

SMALL GAPS OF CIRCULAR β -ENSEMBLE

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In this article, we study the smallest gaps of the circular β -ensemble ($C\beta E$) on the unit circle, where β is any positive integer. The main result is that the smallest gaps, after being normalized by $n^{\frac{\beta+2}{\beta+1}}$, will converge in distribution to a Poisson point process with some explicit intensity. And thus one can derive the limiting density of the k th smallest gap, which is proportional to $x^{k(\beta+1)-1}e^{-x^{\beta+1}}$. In particular, the results apply to the classical COE, CUE and CSE in random matrix theory. The essential part of the proof is to derive several identities and inequalities regarding the Selberg integral, which should have their own interest.

1. Introduction. The extreme spacings of random point processes are important quantities in probability and statistical physics. In random matrix theory, the limiting densities of the smallest gaps of CUE and GUE were first derived by Vinson [16]; by a different method, Soshnikov investigated the smallest gaps for the determinantal point processes on the real line with translation invariant kernels [15]; Soshnikov's technique was adapted by Ben Arous–Bourgade in [4] where they studied the joint density of the smallest gaps of CUE and GUE (which are both determinantal point processes) and proved that the smallest gaps, after being normalized by $n^{4/3}$, will tend to a Poisson point process and the k th smallest gap has the limiting density proportional to $x^{3k-1}e^{-x^3}$. Their results are further generalized by Figalli–Guionnet to some invariant multimatrix Hermitian matrices in [8]. The results about the smallest gaps for random matrices with complex Ginibre, Wishart and universal Unitary ensembles are derived in [12].

Regarding the largest gaps, the decay order $\sqrt{32 \log n}/n$ of the largest gaps of CUE and GUE (in the bulk regime) was predicted by Vinson in [16] and proved by Ben Arous–Bourgade in [4], and this result is further generalized by Figalli–Guionnet in [8]. Recently, the fluctuations of the largest gaps of CUE and GUE have been derived in [7], and it's proved that the largest gaps, after being normalized, will also converge in distribution to a Poisson point process.

In this paper, we will derive the limiting distribution of the smallest gaps of $C\beta E$ where β is any positive integer. Our results confirm the (numerical) predictions in physics [13] and recover Ben Arous–Bourgade's result in the case of CUE (where $\beta = 2$). But our proof is different and technical. One can not make use of the structure of the determinantal point processes any more (for example, when $\beta = 1, 4$, they are Pfaffian point processes other than the determinantal point processes, we refer to Sections 3.9 and 4.2 in [1] for the definitions and the structures of the Pfaffian point process and the determinantal point process and their applications in random matrix theory), and we have to start from the Selberg integral to get the estimates regarding the point correlation functions, where we need to derive several asymptotic limits and inequalities (such as Lemma 1.1 and the crucial Lemma 1.4) which should have their own interest in Selberg integral theory. The method developed in this paper

is also adapted in [6] where we can derive the limiting distribution of the smallest gaps of GOE.

In [10], the authors proved that our results in [6, 7] are universal for both smallest gaps and largest gaps in the bulk of the general Hermitian and symmetric Wigner matrices with assumptions.

1.1. *Main results.* In the theory of random matrices, Dyson introduced three ensembles of random unitary matrices to study the energy-level behaviors in complex quantum systems [5]. They are the circular orthogonal ensemble (COE) on symmetric unitary matrices, the circular unitary ensemble (CUE) on unitary matrices, and the circular symplectic ensemble (CSE) on self dual unitary quaternionic matrices. The circular β -ensemble ($\beta > 0$) is a generalization of these three ensembles. It is a point process on the unit circle and the joint density of the eigenangles $\theta_j \in [-\pi, \pi)$, $1 \leq j \leq n$ with respect to the Lebesgue measure is

$$(1) \quad J_\beta(\theta_1, \dots, \theta_n) = \frac{1}{C_{\beta,n}} \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

Here, the partition function

$$C_{\beta,n} := \int_{-\pi}^\pi d\theta_1 \cdots \int_{-\pi}^\pi d\theta_n \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^\beta$$

is derived by the Selberg integral and it reads

$$C_{\beta,n} = (2\pi)^n \frac{\Gamma(1 + \beta n/2)}{(\Gamma(1 + \beta/2))^n}.$$

In particular, the joint density of the eigenangles of COE, CUE and CSE is given by J_1 , J_2 and J_4 respectively (see Chapter 2 in [9] for the proofs).

One interpretation of the density $J_\beta(\theta_1, \dots, \theta_n)$ is as the Boltzmann factor for a classical gas at inverse temperature β with potential energy

$$- \sum_{1 \leq j < k \leq n} \ln |e^{i\theta_j} - e^{i\theta_k}|.$$

Because of the pairwise logarithmic repulsion, such a classical gas is referred to as a log-gas. This interpretation allows for a number of properties of correlations and distributions to be anticipated using arguments based on macroscopic electrostatics [9].

We will need the following partition functions for the two-component log-gas where the system consists of n_1 particles with charge $q = 1$ and n_2 particles with charge $q = 2$:

$$(2) \quad C_{\beta,n_1,n_2} := \int_{-\pi}^\pi d\theta_1 \cdots \int_{-\pi}^\pi d\theta_{n_1+n_2} \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^{q_j q_k \beta}$$

and

$$(3) \quad C_{\beta,n_1,n_2}(I) := \int_{(-\pi,\pi)^{n_1} \times I^{n_2}} d\theta_1 \cdots d\theta_{n_1+n_2} \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^{q_j q_k \beta},$$

where $q_j = 1$ for $1 \leq j \leq n_1$ and $q_j = 2$ for $n_1 + 1 \leq j \leq n_1 + n_2$.

We also need the following partition function for the system of the two-component log-gas with n_1 particles with charge $q = 1$ and one particle with charge $q = k$:

$$(4) \quad C_{\beta,n_1,(k)} := \int_{-\pi}^\pi d\theta_1 \cdots \int_{-\pi}^\pi d\theta_{n_1+1} \prod_{j < l} |e^{i\theta_j} - e^{i\theta_l}|^{q_j q_l \beta}$$

with $q_j = 1$ for $1 \leq j \leq n_1$ and $q_{n_1+1} = k$, then we have

$$C_{\beta, n_1, (2)} = C_{\beta, n_1, 1}$$

and the following results.

LEMMA 1.1. For $0 < k \leq n, \beta \geq 1$, we have

$$C_{\beta, n-k, (k)} \leq C_{\beta, n} (n\beta)^{k(k-1)\beta/2},$$

and

$$\lim_{n \rightarrow +\infty} \frac{C_{\beta, n-2, 1}}{C_{\beta, n} n^\beta} = A_\beta, \quad \lim_{n \rightarrow +\infty} \frac{C_{\beta, n-k, (k)}}{C_{\beta, n} n^{k(k-1)\beta/2}} = A_{\beta, k},$$

where

$$A_{\beta, k} = \frac{(2\pi)^{1-k} (\Gamma(\beta/2 + 1))^k}{\Gamma(k\beta/2 + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma((k+j)\beta/2 + 1)} (\beta/2)^{k(k-1)\beta/2}$$

and

$$A_\beta = A_{\beta, 2} = (2\pi)^{-1} \frac{(\beta/2)^\beta (\Gamma(\beta/2 + 1))^3}{\Gamma(3\beta/2 + 1) \Gamma(\beta + 1)}.$$

Now we consider the following point process on \mathbb{R}^2 :

$$(5) \quad \chi^{(n, \gamma)} = \sum_{i=1}^n \delta_{(n^\gamma(\theta_{(i+1)} - \theta_{(i)}), \theta_{(i)})}, \quad \chi^{(n)} = \chi^{(n, \gamma)} \Big|_{\gamma = \frac{\beta+2}{\beta+1}},$$

where $\gamma > 0, \theta_{(i)} (1 \leq i \leq n)$ is the increasing rearrangement of $\theta_i (1 \leq i \leq n)$ and $\theta_{(i+n)} = \theta_{(i)} + 2\pi$, i.e. the indices are modulo n , then we have the following main result.

THEOREM 1.1. For $C\beta E$ where β is a positive integer, the point process $\chi^{(n)}$ will converge in distribution to a Poisson point process χ as $n \rightarrow +\infty$ with intensity

$$\mathbb{E}\chi(A \times I) = \frac{A_\beta |I|}{2\pi} \int_A u^\beta du,$$

where $A \subset \mathbb{R}_+$ is any bounded Borel set, $I \subseteq (-\pi, \pi)$ and $|I|$ is the Lebesgue measure of I . In particular, the result holds for COE, CUE and CSE with

$$A_1 = \frac{1}{24}, \quad A_2 = \frac{1}{24\pi}, \quad A_4 = \frac{1}{270\pi},$$

respectively.

As a direct consequence, we can derive the limiting distribution of the k th smallest gap.

COROLLARY 1.1. Let t_k be the k th smallest gap of $C\beta E$ and we define

$$\tau_k = n^{(\beta+2)/(\beta+1)} \times (A_\beta / (\beta + 1))^{1/(\beta+1)} t_k,$$

then we have

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_k \in A) = \int_A \frac{\beta + 1}{(k - 1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}} dx$$

for any bounded interval $A \subset \mathbb{R}_+$.

The above results generalize the previous results derived in [4, 16] for CUE with $\beta = 2$.

1.2. *Factorial moments and correlation functions.* We first review some basic concepts about the factorial moments and the correlation functions of a point process, we refer to the survey [14] for more details. Let

$$X = \sum_i \delta_{X_i}$$

be a simple point process on \mathbb{R} , and we consider the point process

$$(6) \quad X^{(k)} = \sum_{i_1, \dots, i_k \text{ all distinct}} \delta_{(X_{i_1}, \dots, X_{i_k})}$$

on \mathbb{R}^k . One can define a measure m_k on \mathbb{R}^k by

$$(7) \quad m_k(A) = \mathbb{E}(X^{(k)}(A))$$

for any Borel set A in \mathbb{R}^k . In particular, the factorial moment reads

$$(8) \quad m_k(B^k) = \mathbb{E}\left(\frac{(X(B)!) }{(X(B) - k)!}\right),$$

where B is a Borel set in \mathbb{R} .

If m_k is absolutely continuous with respect to the Lebesgue measure, then there exists a function f_k on \mathbb{R}^k such that for any Borel sets B_1, \dots, B_k in \mathbb{R} , we have

$$(9) \quad m_k(B_1 \times \dots \times B_k) = \int_{B_1 \times \dots \times B_k} f_k(x_1, \dots, x_k) dx_1 \cdots dx_k.$$

The function f_k is called the k -point correlation function of the point process. Note that f_k is not a probability density, but it admits the following probabilistic interpretation: for distinct points x_1, \dots, x_k in \mathbb{R} , if $[x_i, x_i + dx_i], i = 1, \dots, k$ are neighbourhoods of x_i , then $f_k(x_1, \dots, x_k) dx_1 \cdots dx_k$ is the probability of the event that each set $[x_i, x_i + dx_i]$ contains a particle.

In particular, (8) and (9) imply that the k th factorial moment of a point process and the k -point correlation function satisfy

$$(10) \quad m_k(B^k) = \mathbb{E}\left(\frac{(X(B)!) }{(X(B) - k)!}\right) = \int_{B^k} f_k(x_1, \dots, x_k) dx_1 \cdots dx_k,$$

where B is a Borel set in \mathbb{R} .

If X is a determinantal point process, then the k -point correlation function has the representation

$$(11) \quad f_k(x_1, \dots, x_k) = \det[K(x_i, x_j)]_{1 \leq i, j \leq k},$$

where $K(x, y)$ is some symmetric kernel. For example, in the case of CUE which is a Haar measure on the unitary group $U(n)$, one can prove that it is a determinantal point process and the k -point correlation function is

$$f_k(\theta_1, \dots, \theta_k) = \det[K_n(\theta_i - \theta_j)]_{1 \leq i, j \leq k}, \quad K_n(\theta) = \frac{1}{2\pi} \frac{\sin(n\theta/2)}{\sin(\theta/2)}.$$

More properties regarding the correlation functions of determinantal point processes can be found in [1, 14].

1.3. *Strategy and key lemmas.* Now we explain the main steps to prove Theorem 1.1. As in [4, 15], we still need to prove the convergence of the factorial moments of $\chi^{(n)}$ in order to prove its convergence to the Poisson point process, but the proof follows a quite different way. For the determinantal point processes considered in [4, 15], the factorial moments are the integrations of the correlation functions (see (10)) which can be further expressed by the symmetric kernels as in (11), therefore, the limits of the factorial moments can be derived directly by taking the limits of the kernels, and one can also use the Hadamard–Fischer inequality to control all the estimates. But for general C β E which are not determinantal point processes, one can only express the point correlation functions and the factorial moments as the integration of the joint density by the definitions (6), (7) and (8). This causes many difficulties and all the proofs require delicate estimates of the integrals.

By the moment method (see Proposition 2.1 in [4]), Theorem 1.1 will be proved if we can prove the following convergence of the factorial moment:

$$(12) \quad \lim_{n \rightarrow +\infty} \mathbb{E} \left(\frac{(\chi^{(n)}(A \times I))!}{(\chi^{(n)}(A \times I) - k)!} \right) = \left(\int_A u^\beta du \right)^k \left(\frac{|I|A^\beta}{2\pi} \right)^k$$

for any fixed positive integer k , where $A \subset \mathbb{R}_+$ is any bounded interval and $I \subseteq (-\pi, \pi)$.

We will not prove this convergence directly. We will study the following auxiliary point process instead. We now introduce $\theta_{i,j} = \theta_i - \theta_j$ for $\theta_i > \theta_j$, $\theta_{i,j} = \theta_i - \theta_j + 2\pi$ for $\theta_i < \theta_j$. For any $\gamma > 0$, we define

$$(13) \quad \theta_{i,j,\gamma} = (n^\gamma \theta_{i,j}, \theta_j)$$

and

$$(14) \quad \tilde{\chi}^{(n,\gamma)} = \sum_{i \neq j} \delta_{\theta_{i,j,\gamma}}, \quad \tilde{\chi}^{(n)} = \tilde{\chi}^{(n,\gamma)} \Big|_{\gamma = \frac{\beta+2}{\beta+1}},$$

i.e., $\tilde{\chi}^{(n)}$ is the point process of all (normalized) spacings, then we have

$$(15) \quad \chi^{(n)} \leq \tilde{\chi}^{(n)}.$$

In fact, we can rewrite

$$(16) \quad \tilde{\chi}^{(n,\gamma)} = \sum_{j=1}^{n-1} \tilde{\chi}^{(n,\gamma,j)}$$

such that

$$(17) \quad \tilde{\chi}^{(n,\gamma,j)} = \sum_{i=1}^n \delta_{(n^\gamma(\theta_{(i+j)} - \theta_{(i)}), \theta_{(i)})}.$$

Then we have

$$\tilde{\chi}^{(n,\gamma,1)} = \chi^{(n,\gamma)} \quad \text{and} \quad 0 \leq \tilde{\chi}^{(n,\gamma,j)}(B) \leq n$$

for every Borel set $B \subset \mathbb{R}^2$.

We need the following lemma which indicates that there is no successive smallest gaps. Such property is also considered in [4, 15] for the determinantal point processes.

LEMMA 1.2. *For any bounded interval $A \subset \mathbb{R}_+$ and $I \subseteq (-\pi, \pi)$, we have $\chi^{(n)}(A \times I) - \tilde{\chi}^{(n)}(A \times I) \rightarrow 0$ in probability as $n \rightarrow +\infty$.*

The proof of Lemma 1.2 relies on the estimates of the integration of the 3-point correlation function, which can be derived for the determinantal point processes by estimating the symmetric kernels directly. But in our case, we can only express the correlation functions as the integrations of the joint density, and thus we will need several integral inequalities (see Lemma 4.1 in §4), which are the basic estimates that will be applied many times in the whole proof.

The significance of Lemma 1.2 is that, instead of proving the convergence of the factorial moment of the point process of the smallest spacings $\chi^{(n)}$ (see (12)), it's enough to prove the following convergence of the factorial moment of $\tilde{\chi}^{(n)}$ of all (normalized) spacings:

$$(18) \quad \lim_{n \rightarrow +\infty} \mathbb{E} \left(\frac{(\tilde{\chi}^{(n)}(A \times I))!}{(\tilde{\chi}^{(n)}(A \times I) - k)!} \right) = \left(\int_A u^\beta du \right)^k \left(\frac{|I|A_\beta}{2\pi} \right)^k$$

for any fixed k . And it is much easier to study the properties of the point process of all spacings $\tilde{\chi}^{(n)}$ compared with only the smallest spacings $\chi^{(n)}$. Actually, (18) is the direct consequence of the following Lemma 1.3 and Lemma 1.4.

LEMMA 1.3. *For any bounded interval $A \subset \mathbb{R}^+$, $I \subseteq (-\pi, \pi)$ and any positive integer $k \geq 1$, we have*

$$\mathbb{E} \left(\frac{(\tilde{\chi}^{(n)}(A \times I))!}{(\tilde{\chi}^{(n)}(A \times I) - k)!} \right) - \left(\int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}n^{k\beta}} \rightarrow 0$$

as $n \rightarrow +\infty$.

To prove Lemma 1.3, we will introduce another auxiliary point process

$$\rho^{(k,n,\gamma)} = \sum_{i_1, \dots, i_{2k} \text{ all distinct}} \delta_{(\theta_{i_1, i_2, \gamma}, \dots, \theta_{i_{2k-1}, i_{2k}, \gamma})}, \quad \rho^{(k,n)} = \rho^{(k,n,\gamma)} \Big|_{\gamma = \frac{\beta+2}{\beta+1}}$$

where $\theta_{i_{2j-1}, i_{2j}, \gamma}$ ($1 \leq j \leq k$) is defined in (13).

Regarding $\rho^{(k,n)}$, we will see that the expectation of $\rho^{(k,n)}$ will converge to the k th factorial moment of $\tilde{\chi}^{(n)}$. Lemma 1.3 is the consequence of the following two convergences:

$$\lim_{n \rightarrow +\infty} \left(\mathbb{E} \frac{(\tilde{\chi}^{(n)}(A \times I))!}{(\tilde{\chi}^{(n)}(A \times I) - k)!} - \mathbb{E} \rho^{(k,n)}((A \times I)^k) \right) = 0$$

and

$$\lim_{n \rightarrow +\infty} \left(\mathbb{E}(\rho^{(k,n)}((A \times I)^k)) - \left(\int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}n^{k\beta}} \right) = 0.$$

Here, the second convergence is the most significant part of the whole proof, it implies that one can bound $\mathbb{E}(\rho^{(k,n)}((A \times I)^k))$ by the quotient of the partition functions $C_{\beta,n-2k,k}(I)/(C_{\beta,n}n^{k\beta})$, therefore, the problem regarding the smallest gaps in nature is just a problem about the integral estimates. To be more precise, one of the crucial ideas of the whole method is that one can bound $\mathbb{E}(\rho^{(k,n)}((A \times I)^k))$, which can be expressed in terms of the integration of the joint density of the one-component log-gas (see (44)), by the generalized partition function of the two-component log-gas (see (45) together with Lemma 6.2).

The intuitive idea of the whole proof is natural: for a pair of two particles with charge 1 of the smallest gap of $C_\beta E$, these two particles will tend to a “double particle” with charge 2 in the limit, therefore, if there are k -pair of such particles of smallest gaps among n particles in the one-component log-gas, then such system can be approximated by the two-component log-gas with $n - 2k$ particles with charge 1 and k particles with charge 2. Therefore, one needs to compare the partition function of the one-component log-gas with the partition function of the two-component log-gas as in the following lemma.

LEMMA 1.4. For any interval $I \subseteq (-\pi, \pi)$ and any positive integer $k \geq 1$, we have

$$\lim_{n \rightarrow +\infty} \frac{C_{\beta, n-2k, k}(I)}{C_{\beta, n} n^{k\beta}} = \left(\frac{|I| A_\beta}{2\pi} \right)^k.$$

The convergence for $k = 1$ is guaranteed by Lemma 1.1. We will not prove this convergence directly for $k \geq 2$. Actually, we only need to prove the following inequality for $k = 2$;

$$(19) \quad \limsup_{n \rightarrow +\infty} \frac{C_{\beta, n-4, 2}(I)}{C_{\beta, n} n^{2\beta}} \leq \left(\frac{|I| A_\beta}{2\pi} \right)^2.$$

And in Section 8, we can prove Lemma 1.4 for $k = 2$ by the upper bound (19), Hölder inequality and the powerful result of the convergence of the factorial moments in Lemma 1.3 which is valid for any fixed intervals A and I . The proof of Lemma 1.4 for all $k \geq 3$ follows by induction and Lemma 1.3 again.

The proof of the upper bound (19) is complicated and it will be proved in Section 7 based on the properties of the Selberg integral and the generalized hypergeometric functions.

The method developed in this article is further applied to derive the limiting distribution of the smallest gaps of GOE in [6]. Actually our method is quite general, it can be used to prove that of $G\beta E$ and more general ensembles. In all cases, as indicated by the intuitive idea mentioned above, one of the main difficulties is to prove the analogue asymptotic limit as in Lemma 1.4, that is, one has to prove the asymptotic limit of the quotient of the partition functions of the two-component log-gas and the one-component log-gas, once this is done, the point process of the smallest gaps can be proved to be converging to some Poisson point process, and hence the limiting density of the k th smallest gap can be derived.

As a final remark, we also conjecture that Theorem 1.1 must be true for any $\beta > 0$, but our method only works for the positive integer β . This is because, in the proof of the upper bound (19), we use the properties about the generalized hypergeometric functions that have been proved to be true for positive integer β . As explained above, if one can prove Lemma 1.4 for every $\beta > 0$ by other methods without using the properties of generalized hypergeometric functions, then Theorem 1.1 will hold for every $\beta > 0$.

2. Proof of Lemma 1.1. Now we give the proof of Lemma 1.1, which is based on the Selberg integral. We refer to Selberg’s original method [11] and Aomoto’s method [2, 3] for the proof of the Selberg integral. We also refer to Chapter 4 in [9] for other proofs and the applications of the Selberg integral in the theory of random matrices.

PROOF. We can write

$$\begin{aligned} C_{\beta, n_1, 1} &= \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n_1+1} \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{1 \leq j \leq n_1} |e^{i\theta_j} - e^{i\theta_{n_1+1}}|^{2\beta} \\ &= \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n_1+1} \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{1 \leq j \leq n_1} |e^{i\theta_j} + 1|^{2\beta} \\ &= (2\pi)^{n_1+1} M_{n_1}(\beta, \beta, \beta/2), \end{aligned}$$

here we changed variables $\theta_j \mapsto \theta_j + \theta_{n_1+1} \pm \pi$ ($1 \leq j \leq n_1$) and used the following formula (see (4.4) in [9]):

$$\begin{aligned}
 M_n(a, b, \lambda) &:= \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_n \prod_{l=1}^n e^{\pi i \theta_l (a-b)} |1 + e^{2\pi i \theta_l}|^{a+b} \\
 (20) \quad &\times \prod_{1 \leq j < k \leq n} |e^{2\pi i \theta_j} - e^{2\pi i \theta_k}|^{2\lambda} \\
 &= \prod_{j=0}^{n-1} \frac{\Gamma(\lambda j + a + b + 1) \Gamma(\lambda(j + 1) + 1)}{\Gamma(\lambda j + a + 1) \Gamma(\lambda j + b + 1) \Gamma(1 + \lambda)}.
 \end{aligned}$$

Similarly, we have

$$(21) \quad C_{\beta, n_1, 1}(I) = (2\pi)^{n_1} |I| M_{n_1}(\beta, \beta, \beta/2) = (2\pi)^{-1} |I| C_{\beta, n_1, 1}$$

and

$$(22) \quad C_{\beta, n_1, (k)} = (2\pi)^{n_1+1} M_{n_1}(k\beta/2, k\beta/2, \beta/2).$$

For every positive integer k , we have

$$\begin{aligned}
 M_n(k\lambda, k\lambda, \lambda) &= \prod_{j=0}^{n-1} \frac{\Gamma(\lambda(j + 2k) + 1) \Gamma(\lambda(j + 1) + 1)}{(\Gamma(\lambda(j + k) + 1))^2 \Gamma(1 + \lambda)} \\
 &= \frac{1}{\Gamma(\lambda + 1)^n} \prod_{j=k}^{2k-1} \frac{\Gamma(\lambda(n + j) + 1)}{\Gamma(j\lambda + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\lambda + 1)}{\Gamma(\lambda(n + j) + 1)},
 \end{aligned}$$

thus we have

$$\begin{aligned}
 C_{\beta, n_1, (k)} &= (2\pi)^{n_1+1} M_{n_1}(k\beta/2, k\beta/2, \beta/2) \\
 &= \frac{(2\pi)^{n_1+1}}{(\Gamma(\beta/2 + 1))^{n_1}} \prod_{j=k}^{2k-1} \frac{\Gamma(\beta(n_1 + j)/2 + 1)}{\Gamma(j\beta/2 + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma(\beta(n_1 + j)/2 + 1)}.
 \end{aligned}$$

And for $n_1 = n - k > 0$, we have

$$\begin{aligned}
 \frac{C_{\beta, n-k, (k)}}{C_{\beta, n}} &= \frac{(2\pi)^{1-k} (\Gamma(\beta/2 + 1))^k}{\Gamma(n\beta/2 + 1)} \prod_{j=1}^{2k-1} \left(\frac{\Gamma(\beta(n_1 + j)/2 + 1)}{\Gamma(j\beta/2 + 1)} \right)^{\text{sgn}(j-k)} \\
 &= \frac{(2\pi)^{1-k} (\Gamma(\beta/2 + 1))^k}{\Gamma(k\beta/2 + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(\beta(n + j)/2 + 1) \Gamma(j\beta/2 + 1)}{\Gamma((k + j)\beta/2 + 1) \Gamma(\beta(n - j)/2 + 1)}.
 \end{aligned}$$

As $\ln \Gamma(x)$ is convex for $x > 0$, we have

$$(\Gamma(\beta/2 + 1))^k \leq \Gamma(k\beta/2 + 1).$$

For $n > k - 1 \geq j \geq 1$, we have $k\beta/2 \geq 1, \beta j \geq 1$,

$$\frac{\Gamma(j\beta/2 + 1)}{\Gamma((k + j)\beta/2 + 1)} \leq \left(\frac{\Gamma(j\beta/2 + 1)}{\Gamma(j\beta/2 + 2)} \right)^{k\beta/2} = \left(\frac{1}{j\beta/2 + 1} \right)^{k\beta/2} \leq 1,$$

and

$$\frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(\beta(n - j)/2 + 1)} \leq \left(\frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(\beta(n + j)/2)} \right)^{\beta j} = (\beta(n + j)/2)^{\beta j} \leq (n\beta)^{\beta j},$$

therefore, we have

$$\frac{C_{\beta,n-k,(k)}}{C_{\beta,n}} \leq (2\pi)^{1-k} \prod_{j=1}^{k-1} (n\beta)^{\beta j} = (2\pi)^{1-k} (n\beta)^{k(k-1)\beta/2},$$

which will imply the first inequality. Using convexity of $\ln \Gamma(x)$, we also have

$$(\beta(n-j)/2 + 1)^{\beta j} \leq \frac{\Gamma(\beta(n+j)/2 + 1)}{\Gamma(\beta(n-j)/2 + 1)} \leq (\beta(n+j)/2)^{\beta j},$$

which implies

$$\lim_{n \rightarrow +\infty} \frac{\Gamma(\beta(n+j)/2 + 1)}{\Gamma(\beta(n-j)/2 + 1)n^{\beta j}} = (\beta/2)^{\beta j}.$$

And thus we have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{C_{\beta,n-k,(k)}}{C_{\beta,n}n^{k(k-1)\beta/2}} \\ &= \frac{(2\pi)^{1-k} (\Gamma(\beta/2 + 1))^k}{\Gamma(k\beta/2 + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma((k+j)\beta/2 + 1)} \cdot \lim_{n \rightarrow +\infty} \prod_{j=1}^{k-1} \frac{\Gamma(\beta(n+j)/2 + 1)}{\Gamma(\beta(n-j)/2 + 1)n^{\beta j}} \\ &= \frac{(2\pi)^{1-k} (\Gamma(\beta/2 + 1))^k}{\Gamma(k\beta/2 + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma((k+j)\beta/2 + 1)} \prod_{j=1}^{k-1} (\beta/2)^{\beta j} \\ &=: A_{\beta,k}. \end{aligned}$$

As $C_{\beta,n_1,(2)} = C_{\beta,n_1,1}$, we have

$$\lim_{n \rightarrow +\infty} \frac{C_{\beta,n-2,1}}{C_{\beta,n}n^\beta} = \lim_{n \rightarrow +\infty} \frac{C_{\beta,n-2,(2)}}{C_{\beta,n}n^\beta} = A_{\beta,2},$$

and the expression of $A_\beta = A_{\beta,2}$ follows directly from that of $A_{\beta,k}$. \square

3. One more auxiliary point process. Now we introduce another auxiliary point process as

$$(23) \quad \rho^{(k,n,\gamma)} = \sum_{i_1, \dots, i_{2k} \text{ all distinct}} \delta_{(\theta_{i_1, i_2, \gamma}, \dots, \theta_{i_{2k-1}, i_{2k}, \gamma})}$$

and we define

$$(24) \quad \rho^{(k,n)} = \rho^{(k,n,\gamma)} \Big|_{\gamma = \frac{\beta+2}{\beta+1}},$$

where $\theta_{i_{2j-1}, i_{2j}, \gamma}$ ($1 \leq j \leq k$) is defined in (13).

We need the following lemma which will be used to prove that the expectation of the random variable $\rho^{(k,n)}((A \times I)^k)$ converges to the factorial moment of $\tilde{\chi}^{(n)}(A \times I)$ for any bounded interval $A \subset \mathbb{R}^+$ and $I \subseteq (-\pi, \pi)$ (see (39)).

LEMMA 3.1. *For any bounded intervals $A \subset \mathbb{R}_+$ and $I \subseteq (-\pi, \pi)$, let $B = A \times I$, then we have*

$$\rho^{(k,n,\gamma)}(B^k) \leq \frac{(\tilde{\chi}^{(n,\gamma)}(B))!}{(\tilde{\chi}^{(n,\gamma)}(B) - k)!}, \quad \gamma > 0.$$

Let c_1 be such that $A \subset (0, c_1)$, $c_n = c_1 n^{-\frac{\beta+2}{\beta+1}}$ and

$$a = \max\{i - j : i, j \in \mathbb{Z}, \theta_{(i)} - \theta_{(j)} \leq 2c_n\},$$

if $c_n \in (0, 1)$, then we have

$$0 \leq \frac{(\tilde{\chi}^{(n)}(B))!}{(\tilde{\chi}^{(n)}(B) - k)!} - \rho^{(k,n)}(B^k) \leq k(k - 1)(a - 1)(\tilde{\chi}^{(n)}(B))^{k-1}$$

and

$$\rho^{(k,n)}(B^k) \geq (\tilde{\chi}^{(n)}(B))^k - k(k - 1)a(\tilde{\chi}^{(n)}(B))^{k-1}.$$

PROOF. We denote

$$\begin{aligned} X_1 &= \{(i_1, \dots, i_{2k}) : i_j \in \mathbb{Z}, 1 \leq i_j \leq n, \forall 1 \leq j \leq 2k, \\ &\quad i_{2j-1} \neq i_{2j}, \forall 1 \leq j \leq k, \{i_{2j-1}, i_{2j}\} \neq \{i_{2l-1}, i_{2l}\}, \forall 1 \leq j < l \leq k\}, \\ X_2 &= \{(i_1, \dots, i_{2k}) : i_j \in \mathbb{Z}, 1 \leq i_j \leq n, \forall 1 \leq j \leq 2k, \\ &\quad i_j \neq i_l, \forall 1 \leq j < l \leq 2k\}, \\ Y_{j,l} &= \{(i_1, \dots, i_{2k}) : \{i_{2j-1}, i_{2j}\} \cap \{i_{2l-1}, i_{2l}\} \neq \emptyset\}, \end{aligned}$$

then we have $X_2 \subseteq X_1$ and $X_1 \setminus X_2 = \bigcup_{1 \leq j < l \leq k} Y_{j,l}$. Let

$$\begin{aligned} X_{j,B} &= \{(i_1, \dots, i_{2k}) \in X_j : \theta_{i_{2j-1}, i_{2j}, \gamma} \in B, \forall 1 \leq j \leq k\}, \quad j = 1, 2, \\ Y_{j,l,B} &= \{(i_1, \dots, i_{2k}) \in Y_{j,l} : \theta_{i_{2j-1}, i_{2j}, \gamma} \in B, \forall 1 \leq j \leq k\}, \end{aligned}$$

then we have

$$(25) \quad \rho^{(k,n,\gamma)}(B^k) = |X_{2,B}|, \quad X_{2,B} \subseteq X_{1,B}, \quad |X_{1,B}| = \frac{(\tilde{\chi}^{(n,\gamma)}(B))!}{(\tilde{\chi}^{(n,\gamma)}(B) - k)!},$$

which gives the first inequality, here $|X|$ is the cardinality of the set X .

We also have $X_{1,B} \setminus X_{2,B} = \bigcup_{1 \leq j < l \leq k} Y_{j,l,B}$ and by symmetry $|Y_{j,l,B}| = |Y_{1,2,B}|$ for $1 \leq j < l \leq k$, therefore

$$(26) \quad |X_{1,B}| - |X_{2,B}| \leq \sum_{1 \leq j < l \leq k} |Y_{j,l,B}| = k(k - 1)|Y_{1,2,B}|/2.$$

Now we assume $\gamma = \frac{\beta+2}{\beta+1}$. If $a = 0$, then we have $\theta_{j,l} \geq n^{-\gamma}(2c_n) = 2c_1$ for every $1 \leq j < l \leq n$, thus $\theta_{j,l,\gamma} \notin B$, and $\tilde{\chi}^{(n)}(B) = \rho^{(k,n)}(B^k) = 0$; if $k = 1$, then by definition $\tilde{\chi}^{(n)}(B) = \rho^{(k,n)}(B^k)$. Thus, the second and third inequalities are clearly true in these two trivial cases, for the rest, we only need to consider the case $a > 0, k > 1$. The key point is to estimate $|Y_{1,2,B}|$.

For fixed $\theta_{i_1, i_2, \gamma} \in B$, we will show that there are at most $2(a - 1)$ choices of (i_3, i_4) to satisfy $(i_1, \dots, i_{2k}) \in Y_{1,2,B}$. Let

$$\begin{aligned} T_j &= \{l : l \neq i_j, \theta_{i_j, l, \gamma} \in B\} \cup \{l : l \neq i_j, \theta_{l, i_j, \gamma} \in B\}, \\ T'_j &= \{l : l \neq i_j, \theta_{i_j, l} \in (0, c_n)\} \cup \{l : l \neq i_j, \theta_{l, i_j} \in (0, c_n)\}, \quad j = 1, 2. \end{aligned}$$

Then we have $T_j \subseteq T'_j$, since $\theta_{j,l,\gamma} \in B$ implies $n^\gamma \theta_{j,l} \in A \subset (0, c_1)$ and $\theta_{j,l} \in (0, n^{-\gamma} c_1) = (0, c_n)$. Assume $\theta_{i_1} = \theta_{(p)}$ then we have

$$\begin{aligned} \{\theta_l : l \in T'_1 \cup \{i_1\}\} &= \{\theta_{(q)} \pmod{2\pi} : |\theta_{(q)} - \theta_{(p)}| < c_n\} \\ &= \{\theta_{(q)} \pmod{2\pi} : r \leq q \leq s\}, \end{aligned}$$

for some $r, s \in \mathbb{Z}$ such that $|\theta_{(r)} - \theta_{(p)}| < c_n, |\theta_{(s)} - \theta_{(p)}| < c_n$, therefore $|\theta_{(r)} - \theta_{(s)}| < 2c_n$, and by the definition of a we have $s - r \leq a$. Since $i_1 \notin T'_1$, we have

$$\begin{aligned} |T'_1| + 1 &= |\{\theta_l : l \in T'_1 \cup \{i_1\}\}| \\ &= |\{\theta_{(q)} \pmod{2\pi} : r \leq q \leq s\}| \\ &\leq s - r + 1 \leq a + 1, \end{aligned}$$

and thus $|T_1| \leq |T'_1| \leq a$. Similarly, we have $|T_2| \leq |T'_2| \leq a$.

Now for $\theta_{i_1, i_2, \gamma} \in B$, by definition we have $i_2 \in T_1$ and $i_1 \in T_2$.

If $\theta_{i_3, i_4, \gamma} \in B, \{i_1, i_2\} \cap \{i_3, i_4\} \neq \emptyset, \{i_1, i_2\} \neq \{i_3, i_4\}$, then we must have $\{i_3, i_4\} = \{i_1, l\}, l \in T_2 \setminus \{i_1\}$ or $\{i_3, i_4\} = \{i_2, l\}, l \in T_1 \setminus \{i_2\}$, and the order of i_3, i_4 is uniquely determined. In fact, by the definition of $\theta_{i, j}$, we have $\theta_{i_3, i_4} + \theta_{i_4, i_3} = 2\pi$, if $\theta_{i_3, i_4, \gamma} \in B, \theta_{i_4, i_3, \gamma} \in B$ then we have $n^\gamma \theta_{i_3, i_4}, n^\gamma \theta_{i_4, i_3} \in A \subset (0, c_1)$, and $\theta_{i_3, i_4} + \theta_{i_4, i_3} < 2n^{-\gamma} c_1 = 2c_n < 2\pi$, a contradiction.

Thus for $\theta_{i_1, i_2, \gamma} \in B$, the number of (i_3, i_4) satisfying $\theta_{i_3, i_4, \gamma} \in B, \{i_1, i_2\} \cap \{i_3, i_4\} \neq \emptyset, \{i_1, i_2\} \neq \{i_3, i_4\}$ is at most $|T_2 \setminus \{i_1\}| + |T_1 \setminus \{i_2\}| = |T_2| - 1 + |T_1| - 1 \leq 2(a - 1)$. Now there are $\tilde{\chi}^{(n)}(B)$ choices of (i_1, i_2) , for fixed (i_1, i_2) there are at most $2(a - 1)$ choices of (i_3, i_4) and $\tilde{\chi}^{(n)}(B)$ choices of $(i_{2l-1}, i_{2l}), 3 \leq l \leq k$, to satisfy $(i_1, \dots, i_{2k}) \in Y_{1,2,B}$, thus we have

$$|Y_{1,2,B}| \leq \tilde{\chi}^{(n)}(B) \times 2(a - 1) \times \tilde{\chi}^{(n)}(B)^{k-2} = 2(a - 1)\tilde{\chi}^{(n)}(B)^{k-1}.$$

By (25) and (26), we have

$$\begin{aligned} 0 &\leq \frac{(\tilde{\chi}^{(n)}(B))!}{(\tilde{\chi}^{(n)}(B) - k)!} - \rho^{(k,n)}(B^k) \\ &= |X_{1,B}| - |X_{2,B}| \leq k(k - 1)|Y_{1,2,B}|/2 \\ &\leq k(k - 1)(a - 1)(\tilde{\chi}^{(n)}(B))^{k-1}, \end{aligned}$$

which is the second inequality.

The third inequality follows from the second inequality and the fact that

$$\begin{aligned} \frac{(\tilde{\chi}^{(n)}(B))!}{(\tilde{\chi}^{(n)}(B) - k)!} &= \prod_{j=0}^{k-1} (\tilde{\chi}^{(n)}(B) - j) \\ &= (\tilde{\chi}^{(n)}(B))^k \prod_{j=0}^{k-1} (1 - j/\tilde{\chi}^{(n)}(B)) \\ &\geq (\tilde{\chi}^{(n)}(B))^k \left(1 - \sum_{j=0}^{k-1} j/\tilde{\chi}^{(n)}(B)\right) \\ &= (\tilde{\chi}^{(n)}(B))^k - k(k - 1)(\tilde{\chi}^{(n)}(B))^{k-1}/2, \end{aligned}$$

this completes the proof. \square

4. Integral inequalities. In this section, we will first prove Lemma 4.1 which is roughly about the integration of the joint density on the small tubes around one variable. Then we can derive several integral inequalities about the two-component log-gas. We will use these inequalities many times during the whole proof.

4.1. *Integral lemma.* We first prove the following integral lemma.

LEMMA 4.1. *Let m, n, β be positive integers with $m \leq n$. Given c such that $n\beta c \in (0, 1)$ and $\theta_j \in \mathbb{R}, j = 1, \dots, m$, we define*

$$F(x) = \prod_{j=1}^m (e^{ix} - e^{i\theta_j}),$$

then we have

$$\begin{aligned} & \left(\frac{\sin(c/2)}{c/2}\right)^\beta \cos(n\beta c) \frac{c^{\beta+1}}{\beta+1} \int_{-\pi}^\pi dx_1 |F(x_1)|^{2\beta} \\ & \leq \int_{-\pi}^\pi dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta \\ & \leq \frac{c^{\beta+1}}{\beta+1} \int_{-\pi}^\pi dx_1 |F(x_1)|^{2\beta}, \end{aligned}$$

and for $k \geq 1$, we have

$$\begin{aligned} & \int_{-\pi}^\pi dx_1 \int_{(x_1, x_1+c)^{k-1}} dx_2 \cdots dx_k \prod_{1 \leq j < l \leq k} |e^{ix_j} - e^{ix_l}|^\beta \prod_{j=1}^k |F(x_j)|^\beta \\ & \leq c^{\beta k(k-1)/2 + k-1} \int_{-\pi}^\pi dx_1 |F(x_1)|^{k\beta}. \end{aligned}$$

For intervals $A \subset (0, c), I \subset (-\pi, \pi)$, we denote

$$\varphi(\beta, A) := \int_A |1 - e^{iu}|^\beta du,$$

then we have

$$\begin{aligned} & \left| \int_I dx_1 \int_{x_1+A} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta - \varphi(\beta, A) \int_I dx_1 |F(x_1)|^{2\beta} \right| \\ & \leq \varphi(\beta, A)(n\beta c) \int_{-\pi}^\pi dx_1 |F(x_1)|^{2\beta} \end{aligned}$$

and

$$\left(\frac{\sin(c/2)}{c/2}\right)^\beta \int_A u^\beta du \leq \varphi(\beta, A) \leq \int_A u^\beta du.$$

PROOF. We can write

$$F(x)^\beta = \sum_{j=0}^{m\beta} a_j e^{ijx}.$$

We change variables $x_2 = x_1 + t$ to get

$$\begin{aligned} (27) \quad & \int_{-\pi}^\pi dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta \\ & = \int_0^c dt \int_{-\pi}^\pi |1 - e^{it}|^\beta |F(x_1)|^\beta |F(x_1+t)|^\beta dx_1. \end{aligned}$$

As

$$F(x_1)^\beta = \sum_{j=0}^{m\beta} a_j e^{ijx_1}, \quad F(x_1 + t)^\beta = \sum_{j=0}^{m\beta} a_j e^{ijt} e^{ijx_1},$$

by Parseval's theorem, we have

$$\int_{-\pi}^{\pi} \overline{F(x_1)^\beta} F(x_1 + t)^\beta dx_1 = 2\pi \sum_{j=0}^{m\beta} \overline{a_j} a_j e^{ijt} = 2\pi \sum_{j=0}^{m\beta} |a_j|^2 e^{ijt}$$

and

$$\int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1 = \int_{-\pi}^{\pi} |F(x_1)^\beta|^2 dx_1 = 2\pi \sum_{j=0}^{m\beta} |a_j|^2.$$

Thus for $t \in (0, c)$, $0 \leq j \leq m\beta \leq n\beta$, we have $0 \leq jt \leq n\beta c < 1$ and

$$\begin{aligned} & \int_{-\pi}^{\pi} |F(x_1)|^\beta |F(x_1 + t)|^\beta dx_1 \\ & \geq \operatorname{Re} \int_{-\pi}^{\pi} \overline{F(x_1)^\beta} F(x_1 + t)^\beta dx_1 \\ (28) \quad & = 2\pi \sum_{j=0}^{m\beta} |a_j|^2 (\cos jt) \\ & \geq 2\pi \sum_{j=0}^{m\beta} |a_j|^2 \cos(n\beta c) \\ & = \cos(n\beta c) \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1, \end{aligned}$$

integrating for $t \in (0, c)$ gives

$$\begin{aligned} & \int_{-\pi}^{\pi} dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta \\ & \geq \int_0^c dt |1 - e^{it}|^\beta \cos(n\beta c) \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1. \end{aligned}$$

As $(\sin x)/x$ is decreasing for $x \in (0, 1)$ and $0 < c \leq n\beta c < 1$, we further have

$$\begin{aligned} \int_0^c dt |1 - e^{it}|^\beta & = \int_0^c dt |2 \sin(t/2)|^\beta \\ & \geq \int_0^c dt \left| t \frac{\sin(c/2)}{c/2} \right|^\beta = \frac{c^{\beta+1}}{\beta + 1} \left(\frac{\sin(c/2)}{c/2} \right)^\beta. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \int_{-\pi}^{\pi} dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta \\ & \geq \frac{c^{\beta+1}}{\beta + 1} \left(\frac{\sin(c/2)}{c/2} \right)^\beta \cos(n\beta c) \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1, \end{aligned}$$

which is the lower bound in the first inequality.

On the other hand, since F is 2π -periodic, for $t \in (0, c)$, we have

$$\begin{aligned} 0 &\leq \int_{-\pi}^{\pi} \left| |F(x_1)|^\beta - |F(x_1+t)|^\beta \right|^2 dx_1 \\ &= \int_{-\pi}^{\pi} (|F(x_1)|^{2\beta} + |F(x_1+t)|^{2\beta}) dx_1 - 2 \int_{-\pi}^{\pi} |F(x_1)|^\beta |F(x_1+t)|^\beta dx_1 \\ &= 2 \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1 - 2 \int_{-\pi}^{\pi} |F(x_1)|^\beta |F(x_1+t)|^\beta dx_1, \end{aligned}$$

which implies

$$(29) \quad \int_{-\pi}^{\pi} |F(x_1)|^\beta |F(x_1+t)|^\beta dx_1 \leq \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1,$$

and using (28) and $2 - 2 \cos(n\beta c) \leq (n\beta c)^2$, we also have

$$(30) \quad \int_{-\pi}^{\pi} \left| |F(x_1)|^\beta - |F(x_1+t)|^\beta \right|^2 dx_1 \leq (n\beta c)^2 \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1.$$

By (27) and (29), we have

$$\begin{aligned} &\int_{-\pi}^{\pi} dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta \\ &\leq \int_0^c dt |1 - e^{it}|^\beta \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1 \\ &\leq \int_0^c dt |t|^\beta \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1 \\ &= \frac{c^{\beta+1}}{\beta+1} \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{2\beta}, \end{aligned}$$

which gives the upper bound in the first inequality.

If $x_j \in (x_1, x_1 + c)$ for $1 < j \leq k$, then we have $|e^{ix_j} - e^{ix_l}| \leq |x_j - x_l| < c$ for $1 \leq j < l \leq k$, therefore,

$$\begin{aligned} &\int_{-\pi}^{\pi} dx_1 \int_{(x_1, x_1+c)^{k-1}} dx_2 \cdots dx_k \prod_{1 \leq j < l \leq k} |e^{ix_j} - e^{ix_l}|^\beta \prod_{j=1}^k |F(x_j)|^\beta \\ &\leq \int_{-\pi}^{\pi} dx_1 \int_{(x_1, x_1+c)^{k-1}} dx_2 \cdots dx_k \prod_{1 \leq j < l \leq k} c^\beta \prod_{j=1}^k |F(x_j)|^\beta \\ &= c^{\beta k(k-1)/2} \int_{(0,c)^{k-1}} dt_2 \cdots dt_k \int_{-\pi}^{\pi} dx_1 \prod_{j=1}^k |F(x_1 + t_j)|^\beta \\ &\leq \frac{c^{\beta k(k-1)/2}}{k} \int_{(0,c)^{k-1}} dt_2 \cdots dt_k \int_{-\pi}^{\pi} dx_1 \sum_{j=1}^k |F(x_1 + t_j)|^{k\beta} \\ &= \frac{c^{\beta k(k-1)/2}}{k} \sum_{j=1}^k \int_{(0,c)^{k-1}} dt_2 \cdots dt_k \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{k\beta} \\ &= c^{\beta k(k-1)/2 + k-1} \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{k\beta}, \end{aligned}$$

which is the second inequality, here we denote $t_1 = 0$.

By changing variables, the definition of $\varphi(\beta, A)$, Hölder inequality and (30), we have

$$\begin{aligned} & \left| \int_I dx_1 \int_{x_1+A} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta - \varphi(\beta, A) \int_I dx_1 |F(x_1)|^{2\beta} \right| \\ &= \left| \int_A du \int_I dx_1 |1 - e^{iu}|^\beta |F(x_1)|^\beta |F(x_1 + u)|^\beta - \int_A |1 - e^{iu}|^\beta du \int_I dx_1 |F(x_1)|^{2\beta} \right| \\ &\leq \int_A du \int_I dx_1 |1 - e^{iu}|^\beta |F(x_1)|^\beta ||F(x_1 + u)|^\beta - |F(x_1)|^\beta| \\ &\leq \int_A du |1 - e^{iu}|^\beta \left(\int_I dx_1 |F(x_1)|^{2\beta} \right)^{\frac{1}{2}} \times \left(\int_I dx_1 ||F(x_1 + u)|^\beta - |F(x_1)|^\beta|^2 \right)^{\frac{1}{2}} \\ &\leq \int_A du |1 - e^{iu}|^\beta \left(\int_{-\pi}^\pi dx_1 |F(x_1)|^{2\beta} \right)^{\frac{1}{2}} \left((n\beta c)^2 \int_{-\pi}^\pi |F(x_1)|^{2\beta} dx_1 \right)^{\frac{1}{2}} \\ &= \varphi(\beta, A)(n\beta c) \int_{-\pi}^\pi dx_1 |F(x_1)|^{2\beta}, \end{aligned}$$

which is the third inequality.

As $(\sin x)/x$ is decreasing for $x \in (0, 1)$ and

$$A \subset (0, c) \subset (0, 1),$$

we have

$$\begin{aligned} \varphi(\beta, A) &= \int_A |1 - e^{iu}|^\beta du \\ &= \int_A |2 \sin(u/2)|^\beta du \\ &\geq \int_A \left| u \frac{\sin(c/2)}{c/2} \right|^\beta du = \left(\frac{\sin(c/2)}{c/2} \right)^\beta \int_A u^\beta du, \end{aligned}$$

and as $|1 - e^{iu}| \leq u$, we also have

$$\varphi(\beta, A) = \int_A |1 - e^{iu}|^\beta du \leq \int_A u^\beta du,$$

which gives the fourth inequality. This completes the proof. \square

4.2. *Inequalities regarding two-component log-gas.* Let $B = (0, c_0) \times (-\pi, \pi)$, $n > 2k$, by the definition of $\rho^{(k,n,\gamma)}$ (recall (23)), we have

$$(31) \quad \mathbb{E} \rho^{(k,n,\gamma)}(B^k) = \frac{n!}{(n - 2k)!} \int_{\Sigma_{n,k,c}} J_\beta(\theta_1, \dots, \theta_n) d\theta_1 \cdots d\theta_n \Big|_{c=c_0/n^\gamma},$$

here

$$(32) \quad \Sigma_{n,k,c} = \{(\theta_1, \dots, \theta_n) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n - k, \theta_j - \theta_{j-k} \in (0, c), \forall n - k < j \leq n\}.$$

For $0 \leq l \leq k$, with the same assumptions as in Lemma 4.1, we denote

$$(33) \quad E_{\beta,n,k,l}(c) := \int_{\Sigma_{n-l,k-l,c}} d\theta_1 \cdots d\theta_{n-l} \prod_{j < m} |e^{i\theta_j} - e^{i\theta_m}|^{q_j q_m \beta} \Big|_{q_s=1+\chi_{\{s \leq l\}}}.$$

Then we have

$$\int_{\Sigma_{n,k,c}} J_\beta(\theta_1, \dots, \theta_n) d\theta_1 \cdots d\theta_n = \frac{E_{\beta,n,k,0}(c)}{C_{\beta,n}},$$

and by definition we can check that

$$E_{\beta,n,k,k}(c) = C_{\beta,n-2k,k}.$$

We need to show that (for $0 < n\beta c < 1$)

$$(34) \quad \left(\frac{\sin(c/2)}{c/2}\right)^\beta \cos(n\beta c) \frac{c^{\beta+1}}{\beta+1} \leq \frac{E_{\beta,n,k,l-1}(c)}{E_{\beta,n,k,l}(c)} \leq \frac{c^{\beta+1}}{\beta+1}.$$

In fact, after changing the order of variables, we can write

$$\begin{aligned} E_{\beta,n,k,l-1}(c) &= \int_{\Sigma_{n-l-1,k-l,c}} d\theta_1 \cdots d\theta_{n-l-1} \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}|^{q_j q_m \beta} \\ &\quad \times \int_{-\pi}^\pi dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta \\ &\quad \times \prod_{j=1}^2 \prod_{m=1}^{n-l-1} |e^{ix_j} - e^{i\theta_m}|^{q_m \beta} \Big|_{q_s=1+\chi_{\{s \leq l-1\}}}, \end{aligned}$$

and

$$\begin{aligned} E_{\beta,n,k,l}(c) &= \int_{\Sigma_{n-l-1,k-l,c}} d\theta_1 \cdots d\theta_{n-l-1} \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}|^{q_j q_m \beta} \\ &\quad \times \int_{-\pi}^\pi dx_1 \prod_{m=1}^{n-l-1} |e^{ix_1} - e^{i\theta_m}|^{2q_m \beta} \Big|_{q_s=1+\chi_{\{s \leq l-1\}}}, \end{aligned}$$

then (34) is the direct consequence of Lemma 4.1 by taking

$$F(x) = \prod_{m=1}^{n-l-1} (e^{ix} - e^{i\theta_m})^{q_m}.$$

By (34) we finally have the following two estimates:

$$(35) \quad E_{\beta,n,k,l}(c) \leq \left(\frac{c^{\beta+1}}{\beta+1}\right)^{k-l} E_{\beta,n,k,k}(c) = \left(\frac{c^{\beta+1}}{\beta+1}\right)^{k-l} C_{\beta,n-2k,k}$$

and

$$(36) \quad \left(\frac{\sin(c/2)}{c/2}\right)^{k\beta} (\cos(n\beta c))^k \left(\frac{c^{\beta+1}}{\beta+1}\right)^k C_{\beta,n-2k,k} \leq E_{\beta,n,k,0}(c).$$

5. No successive small gaps. In this section, we will prove Lemma 1.2 which implies that there is no successive smallest gaps. We first need the following estimate.

LEMMA 5.1. For $B = (0, c_0) \times (-\pi, \pi)$, $n \geq k > 1$, $n^{1-\gamma} \beta c_0 \in (0, 1)$, we have

$$\mathbb{E} \tilde{\chi}^{(n,\gamma,k-1)}(B) \leq n(n^{1-\gamma} \beta c_0)^{\beta k(k-1)/2+k-1}.$$

PROOF. We consider the point process

$$\xi^{(n)} = \sum_{i=1}^n \delta_{\theta_i}, \quad \xi^{(n,k)} = \sum_{i_1, \dots, i_k \text{ all distinct}} \delta_{(\theta_{i_1}, \dots, \theta_{i_k})}.$$

For $B = (0, c_0) \times (-\pi, \pi)$, $n \geq k > 1$, let $c_n = c_0/n^\gamma$, then we have

$$\tilde{\chi}^{(n,\gamma,j)}(B) = \sum_{i=1}^n \mathbf{1}_{\xi^{(n)}(\theta_i+(0,c_n)) \geq j} \leq \frac{1}{j!} \xi^{(n,j+1)}(\Lambda_{j+1,c_n}),$$

here, the angles are modulo 2π , $\mathbf{1}$ is the indicator of an event and we define

$$\Lambda_{k,c} = \{(\theta_1, \dots, \theta_k) : \theta_1 \in (-\pi, \pi), \theta_j - \theta_1 \in (0, c), \forall 1 < j \leq k\}.$$

Let

$$\Lambda_{k,c,n} = \{(\theta_1, \dots, \theta_n) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n - k + 1, \theta_j - \theta_{n-k+1} \in (0, c), \forall n - k + 1 < j \leq n\},$$

then by Lemma 1.1 and Lemma 4.1, we have

$$\begin{aligned} & \mathbb{E} \tilde{\chi}^{(n,\gamma,k-1)}(B) \\ & \leq \frac{1}{(k-1)!} \mathbb{E} \xi^{(n,k)}(\Lambda_{k,c_n}) \\ & = \frac{1}{(k-1)!} \frac{n!}{(n-k)!} \int_{\Lambda_{k,c_n,n}} J_\beta(\theta_1, \dots, \theta_n) d\theta_1 \cdots d\theta_n \\ & = \frac{1}{(k-1)!} \frac{n!}{(n-k)!} \frac{1}{C_{\beta,n}} \int_{-\pi}^\pi d\theta_1 \cdots \int_{-\pi}^\pi d\theta_{n-k} \prod_{1 \leq j < m \leq n-k} |e^{i\theta_j} - e^{i\theta_m}|^\beta \\ & \quad \times \int_{\Lambda_{k,c_n}} dx_1 \cdots dx_k \prod_{1 \leq j < m \leq k} |e^{ix_j} - e^{ix_m}|^\beta \prod_{j=1}^k \prod_{m=1}^{n-k} |e^{ix_j} - e^{i\theta_m}|^\beta \\ & \leq \frac{n^k}{(k-1)!} \frac{1}{C_{\beta,n}} \int_{-\pi}^\pi d\theta_1 \cdots \int_{-\pi}^\pi d\theta_{n-k} \prod_{1 \leq j < m \leq n-k} |e^{i\theta_j} - e^{i\theta_m}|^\beta \\ & \quad \times c_n^{\beta k(k-1)/2+k-1} \int_{-\pi}^\pi dx_1 \prod_{m=1}^{n-k} |e^{ix_1} - e^{i\theta_m}|^{k\beta} \\ & = \frac{n^k}{(k-1)!} \frac{C_{\beta,n-k,(k)}}{C_{\beta,n}} c_n^{\beta k(k-1)/2+k-1} \\ & \leq \frac{n^k}{(k-1)!} (n\beta)^{k(k-1)\beta/2} c_n^{\beta k(k-1)/2+k-1} \\ & = \frac{n(n\beta c_n)^{\beta k(k-1)/2+k-1}}{(k-1)! \beta^{k-1}} \\ & \leq n(n\beta c_n)^{\beta k(k-1)/2+k-1} \\ & = n(n^{1-\gamma} \beta c_0)^{\beta k(k-1)/2+k-1}, \end{aligned}$$

this completes the proof. \square

Now we can give the proof of Lemma 1.2.

PROOF. Let c be such that $A \subset (0, c)$, and $B = (0, c) \times (-\pi, \pi)$, $\gamma = \frac{\beta+2}{\beta+1}$. Then by the definitions (5) and (14), $\chi^{(n)}(A \times I) - \tilde{\chi}^{(n)}(A \times I) \neq 0$ implies $\tilde{\chi}^{(n,\gamma,j)}(A \times I) > 0$ for some

$j > 1$, and thus we must have $\tilde{\chi}^{(n,\gamma,2)}(B) > 0$. Since $\gamma > 1$, for n large enough we have $n^{1-\gamma} \beta c \in (0, 1)$, and by Lemma 5.1 with $k = 3$, we have

$$\begin{aligned} \mathbb{P}(\chi^{(n)}(A \times I) - \tilde{\chi}^{(n)}(A \times I) \neq 0) &\leq \mathbb{P}(\tilde{\chi}^{(n,\gamma,2)}(B) > 0) \\ &\leq \mathbb{E}(\tilde{\chi}^{(n,\gamma,2)}(B)) \\ &\leq n(n^{1-\gamma} \beta c)^{3\beta+2} \\ &= n(n^{-\frac{1}{\beta+1}} \beta c)^{3\beta+2} \rightarrow 0, \end{aligned}$$

this completes the proof. \square

6. Proof of Lemma 1.3. In this section, we will prove Lemma 1.3.

6.1. *Uniform boundedness.* We need the following lemma which will be applied in the proofs of Lemma 1.3 and Lemma 1.4.

LEMMA 6.1.

$$(37) \quad \limsup_{n \rightarrow +\infty} \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}} < +\infty.$$

PROOF. Let c_0 be fixed such that $\beta c_0 \in (0, 1)$ and $B = (0, c_0) \times (-\pi, \pi)$. Thanks to the integral expression of $\mathbb{E}\rho^{(k,n,\gamma)}(B^k)$ in (31), the definition of $E_{\beta,n,k,l}$ (33) and the upper bound (36), with $\gamma = 1$, we have

$$\begin{aligned} \mathbb{E}\rho^{(k,n,1)}(B^k) &= \frac{n!}{(n-2k)!} \frac{E_{\beta,n,k,0}(c)}{C_{\beta,n}} \Big|_{c=c_0/n} \\ &\geq \frac{n!}{(n-2k)!} \frac{C_{\beta,n-2k,k}}{C_{\beta,n}} \times \left(\frac{\sin(c/2)}{c/2}\right)^{k\beta} (\cos(n\beta c))^k \left(\frac{c^{\beta+1}}{\beta+1}\right)^k \Big|_{c=c_0/n} \\ &= \frac{n!n^{-k}}{(n-2k)!} \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}} \left(\frac{\sin(c_0/(2n))}{c_0/(2n)}\right)^{k\beta} (\cos(\beta c_0))^k \left(\frac{c_0^{\beta+1}}{\beta+1}\right)^k. \end{aligned}$$

By the first inequality in Lemma 3.1, we have

$$\rho^{(k,n,1)}(B^k) \leq \frac{(\tilde{\chi}^{(n,1)}(B))!}{(\tilde{\chi}^{(n,1)}(B) - k)!} \leq (\tilde{\chi}^{(n,1)}(B))^k,$$

which implies

$$\begin{aligned} &\limsup_{n \rightarrow +\infty} \mathbb{E}(n^{-1} \tilde{\chi}^{(n,1)}(B))^k \\ &\geq \limsup_{n \rightarrow +\infty} n^{-k} \mathbb{E}\rho^{(k,n,1)}(B^k) \\ &\geq \lim_{n \rightarrow +\infty} \frac{n!n^{-2k}}{(n-2k)!} \limsup_{n \rightarrow +\infty} \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}} (\cos(\beta c_0))^k \left(\frac{c_0^{\beta+1}}{\beta+1}\right)^k \\ &= \limsup_{n \rightarrow +\infty} \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}} \left(\frac{c_0^{\beta+1} \cos(\beta c_0)}{\beta+1}\right)^k. \end{aligned}$$

Thus, to prove (37), we only need to prove

$$(38) \quad \limsup_{n \rightarrow +\infty} \mathbb{E}(n^{-1} \tilde{\chi}^{(n,1)}(B))^k < +\infty.$$

As $\tilde{\chi}^{(n,\gamma)} = \sum_{j=1}^{n-1} \tilde{\chi}^{(n,\gamma,j)}$, by Lemma 5.1 (since $\beta c_0 \in (0, 1)$), we have

$$\mathbb{E}(n^{-1} \tilde{\chi}^{(n,1,j)}(B)) \leq (\beta c_0)^{\beta j(j+1)/2+j} \leq (\beta c_0)^j.$$

Using $0 \leq \tilde{\chi}^{(n,1,j)}(B) \leq n$, we have

$$\mathbb{E}(n^{-1} \tilde{\chi}^{(n,1,j)}(B))^k \leq \mathbb{E}(n^{-1} \tilde{\chi}^{(n,1,j)}(B)) \leq (\beta c_0)^j.$$

By Minkowski inequality, we finally have

$$\begin{aligned} (\mathbb{E}(n^{-1} \tilde{\chi}^{(n,1)}(B))^k)^{1/k} &\leq \sum_{j=1}^{n-1} (\mathbb{E}(n^{-1} \tilde{\chi}^{(n,1,j)}(B))^k)^{1/k} \\ &\leq \sum_{j=1}^{n-1} (\beta c_0)^{j/k} \\ &\leq (1 - (\beta c_0)^{1/k})^{-1}, \end{aligned}$$

thus (38) is true, so is (37). \square

6.2. *Proof of Lemma 1.3.* For $B = A \times I$, we will use Lemma 3.1 to prove that

$$(39) \quad \lim_{n \rightarrow +\infty} \left(\mathbb{E} \frac{(\tilde{\chi}^{(n)}(B))!}{(\tilde{\chi}^{(n)}(B) - k)!} - \mathbb{E} \rho^{(k,n)}(B^k) \right) = 0,$$

and use Lemma 4.1 to prove that

$$(40) \quad \lim_{n \rightarrow +\infty} \left(\mathbb{E} \rho^{(k,n)}(B^k) - \left(\int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} \right) = 0,$$

then Lemma 1.3 follows from (39) and (40), here $\rho^{(k,n)}$ is defined in (24).

Let $A \subset \mathbb{R}_+$ be any bounded interval, $I \subseteq (-\pi, \pi)$ and $B = A \times I$. Let c_1 be such that $A \subset (0, c_1)$, and $B_1 = (0, c_1) \times (-\pi, \pi)$ such that $B \subset B_1$. We denote $\gamma = \frac{\beta+2}{\beta+1}$ and $c_n = c_1/n^\gamma$.

Since $\gamma > 1$, for n large enough we have $n\beta c_n = n^{1-\gamma} \beta c_1 \in (0, 1)$. By the expression of $\mathbb{E} \rho^{(k,n,\gamma)}(B^k)$, $E_{\beta,n,k,l}$ and (35), with $\gamma(\beta + 1) = \beta + 2$, we have

$$\begin{aligned} \mathbb{E} \rho^{(k,n)}(B^k) &\leq \mathbb{E} \rho^{(k,n)}(B_1^k) \\ &= \frac{n!}{(n-2k)!} \frac{E_{\beta,n,k,0}(c_n)}{C_{\beta,n}} \\ &\leq \frac{n!}{(n-2k)!} \frac{C_{\beta,n-2k,k}}{C_{\beta,n}} \left(\frac{c_n^{\beta+1}}{\beta+1} \right)^k \\ &\leq n^{2k} \frac{C_{\beta,n-2k,k}}{C_{\beta,n}} \left(\frac{c_1^{\beta+1}}{\beta+1} \right)^k n^{-\gamma(\beta+1)k} \\ &= n^{2k} \frac{C_{\beta,n-2k,k}}{C_{\beta,n}} \left(\frac{c_1^{\beta+1}}{\beta+1} \right)^k n^{-(\beta+2)k} \\ &= \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}} \left(\frac{c_1^{\beta+1}}{\beta+1} \right)^k. \end{aligned}$$

By Lemma 6.1, we have

$$(41) \quad \limsup_{n \rightarrow +\infty} \mathbb{E} \rho^{(k,n)}(B^k) < +\infty.$$

Let a be defined in Lemma 3.1 and assume n large enough such that $0 < c_n \leq n\beta c_n = n^{1-\gamma} \beta c_1 < 1/4$. By definition, we have $0 \leq a < n$ and $a \geq k$ is equivalent to $\tilde{\chi}^{(n,\gamma,k)}(B_2) > 0$, here, $a, k \in \mathbb{Z}, k > 0$ and $B_2 = (0, 2c_1) \times (-\pi, \pi)$.

By Lemma 5.1 and $(1 - \gamma)(\beta + 1) = -1$, for $1 \leq k < n$, we have

$$\begin{aligned} \mathbb{P}(a \geq k) &= \mathbb{P}(\tilde{\chi}^{(n,\gamma,k)}(B_2) > 0) \\ &\leq \mathbb{E}(\tilde{\chi}^{(n,\gamma,k)}(B_2)) \\ &\leq n(2n^{1-\gamma} \beta c_1)^{k(k+1)\beta/2+k} \\ &= n(2n^{1-\gamma} \beta c_1)^{\beta+1} (2n^{1-\gamma} \beta c_1)^{(k+2)(k-1)\beta/2+k-1} \\ &= (2\beta c_1)^{\beta+1} (2n^{1-\gamma} \beta c_1)^{(k+2)(k-1)\beta/2+k-1} \\ &\leq (2\beta c_1)^{\beta+1} (1/2)^{k-1}. \end{aligned}$$

Since $\mathbb{P}(a \geq k) = 0$ for $k \geq n$, thus

$$\mathbb{P}(a \geq k) \leq (2\beta c_1)^{\beta+1} (1/2)^{k-1}$$

is always true for $k \geq 1$.

The above argument also implies that for $k > 1, k \in \mathbb{Z}$, we must have

$$\lim_{n \rightarrow +\infty} \mathbb{P}(a \geq k) = 0.$$

And by dominated convergence theorem, we can further deduce that

$$(42) \quad \lim_{n \rightarrow +\infty} \mathbb{E}(a - 1)_+^p = 0, \quad \forall p \in (0, +\infty),$$

here, $f_+ = \max(f, 0)$.

By Lemma 3.1, for any $k \geq 1$, we have $(\tilde{\chi}^{(n)}(B))^k \leq 2\rho^{(k,n)}(B^k)$ or $(\tilde{\chi}^{(n)}(B))^k \leq 2k(k - 1)a(\tilde{\chi}^{(n)}(B))^{k-1}$, therefore, we have

$$(\tilde{\chi}^{(n)}(B))^k \leq \max(2\rho^{(k,n)}(B^k), (2k(k - 1)a)^k)$$

and

$$\mathbb{E}(\tilde{\chi}^{(n)}(B))^k \leq 2\mathbb{E}(\rho^{(k,n)}(B^k)) + (2k(k - 1))^k \mathbb{E}(a^k).$$

By (41) and (42), we have

$$(43) \quad \limsup_{n \rightarrow +\infty} \mathbb{E}(\tilde{\chi}^{(n)}(B))^k < +\infty.$$

By Lemma 3.1, Hölder inequality, (42) and (43), we have

$$\begin{aligned} 0 &\leq \mathbb{E} \left(\frac{(\tilde{\chi}^{(n)}(B))!}{(\tilde{\chi}^{(n)}(B) - k)!} - \rho^{(k,n)}(B^k) \right) \\ &\leq k(k - 1) \mathbb{E}((a - 1)_+ (\tilde{\chi}^{(n)}(B))^{k-1}) \\ &\leq k(k - 1) (\mathbb{E}((a - 1)_+^k))^{1/k} (\mathbb{E}(\tilde{\chi}^{(n)}(B))^k)^{1-1/k} \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$, which implies (39).

For $B = A \times I, n > 2k, \gamma > 0$, we have

$$(44) \quad \mathbb{E}_\rho^{(k,n,\gamma)}(B^k) = \frac{n!}{(n-2k)!} \int_{\Sigma_{n,k,cA,I}} J_\beta(\theta_1, \dots, \theta_n) d\theta_1 \cdots d\theta_n \Big|_{c=n^{-\gamma}},$$

here,

$$\Sigma_{n,k,A,I} = \{(\theta_1, \dots, \theta_n) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n-2k, \theta_{j-k} \in I, \theta_j - \theta_{j-k} \in A, \forall n-k < j \leq n\}.$$

We denote

$$\Sigma_{n,k,A,I,l} = \{(\theta_1, \dots, \theta_{n-l}) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n-2k, \theta_j \in I, \forall n-2k < j \leq n-k, \theta_j - \theta_{j-k+l} \in A, \forall n-k < j \leq n-l\}$$

and

$$E_{\beta,n,k,l}(A, I) := \int_{\Sigma_{n,k,A,I,l}} d\theta_1 \cdots d\theta_{n-l} \prod_{j < p} |e^{i\theta_j} - e^{i\theta_p}|^{q_j q_p \beta}$$

with $q_s = 1 + \chi_{\{n-2k < s \leq n-2k+l\}}$, then we have

$$(45) \quad \int_{\Sigma_{n,k,A,I}} J_\beta(\theta_1, \dots, \theta_n) d\theta_1 \cdots d\theta_n = \frac{E_{\beta,n,k,0}(A, I)}{C_{\beta,n}}$$

and

$$E_{\beta,n,k,k}(A, I) = C_{\beta,n-2k,k}(I).$$

We need some inequalities similar to (34).

LEMMA 6.2. $A \subset (0, c)$ and $I \subseteq (-\pi, \pi), n\beta c \in (0, 1), n > 2k, n, \beta, k$ are positive integers, then we have

$$\begin{aligned} & \left| E_{\beta,n,k,0}(A, I) - \left(\int_A u^\beta du \right)^k C_{\beta,n-2k,k}(I) \right| \\ & \leq (kn\beta c + \beta kc^2/24) \left(\frac{c^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k}. \end{aligned}$$

PROOF. As before, after changing the order of variables, we can write

$$\begin{aligned} E_{\beta,n,k,l-1}(A, I) &= \int_{\Sigma_{n-2,k-1,A,I,l-1}} d\theta_1 \cdots d\theta_{n-l-1} \Delta^\beta \\ & \times \int_I dx_1 \int_{x_1+A} dx_2 |e^{ix_1} - e^{ix_2}|^\beta \prod_{j=1}^2 \prod_{m=1}^{n-l-1} |e^{ix_j} - e^{i\theta_m}|^{q_m \beta} \end{aligned}$$

and

$$\begin{aligned} E_{\beta,n,k,l}(A, I) &= \int_{\Sigma_{n-2,k-1,A,I,l-1}} d\theta_1 \cdots d\theta_{n-l-1} \Delta^\beta \\ & \times \int_I dx_1 \prod_{m=1}^{n-l-1} |e^{ix_1} - e^{i\theta_m}|^{2q_m \beta}, \end{aligned}$$

here,

$$\Delta = \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}|^{q_j q_m}, \quad q_s = 1 + \chi_{\{n-2k < s < n-2k+l\}}.$$

By Lemma 4.1, $\Sigma_{n-2,k-1,A,I,l-1} \subset \Sigma_{n-l-1,k-l,c}$ and (35), we have

$$\begin{aligned}
 & |E_{\beta,n,k,l-1}(A, I) - \varphi(\beta, A)E_{\beta,n,k,l}(A, I)| \\
 & \leq \varphi(\beta, A)(n\beta c) \times \int_{\Sigma_{n-2,k-1,A,I,l-1}} d\theta_1 \cdots d\theta_{n-l-1} \Delta^\beta \int_{-\pi}^\pi dx_1 \prod_{m=1}^{n-l-1} |e^{ix_1} - e^{i\theta_m}|^{2q_m\beta} \\
 & \leq \varphi(\beta, A)(n\beta c) \int_{\Sigma_{n-l-1,k-l,c}} d\theta_1 \cdots d\theta_{n-l-1} \int_{-\pi}^\pi dx_1 \\
 & \quad \times \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}|^{q_j q_m \beta} \prod_{m=1}^{n-l-1} |e^{ix_1} - e^{i\theta_m}|^{2q_m\beta} \Big|_{q_s=1+\chi_{\{0 < s-n+2k < l\}}} \\
 & = \varphi(\beta, A)(n\beta c) \int_{\Sigma_{n-l,k-l,c}} d\theta_1 \cdots d\theta_{n-l} \\
 & \quad \times \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}|^{q_j q_m \beta} \Big|_{q_s=1+\chi_{\{0 < s-n+2k \leq l\}}} \\
 & = \varphi(\beta, A)(n\beta c) E_{\beta,n,k,l}(c) \\
 & \leq \varphi(\beta, A)(n\beta c) \left(\frac{c^{\beta+1}}{\beta+1}\right)^{k-l} C_{\beta,n-2k,k},
 \end{aligned}$$

where $\varphi(\beta, A)$ is as in Lemma 4.1 and $E_{\beta,n,k,l}(c)$ is as in (33).

Therefore (using Lemma 4.1), we have

$$\begin{aligned}
 & |E_{\beta,n,k,0}(A, I) - \varphi(\beta, A)^k E_{\beta,n,k,k}(A, I)| \\
 & \leq \sum_{l=1}^k \varphi(\beta, A)^{l-1} |E_{\beta,n,k,l-1}(A, I) - \varphi(\beta, A)E_{\beta,n,k,l}(A, I)| \\
 & \leq \sum_{l=1}^k \varphi(\beta, A)^l (n\beta c) \left(\frac{c^{\beta+1}}{\beta+1}\right)^{k-l} C_{\beta,n-2k,k} \\
 & \leq \sum_{l=1}^k (n\beta c) \left(\frac{c^{\beta+1}}{\beta+1}\right)^k C_{\beta,n-2k,k} \\
 & = (kn\beta c) \left(\frac{c^{\beta+1}}{\beta+1}\right)^k C_{\beta,n-2k,k}.
 \end{aligned}$$

As $1 \geq \frac{\sin x}{x} \geq 1 - x^2/6 > 0$ for $x \in (0, 1)$, and by Lemma 4.1, we have

$$\begin{aligned}
 0 & \leq \left(\int_A u^\beta du\right)^k - \varphi(\beta, A)^k \\
 & \leq \left(\int_A u^\beta du\right)^k \left(1 - \left(\frac{\sin(c/2)}{c/2}\right)^{\beta k}\right) \\
 & \leq \left(\frac{c^{\beta+1}}{\beta+1}\right)^k (1 - (1 - c^2/24)^{\beta k}) \leq \left(\frac{c^{\beta+1}}{\beta+1}\right)^k \beta k c^2/24.
 \end{aligned}$$

By definition, we have

$$0 \leq E_{\beta,n,k,k}(A, I) = C_{\beta,n-2k,k}(I) \leq C_{\beta,n-2k,k},$$

therefore, we have

$$\begin{aligned} & \left| E_{\beta,n,k,0}(A, I) - \left(\int_A u^\beta du \right)^k C_{\beta,n-2k,k}(I) \right| \\ & \leq |E_{\beta,n,k,0}(A, I) - \varphi(\beta, A)^k E_{\beta,n,k,k}(A, I)| \\ & \quad + \left| \left(\int_A u^\beta du \right)^k - \varphi(\beta, A)^k \right| C_{\beta,n-2k,k}(I) \\ & \leq (kn\beta c) \left(\frac{c^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k} + \left(\frac{c^{\beta+1}}{\beta+1} \right)^k (\beta kc^2/24) C_{\beta,n-2k,k}, \end{aligned}$$

which completes the proof. \square

Now we are ready to prove (40). By the expression of $\mathbb{E}\rho^{(k,n,\gamma)}(B^k)$ with $\gamma = \frac{\beta+2}{\beta+1}$ (see (44)), the definition of $E_{\beta,n,k,l}(A, I)$ (see (45)) and changing variables, we have

$$\begin{aligned} & \mathbb{E}(\rho^{(k,n)}((A \times I)^k)) - \left(\int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} \\ & = \frac{n!}{(n-2k)!} \frac{E_{\beta,n,k,0}(n^{-\gamma} A, I)}{C_{\beta,n}} - \left(\int_{n^{-\gamma} A} u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta-k(\beta+1)\gamma}} \\ & = \frac{n^{2k}}{C_{\beta,n}} \left(E_{\beta,n,k,0}(n^{-\gamma} A, I) - \left(\int_{n^{-\gamma} A} u^\beta du \right)^k C_{\beta,n-2k,k}(I) \right) \\ & \quad - \left(n^{2k} - \frac{n!}{(n-2k)!} \right) \frac{E_{\beta,n,k,0}(n^{-\gamma} A, I)}{C_{\beta,n}}. \end{aligned}$$

We first notice that

$$\begin{aligned} 0 & \leq n^{2k} - \frac{n!}{(n-2k)!} \\ & = n^{2k} - \prod_{j=0}^{2k-1} (n-j) \\ & = n^{2k} - n^{2k} \prod_{j=0}^{2k-1} (1-j/n) \\ & \leq n^{2k} - n^{2k} \left(1 - \sum_{j=0}^{2k-1} j/n \right) \\ & = n^{2k} \sum_{j=0}^{2k-1} j/n \\ & = n^{2k-1} k(2k-1). \end{aligned}$$

As $n^{-\gamma} A \subset (0, n^{-\gamma} c_1)$, $\Sigma_{n-2,k-1,n^{-\gamma} A, l, l-1} \subset \Sigma_{n-l-1, k-l, n^{-\gamma} c_1}$, for n large enough we have $n^{1-\gamma} \beta c_1 \in (0, 1)$, then we infer from (35) that

$$0 \leq E_{\beta,n,k,0}(n^{-\gamma} A, I) \leq E_{\beta,n,k,0}(n^{-\gamma} c_1) \leq C_{\beta,n-2k,k} \left(\frac{(n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k.$$

Therefore, we have

$$\begin{aligned}
 0 &\leq \left(n^{2k} - \frac{n!}{(n-2k)!} \right) \frac{E_{\beta,n,k,0}(n^{-\gamma} A, I)}{C_{\beta,n}} \\
 &\leq n^{2k-1} k(2k-1) \frac{C_{\beta,n-2k,k}}{C_{\beta,n}} \left(\frac{(n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k \\
 &= n^{2k-1} k(2k-1) \frac{C_{\beta,n-2k,k}}{C_{\beta,n}} \left(\frac{n^{-(\beta+2)} c_1^{\beta+1}}{\beta+1} \right)^k \\
 &= n^{-1} k(2k-1) \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}} \left(\frac{c_1^{\beta+1}}{\beta+1} \right)^k.
 \end{aligned}$$

By Lemma 6.2, we have

$$\begin{aligned}
 &\frac{n^{2k}}{C_{\beta,n}} \left| E_{\beta,n,k,0}(n^{-\gamma} A, I) - \left(\int_{n^{-\gamma} A} u^\beta du \right)^k C_{\beta,n-2k,k}(I) \right| \\
 &\leq \frac{n^{2k}}{C_{\beta,n}} (kn\beta c + \beta k c^2/24) \left(\frac{c^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k} \Big|_{c=n^{-\gamma} c_1} \\
 &= \frac{n^{2k}}{C_{\beta,n}} (kn^{1-\gamma} \beta c_1 + \beta kn^{-2\gamma} c_1^2/24) \left(\frac{n^{-(\beta+2)} c_1^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k} \\
 &= (kn^{1-\gamma} \beta c_1 + \beta kn^{-2\gamma} c_1^2/24) \left(\frac{c_1^{\beta+1}}{\beta+1} \right)^k \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 &\left| \mathbb{E}(\rho^{(k,n)}((A \times I)^k)) - \left(\int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} \right| \\
 &\leq (kn^{1-\gamma} \beta c_1 + \beta kn^{-2\gamma} c_1^2/24 + n^{-1} k(2k-1)) \left(\frac{c_1^{\beta+1}}{\beta+1} \right)^k \frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}}.
 \end{aligned}$$

Now (40) follows from (37) of the uniform boundedness of $\frac{C_{\beta,n-2k,k}}{C_{\beta,n} n^{k\beta}}$ and the fact that

$$\lim_{n \rightarrow +\infty} (kn^{1-\gamma} \beta c_1 + \beta kn^{-2\gamma} c_1^2/24 + n^{-1} k(2k-1)) = 0.$$

7. Proof of the upper bound (19). Now we consider (19). We will make use of several formulas, especially these on the generalized hypergeometric functions ${}_2F_1^{(\alpha)}$, and we refer to Chapter 13 of [9] for more details.

By definition, we can rewrite the two-component log-gas as

$$(46) \quad C_{\beta,n_1,2}(I) = \int_{I^2} dr_1 dr_2 |e^{ir_1} - e^{ir_2}|^{4\beta} I_{n_1,2}(\beta; r_1, r_2),$$

here

$$\begin{aligned}
 I_{n_1,2}(\beta; r_1, r_2) &:= \int_{(-\pi, \pi)^{n_1}} d\theta_1 \cdots d\theta_{n_1} \\
 &\quad \times \prod_{j=1}^{n_1} \prod_{k=1}^2 |1 - e^{i(\theta_j - r_k)}|^{2\beta} \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta.
 \end{aligned}$$

We will see that the uniform upper bound (19) is a direct consequence of the following Lemma 7.1, together with the integral expression (46) (with $n_1 = n - 4$) and Fatou’s lemma.

LEMMA 7.1. *There exists a constant C depending only on β such that*

$$I_{n-4,2}(\beta; r_1, r_2) |e^{ir_1} - e^{ir_2}|^{4\beta} \leq CC_{\beta,n} n^{2\beta}, \quad \forall n > 4, r_1, r_2 \in [-\pi, \pi],$$

and

$$\limsup_{n \rightarrow +\infty} C_{\beta,n}^{-1} n^{-2\beta} I_{n-4,2}(\beta; r_1, r_2) |e^{ir_1} - e^{ir_2}|^{4\beta} \leq (2\pi)^{-2} A_{\beta}^2.$$

We need to prove several estimates in order to prove Lemma 7.1. By Proposition 13.1.2 in [9], we have the following relation between the generalized hypergeometric function ${}_2F_1^{(\alpha)}$ and the Selberg type integrals:

$$\begin{aligned} & \frac{1}{M_n(a, b, 1/\alpha)} \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_n \prod_{l=1}^n \left(e^{\pi i \theta_l (a-b)} |1 + e^{2\pi i \theta_l}|^{a+b} \right. \\ & \left. \times \prod_{l'=1}^m (1 + t_{l'} e^{2\pi i \theta_l}) \right) \prod_{1 \leq j < k \leq n} |e^{2\pi i \theta_j} - e^{2\pi i \theta_k}|^{2/\alpha} \\ (47) \quad & = {}_2F_1^{(1/\alpha)}(-n, \alpha b; -(n-1) - \alpha(1+a); t_1, \dots, t_m) \\ & = \frac{{}_2F_1^{(1/\alpha)}(-n, \alpha b; \alpha(a+b+m); 1-t_1, \dots, 1-t_m)}{{}_2F_1^{(1/\alpha)}(-n, \alpha b; \alpha(a+b+m); (1)^m)}, \end{aligned}$$

here, $M_n(a, b, 1/\alpha)$ is defined as in (20) and we have used the following formula (Proposition 13.1.7 in [9]):

$$\begin{aligned} & {}_2F_1^{(\alpha)}(a, b; c; t_1, \dots, t_m) \\ & = \frac{{}_2F_1^{(\alpha)}(a, b; a+b+1+(m-1)/\alpha-c; 1-t_1, \dots, 1-t_m)}{{}_2F_1^{(\alpha)}(a, b; a+b+1+(m-1)/\alpha-c; (1)^m)}. \end{aligned}$$

By Proposition 13.1.4 in [9], we have

$$\begin{aligned} & \frac{1}{S_n(\lambda_1, \lambda_2, 1/\alpha)} \int_0^1 dx_1 \cdots \int_0^1 dx_n \prod_{l=1}^n x_l^{\lambda_1} (1-x_l)^{\lambda_2} (1-sx_l)^{-r} \\ (48) \quad & \times \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2/\alpha} \\ & = {}_2F_1^{(\alpha)}\left(r, \frac{1}{\alpha}(n-1) + \lambda_1 + 1; \frac{2}{\alpha}(n-1) + \lambda_1 + \lambda_2 + 2; (s)^n\right), \end{aligned}$$

here, by (4.1) and (4.3) in [9], the Selberg integral is

$$\begin{aligned} & S_n(\lambda_1, \lambda_2, \lambda) := \int_0^1 dt_1 \cdots \int_0^1 dt_n \prod_{l=1}^n t_l^{\lambda_1} (1-t_l)^{\lambda_2} \prod_{1 \leq j < k \leq n} |t_j - t_k|^{2\lambda} \\ (49) \quad & = \prod_{j=0}^{n-1} \frac{\Gamma(\lambda_1 + 1 + j\lambda) \Gamma(\lambda_2 + 1 + j\lambda) \Gamma(1 + (j+1)\lambda)}{\Gamma(\lambda_1 + \lambda_2 + 2 + (n+j-1)\lambda) \Gamma(1 + \lambda)}. \end{aligned}$$

Now we change variables $\theta_j \mapsto \theta_j + r_1 \pm \pi$ to obtain

$$I_{n_1,2}(\beta; r_1, r_2) = \int_{(-\pi,\pi)^{n_1}} d\theta_1 \cdots d\theta_{n_1} \prod_{j=1}^{n_1} (|1 + e^{i\theta_j}|^{2\beta} |1 + e^{i(\theta_j+r_1-r_2)}|^{2\beta}) \\ \times \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

For positive integer β , we have

$$|1 + e^{i(\theta_j+r_1-r_2)}|^{2\beta} = e^{-i\beta(\theta_j+r_1-r_2)} (1 + e^{i(\theta_j+r_1-r_2)})^{2\beta},$$

which shows

$$I_{n_1,2}(\beta; r_1, r_2) = e^{-i\beta n_1(r_1-r_2)} \int_{(-\pi,\pi)^{n_1}} d\theta_1 \cdots d\theta_{n_1} \\ \times \prod_{j=1}^{n_1} (e^{-i\beta\theta_j} |1 + e^{i\theta_j}|^{2\beta} (1 + e^{i(\theta_j+r_1-r_2)})^{2\beta}) \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

Comparing with (47) and changing variables $\theta_j \mapsto 2\pi\theta_j$, this integral is of the type therein with

$$n = n_1, \quad m = 2\beta, \quad a - b = -2\beta, \quad a + b = 2\beta, \quad 2/\alpha = \beta,$$

and

$$t_k = t := e^{i(r_1-r_2)} \quad \text{for } 1 \leq k \leq m.$$

Thus (47) implies that $I_{n_1,2}$ is proportional to

$$t^{-\beta n_1} {}_2F_1^{(\beta/2)}(-n_1, 4; 8; ((1-t))^{2\beta}),$$

and by (20), (47), ${}_2F_1^{(\beta/2)}$ equals to 1 at the origin, thus by considering the case of $t_k = t = 1$ ($1 \leq k \leq 2\beta$) for $r_1 = r_2$, we will have

$$(50) \quad I_{n_1,2}(\beta; r_1, r_2) = I_{n_1,2}(\beta; r_1, r_1) t^{-\beta n_1} {}_2F_1^{(\beta/2)}(-n_1, 4; 8; ((1-t))^{2\beta}),$$

where

$$(51) \quad I_{n_1,2}(\beta; r_1, r_1) = \int_{(-\pi,\pi)^{n_1}} d\theta_1 \cdots d\theta_{n_1} \prod_{j=1}^{n_1} |1 + e^{i\theta_j}|^{4\beta} \\ \times \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta \\ = (2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2).$$

Comparison with (48) shows that ${}_2F_1^{(\beta/2)}$ is of the type therein with

$$r = -n_1, \quad \alpha = \beta/2, \quad n = 2\beta, \\ \lambda_1 = \lambda_2 = 4 - \frac{1}{\alpha}(n-1) - 1 = \frac{2}{\beta} - 1, \quad s = 1 - t,$$

thus by (48), we have

$$\begin{aligned}
 & {}_2F_1^{(\beta/2)}(-n_1, 4; 8; ((1-t)^{2\beta}) \\
 &= \frac{1}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \\
 (52) \quad & \times \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \prod_{j=1}^{2\beta} u_j^{2/\beta-1} (1-u_j)^{2/\beta-1} (1-(1-t)u_j)^{n_1} \\
 & \times \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta}.
 \end{aligned}$$

Using (50), (51), (52), we have

$$\begin{aligned}
 & I_{n_1,2}(\beta; r_1, r_2) \\
 (53) \quad &= \frac{(2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2) t^{-\beta n_1}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \\
 & \times \prod_{j=1}^{2\beta} u_j^{2/\beta-1} (1-u_j)^{2/\beta-1} (1-(1-t)u_j)^{n_1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta}.
 \end{aligned}$$

Now we rewrite (53) as

$$I_{n_1,2}(\beta; r_1, r_2) = \frac{(2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2) t^{-\beta n_1}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} F_{n_1,\beta}(t),$$

here $t = e^{i(r_1-r_2)}$ and we denote

$$\begin{aligned}
 F_{n_1,\beta}(t) &:= \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \\
 & \times \prod_{j=1}^{2\beta} u_j^{2/\beta-1} (1-u_j)^{2/\beta-1} (1-(1-t)u_j)^{n_1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta},
 \end{aligned}$$

then $F_{n_1,\beta}$ is an analytic function (in fact a polynomial) of t . As $|1 - (1-t)u_j| = |1 - u_j + tu_j| \leq |1 - u_j| + |tu_j| = 1$ for $u_j \in [0, 1]$, $|t| = 1$, we have

$$\begin{aligned}
 |F_{n_1,\beta}(t)| &\leq \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \\
 & \times \prod_{j=1}^{2\beta} u_j^{2/\beta-1} (1-u_j)^{2/\beta-1} |1 - (1-t)u_j|^{n_1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta} \\
 &\leq \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \prod_{j=1}^{2\beta} u_j^{2/\beta-1} (1-u_j)^{2/\beta-1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta} \\
 &= S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta),
 \end{aligned}$$

which together with (22) implies

$$\begin{aligned}
 (54) \quad I_{n_1,2}(\beta; r_1, r_2) &= \frac{(2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2)}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} |F_{n_1,\beta}(t)| \\
 &\leq (2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2) = (2\pi)^{-1} C_{\beta,n_1,(4)}.
 \end{aligned}$$

Changing variables $u_j \mapsto t_j/(1+t_j)$, we obtain

$$\begin{aligned}
 F_{n_1,\beta}(t) &= \int_{(0,+\infty)^{2\beta}} \frac{dt_1 \cdots dt_{2\beta}}{(1+t_1)^2 \cdots (1+t_{2\beta})^2} \\
 &\quad \times \prod_{j=1}^{2\beta} \frac{t_j^{2/\beta-1}}{(1+t_j)^{2(2/\beta-1)}} \left(\frac{1+tt_j}{1+t_j}\right)^{n_1} \prod_{1 \leq j < k \leq 2\beta} \left| \frac{t_j - t_k}{(1+t_j)(1+t_k)} \right|^{4/\beta} \\
 &= \int_{(0,+\infty)^{2\beta}} dt_1 \cdots dt_{2\beta} \prod_{j=1}^{2\beta} \frac{t_j^{2/\beta-1} (1+tt_j)^{n_1}}{(1+t_j)^{2(2/\beta-1)+2+n_1+4/\beta \cdot (2\beta-1)}} \\
 &\quad \times \prod_{1 \leq j < k \leq 2\beta} |t_j - t_k|^{4/\beta}.
 \end{aligned}$$

Since $2(2/\beta - 1) + 2 + 4/\beta \cdot (2\beta - 1) = 4/\beta + 8 - 4/\beta = 8$, we have

$$\begin{aligned}
 F_{n_1,\beta}(-z^2) &= \int_{(0,+\infty)^{2\beta}} dt_1 \cdots dt_{2\beta} \prod_{j=1}^{2\beta} \frac{t_j^{2/\beta-1} (1-z^2t_j)^{n_1}}{(1+t_j)^{8+n_1}} \\
 &\quad \times \prod_{1 \leq j < k \leq 2\beta} |t_j - t_k|^{4/\beta}.
 \end{aligned}$$

For $z \in (0, +\infty)$, a simple changing of variables $zt_j \mapsto s_j$ shows that

$$\begin{aligned}
 F_{n_1,\beta}(-z^2) &= z^{-8\beta} \int_{(0,+\infty)^{2\beta}} ds_1 \cdots ds_{2\beta} \prod_{j=1}^{2\beta} \frac{s_j^{2/\beta-1} (1-zs_j)^{n_1}}{(1+z^{-1}s_j)^{8+n_1}} \\
 &\quad \times \prod_{1 \leq j < k \leq 2\beta} |s_j - s_k|^{4/\beta}.
 \end{aligned}$$

Since both sides are analytic functions of z for $\text{Re } z > 0$, this identity is always true for $\text{Re } z > 0$, moreover, we can decompose $(0, +\infty)$ into $(0, 1] \cup [1, +\infty)$ and use the symmetry of s_j to obtain

$$(55) \quad F_{n_1,\beta}(-z^2) = z^{-8\beta} \sum_{l=0}^{2\beta} \binom{2\beta}{l} F_{n_1,\beta,l}(z), \quad \text{Re } z > 0,$$

where

$$\begin{aligned}
 F_{n_1,\beta,l}(z) &:= \int_{(0,1]^l \times [1,+\infty)^{2\beta-l}} ds_1 \cdots ds_{2\beta} \prod_{j=1}^{2\beta} \frac{s_j^{2/\beta-1} (1-zs_j)^{n_1}}{(1+z^{-1}s_j)^{8+n_1}} \\
 &\quad \times \prod_{1 \leq j < k \leq 2\beta} |s_j - s_k|^{4/\beta}.
 \end{aligned}$$

The changing of variables $s_j \mapsto s_j^{-1}$ for $l < j \leq 2\beta$ shows that

$$\begin{aligned}
 F_{n_1,\beta,l}(z) &= \int_{(0,1]^{2\beta}} ds_1 \cdots ds_{2\beta} \prod_{j=1}^l \frac{s_j^{2/\beta-1} (1-zs_j)^{n_1}}{(1+z^{-1}s_j)^{8+n_1}} \\
 &\quad \times \prod_{j=l+1}^{2\beta} \frac{s_j^{-2/\beta+1} (1-zs_j^{-1})^{n_1}}{(1+z^{-1}s_j^{-1})^{8+n_1} s_j^2} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta} \prod_{l < j < k \leq 2\beta} |s_j^{-1} - s_k^{-1}|^{4/\beta}
 \end{aligned}$$

$$\begin{aligned} & \times \prod_{j=1}^l \prod_{k=l+1}^{2\beta} |s_j - s_k^{-1}|^{4/\beta} \\ &= \int_{(0,1]^{2\beta}} ds_1 \cdots ds_{2\beta} \prod_{j=1}^l \frac{s_j^{2/\beta-1} (1 - zs_j)^{n_1}}{(1 + z^{-1}s_j)^{8+n_1}} \prod_{j=l+1}^{2\beta} \frac{s_j^a (s_j - z)^{n_1}}{(s_j + z^{-1})^{8+n_1}} \\ & \times \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta} \prod_{l < j < k \leq 2\beta} |s_j - s_k|^{4/\beta} \prod_{j=l+1}^l \prod_{k=l+1}^{2\beta} |1 - s_j s_k|^{4/\beta}, \end{aligned}$$

here, $a = -2/\beta + 1 + 8 - 2 - 4/\beta \cdot (2\beta - 1) = 2/\beta - 1$. For $z = e^{i\theta}$, $\theta \in (-\pi/2, \pi/2)$ i.e., $\text{Re } z > 0$, and for $s > 0$, we have $|1 + z^{-1}s|^2 = |s + z^{-1}|^2 = 1 + s^2 + 2s \cos \theta > 1$ and $|1 - zs| = |s - z|$, therefore, we have

$$\begin{aligned} |F_{n_1, \beta, l}(e^{i\theta})| &\leq \int_{(0,1]^{2\beta}} ds_1 \cdots ds_{2\beta} \prod_{j=1}^{2\beta} \frac{s_j^{2/\beta-1} |1 - e^{i\theta} s_j|^{n_1}}{|1 + e^{-i\theta} s_j|^{n_1}} \\ (56) \quad &\times \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta} \prod_{l < j < k \leq 2\beta} |s_j - s_k|^{4/\beta} \\ &= F_{n_1, \beta, (l)}(\theta) F_{n_1, \beta, (2\beta-l)}(\theta), \end{aligned}$$

here, we used $|1 - s_j s_k| \leq 1$ and we denote

$$F_{n_1, \beta, (l)}(\theta) := \int_{(0,1]^l} ds_1 \cdots ds_l \prod_{j=1}^l \frac{s_j^{2/\beta-1} |1 - e^{i\theta} s_j|^{n_1}}{|1 + e^{-i\theta} s_j|^{n_1}} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta}.$$

As $\frac{1-s}{1+s} \leq e^{-2s}$ for $s \in (0, 1)$, we have

$$\frac{|1 - e^{i\theta} s|^{n_1}}{|1 + e^{-i\theta} s|^{n_1}} = \left| \frac{1 + s^2 - 2s \cos \theta}{1 + s^2 + 2s \cos \theta} \right|^{n_1/2} \leq e^{-\frac{2sn_1 \cos \theta}{1+s^2}},$$

which implies

$$\begin{aligned} F_{n_1, \beta, (l)}(\theta) &\leq \int_{(0,1]^l} ds_1 \cdots ds_l \prod_{j=1}^l s_j^{2/\beta-1} e^{-\frac{2s_j n_1 \cos \theta}{1+s_j^2}} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta} \\ &\leq \int_{(0,1]^l} ds_1 \cdots ds_l \prod_{j=1}^l s_j^{2/\beta-1} e^{-s_j n_1 \cos \theta} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta}. \end{aligned}$$

We denote

$$J_{n, \beta}(z) := \int_{(0, +\infty)^n} \prod_{j=1}^n t_j^{2/\beta-1} e^{-zt_j} \prod_{1 \leq j < k \leq n} |t_j - t_k|^{4/\beta} dt_1 \cdots dt_n,$$

then we have

$$J_{n, \beta}(z) = z^{-2n^2/\beta} J_{n, \beta}(1).$$

According to Proposition 4.7.3 in [9], we have the explicit evaluation

$$(57) \quad J_{n, \beta}(1) = \prod_{j=1}^n \frac{\Gamma(1 + 2j/\beta) \Gamma(2j/\beta)}{\Gamma(1 + 2/\beta)}.$$

By the definition of $J_{n,\beta}$, we have the upper bound

$$(58) \quad F_{n_1,\beta,(l)}(\theta) \leq J_{l,\beta}(n_1 \cos \theta) = (n_1 \cos \theta)^{-2l^2/\beta} J_{l,\beta}(1).$$

We change of variables $n_1 s_j \mapsto t_j$ to get

$$\begin{aligned} F_{n_1,\beta,(l)}(\theta) &\leq n_1^{-2l^2/\beta} \int_{(0,n_1)^l} dt_1 \cdots dt_l \\ &\quad \times \prod_{j=1}^l t_j^{2/\beta-1} e^{-\frac{2t_j \cos \theta}{1+t_j^2 n_1^{-2}}} \prod_{1 \leq j < k \leq l} |t_j - t_k|^{4/\beta}. \end{aligned}$$

By the dominated convergence theorem, we further have

$$\begin{aligned} \limsup_{n_1 \rightarrow +\infty} n_1^{2l^2/\beta} F_{n_1,\beta,(l)}(\theta) &\leq \int_{(0,+\infty)^l} dt_1 \cdots dt_l \prod_{j=1}^l t_j^{2/\beta-1} e^{-2t_j \cos \theta} \\ &\quad \times \prod_{1 \leq j < k \leq l} |t_j - t_k|^{4/\beta} \\ &= J_{l,\beta}(2 \cos \theta) = (2 \cos \theta)^{-2l^2/\beta} J_{l,\beta}(1). \end{aligned}$$

Therefore, we have

$$(59) \quad \limsup_{n_1 \rightarrow +\infty} (2n_1 \cos \theta)^{2l^2/\beta} F_{n_1,\beta,(l)}(\theta) \leq J_{l,\beta}(1).$$

7.1. *Proof of Lemma 7.1.* Now we are ready to prove Lemma 7.1.

PROOF. If $|e^{ir_1} - e^{ir_2}| \leq n^{-1}$, then by (54) with $n_1 = n - 4$ and Lemma 1.1, we have the first inequality,

$$I_{n-4,2}(\beta; r_1, r_2) |e^{ir_1} - e^{ir_2}|^{4\beta} \leq (2\pi)^{-1} C_{\beta,n-4,(4)} n^{-4\beta} \leq C C_{\beta,n} n^{2\beta}.$$

If $|e^{ir_1} - e^{ir_2}| \geq n^{-1}$, as $t = e^{i(r_1-r_2)}$, we have $|t - 1| = |e^{ir_1} - e^{ir_2}| \geq n^{-1}$ and we can write $t = -e^{2i\theta}$ for some $\theta \in (-\pi/2, \pi/2)$, then by (22) and (54), we have

$$\begin{aligned} I_{n-4,2}(\beta; r_1, r_2) |e^{ir_1} - e^{ir_2}|^{4\beta} &= \frac{(2\pi)^{n_1} M_{n-4}(2\beta, 2\beta, \beta/2)}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} |F_{n-4,\beta}(t)| |1 - t|^{4\beta} \\ &= \frac{(2\pi)^{-1} C_{\beta,n-4,(4)} |1 - t|^{4\beta}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} |F_{n-4,\beta}(t)|. \end{aligned}$$

By (55) and (56), we have

$$\begin{aligned} |F_{n-4,\beta}(t)| &\leq \sum_{l=0}^{2\beta} \binom{2\beta}{l} |F_{n-4,\beta,l}(e^{i\theta})| \\ &\leq \sum_{l=0}^{2\beta} \binom{2\beta}{l} F_{n-4,\beta,(l)}(\theta) F_{n-4,\beta,(2\beta-l)}(\theta), \end{aligned}$$

thus we have

$$I_{n-4,2}(\beta; r_1, r_2) |e^{ir_1} - e^{ir_2}|^{4\beta} \leq \sum_{l=0}^{2\beta} I_{n-4,2}^{(l)}(\beta; r_1, r_2),$$

where

$$I_{n-4,2}^{(l)}(\beta; r_1, r_2) = \frac{(2\pi)^{-1}C_{\beta,n-4,(4)}|1-t|^{4\beta}}{S_{2\beta}(2/\beta-1, 2/\beta-1, 2/\beta)} \binom{2\beta}{l} F_{n-4,\beta,(l)}(\theta)F_{n-4,\beta,(2\beta-l)}(\theta).$$

As $t = -e^{2i\theta}$, we know that $|1-t| = 2\cos\theta \geq n^{-1}$, by (58) and Lemma 1.1 we have

$$\begin{aligned} I_{n-4,2}^{(l)}(\beta; r_1, r_2) &\leq \frac{CC_{\beta,n}n^{6\beta}(2\cos\theta)^{4\beta}}{S_{2\beta}(2/\beta-1, 2/\beta-1, 2/\beta)} \binom{2\beta}{l} \\ &\quad \times (n_1\cos\theta)^{-2l^2/\beta} J_{l,\beta}(1)(n_1\cos\theta)^{-2(2\beta-l)^2/\beta} J_{2\beta-l,\beta}(1) \\ &\leq CC_{\beta,n}n^{2\beta}(2n\cos\theta)^{4\beta}(n_1\cos\theta)^{-2l^2/\beta-2(2\beta-l)^2/\beta} \\ &\leq CC_{\beta,n}n^{2\beta}(n_1\cos\theta)^{4\beta}(n_1\cos\theta)^{-4(\beta^2+(\beta-l)^2)/\beta} \\ &= CC_{\beta,n}n^{2\beta}(n_1\cos\theta)^{-4(\beta-l)^2/\beta} \leq CC_{\beta,n}n^{2\beta}, \end{aligned}$$

here $n_1 = n - 4$, $n_1 \cos\theta = n_1|1-t|/2 \geq n_1/(2n) \geq 1/10$, and C is a constant depending only on β, l . Summing up, we now conclude the first inequality.

Now we consider the second inequality regarding the limit superior. If $|e^{ir_1} - e^{ir_2}| = 0$, then the result is clearly true. If $|e^{ir_1} - e^{ir_2}| > 0$, then we can write $t = e^{i(r_1-r_2)} = -e^{2i\theta}$ for some $\theta \in (-\pi/2, \pi/2)$, and $|1-t| = 2\cos\theta$. Recall that

$$0 \leq I_{n-4,2}^{(l)}(\beta; r_1, r_2) \leq CC_{\beta,n}n^{2\beta}(n_1\cos\theta)^{-4(\beta-l)^2/\beta}, \quad n_1 = n - 4,$$

then for $l \neq \beta$, we have

$$\lim_{n \rightarrow +\infty} C_{\beta,n}^{-1}n^{-2\beta} I_{n-4,2}^{(l)}(\beta; r_1, r_2) = 0,$$

thus

$$\begin{aligned} (60) \quad &\limsup_{n \rightarrow +\infty} C_{\beta,n}^{-1}n^{-2\beta} I_{n-4,2}(\beta; r_1, r_2) |e^{ir_1} - e^{ir_2}|^{4\beta} \\ &\leq \limsup_{n \rightarrow +\infty} C_{\beta,n}^{-1}n^{-2\beta} I_{n-4,2}^{(\beta)}(\beta; r_1, r_2). \end{aligned}$$

Notice that

$$I_{n-4,2}^{(\beta)}(\beta; r_1, r_2) = \frac{(2\pi)^{-1}C_{\beta,n-4,(4)}|1-t|^{4\beta}}{S_{2\beta}(2/\beta-1, 2/\beta-1, 2/\beta)} \binom{2\beta}{\beta} |F_{n-4,\beta,(\beta)}(\theta)|^2,$$

we have

$$\begin{aligned} &C_{\beta,n}^{-1}n^{-2\beta} I_{n-4,2}^{(\beta)}(\beta; r_1, r_2) \\ &= \frac{(2\pi)^{-1}C_{\beta,n}^{-1}n^{-2\beta}C_{\beta,n-4,(4)}(2\cos\theta)^{4\beta}}{S_{2\beta}(2/\beta-1, 2/\beta-1, 2/\beta)} \binom{2\beta}{\beta} |F_{n-4,\beta,(\beta)}(\theta)|^2 \\ &= \frac{C_{\beta,n-4,(4)}}{C_{\beta,n}n^{6\beta}} \frac{(2\pi)^{-1}(2n\cos\theta)^{4\beta}}{S_{2\beta}(2/\beta-1, 2/\beta-1, 2/\beta)} \binom{2\beta}{\beta} |F_{n-4,\beta,(\beta)}(\theta)|^2. \end{aligned}$$

Therefore, by (59), Lemma 1.1 and Lemma 7.2 below, we have

$$\begin{aligned} &\limsup_{n \rightarrow +\infty} C_{\beta,n}^{-1}n^{-2\beta} I_{n-4,2}^{(\beta)}(\beta; r_1, r_2) \\ &= \lim_{n \rightarrow +\infty} \frac{C_{\beta,n-4,(4)}}{C_{\beta,n}n^{6\beta}} \binom{2\beta}{\beta} \end{aligned}$$

$$\begin{aligned} & \times \frac{(2\pi)^{-1}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \left| \limsup_{n \rightarrow +\infty} (2n \cos \theta)^{2\beta} F_{n-4, \beta, (\beta)}(\theta) \right|^2 \\ & \leq \frac{(2\pi)^{-1} A_{\beta,4}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \binom{2\beta}{\beta} |J_{\beta, \beta}(1)|^2 = (2\pi)^{-2} A_{\beta}^2. \end{aligned}$$

This, together with (60), will complete the proof of Lemma 7.1 provided Lemma 7.2. \square

Now we prove the following identity to complete Lemma 7.1.

LEMMA 7.2. *It holds that*

$$(2\pi) A_{\beta,4} \binom{2\beta}{\beta} \frac{|J_{\beta, \beta}(1)|^2}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} = A_{\beta}^2.$$

PROOF. Notice that the Selberg integral

$$\begin{aligned} S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta) &= \prod_{j=0}^{2\beta-1} \frac{(\Gamma(2(j+1)/\beta))^2 \Gamma(1+2(j+1)/\beta)}{\Gamma(2(2\beta+j+1)/\beta) \Gamma(1+2/\beta)} \\ &= \prod_{j=1}^{2\beta} \frac{(\Gamma(2j/\beta))^2 \Gamma(1+2j/\beta)}{\Gamma(2j/\beta+4) \Gamma(1+2/\beta)} \\ &= \prod_{j=1}^{2\beta} \frac{(\Gamma(2j/\beta))^2}{\prod_{k=1}^3 (2j/\beta+k) \Gamma(1+2/\beta)}, \end{aligned}$$

that

$$\prod_{j=1}^{2\beta} \prod_{k=1}^3 (2j/\beta+k) = (2/\beta)^{6\beta} \prod_{k=1}^3 \prod_{j=1}^{2\beta} (j+k\beta/2) = (2/\beta)^{6\beta} \prod_{k=1}^3 \frac{\Gamma(1+(k+4)\beta/2)}{\Gamma(1+k\beta/2)},$$

and that (using (57))

$$\begin{aligned} \prod_{j=1}^{2\beta} \frac{(\Gamma(2j/\beta))^2}{\Gamma(1+2/\beta)} &= \prod_{j=1}^{\beta} \prod_{k=0}^1 \frac{(\Gamma(2(j+k\beta)/\beta))^2}{\Gamma(1+2/\beta)} \\ &= \prod_{j=1}^{\beta} \frac{(\Gamma(2j/\beta) \Gamma(2j/\beta+2))^2}{(\Gamma(1+2/\beta))^2} \\ &= |J_{\beta, \beta}(1)|^2 \prod_{j=1}^{\beta} (2j/\beta+1)^2 \\ &= |J_{\beta, \beta}(1)|^2 (2/\beta)^{2\beta} \frac{(\Gamma(1+3\beta/2))^2}{(\Gamma(1+\beta/2))^2}, \end{aligned}$$

we have

$$\begin{aligned} & (2\pi) A_{\beta,4} \binom{2\beta}{\beta} \frac{|J_{\beta, \beta}(1)|^2}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \\ &= (2\pi) A_{\beta,4} \frac{\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2} (2/\beta)^{4\beta} \frac{(\Gamma(1+\beta/2))^2}{(\Gamma(1+3\beta/2))^2} \prod_{k=1}^3 \frac{\Gamma(1+(k+4)\beta/2)}{\Gamma(1+k\beta/2)}, \end{aligned}$$

as in Lemma 1.1,

$$A_{\beta,4} = \frac{(2\pi)^{-3}(\Gamma(\beta/2 + 1))^4}{\Gamma(2\beta + 1)}(\beta/2)^{6\beta} \prod_{j=1}^3 \frac{\Gamma(j\beta/2 + 1)}{\Gamma((4 + j)\beta/2 + 1)},$$

we have

$$\begin{aligned} (2\pi)A_{\beta,4} &\left(\frac{2\beta}{\beta}\right) \frac{|J_{\beta,\beta}(1)|^2}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \\ &= \frac{(2\pi)^{-2}(\Gamma(\beta/2 + 1))^4}{\Gamma(2\beta + 1)}(\beta/2)^{2\beta} \frac{\Gamma(1 + 2\beta)}{(\Gamma(1 + \beta))^2} \frac{(\Gamma(1 + \beta/2))^2}{(\Gamma(1 + 3\beta/2))^2} \\ &= \frac{(2\pi)^{-2}(\Gamma(\beta/2 + 1))^6}{(\Gamma(1 + \beta))^2(\Gamma(1 + 3\beta/2))^2}(\beta/2)^{2\beta} = A_{\beta}^2, \end{aligned}$$

this completes the proof. \square

8. Proof of Lemma 1.4. Now we give the proof of Lemma 1.4 which will complete the proof Theorem 1.1.

PROOF. As $C_{\beta,n-2,1}(I) = |I|C_{\beta,n-2,1}/(2\pi)$ (recall (21)), by Lemma 1.1, we have

$$(61) \quad \lim_{n \rightarrow +\infty} \frac{C_{\beta,n-2,1}(I)}{C_{\beta,n}n^{\beta}} = \frac{|I|}{2\pi} \lim_{n \rightarrow +\infty} \frac{C_{\beta,n-2,1}}{C_{\beta,n}n^{\beta}} = \frac{|I|A_{\beta}}{2\pi},$$

that is, Lemma 1.4 is true for $k = 1$. Now we assume $|I| > 0$, then for every $\lambda > 0$, we can find $A = (0, a(\lambda))$ such that

$$\lambda = \int_A u^{\beta} du \times \frac{|I|A_{\beta}}{2\pi}.$$

We denote

$$X_n := \tilde{\chi}^{(n)}(A \times I),$$

then by Lemma 1.3 with $k = 1$ and (61), we have

$$\lim_{n \rightarrow +\infty} \mathbb{E}X_n = \lim_{n \rightarrow +\infty} \left(\int_A u^{\beta} du \right) \frac{C_{\beta,n-2,1}(I)}{C_{\beta,n}n^{\beta}} = \lambda;$$

and with $k = 2$ in Lemma 1.3, we have

$$\liminf_{n \rightarrow +\infty} \mathbb{E}(X_n(X_n - 1)) = \liminf_{n \rightarrow +\infty} \left(\int_A u^{\beta} du \right)^2 \frac{C_{\beta,n-4,2}(I)}{C_{\beta,n}n^{2\beta}}.$$

On the other hand, by Hölder inequality, we have $\mathbb{E}(X_n)^2 \geq (\mathbb{E}X_n)^2$ and $\mathbb{E}(X_n(X_n - 1)) \geq (\mathbb{E}X_n)^2 - (\mathbb{E}X_n)$, and thus we have

$$\liminf_{n \rightarrow +\infty} \mathbb{E}(X_n(X_n - 1)) \geq \liminf_{n \rightarrow +\infty} ((\mathbb{E}X_n)^2 - (\mathbb{E}X_n)) = \lambda^2 - \lambda.$$

Therefore, we have

$$\liminf_{n \rightarrow +\infty} \frac{C_{\beta,n-4,2}(I)}{C_{\beta,n}n^{2\beta}} \geq \left(\int_A u^{\beta} du \right)^{-2} (\lambda^2 - \lambda) = (1 - \lambda^{-1}) \left(\frac{|I|A_{\beta}}{2\pi} \right)^2.$$

Letting $\lambda \rightarrow +\infty$, we have

$$\liminf_{n \rightarrow +\infty} \frac{C_{\beta,n-4,2}(I)}{C_{\beta,n}n^{2\beta}} \geq \left(\frac{|I|A_{\beta}}{2\pi} \right)^2,$$

which along with (19) gives Lemma 1.4 for $k = 2$.

Moreover, since

$$\mathbb{E}(X_n - \lambda)^2 = \mathbb{E}(X_n(X_n - 1)) - (2\lambda - 1)(\mathbb{E}X_n) + \lambda^2,$$

by Lemmas 1.3 and (19), we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{E}(X_n(X_n - 1)) &= \limsup_{n \rightarrow +\infty} \left(\int_A u^\beta du \right)^2 \frac{C_{\beta,n-4,2}(I)}{C_{\beta,n}n^{2\beta}} \\ &\leq \left(\int_A u^\beta du \right)^2 \left(\frac{|I|A^\beta}{2\pi} \right)^2 = \lambda^2, \end{aligned}$$

and thus we have

$$\limsup_{n \rightarrow +\infty} \mathbb{E}(X_n - \lambda)^2 \leq \lambda^2 - (2\lambda - 1)\lambda + \lambda^2 = \lambda.$$

Now we denote by C a constant independent of n, λ , which may be different from line to line. As $X_n^k \leq \frac{2X_n!}{(X_n-k)!} + C$ ($-C$ can be chosen as the lower bound of the polynomial $2x(x-1)\cdots(x-k+1) - x^k$ for $x \geq 0$), by Lemma 1.3 and Lemma 6.1, we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{E}(X_n^k) &\leq 2 \limsup_{n \rightarrow +\infty} \mathbb{E}\left(\frac{X_n!}{(X_n - k)!}\right) + C \\ &\leq 2 \limsup_{n \rightarrow +\infty} \left(\int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}n^{k\beta}} + C \\ &\leq 2 \left(\int_A u^\beta du \right)^k \limsup_{n \rightarrow +\infty} \frac{C_{\beta,n-2k,k}}{C_{\beta,n}n^{k\beta}} + C \\ &\leq C \left(\int_A u^\beta du \right)^k + C \leq C\lambda^k + C. \end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned} \mathbb{E}\left(\frac{(X_n - \lambda)^2 X_n!}{(X_n - k + 1)!}\right) &\leq \mathbb{E}((X_n - \lambda)^2 X_n^{k-1}) \\ &\leq (\mathbb{E}(X_n - \lambda)^2)^{\frac{1}{2}} (\mathbb{E}((X_n - \lambda)^2 X_n^{2k-2}))^{\frac{1}{2}} \\ &\leq (\mathbb{E}(X_n - \lambda)^2)^{\frac{1}{2}} (\mathbb{E}(X_n^{2k} + \lambda^2 X_n^{2k-2}))^{\frac{1}{2}}, \end{aligned}$$

and thus for any positive integer k , we have

$$\begin{aligned} (62) \quad &\limsup_{n \rightarrow +\infty} \mathbb{E}\left(\frac{(X_n - \lambda)^2 X_n!}{(X_n - k + 1)!}\right) \\ &\leq \left(\limsup_{n \rightarrow +\infty} \mathbb{E}(X_n - \lambda)^2 \right)^{\frac{1}{2}} \left(\limsup_{n \rightarrow +\infty} \mathbb{E}(X_n^{2k} + \lambda^2 X_n^{2k-2}) \right)^{\frac{1}{2}} \\ &\leq \lambda^{\frac{1}{2}} ((C\lambda^{2k} + C) + \lambda^2(C\lambda^{2k-2} + C))^{\frac{1}{2}} \leq C\lambda^{\frac{1}{2}}(\lambda^k + 1). \end{aligned}$$

Now we can prove the lemma by induction. Assume $j \geq 2$ and Lemma 1.4 is true for $k = j - 1, j$, then by Lemma 1.3, we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{E}\left(\frac{X_n!}{(X_n - k)!}\right) &= \lim_{n \rightarrow +\infty} \left(\int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}n^{k\beta}} \\ &= \left(\int_A u^\beta du \right)^k \left(\frac{|I|A^\beta}{2\pi} \right)^k = \lambda^k, \quad k = j - 1, j. \end{aligned}$$

We note that $(X_n - \lambda)^2 = (X_n - k)(X_n - k - 1) - (2\lambda - 2k - 1)(X_n - k) + (\lambda - k)^2$, then for any integer $k \geq 2$, we have the identity

$$(63) \quad \frac{(X_n - \lambda)^2 X_n!}{(X_n - k)!} = \frac{X_n!}{(X_n - k - 2)!} - \frac{(2\lambda - 2k - 1)X_n!}{(X_n - k - 1)!} + \frac{(\lambda - k)^2 X_n!}{(X_n - k)!}.$$

Now by induction, (62), (63) and Lemma 1.3, we have

$$\begin{aligned} C\lambda^{\frac{1}{2}}(\lambda^j + 1) &\geq \limsup_{n \rightarrow +\infty} \mathbb{E} \left(\frac{(X_n - \lambda)^2 X_n!}{(X_n - j + 1)!} \right) \\ &= \limsup_{n \rightarrow +\infty} \mathbb{E} \left(\frac{X_n!}{(X_n - j - 1)!} - \frac{(2\lambda - 2j + 1)X_n!}{(X_n - j)!} + \frac{(\lambda - j + 1)^2 X_n!}{(X_n - j + 1)!} \right) \\ &= \limsup_{n \rightarrow +\infty} \mathbb{E} \left(\frac{X_n!}{(X_n - j - 1)!} \right) - (2\lambda - 2j + 1)\lambda^j + (\lambda - j + 1)^2 \lambda^{j-1} \\ &= \limsup_{n \rightarrow +\infty} \left(\int_A u^\beta du \right)^k \frac{C_{\beta, n-2k, k}(I)}{C_{\beta, n} n^{k\beta}} - (\lambda^2 - (j - 1)^2 - \lambda)\lambda^{j-1}, \end{aligned}$$

where we denote $k = j + 1$ in the last line. Therefore, as λ large enough, we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{C_{\beta, n-2k, k}(I)}{C_{\beta, n} n^{k\beta}} &\leq \left(\int_A u^\beta du \right)^{-k} (\lambda^{j+1} + C\lambda^{\frac{1}{2}}(\lambda^j + 1)) \\ &= \left(\frac{|I|A_\beta}{2\pi} \right)^k (1 + C\lambda^{-\frac{1}{2}} + C\lambda^{-j-\frac{1}{2}}). \end{aligned}$$

Letting $\lambda \rightarrow +\infty$, we have

$$\limsup_{n \rightarrow +\infty} \frac{C_{\beta, n-2k, k}(I)}{C_{\beta, n} n^{k\beta}} \leq \left(\frac{|I|A_\beta}{2\pi} \right)^k.$$

Similarly, as $\frac{(X_n - \lambda)^2 X_n!}{(X_n - j + 1)!} \geq 0$, by induction and Lemma 1.3 again, we have

$$\begin{aligned} 0 &\leq \liminf_{n \rightarrow +\infty} \mathbb{E} \left(\frac{X_n!}{(X_n - j - 1)!} - \frac{(2\lambda - 2j + 1)X_n!}{(X_n - j)!} + \frac{(\lambda - j + 1)^2 X_n!}{(X_n - j + 1)!} \right) \\ &= \liminf_{n \rightarrow +\infty} \left(\int_A u^\beta du \right)^k \frac{C_{\beta, n-2k, k}(I)}{C_{\beta, n} n^{k\beta}} - (\lambda^2 - (j - 1)^2 - \lambda)\lambda^{j-1}, \end{aligned}$$

where $k = j + 1$ again. Therefore, we have

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{C_{\beta, n-2k, k}(I)}{C_{\beta, n} n^{k\beta}} &\geq \left(\int_A u^\beta du \right)^{-k} (\lambda^2 - (j - 1)^2 - \lambda)\lambda^{j-1} \\ &= \left(\frac{|I|A_\beta}{2\pi} \right)^k (1 - \lambda^{-1} - (j - 1)^2 \lambda^{-2}). \end{aligned}$$

Letting $\lambda \rightarrow +\infty$ again, we have

$$\liminf_{n \rightarrow +\infty} \frac{C_{\beta, n-2k, k}(I)}{C_{\beta, n} n^{k\beta}} \geq \left(\frac{|I|A_\beta}{2\pi} \right)^k,$$

thus Lemma 1.4 is also true for $k = j + 1$. This completes the proof of Lemma 1.4, and thus we complete the proof Theorem 1.1. \square

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