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A SHAPE THEOREM FOR THE ORTHANT MODEL

BY MARK HOLMES[1] AND THOMAS S. SALISBURY[2]

[1]*School of Mathematics and Statistics, University of Melbourne, holmes.m@unimelb.edu.au*
[2]*Deptartment of Mathematics and Statistics, York University, salt@yorku.ca*

We study a particular model of a random medium, called the orthant model, in general dimensions $d \geq 2$. Each site $x \in \mathbb{Z}^d$ independently has arrows pointing to its positive neighbours $x + e_i$, $i = 1, \dots, d$ with probability p and, otherwise, to its negative neighbours $x - e_i$, $i = 1, \dots, d$ (with probability $1 - p$). We prove a shape theorem for the set of sites reachable by following arrows, starting from the origin, when p is large. The argument uses subadditivity, as would be expected from the shape theorems arising in the study of first passage percolation. The main difficulty to overcome is that the primary objects of study are not stationary which is a key requirement of the subadditive ergodic theorem.

1. The model and main results. Fix $d \geq 2$, and set $[d] = \{1, 2, \dots, d\}$. Let $\mathcal{E}_+ = \{e_i\}_{i \in [d]}$ denote the set of canonical basis vectors for $\mathbb{Z}^d$, and let $\mathcal{E}_- = \{-e_i\}_{i \in [d]}$ and $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$. Let o denote the origin in $\mathbb{Z}^d$. Let μ be a probability measure on the power set of $\mathcal{E}$.

Let $(\mathcal{G}_x)_{x \in \mathbb{Z}^d}$ be i.i.d. with law μ. This induces a random directed graph on $\mathbb{Z}^d$—insert arrows from x to each of the vertices $\{x + e : e \in \mathcal{G}_x\}$. These kinds of models are called *degenerate random environments* [9, 11], and their study is motivated by the fact that they lay the foundation for understanding the behaviour of *random walks in nonelliptic random environments* (see [10, 12] for the nonelliptic setting and [17] for the general theory in the uniformly elliptic setting). We are interested in the set of vertices $\mathcal{C}_x \subset \mathbb{Z}^d$ that can be reached from x by following these arrows as well as the sets $\mathcal{B}_x = \{y \in \mathbb{Z}^d : x \in \mathcal{C}_y\}$ and $\mathcal{M}_x = \mathcal{C}_x \cap \mathcal{B}_x$.

Our goal is to obtain a shape theorem for $\mathcal{C}_o$, under a particular choice of μ (to be described below); see Theorem 1. As one would expect, this is proved using the subadditive ergodic theorem.

The key technical problem for us to overcome is that the basic object of study (labelled $\beta_n(u)$ below) fails to have the stationarity properties required by the subadditive ergodic theorem because of the special role of the origin o (for further discussion, see Section 2.1). Our approach is to find a substitute quantity that does have the required stationarity and then bootstrap our way from that to $\beta_n(u)$ using geometric arguments and estimates based on large deviations and the enumeration of self-avoiding walks. We hope that this approach will be useful in other situations, where the subadditive ergodic theorem does not directly apply.

We will now specialize to two models, which we call the *orthant model* and the *half-orthant model*. To avoid confusion, from this point onward notation such as $\mathcal{C}_o$, $\mathcal{B}_o$, $\mathcal{M}_o$, $\mathcal{G}_o$, L_o will refer to the half-orthant model, whereas the corresponding objects for the orthant model will be labelled $\mathcal{C}_o^*$, $\mathcal{B}_o^*$, etc.

The *orthant model* is an elegant example of this class of models in which $\mu(\{\mathcal{E}_+\}) = p = 1 - \mu(\{\mathcal{E}_-\})$. When $d = 2$, this model is dual to oriented site percolation (OTSP) on the triangular lattice [9, 11]. This fact has been exploited to prove a shape theorem for $\mathcal{C}_o^*$ in two

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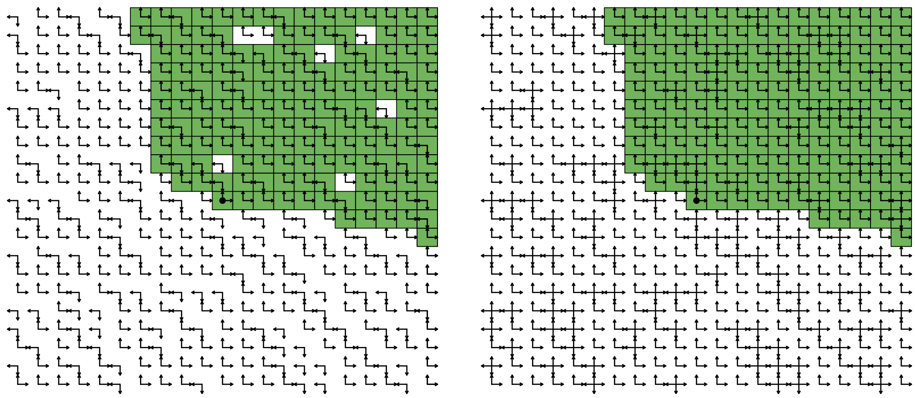

FIG. 1. *Coupled realisations of finite parts of the sets $\mathcal{C}_o^*$ and $\mathcal{C}_o$ for the orthant and half-orthant models, respectively, with $p = 0.7$ and $d = 2$. Note that the boundaries of the two shaded clusters are the same* (*see Theorem **L***).

dimensions as well as to give improved estimates of the critical parameter for OTSP. The left side of Figure 1 shows an example of $\mathcal{C}_o^*$ when $d = 2$. Figure 2 shows an example of $\mathcal{C}_o^*$ when $d = 3$.

Let $\Omega_+ = \{x \in \mathbb{Z}^d : \mathcal{G}_x^* = \mathcal{E}_+\}$ and $\Omega_- = \mathbb{Z}^d \setminus \Omega_+$. The orthant model has the property that $\mathcal{C}_o^*$ is almost surely infinite (this means that a random walk on the random directed graph $\mathcal{G}^* = (\mathcal{G}_x^*)_{x \in \mathbb{Z}^d}$ will visit infinitely many sites), since, for example, it contains an infinite path consisting entirely of e_1 steps from Ω_+ sites and $-e_2$ steps from Ω_- sites. It also has the property that $\mathcal{C}_o^*$ is nonmonotone in p under the standard coupling of environments (as in site percolation); see, for example, (1) below. Nevertheless it has recently been proved [13] that, for each $d \geq 2$, there is a phase transition in the structure of $\mathcal{C}_o^*$ as p varies. In order to state this result, we define the *half-orthant model* to be the model with $\mu(\{\mathcal{E}_+\}) = p = 1 - \mu(\{\mathcal{E}\})$. The orthant and half-orthant models can be realized on the same probability space in such a way that $\mathcal{C}_o^*(p) \subset \mathcal{C}_o(p)$ for every $p \in [0, 1]$ as follows: Let $(U_x)_{x \in \mathbb{Z}^d}$ be i.i.d. standard uniform random variables, and set

$$\mathcal{G}_x(p) = \mathcal{E}_+ \iff \mathcal{G}_x^*(p) = \mathcal{E}_+ \iff U_x \leq p. \tag{1}$$

For $x \in \mathbb{Z}^d$ let $L_x^* := \inf\{k \in \mathbb{Z} : x + ke_1 \in \mathcal{C}_o^*\}$ and $L_x := \inf\{k \in \mathbb{Z} : x + ke_1 \in \mathcal{C}_o\}$. Then, trivially, $L_x^* \geq L_x$ for every x.

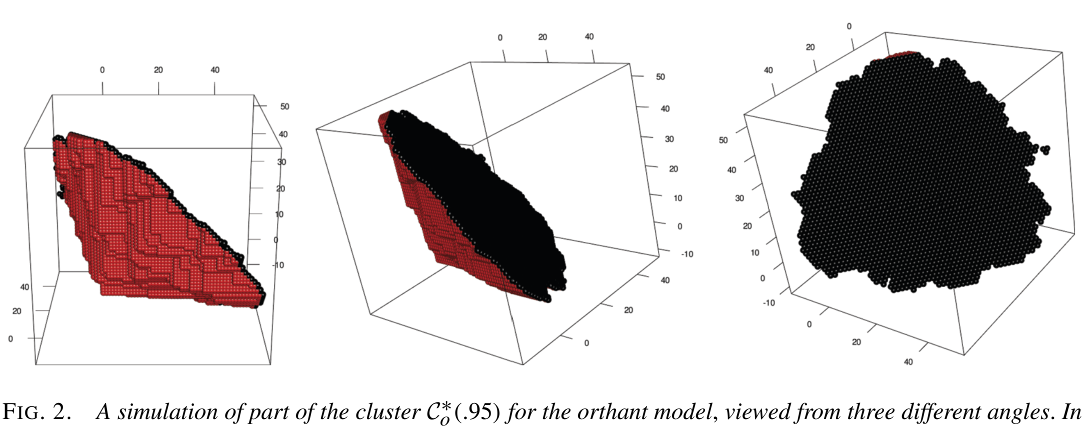

FIG. 2. *A simulation of part of the cluster $\mathcal{C}_o^*(.95)$ for the orthant model, viewed from three different angles. In each case the black/dark vertices are a cross-section where the sum of coordinates is equal to* 50.

For $z \in \mathbb{Z}^d$, we define $z_{\{1+\}} = \{z + ke_1 : k \in \mathbb{Z}_+\}$ and for $A \subset \mathbb{Z}^d$

$$A_{\{1+\}} = \bigcup_{z \in A} z_{\{1+\}}. \tag{2}$$

The following result is a special case of a theorem from [13] and shows that the "left boundaries" of $\mathcal{C}_o^*$ and $\mathcal{C}_o$ are the same. Since $\mathcal{C}_o(p)$ is monotone decreasing in p, this gives a monotonicity result for the left boundary of $\mathcal{C}_o^*(p)$.

THEOREM ***L***. *Under the coupling* (1), *for each* $x \in \mathbb{Z}^d$ *and* $p \in (0, 1)$, $L_x = L_x^* \in [-\infty, \infty)$ *and* $(\mathcal{C}_o^*(p))_{\{1+\}} = \mathcal{C}_o(p)$, *a.s.*

Of course, a similar statement holds in the direction of any e_j, not just in direction e_1, by symmetry of the orthant and half-orthant model.

The following result (also a special case of a theorem from [13]) then reveals a phase transition for the half-orthant model.

THEOREM ***C***. *There exists* $p_c = p_c(d) \in (\frac{1}{2}, 1)$ *such that*:

if $p < p_c$ *then* $\mathcal{C}_o(p) = \mathbb{Z}^d$ *almost surely, and*
if $p > p_c$ *then* $L_x(p) \in \mathbb{R}$ *for every* $x \in \mathbb{Z}^d$ *almost surely* (*so* $\mathcal{C}_o(p) \neq \mathbb{Z}^d$).

These results have immediate consequences for the orthant model with $0 < p < 1$, which we now describe. Suppose first that $1 - p_c < p < p_c$. When $d = 2$, $\mathcal{C}_o^*$ is all of $\mathbb{Z}^2$, with the exception of of a collection of finite holes. More precisely, all connected components of $\mathbb{Z}^2 \setminus \mathcal{C}_o^*$ are finite, so filling in the holes gives back $\mathbb{Z}^2$. We do not know if the same finiteness statement holds when $d > 2$, but what we can say is that, when $1 - p_c < p < p_c$, the cluster $\mathcal{C}_o^*$ intersects any half-infinite line in $\mathbb{Z}^d$ (parallel to one of the axes) infinitely often.

In contrast, when $p > p_c$, the region to the left of $\mathcal{C}_o^*$ is infinite (i.e., all L_x^* are finite). While when $p < 1 - p_c$, the region to the right of $\mathcal{C}_o^*$ is infinite (i.e., all $\sup\{k \in \mathbb{Z} : x + ke_1 \in \mathcal{C}_o^*\}$ are finite).

For the proofs, when $\frac{1}{2} \leq p < 1$, the above statements follow immediately from Theorems ***L*** and ***C***. To obtain the corresponding statements for $0 < p \leq \frac{1}{2}$, let $\mathbf{1} = \sum_{e \in \mathcal{E}_+} e$, and observe that the orthant model is distributionally symmetric under the map $p \mapsto 1 - p$ followed by reflection across the plane $\{x \in \mathbb{Z}^d : x \cdot \mathbf{1} = 0\}$. In other words, the orthant model with parameter $1 - p$ is an orthant model with parameter p but in the "reverse direction."

Our main result, Theorem 1 below, proves a shape theorem for $\mathcal{C}_o$ when p is large. In view of Theorem ***L***, this immediately implies a shape theorem for the boundary of $\mathcal{C}_o^*$. For a set $H \subset \mathbb{R}^d$ and $s \geq 0$, we let $sH = \{sx : x \in H\}$. Recall that a cone is any subset C of $\mathbb{R}^d$ such that $sC = C$ for any $s > 0$. We will prove that the region $n^{-1}\mathcal{C}_o$ for large n is asymptotically a convex cone symmetric under permutations of the axes. In particular, $\mathcal{C}_o$ is "almost" contained in the half space $\{x \in \mathbb{Z}^d : x \cdot \mathbf{1} > 0\}$; see Lemma 1. This cone could be described as $\bigcup_{s \geq 0} s\chi$, where $\chi = C \cap \mathcal{S}_1$ is a convex subset of $\mathcal{S}_1 := \{x \in \mathbb{R}^d : x \cdot \mathbf{1} = 1\}$. We call χ the *shape* of C.

We will let $\mathbb{P}_p$ denote the law of the half-orthant model with p fixed. For $u \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, let (see Figure 3)

$$\beta_n(u) = \inf\{k \in \mathbb{Z} : k\mathbf{1} + nu \in \mathcal{C}_o\}.$$

Let $O_r = \{x \in \mathbb{R}^d : \|x\|_\infty \leq r\}$. The following is our main result.

THEOREM 1. *Fix* $d \geq 2$. *For the half-orthant model there is a* $p_1 < 1$ *such that the following hold for* $p > p_1$:

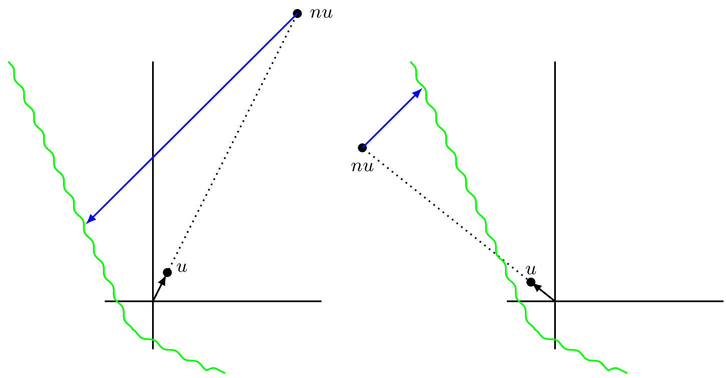

FIG. 3. *Illustrations of $\beta_n(u)$ in two dimensions for two choices of u. The squiggly (green) line is the boundary of $\mathcal{C}_o$. The (blue) line from nu to the squiggly line is $\beta_n(u)\mathbf{1}$. In the first case $\beta_n(u) < 0$, while in the second case $\beta_n(u) > 0$.*

(a) *For $u \in \mathbb{Z}^d$, there is a deterministic $\gamma(u)$ such that $\frac{\beta_n(u)}{n} \to \gamma(u)$, $\mathbb{P}_p$-a.s.;*

(b) *$\gamma(u+w) \le \gamma(u) + \gamma(w)$; $\gamma(ru) = r\gamma(u)$; $\gamma(u + r\mathbf{1}) = \gamma(u) - r$ for $r \in \mathbb{Z}_+$ and $u, w \in \mathbb{Z}^d$; γ is symmetric under permutation of coordinates; $\gamma(u) \ge 0$ if $u \cdot \mathbf{1} \le 0$; $\gamma(u) \le 0$ if u lies in the positive orthant.*

(c) *γ extends to be a Lipschitz map $\mathbb{R}^d \to \mathbb{R}$ with these same properties but for $r \in [0, \infty)$ and $u, w \in \mathbb{R}^d$.*

(d) *The set $C := \{z \in \mathbb{R}^d : \gamma(z) \le 0\}$ is a closed convex cone which is symmetric under permutations of the coordinates, contains the positive orthant and is contained in the half-space $\{z : z \cdot \mathbf{1} \ge 0\}$.*

(e) *$\frac{1}{n}\mathcal{C}_o \to C$ in the sense that, for every $\epsilon > 0$ and every $r < \infty$, the following holds $\mathbb{P}_p$-a.s. for sufficiently large (random) n:*

$$\left(O_r \cap \frac{1}{n}\mathcal{C}_o\right) \subset O_\epsilon + C \quad \text{and} \quad O_r \cap C \subset O_\epsilon + \frac{1}{n}\mathcal{C}_o.$$

When $p = 1$, the shape χ of C is the simplex $\{x \in \mathbb{R}^d_+ : x \cdot \mathbf{1} = 1\}$, and, in particular, when $d = 3$, then χ is a (filled) triangle. As we decrease p, the triangle becomes rounded; see, for example, the third picture in Figure 2. Note that (e) implies convergence in the pointed Gromov–Hausdorff metric; see [2].

As remarked earlier, we will analyze $\beta_n(u)$ using subadditivity. The special role of the origin creates a lack of stationarity, so these subadditivity arguments are far from routine (see Section 2.1 for more on this point). Circumventing this obstacle is the main technical contribution of this paper. In doing so, we will rely on the exponential decay of certain probabilities, which is the main reason why we only have a proof for large p, rather than for all $p > p_c$. In other words, the following remains open:

OPEN PROBLEM 1. Prove that Theorem 1 holds for all $p > p_c$.

Note that [6] uses a block argument to go from large p to all $p > p_c$ in the setting of oriented percolation in two dimensions. In the context of Theorem 1, it would be interesting to find some analogue to this approach.

According to [13], there are phase transitions for more general models of degenerate random environments. We introduced L_x above, which is analogous to $\beta_1(u)$, except that e_1 plays the role of $\mathbf{1}$. Under Condition 2 of [13], one could attempt to formulate shape theorems for such general models, in which case it may be more natural to use the L_x's. We will, therefore, record the following variation of (a) of Theorem 1 (still in the setting of the half-orthant model). Let Z denote the discrete hyperplane $\{y \in \mathbb{Z}^d : y \cdot e_1 = 0\}$.

COROLLARY 1. *Assume the conditions of Theorem 1, and let $p > p_1$. For each $v \in Z$, there exists a deterministic $\zeta(v) \in \mathbb{R}$ such that*

$$n^{-1} L_{nv} \to \zeta(v), \quad \mathbb{P}_p\text{- almost surely as } n \to \infty. \tag{3}$$

OPEN PROBLEM 2. Prove a version of (3) (or of Theorem 1) for more general degenerate random environments, for example, assuming Condition 2 of [13].

A comparison with the shape theorems of first passage percolation might suggest that the convergence in (e) of Theorem 1 (in which ϵ is fixed) fails to capture the fine structure of $\frac{1}{n}\mathcal{C}_o$. The following is intended to show that this is not actually the case. For $\delta > 0$, let $C_\delta = \{z \in \mathbb{R}^d : \gamma(z) \le -\delta\}$.

COROLLARY 2. *Assume the conditions of Theorem 1, and let $p > p_1$. Then, for every $\delta > 0$ and every $R < \infty$, the following holds $\mathbb{P}_p$-a.s. for sufficiently large (random) n:*

$$O_R \cap C_\delta \cap \frac{1}{n}\mathbb{Z}^d \subset \frac{1}{n}\mathcal{C}_o.$$

In other words, $\frac{1}{n}\mathcal{C}_o$ fills out the available lattice points of C, away from the boundary of C.

All of the above results concern the forward clusters $\mathcal{C}_o$ or $\mathcal{C}_o^*$. A crucial difference between forward and backward clusters is that $\mathcal{B}_o^*$ can be finite for the orthant model. For example, if $e_i \in \Omega_+$ and $-e_i \in \Omega_-$ for each $i \in [d]$ (this has positive probability for any $p \in (0,1)$), then there are no arrows pointing to the origin, so $\mathcal{B}_o^* = \{o\}$. For the half-orthant model, $\mathcal{B}_o$ will be infinite, since it contains $-\mathbb{Z}_+ e_1$.

For $x \in \mathbb{Z}^d$, let $R_x(p) = \sup\{k \in \mathbb{Z} : x + ke_1 \in \mathcal{B}_o(p)\}$. The following theorem is proved in [13].

THEOREM 2. *For the half-orthant model, let p_c be as in Theorem **C**. Then,*

if $p < p_c$ then $\mathcal{B}_o = \mathbb{Z}^d$ almost surely, and
if $p > p_c$ then R_x is finite for every $x \in \mathbb{Z}^d$.

The proof of Theorem 1 can be adapted to work for the cluster $\mathcal{B}_o$, with $\hat{\beta}_n(u) = \sup\{k \in \mathbb{Z} : k\mathbf{1} + nu \in \mathcal{B}_o\}$.

THEOREM 3. *Fix $d \ge 2$. For the half-orthant model, with $p_1 < 1$ and γ, C as in Theorem 1, the following hold for $p > p_1$, $\mathbb{P}_p$-a.s.:*

(a) *For $u \in \mathbb{Z}^d$, $\frac{\hat{\beta}_n(u)}{n} \to -\gamma(-u)$;*

(e) *$\frac{1}{n}\mathcal{B}_o \to -C$ in the sense that, for every $\epsilon > 0$ and every $r < \infty$, the following holds for sufficiently large (random) n:*

$$\left(O_r \cap \frac{1}{n}\mathcal{B}_o\right) \subset O_\epsilon + -C \quad \text{and} \quad O_r \cap -C \subset O_\epsilon + \frac{1}{n}\mathcal{B}_o.$$

Section 2 is devoted to the proof of Theorem 1. Section 3 verifies the lemmas required for that proof. Corollaries 1 and 2 are proved in Section 4. The proof of Theorem 3 is omitted.

2. Proof of Theorem 1. The results about limit cones in [11] used oriented percolation in $\mathbb{Z}^2$ and, therefore, subadditivity in an indirect way (see [6]). The higher-dimensional analogue in Theorem 1 will directly rely on subadditivity, borrowing from the approach taken with the shape theorems of first passage percolation; see [2]. In particular, our goal is to prove a shape theorem analogous to that of Cox and Durrett [4].

Recall that we are working with the half-orthant model, where $\mu(\{\mathcal{E}_+\}) = p = 1 - \mu(\{\mathcal{E}\})$. We will prove Theorem 1 via a sequence of lemmas.

Recall that $\mathbf{1} = e_1 + \cdots + e_d$. Our notation for coordinates will be that $x^{[i]} = x \cdot e_i$ for $i \in [d]$. For $0 \le \eta \le 1$, consider the cone

$$K_\eta = \{x \in \mathbb{R}^d : x \cdot \mathbf{1} \ge \eta \|x\|_1\}.$$

The case $\eta = 0$ is a half-space, while $\eta = 1$ is the positive orthant. The following result (together with a simple application of the Borel–Cantelli lemma) shows that, for each $\eta \in [0, 1)$, if p is sufficiently large, then $\mathcal{C}_o \subset K_\eta - M\mathbf{1}$ for some random $M > 0$ almost surely.

LEMMA 1. *There exists $\theta(d) > 1$ such that the following holds. For $\eta \in [0, 1)$ there is a $p_0(\eta, d) < 1$ for which $p > p_0$ implies that there exists $c_1 = c_1(\eta, d) > 0$ such that $\mathbb{P}_p(\mathcal{C}_o \not\subset K_\eta - m\mathbf{1}) \le c_1\theta^{-md}$ for each $m \in \mathbb{Z}_+$.*

This result is a reformulation of Theorem 4.2 of [9]. We will nevertheless give the full (but short) proof in Section 3, because elsewhere in this paper a similar argument will be needed in other settings.

A similar result holds for the cluster $\mathcal{B}_o$ which then implies that the bi-connected cluster $\mathcal{M}_o = \mathcal{C}_o \cap \mathcal{B}_o$ is finite whenever $p > p_0(\eta, d)$ for some $\eta > 0$.

OPEN PROBLEM 3. For the orthant or half-orthant model, show that $\mathcal{M}_o$ is a.s. finite, whenever $p > p_c$, where p_c is as in Theorem ***C***.

Before we continue with our sequence of lemmas, note that, for every $n \in \mathbb{N}$,

$$\beta_n(u + r\mathbf{1}) = \beta_n(u) - nr, \quad \text{for any } r \in \mathbb{Z}. \tag{4}$$

So if (a) of the Theorem holds for u, then it also holds for any $u + r\mathbf{1}$ ($r \in \mathbb{Z}$) with $\gamma(u + r\mathbf{1}) = \gamma(u) - r$. Since $o \in \mathcal{C}_o$, we know $\beta_1(o) \le 0$. By Lemma 1, $\beta_1(o)$ is an integrable random variable. By definition, $\beta_n(o) = \beta_1(o)$, so, in fact

$$\gamma(o) = \lim_{n \to \infty} \frac{1}{n}\beta_n(o) = 0. \tag{5}$$

Therefore, part (a) of the theorem holds for $u = o$. By the above remark it also holds for u any multiple of $\mathbf{1}$, with $\gamma(r\mathbf{1}) = -r$.

Therefore, to prove part (a) of the theorem we may assume u is not of the form $j\mathbf{1}$. Then, there exists $v \in \mathbb{R}^d$ such that

$$u \cdot v > 0 \quad \text{and} \quad v \cdot \mathbf{1} = 0 \tag{6}$$

(e.g., we may take v to be the projection of u onto the hyperplane orthogonal to $\mathbf{1}$, i.e., $v = u - \frac{u \cdot \mathbf{1}}{d}\mathbf{1}$). Fix any such v. Then, we may define $\sigma = \sigma(u, v) > 0$ by

$$\sigma = \frac{u \cdot v}{\|u\|_1 \|v\|_\infty}. \tag{7}$$

Define $\Lambda_{u,v}(m, n) = \{z \in \mathbb{Z}^d : mu \cdot v \le z \cdot v < nu \cdot v\}$. In other words, $\Lambda_{u,v}(m, n)$ is a slab in $\mathbb{Z}^d$, running orthogonal to v and containing mu and nu on its boundary; see Figure 4. Note

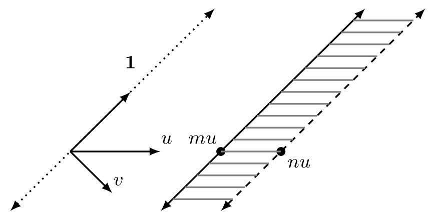

FIG. 4. *An illustration of $\Lambda_{u,v}(m,n)$ in two dimensions. Here, $\Lambda_{u,v}(m,n)$ is the hatched region, including the solid line but not the dashed line. $\Lambda_{u,v}(m)$ is the region from the dotted line to the solid line, including the former but not the latter. $\Lambda_{u,v}(n,\infty)$ is the region to the right of the dashed line.*

that if x is any point of $\Lambda_{u,v}(m,n)$, then $x+k\mathbf{1}\in\Lambda_{u,v}(m,n)$ for every $k\in\mathbb{Z}$. We also define $\Lambda_{u,v}(-\infty,n)=\{z\in\mathbb{Z}^d : z\cdot v<nu\cdot v\}$ and $\Lambda_{u,v}(n,\infty)=\{z\in\mathbb{Z}^d : z\cdot v\geq nu\cdot v\}$.

For $\Lambda\subset\mathbb{Z}^d$ and $x\in\Lambda$, let $\mathcal{C}_x[\Lambda]$ be x together with the set of $y\in\mathbb{Z}^d$, we can reach from x by following arrows (consistent with the environment) that start in Λ. Note that y itself need not be in Λ. For $n\in\mathbb{N}$, set $B_n(u,v)=\inf\{k : k\mathbf{1}+nu\in\mathcal{C}_o[\Lambda_{u,v}(0,n)]\}$. This is $>-\infty$ by Lemma 1 but (as remarked in the proof of Lemma 2) could $=+\infty$.

Recall the notation $p_0(\eta,d)$ from Lemma 1, and, henceforth, write $p_0(d)$ for $p_0(0,d)$. Write $k^+=\max\{k,0\}$ and $k^-=\max\{-k,0\}$ for $k\in\mathbb{R}$.

LEMMA 2. *Assume* (6) *and that $p>p_0(d)$. There exist constants Γ and n_0 such that*

$$\mathbb{E}_p\big[\big(B_n(u,v)\big)^-\big]\leq\mathbb{E}_p\big[\big(\beta_n(u)\big)^-\big]\leq\Gamma+\frac{n}{d}\|u\|_1\quad\text{for each } n\in\mathbb{N},\text{ and}$$

$$\big(\beta_n(u)\big)^+\leq\big(B_n(u,v)\big)^+\leq n\|u\|_1\quad\text{for } n\geq n_0.$$

Here, Γ depends only on d, but n_0 may depend on the choices of u and v (but not p). There exists an integrable random variable $Y\geq 0$ such that $\sup_n\frac{|\beta_n(u)|}{n}\leq Y$. For $n\geq n_0(u,v)$, $B_n(u,v)$ has a moment generating function that is finite in a neighbourhood of 0.

Lemma 2 will provide us with bounds that, together with a subadditivity argument, give the following result.

LEMMA 3. *For any $u\in\mathbb{Z}^d\setminus(\mathbb{Z}\mathbf{1})$, there is a constant $\gamma(u)\in\mathbb{R}$ such that, for all $p>p_0(d)$ and all $v\in\mathbb{R}^d$ satisfying* (6), *$\mathbb{P}_p$-almost surely,*

$$\gamma(u):=\lim_{n\to\infty}\frac{B_n(u,v)}{n}. \tag{8}$$

Moreover, $n^{-1}B_n(u,v)\to\gamma(u)$ in L^1, and $\gamma(u)=\inf_{n\geq 1}n^{-1}\mathbb{E}_p[B_n(u,v)]$.

Although the proof of Lemma 3 will leave open the possibility that the $\gamma(u)$ in Lemma 3 above may depend on the choice of v, in fact, it will be shown in the proof of Theorem 1(a) below that it does not.

We will need the following large deviation-type estimate.

LEMMA 4. *Assume that u and v satisfy* (6). *For any $p>p_0(d)$ and $\delta>0$, there is a $\bar{c}>0$ and an n_1 such that $\mathbb{P}_p(B_n(u,v)\geq n(\gamma(u)+4\delta))\leq e^{-n\bar{c}}$ whenever $n\geq n_1$.*

Since $\beta_n(u)\leq B_n(u,v)$, it is clear that

$$\limsup_{n\to\infty}\frac{\beta_n(u)}{n}\leq\gamma(u). \tag{9}$$

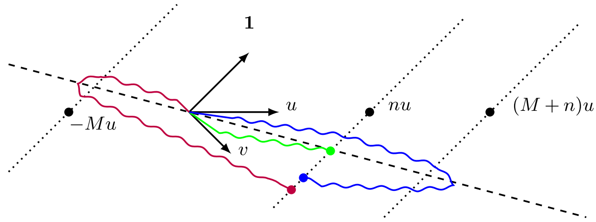

FIG. 5. *A schematic illustrating the events A'_n and A''_n. The dashed line lies in direction $u+\gamma(u)\mathbf{1}$. The squiggly (green) curve in the middle represents a path from o to $nu+B_n(u,v)\mathbf{1}$. $A''_n(M)$ is the event that a path can be found such as the squiggly (blue) one on the right, that reaches significantly below the middle path by first travelling into $\Lambda_{u,v}(M+n,\infty)$. $A'_n(M)$ is the event that a path, such as the squiggly (purple) one on the left, can be found, that also reaches well below the middle path, this time by first visiting $\Lambda_{u,v}(-\infty,-M)$.*

To go further, we must address the possibility of paths that cross on either the positive or negative side of $\Lambda_{u,v}(0,n)$ and then backtrack to get lower than $n\gamma(u)$.

For $M, n\in\mathbb{N}$, let $A'_n(M)$ denote the event that there exists a self-avoiding path consistent with the environment, running from o to some point $k\mathbf{1}+nu$ with $k<n\gamma(u)$ that hits $\Lambda_{u,v}(-\infty,-M)$; see Figure 5. Similarly, let $A''_n(M)$ be the event that there exists a self-avoiding path consistent with the environment, running from o to some point $k\mathbf{1}+nu$ with $k<n\gamma(u)$ that hits $\Lambda_{u,v}(M+n,\infty)$; see Figure 5.

Finally, let $\hat{A}_n$ be the event that there is a path from o to some $k\mathbf{1}$ with $k<0$ that is consistent with the environment and which reaches $\Lambda_{u,v}(n,\infty)$; see Figure 6.

LEMMA 5. *Assume that u and v satisfy (6). There exist $c=c(u,v)>1$ and $p_1=p_1(d)\in[p_0(d),1)$ such that $\mathbb{P}_p(A'_n(\lfloor cn\rfloor)$ i.o.$)=\mathbb{P}_p(A''_n(\lfloor cn\rfloor)$ i.o.$)=\mathbb{P}_p(\hat{A}_n$ i.o.$)=0$ whenever $p>p_1$.*

Let $c>1$ be as in Lemma 5. Define, for $n\in\mathbb{N}$,

$$\beta_n^0(u,v)=\inf\{k:k\mathbf{1}+nu\in\mathcal{C}_o[\Lambda_{u,v}(-\infty,n)]\},\tag{10}$$

$$\beta_n^1(u,v)=\inf\{k:k\mathbf{1}+nu\in\mathcal{C}_o[\Lambda_{u,v}(-\infty,\lfloor nc\rfloor+n)]\},\tag{11}$$

so $\beta_n(u)\le\beta_n^1(u,v)\le\beta_n^0(u,v)\le B_n(u,v)$, and, therefore, $\limsup\frac{\beta_n^1(u,v)}{n}\le\limsup\frac{\beta_n^0(u,v)}{n}\le\gamma(u)$.

LEMMA 6. *For $p>p_1(d)$, $\frac{\beta_n^0(u,v)}{n}\to\gamma(u)$ $\mathbb{P}_p$-a.s. and in L^1.*

LEMMA 7. *For $p>p_1(d)$, $\frac{\beta_n^1(u,v)}{n}\to\gamma(u)$ a.s.*

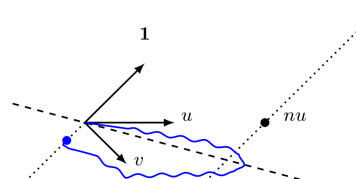

FIG. 6. *A schematic illustrating the event $\hat{A}_n$. The dashed line lies in direction $u+\gamma(u)\mathbf{1}$. $\hat{A}_n$ is the event that a path can be found such as the squiggly (blue) one that reaches significantly below o by first travelling into $\Lambda_{u,v}(n,\infty)$.*

The above lemmas will be proved in Section 3. Assuming the above lemmas, we are now ready to prove our main result.

PROOF OF THEOREM 1. (a): Let $\epsilon > 0$ and $M_n = \lfloor nc \rfloor$, where c is the constant of Lemma 5. If $\frac{\beta_n(u)}{n} < \gamma(u) - 2\epsilon$, then there is a self-avoiding path from o to some point $k\mathbf{1} + nu$ with $k < n(\gamma(u) - 2\epsilon)$. Such a path must either stay in $\Lambda_{u,v}(-\infty, M_n + n)$ or it must reach $\Lambda_{u,v}(M_n + n, \infty)$ (i.e., $A''_n(M_n)$ occurs). By Lemmas 7 and 5 this is only possible for finitely many n. Therefore, $\frac{\beta_n(u)}{n} \geq \gamma(u) - 2\epsilon$ for all sufficiently large n a.s. Together with (9), this verifies (a), and, in particular, it shows that $\gamma(u)$ does not depend on the choice of v satisfying (6).

(b): Let $r \in \mathbb{N}$, and let $m = nr$. From the definition of $\beta_n(u)$, we have that $\beta_{m/r}(ru) = \beta_m(u)$. From this and part (a) it follows that

$$\gamma(ru) = r\gamma(u) \tag{12}$$

for $r \in \mathbb{N}$. This holds for $r = 0$ as well by (5). The assertion that $\gamma(u + s\mathbf{1}) = \gamma(u) - s$ for all $s \in \mathbb{Z}$ was verified in the paragraph prior to (5).

Symmetry of γ under coordinate permutations is also straightforward, and the statement that $\gamma(u) \leq 0$ when u lies in the positive orthant follows from the fact that $\mathcal{C}_o$ contains this orthant. Setting $\eta = 0$ in Lemma 1 shows that if $p > p_0(d)$, then there is an $A \geq 0$ such that $\mathcal{C}_o \subset K_0 - A\mathbf{1}$. If $j\mathbf{1} + nu \in K_0 - A\mathbf{1}$, then $[(A + j)\mathbf{1} + nu] \cdot \mathbf{1} \geq 0$, so $j \geq -A - \frac{n}{d} u \cdot \mathbf{1}$. This proves that $\gamma(u) \geq -\frac{1}{d} u \cdot \mathbf{1}$, so $\gamma(u) \geq 0$ whenever $u \cdot \mathbf{1} \leq 0$.

The proof that γ is subadditive requires more care. We start by showing that $\gamma(u + w) \leq \gamma(u) + \gamma(w)$ in the case that there is a $v \in \mathbb{R}^d$ such that $u \cdot v > 0$, $w \cdot v > 0$, and $v \cdot \mathbf{1} = 0$. Let

$$\hat{\Lambda}_{u,v,w}(n) = \{z \in \mathbb{Z}^d : nu \cdot v \leq z \cdot v < n(u + w) \cdot v\}, \tag{13}$$

and take (for $n \in \mathbb{N}$)

$$\hat{B}_n(u, v, w) = \inf\{k : (B_n(u, v) + k)\mathbf{1} + n(u + w) \in \mathcal{C}_{B_n(u,v)\mathbf{1}+nu}[\hat{\Lambda}_{u,v,w}(n)]\}. \tag{14}$$

Because $\Lambda_{u,v}(0, n)$ and $\hat{\Lambda}_{u,v,w}(n)$ do not overlap, we know that the environments in $\hat{\Lambda}_{u,v,w}(n)$ are independent of $B_n(u, v)$. Therefore, $\hat{B}_n(u, v, w)$ has the same law as $B_n(w)$, and it follows from Lemma 3 that $\frac{1}{n}\hat{B}_n(u, v, w)$ converges in probability to $\gamma(w)$. Of course, $\frac{1}{n}B_n(u, v)$ and $\frac{1}{n}B_n(u + w, v)$ converge in probability to $\gamma(u)$ and $\gamma(u + w)$. Now, $\hat{\Lambda}_{u,v,w}(n) \cup \Lambda_{u,v}(0, n) \subset \Lambda_{u+w,v}(0, n)$, so by concatenating paths we know that

$$B_n(u + w, v) \leq B_n(u, v) + \hat{B}_n(u, v, w).$$

Therefore, $\gamma(u + w) \leq \gamma(u) + \gamma(w)$.

It remains to show subadditivity when such a v does not exist. We start with the case $w = -u$ and show that

$$\gamma(u) + \gamma(-u) \geq 0 \quad [\text{which} = \gamma(u + (-u))]. \tag{15}$$

Suppose that (15) fails. Let $\epsilon > 0$ be such that $\gamma(u) + \gamma(-u) + 2\epsilon < 0$, and $n_\epsilon \in \mathbb{N}$ be such that $n_\epsilon > -2/(\gamma(u) + \gamma(-u) + 2\epsilon)$. Choose v so $u \cdot v > 0$ and $v \cdot \mathbf{1} = 0$. For $n \gg n_\epsilon$, set $k = \lceil n(\gamma(u) + \epsilon)\rceil$, $j = \lceil n(\gamma(-u) + \epsilon)\rceil$, $z = nu + k\mathbf{1}$ and $y = (j + k)\mathbf{1}$. Then, since $\mathcal{C}_o$ has no holes, with high probability there is a path in $\mathcal{C}_o$ from o to z, and a path in $\mathcal{C}_z$ from z to y. Concatenating them gives a path from o to y that reaches $\Lambda_{u,v}(n, \infty)$. But $j + k \leq n[\gamma(u) + \gamma(-u) + 2\epsilon] + 2 < 0$, and, therefore, $\mathbb{P}_p(\hat{A}_n) \to 1$ which contradicts Lemma 5. Thus, (15) holds.

Now, consider the general case that $u, w \in \mathbb{Z}^d$, yet no v exists as above. Write $u = s\mathbf{1} + u'$ and $w = t\mathbf{1} + w'$ where u' and w' are $\perp \mathbf{1}$. Then, $s = \frac{1}{d} u \cdot \mathbf{1}$, so $du' \in \mathbb{Z}^d$ and, likewise,

$dw' \in \mathbb{Z}^d$. The half-spaces $\{z : z \cdot u' > 0\}$ and $\{z : z \cdot w' > 0\}$ do not intersect, as if they did, we could find a v in their intersection, that is, $\perp \mathbf{1}$. Therefore, in fact, u' and w' point in opposite directions. Thus, there is an $x \in \mathbb{Z}^d$ and $i, j \in \mathbb{Z}$ such that $du' = ix$ and $dw' = jx$. By interchanging u and w if necessary, we may assume that $|i| \le |j|$. Replacing x by $-x$ if necessary, we may also assume that $j \ge 0 \ge i$. Then, using the fact that $i + j \in \mathbb{Z}_+$ we get

$$\begin{aligned}\gamma(du + dw) &= \gamma(du' + dw') - d(s+t)\\ &= \gamma((i+j)x) - d(s+t)\\ &= (i+j)\gamma(x) - d(s+t)\\ &= i\gamma(x) - ds + j\gamma(x) - dt\\ &= -\gamma(-ix) - ds + \gamma(jx) - dt\\ &\le \gamma(ix) - ds + \gamma(jx) - dt\\ &= \gamma(du) + \gamma(dw).\end{aligned}$$

Therefore, $\gamma(u + w) \le \gamma(u) + \gamma(w)$, as required.

(c) and (d): It is now simple to extend the definition of $\gamma(u)$ to $u \in \mathbb{Q}^d$ by taking the limit over those n for which $nu \in \mathbb{Z}^d$. Then, some algebra shows that (b) can be extended to $u, w \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+$.

From subadditivity we see that

$$\left|\gamma(w) - \gamma(u)\right| \le \max\left(\left|\gamma(u-w)\right|, \left|\gamma(w-u)\right|\right).$$

Therefore, Lemma 2 (which applies only to $\mathbb{Z}^d$) together with (12) implies that, for $u, w \in \mathbb{Q}^d$,

$$\left|\gamma(u) - \gamma(w)\right| \le \|u - w\|_1. \tag{16}$$

This, in turn, implies the existence of an extension of γ to $\mathbb{R}^d$ that is uniformly Lipschitz (for $u \in \mathbb{R}^d$, take $\gamma(u)$ to be the limit of a sequence $\gamma(u_m)$ where u_m are rationals approaching u). One can now verify that (b) holds with $u, w \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$, by taking limits via sequences of rationals u_m, w_m, r_m converging to u, w, r, respectively. This shows (c). Now, (d) also follows in turn, using the properties (b) now established on $\mathbb{R}^d$ and $\mathbb{R}$: C is closed because γ is continuous; it is a cone because if $x \in C$ and $r \ge 0$, then $\gamma(rx) = r\gamma(x) \le 0$, so $rx \in C$ (so C is the cone with shape $\chi = \{x \in C : x \cdot \mathbf{1} = 1\}$); C is convex because if $x, y \in C$ and $s \in (0, 1)$, then $sx \in C$ and $(1-s)y \in C$ and $\gamma(sx + (1-s)y) \le \gamma(sx) + \gamma((1-s)y) \le 0$, so $sx + (1-s)y \in C$. The remaining assertions of (d) are straightforward.

(e): Fix r and ϵ and an integer $m > \frac{4r}{\epsilon}$. The set $\mathcal{U} := O_{m+2}$ is finite, so by part (a) we may find a random J such that if $j \ge J$, then $|\frac{\beta_j(u)}{j} - \gamma(u)| < \frac{1}{2}$ for every $u \in \mathcal{U}$. We will show the assertion of (e) for every n such that

$$n \ge \max\left(\frac{Jm}{r}, \frac{4}{\epsilon}\right), \tag{17}$$

so suppose n satisfies (17). Set $k = \lceil \frac{nr}{m} \rceil \ge \frac{nr}{m} \ge J$.

We first show that $O_r \cap C \subset O_\epsilon + n^{-1}\mathcal{C}_o$. Let $z \in O_r \cap C$, so $\gamma(z) \le 0$. We know that if some $y \in \mathbb{R}^d$ satisfies $\|y\|_\infty \le m$, then there is a $u \in \mathcal{U}$ with $u^{[i]} \le y^{[i]} \le u^{[i]} + 1$ for every $i \in [d]$, and, moreover, that $u' = u + 2\mathbf{1} \in \mathcal{U}$. Since $\|\frac{n}{k}z\|_\infty \le \frac{nr}{k} \le m$, we may choose u and u' as above, for $y = \frac{n}{k}z$. In particular,

$$\left\|\frac{k}{n}u' - z\right\|_\infty = \frac{k}{n}\left\|u' - \frac{n}{k}z\right\|_\infty \le \frac{2k}{n} \le \frac{2}{n}\left(\frac{nr}{m} + 1\right) < \epsilon$$

by the choice of m and the fact that $\frac{1}{n} < \frac{\epsilon}{4}$. So we need only show that $ku' \in \mathcal{C}_o$.

Now, $\gamma(z) \le 0$, so $\gamma(\frac{n}{k}z) \le 0$, and, because $u + \mathbf{1} - \frac{n}{k}z$ lies in the positive quadrant, also $\gamma(u + \mathbf{1} - \frac{n}{k}z) \le 0$. By subadditivity it follows that $\gamma(u + \mathbf{1}) \le 0$, and, hence, $\gamma(u') \le -1$. Because $k \ge J$, our choice of J implies that $\beta_k(u') < 0$, so, in fact, $ku' \in \mathcal{C}_o$, as required.

Conversely, consider $z \in O_r \cap \frac{1}{n}\mathcal{C}_o$ so that, by definition, there is a path consistent with the environment from o to nz. Construct $u, u' \in \mathcal{U}$, as above, so $u^{[i]} \le (\frac{n}{k}z)^{[i]} \le u^{[i]} + 1$ for each $i \in [d]$. Therefore, $k(u + \mathbf{1}) - nz$ belongs to the positive orthant, so there is also a path consistent with the environment running from nz to $k(u + \mathbf{1})$. Concatenating the two paths, we see that $\beta_k(u + \mathbf{1}) = \beta_1(k(u + \mathbf{1})) \le 0$. Recalling (4), it follows that

$$\frac{\beta_k(u')}{k} = \frac{\beta_k(u + \mathbf{1})}{k} - 1 \le -1.$$

Since $k \ge J$, we obtain that $\gamma(u') \le -\frac{1}{2} < 0$. Therefore, $u' \in C$ and, in turn, $\frac{k}{n}u' \in C$. As above, $\|\frac{k}{n}u' - z\|_\infty < \epsilon$, and the proof is complete. □

2.1. *Further discussion.* While the approach we have taken bears some relation to the subadditivity arguments of first passage percolation, it is in many ways closer to the approach used for oriented percolation (see [6]). For example, consider an oriented percolation model where vertices $x \in \mathbb{Z}^2$ connect at random to one or both of $x + e_1 + e_2$ or $x + e_1 - e_2$. The analogue of our $\beta_n(u)$ is then $\beta_n^{\text{op}} = \inf\{k : o \text{ connects to } (n, k)\}$.

Subadditivity for β_n^{op} is easily obtained (as we saw it was for $\beta_n(u)$). The difference is that, for oriented percolation, the orientation guarantees that only new regions get explored which immediately implies independence properties. This, in turn, gives rise to the stationarity required by the subadditive ergodic theorem. In our setting, paths can wander away and then return, so independence fails. This means that it is not trivial to rule out the special role of the origin being felt at large distances. In other words, conditioning on having o connect to a distant vertex x could, in principle, bias connections from x in comparison to connections from a typical vertex.

Our approach has been to find analogues of the $\beta_n(u)$ that restrict the ability of paths to wander and thus restore independence. Then, we must control the probabilities of wandering in order to show that the $\beta_n(u)$ behave in the same way as these analogues.

There are actually two natural kinds of shape theorems that one might ask about with these kinds of models—corresponding, respectively, to *geometric* vs. *metric* properties of $\mathcal{C}_o$. The former, which is the subject of this paper, also occurs with oriented percolation (as indicated above) and, for us, is only relevant with $p > p_c$. This kind of result seeks to describe the boundary of the cluster, seen from far away.

The metric approach, in contrast, studies $T(x, y)$, defined as the minimal number of steps it takes, consistent with the environment, to travel from x to y. As with first passage percolation or supercritical percolation, one might attempt to obtain a shape theorem for the set of vertices reachable in n or fewer steps by considering the asymptotics of $\frac{1}{n}T(x, nu)$; see [14] or [2] for details or [5] for a very general treatment of such problems. The two approaches are, of course, related, since for the latter to have a finite limit we must have $nu \in \mathcal{C}_o$ for all n sufficiently large. As we have seen, the latter is true for every $u \in \mathbb{Z}^d$ when $p < p_c$ but is a restriction on u when $p > p_c$.

The problem of having some $T(x, y) = \infty$ can also arise in first passage percolation; see [16] and [3], building on earlier work of [1, 7], and [8], §5g. But there this is a local problem, arising because there are isolated "bubbles" within $\mathbb{Z}^d$ that cannot be reached. A similar issue would arise if we studied the corresponding $T^*(x, y)$ for the orthant model. On the left side of Figure 1, for example, $T^*(o, y) = \infty$ if y belongs to one of the cavities within $\mathcal{C}_o^*$.

In our setting a different problem arises, even for the half-orthant model, when $p > p_c$. Namely, that $T(o, y) = \infty$ for most $y \in \mathbb{Z}^d$ and we have very little control of $T(o, y)$ as y

approaches the boundary of $\mathcal{C}_o$. We hope that combining the current results with arguments of first-passage percolation may make it possible to prove a metric shape theorem for our models.

3. Proofs of lemmas. Recall that $\Omega_+ = \{x \in \mathbb{Z}^d : \mathcal{G}_x = \mathcal{E}_+\}$ and $\Omega_- = \mathbb{Z}^d \setminus \Omega_+$.

PROOF OF LEMMA 1. We let the constant c vary from line to line. There is a $\theta > 1$ such that the number of self-avoiding paths in $\mathbb{Z}^d$ from o of length n is, at most, θ^n. For $m \in \mathbb{Z}_+$, let $\partial_{\eta,m}$ be the set of vertices x adjacent to some vertex in $K_\eta - m\mathbf{1}$ but not in $K_\eta - m\mathbf{1}$.

If $\mathcal{C}_o \not\subset K_\eta - m\mathbf{1}$, there is an $x \in \partial_{\eta,m}$ and a self-avoiding path ω from o to x consistent with the environment. Set $x^+ = \sum_{i=1}^d \max\{x \cdot e_i, 0\}$ and $x^- = -\sum_{i=1}^d \min\{x \cdot e_i, 0\}$. For some $k \geq 0$, ω takes $k + x^+$ steps in a direction from $\mathcal{E}_+$, and $k + x^-$ steps in a direction from $\mathcal{E}_-$. Therefore, at least $k + x^-$ vertices come from Ω_-. If $\theta^2(1-p) < 1$, then we obtain

$$\begin{aligned}\mathbb{P}_p(\mathcal{C}_o \not\subset K_\eta - m\mathbf{1}) &\leq \sum_{x\in\partial_{\eta,m}} \sum_{k=0}^{\infty} \theta^{\|x\|_1+2k}(1-p)^{k+x^-}\\ &\leq c \sum_{x\in\partial_{\eta,m}} \theta^{\|x\|_1}(1-p)^{x^-}.\end{aligned}$$

For $x = y - m\mathbf{1}$, we have $x \cdot \mathbf{1} = y \cdot \mathbf{1} - md$ and $\|y\|_1 \leq \|x\|_1 + md$. If $x \notin K_\eta - m\mathbf{1}$, then $y \notin K_\eta$, so

$$\begin{aligned}x^+ - x^- = x \cdot \mathbf{1} = y \cdot \mathbf{1} - md &< \eta\|y\|_1 - md \leq \eta\|x\|_1 - md(1-\eta)\\ &= \eta(x^+ + x^-) - md(1-\eta).\end{aligned}$$

Therefore, $x^+ < \frac{1+\eta}{1-\eta}x^- - md$ so $\|x\|_1 = x^+ + x^- < \frac{2}{1-\eta}x^- - md$. The number of such x with $x^- = j$ is, therefore, at most, cj^d, uniformly in m. Therefore,

$$\mathbb{P}_p(\mathcal{C}_o \not\subset K_\eta - m\mathbf{1}) \leq c\theta^{-md} \sum_{j=0}^{\infty} \bigl[\theta^{\frac{2}{1-\eta}}(1-p)\bigr]^j j^d.$$

Provided $p > 1 - \theta^{-\frac{2}{1-\eta}}$, this gives us a bound $c\theta^{-md}$, as claimed. $\square$

PROOF OF LEMMA 2. For the first inequality, take $\eta = 0$ in Lemma 1 to see that if $p > p_0(d)$, then there is an integrable $A \geq 0$ such that $\mathcal{C}_o \subset K_0 - A\mathbf{1}$. Thus, if $k = \inf\{j \in \mathbb{Z} : j\mathbf{1} + nu \in K_0 - A\mathbf{1}\}$, it follows that $B_n(u,v) \geq \beta_n(u) \geq k$ and so $(B_n(u,v))^- \leq (\beta_n(u))^- \leq k^-$. But $j\mathbf{1} + nu \in K_0 - A\mathbf{1} \Leftrightarrow [(A+j)\mathbf{1} + nu] \cdot \mathbf{1} \geq 0 \Leftrightarrow j \geq -A - \frac{n}{d}\sum u_i$. If this holds, then $j \geq -A - \frac{n}{d}\|u\|_1$, so, in particular, $k \geq -A - \frac{n}{d}\|u\|_1$ and, therefore,

$$(\beta_n(u))^- \leq A + \frac{n}{d}\|u\|_1. \tag{18}$$

Therefore, the desired inequality holds, with $\Gamma = \mathbb{E}_p[A]$.

We know that $\mathcal{C}_o$ contains the positive orthant, so $k\mathbf{1} + nu \in \mathcal{C}_o$ whenever all its coordinates are nonnegative. In other words,

$$\beta_n(u) \leq n \max_i\{(u^{[i]})^-\} \leq n\|u\|_1 \tag{19}$$

for every n. Combined with (18), this shows that the $\sup_n \frac{|\beta_n(u)|}{n} \leq Y$, where $Y = A + (1 + 1/d)\|u\|_1$ is integrable.

To complete the second set of inequalities of the lemma, we use the same argument that gave us (19) but this time for $B_n(u,v)$. To make this work, we must show that if $x = k\mathbf{1} + nu$

lies in the positive orthant, then it also lies in $\mathcal{C}_o[\Lambda_{u,v}(0,n)]$. We will be able to carry this out for $n \ge n_0$, where n_0 is chosen so that $n_0 u \cdot v > 2\|v\|_\infty$. [1]

We will construct a path $y_0, y_1, \dots, y_m$ from $y_0 = o$ to $y_m = x$, where at each stage $y_{j+1} = y_j + e_{i_j}$ for some $i_j \in [d]$. The i_j's will be chosen so that $0 \le y_j \cdot v < nu \cdot v$ (for $j \ne m$) and $0 \le y_j^{[i]} \le x^{[i]}$ for each i, j. This shows that $x \in \mathcal{C}_o[\Lambda_{u,v}(0,n)]$.

Suppose we've got as far as y_j, and $y_j \ne x$. Consider $I = \{i \in [d] : y_j^{[i]} < x^{[i]}\}$ which must be nonempty (since $y_j \ne x$). If

$$\|v\|_\infty \le y_j \cdot v < nu \cdot v - \|v\|_\infty, \tag{20}$$

then choose any $i \in I$ as i_j. Since

$$0 \le y_j \cdot v - \|v\|_\infty \le (y_j + e_i) \cdot v \le y_j \cdot v + \|v\|_\infty < nu \cdot v,$$

the iteration continues.

If (20) fails, then either $y_j \cdot v < \|v\|_\infty$ or $y_j \cdot v \ge nu \cdot v - \|v\|_\infty$. In the first case, choose i_j to be any $i \in I$ with $e_i \cdot v > 0$. Note that there must be some such i, as otherwise

$$\begin{aligned} x \cdot v &= \sum_{i \in I} x^{[i]} v^{[i]} + \sum_{i \notin I} y_j^{[i]} v^{[i]} \\ &\le \sum_{i \in I} y_j^{[i]} v^{[i]} + \sum_{i \notin I} y_j^{[i]} v^{[i]} \\ &= y_j \cdot v \le \|v\|_\infty \end{aligned}$$

which contradicts the choice of n_0 (recall that $x \cdot v = nu \cdot v$). Then, $0 \le y_j \cdot v < (y_j + e_i) \cdot v \le 2\|v\|_\infty < nu \cdot v$, because $n \ge n_0$, and, again, the iteration continues. In the second case, choose $i \in I$ with $e_i \cdot v \le 0$, if possible. If there is indeed such an i, then $0 < nu \cdot v - 2\|v\|_\infty \le (y_j + e_i) \cdot v \le y_j \cdot v < nu \cdot v$ (again because $n \ge n_0$), and, once more, the iteration continues. Finally, if in the second case all $i \in I$ have $e_i \cdot v > 0$, then choose any $i \in I$ as i_j. As long as $j < m$, x is y_j plus a sum of multiple such e_i, so we'll have $0 \le y_j \cdot v < (y_j + e_i) \cdot v < x \cdot v = nu \cdot v$, and, again, the iteration continues.

From the above we have $|B_n(u,v)| \le A + n\|u\|_1$. Finiteness of the moment generating function of $B_n(u,v)$ now follows, because Lemma 1 establishes an exponential bound for the tail of the random variable A. $\square$

PROOF OF LEMMA 3. For $n \in \mathbb{N}$, let $B_{0,n}(u,v) = B_n(u,v)$. For $n > m > 0$, let

$$B_{m,n}(u,v) = \inf\{k : (B_m(u,v) + k)\mathbf{1} + nu \in \mathcal{C}_{B_m(u,v)\mathbf{1}+mu}[\Lambda_{u,v}(m,n)]\}. \tag{21}$$

Then,

$$B_n(u,v) \le B_m(u,v) + B_{m,n}(u,v), \tag{22}$$

because concatenating the given paths from o to $B_m(u,v)\mathbf{1} + mu$ and from there to $(B_m(u,v) + B_{m,n}(u,v))\mathbf{1} + nu$ produces a path from o to the latter point that also stays in $\Lambda_{u,v}(0,n)$.

We will apply the subadditive ergodic theorem (in a version due to Liggett [15]) to the $B_n(u,v)$. The stationarity properties required there follow easily from our construction and an independence argument. Specifically, the fact that $\Lambda_{u,v}(0,m)$ and $\Lambda_{u,v}(m,n)$ are disjoint

[1] To show that such a restriction on n is actually needed, suppose $d = 3$, $v \cdot e_2 < 0$, $v \cdot e_3 < 0$, and $v \cdot e_1 > -v \cdot e_2 > 0$. Take $u = -100\mathbf{1} + e_1 + e_2$, $k = 100$ and $n = 1$. Then, $x = e_1 + e_2$ and $u \cdot v = (e_1 + e_2) \cdot v > 0$. But if $o \in \Omega_+$ (so $\mathcal{G}_o = \mathcal{E}_+$), then we are unable to start a path that reaches x via $\Lambda_{u,v}(0,1)$, since any initial step e_i will exit $\Lambda_{u,v}(0,1)$.

means that their environments are independent. Therefore, if we shift the environments of $\Lambda_{u,v}(m,n)$ by a vector $Z\mathbf{1}$, where Z is determined by the environments of $\Lambda_{u,v}(0,m)$, then we get an environment whose law is the same and which is still independent of the environments of $\Lambda_{u,v}(0,m)$. With $Z = B_m(u,v)$, it is this shifted environment that determines $B_{m,n}(u,v)$. Therefore, $B_{m,n}(u,v)$ is independent of $B_m(u,v)$.

It follows that for every $k \geq 1$ and $m \geq 0$, B_k and any $B_{m,m+k}$ have the same distribution. The remainder of hypotheses (1.8) and (1.9) of [15] follow similarly. Note that later on, in the proof of Lemma 6], we will encounter a similar situation where the analogous condition holds only for $m \geq 1$ but not for $m = 0$. This means that the result of [15] won't apply there, something we'll have to work around when we get to that point.

It remains only to check the moment hypotheses in [15] which are that $\mathbb{E}[|B_n(u,v)|] < \infty$ for every n and $\mathbb{E}[B_n(u,v)] \geq -cn$ for some c. In fact, it is easily seen that the argument there works if we assume instead that these inequalities hold for $n \geq n_0$, where n_0 is fixed. The one point in Liggett's proof that isn't a trivial change comes five lines after (2.4a), where he uses that $\gamma_n \leq k\gamma_m + \gamma_\ell$, where ℓ is small and γ_ℓ is finite. But we can get around that by using $\gamma_n \leq (k-i)\gamma_m + \gamma_{im+\ell}$ for some i chosen so $\gamma_{im+\ell} < \infty$. The above inequalities then follow from Lemma 2. A similar remark applies to the argument in [15] which shows that $\gamma(u) = \inf_{n\geq 1} n^{-1}\mathbb{E}_p[B_n(u,v)]$.

Applying the subadditive ergodic theorem, we have obtained that there is a deterministic $\gamma(u) \in (-\infty,\infty)$ such that $\frac{B_n(u,v)}{n} \to \gamma(u)$ a.s. and in L^1. □

PROOF OF LEMMA 4. Since $\lim \frac{\mathbb{E}_p[B_n(u,v)]}{n} = \inf \frac{\mathbb{E}_p[B_n(u,v)]}{n} = \gamma(u)$ by Lemma 3, we may choose j_0 (nonrandom) so large that $j_0\gamma(u) \leq \mathbb{E}_p[B_{j_0}(u,v)] \leq j_0(\gamma(u)+\delta)$. We may also ensure that $j_0 \geq n_0$, where n_0 is as in Lemma 2. Take n_1' sufficiently large that $n_1' \geq j_0$ and $n_1'\delta > 2j_0\|u\|_1$.

For $n \geq n_1'$, write $n = j_0K + j$ where $j_0 \leq j < 2j_0$. Set $Y_k = B_{(k-1)j_0,kj_0}(u,v)$ and $\bar{y} = \mathbb{E}_p[Y_1]$. Then,

$$B_n(u,v) \leq \sum_{k=1}^{K} Y_k + B_{j_0K,n}(u,v)$$

where the Y_k are independent and identically distributed. Because $n - Kj_0 \geq j_0 \geq n_0$, Lemma 2 implies that $B_{j_0K,n}(u,v) \leq 2j_0\|u\|_1 < n\delta$. Therefore, whenever $B_n(u,v) \geq n(\gamma(u)+4\delta)$, we must also have $\sum_{k=1}^K Y_k \geq n(\gamma(u)+3\delta)$.

First, suppose $\gamma(u) \geq 0$. Then, $n(\gamma(u)+3\delta) \geq Kj_0(\gamma(u)+3\delta) \geq Kj_0(\gamma(u)+2\delta) \geq K(\bar{y}+j_0\delta)$. Now, assume instead that $\gamma(u) < 0$. Since $n\delta > 2j_0\|u\|_1 \geq -2j_0\gamma(u)$, and $n \leq j_0(K+2)$, we have that $n(\gamma(u)+3\delta) \geq j_0(K+2)\gamma(u) + 3n\delta \geq Kj_0(\gamma(u)+2\delta) + [n\delta + 2j_0\gamma(u)] \geq K(\bar{y}+j_0\delta)$. Therefore, in either case, whenever $B_n(u,v) \geq n(\gamma(u)+4\delta)$, we must also have $\sum_{k=1}^K Y_k \geq K(\bar{y}+j_0\delta)$.

Lemma 2 shows that the moment generating function ψ of the Y_k is finite in a neighbourhood of 0. Standard large deviations estimates then give the existence of a $t > 0$ such that $\mathbb{P}_p(\sum_{k=1}^K Y_k \geq K(\bar{y}+j_0\delta)) < e^{-tK}$ for all K sufficiently large (i.e., whenever $n \geq n_1$, for some choice of $n_1 \geq n_1'$ that makes $K \geq j_0$ as well as $K \geq 2$).

Setting $\bar{c} = \frac{t}{2j_0}$, we use the bound $n \leq j_0K + 2j_0 = j_0(K+2)$ to conclude that $\bar{c}n \leq \frac{t(K+2)}{2} = tK(\frac{1}{2}+\frac{1}{K}) \leq tK$ (the latter since $K \geq 2$), and the conclusion of the lemma holds. □

PROOF OF LEMMA 5. We will estimate $\mathbb{P}_p(A_n'(M))$ (where $M = \lfloor cn \rfloor$ for some $c > 1$ to be chosen later) using the argument of Lemma 1; see Figure 5. On the event $A_n'(M)$, we have a self-avoiding path ω (consistent with the environment) from o to a point $x =$

$j\mathbf{1} + nu$ with $j < n\gamma(u)$ that reaches some point $y_0 \in \mathbb{Z}^d$ with $y_0 \cdot v \le -Mu \cdot v$. Recall that $x^+ = \sum_{i=1}^d \max\{x \cdot e_i, 0\}$ and $x^- = -\sum_{i=1}^d \min\{x \cdot e_i, 0\}$. The path takes $k + x^+$ steps using directions in $\mathcal{E}_+$ and $k + x^-$ steps using directions in $\mathcal{E}_-$, for some $k \ge 0$. For a given k and x, the probability that such a path ω exists is, at most, $\theta^{\|x\|_1 + 2k}(1-p)^{k+x^-}$.

Choose $0 < \alpha < 1$, and set $N = \lfloor n^\alpha \rfloor$. Set $\eta = 0$, and assume $p > p_0(d)$. Lemma 1 implies that there is an event G of probability, at most, $c_1\theta^{-Nd}$ such that off G we have $x \in K_0 - N\mathbf{1}$. Therefore, $x \cdot \mathbf{1} \ge -Nd$, and so

$$jd + n\|u\|_1 \ge (j\mathbf{1} + nu) \cdot \mathbf{1} \ge -Nd. \tag{23}$$

Let $n \ge n_0$, where n_0 is the constant in Lemma 2. Then, since $n^{-1}\beta_n(u) \le \|u\|_1$ for $n \ge n_0$ (by Lemma 2), we have that $j \le n\|u\|_1$. Choose $n_0' \ge n_0$ sufficiently large to make $Nd \le (d-1)n\|u\|_1$ when $n \ge n_0'$. Together with (23), this gives

$$|j| \le n\|u\|_1 \quad \text{for } n \ge n_0'. \tag{24}$$

It follows that, for $n \ge n_0'$, we have

$$\|x\|_1 \le n\|u\|_1 + |j|d \le n\|u\|_1(1+d). \tag{25}$$

Therefore, for fixed k and x as above, the probability that such a path ω exists is, at most, $\theta^{n\|u\|_1(1+d)+2k}(1-p)^k$.

It takes at least $\|y_0\|_1$ steps for the walk ω to reach $y_0 \in H := \{y \in \mathbb{Z}^d : y \cdot v \le -Mu \cdot v\}$. But $\|y_0\|_1 \cdot \|v\|_\infty \ge -y_0 \cdot v \ge Mu \cdot v$. Therefore, $\|y_0\|_1 \ge M\sigma\|u\|_1$. From there ω must travel back and reach the half-space $H' := \{y' : y' \cdot v \ge nu \cdot v\}$, and, since $\|y' - y_0\|_1\|v\|_\infty \ge (y' - y_0) \cdot v \ge (n+M)u \cdot v$ for $y' \in H'$, this takes at least $\sigma(n+M)\|u\|_1$ steps. It follows that the total length $2k + \|x\|_1$ of the path ω is, at least, $\sigma(n+2M)\|u\|_1$. Therefore, $k \ge \frac{1}{2}(\sigma(n+2M)\|u\|_1 - \|x\|_1)$. For $k \ge n_0'$, (25) implies that

$$k \ge \frac{1}{2}\big(\sigma(n+2M) - n(1+d)\big)\|u\|_1. \tag{26}$$

Now, choose $c > 1$ so large that $\sigma(n + 2\lfloor cn \rfloor) \ge n(2+d)$ for every $n \ge n_0'$, and let $M = \lfloor cn \rfloor$. By (26) (with $M = \lfloor cn \rfloor$) we have $k \ge \frac{n\|u\|_1}{2}$, whenever $n \ge n_0'$.

By (24) there are, at most, $1 + 2n\|u\|_1$ possible choices for j (and hence x), for any given n. Summing over k and these x's and adding back the probability $\mathbb{P}_p(G) \le c_1\theta^{-Nd}$, we obtain that

$$\begin{aligned} \mathbb{P}_p\big(A_n'(\lfloor cn \rfloor)\big) &\le c_1\theta^{-Nd} + (1 + 2n\|u\|_1)\theta^{n\|u\|_1(1+d)} \sum_{k \ge \frac{n\|u\|_1}{2}} \big[\theta^2(1-p)\big]^k \\ &\le c_1\theta^{-Nd} + \frac{1 + 2n\|u\|_1}{1 - \theta^2(1-p)} \theta^{n\|u\|_1(2+d)}(1-p)^{\frac{n\|u\|_1}{2}} \end{aligned}$$

for $n \ge n_0'$. Recalling that $N = \lfloor n^\alpha \rfloor$, we see that these terms are summable for any $p > p_1$, provided $\theta^{2+d}(1-p_1)^{\frac{1}{2}} < 1$ and $p_1 \ge p_0(d)$. Note that p_1 does not depend on u and v, though c may. This proves the result for A_n'. The claims for A_n'' and $\hat{A}_n$ are proved similarly; see Figures 5 and 6. □

PROOF OF LEMMA 6. Let $p > p_0(d)$, and let $\epsilon > 0$. Fix $c > 1$, as in Lemma 5, and let $M = \lfloor nc \rfloor$, $g = \lceil n(c\gamma(u) + 2\epsilon) \rceil$ and $h = \lfloor n(\gamma(u) - 3\epsilon) \rfloor$. Set $z = -g\mathbf{1} - Mu$. For $n \in \mathbb{N}$, take

$$B_n^0 = B_n^0(u, v) = \inf\{k : k\mathbf{1} + nu \in \mathcal{C}_o[\Lambda_{u,v}(-M, n)]\} \tag{27}$$

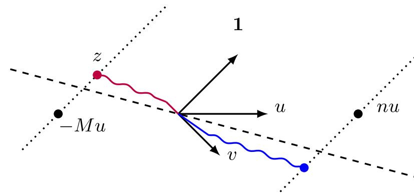

FIG. 7. *A schematic illustrating the event A_n. The dashed line lies in direction $u + \gamma(u)\mathbf{1}$. The squiggly* (*blue*) *curve on the right represents a path from o to $nu + B_n^0\mathbf{1}$ in $\Lambda_{u,v}(-M, n)$. A_n is the event that this curve ends significantly below the dashed line, as shown in the figure. The squiggly* (*purple*) *path on the left connects a point z to o through $\Lambda_{u,v}(-M, n)$, as in* (28), *where z lies significantly above the dashed line. If both left and right paths exist, concatenating them yields a path as in* (29).

and

$$A_n = \{B_n^0 < n(\gamma(u) - 3\epsilon)\}.$$

We will show that

$$\mathbb{P}_p(A_n) \le \mathbb{P}_p(o \notin \mathcal{C}_z[\Lambda_{u,v}(-M, n)]) \tag{28}$$

$$+ \mathbb{P}_p(\exists k\mathbf{1} + nu \in \mathcal{C}_z[\Lambda_{u,v}(-M, n)] \text{ with } k \le h) \tag{29}$$

$$\to 0, \quad \text{as } n \to \infty;$$

see Figure 7. To see the inequality, suppose that A_n holds so that there is a path from o to some $k\mathbf{1} + nu$ with $k \le h$ that stays in $\Lambda_{u,v}(-M, n)$. Either there is no path from z to o that stays in $\Lambda_{u,v}(-M, n)$, or there is such a path, in which case concatenating the two would give a path from z to $k\mathbf{1} + nu$.

Next, we must show that both terms (28) and (29) must $\to 0$. Consider (29). By translation invariance it is equal to

$$\begin{aligned}
&\mathbb{P}_p(\exists k\mathbf{1} + (M + n)u \in \mathcal{C}_o[\Lambda_{u,v}(0, M + n)] \text{ with } k \le g + h) \\
&\quad = \mathbb{P}_p(g + h \ge B_{M+n}(u, v)) \\
&\quad \le \mathbb{P}_p((M + n + 1)\gamma(u) - n\epsilon + 1 \ge B_{M+n}(u, v)) \\
&\quad \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

since

$$g + h \le n(c\gamma(u) + 2\epsilon) + 1 + n(\gamma(u) - 3\epsilon) \le (M + n + 1)\gamma(u) - n\epsilon + 1.$$

Turning to (28), we have (as in the proof of Lemma 2) that if $n \ge n_0$ and $y = k\mathbf{1}$ for some $k \le 0$, then $o \in \mathcal{C}_y[\Lambda_{u,v}(-M, n)]$. Therefore, translation invariance shows that, for $n \ge n_0$,

$$\begin{aligned}
\mathbb{P}_p(o \in \mathcal{C}_z[\Lambda_{u,v}(-M, n)]) &\ge \mathbb{P}_p(\exists k\mathbf{1} \in \mathcal{C}_z[\Lambda_{u,v}(-M, n)] \text{ with } k \le 0) \\
&= \mathbb{P}_p(\exists k\mathbf{1} + Mu \in \mathcal{C}_o[\Lambda_{u,v}(0, M + n)] \text{ with } k \le g) \\
&\ge \mathbb{P}_p(\exists k\mathbf{1} + Mu \in \mathcal{C}_o[\Lambda_{u,v}(0, M)] \text{ with } k \le g) \\
&= \mathbb{P}_p(g \ge B_M(u, v)).
\end{aligned}$$

Because $g \ge n(c\gamma(u) + \epsilon) \ge M\gamma(u) + n\epsilon$, this probability is, at least,

$$\mathbb{P}_p(M\gamma(u) + n\epsilon \ge B_M(u, v)) \to 1, \quad \text{as } n \to \infty.$$

Therefore, $\mathbb{P}_p(o \in \mathcal{C}_z[\Lambda_{u,v}(-M, n)]) \to 1$. This shows (28), and so $\mathbb{P}_p(A_n) \to 0$, as claimed.

We conclude from (28)–(29) and Lemma 5 that if $p > p_1$, then $\mathbb{P}_p(A_n \cup A_n') \to 0$. Off this set, there can be no self-avoiding path ω consistent with the environment that stays in

$\Lambda_{u,v}(-\infty, n)$ and which runs from o to some point $k\mathbf{1} + nu$ with $k < n(\gamma(u) - 3\epsilon)$. For if ω stays in $\Lambda_{u,v}(-M, n)$, then A_n would hold, and if it leaves $\Lambda_{u,v}(-M, n)$, then it enters $\Lambda_{u,v}(-\infty, -M)$, and A'_n would hold.

We conclude that off $A_n \cup A'_n$ we have $\frac{\beta_n^0(u,v)}{n} \ge \gamma(u) - 3\epsilon$. Combined with Lemma 3 and the fact that $\beta_n^0(u, v) \le B_n(u, v)$, this means that $\frac{\beta_n^0(u,v)}{n} \to \gamma(u)$ in probability. L^1 convergence now also follows because $\frac{\beta_n(u)}{n} \le \frac{\beta_n^0(u,v)}{n} \le \frac{B_n(u,v)}{n}$ and the latter $\to \gamma(u)$ in L^1, while, by Lemma 5, the former is bounded below by an integrable random variable.

To extend the convergence in probability to almost surely convergence, we must carefully examine the proof of the version of the subadditive ergodic theorem given in [15]. Define, for $0 < m < n$,

$$\tilde{B}_{m,n}(u, v) = \inf\{k : (\beta_m^0(u, v) + k)\mathbf{1} + nu \in \mathcal{C}_{\beta_m^0(u,v)\mathbf{1}+mu}[\Lambda_{u,v}(m, n)]\}.$$

Examining (21), we see that $\tilde{B}_{m,n}(u, v)$ has the same distribution as $B_{m,n}(u, v)$.

We also know that, for $0 < m < n$,

$$\beta_n^0(u, v) \le \beta_m^0(u, v) + \tilde{B}_{m,n}(u, v).$$

The required moment bounds (for $n \ge n_0$) follow from Lemma 2, as in the proof of Lemma 3. We know that the joint distribution of $(\tilde{B}_{m,m+k}(u, v); k \ge 1)$ does not depend on $m \ge 1$ (as in hypothesis (1.8) of [15]). We also know that, for each $k \ge 1$, $(\tilde{B}_{nk,(n+1)k}(u, v); n \ge 1)$ is a stationary process (as in hypothesis (1.9) of [15]). What is not true is that $\beta_n^0(u, v)$ and $\tilde{B}_{m,m+n}(u, v)$ have the same law. In other words, hypothesis (1.8) of [15] does not apply in the present setting.

To get around this, set $\gamma_n^0 = \mathbb{E}_p[\beta_n^0(u, v)]$ and $\gamma_n = \mathbb{E}_p[B_n(u, v)]$. From (22) (and the fact that $\mathbb{E}_p[B_{m,n}(u, v)] = \mathbb{E}_p[B_{n-m}(u, v)] = \gamma_{n-m}$) we know that γ_n is a subadditive sequence, and, therefore, $\lim \frac{\gamma_n}{n} = \inf \frac{\gamma_n}{n} = \gamma(u)$. We also have $\gamma_n^0 \le \gamma_n$, and $\gamma_{m+n}^0 \le \gamma_m^0 + \gamma_n$ for every m and n. Since also $\lim \frac{\gamma_n^0}{n} = \gamma(u)$, we conclude that

$$\gamma(u) = \lim_{n\to\infty} \frac{\gamma_n^0}{n} = \inf_{n\ge1} \frac{\gamma_n}{n}$$

which is a replacement for (2.1) of [15].

It turns out that this is enough to make the rest of Liggett's proof work for us. In other words, once the above analogue to his (2.1) is shown, then the arguments for his (2.2), (2.3) and (2.4) all follow as written. From which we conclude that if $p > p_1$, then $\frac{\beta_n^0(u,v)}{n} \to \gamma(u)$ a.s. $\square$

PROOF OF LEMMA 7. Let $\epsilon > 0$. Let c be as in Lemma 5, $\bar{c}$ be as in Lemma 4, and $c_1 = c_1(0, d)$ be as in Lemma 1. Take $\delta < \frac{\epsilon}{1+5c}$ and $M = \lfloor nc \rfloor$. For $i \le n(\gamma(u) - \epsilon)$, let $w_i = i\mathbf{1} + nu$. Define the event $C_i = \{\nexists k \le (M + n)(\gamma(u) - \delta)$ with $k\mathbf{1} + (M + n)u \in \mathcal{C}_{w_i}[\Lambda_{u,v}(n, M + n)]\}$. By translation invariance,

$$\begin{aligned}
\mathbb{P}_p(C_i) &= \mathbb{P}_p\bigl(B_M(u, v) \\
&\qquad \ge (M + n)\bigl(\gamma(u) - \delta\bigr) - i\bigr) \\
&\qquad \le \mathbb{P}_p\bigl(B_M(u, v) \ge M\bigl(\gamma(u) + 4\delta\bigr)\bigr) \le e^{-M\bar{c}},
\end{aligned}$$

since

$$\begin{aligned}
(M + n)\bigl(\gamma(u) - \delta\bigr) - i &\ge (M + n)\bigl(\gamma(u) - \delta\bigr) - n\bigl(\gamma(u) - \epsilon\bigr) \\
&= M\gamma(u) - (M + n)\delta + n\epsilon
\end{aligned}$$

$$\geq M\gamma(u) + \delta\big(n(1+5c) - (M+n)\big)$$
$$\geq M\big(\gamma(u) + 4\delta\big).$$

Define the event

$$D_n = \{\beta^0_{M+n}(u,v) > (M+n)\big(\gamma(u)-\delta\big) \text{ but } \beta^1_n(u,v) \leq n\big(\gamma(u)-\epsilon\big)\}.$$

On D_n there exists a path in $\mathcal{C}_o[\Lambda_{u,v}(-\infty, M+n)]$ from o to some $i_0\mathbf{1} + nu$ with $i_0 \leq n(\gamma(u)-\epsilon)$. Choose $0 < \alpha < 1$, and set $N = \lfloor n^\alpha \rfloor$. Then, either $i_0 < -N - \frac{n\|u\|_1}{d}$ or (by Lemmas 2 and 3) $-N - \frac{n\|u\|_1}{d} \leq i_0 \leq n(\gamma(u)-\epsilon) < n\gamma(u) \leq n\|u\|_1$.

In the first case, $[(N+i_0)\mathbf{1} + nu] \cdot \mathbf{1} = (N+i_0)d + nu \cdot \mathbf{1} < -n\|u\|_1 + n\|u\|_1 = 0$, so $(N+i_0)\mathbf{1} + nu \notin K_0$. In other words, $i_0\mathbf{1} + nu \notin K_0 - N\mathbf{1}$ which by Lemma 1 has probability, at most, $c_1\theta^{-Nd}$.

There cannot be a path in $\Lambda_{u,v}(n, M+n)$ from w_{i_0} to any $k\mathbf{1} + (M+n)u$ with $k \leq (M+n)(\gamma(u)-\delta)$, because concatenating the two paths would make $\beta^0_{M+n}(u,v) \leq (M+n)(\gamma(u)-\delta)$. Therefore,

$$\mathbb{P}_p(D_n) \leq c_1\theta^{-Nd} + \sum_{i=-N-\frac{n\|u\|_1}{d}}^{n\|u\|_1} \mathbb{P}_p(C_i) \leq c_1\theta^{-Nd} + \left[1 + N + n\|u\|_1\left(1 + \frac{1}{d}\right)\right]e^{-M\bar{c}}.$$

Summing over n shows that $\sum_n \mathbb{P}_p(D_n) < \infty$. Lemma 6 tells us that if $p > p_1$, then $\beta^0_{M+n}(u,v) > (M+n)(\gamma(u)-\delta)$, eventually, and, therefore, $\beta^1_n(u,v) \geq n(\gamma(u)-\epsilon)$ for all but finitely many n. In other words, $\frac{1}{n}\beta^1_n(u,v) \to \gamma(u)$ a.s. □

4. Proofs of corollaries.

PROOF OF COROLLARY 2. Fix $\delta > 0$ and $R < \infty$. Set $\epsilon = \frac{\delta}{d} \wedge 1$ and $r = R + 1$. Choose n so large that the inclusions of (e) of Theorem 1 hold for n, with this ϵ and r.

Let $z \in O_R \cap C_\delta \cap \frac{1}{n}\mathbb{Z}^d$. Then, $\|\epsilon\mathbf{1}\|_1 \leq \delta$, so by (16) and the fact that $\gamma(z) \leq -\delta$, we see that $\gamma(z') \leq 0$, where $z' = z - \epsilon\mathbf{1}$. Thus, $z' \in C$. Since $z \in O_R$ and $\epsilon \leq 1$, we know that $z' \in O_r$. So by choice of n, there is a $w \in \frac{1}{n}\mathcal{C}_o$ such that $\|w - z'\|_\infty < \epsilon$. Therefore, $w' = \epsilon\mathbf{1} - (w - z')$ lies in the positive orthant, and we have $z = w + w'$. That implies that o connects to nw and nw connects to nz, so $z \in \frac{1}{n}\mathcal{C}_o$, as claimed. □

We know that $\gamma(e_1) \leq 0$. But it will be useful to know that this inequality is strict.

LEMMA 8. *Assume the conditions of Theorem 1, and let $p > p_1$. Then, $\gamma(e_1) < 0$.*

PROOF. Let $n, k > 0$. Consider a path from o, constructed as follows. At sites in $\mathcal{E}_+$, follow e_1. At the first n sites in $\mathcal{E}_-$, follow $-e_2$. At the next n sites in $\mathcal{E}_-$, follow $-e_3$. Do the same in turn for $-e_4, \dots, -e_d$ till n steps have been taken in the direction of each. After that, follow e_1 at sites in $\mathcal{E}_-$.

This path will reach $n(ke_1 - \mathbf{1})$, provided there are at least $n(d-1)$ sites in $\mathcal{E}_-$ among the first $n[(k-1)+d-1]$ sites visited. Choose k so that $(1-p)[k+d-2] > d-1$. Then, the law of large numbers implies that the above event has probability $\to 1$, when $n \to \infty$.

In other words, $\beta_n(ke_1 - \mathbf{1}) \leq 0$ with probability $\to 1$. Since $\frac{1}{n}\beta_n(ke_1 - \mathbf{1}) \to \gamma(ke_1 - 1)$ in probability, we get that $\gamma(ke_1 - \mathbf{1}) \leq 0$. Therefore,

$$\gamma(e_1) = \frac{1}{k}\gamma(ke_1) = \frac{1}{k}\big[\gamma(ke_1 - \mathbf{1}) - 1\big] \leq -\frac{1}{k} < 0.$$ □

PROOF OF COROLLARY 1. Fix $w \in \mathbb{Z}^d$. Let $\zeta(w) = \inf\{t \in \mathbb{R} : \gamma(w+te_1) \le 0\}$. If $s < t$, then

$$\gamma(w+te_1) \le \gamma(w+se_1) + \gamma\big((t-s)e_1\big) = \gamma(w+se_1) + (t-s)\gamma(e_1) < \gamma(w+se_1)$$

by Lemma 8. So γ is strictly decreasing and continuous, and, therefore,

$$\gamma(w+te_1) \le 0 \quad \Rightarrow \quad t \ge \zeta(w) \tag{30}$$

$$\gamma(w+te_1) \ge 0 \quad \Rightarrow \quad t \le \zeta(w). \tag{31}$$

Set $s = \liminf \frac{L_{nw}}{n}$ and $S = \limsup \frac{L_{nw}}{n}$, and let $M \in \mathbb{Z}$. We will show that $S \wedge M \le \zeta(w) \le s$, from which the desired result is immediate.

Consider the upper inequality first. Lemma 1 implies that $s > -\infty$. There is nothing to show if $s = \infty$, so assume $s \in \mathbb{R}$. Then, we may find a sequence n_k such that $\frac{1}{n_k} L_{n_k w} \to s$. Therefore, $n_k w + L_{n_k w} e_1 \in \mathcal{C}_o$. Set $r = \|w\|_\infty + |s| + 1 < \infty$. Then, for any $\epsilon > 0$, (e) of Theorem 1 shows that, for k sufficiently large,

$$w + \frac{1}{n_k} L_{n_k w} e_1 \in O_r \cap \frac{1}{n_k}\mathcal{C}_o \subset O_\epsilon + C.$$

The latter is closed, so, taking limits, we have $w + se_1 \in O_\epsilon + C$ for every $\epsilon > 0$. Since C is closed, in fact, $w + se_1 \in C$, so $\gamma(w+se_1) \le 0$. By (30) we get that $s \ge \zeta(w)$, as claimed.

To show the lower inequality, we argue similarly. If $S = -\infty$ there is nothing to show, so assume this is not the case. Let $\frac{1}{n_k} L_{n_k w} \to S$. Therefore, $n_k w + \min(L_{n_k w} - 1, n_k M) e_1 \notin \mathcal{C}_o$. Set $R = \|w\|_\infty + |S \wedge M| + 1 < \infty$. Then, for k sufficiently large,

$$w + \frac{1}{n_k} \min(L_{n_k w} - 1, n_k M) e_1 \in \left[O_R \cap \frac{1}{n_k}\mathbb{Z}^d \right] \setminus \frac{1}{n_k}\mathcal{C}_o.$$

If $\delta > 0$, then, for k sufficiently large, Corollary 2 implies that the above point $\notin C_\delta$. In other words, $\gamma(w + \frac{1}{n_k}\min(L_{n_k w} - 1, n_k M)e_1) \ge -\delta$. Taking limits, we get that $\gamma(w + (S \wedge M)e_1) \ge -\delta$. Since $\delta > 0$ was arbitrary, in fact, $\gamma(w + (S \wedge M)e_1) \ge 0$. By (31) we obtain that $S \wedge M \le \zeta(w)$, as claimed, and we are done. □

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