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Exact rate of convergence of the expected W_2 distance between the empirical and true Gaussian distribution

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Abstract

We study the Wasserstein distance W_2 for Gaussian samples. We establish the exact rate of convergence $\sqrt{\log\log n/n}$ of the expected value of the W_2 distance between the empirical and true $c.d.f.$'s for the normal distribution. We also show that the rate of weak convergence is unexpectedly $1/\sqrt{n}$ in the case of two correlated Gaussian samples.

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1 Introduction

In this article we investigate in details the asymptotic behaviour of the quadratic Wasserstein distance between the empirical cumulative distribution function ($c.d.f.$) of a sample $X_1, \ldots, X_n$ of independent standard Gaussian random variables denoted by $\mathbb{F}_n$ and the standard normal $c.d.f.$ denoted by Φ. Thus we consider the random variable

$$W_2^2(\mathbb{F}_n, \Phi) = \int_0^1 |\mathbb{F}_n^{-1}(u) - \Phi^{-1}(u)|^2 du.$$

More precisely we are interested in the exact rate of convergence of $\mathbb{E}\left(W_2^2(\mathbb{F}_n, \Phi)\right)$. Define $h(u) = \Phi' \circ \Phi^{-1}(u)$ for $u \in (0,1)$. First note that Corollary 19 in [1] does not apply in this specific case where $b = 2$, and indeed we almost surely have $\lim_{n\to+\infty} nW_2^2(\mathbb{F}_n, \Phi) = +\infty$. Secondly, to our knowledge the most precise result about the behaviour of $W_2(\mathbb{F}_n, \Phi)$ is given by Theorem 4.6 (ii) in [9] which implies, as $n \to +\infty$, the convergence in distribution

$$nW_2^2(\mathbb{F}_n, \Phi) - \int_{1/n}^{1-1/n} \frac{u(1-u)}{h^2(u)} du \to \int_0^1 \frac{\mathbb{B}^2(u) - \mathbb{E}\left(\mathbb{B}^2(u)\right)}{h^2(u)} du, \tag{1.1}$$

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where $\mathbb{B}$ is a standard Brownian bridge. This is not enough to control $n\mathbb{E}(W_2^2(\mathbb{F}_n,\Phi))$ since the deterministic centering integral is diverging. In [4] specific bounds on $n\mathbb{E}(W_p^p(\mathbb{F}_n,F))$ are given for $\log$-concave distribution F. In the standard Gaussian case Corollary 6.14 of [4] reads

$$c\frac{\log\log n}{n} \leqslant \mathbb{E}\left(W_2^2(\mathbb{F}_n,\Phi)\right) \leqslant C\frac{\log\log n}{n} \tag{1.2}$$

where $0<c<C<+\infty$. The main achievement below is to compute the exact asymptotic constant in (1.2). As far as we know this is the first result of this kind.

In the spirit of [1] we moreover extend the investigations in the one sample case to the two correlated samples case. More precisely, we study the random quantity $W_2^2(\mathbb{F}_n,\mathbb{G}_n)$ where $\mathbb{F}_n,\mathbb{G}_n$ are the marginal empirical $c.d.f.$ obtained from a n-sample $(X_i,Y_i)_{1\leqslant i\leqslant n}$ of standard Gaussian couples with correlation ρ. If the Gaussian marginals Φ_X and Φ_Y were not identical the general Theorem 14 in [2] would imply the convergence in distribution

$$\sqrt{n}\big(W_2^2(\mathbb{F}_n,\mathbb{G}_n)-W_2^2(\Phi_X,\Phi_Y)\big)\to\mathcal{N}\big(0,\sigma^2(\Sigma)\big) \tag{1.3}$$

where Σ is the covariance matrix of (X_1,Y_1) and $\sigma^2(\Sigma)$ has a closed form expression that explicitly depends on Σ. In particular, Corollary 18 of [2] shows that for two independent samples from two distinct Gaussian distributions $\mathcal{N}(\nu,\zeta^2)$ and $\mathcal{N}(\mu,\xi^2)$ it holds $\sigma^2(\Sigma)=4(\zeta^2+\xi^2)(\nu-\mu)^2+2(\zeta^2+\xi^2)(\zeta-\xi)^2$.

Surprisingly, the second result below establishes that whenever the marginals are the same, $\Phi_X=\Phi_Y=\Phi$, and the samples are not independent, that is $\rho\neq 0$, the rate of weak convergence of $W_2^2(\mathbb{F}_n,\mathbb{G}_n)$ is $1/n$ and the limiting distribution is a slight variation of the one given at Theorem 11 in [1], even if the sufficient condition of the latter result is not satisfied.

2 The results

First we provide the limiting constant in (1.2).

Theorem 2.1. *Let $\mathbb{F}_n$ be the empirical $c.d.f.$ of an $i.i.d.$ standard normal sample of size n and Φ the $c.d.f.$ of the standard normal distribution. Then it holds*

$$\lim_{n\to+\infty}\frac{n}{\log\log n}\mathbb{E}\left(W_2^2(\mathbb{F}_n,\Phi)\right)=1,$$
$$\lim_{n\to+\infty}\sqrt{\frac{n}{\log\log n}}\mathbb{E}\left(W_2(\mathbb{F}_n,\Phi)\right)=1.$$

Remark 2.2. This result is consistent with (1.1) and the fact that, by [3], we have

$$\int_{1/n}^{1-1/n}\frac{u(1-u)}{h^2(u)}du=\log\log n+\log 2+\gamma_0+o(1)$$

which implies that $\frac{n}{\log\log n}W_2^2(\mathbb{F}_n,\Phi)\to 1$ in probability.

Remark 2.3. In the case of a sample of unstandardized normal random variables with variance σ^2 the expected W_2-distance between the empirical and the true distribution has the same rate as above and limiting constants σ^2 and σ, respectively.

Remark 2.4. If $\mathbb{G}_n$ is a second empirical $c.d.f.$ independent of $\mathbb{F}_n$ and build from another sample we see that $\mathbb{E}\left(W_2^2(\mathbb{F}_n,\mathbb{G}_n)\right)=\mathbb{E}\left(W_2^2(\mathbb{F}_n,\Phi)\right)+\mathbb{E}\left(W_2^2(\mathbb{G}_n,\Phi)\right)$ since $\mathbb{E}(\int_0^1(\mathbb{F}_n^{-1}(u)-\Phi^{-1}(u))du)=0$. Therefore, in this independent case we have

$$\lim_{n\to+\infty}\frac{n}{\log\log n}\mathbb{E}\left(W_2^2(\mathbb{F}_n,\mathbb{G}_n)\right)=2$$

which is in contrast with the forthcoming dependent sample case.

Second, in the setting of [2] and [1] we also get the rate of weak convergence in the two correlated samples case.

Theorem 2.5. *Let $\mathbb{F}_n$ and $\mathbb{G}_n$ denote the marginal empirical $c.d.f.$ of a size n $i.i.d.$ sample of correlated bivariate standard normal with covariance ρ, $0 < |\rho| < 1$. Let*

$$C_\rho(u,v) = \mathbb{P}(X \leqslant \Phi^{-1}(u), Y \leqslant \Phi^{-1}(v)), \quad u,v \in (0,1),$$
$$\mathcal{G}(u) = \frac{\mathbb{B}^X(u)}{h(u)} - \frac{\mathbb{B}^Y(u)}{h(u)}, \quad u \in (0,1),$$

where $(\mathbb{B}^X, \mathbb{B}^Y)$ are two standard Brownian bridges with cross covariance

$$\mathrm{Cov}(\mathbb{B}^X(u), \mathbb{B}^Y(v)) = C_\rho(u,v) - uv, \ u,v \in (0,1)\,.$$

Then we have the convergence in distribution

$$nW_2^2(\mathbb{F}_n, \mathbb{G}_n) \to ||\mathcal{G}||_2^2 = \int_0^1 \mathcal{G}(u)^2 du$$

and the limiting random variable is almost surely finite with finite expectation.

Remark 2.6. By Theorem 2.5 it holds $\sqrt{n}W_2(\mathbb{F}_n, \mathbb{G}_n) \to ||\mathcal{G}||_2$ with a CLT rate and a non degenerate limiting distribution with finite variance. This was not expected since in the case of two independent samples, that is $\rho = 0$, it holds

$$\mathbb{E}(||\mathcal{G}||_2^2) = \int_0^1 \mathbb{E}(\mathcal{G}(u)^2) du = 2\int_0^1 \frac{u(1-u)}{h^2(u)} du = +\infty$$

which proves by Theorem 1.3 of [8] that $\mathbb{P}(||\mathcal{G}||_2 = +\infty) = 1$, and is consistent with the similar case where $\mathbb{G}_n$ is replaced with Φ as shown by Theorem 2.1.

Remark 2.7. Theorem 2.5 is an extension of Theorem 11 in [1] for Gaussian correlated samples that proves that the dependency between two $i.i.d.$ samples expressed through the joint law may influence the rate of convergence of $W_2^2(\mathbb{F}_n, \mathbb{G}_n)$ if the marginal distributions are the same. In the general CLT formulated at Theorem 14 of [2], only the limiting finite variance of $\sqrt{n}(W_2^2(\mathbb{F}_n, \mathbb{G}_n) - W_2^2(\Phi_X, \Phi_Y))$ was affected by the joint law if the marginal distributions are different, not the rate $1/\sqrt{n}$ as recalled at (1.3) above.

3 Proofs

3.1 Preliminaries

Note that the density quantile function $h(u) = \Phi' \circ \Phi^{-1}(u)$ is symmetric on $(0,1)$ about $u = 1/2$. Straightforward computations yield, as $x \to +\infty$,

$$\psi(x) = -\log(1 - \Phi(x)) = \frac{x^2}{2} + \log x + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{x^2}\right),$$
$$\psi^{-1}(x) = \sqrt{2\left(x - \frac{1}{2}\log x - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log 2 + O\left(\frac{\log x}{x}\right)\right)}.$$

As a consequence, we have, as $u \to 1$,

$$\Phi^{-1}(u) = \psi^{-1}\left(\log\left(\frac{1}{1-u}\right)\right)$$
$$= \sqrt{2\left(\log\left(\frac{1}{1-u}\right) - \frac{1}{2}\log\log\left(\frac{1}{1-u}\right) - \frac{1}{2}\log(4\pi) + O\left(\frac{\log\log(1/(1-u))}{\log(1/(1-u))}\right)\right)}, \quad (3.1)$$

and

$$h(u) = \Phi' \circ \Phi^{-1}(u) = \sqrt{2}(1-u)\sqrt{\log\left(\frac{1}{1-u}\right)}\left(1 + O\left(\frac{\log\log\left(1/(1-u)\right)}{\log\left(1/(1-u)\right)}\right)\right). \quad (3.2)$$

Let us extend the results concerning the first and second moments of the extreme order statistics of a Gaussian sample stated at page 376 in [6].

Lemma 3.1. *Let $Z_1 \leqslant \cdots \leqslant Z_n$ denote the order statistics of $X_1, ..., X_n$. Let $1 \leqslant \theta \leqslant 2$ and $C > 0$. For any $k \leqslant C(\log n)^\theta$ it holds*

$$\mathbb{E}\left(Z_{n-k}\right) = \sqrt{2\log n} - \frac{\log\log n + 2(s^1_{k+1} - \gamma_0) + \log(4\pi)}{\sqrt{8\log n}} + O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right),$$
$$\mathbb{V}\left(Z_{n-k}\right) = \frac{\pi^2/6 - s^2_{k+1}}{2\log n} + O\left(\frac{1}{(\log n)^2}\right),$$

where, for $k > 0$, $s^1_k = \sum_{j=1}^k 1/j$, $s^2_k = \sum_{j=1}^k 1/j^2$ and γ_0 is the Euler constant.

Proof of Lemma 3.1. Following [6], let $\xi_{n-k+1} = n(1 - \Phi(Z_{n-k+1}))$ for $k \geqslant 1$. Since the random variables $\xi_1/n < ... < \xi_n/n$ are the order statistics of n independent uniform random variables, we see that ξ_{n-k+1} has density

$$f_{\xi_{n-k+1}}(x) = \binom{n-1}{k-1}\left(\frac{x}{n}\right)^{k-1}\left(1 - \frac{x}{n}\right)^{n-k} 1_{[0,n]}(x).$$

Step 1. Write $\Gamma(k) = (k-1)!$ and observe that

$$\binom{n-1}{k-1}\left(\frac{1}{n}\right)^{k-1} = \exp\left(\sum_{j=1}^{k-1}\log\left(1 - \frac{j}{n}\right)\right)\frac{1}{\Gamma(k)} = \left(1 + O\left(\frac{(\log n)^{3\theta}}{n}\right)\right)\frac{1}{\Gamma(k)}$$

since we have

$$-\sum_{j=1}^k \frac{j}{n} - \sum_{j=1}^k \left(\frac{j}{n}\right)^2 \leqslant \sum_{j=1}^k \log\left(1 - \frac{j}{n}\right) \leqslant -\sum_{j=1}^k \frac{j}{n}$$

$$\max_{1\leqslant k\leqslant C(\log n)^\theta}\left|\sum_{j=1}^k \log\left(1 - \frac{j}{n}\right) + \sum_{j=1}^k \frac{j}{n}\right| \leqslant \frac{1}{n}\sum_{j=1}^{\left[C(\log n)^\theta\right]} \frac{j^2}{n} = O\left(\frac{(\log n)^{3\theta}}{n}\right).$$

Step 2. For $k \geqslant 1$ we have

$$\begin{aligned}
\mathbb{E}\left(Z_{n-k+1}\right) &= \mathbb{E}\left(\Phi^{-1}\left(1 - \frac{\xi_{n-k+1}}{n}\right)\right) \\
&= \frac{(n-1)...(n-k+1)}{\Gamma(k)}\int_0^n \left(\frac{x}{n}\right)^{k-1}\left(1 - \frac{x}{n}\right)^{n-k}\Phi^{-1}\left(1 - \frac{x}{n}\right)dx \\
&= \left(1 - \frac{1}{n}\right)...\left(1 - \frac{k-1}{n}\right)\int_0^n \frac{x^{k-1}}{\Gamma(k)}\left(1 - \frac{x}{n}\right)^{n-k}\Phi^{-1}\left(1 - \frac{x}{n}\right)dx \\
&= \exp\left(-\frac{s^1_k}{n} - \frac{s^2_k}{2n^2}(1 + o(1))\right)(E_{1,n} + E_{2,n})
\end{aligned}$$

where, for $p > \theta + 1$, $x(n) = (\log n)^p$ and $f_{\Gamma(k)}(x) = \dfrac{x^{k-1}}{\Gamma(k)}e^{-x}$ for $x > 0$,

$$E_{1,n} = (1 + o(1))\int_0^{x(n)} \Phi^{-1}\left(1 - \frac{x}{n}\right) f_{\Gamma(k)}(x)dx,$$
$$E_{2,n} = \int_{x(n)}^n \Phi^{-1}\left(1 - \frac{x}{n}\right)\frac{x^{k-1}}{\Gamma(k)}\left(1 - \frac{x}{n}\right)^{n-k}dx.$$

Assume that $k \leqslant C(\log n)^\theta$. By (3.1) it holds, for some $K > 0$ and all n large enough,

$$\begin{aligned}
|E_{2,n}| &\leqslant \int_{x(n)}^{n} x^{k-1} \left(1-\frac{x}{n}\right)^{n-k} \left|\Phi^{-1}\left(1-\frac{x}{n}\right)\right| dx \\
&\leqslant K\sqrt{\log n} \int_{x(n)}^{n/2} \exp\left(-(n-k)\frac{x}{n} + (k-1)\log x\right) dx \\
&+ K\int_{n/2}^{n} x^{k-1}\left(1-\frac{x}{n}\right)^{n-k} \sqrt{\log\left(\frac{1}{1-x/n}\right)} dx \\
&\leqslant K\exp\left(-x(n) + \frac{C(\log n)^\theta + \log\log n}{2} + C(\log n)^{\theta+1}\right) \\
&+ K\int_{n/2}^{n} x^{k-1}\left(1-\frac{x}{n}\right)^{n-k-1} dx \\
&\leqslant K\exp\left(-\frac{(\log n)^p}{2}\right) + Kn^k\left(\frac{1}{2}\right)^{n-k-1} \\
&\leqslant K\exp\left(-(1+o(1))(\log n)^p\right).
\end{aligned}$$

Now turn to

$$\int_0^{x(n)} \Phi^{-1}\left(1-\frac{x}{n}\right)\binom{n-1}{k-1}\left(\frac{x}{n}\right)^{k-1}\left(1-\frac{x}{n}\right)^{n-k} dx$$

where, for $0 < x < x(n)$, we have, by (3.1),

$$\begin{aligned}
\Phi^{-1}(1-x/n) &= \sqrt{2\left(\log(n/x) - \frac{1}{2}\log\log(n/x) - \frac{1}{2}\log(4\pi) - O\left(\frac{\log\log(n/x(n))}{\log(n/x(n))}\right)\right)} \\
&= \sqrt{2\log n} - \frac{2\log x + \log\log(n/x) + \log(4\pi)}{2\sqrt{2\log n}} + O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right) \qquad (3.3)
\end{aligned}$$

which is integrable near 0 with respect to the above density since

$$0 < \log(\log n - \log x) = \log\log n + \log\left(1 - \frac{\log x}{\log n}\right) \leqslant \log\log n + \left|\frac{\log x}{\log n}\right| \qquad (3.4)$$

and $\log x, (\log x)^2$ are integrable with respect to any Gamma distribution. Hence

$$\begin{aligned}
&E_{1,n} \\
&= \int_0^{x(n)} \left(\sqrt{2\log n} - \frac{2\log x + \log\log n + \log(4\pi)}{\sqrt{8\log n}} + O\left(\frac{(\log\log n + |\log x|)^2}{(\log n)^{3/2}}\right)\right) f_{\Gamma(k)}(x)dx \\
&= O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right) + \int_0^{x(n)} \left(\sqrt{2\log n} - \frac{2\log x + \log\log n + \log(4\pi)}{\sqrt{8\log n}}\right) f_{\Gamma(k)}(x)dx \\
&= O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right) + \int_0^{+\infty} \left(\sqrt{2\log n} - \frac{2\log x + \log\log n + \log(4\pi)}{\sqrt{8\log n}}\right) f_{\Gamma(k)}(x)dx
\end{aligned}$$

since we have $x(n) = (\log n)^p$, $p > 1$ thus, for any $s > 1$,

$$\int_{x(n)}^{+\infty} f_{\Gamma(k)}(x)dx = o\left(\frac{1}{n^s}\right), \quad \int_{x(n)}^{+\infty} \log x f_{\Gamma(k)}(x)dx = o\left(\frac{1}{n^s}\right).$$

and moreover – see [6] – it holds

$$\int_0^{+\infty} \log x \; f_{\Gamma(k)}(x)dx = s_{k+1}^1 - \gamma_0,$$

which yields the conclusion.

Similar computations give the claimed result for the variance. More precisely in the step 2 when substituing $\Phi^{-1}\left(1-\frac{x}{n}\right)^2$ to $\Phi^{-1}\left(1-\frac{x}{n}\right)$ in $E_{1,n}$ and $E_{2,n}$ it again appears that we can only consider integrals up to $x(n)$. Then it remains to compute, by substituing the expression of $\mathbb{E}(Z_{n-k})$ and using equation (3.3) for $\Phi^{-1}\left(1-\frac{x}{n}\right)$:

$$\begin{aligned}
&\int_0^{x(n)} \left(\Phi^{-1}\left(1-\frac{x}{n}\right)-\mathbb{E}(Z_{n-k})\right)^2 \binom{n-1}{k-1}\left(\frac{x}{n}\right)^{k-1}\left(1-\frac{x}{n}\right)^{n-k} dx \\
= &\int_0^{x(n)} \left(-\frac{2(\log x-(s^1_{k+1}-\gamma_0))}{2\sqrt{2\log n}}+\frac{-\log\log(n/x)+\log\log n}{2\sqrt{2\log n}}+O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right)\right)^2 \\
\times &\binom{n-1}{k-1}\left(\frac{x}{n}\right)^{k-1}\left(1-\frac{x}{n}\right)^{n-k} dx.
\end{aligned}$$

We conclude along the same lines as above by the upper bound (3.4) and the fact that the variance of the logarithm of a variable with distribution $\Gamma(k)$ is $\pi^2/6-s^2_{k+1}$. □

3.2 Proof of Theorem 2.1

We intend to mimic the sheme of proof worked out in [2] and [1] – specialized to the simpler case of the distance between the empirical and true c.d.f.'s instead of two correlated empirical ones. However all arguments have to be reconsidered since the almost sure controls by means of the law of the iterated logarithm and strong approximations can not be turned easily into L_1 controls. Indeed, what happens now is that the main part of the random integral we consider is also built from the extreme parts rather than the inner part only. Moreover, only a very short extreme interval can be neglected and the remainder extreme intervals define a divergent integral to be precisely evaluated as a series. This is why the expectation rate is no more a CLT rate. Note that the $\log\log n$ in this paper only comes from the primitive of $u(1-u)/h(u)^2$. Introduce the following decomposition, for $C>0$, $\gamma>1$ and $1<\theta\leqslant 2$,

$$A_n=\int_{1-1/(n(\log n)^\gamma)}^1 \left(Z_n-\Phi^{-1}(u)\right)^2 du,\quad B_n=\int_{1-1/n}^{1-1/(n(\log n)^\gamma)} \left(Z_n-\Phi^{-1}(u)\right)^2 du,$$

$$C_n=\int_{1-[C(\log n)^\theta]/n}^{1-1/n} \left(\mathbb{F}_n^{-1}(u)-\Phi^{-1}(u)\right)^2 du,\quad D_n=\int_{1/2}^{1-\left[C(\log n)^\theta\right]/n} \left(\mathbb{F}_n^{-1}(u)-\Phi^{-1}(u)\right)^2 du.$$

Step 1. We have, for $\gamma>1$,

$$\frac{nA_n}{\log\log n}\leqslant \frac{2Z_n^2}{(\log n)^\gamma\log\log n}+\frac{2n}{\log\log n}\int_{1-1/(n(\log n)^\gamma)}^1 \left(\Phi^{-1}(u)\right)^2 du$$

where

$$\lim_{n\to+\infty}\frac{\mathbb{E}\left(Z_n^2\right)}{(\log n)^\gamma\log\log n}=0$$

and

$$\begin{aligned}
\int_{1-1/(n(\log n)^\gamma)}^1 \left(\Phi^{-1}(u)\right)^2 du &= \int_{1-1/(n(\log n)^\gamma)}^1 2\log\left(\frac{1}{1-u}\right)(1+o(1-u))^2\, du \\
&= \left[-2(1-u)\log\left(\frac{1}{1-u}\right)\right]_{1-1/(n(\log n)^\gamma)}^1 = O\left(\frac{1}{n(\log n)^{\gamma-1}}\right)
\end{aligned}$$

hence

$$\lim_{n\to+\infty}\frac{n\mathbb{E}\left(A_n\right)}{\log\log n}=0.$$

Step 2. Notice that for all $u \in [1-1/n, 1-1/(n(\log n)^\gamma)]$, we have

$$\Phi^{-1}(u) = \sqrt{2\log n} + O\left(\frac{\log\log n}{\sqrt{\log n}}\right).$$

Next observe that

$$\begin{aligned}\mathbb{E}(B_n) &= \frac{\mathbb{V}(Z_n)}{n}\left(1 - \frac{1}{(\log n)^\gamma}\right) + \int_{1-1/n}^{1-1/(n(\log n)^\gamma)} \left(\mathbb{E}(Z_n) - \Phi^{-1}(u)\right)^2 du \\ &= O\left(\frac{1}{n\log n}\right) + O\left(\frac{(\log\log n)^2}{n\log n}\right),\end{aligned}$$

hence

$$\lim_{n\to+\infty} \frac{n\mathbb{E}(B_n)}{\log\log n} = 0.$$

Step 3. Start with

$$C_n = \sum_{k=1}^{\left[C(\log n)^\theta\right]} \int_{1-(k+1)/n}^{1-k/n} \left(Z_{n-k} - \Phi^{-1}(u)\right)^2 du.$$

Recall that

$$s_k^1 - \gamma_0 = \log k + \frac{1}{2k} + O\left(\frac{1}{k^2}\right).$$

Now, for $1 \leqslant k \leqslant \left[C(\log n)^\theta\right]$ and $u \in [1-(k+1)/n, 1-k/n]$ we have

$$\begin{aligned}\Phi^{-1}(u) &= \sqrt{2\left(\log(1-u) - \frac{1}{2}\log\log(1-u) - \frac{1}{2}\log(4\pi) - O\left(\frac{\log\log n}{\log n}\right)\right)} \\ &= \sqrt{2\log n} - \frac{2\log k + \log\log n + \log(4\pi)}{\sqrt{8\log n}} + O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right)\end{aligned}$$

thus, by Lemma 3.1, we have, uniformly in k,

$$\mathbb{V}(Z_{n-k}) = \frac{\pi^2/6 - s_{k+1}^2}{2\log n} + O\left(\frac{1}{(\log n)^2}\right)$$

then

$$\begin{aligned}&\mathbb{E}\left(\left(Z_{n-k} - \Phi^{-1}(u)\right)^2\right) \\ &= \mathbb{V}(Z_{n-k}) + \left(\mathbb{E}(Z_{n-k}) - \Phi^{-1}(u)\right)^2 \\ &= \frac{\pi^2/6 - s_{k+1}^2}{2\log n} + O\left(\frac{1}{(\log n)^2}\right) + \left(\frac{\log k - (s_{k+1}^1 - \gamma_0)}{\sqrt{2\log n}} + O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right)\right)^2 \\ &= \frac{\pi^2/6 - s_{k+1}^2}{2\log n} + O\left(\frac{1}{(\log n)^2}\right) + \left(\frac{1+O(1/k)}{2k\sqrt{2\log n}} + O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right)\right)^2.\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\mathbb{E}\,(C_n)\\
&= \frac{1}{n}\sum_{k=1}^{\left[C(\log n)^\theta\right]} \frac{\pi^2/6 - s_{k+1}^2}{2\log n} + O\left(\frac{1}{n(\log n)^{2-\theta}}\right)\\
&\quad + \frac{1}{n}\sum_{k=1}^{\left[C(\log n)^\theta\right]} \left(\frac{1+O(1/k)}{2k\sqrt{2\log n}} + O\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right)\right)^2\\
&= O\left(\frac{(\log n)^{\theta/2}}{n\log n}\right) + \frac{1}{n}\sum_{k=\left[(\log n)^{\theta/2}\right]}^{\left[C(\log n)^\theta\right]} \frac{\pi^2/6 - s_{k+1}^2}{\log n} + O\left(\frac{1}{n\log n}\right) + O\left(\frac{(\log\log n)^3}{n(\log n)^{3-\theta}}\right)\\
&\leqslant \frac{C(\log n)^\theta}{n\log n}\sum_{j=\left[(\log n)^{\theta/2}\right]}^{+\infty} \frac{1}{j^2} + O\left(\frac{(\log n)^{\theta/2}}{n\log n}\right)\\
&= O\left(\frac{(\log n)^{\theta/2}}{n\log n}\right).
\end{aligned}$$

Thus, for any $\theta \leqslant 2$ we have

$$\lim_{n\to+\infty} \frac{n\mathbb{E}(C_n)}{\log\log n} = 0.$$

Step 4. Now we compute the limit of the main deterministic contribution to the main stochastic term D_n, namely

$$D_{1,n} = \int_{1/2}^{1-\left[C(\log n)^\theta\right]/n} \frac{u(1-u)}{h^2(u)}du.$$

Let v_n be such that $\log v_n = (\log n)^{\varepsilon_n}$, $\lim_{n\to+\infty}\varepsilon_n = 0$, $\lim_{n\to+\infty}\varepsilon_n\log\log n = +\infty$. By using (3.2) it holds

$$\begin{aligned}
&\frac{1}{\log\log n}\int_{1-1/v_n}^{1-\left[C(\log n)^\theta\right]/n} \frac{u(1-u)}{h^2(u)}du\\
&= \frac{1+o(1)}{2\log\log n}(\log(\log n - \log(\left[C(\log n)^\theta\right])) - \log\log v_n)\\
&= \frac{1+o(1)}{2\log\log n}\log\left(\frac{(1+o(1))\log n}{\log v_n}\right)\\
&= \frac{1+o(1)}{2}(1-\varepsilon_n)
\end{aligned}$$

and

$$\frac{1}{\log\log n}\int_{1/2}^{1-1/v_n} \frac{u(1-u)}{h^2(u)}du \leqslant \frac{1+o(1)}{2\log\log n}(\log\log v_n) = \frac{1+o(1)}{2}\varepsilon_n.$$

Therefore

$$\lim_{n\to+\infty} \frac{D_{1,n}}{\log\log n} = \frac{1}{2}. \tag{3.5}$$

Compared with the result of [3] recalled at Remark 2.2 the truncation at level $1/v_n$ instead of $1/n$ preserves the same first order.

Step 5. To show that $\mathbb{E}(D_n)$ behaves as $D_{1,n} + o(1)$ we proceed as in [2] with strong approximation arguments. First, we substitute the uniform quantile process to the general quantile process with a sharp control of the expectation of the random error

terms in the Taylor Lagrange expansion. For short, write $d_n = \left[C(\log n)^\theta\right]/n$ and $\beta_n^X(u) = \sqrt{n}(\mathbb{F}_n^{-1}(u) - \Phi^{-1}(u))$ so that

$$\frac{nD_n}{\log\log n} = \frac{1}{\log\log n}\int_{1/2}^{1-d_n} (\beta_n^X(u))^2 du.$$

Defining $U_i = \Phi(X_i)$ which is uniform on $(0,1)$ we obviously have $U_{(i)} = \Phi(X_{(i)})$. Let denote $\mathbb{F}_n^U$ the uniform empirical $c.d.f.$ associated to the U_i and define the underlying uniform quantile process to be

$$\beta_n^U(u) = \sqrt{n}((\mathbb{F}_n^U)^{-1}(u) - u) = \sqrt{n}(\Phi(\mathbb{F}_n^{-1}(u)) - u).$$

Thus for all $1/2 \leqslant u \leqslant 1 - d_n$ there exists a random u^* such that $|u - u^*| \leqslant \left|\beta_n^U(u)\right|/\sqrt{n}$ and

$$\begin{aligned}
\beta_n^X(u)h(u) &= \sqrt{n}(\mathbb{F}_n^{-1}(u) - \Phi^{-1}(u))h(u) \\
&= \sqrt{n}(\Phi^{-1}(\Phi(\mathbb{F}_n^{-1}(u))) - \Phi^{-1}(u))h(u) \\
&= \sqrt{n}\left(\frac{\Phi(\mathbb{F}_n^{-1}(u)) - u}{h(u)} + \frac{h'(u^*)}{2h^2(u^*)}(\Phi(\mathbb{F}_n^{-1}(u)) - u)^2\right)h(u) \\
&= \beta_n^U(u) + r_n(u)
\end{aligned}$$

with

$$\begin{aligned}
r_n(u) &= \frac{1}{2\sqrt{n}}\left(\beta_n^U(u)\right)^2 \frac{h'(u^*)}{h(u^*)}\frac{h(u)}{h(u^*)} \\
&= \frac{1}{2\sqrt{n}}\left(\frac{\beta_n^U(u)}{\sqrt{1-u}}\right)^2\left(\frac{1-u}{1-u^*}\right)\left((1-u^*)\frac{\Phi''(\Phi^{-1}(u^*))}{\Phi'^2(\Phi^{-1}(u^*))}\right)\frac{h(u)}{h(u^*)}.
\end{aligned}$$

We study

$$\frac{nD_n}{\log\log n} = \frac{1}{\log\log n}\int_{1/2}^{1-d_n} (\beta_n^U(u) + r_n(u))^2 \frac{du}{h(u)^2}.$$

Since we have

$$\sup_{0<u<1} u(1-u)\frac{\left|\Phi''(\Phi^{-1}(u))\right|}{\Phi'^2(\Phi^{-1}(u))} = 1$$

it holds, by Lemma 6.1.1 in [7],

$$0 \leqslant \frac{h(u)}{h(u^*)} \leqslant \frac{\max(u,u^*)}{\min(u,u^*)}\frac{1-\min(u,u^*)}{1-\max(u,u^*)}.$$

Now we introduce the sequence of events, with $0 < \varepsilon < 1$,

$$\mathcal{A}_n = \left\{\left|\frac{\beta_n^U(u)}{\sqrt{u(1-u)}}\right| \leqslant (1-\varepsilon)\sqrt{n(1-u)},\ d_n < u < 1 - d_n\right\}. \tag{3.6}$$

On the event $\mathcal{A}_n$ we have the following control of u^*,

$$\frac{\max(u,u^*)}{\min(u,u^*)}\frac{1-\min(u,u^*)}{1-\max(u,u^*)} \leqslant \frac{4}{\varepsilon^2}$$

since, for instance,

$$0 \leqslant \frac{1-u}{1-u^*} \leqslant 1 + \frac{u^*-u}{1-u-(u^*-u)} \leqslant 1 + \frac{\left|\frac{\beta_n^U(u)}{\sqrt{u(1-u)}}\frac{1}{\sqrt{n(1-u)}}\right|}{1-\left|\frac{\beta_n^U(u)}{\sqrt{u(1-u)}}\frac{1}{\sqrt{n(1-u)}}\right|} \leqslant \frac{2}{\varepsilon},$$

$$0 \leqslant \frac{u}{u^*} = 1 + \frac{u - u^*}{u + u^* - u} \leqslant 1 + \frac{\left| \frac{\beta_n^U(u)}{\sqrt{u(1-u)}} \frac{1}{\sqrt{n(1-u)}} \right|}{1 - \left| \frac{\beta_n^U(u)}{\sqrt{u(1-u)}} \frac{1}{\sqrt{n(1-u)}} \right|} \leqslant \frac{2}{\varepsilon},$$

and the same holds for the reverse ratios. Hence we have

$$1_{\mathcal{A}_n} r_n(u) \leqslant \frac{4}{\varepsilon^3 \sqrt{n}} \left(\frac{\beta_n^U(u)}{\sqrt{1-u}} \right)^2$$

thus

$$\mathbb{E}\left(\int_{1/2}^{1-d_n} 1_{\mathcal{A}_n} \frac{r_n(u)^2}{h(u)^2} du \right) \leqslant \int_{1/2}^{1-d_n} \frac{16}{\varepsilon^6 n(1-u)} \mathbb{E}\left(\frac{\beta_n^U(u)}{\sqrt{1-u}} \right)^4 \frac{1-u}{h(u)^2} du.$$

By Lemma 3.2 below and (3.5) we have, when $\theta = 2$,

$$\sup_{1/2<u<1-d_n} \mathbb{E}\left(\frac{\beta_n^U(u)}{\sqrt{1-u}} \right)^4 = O(1), \quad \int_{1/2}^{1-d_n} \frac{1-u}{h(u)^2} du = O(\log\log n). \tag{3.7}$$

It ensues

$$\mathbb{E}\left(\int_{1/2}^{1-d_n} 1_{\mathcal{A}_n} \frac{r_n(u)^2}{h(u)^2} du \right) = O\left(\frac{\log\log n}{(\log n)^2} \right).$$

By using the Cauchy-Schwartz inequality we easily get

$$\lim_{n \to +\infty} \mathbb{E}\left(\int_{1/2}^{1-d_n} 1_{\mathcal{A}_n} \frac{\beta_n^U(u) r_n(u)}{h(u)^2} du \right) = 0,$$

since by (3.7) we have, again for $\theta = 2$,

$$\int_{1/2}^{1-d_n} \frac{\mathbb{E}(\beta_n^U(u)^2)}{h(u)^2} du = O(\log\log n).$$

Step 6. Next we evaluate the probability of the rare event $\mathcal{A}_n^c$ from (3.6). To this aim we work on the KMT probability space where we can define a sequence $\mathbb{B}_n$ of standard Brownian bridges approximating the processes β_n^U in such a way that the error process $w_n = \beta_n^U - \mathbb{B}_n$ satisfies, for universal positive constants c_1, c_2, c_3 and all $x > 0, n \geqslant 1$,

$$\mathbb{P}\left(\sup_{0<u<1} |w_n(u)| > \frac{c_1}{\sqrt{n}} (x + \log n) \right) \leqslant c_2 \exp(-c_3 x). \tag{3.8}$$

Hence we have

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_n^c) &= \mathbb{P}\left(\exists u \in [1/2, 1-d_n], \left|\frac{\beta_n^U(u)}{\sqrt{u(1-u)}}\right| > (1-\varepsilon)\sqrt{n(1-u)}\right) \\
&\leqslant \mathbb{P}\left(\sup_{1/2<u<1-d_n}\left|\frac{\beta_n^U(u)}{\sqrt{u(1-u)}}\right| > (1-\varepsilon)(\log n)^{\theta/2}\right) \\
&\leqslant \mathbb{P}\left(\left\{\sup_{1/2<u<1-d_n}\left|\frac{\mathbb{B}_n(u)}{\sqrt{u(1-u)}}\right| > \frac{1-\varepsilon}{2}(\log n)^{\theta/2}\right\}\dots\right. \\
&\quad \left.\dots\cap\left\{\sup_{1/2<u<1-d_n}\left|\frac{w_n(u)}{\sqrt{u(1-u)}}\right| \leqslant \frac{1-\varepsilon}{2}(\log n)^{\theta/2}\right\}\right) \\
&\quad + \mathbb{P}\left(\sup_{1/2<u<1-d_n}\left|\frac{w_n(u)}{\sqrt{u(1-u)}}\right| > \frac{(1-\varepsilon)}{2}(\log n)^{\theta/2}\right) \\
&\leqslant \mathbb{P}\left(\sup_{1/2<u<1-d_n}\left|\frac{\mathbb{B}_n(u)}{\sqrt{u(1-u)}}\right| > \frac{1-\varepsilon}{2}(\log n)^{\theta/2}\right) \\
&\quad + \mathbb{P}\left(\sup_{1/2<u<1-d_n}|w_n(u)| > \sqrt{C}\frac{1-\varepsilon}{2}\frac{(\log n)^{\theta}}{\sqrt{n}}\right).
\end{aligned}$$

Recall that $1 < \theta \leqslant 2$. By the theorem of Borell-Sudakov (see [5], [10]) and (3.8) we obtain, for any $\gamma > 2$, the constant C fixed as large as needed and all n large enough,

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_n^c) &\leqslant \exp\left(-\frac{(1-\varepsilon)^2(\log n)^{\theta}}{8\sup_{1/2<u<1-d_n}(\mathsf{Var}(B_n(u)/\sqrt{u(1-u)})}\right) + c_2\exp\left(-c_3(\log n)^{\theta}\right) \\
&\leqslant \exp\left(-\frac{(1-\varepsilon)^2}{8}(\log n)^{\theta}\right) + c_2\exp\left(-c_3(\log n)^{\theta}\right) \\
&\leqslant \frac{1}{n^{\gamma}}.
\end{aligned}$$

Therefore we get, for any $0 < b < \gamma/2 - 1$,

$$\begin{aligned}
&\mathbb{E}\left(1_{\mathcal{A}_n^c}\int_{1/2}^{1-d_n} n(F_n^{-1}(u) - \Phi^{-1}(u))^2 du\right) \\
&\leqslant \mathbb{P}\left(\mathcal{A}_n^c\right) 2n\int_0^1 \Phi^{-1}(u)^2 du + 2\mathbb{E}\left(1_{\mathcal{A}_n^c} n Z_n^2\right) \\
&\leqslant 2n\mathbb{P}\left(\mathcal{A}_n^c\right) + \sqrt{\mathbb{P}\left(A_n^c\right) n^2\mathbb{E}\left(Z_n^4\right)} = O\left(\frac{1}{n^b}\right).
\end{aligned}$$

Step 7. It remains to study

$$\frac{1}{\log\log n}\int_{1/2}^{1-d_n}\mathbb{E}(\beta_n^U)^2\frac{du}{h(u)^2}.$$

At this stage the approximation bounds play a crucial role and there is no room for relaxing the trimming constraints. To be more specific the only allowed choice $\theta \leqslant 2$ is $\theta = 2$. Choose an arbitrarily large constant $C > 0$. Given any $0 < \eta < 1$, consider the sequence of events

$$\mathcal{B}_n = \left\{|w_n(u)| < \eta\sqrt{u(1-u)}, \frac{1}{2} < u < 1 - d_n\right\}.$$

By (3.8), for any $k_1 > 0$ there exists $C = C_\eta > (1 + k_1/c_3)^2/\eta^2 > 0$ and $n_0 > 0$ large enough such that for all $n > n_0$ we have

$$\begin{aligned}
1 - \mathbb{P}(\mathcal{B}_n) &\leqslant \mathbb{P}\left(\sup_{1/2<u<1-d_n} |w_n(u)| > \eta \sqrt{\frac{C_\eta}{n}} (\log n)^{\theta/2} \right) \\
&\leqslant \mathbb{P}\left(\sup_{0<u<1} |w_n(u)| > \frac{c_1}{\sqrt{n}} \left((\eta\sqrt{C_\eta} - 1) \log n + \log n \right) \right) \\
&\leqslant c_2 \exp(-c_3(\eta\sqrt{C_\eta} - 1) \log n) \\
&\leqslant \frac{1}{n^{k_1}}.
\end{aligned}$$

Lemma 3.2. *For any $p \geqslant 1$ there exist constants $C > 0$ and κ_p such that we have, for $d_n = [C\frac{(\log n)^2}{n}]$ and all n large enough,*

$$\sup_{d_n<u<1-d_n} \mathbb{E}\left(\frac{|w_n(u)|}{\sqrt{u(1-u)}} \right)^p < 2\eta^p, \quad \sup_{d_n<u<1-d_n} \mathbb{E}\left(\frac{|\beta_n^U(u)|}{\sqrt{u(1-u)}} \right)^p < \kappa_p.$$

Proof of Lemma 3.2. Start with

$$\mathbb{E}\left(\frac{|w_n(u)|}{\sqrt{u(1-u)}} \right)^p \leqslant \eta^p + \mathbb{E}\left(1_{\mathcal{B}_n^c} \frac{|w_n(u)|}{\sqrt{u(1-u)}} \right)^p$$

then set, for $k \geqslant 0$,

$$\begin{aligned}
\mathcal{F}_n &= \left\{ \left| \frac{B_n(u)}{\sqrt{u(1-u)}} \right| < n : d_n < u < 1 - d_n \right\}, \\
\mathcal{F}_n^c &\subset \bigcup\nolimits_{k\in\mathbb{N}} \mathcal{F}_{n,k}, \\
\mathcal{F}_{n,k} &= \left\{ n + k \leqslant \sup_{0<u<1} \left| \frac{B_n(u)}{\sqrt{u(1-u)}} \right| < n + k + 1 \right\}.
\end{aligned}$$

Since $\left|\beta_n^U(u)/\sqrt{u(1-u)}\right| \leqslant n$ for $d_n < u < 1 - d_n$ and all n large enough, we have

$$1_{\mathcal{F}_{n,k}} \sup_{d_n<u<1-d_n} \frac{|w_n(u)|}{\sqrt{u(1-u)}} \leqslant 2n + k + 1, \quad 1_{\mathcal{F}_n} \sup_{d_n<u<1-d_n} \frac{|w_n(u)|}{\sqrt{u(1-u)}} \leqslant 2n.$$

By Sudakov-Borell theorem it holds $\mathbb{P}(\mathcal{F}_{n,k}) \leqslant \exp\left(-(n+k)^2/2\right)$ whereas $\mathbb{P}(\mathcal{B}_n^c) < 1/n^{k_1}$. Hence by choosing $k_1 > p$ it holds

$$\begin{aligned}
\mathbb{E}\left(1_{\mathcal{B}_n^c} \frac{|w_n(u)|}{\sqrt{u(1-u)}} \right)^p &\leqslant \left(\sum_{k\in\mathbb{N}} (2n + k + 1)^p \mathbb{P}(\mathcal{F}_{n,k}) \right) + \mathbb{E}\left((2n)^p 1_{\mathcal{F}_n \cap \mathcal{B}_n^c} \right) \\
&\leqslant \sum_{k\in\mathbb{N}} (2n + k + 1)^p \exp\left(-(n+k)^2/2\right) + \frac{(2n)^p}{n^{k_1}} \\
&= o(1),
\end{aligned}$$

which proves the first claimed upper bound. Since

$$\mathbb{E}\left(|B_n(u)|/\sqrt{u(1-u)} \right)^p < +\infty$$

doesn't depend on n the second expectation bound follows. □

By Lemma 3.2 we get

$$\frac{1}{\log\log n}\mathbb{E}\left(\int_{1/2}^{1-d_n}(w_n(u))^2\frac{du}{h(u)^2}\right)$$
$$=\frac{1}{\log\log n}\mathbb{E}\left(\int_{1/2}^{1-d_n}\left(\frac{w_n(u)}{\sqrt{u(1-u)}}\right)^2\frac{u(1-u)}{h(u)^2}du\right)=O(\eta^2)$$

and, by (3.5),

$$\frac{1}{\log\log n}\mathbb{E}\left(\left|\int_{1/2}^{1-d_n}\frac{w_n(u)\mathbb{B}_n(u)}{h(u)^2}du\right|\right)$$
$$\leqslant\sqrt{\frac{1}{\log\log n}\mathbb{E}\left(\int_{1/2}^{1-d_n}\frac{w_n(u)^2}{h(u)^2}du\right)}\sqrt{\frac{1}{\log\log n}\mathbb{E}\left(\int_{1/2}^{1-d_n}\frac{(\mathbb{B}_n(u))^2}{h(u)^2}du\right)}$$
$$=O(\eta)\sqrt{\frac{1}{\log\log n}\int_{1/2}^{1-d_n}\frac{u(1-u)}{h(u)^2}du}.$$

By choosing η as small as desired, the first assertion of Theorem 2.1 is proved.

Step 8. The sequence $\sqrt{n/\log\log n}W_2(\mathbb{F}_n,\Phi)$ is bounded in L^2, thus uniformly integrable, and from (1.1) (see [9]) converges in probability to 1. Thus the convergence holds in L^1, which establishes the second assertion of Theorem 2.1. □

3.3 Proof of Theorem 2.5

In Theorem 11 of [1] we proved that $nW_2^2(\mathbb{F}_n,\mathbb{G}_n)$ converges in distribution to

$$\|\mathcal{G}\|_2^2=\int_0^1\left(\frac{\mathbb{B}^X(u)}{h(u)}-\frac{\mathbb{B}^Y(u)}{h(u)}\right)^2du$$

under assumptions on the common probability distribution F of the samples ensuring that $\sqrt{n}(\mathbb{F}_n^{-1}(u)-F^{-1}(u))$ and $\sqrt{n}(\mathbb{G}_n^{-1}(u)-G^{-1}(u))$ can be simultaneously approximated on a suitable sub-interval of $[0,1]$ by $\mathbb{B}^X(u)/h(u)$ and $\mathbb{B}^Y(u)/h(u)$ respectively. Here $\mathbb{B}^X(u)$ and $\mathbb{B}^Y(u)$ are two standard Brownian bridges coupled to the marginal samples respectively, and are then correlated together as mentionned at Theorem 2.5 if the two samples are. In [1] the imposed assumptions for the Gaussian approximation concerned the tail of F with respect to the cost function, and the integrability condition

$$\int_0^1\frac{u(1-u)}{h^2(u)}du<+\infty$$

was morerover required. Under the latter condition, the expectation of $\|\mathcal{G}\|_2^2$ is finite since it is bounded by $4\int_0^1 u(1-u)/h^2(u)du$. Now, this upper bound is appropriated to the independent case whereas in our currently dependent case the sample is Gaussian and

$$\mathbb{E}(\|\mathcal{G}\|_2^2)=2\int_0^1\frac{u-C_\rho(u)}{h^2(u)}du$$

which we shall next prove to be finite if $0<|\rho|<1$. Then, as the tail conditions of Theorem 11 in [1] are satisfied by the Gaussian distribution $F=G=\Phi$, the weak convergence of $nW_2^2(\mathbb{F}_n,\mathbb{G}_n)$ is easily established by a straightforward adaptation of the proof of the latter theorem. This long and technical proof is thus omitted. Notice that in

the case $\rho = 0$ we have $\mathbb{E}(\|\mathcal{G}\|_2^2) = 2\int_0^1 u(1-u)/h^2(u)du = +\infty$ and therefore by [8] the random variable $\|\mathcal{G}\|_2^2 = +\infty$ $a.s.$ and $nW_2^2(\mathbb{F}_n, \mathbb{G}_n)$ do not weakly converges.

Let us prove that

$$\int_0^1 \frac{u - C_\rho(u)}{h^2(u)} du < +\infty.$$

Notice that for $a > 0$, as $u \to 1$,

$$1 - \Phi(a\Phi^{-1}(u)) = (4\pi)^{\frac{1-a^2}{2}} \frac{(1-u)^{a^2}}{a(\log(\frac{1}{1-u}))^{\frac{1-a^2}{2}}} \left(1 + O\left(\frac{\log\log(\frac{1}{1-u})}{\log(\frac{1}{1-u})}\right)\right).$$

First assume that $-1 < \rho < 0$. It holds

$$\begin{aligned}
u - C_\rho(u) &= u - \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(u)} \exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right) dxdy \\
&= u - \int_{-\infty}^{\Phi^{-1}(u)} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \Phi\left(\frac{\Phi^{-1}(u) - \rho y}{\sqrt{1-\rho^2}}\right) dy \\
&= u - \int_0^u \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right) dv = \int_0^u \left(1 - \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv \\
&= \int_0^u \left(1 - \Phi\left(\Phi^{-1}(u)\sqrt{\frac{1-\rho}{1+\rho}} + \frac{\rho(\Phi^{-1}(u) - \Phi^{-1}(v))}{\sqrt{1-\rho^2}}\right)\right) dv \\
&\leqslant u\left(1 - \Phi\left(\Phi^{-1}(u)\sqrt{\frac{1-\rho}{1+\rho}}\right)\right) = O\left(\frac{u(1-u)^{\frac{1-\rho}{1+\rho}}}{(\log(\frac{1}{1-u}))^{\frac{2\rho}{1+\rho}}}\right), \quad u \to 1,
\end{aligned}$$

which proves that $(u - C_\rho(u))/h^2(u)$ is integrable near 1 since $-1 < \rho < 0$. By symmetry the same holds near 0.

Next the case $0 < \rho < 1$ near 1 follows from the equality

$$\begin{aligned}
u - C_\rho(u) &= \int_0^u \left(1 - \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv \\
&= \int_0^{\frac{1}{2}} \left(1 - \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv + \int_{\frac{1}{2}}^u \left(1 - \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv.
\end{aligned}$$

Then we get, for the first term, the upper bound

$$\int_0^{\frac{1}{2}} \left(1 - \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv \leqslant \frac{1}{2}\left(1 - \Phi\left(\frac{\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right)$$

that is, up to a logarithmic factor, of order $(1-u)^{\frac{1}{1-\rho^2}}$ as $u \to 1$.

The second term needs more attention. First we choose $0 < \alpha < 1$ such that for all $v \in [1/2, 1 - (1-u)^{\alpha^2}]$ we have, for u close to 1 and η arbitrarily small, $\Phi^{-1}(v) \leqslant (\alpha + \eta)\Phi^{-1}(u)$ and $1 - \alpha\rho > \sqrt{1-\rho^2}$. We take $\alpha < (1 - \sqrt{1-\rho^2})/\rho$, which is actually less than ρ and we have for u close enough to 1,

$$\Phi^{-1}(v) \leqslant \Phi^{-1}(1 - (1-u)^{\alpha^2}) \leqslant (\alpha + \eta)\Phi^{-1}(u).$$

Thus it comes

$$\int_{\frac{1}{2}}^{1-(1-u)^{\alpha^2}} \left(1 - \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv \leqslant \frac{1}{2}\left(1 - \Phi\left(\frac{(1 - (\alpha+\eta)\rho)\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right)$$

that is, up to a logarithmic factor, of order $(1-u)^{\frac{(1-(\alpha+\eta)\rho)^2}{1-\rho^2}}$, with $\frac{(1-(\alpha+\eta)\rho)^2}{1-\rho^2} > 1$ for u close enough to 1.

It remains to study

$$\int_{1-(1-u)^{\alpha^2}}^{u} \left(1-\Phi\left(\frac{\Phi^{-1}(u)-\rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv.$$

Recall that for $x>0$, $1-\Phi(x) \leqslant \dfrac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}$. Thus we have

$$\begin{aligned}
&\int_{1-(1-u)^{\alpha^2}}^{u} \left(1-\Phi\left(\frac{\Phi^{-1}(u)-\rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\right) dv \\
&\leqslant \int_{1-(1-u)^{\alpha^2}}^{u} \frac{e^{-\frac{1}{2}\frac{\Phi^{-1}(u)^2}{1-\rho^2}} e^{-\frac{1}{2}\frac{\rho^2\Phi^{-1}(v)^2}{1-\rho^2}} e^{\frac{\rho\Phi^{-1}(u)\Phi^{-1}(v)}{1-\rho^2}}}{\sqrt{2\pi}\frac{\Phi^{-1}(u)-\rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}} dv \\
&\leqslant \frac{e^{-\frac{1}{2}\frac{\Phi^{-1}(u)^2}{1-\rho^2}}}{\sqrt{2\pi}\Phi^{-1}(u)(1-\rho)} \int_{1-(1-u)^{\alpha^2}}^{u} \sqrt{1-\rho^2} e^{-\frac{1}{2}\frac{\rho^2\Phi^{-1}(v)^2}{1-\rho^2}} e^{\frac{\rho\Phi^{-1}(u)\Phi^{-1}(v)}{1-\rho^2}} dv \\
&= \frac{e^{-\frac{1}{2}\frac{\Phi^{-1}(u)^2}{1-\rho^2}}}{\sqrt{2\pi}\Phi^{-1}(u)(1-\rho)} \int_{\Phi^{-1}(1-(1-u)^{\alpha^2})}^{\Phi^{-1}(u)} \sqrt{1-\rho^2} e^{-\frac{1}{2}\frac{y^2}{1-\rho^2}} e^{\frac{\rho\Phi^{-1}(u)y}{1-\rho^2}} \frac{dy}{\sqrt{2\pi}} \\
&= \frac{e^{-\frac{1}{2}\frac{\Phi^{-1}(u)^2}{1-\rho^2}}}{\sqrt{2\pi}\Phi^{-1}(u)(1-\rho)} \int_{\Phi^{-1}(1-(1-u)^{\alpha^2})-\rho\Phi^{-1}(u)}^{\Phi^{-1}(u)-\rho\Phi^{-1}(u)} \sqrt{1-\rho^2} e^{-\frac{1}{2}\frac{z^2}{1-\rho^2}} e^{\frac{1}{2}\frac{\rho^2\Phi^{-1}(u)^2}{1-\rho^2}} \frac{dz}{\sqrt{2\pi}} \\
&= \frac{e^{-\frac{1}{2}\Phi^{-1}(u)^2}}{\sqrt{2\pi}\Phi^{-1}(u)(1-\rho)} \int_{\Phi^{-1}(1-(1-u)^{\alpha^2})-\rho\Phi^{-1}(u)}^{\Phi^{-1}(u)-\rho\Phi^{-1}(u)} \sqrt{1-\rho^2} e^{-\frac{1}{2}\frac{z^2}{1-\rho^2}} \frac{dz}{\sqrt{2\pi}} = O\left(\frac{1-u}{\log(\frac{1}{1-u})}\right)
\end{aligned}$$

since $\alpha < \frac{1}{\rho}(1-\sqrt{1-\rho^2}) < \rho$ and $\Phi^{-1}(1-(1-u)^{\alpha^2}) \leqslant (\alpha+\eta)\Phi^{-1}(u)$, with η arbitrarily small by choosing u close to 1. Therefore this term is $O((1-u)/\log(1/(1-u)))$ near 1.

Now collecting the previous results, as $u \to 1$ we finally obtain

$$\frac{u-C_\rho(u)}{h^2(u)} = O\left(\frac{1}{(1-u)\log^2(\frac{1}{1-u})}\right)$$

which proves that it is integrable near 1. By symmetry the same holds near 0. We conclude that $(u-C_\rho(u))/h^2(u)$ is integrable on $(0,1)$.

References

[1] P. Berthet and J-C. Fort. Weak convergence of Wasserstein type distances. *hal-01838700v2, 2018 and arXiv:1911.02389*, 2019.

[2] P. Berthet, J-C. Fort, and T. Klein. A central limit theorem for Wasserstein type distances between two different real distributions. *To appear in Ann. Inst. Henri Poincaré Probab. Stat., hal-01526879*, 2018.

[3] P. J. Bickel and W. R. van Zwet. Asymptotic expansions for the power of distribution free tests in the two-sample problem. *Ann. Statist.*, 6(5):937–1004, 1978. MR-0499567

[4] S. G. Bobkov and M. Ledoux. One-dimensional empirical measures, order statistics and Kantorovich transport distances. *To appear in: Memoirs of the AMS*, Preprint 2016. MR-4028181

[5] C. Borell. The Brunn-Minkowski inequality in Gauss space. *Invent. Math.*, 30(2):207–216, 1975. MR-0399402

[6] H. Cramer. *Mathematical methods of statistics*. Princeton Mathematical Series; 9. 1945. MR-0016588

[7] M. Csörgö and L. Horváth. *Weighted approximations in probability and statistics*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. 1993. MR-1215046

[8] M. Csörgö, L. Horváth, and Q.-M. Shao. Convergence of integrals of uniform empirical and quantile processes. *Stochastic Processes and their Applications*, 45(2):283–294, 1993. MR-1208874

[9] E. del Barrio, E. Giné, and F. Utzet. Asymptotics for L_2 functionals of the empirical quantile process, with applications to tests of fit based on weighted Wasserstein distances. *Bernoulli*, 11(1):131–189, 2005. MR-2121458

[10] M. Ledoux and M. Talagrand. *Probability in Banach spaces*. Classics in Mathematics. Springer-Verlag, Berlin, 2011. MR-2814399

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