1}{L}g(w) < \varrho \Longrightarrow f(w) < aL$; by continuity of f and g, we obtain (by taking $K = L\varrho a$)

$$\forall K > 0\ \forall w \in \mathcal{Y} : \quad g(w) \leq \frac{K}{a} \quad \Longrightarrow \quad \frac{f(w)}{K} \leq \frac{1}{\varrho}.$$

Let $\varepsilon > 0$. Then, there exists an $L > 0$ and $w_L \in \mathcal{Y}$ such that $f(w_L) \geq aL$ and $\frac{1}{L} g(w_L) \leq \varrho + \varepsilon$. Then, with $K = La(\varrho + \varepsilon)$, we have for $w = w_L$ that $\frac{g(w)}{K} \leq \frac{1}{a}$ and $\frac{f(w)}{K} \geq \frac{1}{\varrho+\varepsilon}$. So that $\sup_{K>0} \sup_{\substack{w \in \mathcal{Y} \\ g(w) \leq \frac{K}{a}}} \frac{f(w)}{K} = \frac{1}{\varrho}$. □

PROOF OF THEOREM 2.6. By (46) and Lemma 10.5 (for $a = 1$), we have

$$\frac{2}{\varrho_n} = 4 \sup_{L>0} \sup_{\substack{V \in C_c^\infty(Q_L) \\ \|V\|_{L^2}^2 \leq 1}} \sup_{\substack{F \sqsubset C_c^\infty(Q_L) \\ \dim F = n}} \inf_{\substack{\psi \in F: \\ \|\psi\|_{L^2}^2 = 1}} \int_{Q_L} -|\nabla \psi|^2 + V\psi^2,$$

from which (1) follows. By Cauchy–Schwarz, for $\psi \in C_c^\infty(\mathbb{R}^2)$, the supremum of $\int V\psi^2$ with respect to $V \in C_c^\infty(\mathbb{R}^2)$ with L^2 norm equal to 1 is attained at $V = \frac{\psi^2}{\|\psi^2\|_{L^2}}$; therefore, this supremum equals $\|\psi\|_{L^4}^2$, and, hence, we derive the first equality in (2). In Lemma 10.4 we have already seen that $\frac{2}{\rho_1} \leq \chi$. For the other inequality we refer to [6], Theorem C.1 (basically the trick is to replace "ψ" by "$\lambda f(\lambda \cdot)$" and optimise over $\lambda > 0$ first, then over $f \in L^2$ with $\|f\|_{L^2} = 1$). □

11. Convergence of Gaussians. In this section we prove the convergence of Gaussians mentioned in Section 6 and Section 8. We bundle the proofs together in a general setting, as they rely on similar techniques.

For $r \geq 1$, we let $X_{k,r}^\varepsilon$ and $Y_{k,r}^\varepsilon$ be centered Gaussian variables for $k \in \mathbb{N}_0^d$, $\varepsilon > 0$ such that every finite subset of $\{Y_{k,r}^\varepsilon : k \in \mathbb{N}_0^d, \varepsilon > 0\} \cup \{X_{k,r}^\varepsilon : k \in \mathbb{N}_0^d, \varepsilon > 0\}$ is jointly Gaussian for all $r \geq 1$. We write

(80) $$\xi_{r,\varepsilon} = \sum_{k \in \mathbb{N}_0^d} Y_{k,r}^\varepsilon \mathfrak{n}_{k,r}, \qquad \theta_{r,\varepsilon} = \sum_{k \in \mathbb{N}_0^d} X_{k,r}^\varepsilon \mathfrak{n}_{k,r}.$$

Also, we introduce the notation

$$\rho^\odot : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \qquad \rho^\odot(x,y) = \sum_{\substack{i,j \in \mathbb{N}_{-1} \\ |i-j| \leq 1}} \rho_i(x)\rho_j(y),$$

$$\Theta_{r,\varepsilon} = \theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon} - \mathbb{E}[\theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon}],$$

$$\Xi_{r,\varepsilon} = \xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon} - \mathbb{E}[\xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon}].$$

LEMMA 11.1. *Let $d = 2$. Write $F_{r,\varepsilon}(k,l) = \mathbb{E}[X_{k,r}^\varepsilon X_{l,r}^\varepsilon]$. Let $I \subset [1,\infty)$. Suppose that*

$$\forall \delta > 0\ \exists C > 0\ \forall r \in I\ \forall k,l \in \mathbb{N}_0^2\ \forall \varepsilon > 0 :$$

(81) $$|F_{r,\varepsilon}(k,l)| \leq C \prod_{i=1}^d (1 + |k_i - l_i|)^{\delta - 1}.$$

For all $\gamma \in (0,1)$, there exists a $\mathfrak{C} > 0$ such that, for all $r \in I$, $i \in \mathbb{N}_{-1}$, $\varepsilon > 0$, $x \in Q_r$,

(82) $$\mathbb{E}[|\Delta_i \theta_{r,\varepsilon}|(x)^2] \leq \mathfrak{C} r^{2\gamma} 2^{(2+\gamma)i}, \qquad \mathbb{E}[|\Delta_i \Theta_{r,\varepsilon}|(x)^2] \leq \mathfrak{C} r^{2\gamma} 2^{\gamma i}.$$

PROOF. This follows from Lemma 11.5 and Lemma 11.12. □

Observe that

$$\theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon} - \xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon} = \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^{\odot}(\frac{k}{r},\frac{l}{r})}{1+\frac{\pi^2}{r^2}|l|^2} \mathfrak{n}_{k,r}\mathfrak{n}_{l,r}\big[X_{k,r}^{\varepsilon}X_{l,r}^{\varepsilon} - Y_{k,r}^{\varepsilon}Y_{l,r}^{\varepsilon}\big]. \tag{83}$$

THEOREM 11.2. *Let $d=2$, $I\subset[1,\infty)$. We write $\mathfrak{R}=\{(k,l)\in\mathbb{N}_0^2\times\mathbb{N}_0^2 : k_1\neq l_1, k_2\neq l_2\}$. Let $G_{r,\varepsilon}(k,l)=\mathbb{E}[X_{k,r}^{\varepsilon}X_{l,r}^{\varepsilon} - Y_{k,r}^{\varepsilon}Y_{l,r}^{\varepsilon}]$. Consider the following conditions*:

$$\forall k\in\mathbb{N}_0^2\ \forall r\in I;\quad \mathbb{E}\big[\big|X_{k,r}^{\varepsilon} - Y_{k,r}^{\varepsilon}\big|^2\big] \xrightarrow{\varepsilon\downarrow 0} 0, \tag{84}$$

$$\forall r\in I\ \forall\delta>0\ \exists C>0\ \exists\varepsilon_0>0\ \forall\varepsilon\in(0,\varepsilon_0)\ \forall k,l\in\mathbb{N}_0^2:$$

$$\big|G_{r,\varepsilon}(k,l)\big| \le C\begin{cases}\displaystyle\prod_{i=1}^{2}\frac{1}{1+|k_i-\frac{r}{\varepsilon}|)^{1-\delta}} + \frac{1}{1+|l_i-\frac{r}{\varepsilon}|)^{1-\delta}}, & (k,l)\in\mathfrak{R},\\ \displaystyle\sum_{i=1}^{2}\frac{1}{1+|k_i-\frac{r}{\varepsilon}|)^{1-\delta}} + \frac{1}{1+|l_i-\frac{r}{\varepsilon}|)^{1-\delta}}, & (k,l)\in\mathbb{N}_0^2\times\mathbb{N}_0^2\setminus\mathfrak{R}.\end{cases} \tag{85}$$

(a) *Suppose that* (84) *holds and that* (81) *holds for $F_{r,\varepsilon}(k,l)$ being either $\mathbb{E}[X_{k,r}^{\varepsilon}X_{l,r}^{\varepsilon}]$, $\mathbb{E}[X_{k,r}^{\varepsilon}Y_{l,r}^{\varepsilon}]$ or $\mathbb{E}[Y_{k,r}^{\varepsilon}Y_{l,r}^{\varepsilon}]$. Then, for $r\in I$, $\alpha<-1$, in $\mathfrak{X}_n^{\alpha}$ we have*

$$(\theta_{r,\varepsilon}-\xi_{r,\varepsilon},\Theta_{r,\varepsilon}-\Xi_{r,\varepsilon}) \xrightarrow{\mathbb{P}} 0.$$

(b) *Suppose* (85) *holds. Then, $\mathbb{E}[\theta_{r,\varepsilon}\odot\sigma(\mathrm{D})\theta_{r,\varepsilon} - \xi_{r,\varepsilon}\odot\sigma(\mathrm{D})\xi_{r,\varepsilon}]\to 0$ in $\mathcal{C}_{\mathfrak{n}}^{-\gamma}$ for all $\gamma>0$ and $r\in I$.*

Consequently, if the above assumptions in (a) *and* (b) *hold, then with $c=0$, for $r\in I$, $\alpha<-1$, in $\mathfrak{X}_n^{\alpha}$*

$$\big(\theta_{r,\varepsilon}-\xi_{r,\varepsilon},\theta_{r,\varepsilon}\odot\sigma(\mathrm{D})\theta_{r,\varepsilon}-\xi_{r,\varepsilon}\odot\sigma(\mathrm{D})\xi_{r,\varepsilon}\big) \xrightarrow{\mathbb{P}} (0,c). \tag{86}$$

PROOF. (a) We use Lemma 6.10(b). By Lemma 11.1 we obtain (53) for $\zeta=\theta_{r,\varepsilon}-\xi_{r,\varepsilon}$ with $a=2+\gamma$ and for $\zeta=\Theta_{r,\varepsilon}-\Xi_{r,\varepsilon}$ with $a=2\gamma$ for $\gamma\in(0,1)$. (84) implies that $\mathbb{E}[|\langle\theta_{r,\varepsilon}-\xi_{r,\varepsilon},\mathfrak{n}_{k,r}\rangle|^2]\to 0$, that is, (55) holds for $\zeta_\varepsilon=\theta_{r,\varepsilon}-\xi_{r,\varepsilon}$. In Lemma 11.13 we show that (55) holds for $\zeta_\varepsilon=\Theta_{r,\varepsilon}-\Xi_{r,\varepsilon}$.
(b) is shown in Lemma 11.14. □

THEOREM 11.3. *Let $\tau\in C_c^{\infty}(\mathbb{R}^2,[0,1])$ and $\tau':\mathbb{R}^2\to[0,1]$ be compactly supported functions. Suppose τ and τ' are equal to* 1 *on a neighbourhood of* 0:

(a) *For all $r\ge 1$* (86) *holds with $c=c_{\tau'}-c_{\tau}$ in case $X_{k,r}^{\varepsilon}=\tau'(\frac{\varepsilon}{r}k)Z_k$ and $Y_{k,r}=\tau(\frac{\varepsilon}{r}k)Z_k$.*
(b) *Let $L>r\ge 1$ and $y\in\mathbb{R}^2$ be such that $y+Q_r\subset Q_L$. With $\mathcal{W}$ as in* 6.2, *for*

$$\mathcal{Z}_m=\langle\mathcal{W},\mathfrak{n}_{m,L}\rangle,\qquad Z_k=\langle\mathcal{W},\mathcal{T}_y\mathfrak{n}_{k,r}\rangle=\sum_{m\in\mathbb{N}_0^2}\mathcal{Z}_m\langle\mathfrak{n}_{m,L},\mathcal{T}_y\mathfrak{n}_{k,r}\rangle_{L^2(Q_r)},$$

$$X_{k,r}^{\varepsilon}=\sum_{m\in\mathbb{N}_0^2}\mathbb{1}_{(-1,1)^2}\bigg(\frac{\varepsilon}{L}m\bigg)\mathcal{Z}_m\langle\mathfrak{n}_{m,L},\mathcal{T}_y\mathfrak{n}_{k,r}\rangle_{L^2(Q_r)},\qquad Y_{k,r}^{\varepsilon}=\mathbb{1}_{(-1,1)^2}\bigg(\frac{\varepsilon}{r}k\bigg)Z_k.$$

(86) *holds with $c=0$.*

PROOF. (a) That (84) holds is clear. As $|\mathbb{E}[X^\varepsilon_{k,r}X^\varepsilon_{l,r}]| \vee |\mathbb{E}[X^\varepsilon_{k,r}Y^\varepsilon_{l,r}]| \vee |\mathbb{E}[Y^\varepsilon_{k,r}Y^\varepsilon_{l,r}]| \leq 2\delta_{k,l}$, (81) also holds for each of those expectations, and, thus, the conditions of Theorem 11.2(a) hold. Therefore, it is sufficient to show that $\mathbb{E}[\theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon} - \xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon}] \xrightarrow{\mathbb{P}} c_{\tau'} - c_\tau$ in $\mathcal{C}_{\mathfrak{n}}^{-\gamma}$ for all $\gamma > 0$. This follows by Lemma 6.15 and Lemma 6.16, as they show that $\mathbb{E}[\theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon}] - c_\varepsilon - c_{\tau'}$ and $\mathbb{E}[\xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon}] - c_\varepsilon - c_\tau$ converge to the same limit in $\mathcal{C}_{\mathfrak{n}}^{-\gamma}$.

(b) We prove this in Theorem 12.1. □

11.1. *Terms in the first Wiener chaos.*

LEMMA 11.4. *Consider the setting of 6.3, that is, $Y^\varepsilon_{k,r} = \tau(\frac{\varepsilon}{r}k)Z_k$ for i.i.d. standard normal random variables $(Z_k)_{k\in\mathbb{N}_0^d}$ and $\tau \in C_c^\infty(\mathbb{R}^2, [0,1])$. For all $\gamma \in (0,1)$, there exists a $C > 0$ such that, for all $r \geq 1$, $i \in \mathbb{N}_{-1}$, $\varepsilon, \delta > 0$, $x \in Q_r$,*

$$\mathbb{E}\big[\big|\Delta_i(\xi_{r,\varepsilon} - \xi_{r,\delta})(x)\big|^2\big] \leq C2^{(d+2\gamma)i}|\varepsilon - \delta|^\gamma. \tag{87}$$

PROOF. Let $\gamma \in (0,1)$. As $\Delta_i(\xi_{r,\varepsilon} - \xi_{r,\delta})(x) = \sum_{k\in\mathbb{N}_0^2} \rho_i(\frac{k}{r})(\tau(\varepsilon\frac{k}{r}) - \tau(\delta\frac{k}{r}))Z_k\mathfrak{n}_{k,r}(x)$, and $\|\mathfrak{n}_{k,r}\|_\infty^2 \leq (\frac{2}{r})^d$, by (60) we have

$$\mathbb{E}\big[\big|\Delta_i(\xi_{r,\varepsilon} - \xi_{r,\delta})(x)\big|^2\big] \lesssim r^{-d}2^{(d+2\gamma)i} \sum_{k\in\frac{1}{r}\mathbb{N}_0^d} \frac{(\tau(\varepsilon k) - \tau(\delta k))^2}{(1+|k|)^{d+2\gamma}}.$$

As $|\tau(\varepsilon k) - \tau(\delta k)| \leq \|\nabla\tau\|_\infty|\varepsilon - \delta||k|$ and $\|\tau\|_\infty = 1$,

$$\big(\tau(\varepsilon k) - \tau(\delta k)\big)^2 \lesssim \|\nabla\tau\|_\infty^\gamma|\varepsilon - \delta|^\gamma|k|^\gamma. \tag{88}$$

Therefore, as $\sum_{k\in\frac{1}{r}\mathbb{Z}^d} r^{-d}\frac{|k|^\gamma}{(1+|k|)^{d+2\gamma}} < \infty$, we obtain (87). □

LEMMA 11.5. *Suppose that (81) holds for $F_{r,\varepsilon}(k,l) = \mathbb{E}[X^\varepsilon_{k,r}X^\varepsilon_{l,r}]$. For all $\gamma \in (0,1)$, there exists a $C > 0$ (independent of r) such that, for all $i \in \mathbb{N}_{-1}$, $\varepsilon > 0$, $x \in Q_r$*

$$\mathbb{E}\big[\big|\Delta_i\theta_{r,\varepsilon}(x)\big|^2\big] \leq Cr^{d\gamma}2^{(d+\gamma)i}. \tag{89}$$

PROOF. By (60) $2^{-\beta i}\|\Delta_i\mathfrak{n}_{k,r}\|_{L^\infty} \lesssim r^{-\frac{d}{2}}(1 + |\frac{k}{r}|)^{-\beta} \leq r^{-\frac{d}{2}}\prod_{i=1}^d(\frac{1}{r} + \frac{k_i}{r})^{-\frac{\beta}{d}}$. Let $\delta > 0$ be such that $\delta < \gamma$ (so that in particular $\delta < \frac{1+\gamma}{2}$). As $|\mathbb{E}[X^\varepsilon_{k,r}X^\varepsilon_{l,r}]| \lesssim \prod_{i=1}^d(1 + |k_i - l_i|)^{\delta-1} = \prod_{i=1}^d r^{\delta-1}(\frac{1}{r} + |\frac{k_i}{r} - \frac{l_i}{r}|)^{\delta-1}$, we have, by using Lemma 11.7,

$$\begin{aligned}
&2^{-(d+d\gamma)i}\mathbb{E}\big[\big|\Delta_i\theta_{r,\varepsilon}(x)\big|^2_{L^\infty}\big] \\
&\quad\lesssim \Bigg(\sum_{k,l\in\frac{1}{r}\mathbb{N}_0} \frac{1}{(\frac{1}{r}+k)^{\frac{1+\gamma}{2}}} \frac{1}{(\frac{1}{r}+l)^{\frac{1+\gamma}{2}}} \frac{r^{\delta-2}}{(\frac{1}{r}+|k-l|)^{1-\delta}}\Bigg)^d \\
&\quad\lesssim \Bigg(\sum_{k\in\frac{1}{r}\mathbb{N}_0} \frac{r^{\delta-1}}{(\frac{1}{r}+k)^{\frac{1+\gamma}{2}}} \frac{1}{(\frac{1}{r}+k)^{\frac{1+\gamma}{2}-\delta}}\Bigg)^d \lesssim \Bigg(r^\delta\Big(\frac{1}{r}\Big)^{\delta-\gamma}\Bigg)^d \lesssim r^{d\gamma}.
\end{aligned}$$

□

In the following two lemmas we present tools to bound sums by integrals which will be frequently used.

LEMMA 11.6. *Let $M \in \mathbb{N}$ and $f : [0, M] \to \mathbb{R}$ be a decreasing measurable function. Then, $\sum_{m=1}^{M} f(m) \le \int_0^M f(x)\,\mathrm{d}x \le \sum_{m=0}^{M-1} f(m)$. If f instead is increasing, then $\sum_{m=0}^{M-1} f(m) \le \int_0^M f(x)\,\mathrm{d}x \le \sum_{m=1}^{M} f(m)$.*

LEMMA 11.7. *Let $\gamma, \delta > 0$ be such that $\delta < \gamma < 1$. There exists a $C > 0$ such that, for all $r \ge 1$, $b > 0$ and $u, v \in \mathbb{R}$,*

$$\sum_{k\in\mathbb{N}_0} \frac{1}{r} \frac{1}{(b + |\frac{k}{r} - u|)^{\gamma}} \frac{1}{(b + |\frac{k}{r} - v|)^{1-\delta}} \le C\big(b + |u - v|\big)^{\delta-\gamma}, \tag{90}$$

and for all $l \in \mathbb{R}^2$,

$$\sum_{k\in\frac{1}{r}\mathbb{N}_0^2} \frac{1}{r^2} \frac{\rho^{\odot}(k, l)}{(1 + |k - l|)^{\gamma}} \le C\big(1 + |l|\big)^{2-\gamma}. \tag{91}$$

PROOF. We can bound both sums by "their corresponding integral" by observing the following. For $k \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$ with $|x - \frac{k}{r}|_\infty < \frac{1}{2r}$ and thus $|x - \frac{k}{r}| \le \frac{\sqrt{d}}{2r}$, for $u \in \mathbb{R}^d$,

$$|x - u| \le \left|\frac{k}{r} - u\right| + \left|x - \frac{k}{r}\right| \le \left|\frac{k}{r} - u\right| + \frac{\sqrt{d}}{2r}. \tag{92}$$

So that

$$\frac{1}{(b + |\frac{k}{r} - u|)^{\gamma}} \le \frac{(b + \frac{\sqrt{d}}{2r})^{\gamma}}{(b + \frac{\sqrt{d}}{2r} + |\frac{k}{r} - u|)^{\gamma}} \le \frac{(b + \frac{\sqrt{d}}{2})^{\gamma}}{(b + |x - u|)^{\gamma}}.$$

Then, (90) follows by Lemma B.1 and by Lemma 11.9 we have $\sum_{k\in\frac{1}{r}\mathbb{N}_0^2} \frac{1}{r^2} \frac{\rho^{\odot}(k,l)}{(1+|k-l|)^{\gamma}} \lesssim 1 + 2\pi \int_{\frac{1}{c}|l|}^{c|l|} \frac{x}{(1+|x-|l||)^{\gamma}}\,\mathrm{d}x \lesssim (1 + |l|)^{2-\gamma}$. □

11.2. *Terms in the second Wiener chaos*. In order to bound terms in the second Wiener chaos, that is, $\Xi_{r,\varepsilon}$, $\Theta_{r,\varepsilon}$ and $\mathbb{E}[\theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon} - \xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon}]$, we start by presenting auxiliary lemma's and observations.

THEOREM 11.8 (Wick's theorem, [18], Theorem 1.28). *Let A, B, C, D be jointly Gaussian random variables. Then,*

$$\mathbb{E}[ABCD] = \mathbb{E}[AB]\mathbb{E}[CD] + \mathbb{E}[AC]\mathbb{E}[BD] + \mathbb{E}[AD]\mathbb{E}[BC].$$

LEMMA 11.9. *There exist $b > 0$ and $c > 1$ such that*

$$\operatorname{supp} \rho^{\odot} \subset B(0, b)^2 \cup \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \frac{1}{c}|x| \le |y| \le c|x|\right\}$$

Consequently, uniformly in $x, y \in \mathbb{R}^d$

$$\frac{\rho^{\odot}(x, y)}{(1 + |x|^2)} \approx \frac{\rho^{\odot}(x, y)}{(1 + |y|^2)}. \tag{93}$$

PROOF. Let $0 < a < b$ be such that $\operatorname{supp} \rho_0 \subset \{x \in \mathbb{R}^d : a \le |x| \le b\}$ and $\operatorname{supp} \rho_{-1} \subset B(0, b)$. Let $i, j \in \mathbb{N}_{-1}$ and $x, y \in \mathbb{R}^2$ be such that $\rho_i(x)\rho_j(y) \ne 0$. If $i, j \in \{-1, 0\}$, then $x, y \in B(0, b)$. Suppose $i, j \ge 0$ and $|i - j| \le 1$. Then, $|x| \in [2^i a, 2^i b]$ and $|y| \in [2^j a, 2^j b] \subset [2^{i-1}a, 2^{i+1}b]$. This in turn implies

$$\frac{a}{2b}|x| \le \frac{a}{2b}2^i b = 2^{i-1}a \le |y| \le 2^{i+1}b \le \frac{2b}{a}2^i a \le \frac{2b}{a}|x|.$$

□

11.10. Let $k, l, z \in \mathbb{N}_0^d$. We write $\mathfrak{n}_k = \mathfrak{n}_{k,r}$ here. By (38) (and using (26)) and as $\mathfrak{n}_{\mathfrak{q}\circ k} = \mathfrak{n}_k$ for all $\mathfrak{q} \in \{-1, 1\}^d$,

$$\begin{aligned} \langle \mathfrak{n}_k \mathfrak{n}_l, \mathfrak{n}_z \rangle_{L^2(Q_r)} &= (2r)^{-\frac{d}{2}} \sum_{\mathfrak{p} \in \{-1,1\}^d} \frac{\nu_k \nu_l}{\nu_{k+\mathfrak{p}\circ l}} \langle \mathfrak{n}_{k+\mathfrak{p}\circ l}, \mathfrak{n}_z \rangle_{L^2(Q_r)} \\ &= (2r)^{-\frac{d}{2}} \sum_{\mathfrak{p},\mathfrak{q} \in \{-1,1\}^d} \frac{\nu_k \nu_l}{\nu_{k+\mathfrak{q}\circ\mathfrak{p}\circ l}} \delta_{\mathfrak{q}\circ k+\mathfrak{p}\circ l, z}. \end{aligned} \tag{94}$$

By combining this with (60), using that $|\mathfrak{n}_k(x)| \le (\frac{2}{r})^{-\frac{d}{2}}$, we have, for $x \in (0, r)^d$ and $\gamma > 0$,

$$r^d |\Delta_i(\mathfrak{n}_k \mathfrak{n}_l)(x)| \lesssim \sum_{\mathfrak{p},\mathfrak{q} \in \{-1,1\}^d} \rho_i\left(\frac{\mathfrak{q}\circ k + \mathfrak{p}\circ l}{r}\right) \frac{2^{\gamma i}}{(1 + |\frac{k}{r} - \frac{l}{r}|)^\gamma}. \tag{95}$$

LEMMA 11.11. *Let $d = 2$. Consider the setting of 6.3, as we did in Lemma 11.4. For all $\gamma \in (0, 1)$, there exists a $C > 0$ (independent of r) such that, for all $i \in \mathbb{N}_{-1}$, $\varepsilon, \delta > 0$, $x \in Q_r$,*

$$\mathbb{E}[|\Delta_i(\Xi_{r,\varepsilon} - \Xi_{r,\delta})(x)|^2] \le C|\varepsilon - \delta|^\gamma 2^{2\gamma i}. \tag{96}$$

PROOF. First, observe $\Xi_{r,\varepsilon} = \sum_{k,l \in \mathbb{N}_0^2} \rho^{\odot}(\frac{k}{r}, \frac{l}{r}) \frac{\tau(\varepsilon\frac{k}{r})\tau(\varepsilon\frac{l}{r})}{1+\frac{\pi^2}{r^2}|l|^2} [Z_k Z_l - \delta_{k,l}]\mathfrak{n}_k \mathfrak{n}_l$. By Theorem 11.8 and (95) (as both contributions $\delta_{k,m}\delta_{l,n}$ and $\delta_{k,n}\delta_{m,l}$ can be bounded by the same expression by Lemma 11.9),

$$\begin{aligned} &2^{-2\gamma i}\mathbb{E}[|\Delta_i(\Xi_{r,\varepsilon} - \Xi_{r,\delta})(x)|] \\ &\quad \lesssim \sum_{k,l \in \frac{1}{r}\mathbb{N}_0^2} \frac{1}{r^4} \frac{\rho^{\odot}(k,l)^2}{(1 + \pi^2|l|^2)^2} \frac{[\tau(\varepsilon k)\tau(\varepsilon l) - \tau(\delta k)\tau(\delta l)]^2}{(1 + |k - l|)^{2\gamma}}. \end{aligned}$$

As $2(ab - cd) = (a - c)(b + d) + (a + c)(b - d)$ similar to (88), as in the proof of Lemma 11.4, we obtain

$$|\tau(\varepsilon k)\tau(\varepsilon l) - \tau(\delta k)\tau(\delta l)|^2 \le 4\|\nabla\tau\|_\infty^\gamma |\varepsilon - \delta|^\gamma (|k|^\gamma + |l|^\gamma).$$

Using Lemma 11.9 and (91), we obtain

$$\begin{aligned} 2^{-2\gamma i}\mathbb{E}[\|\Delta_i(\Xi_{r,\varepsilon} - \Xi_{r,\delta})\|_{L^\infty}^2] &\lesssim |\varepsilon - \delta|^\gamma \sum_{l \in \frac{1}{r}\mathbb{N}_0^2} \frac{r^{-4}}{(1 + |l|)^{4-\gamma}} \sum_{k \in \frac{1}{r}\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)^2}{(1 + |k - l|)^{2\gamma}} \\ &\lesssim |\varepsilon - \delta|^\gamma \sum_{l \in \frac{1}{r}\mathbb{N}_0^2} \frac{r^{-2}}{(1 + |l|)^{2+\gamma}}. \end{aligned}$$

□

LEMMA 11.12. *Suppose that (81) holds for $F_{r,\varepsilon}(k, l) = \mathbb{E}[X_{k,r}^\varepsilon X_{l,r}^\varepsilon]$. For all $\gamma \in (0, \infty)$, there exists a $C > 0$ (independent of r) such that, for all $i \in \mathbb{N}_{-1}$, $\varepsilon > 0$,*

$$\mathbb{E}[|\Delta_i \Theta_{r,\varepsilon}(x)|^2] \le Cr^{2\gamma} 2^{\gamma i}. \tag{97}$$

PROOF. First note that $\Theta_{r,\varepsilon} = \sum_{k,l \in \mathbb{N}_0^2} \frac{\rho^{\odot}(\frac{k}{r}, \frac{l}{r})}{1+\pi^2|\frac{l}{r}|^2} \mathfrak{n}_{k,r}\mathfrak{n}_{l,r}[X_{k,r}^\varepsilon X_{l,r}^\varepsilon - \mathbb{E}[X_{k,r}^\varepsilon X_{l,r}^\varepsilon]]$. By Theorem 11.8,

$$\begin{aligned} &\mathbb{E}([X_{k,r}^\varepsilon X_{l,r}^\varepsilon - \mathbb{E}[X_{k,r}^\varepsilon X_{l,r}^\varepsilon]][X_{m,r}^\varepsilon X_{n,r}^\varepsilon - \mathbb{E}[X_{m,r}^\varepsilon X_{n,r}^\varepsilon]]) \\ &\quad = \mathbb{E}[X_{k,r}^\varepsilon X_{m,r}^\varepsilon]\mathbb{E}[X_{l,r}^\varepsilon X_{n,r}^\varepsilon] + \mathbb{E}[X_{k,r}^\varepsilon X_{n,r}^\varepsilon]\mathbb{E}[X_{l,r}^\varepsilon X_{m,r}^\varepsilon]. \end{aligned}$$

By exploiting symmetries using Lemma 11.9 and by (95), we have

$$2^{-2\gamma i}\mathbb{E}\big[\big|\Delta_i\Theta_{r,\varepsilon}(x)\big|^2\big] \lesssim \sum_{k,l,m,n\in\frac{1}{r}\mathbb{N}_0^2} \frac{r^{-4}\rho^{\odot}(k,l)\rho^{\odot}(m,n)|\mathbb{E}[X^{\varepsilon}_{rk}X^{\varepsilon}_{rm}]\mathbb{E}[X^{\varepsilon}_{rl}X^{\varepsilon}_{rn}]|}{(1+|k-l|)^{\gamma}(1+|m-n|)^{\gamma}(1+|l|^2)(1+|m|^2)}.$$

We will bound the $\rho^{\odot}$ function by 1, use the bound (81) for some $\delta>0$ (will be chosen small enough later) and we "separate the dimensions" by using that $1+|k|^2 \gtrsim (1+k_1)(1+k_2)$ and $(1+|k-l|)^{\gamma} \gtrsim (\frac{1}{r}+|k_1-l_1|)^{\frac{\gamma}{2}}(\frac{1}{r}+|k_2-l_2|)^{\frac{\gamma}{2}}$ and obtain

$$\begin{aligned} &2^{-2\gamma i}\mathbb{E}\big[\big|\Delta_i\Theta_{r,\varepsilon}(x)\big|^2\big] \\ &\quad\lesssim \Big(\sum_{k,l,m,n\in\frac{1}{r}\mathbb{N}_0} \frac{r^{2\delta-4}(\frac{1}{r}+|k-m|)^{\delta-1}(\frac{1}{r}+|l-n|)^{\delta-1}}{(\frac{1}{r}+|k-l|)^{\frac{\gamma}{2}}(\frac{1}{r}+|m-n|)^{\frac{\gamma}{2}}(\frac{1}{r}+l)(\frac{1}{r}+m)}\Big)^2. \end{aligned} \tag{98}$$

For $\delta<\frac{\gamma}{2,}$ we have, by Lemma 11.7,

$$\sum_{n\in\frac{1}{r}\mathbb{N}_0} \frac{r^{-1}(\frac{1}{r}+|l-n|)^{\delta-1}}{(\frac{1}{r}+|m-n|)^{\frac{\gamma}{2}}} \vee \sum_{k\in\frac{1}{r}\mathbb{N}_0} \frac{r^{-1}(\frac{1}{r}+|k-m|)^{\delta-1}}{(\frac{1}{r}+|k-l|)^{\frac{\gamma}{2}}} \lesssim \frac{1}{(\frac{1}{r}+|m-l|)^{\frac{\gamma}{2}-\delta}},$$

and for $\delta<\frac{\gamma}{4}$, the square root of the right-hand side of (98) can be bounded by

$$\sum_{m,l\in\frac{1}{r}\mathbb{N}_0} \frac{r^{2\delta-2}}{(\frac{1}{r}+|m-l|)^{\gamma-2\delta}}\frac{1}{\frac{1}{r}+m}\frac{1}{\frac{1}{r}+l} \lesssim \sum_{l\in\frac{1}{r}\mathbb{N}_0}\frac{r^{2\delta-1}}{(\frac{1}{r}+l)^{\gamma-3\delta}}\frac{1}{\frac{1}{r}+l} \lesssim r^{\gamma-2\delta} \lesssim r^{\gamma}.$$

Hence, we obtain (97). □

LEMMA 11.13. *Suppose that* (84) *holds and that* (81) *holds for* $F_{r,\varepsilon}(k,l)$, *being either* $\mathbb{E}[X^{\varepsilon}_{k,r}X^{\varepsilon}_{l,r}]$, $\mathbb{E}[X^{\varepsilon}_{k,r}Y^{\varepsilon}_{l,r}]$ *or* $\mathbb{E}[Y^{\varepsilon}_{k,r}Y^{\varepsilon}_{l,r}]$. *Then*, $\mathbb{E}[|\langle\Theta_{r,\varepsilon}-\Xi_{r,\varepsilon},\mathfrak{n}_z\rangle|^2]\to 0$ *for all* $z\in\mathbb{N}_0^2$.

PROOF. Fix $z\in\mathbb{N}_0^2$. Given a function $H:(\mathbb{N}_0^2)^4\to\mathbb{R}$, let us use the following (formal) notation:

$$\mathfrak{S}(H)=\sum_{k,l,m,n\in\mathbb{N}_0^2}\frac{\rho^{\odot}(k,l)}{1+\frac{\pi^2}{r^2}|l|^2}\frac{\rho^{\odot}(m,n)}{1+\frac{\pi^2}{r^2}|n|^2}\sum_{\mathfrak{p},\mathfrak{r},\mathfrak{q},\mathfrak{s}\in\{-1,1\}^2}\delta_{\mathfrak{r}\circ k+\mathfrak{p}\circ l,z}\delta_{\mathfrak{s}\circ m+\mathfrak{q}\circ n,z}H(k,l,m,n).$$

By (94), as $\frac{1}{4}\le\nu_k\le 1$ for all $k\in\mathbb{N}_0^2$,

$$\mathbb{E}\big[\big|\langle\Theta_{r,\varepsilon}-\Xi_{r,\varepsilon},\mathfrak{n}_z\rangle\big|^2\big]\lesssim\mathfrak{S}(E_{\varepsilon}),$$

where

$$\begin{aligned} E_{\varepsilon}(k,l,m,n)=&\,\mathbb{E}\big([X^{\varepsilon}_{k,r}X^{\varepsilon}_{l,r}-Y^{\varepsilon}_{k,r}Y^{\varepsilon}_{l,r}][X^{\varepsilon}_{m,r}X^{\varepsilon}_{n,r}-Y^{\varepsilon}_{m,r}Y^{\varepsilon}_{n,r}]\big)\\ &-\mathbb{E}[X^{\varepsilon}_{k,r}X^{\varepsilon}_{l,r}-Y^{\varepsilon}_{k,r}Y^{\varepsilon}_{l,r}]\mathbb{E}[X^{\varepsilon}_{m,r}X^{\varepsilon}_{n,r}-Y^{\varepsilon}_{m,r}Y^{\varepsilon}_{n,r}]. \end{aligned}$$

We decompose E_{ε} using Wick's theorem (Theorem 11.8). Let us for a few lines write $A_k=X^{\varepsilon}_{k,r}$ and $B_k=Y^{\varepsilon}_{k,r}$, then we obtain

$$\begin{aligned} &\mathbb{E}\big([A_kA_l-B_kB_l][A_mA_n-B_mB_n]\big)-\mathbb{E}[A_kA_l-B_kB_l]\mathbb{E}[A_mA_n-B_mB_n]\\ &\quad=\mathbb{E}[A_kA_lA_mA_n]-\mathbb{E}[A_kA_lB_mB_n]-\mathbb{E}[B_kB_lA_mA_n]+\mathbb{E}[B_kB_lB_mB_n]\\ &\qquad-\big(\mathbb{E}[A_kA_l]-\mathbb{E}[B_kB_l]\big)\big(\mathbb{E}[A_mA_n]-\mathbb{E}[B_mB_n]\big)\\ &\quad=\mathbb{E}[A_kA_m]\mathbb{E}[A_lA_n]-\mathbb{E}[A_kB_m]\mathbb{E}[A_lB_n]-\mathbb{E}[B_kA_m]\mathbb{E}[B_lA_n]+\mathbb{E}[B_kB_m]\mathbb{E}[B_lB_n]\\ &\qquad+\mathbb{E}[A_kA_n]\mathbb{E}[A_mA_l]-\mathbb{E}[A_kB_n]\mathbb{E}[B_mA_l]-\mathbb{E}[B_kA_n]\mathbb{E}[A_mB_l]+\mathbb{E}[B_kB_n]\mathbb{E}[B_mB_l]. \end{aligned}$$

Observe that

$$\begin{aligned}&\mathbb{E}[A_k A_m]\mathbb{E}[A_l A_n] - \mathbb{E}[A_k B_m]\mathbb{E}[A_l B_n]\\&\quad = \mathbb{E}[A_k A_m]\mathbb{E}\big[A_l(A_n - B_n)\big] - \mathbb{E}\big[A_k(B_m - A_m)\big]\mathbb{E}\big[A_l(B_n - A_n)\big]\\&\qquad - \mathbb{E}\big[A_k(B_m - A_m)\big]\mathbb{E}[A_l A_n].\end{aligned}$$

Hence, as $\mathbb{E}[|A_k - B_k|^2] = \mathbb{E}[|X^{\varepsilon}_{k,r} - Y^{\varepsilon}_{k,r}|^2] \to 0$ by (84), we have $E_\varepsilon(k,l,m,n) \to 0$ for all $k,l,m,n \in \mathbb{N}_0^2$. We show that $\mathfrak{S}(E_\varepsilon)$ converges to zero by a dominated convergence argument. Let us write

$$J(k,l,m,n) := \prod_{i=1}^{2} \big(1 + |k_i - m_i|\big)^{\delta-1}\big(1 + |l_i - n_i|\big)^{\delta-1}$$

and $\tilde{J}(k,l,m,n) = J(k,l,n,m)$. Then, by (81) we have $E_\varepsilon \le J + \tilde{J}$ and, by the symmetries obtained by Lemma 11.9, $\mathfrak{S}(\tilde{J}) \lesssim \mathfrak{S}(J)$. Moreover, by "merging the $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$ and k,l,m,n variables" (in the sense of summing over $k \in \mathbb{Z}^2$ instead of $\mathfrak{q} \circ k$ with $\mathfrak{q} \in \{-1,1\}^2$ and $k \in \mathbb{N}_0^2$), we have

$$\mathfrak{S}(J) \lesssim \sum_{l,n \in \mathbb{Z}^2} \frac{1}{1 + \frac{\pi^2}{r^2}|l|^2} \frac{1}{1 + \frac{\pi^2}{r^2}|n|^2} \prod_{i=1}^{2} \big(1 + |l_i - n_i|\big)^{2\delta-2},$$

which is finite by Lemma 11.7. □

LEMMA 11.14. *If* (85) *holds, then* $\mathbb{E}[\theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon} - \xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon}] \to 0$ *in* $\mathcal{C}_{\mathfrak{n}}^{-\gamma}$ *for all* $\gamma > 0$.

PROOF. Let us abbreviate $G_{r,\varepsilon}(k,l) = \mathbb{E}[X^{\varepsilon}_{k,r}X^{\varepsilon}_{l,r} - Y^{\varepsilon}_{k,r}Y^{\varepsilon}_{l,r}]$. By (83) and (95),

$$\begin{aligned}&\sup_{i \in \mathbb{N}_{-1}} 2^{-\gamma i} \big\| \Delta_i \mathbb{E}\big[\theta_{r,\varepsilon} \odot \sigma(\mathrm{D})\theta_{r,\varepsilon} - \xi_{r,\varepsilon} \odot \sigma(\mathrm{D})\xi_{r,\varepsilon}\big]\big\|_\infty\\&\quad \lesssim \sum_{k,l \in \mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{|G_{r,\varepsilon}(k,l)|}{(1+|k-l|)^\gamma}.\end{aligned}$$

We use (85) and consider the sums over $\mathfrak{R}$ and $\mathbb{N}_0^2 \times \mathbb{N}_0^2 \setminus \mathfrak{R}$ as in Theorem 11.2 separately.

- [Sum over $\mathfrak{R}$] By exploiting symmetries using Lemma 11.9,

$$\sum_{(k,l) \in \mathfrak{R}} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{|G_{r,\varepsilon}(k,l)|}{(1+|k-l|)^\gamma} \lesssim \mathcal{S}_{\varepsilon,1} + \mathcal{S}_{\varepsilon,2},$$

$$\mathcal{S}_{\varepsilon,1} = \sum_{k,l \in \mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{1}{(1+|k-l|)^\gamma} \frac{1}{(1+|\frac{r}{\varepsilon} - l_1|)^{1-\delta}} \frac{1}{(1+|\frac{r}{\varepsilon} - l_2|)^{1-\delta}}$$

$$\mathcal{S}_{\varepsilon,2} = \sum_{k,l \in \mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{1}{(1+|k-l|)^\gamma} \frac{1}{(1+|\frac{r}{\varepsilon} - l_1|)^{1-\delta}} \frac{1}{(1+|\frac{r}{\varepsilon} - k_2|)^{1-\delta}}.$$

By (91), by using that $(1+|l|^2) \ge (1+l_1)(1+l_2)$ and by using (90) with $\delta < \gamma$,

$$\mathcal{S}_{\varepsilon,1} \lesssim \bigg(\sum_{l \in \mathbb{N}_0} \frac{1}{(1+l)^{\frac{\gamma}{2}}} \frac{1}{(1+|\frac{r}{\varepsilon} - l|)^{1-\delta}}\bigg)^2 \lesssim \Big(1 + \frac{r}{\varepsilon}\Big)^{\delta-\gamma} \lesssim \varepsilon^{\gamma-\delta}.$$

For $\mathcal{S}_{\varepsilon,2}$, by Lemma 11.9 there exist $b>0$, $c>1$ such that (using that $|k-l|\geq|k_1-l_1|$)

$$\sum_{k\in\mathbb{N}_0^2}\frac{\rho^{\odot}(k,l)}{(1+|k-l|)^{\gamma}}\frac{1}{(1+|\frac{r}{\varepsilon}-k_2|)^{1-\delta}}$$

$$\lesssim\sum_{\substack{k\in\mathbb{N}_0^2\\|k|\leq b}}\frac{1}{(1+|\frac{r}{\varepsilon}-k_2|)^{1-\delta}}+\sum_{\substack{k_1\in\mathbb{N}_0\\k_1\leq c|l|}}\frac{1}{(1+|k_1-l_1|)^{\gamma}}\sum_{\substack{k_2\in\mathbb{N}_0\\k_2\leq c|l|}}\frac{1}{(1+|\frac{r}{\varepsilon}-k_2|)^{1-\delta}}.$$

We will bound the second sum on the right-hand side by its corresponding integrals (see Lemma 11.6) and will bound these to get a bound on the sum over k. Straightforward calculations show

$$\int_0^{c|l|}\frac{1}{(1+|x-l_1|)^{\gamma}}\,\mathrm{d}x\lesssim(1+|l|)^{1-\gamma}.$$

On the other hand, for $\delta>0$ and $z>0$,

$$\int_0^{z}\frac{1}{1+|\frac{r}{\varepsilon}-x|}\,\mathrm{d}x\lesssim\log\Bigl(1+\frac{r}{\varepsilon}\Bigr)^2(1+z)\lesssim\Bigl(1+\frac{r}{\varepsilon}\Bigr)^{2\delta}(1+z)^{\delta}.$$

Hence, for all $\delta>0$ (we use (90) for the last inequality),

$$\mathcal{S}_{\varepsilon,2}\lesssim\sum_{l\in\mathbb{N}_0^2}\frac{1}{(1+|l|^2)}\frac{1}{(1+|\frac{r}{\varepsilon}-l_1|)^{1-\delta}}(1+l_1+l_2)^{1-\gamma+\delta}\Bigl(1+\frac{1}{\varepsilon}\Bigr)^{2\delta}$$

$$\lesssim\sum_{l\in\mathbb{N}_0^2}\frac{1}{(1+l_1+l_2)^{1+\gamma-\delta}}\frac{1}{(1+|\frac{r}{\varepsilon}-l_1|)^{1-\delta}}\Bigl(1+\frac{1}{\varepsilon}\Bigr)^{2\delta}$$

$$\lesssim\sum_{l_2\in\mathbb{N}_0}\frac{1}{(1+l_2)^{1+\delta}}\sum_{l_1\in\mathbb{N}_0}\frac{1}{(1+l_1)^{\gamma-2\delta}}\frac{1}{(1+|\frac{r}{\varepsilon}-l_1|)^{1-\delta}}\Bigl(1+\frac{1}{\varepsilon}\Bigr)^{2\delta}\lesssim\Bigl(1+\frac{1}{\varepsilon}\Bigr)^{5\delta-\gamma}.$$

Therefore, by choosing $\delta<\frac{\gamma}{5}$, we also obtain $\mathcal{S}_{\varepsilon,2}\to 0$:

• [Sum over $\mathbb{N}_0^2\times\mathbb{N}_0^2\setminus\mathfrak{R}$] Observe that $\mathbb{N}_0^2\times\mathbb{N}_0^2\setminus\mathfrak{R}=\{(k,l)\in\mathbb{N}_0^2\times\mathbb{N}_0^2:\exists i\in\{1,2\}: k_i=l_i\}$. Therefore, again by exploiting symmetries using Lemma 11.9 (we bound the sum over $\mathbb{N}_0^2\times\mathbb{N}_0^2\setminus\mathfrak{R}$ by the sum over all $l\in\mathbb{N}_0^2$, $k_2\in\mathbb{N}_0$ and take $k_1=l_1$) and using (90) for $\delta<\frac{\gamma}{2}$,

$$\sum_{(k,l)\in\mathbb{N}_0^2\times\mathbb{N}_0^2\setminus\mathfrak{R}}\frac{\rho^{\odot}(k,l)}{(1+|l|^2)}\frac{|G_{r,\varepsilon}(k,l)|}{(1+|k-l|)^{\gamma}}$$

$$\lesssim\sum_{l\in\mathbb{N}_0^2}\frac{1}{(1+|l|^2)}\sum_{k_2\in\mathbb{N}_0}\frac{1}{(1+|k_2-l_2|)^{\gamma}}\frac{1}{1+|\frac{r}{\varepsilon}-k_2|}$$

$$\lesssim\sum_{l\in\mathbb{N}_0^2}\frac{1}{(1+l_1)^{1+\delta}}\frac{1}{(1+l_2)^{1-\delta}}\frac{1}{(1+|\frac{r}{\varepsilon}-l_2|)^{\gamma-\delta}}\lesssim\Bigl(1+\frac{1}{\varepsilon}\Bigr)^{2\delta-\gamma}.$$

□

12. Proof of Theorem 11.3(b). In this section we consider $d=2$, $L>r\geq 1$ and $y\in\mathbb{R}^2$ such that $y+Q_r\subset Q_L$. We write $\tau=\mathbb{1}_{(-1,1)^2}$. We consider $X_{k,r}^{\varepsilon}$ and $Y_{k,r}^{\varepsilon}$, as in Theorem 11.3(b). For $m,l\in\mathbb{N}_0$ and $z\in[0,L-r]$, we write

(99) $$b_{m,l}^{z}=\langle\mathfrak{n}_{m,L},\mathcal{T}_z\mathfrak{n}_{l,r}\rangle_{L^2([0,r])}.$$

Then, we have

$$X_{k,r}^{\varepsilon} = \sum_{m\in\mathbb{N}_0^2} \tau\left(\frac{\varepsilon}{L}m\right)\mathcal{Z}_m \prod_{i=1}^{2} b_{m_i,k_i}^{y_i}, \qquad Y_{k,r}^{\varepsilon} = \tau\left(\frac{\varepsilon}{r}k\right) \sum_{m\in\mathbb{N}_0^2} \mathcal{Z}_m \prod_{i=1}^{2} b_{m_i,k_i}^{y_i}.$$

And so with $G_{r,\varepsilon}(k,l) = \mathbb{E}[X_{k,r}^{\varepsilon}X_{l,r}^{\varepsilon} - Y_{k,r}^{\varepsilon}Y_{l,r}^{\varepsilon}]$, as in Theorem 11.2

$$G_{r,\varepsilon}(k,l) = \sum_{m\in\mathbb{N}_0^2}\left(\prod_{i=1}^{2} b_{m_i,k_i}^{y_i} b_{m_i,l_i}^{y_i}\right)\left[\tau\left(\frac{\varepsilon}{L}m\right)^2 - \tau\left(\frac{\varepsilon}{r}k\right)\delta_{k,l}\right]. \tag{100}$$

THEOREM 12.1. (84), (81) *and* (85) *hold* (*for* $I = [1, L]$)

PROOF. For (84) we have

$$\begin{aligned}\mathbb{E}\big[\big|\langle\theta_{r,\varepsilon} - \xi_{r,\varepsilon}, \mathfrak{n}_k\rangle\big|^2\big] &= \mathbb{E}\big[\big|X_{k,r}^{\varepsilon} - Y_{k,r}^{\varepsilon}\big|^2\big] \\ &\lesssim \sum_{m\in\mathbb{N}_0^d}\left(\tau\left(\frac{\varepsilon}{L}m\right) - \tau\left(\frac{\varepsilon}{r}k\right)\right)^2 \prod_{i=1}^{2}(b_{m_i,k_i}^{y_i})^2.\end{aligned}$$

By Lebesgue's dominated convergence theorem this converges to zero. (81) follows by Theorem 12.4 by observing that $\mathbb{E}[X_{k,r}^{\varepsilon}Y_{l,r}^{\varepsilon}] = \tau(\frac{\varepsilon}{r}k)\mathbb{E}[X_{k,r}^{\varepsilon}X_{l,r}^{\varepsilon}]$, $\mathbb{E}[Y_{k,r}^{\varepsilon}Y_{l,r}^{\varepsilon}] \le 2\delta_{k,l}$ and that $|\mathbb{E}[X_{k,r}^{\varepsilon}X_{l,r}^{\varepsilon}]| \le \prod_{i=1}^{2}(\sum_{m\in\mathbb{N}_0}|b_{m,k_i}^{y_i}b_{m,l_i}^{y_i}|)$. (85) follows by Lemma 12.7. □

12.2. The estimates (81) and (85) will rely on bounds on $b_{m,l}^z$ for $m, l \in \mathbb{N}_0$ and $z \in [0, L-r]$. Let us calculate $b_{m,l}^z$ here. For notational convenience we put $\frac{\sin(\pi x)}{x}$ and $\frac{1-\cos(\pi x)}{x}$ for $x = 0$ equal to 1 here. By using some trigonometric rules, one can compute that

$$b_{m,l}^z = \sqrt{\frac{r}{L}}\frac{1}{\pi}\nu_m\nu_l\left[f_{m,l}\cos\left(\frac{\pi}{L}mz\right) + g_{m,l}\sin\left(\frac{\pi}{L}mz\right)\right], \tag{101}$$

where

$$f_{m,l} = \sum_{\mathfrak{p}\in\{-1,1\}} \frac{\sin(\pi(\frac{r}{L}m + \mathfrak{p}l))}{\frac{r}{L}m + \mathfrak{p}l}, \qquad g_{m,l} = \sum_{\mathfrak{p}\in\{-1,1\}} \frac{1 - \cos(\pi(\frac{r}{L}m + \mathfrak{p}l))}{\frac{r}{L}m + \mathfrak{p}l}.$$

Let us demonstrate (101) in the easier case $z = 0$. Due to the identities $2\cos(a)\cos(b) = \sum_{\mathfrak{p}\in\{-1,1\}}\cos(a + \mathfrak{p}b)$ and $\sin(\pi(a \pm l)) = (-1)^l\sin(\pi a)$ for $a, b \in \mathbb{R}$ and $l \in \mathbb{Z}$, we obtain

$$\begin{aligned}\langle\mathfrak{n}_{m,L}, \mathfrak{n}_{l,r}\rangle_{L^2([0,r])} &= \frac{2}{\sqrt{Lr}}\nu_m\nu_l\int_0^r \cos\left(\frac{\pi}{L}mx\right)\cos\left(\frac{\pi}{r}lx\right)\mathrm{d}x \\ &= \sqrt{\frac{r}{L}}\frac{1}{\pi}\nu_m\nu_l\sum_{\mathfrak{p}\in\{-1,1\}}\frac{\sin(\pi(\frac{mr}{L} + \mathfrak{p}l))}{\frac{mr}{L} + \mathfrak{p}l}.\end{aligned} \tag{102}$$

As a consequence we obtain the following.

LEMMA 12.3. *There exists a* $C > 0$ (*independent of* r *and* L) *such that, for all* $z \in [0, L-r]$ *and* $m, l \in \mathbb{N}_0$,

$$|b_{m,l}^z| \le C\sqrt{\frac{r}{L}}\frac{1}{1 + |\frac{r}{L}m - l|}. \tag{103}$$

PROOF. This follows from the expression (101) by using that $|\frac{\sin(\pi x)}{x}| \lesssim \frac{1}{1+|x|}$ and $\frac{1-\cos(\pi x)}{x} \lesssim \frac{1}{1+|x|}$. $\square$

THEOREM 12.4. *For all $\delta > 0$, there exists a $C > 0$ (independent of L and r) such that, for all $k, l \in \mathbb{N}_0$ and $z \in [0, L-r]$,*

$$\sum_{m\in\mathbb{N}_0} |b^z_{m,k} b^z_{m,l}| \le C\big(1 + |k-l|\big)^{\delta-1}. \tag{104}$$

PROOF. This follows by Lemma 12.3 and by (90), as $1 + |\frac{r}{L}m - u| \ge (1 + |\frac{r}{L}m - u|)^{1-\frac{\delta}{2}}$ for $\delta > 0$. $\square$

12.5. Let $C > 0$ be as in Lemma 12.3.

(a) For all $z \in [0, L-r]$ and $M, k, l \in \mathbb{N}_0$ such that $\frac{r}{L}M \le l \le k$, by Lemma 11.6

$$\sum_{m=0}^{M-1} |b^z_{m,k} b^z_{m,l}| \le C^2 \int_0^M \frac{1}{(1 + l - \frac{r}{L}x)^2}\,\mathrm{d}x \le C^2 \frac{1}{1 + |l - \frac{r}{L}M|},$$

(b) Similarly, for all $z \in [0, L-r]$ and $M, k, l \in \mathbb{N}_0$ such that $l \le k \le \frac{r}{L}M$,

$$\sum_{m=M}^{\infty} |b^z_{m,k} b^z_{m,l}| \le (C^2 + 1)\frac{1}{1 + |k - \frac{r}{L}M|}.$$

As a consequence of the above and $\sum_{m\in\mathbb{N}_0} b^z_{m,k} b^z_{m,l} = \delta_{k,l}$, we obtain the following lemma.

LEMMA 12.6. *There exists a $C > 0$ such that for all $z \in [0, L-r]$, $M \in [0, \infty)$ and $k, l \in \mathbb{N}_0$: If either $k \ne l$ or $k = l \le \frac{r}{L}M$, then*

$$\frac{1}{C}\bigg|\sum_{m\in\mathbb{N}_0, m\ge M} b^z_{m,k} b^z_{m,l}\bigg| \le \frac{1}{(1 + |k - \frac{r}{L}M|)^{1-\delta}} + \frac{1}{(1 + |l - \frac{r}{L}M|)^{1-\delta}}, \tag{105}$$

and if either $k \ne l$ or $k = l \ge \frac{r}{L}M$,

$$\frac{1}{C}\bigg|\sum_{m\in N_0, m< M} b^z_{m,k} b^z_{m,l}\bigg| \le \frac{1}{(1 + |k - \frac{r}{L}M|)^{1-\delta}} + \frac{1}{(1 + |l - \frac{r}{L}M|)^{1-\delta}}. \tag{106}$$

PROOF. By (103) we may assume $M \in \mathbb{N}_0$. The statements for $k = l$ follow immediately by the bounds in 12.5. For $k \ne l$, we have $\sum_{m\in N_0, m<M} b^z_{m,k} b^z_{m,l} = \sum_{m\in N_0, m\ge M} b^z_{m,k} b^z_{m,l}$ so that the rest follows by 12.5, and by observing that if $l \le \frac{r}{L}M \le k$ that $|\sum_{m\in N_0, m<M} b^z_{m,k} \times b^z_{m,l}| \lesssim \frac{1}{(1+|k-l|)^{1-\delta}}$ by Theorem 12.4 which is less than the right-hand side of both (105) and (106). $\square$

LEMMA 12.7. *Write $G_{r,\varepsilon}(k,l) = \mathbb{E}[X^\varepsilon_{k,r}X^\varepsilon_{l,r} - Y^\varepsilon_{k,r}Y^\varepsilon_{l,r}]$. There exists a $C > 0$ such that, for all $\varepsilon > 0$ and $k, l \in \mathbb{N}_0^2$,*

$$\frac{1}{C}\left|G_{r,\varepsilon}(k,l)\right| \le \begin{cases} \prod_{i=1}^{2} \frac{1}{(1+|k_i - \frac{r}{\varepsilon}|)^{1-\delta}} + \frac{1}{(1+|l_i - \frac{r}{\varepsilon}|)^{1-\delta}} \\ \quad \text{if for } i \in \{1,2\} \text{ either } k_i \neq l_i \\ \quad \text{or } k_i = l_i \ge \frac{r}{\varepsilon}, \\ \frac{1}{(1+|k_i - \frac{r}{\varepsilon}|)^{1-\delta}} + \frac{1}{(1+|l_i - \frac{r}{\varepsilon}|)^{1-\delta}} \\ \quad \text{if either } k_i \neq l_i \text{ or } k_i = l_i \ge \frac{r}{\varepsilon} \\ \quad \text{and } k_{3-i} = l_{3-i} < \frac{r}{\varepsilon}, \\ \frac{1}{(1+|k_1 - \frac{r}{\varepsilon}|)^{1-\delta}} + \frac{1}{(1+|k_2 - \frac{r}{\varepsilon}|)^{1-\delta}} \\ \quad k_i = l_i < \frac{r}{\varepsilon} \text{ for } i \in \{1,2\}. \end{cases} \tag{107}$$

PROOF. Let $(k,l) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2$ be such that $k = l$ with $|k|_\infty < \frac{r}{\varepsilon}$. Then (see (100)),

$$\left|G_{r,\varepsilon}(k,l)\right| = \left|\sum_{m\in\mathbb{N}_0^2:|m|_\infty \ge \frac{L}{\varepsilon}} \prod_{i=1}^{2} b^{y_i}_{m_i,k_i} b^{y_i}_{m_i,l_i}\right| \lesssim \left|\sum_{m\in\mathbb{N}_0, m\ge\frac{L}{\varepsilon}} (b^{y_1}_{m,k_1})^2\right| + \left|\sum_{m\in\mathbb{N}_0, m\ge\frac{L}{\varepsilon}} (b^{y_2}_{m,k_2})^2\right|.$$

If k and l are not like that, then

$$G_{r,\varepsilon}(k,l) = \left(\sum_{m\in\mathbb{N}_0, m<\frac{L}{\varepsilon}} b^{y_1}_{m,k_1} b^{y_1}_{m,l_1}\right)\left(\sum_{m\in\mathbb{N}_0, m<\frac{L}{\varepsilon}} b^{y_2}_{m,k_2} b^{y_2}_{m,l_2}\right),$$

so that the bound (107) follows from Lemma 12.6. □

APPENDIX A: THE MIN-MAX FORMULA FOR SMOOTH POTENTIALS

LEMMA A.1. *Let $f_1, \dots, f_n$ be pairwise orthogonal in H_0^2. There exist pairwise orthogonal $f_{1,k}, \dots, f_{n,k}$ in C_c^∞ for $k \in \mathbb{N}$ such that, for all i,*

$$f_{i,k} \xrightarrow{k\to\infty} f_i \quad \text{in } H_0^2. \tag{108}$$

PROOF. Let $g_{i,k} \in C_c^\infty$ be such that $g_{i,k} \to f_i$ in H_0^2 for all i. By doing a Gram–Schmidt procedure on $g_{1,k}, \dots, g_{n,k}$, we can give the proof by induction. We prove the induction step, assuming that $f_{1,k} = g_{1,k}, \dots, f_{n-1,k} = g_{n-1,k}$ are pairwise independent. We define

$$f_{n,k} = g_{n,k} - \sum_{i=1}^{n-1} \frac{\langle g_{n,k}, f_{i,k}\rangle}{\langle f_{i,k}, f_{i,k}\rangle} f_{i,k}.$$

Then, $f_{n,k}$ is pairwise independent from $f_{1,k}, \dots, f_{n-1,k}$. As for $i \in \{1, \dots, n-1\}$, we have

$$\langle g_{n,k}, f_{i,k}\rangle \to \langle f_n, f_i\rangle = 0;$$

it follows that $f_{n,k} \to f_n$. □

LEMMA A.2. *Let $\zeta \in L^\infty$, $n \in \mathbb{N}$ and $L > 0$. Then, (for notation, see 5.4),*

$$\lambda_n(Q_L, \zeta) = \sup_{\substack{F \sqsubset H_0^2 \\ \dim F = n}} \inf_{\substack{\psi\in F \\ \|\psi\|_{L^2}=1}} \langle \mathscr{H}_\zeta \psi, \psi\rangle = \sup_{\substack{F \sqsubset C_c^\infty \\ \dim F = n}} \inf_{\substack{\psi\in F \\ \|\psi\|_{L^2}=1}} \langle \mathscr{H}_\zeta \psi, \psi\rangle. \tag{109}$$

PROOF. First observe that

$$\lambda_n(Q_L,\zeta) = \sup_{\substack{f_1,\dots,f_n\in H_0^2\\ \langle f_i,f_j\rangle_{H_0^2}=\delta_{ij}}} \inf_{\substack{\psi=\sum_{i=1}^n \alpha_i f_i\\ \alpha_i\in[0,1],\sum_{i=1}^n\alpha_i^2=1}} \langle \mathscr{H}_\zeta \psi,\psi\rangle.$$

Let $f_1,\dots,f_n\in H_0^2$ with $\langle f_i,f_j\rangle_{H_0^2}=\delta_{ij}$. By Lemma A.1 there exist $f_{1,k},\dots,f_{n,k}$ in C_c^∞ with $\langle f_{i,k},f_{j,k}\rangle_{H_0^2}=\delta_{ij}$ (by renormalising) such that (108) holds. Then,

$$\begin{aligned}
&\Bigl| \inf_{\substack{\psi=\sum_{i=1}^n \alpha_i f_i\\ \alpha_i\in[0,1],\sum_{i=1}^n\alpha_i^2=1}} \langle \mathscr{H}_\zeta \psi,\psi\rangle - \inf_{\substack{\psi=\sum_{i=1}^n \alpha_i f_{i,k}\\ \alpha_i\in[0,1],\sum_{i=1}^n\alpha_i^2=1}} \langle \mathscr{H}_\zeta \psi,\psi\rangle\Bigr|\\
&\quad\le \sup_{\substack{\psi=\sum_{i=1}^n \alpha_i f_i,\varphi=\sum_{i=1}^n \alpha_i f_{i,k}\\ \alpha_i\in[0,1],\sum_{i=1}^n\alpha_i^2=1}} \bigl|\langle \mathscr{H}_\zeta \psi,\psi\rangle_{L^2} - \langle \mathscr{H}_\zeta \varphi,\varphi\rangle_{L^2}\bigr|\\
&\quad\lesssim \sup_{\alpha_i\in[0,1],\sum_{i=1}^n\alpha_i^2=1} \Bigl\|\sum_{i=1}^n \alpha_i f_i - \sum_{i=1}^n \alpha_i f_{i,k}\Bigr\|_{H_0^2} \le \sum_{i=1}^n \|f_i - f_{i,k}\|_{H_0^2} \to 0.
\end{aligned}$$

This proves

$$\lambda_n(Q_L,\zeta) = \sup_{\substack{f_1,\dots,f_n\in C_c^\infty\\ \langle f_i,f_j\rangle_{H_0^2}=\delta_{ij}}} \inf_{\substack{\psi=\sum_{i=1}^n \alpha_i f_i\\ \alpha_i\in[0,1],\sum_{i=1}^n\alpha_i^2=1}} \langle \mathscr{H}_\zeta \psi,\psi\rangle \tag{110}$$

and, therefore, (109). □

APPENDIX B: USEFUL BOUND ON AN INTEGRAL

LEMMA B.1. *Let $\gamma,\theta\in(0,1)$ and $\gamma+\theta>1$. There exists a $C>0$ such that, for all $b>0$ and $u\in\mathbb{R}$,*

$$\int_0^\infty \frac{1}{(b+|x-u|)^\gamma}\frac{1}{(b+x)^\theta}\,\mathrm{d}x \le C\bigl(b+|u|\bigr)^{1-\gamma-\theta}. \tag{111}$$

Consequently, there exists a $C>0$ such that, for all $b>0$ and $u,v\in\mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{(b+|x-u|)^\gamma}\frac{1}{(b+|x-v|)^\theta}\,\mathrm{d}x \le C\bigl(b+|u-v|\bigr)^{1-\gamma-\theta}. \tag{112}$$

PROOF. By a simple substitution argument we may assume $b=1$. We have uniformly in $a\in(0,1)$,

$$\int_0^\infty \frac{1}{(a+x)^\gamma}\frac{1}{(1+x)^\theta}\,\mathrm{d}x \le \int_1^\infty \frac{1}{x^{\gamma+\theta}}\,\mathrm{d}x + \int_0^1 \frac{1}{(a+x)^\gamma}\,\mathrm{d}x \lesssim 1+(1+a)^{1-\gamma}\lesssim 1.$$

Hence, for all $u\ge 0$,

$$\begin{aligned}
&\int_u^\infty \frac{1}{(1+x-u)^\gamma}\frac{1}{(1+x)^\theta}\,\mathrm{d}x\\
&\quad= \int_0^\infty \frac{1}{(1+x)^\gamma}\frac{1}{(1+u+x)^\theta}\,\mathrm{d}x \qquad (113)\\
&\quad= (1+u)^{1-\gamma-\theta}\int_0^\infty \frac{1}{(\frac{1}{1+u}+x)^\gamma}\frac{1}{(1+x)^\theta}\,\mathrm{d}x \lesssim (1+u)^{1-\gamma-\theta}.
\end{aligned}$$

On the other hand, we have

$$\int_0^{\frac{u}{2}} \frac{1}{(1+u-x)^\gamma} \frac{1}{(1+x)^\theta} \, \mathrm{d}x \le \left(1+\frac{u}{2}\right)^{-\gamma} \int_0^{\frac{u}{2}} \frac{1}{(1+x)^\theta} \, \mathrm{d}x \lesssim (1+u)^{1-\gamma-\theta},$$

and, similarly, $\int_{\frac{u}{2}}^{u} \frac{1}{(1+u-x)^\gamma} \frac{1}{(1+x)^\theta} \, \mathrm{d}x \lesssim (1+u)^{1-\gamma-\theta}$. In case u is negative, the bound is already proved in (113) (by interchanging θ and γ).

For (112) it is sufficient to observe that

$$\int_v^\infty \frac{1}{(1+|x-u|)^\gamma} \frac{1}{(1+|x-v|)^\theta} \, \mathrm{d}x = \int_0^\infty \frac{1}{(1+|x+v-u|)^\gamma} \frac{1}{(1+x)^\theta} \, \mathrm{d}x,$$
$$\int_{-\infty}^v \frac{1}{(1+|x-u|)^\gamma} \frac{1}{(1+|x-v|)^\theta} \, \mathrm{d}x = \int_0^\infty \frac{1}{(1+|x+u-v|)^\gamma} \frac{1}{(1+x)^\theta} \, \mathrm{d}x. \qquad \square$$

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