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A boundary local time for one-dimensional super-Brownian motion and applications*

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Abstract

For a one-dimensional super-Brownian motion with density $X(t,x)$, we construct a random measure L_t called the boundary local time which is supported on $BZ_t := \partial\{x : X(t,x) = 0\}$, thus confirming a conjecture of Mueller, Mytnik and Perkins [13]. L_t is analogous to the local time at 0 of solutions to an SDE. We establish first and second moment formulas for L_t, some basic properties, and a representation in terms of a cluster decomposition. Via the moment measures and the energy method we give a more direct proof that $\dim(BZ_t) = 2 - 2\lambda_0 > 0$ with positive probability, a recent result of Mueller, Mytnik and Perkins [13], where $-\lambda_0$ is the lead eigenvalue of a killed Ornstein-Uhlenbeck operator that characterizes the left tail of $X(t,x)$. In a companion work [6], the author and Perkins use the boundary local time and some of its properties proved here to show that $\dim(BZ_t) = 2 - 2\lambda_0$ a.s. on $\{X_t(\mathbb{R}) > 0\}$.

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1 Introduction & statement of main results

Super-Brownian motion is a Markov process taking values in the space of finite measures on $\mathbb{R}^d$, $\mathcal{M}_F(\mathbb{R}^d)$, equipped with the topology of weak convergence. We denote this process by $X = (X_t : t \geq 0)$ and denote by $P^X_{X_0}$ and $E^X_{X_0}$, respectively, a probability and its expectation under which X is a super-Brownian motion with initial data $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. In one dimension, X_t is almost surely an absolutely continuous random measure and thus has a density we denote by $X(t,x)$. The density is jointly continuous (and will exist) for $t > 0$, and is continuous with Hölder index $\frac{1}{2} - \epsilon$ in the spatial variable for all $\epsilon > 0$ (see [17], for example, where this is implicit in the proof of Theorem III.4.2). It was shown by Konno and Shiga in [9] and independently by Reimers in [18] that $X(t,x)$ satisfies the following stochastic partial differential equation (SPDE):

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$$\frac{\partial X(t,x)}{\partial t} = \frac{\Delta X(t,x)}{2} + \sqrt{X(t,x)}\dot{W}(t,x), \tag{1.1}$$

where $\dot{W}(t,x)$ is a space-time white noise. For a complete discussion of such equations, including the precise definition of a solution, see [20] and [9].

Before discussing our results, we give a brief introduction to the canonical measure of super-Brownian motion. The canonical measure $\mathbb{N}_0$ is a σ-finite measure on $C([0,\infty),\mathcal{M}_F(\mathbb{R}))\backslash\{0\}$. It describes the behaviour of a single cluster, that is, the descendants of a single ancestor, of super-Brownian motion started at the origin. (Likewise $\mathbb{N}_x$ is a cluster started from x and is just a translation of $\mathbb{N}_0$.) In fact, one way of obtaining $\mathbb{N}_0$ is as a weak limit of branching particle systems starting with a single particle, as in Theorem II.7.3 of [17]. Although $\mathbb{N}_0$ itself is an infinite measure, when restricted to $\{X_t > 0\}$ for $t > 0$ it is finite; in particular we have $\mathbb{N}_0(\{X_t > 0\}) = 2/t$ (see Theorem II.7.2 of [17]). A fact of central importance about the canonical measure is that super-Brownian motion under $P^X_{X_0}$ can be understood as a superposition of canonical clusters. This is discussed in greater detail later on (see (1.15)). We will use the notation X_t and $X(t,x)$ to denote the superprocess and its density, respectively, under both $P^X_{X_0}$ and $\mathbb{N}_0$. The law of the process will always be clear from context. For a complete overview of the canonical measure, including proofs of the properties just stated, see Section II.7 of [17].

In a recent work by Mueller, Mytnik and Perkins [13], the authors studied the small-scale asymptotic behaviour of $X(t,x)$, as well as the boundary of its zero set. We define the random set $Z_t = \{x \in \mathbb{R} : X(t,x) = 0\}$. The boundary of the zero set BZ_t is then defined as

$$BZ_t := \partial Z_t = \{x \in Z_t : (x-\epsilon, x+\epsilon) \cap Z_t^c \neq \emptyset \ \ \forall \epsilon > 0\},$$

where the second equality holds by continuity of the density. The results in [13] involve an eigenvalue $\lambda_0 \in (\frac{1}{2}, 1)$ which we describe in greater detail shortly. The authors of [13] show that the left tail of the distribution of $X(t,x)$ behaves like

$$P^X_{X_0}(0 < X(t,x) < a) \asymp t^{-1/2-\lambda_0}\, a^{2\lambda_0 - 1} \tag{1.2}$$

as $a \downarrow 0$, where $f(a) \asymp g(a)$ means that $f(a)$ is bounded above and below by $cg(a)$ for different constants c. Clearly for the above to be true we must take $t \geq t_0$ for some $t_0 > 0$. The upper bound is uniform in x and the lower bound required a localizing assumption. For details, see Section 4 and in particular Theorem 4.8 of [13]. Let dim(B) denote the Hausdorff dimension of a set $B \subseteq \mathbb{R}$.

Theorem A (Mueller, Mytnik, Perkins [13].) *Under $P^X_{X_0}$, dim$(BZ_t) \leq 2 - 2\lambda_0$ almost surely on $\{X_t > 0\}$ and dim$(BZ_t) \geq 2 - 2\lambda_0$ with positive probability.*

Because $\lambda_0 \in (1/2, 1)$, the dimension satisfies $2 - 2\lambda_0 \in (0,1)$. The lower bound was conjectured to hold with full probability on $\{X_t > 0\}$, implying that dim$(BZ_t) = 2 - 2\lambda_0$ almost surely on $\{X_t > 0\}$. The difficulty in proving that the lower bound for the dimension holds with probability one on $\{X_t > 0\}$ is owing to the delicate nature of the BZ_t. It is not monotone in the initial conditions nor in the measure X_t itself.

We will construct a random measure L_t, which we call the boundary local time of X_t, supported on BZ_t. (See Theorems 1.1 and 1.2.) The existence of L_t was conjectured in Section 5.1 of [13]. Once we have constructed L_t, we use it to give a simpler alternative proof of the lower bound in Theorem A. Our method is to show that L_t has finite p-energy for all $p < 2 - 2\lambda_0$; in particular, see Theorem 1.3 below. In a future work [6], L_t and several of its properties derived here, including Theorem 1.2(a), Proposition 1.6 and Theorem 1.9, will be used to resolve the problem left open in Theorem A and Theorem 1.3, showing that dim$(BZ_t) = 2 - 2\lambda_0$ almost surely on $\{X_t > 0\}$.

We now give a description of λ_0. Define a function $F(x)$ by

$$F(x) := -\log P^X_{\delta_0}\big(\{X(1,x) = 0\}\big) = \mathbb{N}_0\big(\{X(1,x) > 0\}\big) > 0. \tag{1.3}$$

The second equality is standard and is a consequence of (1.14) below. Section 3, from (3.5) to (3.14), provides a thorough overview of F as the limit as $\lambda \to \infty$ of the family of functions $\{V_1^\lambda\}_{\lambda>0}$ which characterize the Laplace transform of the density $X(t,x)$. Let $Af(x) = \frac{1}{2}f''(x) - \frac{x}{2}f'(x)$ denote the infinitesimal generator of a standard, one-dimensional Ornstein-Uhlenbeck process Y. For a bounded, continuous and non-negative function ϕ with limits at infinity (F is such a function), $A^\phi f = Af - \phi f$ is the generator of an Ornstein-Uhlenbeck process with Markovian killing corresponding to ϕ; that is, for a sample path $\{Y_s : s \in [0,\infty)\} \in C([0,\infty);\mathbb{R})$, we define the lifetime of the process as ρ^ϕ, after which it is "killed," or put into an inert cemetery state. The distribution of ρ^ϕ is given by

$$P(\rho > t \,|\, Y) = \exp\left(-\int_0^t \phi(Y_s)\,ds\right) \quad \text{for } t > 0. \tag{1.4}$$

Section 2 develops the relevant theory for these processes and their generators. In particular, Theorem 2.1 states that A^ϕ, taken as an operator on the appropriate Hilbert space, has a countable orthonormal family of eigenfunctions $\{\psi_n^\phi\}_{n=0}^\infty$ with corresponding discrete spectrum $0 \geq -\lambda_0^\phi \geq -\lambda_1^\phi \geq \cdots \to -\infty$. We define $\lambda_0 = \lambda_0^F > 0$. As we have noted, it was shown in [13] that $\lambda_0 \in (1/2, 1)$. Numerical estimates by Zhu [22], for which the stated digits are expected to be accurate, suggest that $\lambda_0 \approx 0.8882$. This implies that the value of $\dim(BZ_t)$ from Theorem A, $2 - 2\lambda_0$, is approximately 0.224. A more detailed discussion of the numerics can be found in the introduction of [6].

The method the authors of [13] used to show (1.2) involved computing the asymptotic behaviour of the Laplace transform of the density. In particular (see Proposition 4.5 of that work),

$$\begin{aligned} &\lim_{\lambda\to\infty} t^{\lambda_0}\,\lambda^{2\lambda_0}\,E_{X_0}^X\left(\int \phi(x)\,X(t,x)\,e^{-\lambda X(t,x)}dx\right) \\ &= c_0 \iint \phi(w_0 + \sqrt{t}z)\exp\left(-\frac{1}{t}\int F(z + t^{-1/2}(w_0 - x_0)\,dX_0(w_0)\right)\psi_0^F(z)\,dm(z)\,dX_0(x_0) \end{aligned} \tag{1.5}$$

for every bounded Borel function ϕ, where $m(dz)$ denotes the unit variance Gaussian measure in one dimension, c_0 is a positive constant and ψ_0^F is the lead eigenfunction of A^F. For a super-Brownian motion with density $X(t,x)$, for $\lambda > 0$ we define the measure $L_t^\lambda \in \mathcal{M}_F(\mathbb{R})$ by $dL_t^\lambda(x) = \lambda^{2\lambda_0}e^{-\lambda X(t,x)}X(t,x)\,dx$. That is, for a bounded measurable function $\phi : \mathbb{R} \to \mathbb{R}$, we define

$$L_t^\lambda(\phi) = \lambda^{2\lambda_0}\int \phi(x)\,X(t,x)\,e^{-\lambda X(t,x)}dx. \tag{1.6}$$

L_t^λ is defined the same way under $P_{X_0}^X$ and $\mathbb{N}_0$. The scaling factor of $\lambda^{2\lambda_0}$ can be deduced from (1.5). The convergence of $E_{X_0}^X(L_t^\lambda(\phi))$ as $\lambda \to \infty$, noted in (1.5), led the authors of [13] to conjecture (Section 5.1 of that reference) that there is a random measure L_t on $\mathbb{R}$ such that $L_t^\lambda \to L_t$ in $\mathcal{M}_F(\mathbb{R})$ in probability. Our main result is the verification of this conjecture. In all that follows, $X_0 \in \mathcal{M}_F(\mathbb{R})$.

Theorem 1.1 (Boundary local time: existence and convergence)**.** *Let $t > 0$. Under both $P_{X_0}^X$ and $\mathbb{N}_0$ there is a random measure $L_t(dx) \in \mathcal{M}_F(\mathbb{R})$, supported on BZ_t, such that $L_t^\lambda \to L_t$ in measure as $\lambda \to \infty$, and there is a sequence $\lambda_n \to \infty$ such that $L_t^{\lambda_n} \to L_t$ a.s. as $n \to \infty$. Moreover, under $P_{X_0}^X$ or $\mathbb{N}_0$, for all bounded and continuous functions ϕ, $L_t^\lambda(\phi) \to L_t(\phi)$ in $\mathcal{L}^2$ as $\lambda \to \infty$.*

By almost sure convergence, we mean that for $P_{X_0}^X$ or $\mathbb{N}_0$-a.e. ω, $L_t^{\lambda_n}(\omega) \to L_t(\omega)$ weakly in $\mathcal{M}_F(\mathbb{R})$. Convergence in measure means with respect to any metric on $\mathcal{M}_F(\mathbb{R})$

which induces the weak topology, e.g. the Wasserstein metric (see for example p. 48 of [16]).

Theorem 1.2 (Properties of L_t)**.** *(a) For all $t > 0$, we have $P^X_{X_0}(L_t > 0 \,|\, X_t > 0) > 0$ and $\mathbb{N}_0(L_t > 0 \,|\, X_t > 0) \geq \frac{1-\lambda_0}{2}$.*
(b) L_t is atomless almost surely under $P^X_{X_0}$ and $\mathbb{N}_0$.

Definition. *L_t is the boundary local time of X_t.*

We note that Z_t will contain intervals, unlike the zero set of a Brownian motion (which is equal to its boundary). It is easy to see that L_t is supported on BZ_t from the fact that as λ gets large, L^λ_t concentrates on $\{x : 0 < X(t,x) = O(\lambda^{-1})\}$, and properties of the weak topology on $\mathcal{M}_F(\mathbb{R})$ (see the proof of Theorem 1.1 in Section 4). For fixed $t > 0$, $x \to X(t,x)$ is a continuous path taking values in $\mathbb{R}^+ = [0,\infty)$. BZ_t is the set of points where this path begins and ends its excursions from 0. As L_t is supported on BZ_t, in this sense L_t is a local time of $x \to X(t,x)$ on these excursion endpoints, and hence the boundary local time of $X(t,\cdot)$.

The existence of a measure supported on BZ_t allows us to use the energy method to study its dimension. We will provide a second moment formula for L_t, with which we compute the expectation of energy integrals of the form

$$\iint |x-y|^{-p} \, dL_t(x) \, dL_t(y). \tag{1.7}$$

If $L_t > 0$ and the above energy is finite, then $\dim(\text{supp}(L_t)) \geq p$ by Frostman's connection between energy integrals and Hausdorff dimension (see Theorem 4.27 of Mörters and Peres [12]). We introduce some notation. For $h : \mathbb{R}^2 \to \mathbb{R}$, define $(L_t \times L_t)(h)$ by

$$(L_t \times L_t)(h) = \iint h(x,y) \, dL_t(x) \, dL_t(y).$$

For $p > 0$, we define $h_p(x,y) = |x-y|^{-p}$. The second moment formula for L_t allows us to establish the following.

Theorem 1.3 (Finite energy and Hausdorff dimension)**.** *Both $E^X_{X_0}((L_t \times L_t)(h_p))$ and $\mathbb{N}_0((L_t \times L_t)(h_p))$ are finite for all $p < 2 - 2\lambda_0$. Moreover, dim$(BZ_t) = 2 - 2\lambda_0$ almost surely on $\{L_t > 0\}$ under both measures.*

The fact that $\dim(BZ_t) \leq 2 - 2\lambda_0$ $P^X_{X_0}$-a.s. is already known from Theorem A, and from this it follows easily under $\mathbb{N}_0$, as we point out in the proof of Theorem 1.3. By the above, the lower bound, ie. $\dim(BZ_t) \geq 2 - 2\lambda_0$, holds with at least the probability that $L_t > 0$, as in Theorem 1.2(a). This plays an important role in Hughes-Perkins [6]; in Theorem 1.2 of [6] we show that with respect to both $P^X_{X_0}$ and $\mathbb{N}_0$, $L_t > 0$ almost surely on $\{X_t > 0\}$, thus improving part (a) of Theorem 1.2 above and establishing almost sure non-degeneracy of L_t. Combined with Theorem 1.3, this will show that $\dim(BZ_t) = 2 - 2\lambda_0$ almost surely on $\{X_t > 0\}$.

There are a number of other potential uses for such a local time. We now discuss some possibilities. By sampling a point from L_t, we are able to "view X_t from the perspective of a typical point in BZ_t." More precisely, one can define $Q_{X_0}((Z, X_t) \in A) = E^X_{X_0}(\int 1_A(z, X_t) \, dL_t(z))$ and study properties of the Palm measure $Q_{X_0}(X_t \in \cdot \,|\, Z = z)$. The behaviour of X_t near BZ_t is complex and there is still much that is not understood about it. For example, the density has an improved modulus of continuity and is nearly Lipzschitz (ie. Hölder $1 - \eta$ for all $\eta > 0$) at points in BZ_t (see Theorem 2.3 of [15]). This suggests that BZ_t would be small, but despite this BZ_t has positive dimension. Constructing and studying the Palm measure described above would give a more structured approach for investigating this phenomenon.

As a local time, L_t has the potential to study pathwise uniqueness in the SPDE (1.1), a problem which remains open, assuming a similar role as that of the semi-martingale local time in the Yamada-Watanabe Theorem for one-dimensional SDEs (see Theorem V.40 of Rogers and Williams [19]). It may also provide insight in the behaviour of some discrete processes; super-Brownian motion in high dimensions is the scaling limit of a number of lattice models and interacting particle systems. In dimension one, it is still the scaling limit of branching random walk (for example see [21] or Theorem II.5.1(iii) of [17]). One could obtain information about the boundaries of such approximating processes by proving a limit theorem establishing weak convergence of the laws of their discrete local times to that of L_t. Of course, L_t allows for us to study BZ_t more directly, as we have done in Theorem 1.3. In fact, with L_t it may be possible to determine the exact Hausdorff measure function of BZ_t.

We now discuss the method of our proof. Upper bounds on second moments of L_t^λ were obtained in Section 5.1 of [13], but in order to establish the existence of L_t we require exact asymptotics, which are more delicate. The main ingredient is the following convergence result. In order to state it we need to introduce some notation. Recall that $m(dx)$ denotes the centred unit variance Gaussian measure. Let $\psi_0 = \psi_0^F$ (the eigenfunction of A^F corresponding to eigenvalue $-\lambda_0$). The constant $C_{1.4}$ is given explicitly in (5.46), and the function ρ is defined in (5.47). The function $V_t^{\infty,\infty}$ is defined in Section 3 as $V_t^{\infty,\infty}(x_1,x_2) = \mathbb{N}_0\left(\{X(t,x_1)>0\}\cup\{X(t,x_2)>0\}\right)$ (see (3.25)). Finally, we will denote by P_x^B and E_x^B the law and associated expectation of a standard Brownian motion B_t with initial value $B_0 = x$.

Theorem 1.4 (Convergence of second moments of L_t^λ). *There exists a constant $C_{1.4} > 0$ and continuous function $\rho : \mathbb{R}\times\mathbb{R} \to (0,1]$ such that for bounded Borel $h : \mathbb{R}^2 \to \mathbb{R}$,*

$$\begin{aligned}
\lim_{\lambda,\lambda'\to\infty} &\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)) \\
&= C_{1.4}^2 \int_0^t (t-s)^{-2\lambda_0} \bigg[\iint E_0^B \Big(h(\sqrt{t-s}\, z_1 + B_s, \sqrt{t-s}\, z_2 + B_s) \Big) \\
&\quad \times \exp\left(-\int_0^s V_{t-u}^{\infty,\infty}(\sqrt{t-s}\, z_1 + B_s - B_u, \sqrt{t-s}\, z_2 + B_s - B_u)\, du \right) \\
&\quad \times \rho(z_1,z_2)\, \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \bigg] ds.
\end{aligned}$$

Moreover, the limit is finite for all bounded h.

That the formula above is finite is not obvious, as $\lambda_0 > 1/2$; we discuss this in more detail shortly. From the above we can deduce that $\{L_t^\lambda(\phi)\}_{\lambda>0}$ is Cauchy in $\mathcal{L}^2(\mathbb{N}_0)$ and therefore has a limit by completeness; in particular see Corollary 4.1 and its proof. We then argue that the limit is in fact the integral with respect to a unique measure, which is L_t. The proof of Theorem 1.4 is long and technical; Section 5 is entirely devoted to it. We use the Laplace functional to obtain a Feynman-Kac type representation for $\mathbb{N}_0(L_t^\lambda(\phi)\, L_t^{\lambda'}(\phi))$ and then establish its convergence. The reason we do so under $\mathbb{N}_0$ is because the Feynman-Kac formulas are simpler in this setting. We now present first and second moment formulas for L_t under $\mathbb{N}_0$; as one would expect, the second moment formula in part (b) agrees with the limit of $\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h))$ given in Theorem 1.4. The terms $C_{1.4}$ and ρ are the same that appeared in that result.

Theorem 1.5 (Moments of L_t under $\mathbb{N}_0$). *(a) For a bounded or non-negative Borel function $\phi : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{N}_0(L_t(\phi)) = C_{1.4}\, t^{-\lambda_0} \int \phi(\sqrt{t}z)\, \psi_0(z)\, dm(z). \tag{1.8}$$

(b) For measurable $h : \mathbb{R}^2 \to \mathbb{R}$, either bounded or non-negative,

$$\begin{aligned}\mathbb{N}_0((L_t \times L_t)(h)) &\\ = C_{1.4}^2 \int_0^t (t-s)^{-2\lambda_0} \Bigg[\iint E_0^B \Big(h(\sqrt{t-s}\, z_1 + B_s, \sqrt{t-s}\, z_2 + B_s) \Big) &\\ \times \exp\left(- \int_0^s V_{t-u}^{\infty,\infty}(\sqrt{t-s}\, z_1 + B_s - B_u, \sqrt{t-s}\, z_2 + B_s - B_u)\, du \right) &\\ \times \rho(z_1, z_2)\, \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \Bigg] ds. & \qquad (1.9)\end{aligned}$$

Moreover, (1.9) *is finite for all bounded h.*

As we noted earlier, finiteness of (1.9) is not obvious since $\lambda_0 > 1/2$ (although it is implicit in the proof of Theorem 1.4), which can make (1.9) hard to use; for applications, the following upper bound for second moments is easier to apply than the exact formula. The value θ is defined as $\theta = \int \psi_0\, dm$. Y is an Ornstein-Uhlenbeck process started at z_1 with corresponding expectation $E_{z_1}^Y$. The exponential term in the first bound of the following proposition can be interpreted as a survival probability of Y, producing a w^{λ_0} term which makes the integral finite. (The proofs of Theorem 1.3 and Theorem 1.2(b) in Section 4 both use this technique.)

Proposition 1.6 (Second moment bounds under $\mathbb{N}_0$)**.** *For a non-negative Borel function $h : \mathbb{R}^2 \to \mathbb{R}$,*

$$\begin{aligned}\mathbb{N}_0((L_t \times L_t)(h)) \le C_{1.4}^2 \int_0^t w^{-2\lambda_0} \Bigg[\iint E_{z_1}^Y \Bigg(\exp\left(- \int_0^{\log(t/w)} F(Y_u)\, du \right) &\\ \times h(\sqrt{t} Y_{\log(t/w)}, \sqrt{t} Y_{\log(t/w)} + \sqrt{w}(z_2 - z_1)) \Bigg) \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \Bigg] dw. & \qquad (1.10)\end{aligned}$$

Moreover,

$$\mathbb{N}_0(L_t(1)^2) \le \frac{C_{1.4}^2 \theta^2}{1-\lambda_0} t^{1-2\lambda_0}. \qquad (1.11)$$

As we have alluded to, applying (1.10) with $h(x,y) = |x-y|^{-p}$ gives an upper bound for the expectation of energy integrals of the form (1.7), which is how we prove Theorem 1.3.

Thus far, we have not commented on the proofs of existence and properties of L_t under $P_{X_0}^X$. The proofs rely on the conditional representation in terms of canonical clusters, which we will discuss shortly. First, in order to keep the moment results together, we state our results regarding the moments of L_t under $P_{X_0}^X$.

Theorem 1.7 (Moments of L_t under $P_{X_0}^X$)**.** *For a bounded or non-negative Borel function $\phi : \mathbb{R} \to \mathbb{R}$,*

$$\begin{aligned}E_{X_0}^X(L_t(\phi)) = C_{1.4}\, t^{-\lambda_0} \iint \phi(x_0 + \sqrt{t} z) \exp\left(-\frac{1}{t} \int F(z + t^{-1/2}(x_0 - y_0)\, dX_0(y_0) \right) &\\ \times \psi_0(z)\, dm(z)\, dX_0(x_0). & \qquad (1.12)\end{aligned}$$

(b) There is a constant $C_{1.7}$ such that

$$E_{X_0}^X(L_t(1)^2) \le C_{1.7} \left(X_0(1)\, t^{1-2\lambda_0} + X_0(1)^2\, t^{-2\lambda_0} \right). \qquad (1.13)$$

We note that the right hand side of (1.12) is equal to that of (1.5), and so was originally computed in Proposition 1.5 of [13] as $\lim_{\lambda \to \infty} E_{X_0}^X(L_t^\lambda(\phi))$. The fact that the same formula gives the mean measure of L_t then follows from the $\mathcal{L}^2$ convergence of $L_t^\lambda(\phi)$, as in Theorem 1.1.

We first establish the existence of L_t, as well as its properties, under the measure $\mathbb{N}_0$, owing to the fact that the second moments of L_t^λ admit simpler formulas in this case. In order to prove the same for super-Brownian motion, we need to use the relationship between super-Brownian motion under $P_{X_0}^X$ and the canonical measure, which we now describe. We recall that $\mathbb{N}_x$ is a σ-finite measure such that $\mathbb{N}_x(\{X_t > 0\}) = 2/t$ which describes the "law" of a single cluster of super-Brownian motion started at x; that is, the descendants of a single ancestor at x. More precisely, super-Brownian motion is a superposition of canonical clusters; for a bounded, non-negative Borel function $\phi : \mathbb{R} \to \mathbb{R}$,

$$E_{X_0}^X \left(\exp\left(-X_t(\phi)\right)\right) = \exp\left(-\iint 1 - e^{-\mu_t(\phi)} d\mathbb{N}_{x_0}(\mu)\, dX_0(x_0)\right). \tag{1.14}$$

This expression for the Laplace functional is in fact a consequence of a distributional equality between super-Brownian motion under $P_{X_0}^X$ and a Poisson point process of canonical clusters. For $X_0 \in \mathcal{M}_F(\mathbb{R})$, let $\mathbb{N}_{X_0}(\cdot) = \int \mathbb{N}_x(\cdot)\, dX_0(x)$ and let Θ_{X_0} be a Poisson point process on $C([0,\infty), \mathcal{M}_F(\mathbb{R}))$ with intensity $\mathbb{N}_{X_0}$. We define a $\mathcal{M}_F(\mathbb{R})$-valued process $(X_t : t \geq 0)$ by

$$X_t(\cdot) = \begin{cases} \int \mu_t(\cdot)\, d\Theta_{X_0}(\mu) & \text{if } t > 0, \\ X_0(\cdot) & \text{if } t = 0. \end{cases} \tag{1.15}$$

By Theorem 4 of Section IV.3 of [10], $(X_t : t \geq 0)$ is a super-Brownian motion with initial measure X_0. The "points" of the point process Θ_{X_0} are the clusters of X. For fixed $t > 0$, (1.15) leads to

$$X_t = \sum_{j \in I_t} \mu_t^j,$$

where $\{\mu_t^j : j \in I_t\}$ are the points of a Poisson point process with finite intensity $\mathbb{N}_{X_0}(\mu_t \in \cdot \,|\, \mu_t > 0\}$. Let $\bar{X}_0(\cdot) = X_0(\cdot)/X_0(1)$. Assuming our probability space is rich enough to allow us to choose random relabellings of these points, by the above we can write

$$X_t = \sum_{i=1}^{N} X_t^i, \tag{1.16}$$

where N is Poisson$(2X_0(1)/t)$ and, given N, $\{X_t^i : i = 1, \ldots, N\}$ are iid with distribution $\mathbb{N}_{\bar{X}_0}(X_t \in \cdot \,|\, X_t > 0)$. We can and do condition on the values of the initial points of the clusters, denoted by $x_1, \ldots, x_N$, which are iid points with distribution $\bar{X}_0$, in which case X_t^i has conditional distribution $\mathbb{N}_{x_i}(X_t \in \cdot \,|\, X_t > 0)$. In order to prove the existence and properties of L_t with respect to a super-Brownian motion X_t, we realize the super-Brownian motion as a point process and express X_t as above. Conditioning on N and applying (1.16), we can write $L_t^\lambda(\phi)$ as

$$L_t^\lambda(\phi) = \lambda^{2\lambda_0} \int \left[\sum_{i=1}^{N} X^i(t,x)\right] e^{-\lambda \sum_{i=1}^{N} X^i(t,x)}\, \phi(x)\, dx.$$

The almost sure existence of boundary local times corresponding to the canonical clusters allows us to take this limit quite easily and so establish that L_t exists under $P_{X_0}^X$ (ie. Theorem 1.1). Furthermore, we obtain a conditional representation for L_t in terms of its clusters; this allows us to transfer the properties of L_t under $\mathbb{N}_0$ to L_t under $P_{X_0}^X$. Let L_t^i denote the boundary local time of X_t^i. In the statement that follows, we assume that we have realized X_t using (1.16).

Theorem 1.8 (Cluster decomposition)**.** *Let X_t be super-Brownian motion under $P^X_{X_0}$ and L_t be its boundary local time. Conditional on N, we have*

$$dL_t(x) = \sum_{i=1}^{N} 1\big(\sum_{j\neq i} X^j(t,x) = 0\big)\, dL^i_t(x)$$
$$= 1(X(t,x)=0) \sum_{j=1}^{N} dL^i_t(x). \tag{1.17}$$

Remark. Given the nature of BZ_t, we expect this behaviour. In the cluster decomposition, each cluster has a boundary local time of its own. Since each is supported on the boundary of its respective zero set, the local time L_t of X_t will be equal to the sum of cluster local times, except the boundary of the zero set of one cluster may be "swallowed" by the support of another, hence the indicator functions.

The idea of representing the boundary local time of X_t in terms of the boundary local time of its clusters is not restricted to a super-Brownian motion and its comprising canonical clusters. The following formulation of the same principle will be useful in Hughes-Perkins [6]. Recall that a sum of independent super-Brownian motions is a super-Brownian motion.

Theorem 1.9 (General cluster decomposition)**.** *Suppose $X^1, \dots, X^n$ are independent super-Brownian motions with corresponding boundary local times L^i_t at time $t > 0$, for $i = 1, \dots, n$. Let $X = \sum_{i=1}^n X^i$ and let L_t be the boundary local time of X_t. Then*

$$dL_t(x) = \sum_{i=1}^{n} 1\big(\sum_{j\neq i} X^j(t,x) = 0\big)\, dL^i_t(x)$$
$$= 1(X(t,x)=0) \sum_{i=1}^{n} dL^i_t(x).$$

One example of superprocesses satisfying the above conditions follows from (III.1.3) of [17]. Let $X_0 \in \mathcal{M}_F(\mathbb{R})$ and suppose that $\{A_1, \dots, A_n\}$ is a Borel partition of $\mathbb{R}$. Define X^i as the contribution to X from ancestors at time 0 which are in A_i. (This makes X^i a super-Brownian motion with initial measure $X_0(\cdot \cap A_i)$; a precise definition of X^i may be given in terms of the historical process as in the above reference.) Then $X = \sum_{i=1}^N X^i$ satisfies the conditions of the above theorem.

Notations. *We will make use of the common convention that C denotes any positive constant whose value is not important. The value of C may change line to line in a derivation; to bring attention to the fact that the constant has changed, we will sometimes label the new constant C'. We write $f \sim g$ if $\lim_x f(x)/g(x) = 1$, where the limit will be clear from context. As the reader has probably inferred, we will write $\mu > 0$ when a measure has positive mass (that is, to indicate that $\mu(1) > 0$). For an interval $I \subseteq \mathbb{R}$, let $C(I, \mathbb{R})$ denote the space of continuous maps from I to $\mathbb{R}$.*

Let S_t denote the semi-group of Brownian motion and p_t the associated heat kernel (the Gaussian density of variance t). Let $\mathcal{N}(x_0, \sigma^2)$ denote the law of a one-dimensional Gaussian with mean x_0 and variance σ^2.

Organization of Paper. The paper is organized as follows. **Section 2** gives a brief overview of the theory of one-dimensional Ornstein-Uhlenbeck processes with Markovian killing. Our method relies on a change of variables which allow us to express certain quantities in terms of eigenvalue problems involving these processes' generators.

Section 3 describes fundamental background connecting the Laplace functional of super-Brownian motion to a family of semi-linear PDEs. We also introduce the families V^λ and $V^{\lambda,\lambda'}$, which play a key role in our analysis.

In **Section 4**, we assume Theorem 1.4 (the $\mathcal{L}^2$-convergence result) and proceed to prove our main results, including existence and properties of L_t and the cluster representations. First we prove the existence of L_t under $\mathbb{N}_0$ (Theorem 1.1 for $\mathbb{N}_0$) and the formulae for its first and second moments (Theorem 1.5). Next, we use the cluster decomposition to prove the existence of L_t under $P^X_{X_0}$ (Theorem 1.1 for $P^X_{X_0}$) and its representation in terms of clusters (Theorems 1.8 and 1.9). We then establish the upper bound on second moments of L_t under $\mathbb{N}_0$ (Proposition 1.6), which allows us to prove the remaining results, including Theorem 1.3, the dimension result.

Section 5 contains the proof of Theorem 1.4, with the proof of one technical lemma given in **Section 6**.

2 Killed Ornstein-Uhlenbeck processes

As above, we define the operator A by $Af(x) = \frac{f''(x)}{2} - \frac{xf'(x)}{2}$. The Markov process generated by A is a one-dimensional Ornstein-Uhlenbeck process with mean zero. We denote this process by Y, denote its law when started at x by P^Y_x with corresponding expectation E^Y_x. For general initial conditions $Y_0 \sim \mu \in \mathcal{M}_1(\mathbb{R})$ (the space of probability measures on $\mathbb{R}$), we write its law as P^Y_μ. Y has a stationary measure, the unit variance Gaussian measure, m. When $Y_0 \sim m$, the process is reversible and can be defined for time values in $\mathbb{R}$. We will denote the law of this stationary process on $\mathbb{R}$ by P^Y.

We now introduce the notions of killing and lifetime for the process $(Y_t : t \geq 0)$. Let $\phi \in C^+([-\infty,\infty],\mathbb{R})$, the space of non-negative continuous functions with limits at $\pm\infty$. Such functions are also bounded. We will call this family of functions killing functions. Let $A^\phi f(x) = Af(x) - f(x)\phi(x)$. A^ϕ is the generator of an Ornstein-Uhlenbeck process subjected to Markovian killing at rate $\phi(Y_t)$. The lifetime of the killed process is $\rho^\phi = \inf\{t > 0 : \int_0^t \phi(Y_s)\,ds > e\}$, where e is an independent Exp(1) random variable. We recall that the distribution of ρ^ϕ is given by (1.4).

The generators A and A^ϕ correspond to strongly continuous contraction semigroups on $\mathcal{L}^2(m)$. The following theorem is proved in [13], where it is stated as Theorem 2.3. We note that the statement of the result in that paper had a misprint when describing the convergence of of the transition densities, which appeared in part (c). We have corrected the statement, which is in part (b) of the following.

Theorem 2.1. *For $\phi \in C^+([-\infty,\infty],\mathbb{R})$, the following statements hold.*
(a) A^ϕ has a complete orthonormal family of C^2 eigenfunctions $\{\psi_n : n \geq 1\}$ in $\mathcal{L}^2(m)$ satisfying $A^\phi\psi_n = -\lambda_n\psi_n$, where $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$. Furthermore, $-\lambda_0$ is a simple eigenvalue and $\psi_0 > 0$.
(b) For $t > 0$, the diffusion Y generated by A^ϕ has a jointly continuous transition density $q_t(x,y)$ with respect to m, given by

$$q_t(x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\psi_n(x)\psi_n(y), \tag{2.1}$$

where the series converges in $\mathcal{L}^2(m \times m)$ and uniformly absolutely on sets of the form $[\epsilon,\infty) \times [-\epsilon^{-1},\epsilon^{-1}]^2$ for all $\epsilon > 0$.
(c) For $0 < \delta < \frac{1}{2}$, there exists a constant $c_\delta > 0$ such that

$$q_t(x,y) \leq c_\delta e^{-\lambda_0 t} e^{\delta(x^2+y^2)} \quad \text{for all } t \geq s^*(\delta), \tag{2.2}$$

where $s^*(\delta) > 0$ is the solution of

$$2\delta = \frac{e^{-s^*/2} - e^{-s^*}}{1 - e^{-s^*}}. \tag{2.3}$$

(d) Denote $\theta = \int \psi_0 \, dm$*. For all* $t \geq 0$ *and* $x \in \mathbb{R}$*,*

$$e^{\lambda_0 t} P_x(\rho^\phi > t) = \theta \psi_0(x) + r(t,x), \tag{2.4}$$

where, for any $\delta > 0$*, there is a constant* $c_\delta > 0$ *such that*

$$\psi_0(x) \leq c_\delta e^{\delta x^2}, \tag{2.5}$$

$$|r(t,x)| \leq c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)t}. \tag{2.6}$$

(e) As $T \to \infty$*,* $P_x(Y \in \cdot \,|\, \rho_\phi > T) \to P_x^{Y,\infty}$ *weakly on* $C([0,\infty), \mathbb{R})$*, where* $P_x^{Y,\infty}$ *is the law of the diffusion with the transition density*

$$\tilde{q}_t(x,y) \equiv q_t(x,y) \frac{\psi_0(y)}{\psi_0(x)} e^{\lambda_0 t} \tag{2.7}$$

with respect to m*.*

The bounds in part (d) of the above easily imply the following estimates, which we will often use. For $0 < \delta < 1/2$, there is a constant $C_\delta > 0$ such that

$$P_x^Y(\rho^\phi > t) \leq C_\delta e^{\delta x^2} e^{-\lambda_0 t} \quad \forall\, x \in \mathbb{R}, t > 0. \tag{2.8}$$

This implies that there is a constant $C > 0$ such that

$$P_m^Y(\rho^\phi > t) \leq C e^{-\lambda_0 t} \quad \forall\, t > 0. \tag{2.9}$$

The following limit result is a simple consequence of the eigenfunction expansion for $q_t(x,y)$.

Lemma 2.2. *For all* $x, y \in \mathbb{R}$*,*

$$\lim_{t \to \infty} e^{\lambda_0 t} q_t(x,y) = \psi_0(x)\psi_0(y).$$

The convergence is uniform on compact sets.

Proof. For all $t > 0$ and $x, y \in \mathbb{R}$, from (2.1), we have

$$e^{\lambda_0 t} q_t(x,y) = \psi_0(x)\psi_0(y) + \sum_{n=1}^{\infty} e^{-(\lambda_n - \lambda_0)t} \psi_n(x)\psi_n(y). \tag{2.10}$$

As we are taking $t \to \infty$ we can restrict to $t \geq 1$, in which case the absolute value of the sum above is bounded above by

$$e^{-(\lambda_1 - \lambda_0)(t-1)} \sum_{n=1}^{\infty} e^{-(\lambda_n - \lambda_0)} |\psi_n(x)\psi_n(y)|.$$

By Theorem 2.1(b) with $t = 1$, the series in the above is convergent, and the convergence is uniform on compact sets. Part (a) of the same theorem states that $-\lambda_0$ is a simple eigenvalue. Hence $\lambda_1 - \lambda_0 > 0$ and the above vanishes as $t \to \infty$; in fact, because the series converges uniformly on compacts to a continuous limit, the above vanishes uniformly on compacts as $t \to \infty$, so (2.10) gives the result. □

It will be useful for us to study the distribution of the process Y when conditioned on survival and its endpoint. Hereafter we assume that Y has killing function $\phi \in C^+([-\infty,\infty], \mathbb{R})$ and we denote its lifetime by ρ. For fixed $T > 0$ and $z \in \mathbb{R}$, consider the

$[0,T]$-indexed inhomogeneous Markov process taking values in $\mathbb{R}$ with transition density (with respect to $dm(y_2)$)

$$\hat{q}_{s,t}(y_1,y_2) = \frac{q_{t-s}(y_1,y_2)\, q_{T-t}(y_2,z)}{q_{T-s}(y_1,z)} \tag{2.11}$$

for $0 \le s < t < T$. (The kernels are degenerate when $t = T$, since $Y_T = z$.) Below we verify that the finite dimensional distributions defined by this transition kernel have an extension to a (necessarily) unique law on $C([0,T],\mathbb{R})$, which we denote by $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$ when the initial point is $x \in \mathbb{R}$, and show that it gives an explicit version of the suggested regular conditional distribution for all $z \in \mathbb{R}$. We then establish that for fixed $S > 0$, $P_x^Y(Y|_{[0,S]} \in \cdot\,|\,\rho > T, Y_T = z)$ converges weakly to $P_x^{Y,\infty}(Y|_{[0,S]} \in \cdot)$ as $T \to \infty$ for all $z \in \mathbb{R}$.

Lemma 2.3. *(a) Let $x \in \mathbb{R}$ and $T > 0$. For all $z \in \mathbb{R}$, the finite dimensional distributions described in* (2.11)*, with initial value x, have a unique extension to $C([0,T],\mathbb{R})$. The resulting laws $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$ are continuous in z and define a regular conditional probability for $Y|_{[0,T]}$ under P_x^Y conditioned on Y_T.*
(b) Let $x, z \in \mathbb{R}$, $S > 0$ be fixed. Then $P_x(Y|_{[0,S]} \in \cdot\,|\,\rho > T, Y_T = z)$ converges weakly on $C([0,S],\mathbb{R})$ to $P_x^{Y,\infty}(Y|_{[0,S]} \in \cdot)$ as $T \to \infty$.

(c) For all $S, K > 0$, $\left\{P_x(Y|_{[0,S]} \in \cdot\,|\,\rho > T, Y_T = z) : |x|, |z| \le K, T \ge S\right\}$ is tight on $C([0,S],\mathbb{R})$.

Before proving the lemma, we make an observation concerning time reversals of Y under $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$. For $T > 0$ and $t \in [0,T]$, define $\hat{Y}_t = Y_{T-t}$. Let $x, z \in \mathbb{R}$. For $0 < t_1 < t_2 < T$ and ϕ_1, ϕ_2 bounded Borel functions, we have

$$\begin{aligned}
&E_x^Y(\phi_1(\hat{Y}_{t_1})\,\phi_2(\hat{Y}_{t_2})\,\big|\,\rho > T, Y_T = z) \\
&\quad= \frac{1}{q_T(x,z)} \iint \phi_1(y_1)\,\phi_2(y_2)\,q_{T-t_2}(x,y_2)\,q_{t_2-t_1}(y_2,y_1)\,q_{t_1}(y_1,z)\,dm(y_1)\,dm(y_2) \\
&\quad= E_z^Y(\phi_1(Y_{t_1})\,\phi_2(Y_{t_2})\,\big|\,\rho > T, Y_T = x),
\end{aligned}$$

where the last equality uses $q_t(x,y) = q_t(y,x)$. The above equality of distributions can be extended to general finite dimensional distributions. Because the extension of the finite dimensional distributions to a law on $C([0,T],\mathbb{R})$ (ie. from Lemma 2.3(a)) is unique, we therefore have that for all $x, z \in \mathbb{R}$,

$$P_x^Y(\hat{Y}|_{[0,T]} \in \cdot\,|\,\rho > T, Y_T = z) = P_z^Y(Y|_{[0,T]} \in \cdot\,|\,\rho > T, Y_T = x). \tag{2.12}$$

As a last note, we will sometimes denote the law $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$ simply by $P_x^Y(\cdot\,|\,Y_T = z)$ when it is clear from context that we are working with the killed process.

Proof of Lemma 2.3. Let $x, z \in \mathbb{R}$ and $T > 0$. We define a distribution, which we denote by $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$, on finite (time-indexed) collections of random variables, which describes the finite dimensional distributions (FDDs) of the inhomogeneous Markov process with transition density (2.11). For $0 = t_0 < t_1 < \cdots < t_n < T$ and bounded, continuous functions $\phi_1, \ldots, \phi_n$, we define the n-dimensional FDD of $(Y_{t_1}, \ldots, Y_{t_n})$ under $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$ as

$$\begin{aligned}
&E_x^Y\left(\prod_{i=1}^n \phi_i(Y_{t_i})\,\middle|\,\rho > T, Y_T = z\right) \\
&\quad= \frac{1}{q_T(x,z)} \int \left[\prod_{i=1}^n \phi_i(y_i) q_{t_n - t_{n-1}}(y_{n-1}, y_n)\right] q_{T-t_n}(y_n, z) \prod_{i=1}^n dm(y_i),
\end{aligned} \tag{2.13}$$

where we use the convention $y_0 = x$. We note that (2.13) also defines the FDDs of a regular conditional distribution of $(Y_t : t \in [0,T])$ under P_x^Y conditioned on $Y_T = z$ (which is why we have used this notation). Thus when we have established that these laws extend to a probability on $C([0,T],\mathbb{R})$, we will have explicitly constructed a version of the regular conditional distribution.

To prove that $P_x^Y(\cdot \,|\, \rho > T, Y_T = z)$ extends to a probability on $C([0,T],\mathbb{R})$, we will establish a tightness criterion. We consider the fourth moments of increments of Y. Let $0 < s < t < T$. Expanding using (2.13), we have

$$\begin{aligned} &E_x^Y((Y_t - Y_s)^4 \,|\, \rho > T, Y_T = z) \\ &\quad = \frac{1}{q_T(x,z)} \iint (y_2 - y_1)^4 q_s(x, y_1)\, q_{t-s}(y_1, y_2)\, q_{T-t}(y_2, z)\, dm(y_1)\, dm(y_2). \end{aligned} \tag{2.14}$$

We now collect some elementary bounds and inequalities which will allow us to obtain a useful upper bound for the above. First, we note that while $q_t(x,y)$ is a transition density with respect to m, it will sometimes be useful to express it as a density with respect to the Lebesgue measure. Since $p_1(\cdot)$ is the density of m, we have

$$q_t(x,y)\, dm(y) = q_t(x,y)\, p_1(y)\, dy. \tag{2.15}$$

We will use a comparison with an un-killed Ornstein-Uhlenbeck process. The transition kernel of a standard Ornstein-Uhlenbeck process is described by, for $0 \le s < t$,

$$(Y_t - Y_s \,|\, Y_s = y) \sim \mathcal{N}(e^{-(t-s)/2} y, 1 - e^{-(t-s)}).$$

Let $k_t(x,y)$ denote the a transition density of an un-killed Ornstein-Uhlenbeck process with respect to Lebesgue measure. Then for $x, y \in \mathbb{R}$ and $t > 0$,

$$k_t(x,y) = \frac{(2\pi)^{-1/2}}{\sqrt{1 - e^{-t}}} \exp\left(-(y - e^{-t/2}x)^2 \big/ 2(1 - e^{-t}) \right). \tag{2.16}$$

The transition densities of the killed Ornstein-Uhlenbeck process are bounded above by those of the un-killed process. This implies that

$$\text{(i) } q_t(x,y)\, dm(y) \le k_t(x,y)\, dy, \qquad \text{(ii) } q_t(x,y)\, p_1(y) \le k_t(x,y). \tag{2.17}$$

It is easy to establish from (2.16) that there is a constant $c > 0$ such that

$$k_t(x,y) \le c p_t(y - x e^{-t/2}) \;\text{ for all } t \le 2 \text{ and } x, y \in \mathbb{R}. \tag{2.18}$$

where we recall that $p_t(\cdot)$ is the Gaussian density of variance t. Let $K > 0$. From (2.17)(ii) and (2.18) it follows that there is a constant $C_1(K)$ such that

$$q_{T-t}(y_2, z) \le k_{T-t}(y_2, z) p_1(z)^{-1} \le \frac{C_1(K)}{\sqrt{T - T'}} \;\; \forall\, y_2 \in \mathbb{R}, z \in [-K, K], \text{ and } t \le T' < T. \tag{2.19}$$

Next, we note that it holds by elementary formulas for moments of Gaussians that there is a constant $c > 0$ such that

$$\int (y_2 - y_1)^4 p_t(y_2)\, dy_2 \le c\,(t^2 + |y_1|^4) \;\; \forall\, y_1 \in \mathbb{R},\, t > 0. \tag{2.20}$$

Finally, observe that $q_T(\cdot,\cdot)$ is bounded below by the transition density of Y with constant killing function $\|\phi\|_\infty$. Thus for all $K > 0$ and $M \ge 1$, from (2.15) we have

$$q_T(x,z) \ge e^{-\|\phi\|_\infty T} k_T(x,z) p_1(z)^{-1} \ge \delta(K,M) \;\; \forall\, x, z \in [-K,K],\; T \in [M^{-1}, M] \tag{2.21}$$

for a sufficiently small constant $\delta(K,M)>0$.

Let $0<T'<T$ and suppose that $0<s<t\le T'$ such that $t-s\le 1$. Let $K>0$ and suppose that $x,z\in[-K,K]$. Using (2.17)(i) to bound $q_s(x,y_1)dm(y_1)$ and $q_{t-s}(y_1,y_2)dm(y_2)$, and (2.19) to bound $q_{T-t}(y_2,z)$, from (2.14) we obtain that

$$\begin{aligned} &E_x^Y((Y_t-Y_s)^4\,|\,\rho>T,Y_T=z)\\ &\quad\le \frac{C_1(K)\,(T-T')^{-1/2}}{q_T(x,z)}\int k_s(x,y_1)\Big[\int (y_2-y_1)^4 k_{t-s}(y_1,y_2)dy_2\Big]dy_1\\ &\quad\le \frac{C_1(K)\,(T-T')^{-1/2}}{q_T(x,z)}\int k_s(x,y_1)\Big[\int c(y_2-y_1)^4 p_{t-s}(y_2-e^{-(t-s)/2}y_1)dy_2\Big]dy_1, \end{aligned} \tag{2.22}$$

where the second inequality uses (2.18). Changing variables and applying (2.20), we obtain that

$$\begin{aligned} \int c(y_2-y_1)^4 p_{t-s}(y_2-e^{-(t-s)/2}y_1)dy_2 &\le C(|y_1|^4(1-e^{-(t-s)/2})^4+(t-s)^2)\\ &\le C(1+|y_1|^4)(t-s)^2, \end{aligned} \tag{2.23}$$

where in the second inequality we have $1-e^{-x}\le x$ for $x\ge 0$ and $(t-s)^4\le (t-s)^2$ (since $t-s\le 1$). Substituting this into (2.22), we obtain that

$$E_x^Y((Y_t-Y_s)^4\,|\,Y_T=z)\le \frac{C_1(K)\,(T-T')^{-1/2}}{q_T(x,z)}\,(t-s)^2\int C\,k_s(x,y_1)\,(|y_1|^4+1)\,dy_1. \tag{2.24}$$

Recall that we have assumed $x,z\in[-K,K]$. By (2.16) it is clear that for $K>0$, the integral is bounded above by some constant $C_2(K)>0$ for all $x\in[-K,K]$ and $s>0$. Using this along with (2.21), with a choice of $M\ge 1$ for which $T\in[M^{-1},M]$, from the above we deduce the following:

$$\begin{aligned} &\text{For all } x,z\in[-K,K],\ 0<s<t\le T' \text{ such that } t-s\le 1,\\ &\quad E_x^Y((Y_t-Y_s)^4\,|\,Y_T=z)\le C_2(K)\,\delta(K,M)^{-1}\,C_1(K)\,(T-T')^{-1/2}\times(t-s)^2. \end{aligned} \tag{2.25}$$

Let $T'=2T/3$. Hereafter we consider increments of size at most $1\wedge T/3$. We have that (2.25) holds for all $0<s<t\le 2T/3$ such that $t-s\le 1\wedge T/3$ with constant $C_2(K)\delta(K,M)^{-1}C_1(K)(T/3)^{-1/2}$. It remains to show that it also holds on $[2T/3,T]$. To do so, we make use of reversibility. Suppose $T/3\le s<t<T$. Then

$$\begin{aligned} &E_x^Y((Y_t-Y_s)^4\,\big|\,Y_T=z)\\ &\quad=\frac{1}{q_T(x,z)}\iint (y_2-y_1)^4\,q_s(x,y_1)\,q_{t-s}(y_1,y_2)\,q_{T-t}(y_2,z)\,dm(y_1)\,dm(y_2)\\ &\quad=E_z^Y((Y_{T-s}-Y_{T-t})^4\,\big|\,Y_T=x), \end{aligned} \tag{2.26}$$

where the last equality uses $q_t(x,y)=q_t(y,x)$ (a consequence of (2.1)) and (2.13). Since $0<T-t<T-s\le 2T/3$, by (2.25) and (2.26) we have that for all $x,z\in[-K,K]$ and $T/3\le s<t<T$ such that $t-s\le 1\wedge T/3$,

$$E_x^Y((Y_t-Y_s)^4\,|\,\rho>T,Y_T=z)=C_2(K)\,\delta(K,T)^{-1}\,C_1(K)\,(T/3)^{-1/2}\times(t-s)^2.$$

Combined with the previous statement that this holds for all $0<s<t\le 2T/3$, we have that

$$\begin{aligned} &\text{For all } x,z\in[-K,K],\ 0<s<t<T \text{ such that } t-s\le 1\wedge T/3,\\ &\quad E_x^Y((Y_t-Y_s)^4\,|\,Y_T=z)\le C_2(K)\,\delta(K,M)^{-1}\,C_1(K)\,(T/3)^{-1/2}\times(t-s)^2. \end{aligned} \tag{2.27}$$

The above proof can be easily modified to obtain the same bound (with a potential change to the constant) for increments in which $s = 0$ or $t = T$, and we omit it. Thus by (2.27) and the Kolmogorov Continuity Theorem, $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$ has a unique extension to a probability on $C([0,T],\mathbb{R})$, also denoted by $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$. As we noted earlier, this gives an explicit construction of the regular conditional distribution $(Y_t : t \in [0,T])$ under P_x^Y given $\rho > T$ and $Y_T = z$, Additionally, suppose that $z_n \to z$ and that $z_n \in [-K,K]$ for all $n \geq 1$. From (2.27), $\{P_x^Y(\cdot|\,\rho > T, Y_T = z_n) : n \geq 1\}$ is tight. It is clear from (2.13) and continuity of $q_t(\cdot,\cdot)$ that the FDDs of $P_x^Y(\cdot\,|\,\rho > T, Y_T = z_n)$ converge to those of $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$. Thus the aforementioned tightness proves that $P_x^Y(\cdot\,|\,\rho > T, Y_T = z_n)$ converges to $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$ as a law on $C([0,T],\mathbb{R})$. Thus we have proved part (a).

Before proving (b), we note the following consequence of (2.27) and its proof. Let $S, K > 0$ and fix $M > 1$ such that $S \in [M^{-1}, M]$. By considering increments of $(Y_s : s \in [0,S])$ but allowing the time T at which we condition $Y_T = z$ to take values in $[S,M]$, we have that

$$\{P_x^Y(Y|_{[0,S]} \in \cdot\,|\,\rho > T, Y_T = z) : |x|,|z| \leq K, T \in [S,M]\} \text{ is tight.} \tag{2.28}$$

Next we turn to part (b). Fix $S > 0$ and $x,z \in \mathbb{R}$. We now check that the FDDs of $(Y_s : s \in [0,S])$ under $P_x^Y(\cdot\,|\,\rho > T, Y_T = z)$ converge to those of $(Y_s : s \in [0,S])$ under $P^{Y,\infty}$ as $T \to \infty$. Let $0 < t_1 < t_2 \leq S$ and let ϕ_1 and ϕ_2 be bounded and continuous functions. Then from (2.13), we have

$$\begin{aligned}
&E_x^Y(\phi_1(Y_{t_1})\,\phi_2(Y_{t_2})\,\big|\,\rho > T, Y_T = z)\\
&= \frac{1}{q_T(x,z)}\iint \phi_1(y_1)\,\phi_2(y_2)\,q_{t_1}(x,y_1)\,q_{t_2-t_1}(y_1,y_2)\,q_{T-t_2}(y_2,z)\,dm(y_1)\,dm(y_2)\\
&= \frac{e^{\lambda_0 t_2}}{e^{\lambda_0 T}q_T(x,z)}\iint \phi_1(y_1)\,\phi_2(y_2)\,q_{t_1}(x,y_1)\,q_{t_2-t_1}(y_1,y_2)\\
&\qquad\qquad\qquad\qquad \times e^{\lambda_0(T-t_2)}q_{T-t_2}(y_2,z)\,dm(y_1)\,dm(y_2).
\end{aligned} \tag{2.29}$$

By Lemma 2.2, we have

$$\lim_{T\to\infty} e^{\lambda_0(T-t)}q_{T-t}(y_2,z) = \psi_0(y_2)\psi_0(z), \quad \lim_{T\to\infty} e^{\lambda_0 T}q_T(x,z) = \psi_0(x)\psi_0(z). \tag{2.30}$$

Moreover, applying (2.2) with $\delta = 1/8$, we have that

$$e^{\lambda_0(T-t)}q_{T-t}(y_2,z) \leq c e^{y_2^2/8 + z^2/8} \quad \forall\, y_2, z \in \mathbb{R}, t \in (0,S] \text{ and } T \geq S + s^*(1/8), \tag{2.31}$$

where $s^*(1/8)$ is as in (2.3). Using (2.31) (replacing t with t_2) and (2.17)(i) we obtain the following bound for the integrand in (2.29):

$$\begin{aligned}
&|\phi_1(Y_{t_1})\,\phi_2(Y_{t_2})\,q_{t_1}(x,y_1)\,q_{t_2-t_1}(y_1,y_2)\,e^{\lambda_0(T-t_2)}q_{T-t_2}(y_2,z)|\,dm(y_1)\,dm(y_2)\\
&\quad \leq c e^{z^2/8}\|\phi_1\|_\infty\|\phi_2\|_\infty e^{y_2^2/8}k_{t_1}(x,y_1)k_{t_2-t_1}(y_1,y_2)\,dy_1\,dy_2
\end{aligned}$$

for all $T \geq S + s^*(1/8)$. By (2.16), $k_{t_1}(x,y_1)$ and $k_{t_2-t_1}(y_1,y_2)$ are Gaussians with variance at most 1, and so a short argument shows that the above quantity is integrable. This allows us to use Dominated Convergence in (2.29), so by (2.30) we have

$$\begin{aligned}
&\lim_{T\to\infty} E_x(\phi_1(Y_{t_1})\,\phi_2(Y_{t_2})\,\big|\,Y_T=z)\\
&=\frac{e^{\lambda_0 t_2}}{\psi_0(x)\psi_0(z)}\iint \phi_1(y_1)\,\phi_2(y_2)\,q_{t_1}(x,y_1)\,q_{t_2-t_1}(y_1,y_2)\,\psi_0(y_2)\,\psi_0(z)dm(y_1)dm(y_2)\\
&=\iint \phi_1(y_1)\,\phi_2(y_2)\,\left[e^{\lambda_0 t_1}q_{t_1}(x,y_1)\frac{\psi_0(y_1)}{\psi_0(x)}\right]\left[e^{\lambda_0(t_2-t_1)}q_{t_2-t_1}(y_1,y_2)\frac{\psi_0(y_1)}{\psi_0(y_2)}\right]dm(y_1)dm(y_2)\\
&=\iint \phi_1(y_1)\,\phi_2(y_2)\,\tilde q_{t_1}(x,y_1)\,\tilde q_{t_2-t_1}(y_1,y_2)dm(y_1)dm(y_2)\\
&=E_x^{Y,\infty}(\phi_1(Y_{t_1})\,\phi_2(Y_{t_2})).
\end{aligned}$$

The above argument can be easily generalized to n-fold FDDs for all $n\geq 2$, (the $\delta=1/8$ in (2.31) can be reduced to handle larger n) and thus we have the desired convergence of the FDDs as $T\to\infty$. In order to obtain weak convergence of the laws on $C([0,S],\mathbb{R})$, we need tightness of the distributions as $T\to\infty$. To prove that the distributions are tight we will analyse the fourth moments of increments, as in (2.14), but first we obtain one more bound. We note that by Lemma 2.2 and joint continuity of $(T,(x,y))\to q_T(x,y)$, it holds that for all $K>0$,

$$e^{\lambda_0 T}q_T(x,z)\geq \delta(K)>0 \quad \forall\, x,z\in[-K,K],\, T\geq 1 \tag{2.32}$$

for sufficiently small $\delta(K)>0$. Let $K>0$ and $x,z\in[-K,K]$. In (2.14), we bound $q_{T-t}(y_2,z)$ above using (2.31) and bound the other transition densities using (2.17), which gives

$$\begin{aligned}
&E_x^Y((Y_t-Y_s)^4\,|\,\rho>T,Y_T=z)\\
&\quad\leq \frac{e^{\lambda_0 t+z^2/8}}{e^{\lambda_0 T}q_T(x,z)}\int k_s(x,y_1)\left[\int (y_2-y_1)^4\,k_{t-s}(y_1,y_2)\,e^{y_2^2/8}dy_2\right]dy_1\\
&\quad\leq e^{\lambda_0 S+K^2/8}\,\delta(K)^{-1}\int k_s(x,y_1)e^{y_1^2/4}\left[\int c\,(y-y_1(1-e^{-(t-s)/2}))^4 p_{t-s}(y)\,e^{y^2/4}\,dy\right]dy_1\\
&\quad\leq e^{\lambda_0 S+K^2/8}\,\delta(K)^{-1}\int c'\,k_s(x,y_1)e^{y_1^2/4}\left[\int c'\,(y-y_1(1-e^{-(t-s)/2}))^4\,p_{2(t-s)}(y)\,dy\right]dy_1
\end{aligned}$$

for all $T\geq S+s^*(1/8)$. In the second inequality we have used (2.32) as well as (2.18) and a change of variables. The third follows from a short calculation and the fact that $t-s\leq 1$. Applying (2.20) to the above and arguing as in (2.23), we obtain that, for all $T\geq S+s^*(1/8)$,

$$\begin{aligned}
&E_x^Y((Y_t-Y_s)^4\,|\,\rho>T,Y_T=z)\\
&\quad\leq e^{\lambda_0 S+K^2/8}\,\delta(K)^{-1}\,(t-s)^2\,C\int k_s(x,y_1)\,e^{y_1^2/4}\,(1+|y_1|^4)\,dy_1,\\
&\quad\leq C_3(S,K)\,(t-s)^2\ \ \forall\, x,z\in[-K,K],\ 0\leq s<t\leq S \text{ such that } t-s\leq 1,
\end{aligned} \tag{2.33}$$

for a constant $C_3(S,K)>0$, where to see that the integral is bounded uniformly for $|x|\leq K$, we use the fact, from (2.16), that $k_s(x,y_1)$ is Gaussian with mean of absolute value bounded above by $|x|$ and variance less than 1. The fact that (2.33) holds for all $T\geq S+s^*(1/8)$ implies that the laws $P_x^Y(Y|_{[0,S]}\in\cdot\,|\,\rho>T,Y_T=z)$ are tight as $T\to\infty$. Combined with the convergence of the FDDs to those of $P_x^{Y,\infty}$, this proves (b).

Observe that (2.33) proves part (c) if we restrict to $T\geq S+s^*(1/8)$. If we choose $M\geq 1$ such that $M^{-1}<S<S+s^*(1/8)<M$, then (2.28) gives tightness of the laws for $T\in[S,S+s^*(1/8)]$. Combining these two cases gives the desired tightness and proves (c). □

3 Some non-linear PDE

Let $\mathcal{B}_{b^+}(\mathbb{R})$ denote the space of bounded, non-negative Borel functions. Recall that S_t denotes the semigroup of Brownian motion. By Theorem III.5 of [10], for $\phi \in \mathcal{B}_{b^+}(\mathbb{R})$, there exists a unique non-negative solution, denoted $V_t^{\phi}(x)$, to the evolution equation

$$V_t = S_t(\phi) - \left(\int_0^t S_{t-s}(V_s^2/2)\, ds\right), \tag{3.1}$$

such that

$$E_{X_0}^X\big(e^{-X_t(\phi)}\big) = e^{-X_0(V_t^{\phi})} \tag{3.2}$$

for all $X_0 \in \mathcal{M}_F(\mathbb{R})$. Applying the above with $X_0 = \delta_x$, (1.14) gives

$$\mathbb{N}_x\big(1 - e^{-X_t(\phi)}\big) = V_t^{\phi}(x). \tag{3.3}$$

We are interested in the case when the initial data is a measure, and also in the differential form of the equation. The integral equation (3.1) has a corresponding PDE, which is the following:

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{V^2}{2} \quad \text{ for } (t,x) \in (0,\infty)\times\mathbb{R}, \quad V_t \to \phi \text{ as } t \downarrow 0. \tag{3.4}$$

In [1], this equation was shown to have a unique $C^{1,2}$ solution when $\phi \in \mathcal{M}_F(\mathbb{R})$, where $V_t \to \phi$ is understood as weak convergence of measures. That is, we identify the function V_t with the measure $V_t(x)dx$, which converges weakly to ϕ. By Lemma 2.1 of [14], the solution of (3.4) is also the unique solution to (3.1). We denote the unique solution to (3.1) and (3.4) by V_t^{ϕ}. Part (d) of the same lemma establishes that if $\phi_n \to \phi$ weakly as $n \to \infty$, then $V_t^{\phi_n}(x) \to V_t^{\phi}(x)$ for all $t > 0, x \in \mathbb{R}$. We note from (3.1) that $V_t^{\phi_n} \le S_t\phi_n \le ct^{-1/2}\phi_n(\mathbb{R})$. Using this and the fact that X_t has a bounded, continuous density, if we approximate measures by functions in $\mathcal{B}_{b^+}(\mathbb{R})$, we can take bounded limits in (3.2) and (3.3) to establish that (3.2) and (3.3) hold for V_t^{ϕ} when $\phi \in \mathcal{M}_F(\mathbb{R})$.

Notation. *As X_t is absolutely continuous, when $\phi \in \mathcal{M}_F(\mathbb{R})$ we interpret $X_t(\phi)$ as $\int X(t,x)\, d\phi(x)$.*

We now state some useful properties of solutions to (3.4). For a proof, see Lemma 2.6 in [14].

Proposition 3.1. *Let $\phi, \psi \in \mathcal{M}_F(\mathbb{R})$.*
(a) (Monotonicity) If $\phi \le \psi$, then $0 \le V_t^{\phi} \le V_t^{\psi}$ for all $t > 0$.
(b) (Sub-additivity) $V_t^{\phi+\psi} \le V_t^{\phi} + V_t^{\psi}$ for all $t > 0$.

Next we fix $\phi = \lambda\delta_x \in \mathcal{M}_F(\mathbb{R})$ for $\lambda > 0$, so that $X_t(\phi) = \lambda X(t,x)$. Denote by V_t^{λ} the unique, non-negative $C^{2,1}$ solution to the initial value problem

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{V^2}{2} \quad \text{ for } (t,x) \in (0,\infty)\times\mathbb{R}, \quad V_t \to \lambda\delta_0 \text{ weakly as } t \downarrow 0. \tag{3.5}$$

This family was originally studied in [7]. It is an exercise to use (3.5) or the scaling properties of super-Brownian motion to show that $V_t^{\lambda}(x)$ satisfies the following space-time scaling relationship. For $\lambda, r > 0$, we have

$$V_t^{\lambda r}(x) = \lambda^2 V_{\lambda^2 t}^{r}(\lambda x). \tag{3.6}$$

By translation invariance in the initial conditions of (3.5), and by (3.2) and (3.3) we have

$$E_{\delta_0}^X\big(e^{-\lambda X(t,x)}\big) = e^{-V_t^{\lambda}(x)}, \tag{3.7}$$

$$\mathbb{N}_0\big(1 - e^{-\lambda X(t,x)}\big) = V_t^{\lambda}(x) \tag{3.8}$$

for all $x \in \mathbb{R}$ and $t > 0$. It is clear from (3.7) that V_t^λ increases to a limit as $\lambda \to \infty$. In the PDE literature this was established in [7], where it was shown that V^λ converges locally uniformly as $\lambda \to \infty$ to a function V_t^∞ on $(0,\infty)\times\mathbb{R}$. Heuristically, V_t^∞ is the solution of (3.5) when $\lambda = +\infty$. Rigorously, it is the unique solution to the following problem:

$$\begin{aligned} &\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{V^2}{2} \qquad \text{for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ &\lim_{t\downarrow 0} V_t(x) = 0 \ \ \forall x \neq 0, \ \ \lim_{t\downarrow 0}\int_{B_\epsilon} V_t(x)\,dx = +\infty \ \ \forall \epsilon > 0, \end{aligned} \tag{3.9}$$

where $B_\epsilon = B(0,\epsilon)$, the ball with radius ϵ centered at the origin. V_t^∞ was introduced and shown to solve (3.9) in [2]; uniqueness of the solution is a consequence of Theorem 3.5 of [11]. Taking $\lambda \to \infty$ in (3.8), we see that V_t^∞ satisfies

$$V_t^\infty(x) = \mathbb{N}_0(\{X(t,x) > 0\}). \tag{3.10}$$

We recall that (see Theorem II.7.2 of [17])

$$\mathbb{N}_0(\{X_t > 0\}) = 2/t. \tag{3.11}$$

Thus (3.10) implies that

$$V_t^\infty(x) \le 2/t \ \ \forall x. \tag{3.12}$$

Taking $\lambda^2 = 1/t$ and letting $r \to \infty$ in (3.6), one obtains that $V_t^\infty(x) = t^{-1}V_1^\infty(t^{-1/2}x)$.

Definition. *Define* $F : \mathbb{R} \to \mathbb{R}^+$ *by*

$$F(x) := V_1^\infty(x). \tag{3.13}$$

It follows that $V_t^\infty(x) = t^{-1}F(t^{-1/2}x)$. It was shown in [2] that F is the solution to an ODE problem. (In fact, their PDEs and ODEs have different (constant) coefficients, but Section 3 of [13] shows that F is a rescaled version of the function they study.) F is the unique solution of

$$\begin{aligned} &(i)\ F''(x) + xF'(x) + F(x)(2 - F(x)) = 0 \\ &(ii)\ F > 0, F \in C^2(\mathbb{R}) \\ &(iii)\ F'(0) = 0, F(x) \sim c_1|x|e^{-x^2/2} \ \text{ as } |x| \to \infty \end{aligned} \tag{3.14}$$

for some $c_1 > 0$. We recall that $f(x) \sim h(x)$ means $f(x)/h(x) \to 1$ as $x \to \infty$. This F is the function we discussed in the introduction, for which $-\lambda_0$ is the lead eigenvalue of the operator A^F. In particular, by evaluating (3.10) at $t = 1$ we can recover (1.3), our preliminary definition.

As part of the proof of Theorem A, the authors of [13] computed the rate of convergence of V_t^λ to V_t^∞. In particular, Proposition 4.6 of that reference states that

$$\sup_{x\in\mathbb{R}}\left[V_t^\infty(x) - V_t^\lambda(x)\right] \le Ct^{-1/2-\lambda_0}\lambda^{1-2\lambda_0} \tag{3.15}$$

for some constant C. (This is closely connected to (1.5).) A similar lower bound with the same power of λ is established in the same proposition, although in this case one must be careful when t is close to zero. We will make frequent use of (3.15) in this work to bound error terms arising when we make approximations to obtain an eigenvalue problem. Let Y be an Ornstein-Uhlenbeck process. We define $Z_T(Y)$ as

$$Z_T(Y) = \exp\left(\int_0^T F(Y_s) - V_1^{e^{s/2}}(Y_s)\,ds\right). \tag{3.16}$$

Since $V_1^{e^{s/2}} \uparrow V_1^\infty = F$ as $s \to \infty$, the integrand converges to zero as $s \to \infty$. As $Z_T(Y)$ is increasing in T, we can define $Z_\infty(Y) := \lim_{T\to\infty} Z_T(Y)$. By (3.15), we can easily deduce that the (monotone) limit

$$Z_\infty(Y) := \lim_{T\to\infty} Z_T(Y) \tag{3.17}$$

exists and is finite, and that moreover there is a constant $C_Z > 0$ such that, uniformly for all Y,

$$Z_T(Y) \le Z_\infty(Y) \le C_Z < \infty \ \ \forall T > 0. \tag{3.18}$$

Finally, we introduce another family of solutions to (3.4), which arise when we compute second moments of L_t^λ; we will evaluate expressions that involve the density at two points $x_1, x_2 \in \mathbb{R}$. Let $V_t^{(\lambda,\lambda'),(x_1,x_2)}$ denote V_t^ϕ when $\phi = \lambda\delta_{x_1} + \lambda'\delta_{x_2} \in \mathcal{M}_F(\mathbb{R})$, so that, by (3.3),

$$V_t^{(\lambda,\lambda'),(x_1,x_2)}(y) = \mathbb{N}_y\big(1 - e^{-\lambda X(t,x_1) - \lambda' X(t,x_2)}\big). \tag{3.19}$$

When evaluating this function at 0, we will denote it by $V_t^{\lambda,\lambda'}(x_1,x_2) = V_t^{(\lambda,\lambda'),(x_1,x_2)}(0)$. In other words,

$$V^{\lambda,\lambda'}(x_1,x_2) = \mathbb{N}_0\big(1 - e^{-\lambda X(t,x_1) - \lambda' X(t,x_2)}\big). \tag{3.20}$$

By (3.19), (3.20) and translation invariance of the canonical measure, these families satisfy

$$V_t^{(\lambda,\lambda'),(x_1,x_2)}(y) = V_t^{(\lambda,\lambda'),(x_1-y,x_2-y)}(0) = V_t^{\lambda,\lambda'}(x_1-y, x_2-y). \tag{3.21}$$

Lastly, as can be readily seen from (3.20) and the symmetry of the canonical measure,

$$V_t^{\lambda,\lambda'}(x_1,x_2) = V_t^{\lambda,\lambda'}(-x_1,-x_2) \tag{3.22}$$

for all $x_1, x_2 \in \mathbb{R}$. This family also satisfies a following scaling relationship which can be derived from studying the associated PDE directly. In particular,

$$V_t^{r\lambda,c\lambda'}(x_1,x_2) = \lambda^2 V_{\lambda^2 t}^{r,c\lambda'/\lambda}(\lambda x_1, \lambda x_2) = (\lambda')^2 V_{(\lambda')^2 t}^{r\lambda/\lambda',c}(\lambda' x_1, \lambda' x_2), \tag{3.23}$$

for all $\lambda, \lambda', r, c > 0$ and $x_1, x_2 \in \mathbb{R}$. Taking limits and applying bounded convergence in (3.2), we see that $V_t^{\lambda,\lambda'}(x_1,x_2)$ has a monotone limit as $\lambda, \lambda' \to \infty$ (by Proposition 3.1(a)). We denote this limit $V_t^{\infty,\infty}(x_1,x_2)$. In agreement with our previous notation we define the following.

Definition. *We define* $F_2 : \mathbb{R}^2 \to \mathbb{R}^+$ *by*

$$F_2(x_1,x_2) := V_1^{\infty,\infty}(x_1,x_2). \tag{3.24}$$

By taking the limit as $\lambda, \lambda' \to \infty$ in (3.20) (and in (3.2) with $\phi = \lambda\delta_{x_1} + \lambda'\delta_{x_2}$) we obtain that

$$V_t^{\infty,\infty}(x_1,x_2) = \mathbb{N}_0(\{X(t,x_1) > 0\} \cup \{X(t,x_2) > 0\}) = -\log P_{\delta_0}^X(X(t,x_1) = X(t,x_2) = 0). \tag{3.25}$$

We conclude by stating a version of (3.15) for the functions $V_t^{\lambda,\lambda'}$.

Lemma 3.2. *There is a positive constant* C *such that for all* $t, \lambda, \lambda' > 0$,

$$\sup_{x_1,x_2\in\mathbb{R}} \Big[V_t^{\infty,\infty}(x_1,x_2) - V_t^{\lambda,\lambda'}(x_1,x_2)\Big] \le Ct^{-1/2-\lambda_0}\left[\lambda^{1-2\lambda_0} + \lambda'^{1-2\lambda_0}\right].$$

Proof. Let $x_1, x_2 \in \mathbb{R}$ and $t, \lambda, \lambda' > 0$. We write

$$\begin{aligned} &V_t^{\infty,\infty}(x_1,x_2) - V_t^{\lambda,\lambda'}(x_1,x_2) \\ &\quad = \Big[V_t^{\infty,\infty}(x_1,x_2) - V_t^{\lambda,\infty}(x_1,x_2)\Big] + \Big[V_t^{\lambda,\infty}(x_1,x_2) - V_t^{\lambda,\lambda'}(x_1,x_2)\Big]. \end{aligned} \tag{3.26}$$

Using (3.25) and (3.20) and taking $\lambda' \to \infty$ in the latter, it follows that the first term in the above is equal to

$$\begin{aligned}
&\mathbb{N}_0\left(1 - 1(X(t,x_1) = X(t,x_2) = 0)\right) - \mathbb{N}_0\left(1 - e^{-\lambda X(t,x_1)} 1(X(t,x_2) = 0)\right) \\
&= \mathbb{N}_0\left(1(X(t,x_2)=0)\left(e^{-\lambda X(t,x_1)} - 1(X(t,x_1)=0)\right)\right) \\
&\leq \mathbb{N}_0\left(e^{-\lambda X(t,x_1)} - 1(X(t,x_1)=0)\right) \\
&= V_t^\infty(x_1) - V_t^\lambda(x_1) \\
&\leq C t^{-1/2-\lambda_0} \lambda^{1-2\lambda_0},
\end{aligned}$$

where the second last line follows from (3.10) and (3.8), and the final inequality is by (3.15). We use similar reasoning to bound the second term of (3.26) by the same expression with λ' replacing λ, which gives the desired result. □

4 Existence and properties of L_t

As stated in the introduction, our method first establishes the existence and properties of L_t under $\mathbb{N}_0$ and then uses the cluster decomposition to establish them under $P^X_{X_0}$. The main ingredient in the proof of Theorem 1.1 is the convergence of second moments of $L_t^\lambda(\phi)$ as $\lambda \to \infty$. For a bounded Borel function ϕ, we show that $\mathbb{N}_0(L_t^\lambda(\phi)^2)$ converges as $\lambda \to \infty$. In fact, we prove convergence of second moments of general functions of two variables. For $h : \mathbb{R}^2 \to \mathbb{R}$ we recall the notation

$$(L_t^\lambda \times L_t^\lambda)(h) = \int h(x,y)\, dL_t^\lambda(x)\, dL_t^\lambda(y).$$

$L_t^\lambda(\phi)^2$ is easily recovered by taking $h(x,y) = \phi(x)\phi(y)$. The following result is the workhorse of this paper.

Theorem 1.4 (Convergence of second moments of L_t^λ)**.** *There exists a constant $C_{1.4} > 0$ and continuous function $\rho : \mathbb{R} \times \mathbb{R} \to (0,1]$ such that for bounded Borel $h : \mathbb{R}^2 \to \mathbb{R}$,*

$$\begin{aligned}
&\lim_{\lambda,\lambda' \to \infty} \mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)) \\
&\quad = C_{1.4}^2 \int_0^t (t-s)^{-2\lambda_0} \Bigg[\iint E_0^B \left(h(\sqrt{t-s}\, z_1 + B_s, \sqrt{t-s}\, z_2 + B_s) \right) \\
&\quad \times \exp\left(-\int_0^s V_{t-u}^{\infty,\infty}(\sqrt{t-s}\, z_1 + B_s - B_u, \sqrt{t-s}\, z_2 + B_s - B_u)\, du \right) \\
&\quad \times \rho(z_1,z_2)\, \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \Bigg] ds.
\end{aligned}$$

Corollary 4.1. *For a bounded Borel function ϕ, $L_t^\lambda(\phi)$ converges in $\mathcal{L}^2(\mathbb{N}_0)$ as $\lambda \to \infty$.*

Proof. Since $\mathcal{L}^2(\mathbb{N}_0)$ is complete, it is enough to show that $\{L_t^\lambda(\phi)\}_{\lambda>0}$ is Cauchy in $\mathcal{L}^2(\mathbb{N}_0)$. For $\lambda, \lambda' > 0$, we have

$$\mathbb{N}_0((L_t^\lambda(\phi) - L_t^{\lambda'}(\phi))^2) = \mathbb{N}_0((L_t^\lambda(\phi)^2) + \mathbb{N}_0((L_t^{\lambda'}(\phi)^2) - 2\mathbb{N}_0(L_t^\lambda(\phi) L_t^{\lambda'}(\phi)).$$

By Theorem 1.4, this converges to 0 as $\lambda, \lambda' \to \infty$. □

The proof of Theorem 1.4 is long and technical. We defer it to Section 5, which is devoted to its proof. For now, we assume the result and use it to establish our other main results, the first being the existence of L_t under $\mathbb{N}_0$.

Proof of Theorem 1.1 for $\mathbb{N}_0$. Fix $t > 0$. Because $X_t = 0$ implies that $L_t^\lambda = 0$ for all $\lambda > 0$, without loss of generality we can work under the finite measure $\mathbb{N}_0(\cdot \cap \{X_t > 0\})$. By Corollary 4.1, for a bounded continuous function ϕ, there exists a random variable $l(t,\phi)$ such that $L_t^\lambda(\phi) \to l(t,\phi)$ in $\mathcal{L}^2(\mathbb{N}_0)$ as $\lambda \to \infty$. It follows that $L_t^\lambda(\phi) \to l(t,\phi)$ in measure. We will now establish that there exists a unique random measure L_t such that the random variable $l(t,\phi)$ is the integral of ϕ with respect to a random measure L_t, ie. $l(t,\phi) = L_t(\phi)$ for all continuous and bounded functions ϕ.

We need to establish that the measures $\{L_t^\lambda : \lambda > 0\}$ are tight $\mathbb{N}_0$-almost surely. To see that this is true, we recall that $X(t,\cdot)$ is compactly supported $\mathbb{N}_0$-a.s., (see Corollary III.1.4 of [17] for the result under $P_{\delta_0}^X$; condition the cluster representation on $N = 1$ to get it for $\mathbb{N}_0$) and hence the mass of X_t is contained in a ball $B(0,R)$ for some $R = R(\omega) > 0$. Since $L_t^\lambda(A) = \lambda^{2\lambda_0} \int_A X(t,x) e^{-\lambda X(t,x)}\,dx$, this implies that the mass of L_t^λ is contained in $B(0,R)$ for all $\lambda > 0$, which implies that $\{L_t^\lambda(\omega) : \lambda > 0\}$ is tight.

Let $\{\phi_n\}_{n=1}^\infty$ be a countable determining class for $\mathcal{M}_F(\mathbb{R})$ consisting of bounded, continuous functions. We choose $\phi_1 = 1$. $\mathcal{L}^1$-boundedness of the total mass and tightness are sufficient conditions for a family in $\mathcal{M}_F(\mathbb{R})$ (with the weak topology) to be relatively compact. By Corollary 4.1, $\{L_t^\lambda(1) : \lambda > 0\}$ is $\mathcal{L}^2(\mathbb{N}_0)$-bounded, and hence $\mathcal{L}^1(\mathbb{N}_0)$-bounded, and so from the above we see that

$$\{L_t^\lambda : \lambda > 0\} \text{ is relatively compact } \mathbb{N}_0\text{-a.s.}$$

As we have noted, $L_t^\lambda(\phi_n) \to l(t,\phi_n)$ in measure as $\lambda \to \infty$. Using the fact that convergence in measure implies almost sure convergence along a subsequence, we can iteratively define subsequences and take a diagonal subsequence $\{\lambda_m\}_{m=1}^\infty$ which satisfies

$$L_t^{\lambda_m}(\phi_n) \to l(t,\phi_n) \text{ as } m \to \infty \text{ for all } n \geq 1 \quad \mathbb{N}_0\text{-a.s.} \tag{4.1}$$

As shown above, $\{L_t^{\lambda_m}\}_{m=1}^\infty$ is relatively compact $\mathbb{N}_0$-almost surely. Combined with (4.1), this means that for $\mathbb{N}_0$-a.a. ω we have the above convergence for all $n \geq 1$ and relative compactness of the measures $\{L_t^{\lambda_m}\}_{m=1}^\infty$. Choose such an ω. By relative compactness of $\{L_t^\lambda\}_{\lambda>0}$, any subsequence of $\{\lambda_m\}_{m=1}^\infty$ admits a further sequence along which the measures converge in the weak topology. It remains to show that all subsequential limits coincide. Suppose $L_t(\omega)$ and $L_t'(\omega)$ are two such limit measures. Since ω has been chosen so that (4.1) holds, we have that $L_t(\omega)(\phi_n) = L_t'(\omega)(\phi_n)$ for all n. Since the family $\{\phi_n\}_{n\geq1}$ are a determining class, this implies that $L_t(\omega) = L_t'(\omega)$. Hence all subsequences admit a further subsequence with the same limit $L_t(\omega)$ in the weak topology. Since the weak topology on $\mathcal{M}_F(\mathbb{R})$ is metrizable, the "every subsequence admits a further converging subsequence" criterion for convergence applies, and we have $L_t^{\lambda_m}(\omega)$ converges to $L_t(\omega) \in \mathcal{M}_F(\mathbb{R})$ as $m \to \infty$. This gives the almost sure convergence along $\{\lambda_m\}_{m=1}^\infty$.

We now check that $L_t^\lambda \to L_t$ in measure as $\lambda \to \infty$. First note that we can restrict to the finite measure $\mathbb{N}_0(\cdot \cap \{X_t > 0\})$, since $L_t^\lambda = L_t = 0$ for all $\lambda > 0$ on $\{X_t = 0\}$. Let $d(\mu,\nu)$ be a metric which metrizes the weak topology on $\mathcal{M}_F(\mathbb{R})$, e.g. the Wasserstein metric (see p. 48 of [16]). If L_t^λ did not converge to L_t in measure, then there would be a sequence $\{\bar\lambda_k\}_{k=1}^\infty$ and $\epsilon, \delta > 0$ such that $\mathbb{N}_0(\{d(L_t^{\bar\lambda_k}, L_t) > \epsilon\} \cap \{X_t > 0\}) > \delta$ for all $k \geq 1$. However, using the previous argument we can obtain a subsequence on which the measures converge to L_t $\mathbb{N}_0(\cdot \cap \{X_t > 0\})$-a.s., which, because $\mathbb{N}_0(\cdot \cap \{X_t > 0\})$ is a finite measure, contradicts the previous statement. Hence we must have that $L_t^\lambda \to L_t$ in measure.

Next, we observe that for continuous and bounded ϕ, $L_t(\phi) = l(t,\phi)$. To see this, recall that $L_t^{\lambda_m}(\phi)$ converges to $l(t,\phi)$ in $\mathcal{L}^2(\mathbb{N}_0)$. As we have just shown that $\lim_{m\to\infty} L_t^{\lambda_m}(\phi) = L_t(\phi)$ $\mathbb{N}_0$-a.s, it must hold that $L_t(\phi) = l(t,\phi)$. This implies that $L_t^\lambda(\phi) \to L_t(\phi)$ in $\mathcal{L}^2(\mathbb{N}_0)$ by Corollary 4.1.

Finally, we verify that L_t is supported on BZ_t. We fix ω outside of a null set such that $L_t^{\lambda_m} \to L_t$ in $\mathcal{M}_F(\mathbb{R})$ as $m \to \infty$. For an open set U, $L_t(U) \leq \liminf_{m\to\infty} L_t^{\lambda_m}(U)$ (a consequence of the Portmanteau theorem). From (1.6), we have $L_t^{\lambda_m}(Z_t) = 0$ for all $m \geq 1$, which implies that $L_t(\mathrm{int}(Z_t)) = 0$. Moreover, $X(t,x) > 0$ implies that $\lambda_m^{2\lambda_0} X(t,x) e^{-\lambda_m X(t,x)} \to 0$ as $m \to \infty$, so for $\epsilon > 0$, $L_t(\{x : X(t,x) > \epsilon\}) = 0$, and hence $L_t(Z_t^c) = 0$. Since $L_t(\mathrm{int}(Z_t) \cup Z_t^c) = 0$, we must have $\mathrm{supp}(L_t) \subseteq BZ_t$. □

Proof of Theorem 1.5. To prove (b), by Theorem 1.4 it is enough to show that $\mathbb{N}_0((L_t \times L_t)(h)) = \lim_{n\to\infty} \mathbb{N}_0((L_t^{\lambda_n} \times L_t^{\lambda_n})(h))$ for a sequence $\lambda_n \to \infty$, which we choose to be the sequence from Theorem 1.1 on which $L_t^{\lambda_n} \to L_t$ almost surely. Because $L_t = 0$ when $X_t = 0$, we can work on the probability measure $\mathbb{N}_0(\cdot \,|\, X_t > 0)$. For bounded and continuous $h : \mathbb{R}^2 \to \mathbb{R}$, $|(L_t^{\lambda_n} \times L_t^{\lambda_n})(h)| \leq \|h\|_\infty L_t^{\lambda_n}(1)^2$. By Theorem 1.1, $L_t^{\lambda_n}(1)$ converges in probability and in $\mathcal{L}^2(\mathbb{N}_0(\cdot \,|\, X_t > 0))$ to $L_t(1)$, which implies that $L_t^{\lambda_n}(1)^2$ and hence $(L_t^{\lambda_n} \times L_t^{\lambda_n})(h)$ are uniformly integrable (see, e.g. Theorem 4.6.3 of [4]). We can therefore exchange limit and expectation, giving

$$\mathbb{N}_0(\lim_{n\to\infty} (L_t^{\lambda_n} \times L_t^{\lambda_n})(h)) = \lim_{n\to\infty} \mathbb{N}_0((L_t^{\lambda_n} \times L_t^{\lambda_n})(h)).$$

Since $L_t^{\lambda_n} \to L_t$ in $\mathcal{M}_F(\mathbb{R})$ and h is bounded and continuous, the integrand on the left hand side is equal to $(L_t \times L_t)(h)$, which gives the result. By a Monotone Class Theorem (e.g. Corollary 4.4 in the Appendix of Ethier and Kurtz [5]), the same holds for all bounded and measurable h.

We now turn to part (a). Let $\phi : \mathbb{R} \to \mathbb{R}$ be bounded and Borel. We recall from the Introduction (see (1.5)) that Proposition 4.5 of [13] states that

$$\begin{aligned} &\lim_{\lambda\to\infty} t^{\lambda_0} E_{X_0}^X(L_t^\lambda(\phi)) \\ &= C_{1.4} \iint \phi(x_0 + \sqrt{t}z) \exp\left(-\frac{1}{t}\int F(z + t^{-1/2}(x_0 - y_0)\, dX_0(y_0)\right) \psi_0(z)\, dm(z)\, dX_0(x_0). \end{aligned} \tag{4.2}$$

(The fact that the constant appearing in Proposition 4.5 of [13] equals $C_{1.4}$ is implicit in the proof.) The proof uses the Palm measure formula for X_t under $P_{X_0}^X$; see Theorem 4.1.3 of Dawson-Perkins [3]. The corresponding Palm measure formula for the superprocess under $\mathbb{N}_0$ is in fact simpler, and the same proof shows that

$$\lim_{\lambda\to\infty} \mathbb{N}_0(L_t^\lambda(\phi)) = C_{1.4}\, t^{-\lambda_0} \int \phi(\sqrt{t}z)\, \psi_0(z)\, dm(z). \tag{4.3}$$

Consider now a bounded and continuous function ϕ; we can also clearly assume that $\phi \geq 0$. By Theorem 1.1 (under $\mathbb{N}_0$), $L_t^\lambda(\phi)$ converges in $\mathcal{L}^2$ with respect to the probability measure $\mathbb{N}_0(X_t \in \cdot \,|\, X_t > 0)$, which implies that it also converges in $\mathcal{L}^1$, allowing us to exchange limit and expectation in (4.3), which gives part (a) for bounded and continuous ϕ. This extends to all bounded and measurable ϕ by a monotone class argument (as above for part (b)). Finally, it is clear that both (a) and (b) hold for general non-negative functions by the Monotone Convergence Theorem. □

We now describe how to ascertain the existence of L_t when X_t is a super-Brownian motion under $P_{X_0}^X$ via the cluster representation. In particular, we recall (1.15) and (1.16). Let $X_0 \in \mathcal{M}_F(\mathbb{R})$ and $t > 0$.

Proof of Theorem 1.1 for $P_{X_0}^X$. Let $N, x_1, \ldots, x_N, X_t^1, \ldots X_t^N$ be as in the cluster decomposition (1.16). For $\lambda > 0$, define the measure L_t^λ via (1.6) using X_t. For $i = 1, \ldots, N$, let $L_t^{i,\lambda}$ denote the measure defined in (1.6) corresponding to X_t^i. By Theorem 1.1 for $\mathbb{N}_0$

and translation invariance, $\mathbb{N}_{x_i}(X_t^i \in \cdot \,|\, X_t^i > 0)$-a.s. there exists L_t^i such that $L_t^{i,\lambda} \to L_t$ in $\mathcal{M}_F(\mathbb{R})$ in measure. Define $L_t \in \mathcal{M}_F(\mathbb{R})$ by (1.17). That is,

$$dL_t(x) = \sum_{i=1}^N 1\big(\sum_{j\neq i} X^j(t,x) = 0\big)\, dL_t^i(x).$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. We will show that

$$L_t^\lambda(\phi) \to L_t(\phi) \text{ in probability as } \lambda \to \infty. \tag{4.4}$$

Once we establish (4.4), the proof of Theorem 1.1 for $\mathbb{N}_0$ applies and shows that $L_t^\lambda \to L_t$ in probability in $\mathcal{M}_F(\mathbb{R})$ as $\lambda \to \infty$. With the exception of $\mathcal{L}^2$ convergence, which we show afterward, this proves Theorem 1.1 for $P_{X_0}^X$.

Turning to (4.4), we will argue conditionally on $(N, x_1, \dots, x_N)$. That is, we argue under the regular conditional distribution for $(X_t^1, \dots, X_t^N)$ given $(N, x_1, \dots, x_N)$. As such, we treat $N \geq 1$ and $x_1, \dots, x_N \in \mathbb{R}$ as fixed, and $X_t^1, \dots, X_t^N$ are independent random measures with respective laws $\mathbb{N}_{x_i}(X_t \in \cdot \,|\, X_t > 0)$ for $i = 1, \dots, N$. Let E denote the expectation of a probability realizing this conditional representation for X_t. Expanding $L_t^\lambda(\phi)$ in terms of the clusters, we have

$$\begin{aligned} L_t^\lambda(\phi) &= \int \lambda^{2\lambda_0} X(t,x) e^{-\lambda X(t,x)}\, \phi(x)\, dx \\ &= \int \lambda^{2\lambda_0} \left[\sum_{i=1}^N X^i(t,x)\right] e^{-\lambda \sum_{i=1}^N X^i(t,x)} \phi(x)\, dx \\ &= \sum_{i=1}^N \int \lambda^{2\lambda_0} X^i(t,x) e^{-\lambda X^i(t,x)} \left[e^{-\lambda \sum_{j\neq i} X^j(t,x)} \phi(x)\right] dx \\ &= \sum_{i=1}^N L_t^{i,\lambda}\big(\phi \cdot e^{-\lambda Z_N^i(t,\cdot)}\big), \end{aligned} \tag{4.5}$$

where we define $Z_N^i(t,x) = \sum_{j\neq i} X^j(t,x)$, in which the indices are understood to sum from 1 to N. Using this notation, $L_t(\phi) = \sum_{i=1}^N L_t^i(\phi \cdot 1(Z_N^i(t,\cdot) = 0))$. Thus by (4.5), to prove (4.4) it is clearly enough to show that for any $1 \leq i \leq N$,

$$L_t^{i,\lambda}\big(\phi \cdot e^{-\lambda Z_N^i(t,\cdot)}\big) \to L_t^i(\phi \cdot 1(Z_N^i(t,\cdot) = 0)) \text{ in probability as } \lambda \to \infty. \tag{4.6}$$

Without loss of generality, assume that $\lambda > 1$. Let $1 \leq \lambda' \leq \lambda$. Then

$$\begin{aligned} &|L_t^{i,\lambda}(\phi \cdot e^{-\lambda Z_N^i(t,\cdot)}) - L_t^i(\phi \cdot 1\big(Z_N^i(t,\cdot) = 0))| \\ &\quad \leq |L_t^{i,\lambda}(\phi \cdot (e^{-\lambda Z_N^i(t,\cdot)} - e^{-\lambda' Z_N^i(t,\cdot)}))| + |L_t^{i,\lambda}(\phi \cdot e^{-\lambda' Z_N^i(t,\cdot)}) - L_t^i(\phi \cdot e^{-\lambda' Z_N^i(t,\cdot)})| \\ &\qquad\quad + |L_t^i(\phi \cdot (e^{-\lambda' Z_N^i(t,\cdot)} - 1(Z_N^i(t,\cdot) = 0)))| \\ &\quad \leq \|\phi\|_\infty |L_t^{i,\lambda}(e^{-\lambda' Z_N^i(t,\cdot)} 1(Z_N^i(t,\cdot) > 0))| + |L_t^{i,\lambda}(\phi \cdot e^{-\lambda' Z_N^i(t,\cdot)}) - L_t^i(\phi \cdot e^{-\lambda' Z_N^i(t,\cdot)})| \\ &\qquad\quad + \|\phi\|_\infty L_t^i(e^{-\lambda' Z_N^i(t,\cdot)} 1(Z_N^i(t,\cdot) > 0)) \\ &\quad =: \|\phi\|_\infty R_1(\lambda', \lambda) + R_2(\phi, \lambda') + \|\phi\|_\infty R_3(\lambda', \lambda). \end{aligned} \tag{4.7}$$

We first consider R_1. Since X_t^i and $Z_N^i(t,\cdot)$ are independent and L_t^λ is a measurable function of X_t^i, conditional on X_t^i we have, for all $\lambda > 1$ and $1 \le \lambda' < \lambda$,

$$\begin{aligned}
E(R_1(\lambda',\lambda)\,|\,X_t^i) &= \int E(e^{-\lambda' Z_N^i(t,x)}1(Z_N^i(t,x)>0))\,|\,X_t^i)\,dL_t^{i,\lambda}(x)\\
&= \int E(e^{-\lambda'\sum_{j\neq i}X^j(t,x)}1(\sum_{j\neq i}X^j(t,x)>0))\,dL_t^{i,\lambda}(x)\\
&\le \sum_{j\neq i}\int \mathbb{N}_{x_j}(e^{-\lambda' X^j(t,x)}1(X^j(t,x)>0)\,|\,X_t^j>0)\,dL_t^{i,\lambda}(x)\\
&= \sum_{j\neq i}\mathbb{N}_{x_j}(X_t^j>0)^{-1}\int \mathbb{N}_{x_j}(e^{-\lambda' X^j(t,x)} - 1(X^j(t,x)=0))\,dL_t^{i,\lambda}(x)\\
&= (t/2)\sum_{j\neq i}\int \mathbb{N}_{x_j}(1-1(X^j(t,x)=0)) - \mathbb{N}_{x_j}(1-e^{-\lambda' X^j(t,x)})\,dL_t^{i,\lambda}(x)\\
&= (t/2)\sum_{j\neq i}\int V_t^{\infty}(x-x_j) - V_t^{\lambda'}(x-x_j)\,dL_t^{i,\lambda}(x),
\end{aligned} \tag{4.8}$$

where in the second last line we have used (3.11), and the last follows from (3.8), (3.10), and translation invariance. We apply (3.15) to the integrand and take the expectation of the above to obtain that

$$\begin{aligned}
E(R_1(\lambda',\lambda)) &\le C\frac{N-1}{2}t^{1/2-\lambda_0}\lambda'^{-(2\lambda_0-1)}\mathbb{N}_{x_i}(L_t^{i,\lambda}(1)\,|\,X_t^i>0)\\
&= C\frac{N-1}{4}t^{3/2-\lambda_0}\lambda'^{-(2\lambda_0-1)}\mathbb{N}_{x_i}(L_t^{i,\lambda}(1)) \qquad \text{(by (3.11))}\\
&\le C(t,N)\lambda'^{-(2\lambda_0-1)},
\end{aligned} \tag{4.9}$$

for all $\lambda > 1$ and $1 \le \lambda' < \lambda$, where the last inequality is by Theorem 1.5(a) and the fact that $L_t^{i,\lambda}(1) \to L_t^i(1)$ in $\mathcal{L}^2(\mathbb{N}_{x_i})$ (from Theorem 1.1). Next we consider R_3. Note that we can expand and bound this term in exactly the same way as we did R_1 in (4.7) but with L_t^i replacing $L_t^{i,\lambda}$. Taking the expectation and proceeding as above then gives

$$E(R_3) \le \frac{N-1}{4}t^{3/2-\lambda_0}\mathbb{N}_{x_i}(L_t^i(1))\lambda'^{-(2\lambda_0-1)}. \tag{4.10}$$

Fix $\delta > 0$. By (4.9) and (4.10) and Markov's inequality there exists $\bar{\lambda}(\delta)$ such that for $\lambda' \ge \bar{\lambda}(\delta)$,

$$P(R_1(\lambda',\lambda) > \delta) + P(R_3(\lambda',\lambda) > \delta) < C'(t,N)\lambda'^{-(2\lambda_0-1)}/\delta. \tag{4.11}$$

Now consider $R_2(\phi)$. Since $\phi \cdot e^{-\lambda' Z_N^i(t,\cdot)}$ is a bounded, continuous function for all $\lambda' \ge 1$, by Theorem 1.1 for $\mathbb{N}_{x_i}$, $R_2(\phi,\lambda') \to 0$ in probability as $\lambda \to \infty$ for all $\lambda' \ge 1$. From this and (4.11) we conclude, by choosing $\lambda' \le \lambda$ sufficiently large, that (4.7) converges to 0 in probability as $\lambda \to \infty$. As we noted in (4.6), this is sufficient to prove the result.

It remains to show that $L_t^\lambda(\phi) \to L_t(\phi)$ in $\mathcal{L}^2(P_{X_0}^X)$ for all continuous and bounded functions ϕ. Let ϕ be such a function, and suppose that X_t is realized as in (1.16) under a probability $P_{X_0}^X$. Under $P_{X_0}^X(\cdot\,|\,N)$, from (4.5) and (1.17) we have

$$\begin{aligned}
(L_t^\lambda(\phi) - L_t(\phi))^2 &= \left(\sum_{i=1}^N L_t^{i,\lambda}(e^{-\lambda Z_N^i(t,\cdot)}\cdot\phi) - L_t^i(1(Z_N^i(t,\cdot)=0)\cdot\phi)\right)^2\\
&\le N\sum_{i=1}^N \left[L_t^{i,\lambda}(e^{-\lambda Z_N^i(t,\cdot)}\cdot\phi) - L_t^i(1(Z_N^i(t,\cdot)=0)\cdot\phi)\right]^2.
\end{aligned} \tag{4.12}$$

We recall that $X_t^1,\dots,X_t^N$ are iid with distribution $\mathbb{N}_{\bar{X}_0}(X_t \in \cdot\,|\,X_t > 0)$, where $\bar{X}_0 = X_0(\cdot)/X_0(1)$ and $\mathbb{N}_{X_0}(\cdot) = \int \mathbb{N}_x(\cdot)dX_0(x)$. This implies that the N summands in (4.12) are identically distributed; in particular, conditional on N we define identically distributed random variables $e_i^{N,\lambda} \geq 0$, for $i = 1, \dots, N$, by

$$e_i^{N,\lambda} = \left[L_t^{i,\lambda}(e^{-\lambda Z_N^i(t,\cdot)} \cdot \phi) - L_t^i(1(Z_N^i(t,\cdot) = 0) \cdot \phi)\right]^2. \tag{4.13}$$

By (4.6), $e_i^{N,\lambda}$ converges to 0 in probability as $\lambda \to \infty$ when conditioned on $(x_1, \dots, x_N)$. However, one can integrate the conditional probabilities over $(x_1, \dots, x_N) \in \mathbb{R}^N$ to determine that

$$e_i^{N,\lambda} \to 0 \text{ in probability under } P_{X_0}^X(\cdot \,|\, N) \text{ as } \lambda \to \infty. \tag{4.14}$$

It is clear from (4.13) that for all $\lambda > 0$,

$$e_i^{N,\lambda} \leq 2\|\phi\|_\infty^2 (L_t^{i,\lambda}(1)^2 + L_t^i(1)^2) \quad \forall\, i = 1, \dots, N, \forall\, N \geq 1. \tag{4.15}$$

By Theorem 1.1 for $\mathbb{N}_0$, $L_t^{i,\lambda}(1)^2 \to L_t^i(1)^2$ in probability under $\mathbb{N}_{\bar{X}_0}(X_t \in \cdot \,|\, X_t > 0)$ and hence under $P_{X_0}^X(\cdot \,|\, N)$. Furthermore, since $L_t^{i,\lambda}(1) \to L_t^i(1)$ in $\mathcal{L}^2(\mathbb{N}_{\bar{X}_0}(\cdot \,|\, X_t > 0))$ (by Theorem 1.1 for $\mathbb{N}_0$), it follows from Cauchy-Schwarz that $L_t^{i,\lambda}(1)^2 \to L_t^i(1)^2$ in $\mathcal{L}^1(\mathbb{N}_{\bar{X}_0}(\cdot \,|\, X_t > 0))$; since X_t^i has distribution $\mathbb{N}_{\bar{X}_0}(X_t \in \cdot \,|\, X_t > 0)$ under $P_{X_0}^X(\cdot \,|\, N)$, this implies $L_t^{i,\lambda}(1)^2 \to L_t^i(1)^2$ in $\mathcal{L}^1(P_{X_0}^X(\cdot \,|\, N))$. Hence $\{2\|\phi\|_\infty (L_t^{i,\lambda}(1)^2 + L_t^i(1)^2) : \lambda \geq 1\}$ is uniformly integrable. Thus by (4.15), $\{e_i^{N,\lambda} : \lambda \geq 1\}$ is uniformly integrable, and by (4.14) we have $\mathcal{L}^1$ convergence. That is,

$$E_{X_0}^X(e_i^{N,\lambda} \,|\, N) \to 0 \text{ as } \lambda \to \infty. \tag{4.16}$$

Conditioning on $N = n$ and summing over $n \in \mathbb{N}$, by (4.12) and Fubini's Theorem we have

$$E_{X_0}^X((L_t^\lambda(\phi) - L_t(\phi))^2) \leq \sum_{n=1}^\infty P_{X_0}^X(N = n)\, n \sum_{i=1}^n E_{X_0}^X(e_i^{n,\lambda} \,|\, N = n).$$

Since $E_{X_0}^X(e_i^{N,\lambda} \,|\, N) \leq 2\|\phi\|_\infty E_{X_0}^X(L_t^{i,\lambda}(1)^2 + L_t(1)^2) \leq C(t,\phi)$ for all $\lambda \geq 1$, for some constant $C(t,\phi) > 0$ (by uniform integrability), the nth term in the above is bounded above by $C(t,\phi)\, P_{X_0}^X(N = n) n^2$. Dominated Convergence therefore allows us to exchange limit and summation in the above, which by (4.16) gives the result. □

Proof of Theorems 1.8 and 1.9. The proof of Theorem 1.8 is in fact implicit in the above proof of Theorem 1.1 for $P_{X_0}^X$. Conditionally on the number of clusters N, L_t was defined under $P_{X_0}^X$ by (1.17), so by construction it has the claimed conditional representation.

The proof of Theorem 1.9 is virtually identical to that of Theorem 1.8, except in this case we already know that L_t exists and $L_t^\lambda \to L_t$ in $\mathcal{M}_F(\mathbb{R})$. One can then decompose X_t in terms of the different contributions and show that L_t has the desired representation using the same argument as appears above, making the obvious changes between the law of super-Brownian motion and canonical measure where necessary. □

As we have commented on, the expression in Theorem 1.4, which is the same as (1.9) in Theorem 1.5(b), is finite for all bounded h, despite the appearance of non-integrability (since $\lambda_0 > 1/2$). Proposition 1.6, which we restate here for convenience, provides a useful upper bound on second moments which is our main tool for studying L_t. The bound is not difficult to obtain. Its derivation relies only on trivial upper bounds and several changes of variables. Recall that E_z^Y denotes the expectation of a standard Ornstein-Uhlenbeck process Y with $Y_0 = z$.

Proposition 1.6 (Second moment bounds under $\mathbb{N}_0$). *For a non-negative Borel function $h : \mathbb{R}^2 \to \mathbb{R}$,*

$$\mathbb{N}_0((L_t \times L_t)(h)) \le C_{1.4}^2 \int_0^t w^{-2\lambda_0} \Big[\iint E_{z_1}^Y \Big(\exp\Big(- \int_0^{\log(t/w)} F(Y_u)\, du \Big) \times h(\sqrt{t} Y_{\log(t/w)}, \sqrt{t} Y_{\log(t/w)} + \sqrt{w}(z_2 - z_1)) \Big) \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \Big] dw. \tag{1.10}$$

Moreover,

$$\mathbb{N}_0(L_t(1)^2) \le \frac{C_{1.4}^2 \theta^2}{1-\lambda_0} t^{1-2\lambda_0}. \tag{1.11}$$

Proof. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be Borel measurable and non-negative. We use the formula for $\mathbb{N}_0((L_t \times L_t)(h))$ given by (1.9). We recall that $\rho(z_1, z_2) \le 1$ and use this bound, and we bound above by using $V_u^{\infty,\infty}(x, y) \ge V_u^\infty(x)$ in the exponential. This gives

$$\begin{aligned} &\mathbb{N}_0((L_t \times L_t)(h)) \\ &\quad \le C_{1.4}^2 \int_0^t (t-s)^{-2\lambda_0} \Big[\iint E_0^B \Big(\exp\Big(- \int_0^s V_{t-u}^\infty(\sqrt{t-s}\, z_1 + B_s - B_u)\, du \Big) \Big) \\ &\qquad \times h(\sqrt{t-s}\, z_1 + B_s, \sqrt{t-s}\, z_2 + B_s)\, \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \Big] ds. \end{aligned}$$

Since $z_1 \sim m$, $\sqrt{t-s}\, z_1$ has a normal distribution with variance $t - s$, and we interpret it as the Brownian increment $B_t - B_s$. Hence the above is equal to

$$\begin{aligned} &C_{1.4}^2 \int_0^t (t-s)^{-2\lambda_0} \Big[\int E_0^B \Big(\exp\Big(- \int_0^s V_{t-u}^\infty(B_t - B_u)\, du \Big) \times h(B_t, \sqrt{t-s}\, z_2 + B_s) \\ &\qquad \psi_0\Big(\frac{B_t - B_s}{\sqrt{t-s}} \Big) \Big) \psi_0(z_2)\, dm(z_2) \Big] ds \\ &= C_{1.4}^2 \int_0^t w^{-2\lambda_0} \Big[\int E_0^W \Big(\exp\Big(- \int_w^t V_u^\infty(W_u)\, du \Big) h(W_t, \sqrt{w}\, z_2 + W_t - W_w) \\ &\qquad \times \psi_0\Big(\frac{W_w}{\sqrt{w}} \Big)\, \psi_0(z_2)\, dm(z_2) \Big) \Big] dw, \end{aligned}$$

where in the second line we have used $w = t - s$ and defined $W_u = B_t - B_{t-u}$. Hence W_u is a standard Brownian motion under P_0^W. Recall that $V_u^\infty(x) = u^{-1} F(u^{-1/2} x)$. Applying this and letting $u = e^r$ in the integral, we obtain that the above is equal to

$$\begin{aligned} &C_{1.4}^2 \int_0^t w^{-2\lambda_0} \Big[\int E_0^W \Big(\exp\Big(- \int_{\log w}^{\log t} F(e^{-r/2} W_{e^r})\, dr \Big) h(W_t, \sqrt{w}\, z_2 + W_t - W_w) \\ &\qquad \times \psi_0\Big(\frac{W_w}{\sqrt{w}} \Big)\, \psi_0(z_2)\, dm(z_2) \Big) \Big] dw. \end{aligned}$$

We now define a stationary Ornstein-Uhlenbeck process Y (with stationary measure m) by $Y_r = e^{-r/2} W_{e^r}$ for $r \in \mathbb{R}$. Recall that we denote its law by E^Y. The above is therefore equal to

$$\begin{aligned} &C_{1.4}^2 \int_0^t w^{-2\lambda_0} \Big[\int E^Y \Big(\exp\Big(- \int_{\log w}^{\log t} F(Y_u)\, du \Big) h(\sqrt{t}\, Y_{\log t}, \sqrt{w}\, z_2 + \sqrt{t}\, Y_{\log t} - \sqrt{w}\, Y_{\log w}) \\ &\qquad \times \psi_0(Y_{\log w})\, \psi_0(z_2)\, dm(z_2) \Big) \Big] dw. \end{aligned}$$

By stationarity of Y, we can shift time by $\log w$ in the above to obtain

$$C_{1.4}^2 \int_0^t w^{-2\lambda_0} \Big[\int E^Y \Big(\exp\Big(- \int_0^{\log(t/w)} F(Y_u)\, du \Big) \\ h(\sqrt{t}\, Y_{\log(t/w)}, \sqrt{w}\, z_2 + \sqrt{t}\, Y_{\log(t/w)} - \sqrt{w}\, Y_0) \times \psi_0(Y_0)\, \psi_0(z_2)\, dm(z_2) \Big) \Big] dw.$$

Y_0 has distribution m, so we condition on the value of Y_0 and call it z_1. This gives the desired expression and proves that (1.10) holds. The proof of (1.11) is a consequence of the following lemma.

Lemma 4.2. *For* $t > 0$*,*

$$\int P_z^Y(\rho^F > t)\, \psi_0(z)\, dm(z) = \theta e^{-\lambda_0 t}.$$

Returning to (1.11), we apply (1.10) with $h = 1$. Separating the integrals, we obtain that

$$\mathbb{N}_0(L_t(1)^2) \le C_{1.4}^2\, \theta \int_0^t w^{-2\lambda_0} \Big(\int P_z^Y\big(\rho^F > \log(t/w)\big)\, \psi_0(z)\, dm(z) \Big) dw,$$

where we have used $\int \psi_0 dm = \theta$. The inequality (1.11) now readily follows from Lemma 4.2, which completes the proof of Proposition 1.6. ☐

Proof of Lemma 4.2. Expanding in terms of the transition densities, we have

$$\int P_z^Y\big(\rho^F > t\big)\, \psi_0(z)\, dm(z) = \int \Big(\int q_t(z,y)\, dm(y) \Big)\, \psi_0(z)\, dm(z) \\ = \langle q_t, 1 \otimes \psi_0 \rangle_{\mathcal{L}^2(m\times m)}, \tag{4.17}$$

where $\langle \cdot\,, \cdot \rangle_{\mathcal{L}^2(m\times m)}$ denotes the inner product on $\mathcal{L}^2(m \times m)$ and $\otimes$ is the tensor product of functions. Recall from that Theorem 2.1(a) that the eigenfunction expansion (2.1) converges in $\mathcal{L}^2(m\times m)$ to $q_t(\cdot,\cdot)$, and that $\|\psi_0\|_{\mathcal{L}^2(m)} = 1$. Thus by the above and Fubini's theorem, (4.17) is equal to

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} \langle \psi_n \otimes \psi_n, 1 \otimes \psi_0 \rangle_{\mathcal{L}^2(m\times m)} = e^{-\lambda_0 t} \langle \psi_0 \otimes \psi_0, 1 \otimes \psi_0 \rangle_{\mathcal{L}^2(m\times m)} \\ = e^{-\lambda_0 t} \int \psi_0^2\, dm \int \psi_0\, dm = \theta e^{-\lambda_0 t},$$

where the first equality follows from orthogonality of the eigenfunctions, which implies that $\int \psi_n \psi_0\, dm = 0$ for all $n \ge 1$. The last line uses $\int \psi_0\, dm = \theta$ and $\int \psi_0^2\, dm = 1$. ☐

We now use the bounds in Proposition 1.6 to derive the remaining properties of L_t and their consequences. In order of presentation, we now prove Theorem 1.3, Theorem 1.7, and Theorem 1.2.

Proof of Theorem 1.3. Recall that for $p > 0$, $h_p(x,y) = |x-y|^{-p}$. We first establish that

$$\mathbb{N}_0((L_t \times L_t)(h_p)) < \infty \tag{4.18}$$

for all $p < 2 - 2\lambda_0$. Applying (1.10) with h_p, we have

$$\begin{aligned}\mathbb{N}_0((L_t \times L_t)(h_p)) \leq &C_{1.4}^2 \int_0^t w^{-2\lambda_0} \bigg[\iint E_{z_1}^Y \bigg(\exp \bigg(- \int_0^{\log(t/w)} F(Y_u)\, du \bigg) \bigg) \\ &\times |\sqrt{w}(z_2 - z_1)|^{-p}\, \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \bigg] dw \\ = &C_{1.4}^2 \int_0^t w^{-2\lambda_0 - p/2} \bigg[\iint E_{z_1}^Y \bigg(\exp \bigg(- \int_0^{\log(t/w)} F(Y_u)\, du \bigg) \bigg) \\ &\times |z_1 - z_1|^{-p} \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \bigg] dw.\end{aligned}$$

Recalling (1.4), the expectation is equal to the survival probability $P_{z_1}^Y(\rho^F > \log(t/w))$, so the above equals

$$C_{1.4}^2 \int_0^t w^{-2\lambda_0 - p/2} \bigg[\iint P_{z_1}^Y\big(\rho^F > \log(t/w)\big) \big|z_1 - z_2\big|^{-p}\, \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2) \bigg] dw.$$

Applying (2.5) and (2.8), both with $\delta = 1/8$, this is bounded above by

$$\begin{aligned}&C \int_0^t w^{-2\lambda_0 - p/2} \bigg[\iint \big|z_1 - z_2\big|^{-p}\, t^{-\lambda_0}\, w^{\lambda_0}\, e^{z_1^2/4}\, e^{z_2^2/8}\, dm(z_1)\, dm(z_2) \bigg] dw \\ &= C(p)\, t^{-\lambda_0} \int_0^t w^{-\lambda_0 - p/2} dw.\end{aligned}$$

The second line follows because the integrand has Gaussian tails in z_1 and z_2 and $p < 2 - 2\lambda_0 < 1$. Finally, the integral in the final line is finite because $-\lambda_0 - p/2 > -\lambda_0 - \lambda_0 + 1 > -1$, which proves (4.18). In fact, we have shown that

$$\mathbb{N}_0((L_t \times L_t)(h_p)) \leq C(p) t^{1 - 2\lambda_0 - p/2}. \tag{4.19}$$

Next, we establish the same under $P_{X_0}^X$. That is, we will show that

$$E_{X_0}^X((L_t \times L_t)(h_p)) < \infty \tag{4.20}$$

for $p < 2 - 2\lambda_0$. We use the cluster decomposition and argue conditionally as in the proof of Theorem 1.1 (for $P_{X_0}^X$) above. Suppose that $P_{X_0}^X$ is a probability under which X_t is realized as in (1.16). Conditioning on $N, x_1, \ldots, x_N$, by (1.17) we have

$$dL_t(x) \leq \sum_{i=1}^N dL_t^i(x).$$

Thus we obtain that

$$\begin{aligned}&\iint |x - y|^{-p} dL_t(x)\, dL_t(y) \\ &\leq \iint |x - y|^{-p} \bigg(\sum_{i=1}^N dL_t^i(x) \bigg) \bigg(\sum_{i=j}^N dL_t^j(y) \bigg) \\ &= \sum_{i=1}^N \iint |x - y|^{-p} dL_t^i(x)\, dL_t^i(y) + \sum_{i=1}^N \sum_{j \neq i} \iint |x - y|^{-p} dL_t^i(x)\, dL_t^j(y).\end{aligned} \tag{4.21}$$

Recall that the X_t^i are independent with distributions $\mathbb{N}_{x_i}(X_t \in \cdot \mid X_t > 0)$. By (3.11) and (4.19), we therefore have

$$\mathbb{N}_{x_i} \bigg(\iint |x - y|^{-p} dL_t^i(x)\, dL_t^i(y) \bigg| X_t^i > 0 \bigg) = C(p) t^{1 - 2\lambda_0 - p/2}\, (2/t)^{-1} =: C_1(p)\, t^{2 - 2\lambda_0 - p/2}, \tag{4.22}$$

which provides a bound for the summands in the first term of (4.21). We now consider the mixed integrals in (4.21), that is, the summands in the second term. Without loss of generality, let $i = 1$ and $j = 2$, and denote their (independent) distributions by $\mathbb{N}^1_{x_1}(X^1_t \in \cdot \mid X^1_t > 0), \mathbb{N}^2_{x_2}(X^2_t \in \cdot \mid X^2_t > 0)$. Because the integrands are non-negative, we can change the order of integration and obtain

$$\begin{aligned}&\mathbb{N}^1_{x_1} \otimes \mathbb{N}^2_{x_2}\left(\iint |x-y|^{-p} dL^1_t(x)\, dL^2_t(y) \,\middle|\, X^1_t > 0, X^2_t > 0 \right) \\ &\quad = \mathbb{N}^1_{x_1}\left(\int \mathbb{N}^2_{x_2}\left(\int |x-y|^{-p} dL^2_t(y) \,\middle|\, X^2_t > 0 \right) dL^1_t(x) \,\middle|\, X^1_t > 0 \right) \qquad (4.23)\end{aligned}$$

To compute the inner expectation we apply translation invariance and (3.11), which gives

$$\begin{aligned}&\mathbb{N}^2_{x_2}\left(\int |y-x|^{-p} dL^2_t(y) \,\middle|\, X^2_t > 0 \right) \\ &\quad = (t/2)\, \mathbb{N}_0\left(\int |y-x|^{-p} dL_t(y-x_2) \right) \\ &\quad = (t/2)\, \mathbb{N}_0\left(\int |y-x+x_2|^{-p} dL_t(y) \right) \\ &\quad = C_{1.4}(t/2) t^{-\lambda_0} \int |\sqrt{t}z - (x-x_2)|^{-p} \psi_0(z)\, dm(z),\end{aligned}$$

where the last line follows from the mean measure formula (1.8). By (2.5) with $\delta = 1/4$, we have that $\psi_0(z_2)\, dm(z_2) \le c\, e^{-z_2^2/4} dz_2$. Thus the above is bounded above by

$$\begin{aligned}&Ct^{1-\lambda_0} \int (|\sqrt{t}z - (x-x_2)|^{-p} \vee 1) e^{-z^2/4}\, dz \\ &\quad = Ct^{1-\lambda_0} \int (|w-(x-x_2)|^{-p} \vee 1) t^{-1/2} e^{-w^2/4t}\, dw \\ &\quad \le C\, t^{1-\lambda_0} t^{-1/2} \int |w-(x-x_2)|^{-p}\, 1_{|w-(x-x_2)|\le 1}\, dw + C\, t^{1-\lambda_0} \int t^{-1/2} e^{-w^2/4t}\, dw \\ &\quad = C'(p) t^{1/2-\lambda_0} + Ct^{1-\lambda_0} < \infty.\end{aligned}$$

By the above bound and another application of (1.8), (4.23) is bounded above by

$$\left[C'(p) t^{1/2-\lambda_0} + Ct^{1-\lambda_0} \right] \mathbb{N}^1_{x_1}(L^1_t(1) \mid X^1_t > 0) =: C_2(p) \left[t^{3/2-2\lambda_0} + t^{2-2\lambda_0} \right]. \qquad (4.24)$$

We note that both (4.22) and (4.24) are independent of the points $x_1, \dots, x_N$. Therefore by these bounds and (4.21) we have shown that

$$E^X_{X_0}((L_t \times L_t)(h_p) \mid N) \le C_1(p) N t^{2-2\lambda_0-p/2} + C_2(p)(N^2 - N) \left[t^{3/2-2\lambda_0} + t^{2-2\lambda_0} \right].$$

Taking the expectation above with respect to N, which we recall is Poisson with mean $2X_0(1)/t$, gives

$$E^X_{X_0}((L_t \times L_t)(h_p)) \le C_1(p) X_0(1)\, t^{1-2\lambda_0-p/2} + C_2(p) X_0(1)^2 \left[t^{-1/2-2\lambda_0} + t^{-2\lambda_0} \right] < \infty, \qquad (4.25)$$

which proves (4.20).

Under both $P^X_{X_0}$ and $\mathbb{N}_0$, we have shown that the p-energy of L_t has finite expectation, and hence L_t has finite p-energy almost surely, for all $p < 2 - 2\lambda_0$. By the energy method (see, for example, Theorem 4.27 of Mörters and Peres [12]), this implies that $\dim(BZ_t) \ge$

$2-2\lambda_0$ a.s. on $\{L_t>0\}$ under $P^X_{X_0}$ and $\mathbb{N}_0$. Combined with Theorem A, this completes the proof of Theorem 1.3 for $P^X_{X_0}$. To see that the upper bound on the dimension holds for $\mathbb{N}_0$ follows from the cluster decomposition. Consider X_t under $P^X_{\delta_0}$. In the cluster decomposition of X_t, the probability that $N=1$ is positive. Conditioning on this event, X_t is equal to X^1_t, which has law $\mathbb{N}_0(X^1_t\in\cdot\,|\,X_t>0)$. Because $\dim(BZ_t)\le 2-2\lambda_0$ a.s. on this event, we therefore have $\mathbb{N}_0\big(\{\dim(BZ_t)\le 2-2\lambda_0\}\,\big|\,X_t>0\big)=1$. In particular, this implies that $\mathbb{N}_0\big(\{\dim(BZ_t)\le 2-2\lambda_0\}\,\big|\,L_t>0\big)=1$, since $\{L_t>0\}\subseteq\{X_t>0\}$. We note that conditioning on the event $\{L_t>0\}$ is valid under both $P^X_{X_0}$ and $\mathbb{N}_0$ by Theorem 1.2(a). This completes the proof. □

Proof of Theorem 1.7. To see part (a), we note that (4.2) gives an expression for $\lim_{\lambda\to\infty}E^X_{X_0}(L^\lambda_t(\phi))$. On the subsequence $\{\lambda_n\}_{n=1}^\infty$ from Theorem 1.1, $L^{\lambda_n}_t(\phi)\to L_t(\phi)$ a.s. for bounded and continuous ϕ, so it is enough to show that $\lim_{n\to\infty}E^X_{X_0}(L^{\lambda_n}_t(\phi))=E^X_{X_0}(\lim_{n\to\infty}L^{\lambda_n}_t(\phi))$. By Theorem 1.1, $L^\lambda_t(\phi)$ converges in $\mathcal{L}^2(P^X_{X_0})$ and hence is bounded in $\mathcal{L}^2(P^X_{X_0})$. It is therefore uniformly integrable, which justifies the above exchange of limit and integration. This proves the result for bounded and continuous ϕ. We extend the moment formula to bounded measurable functions by a Monotone Class Lemma and to non-negative measurable functions by Monotone convergence.

We now prove part (b). Suppose we realize X_t under a probability $P^X_{X_0}$ such that (1.16) holds. Conditionally on N, by (1.17) we have

$$L_t(1)^2\le\left(\sum_{i=1}^N L^i_t(1)\right)^2=\sum_{i=1}^N L^i_t(1)^2+\sum_{i=1}^N\sum_{j\ne i}L^i_t(1)L^j_t(1).$$

The clusters are independent with laws $\mathbb{N}_{\bar X_0}(X^i_t\in\cdot\,|\,X^i_t>0)=(t/2)\mathbb{N}_{\bar X_0}(\{X^i_t>0,X^i_t\in\cdot\})$, the equality by (3.11). Thus, applying Theorem 1.5(a) and Proposition 1.6(b) to the above and using independence, we obtain

$$E^X_{X_0}(L_t(1)^2\,|\,N)\le CN(t/2)t^{1-2\lambda_0}+C(N^2-N)(t/2)^2t^{-2\lambda_0}. \tag{4.26}$$

As in the proof of Theorem 1.3, we take the expectation with respect to N, which has a Poisson$(2X_0(1)/t)$ distribution. This proves part (b). □

It remains to prove Theorem 1.2. We will derive part (a) below using Proposition 1.6; part (b) requires a few lemmas which we now discuss.

We say that L_t has an atom of mass $c>0$ at x if $L_t(\{x\})=c$. We decompose L_t as

$$L_t=\tilde L_t+\nu_t, \tag{4.27}$$

where $\tilde L_t$ is atomless and ν_t is strictly atomic. We begin with an elementary observation which provides an upper bound for the mass of the atoms of a measure. Let $M\in\mathbb{N}$. Let $I^n_1=[-M,-M+2^{-n}]$, and for $k=2,3\dots,2M2^n$, define the dyadic interval $I^n_k=(-M+(k-1)2^{-n},-M+k2^{-n}]$. Then $\{I^n_k:k\le 2M2^n\}$ is a partition of $[-M,M]$ into disjoint intervals of length 2^{-n}. The following lemma is elementary.

Lemma 4.3. *Fix $M\in\mathbb{N}$ and suppose that μ is a finite measure supported on $[-M,M]$ with decomposition $\mu=\rho+\nu$, where ρ is atomless and $\nu=\sum_{i\in I}c_i\delta_{x_i}$ is strictly atomic. Then for every $n\ge 1$,*

$$\sum_{k=1}^{2M2^n}\mu(I^k_n)^2\ge\sum_{i\in I}c_i^2.$$

The next lemma gives an upper bound for the second moment of L_t on a ball. We denote by $B(x,r)$ the ball of radius $r>0$ centred at $x\in\mathbb{R}$. We recall $s^*(\delta)$ from Theorem 2.1(c); in what follows we use $\delta=1/8$, and s^* denotes $s^*(1/8)$.

Lemma 4.4 (Second moments on balls)**.** *There is a constant $C_{4.4} > 0$ and t-dependent constant $C_{4.4}(t) > 0$ such that for all $x \in \mathbb{R}$ and $r < e^{-s^*}t$,*

$$\begin{aligned}\mathbb{N}_0(L_t(B(x,r))^2) &\le C_{4.4}\left[t^{-\lambda_0}r^{2-2\lambda_0}\,P_0^W(W_{4t/3}\in B(x,r)) + t^{-3\lambda_0+1/2}r\,P_0^W(W_t\in B(x,r))\right]\\ &\le C_{4.4}(t)\left[r^{3-2\lambda_0}+r^2\right],\end{aligned}$$

where W is a standard Brownian motion under P_0^W.

We delay the proof of this lemma to the end of the section and first prove Theorem 1.2.

Proof of Theorem 1.2. First consider part (a). For canonical measure, via the second moment method we have

$$\mathbb{N}_0\left(L_t(1)>0\right) \ge \frac{(\mathbb{N}_0(L_t(1)))^2}{\mathbb{N}_0(L_t(1)^2)} \ge \frac{C_{1.4}^2\theta^2 t^{-2\lambda_0}}{C_{1.4}^2\theta^2 t^{1-2\lambda_0}(1-\lambda_0)^{-1}} = \frac{1-\lambda_0}{t},$$

where we recall that $\int \psi_0\,dm = \theta$ and we have used Theorem 1.5(a) and (1.11). We recall that $\mathbb{N}_0\left(X_t>0\right) = 2/t$, which implies that $\mathbb{N}_0\left(L_t>0\,\middle|\,X_t>0\right) \ge \frac{1-\lambda_0}{2}$. This proves the result for $\mathbb{N}_0$.

To see that $P_{X_0}^X(L_t>0)>0$, we realize X_t under $P_{X_0}^X$ via a cluster decomposition. The event that the number of clusters N is exactly one has some positive probability $p>0$; restricted to this event, X_t is equal to a single canonical cluster conditioned on survival (as in the proof of Theorem 1.3), which we just showed has probability at least $\frac{1-\lambda_0}{2}$ that $L_t>0$. Hence $P_{X_0}^X(L_t>0)\ge p\frac{1-\lambda_0}{2}>0$.

We now prove part (b). First consider L_t under $\mathbb{N}_0$ and recall the decomposition (4.27), ie. $L_t = \tilde{L}_t + \nu_t$, the latter strictly atomic. Fix $M \in \mathbb{N}$ and consider the restriction of L_t to $[-M,M]$, ie. $dL_t^{(M)}(x) := 1_{[-M,M]}(x)\,dL_t(x)$, with decomposition $L_t^{(M)} = \tilde{L}_t^{(M)} + \nu_t^{(M)}$. Note that the radius of the dyadic intervals is $r(I_n^k) = r = 2^{-(n+1)}$. By Lemma 4.4, we have

$$\begin{aligned}\mathbb{N}_0\left(\sum_{k=1}^{2M2^n} L_t^{(M)}(I_n^k)^2\right) &= \sum_{k=1}^{2M2^n}\mathbb{N}_0\left(L_t^{(M)}(I_n^k)^2\right)\\ &\le C(t)\,2M2^n\left[(2^{-(n+1)})^{3-2\lambda_0}+(2^{-(n+1)})^2\right]\\ &\le\ C(t)\,2M\left[(2^{-n})^{2-2\lambda_0}+2^{-n}\right]\\ &\to 0\ \text{ as }\ n\to\infty\end{aligned}$$

because $2-2\lambda_0>0$. Moreover, by Lemma 4.3, the first expression is greater than or equal to the expectation (under $\mathbb{N}_0$) of the sum of the squares of the atoms of $L_t^{(M)}$. The above implies that this expectation must in fact be zero, so $\nu_t^{(M)} = 0$ $\mathbb{N}_0$-a.s. As this holds for all M, $\nu_t = 0$ and L_t is atomless under $\mathbb{N}_0$. To obtain the result under $P_{X_0}^X$, we note from the cluster decomposition and (1.17) that (conditionally) L_t is a sum of N measures which are atomless by the above, and hence is atomless. □

Proof of Lemma 4.4. We apply (1.10) with $h(z_1,z_2) = 1_{B(x,r)}(z_1)\,1_{B(x,r)}(z_2)$. This gives

$$\begin{aligned}&\mathbb{N}_0(L_t(B(x,r))^2)\\ &= C\int_0^t w^{-2\lambda_0}\Bigg[\iint E_{z_1}^Y\Bigg(\exp\left(-\int_0^{\log(t/w)}F(Y_u)\,du\right)\\ &\quad\times 1_{B(x,r)}(\sqrt{t}Y_{\log(t/w)})1_{B(x,r)}(\sqrt{t}Y_{\log(t/w)}+\sqrt{w}(z_2-z_1))\Bigg)\psi_0(z_1)\psi_0(z_2)dm(z_1)dm(z_2)\Bigg]dw.\end{aligned} \tag{4.28}$$

We now divide the above into two cases depending on the size of w. We first consider the singular case, where w is small.

Case 1: $w < e^{-s^*}t$**.**

We interpret the exponential in (4.28) as the probability that Y survives until time $\log(t/w)$ when it is subject to Markovian killing with rate $F(Y_u)$. Because this probability is equal to the integral of the transition density over all of $\mathbb{R}$, the portion of the integral corresponding to $w \in [0, e^{-s^*}t]$ equals

$$\begin{aligned} C &\int_0^{e^{-s^*}t} w^{-2\lambda_0} \bigg[\iiint q_{\log(t/w)}(z_1, y)\, 1_{B(x,r)}(\sqrt{t}y)\, 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}(z_2 - z_1)) \\ &\qquad \times \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1)\, dm(z_2)\, dm(y) \bigg] dw. \\ &\le C \int_0^{e^{-s^*}t} w^{-2\lambda_0} \bigg[\iiint e^{-\lambda_0 \log(t/w)}\, e^{z_1^2/8} e^{y^2/8}\, 1_{B(x,r)}(\sqrt{t}y)\, 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}(z_2 - z_1)) \\ &\qquad \times\ \psi_0(z_1)\, \psi_0(z_2)\, dm(z_1) dm(z_2)\, dm(y) \bigg] dw. \\ &\le C t^{-\lambda_0} \int_0^{e^{-s^*}t} w^{-\lambda_0} \int e^{y^2/8} 1_{B(x,r)}(\sqrt{t}y) \\ &\qquad \times \bigg[\iint e^{z_1^2/4} e^{z_2^2/8}\, 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}(z_2 - z_1))\, dm(z_1)\, dm(z_2) \bigg] dm(y)\, dw. \end{aligned} \tag{4.29}$$

The first inequality uses (2.2) with $\delta = 1/8$, which applies because $\log(t/w) > s^*$ for all w in the above integral, and the second uses (2.5), both with $\delta = 1/8$. In the integral in the last line we collect all the Gaussian terms. The square-bracketed term is equal to

$$\begin{aligned} C &\iint 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}(z_2 - z_1))\, e^{-z_1^2/4} e^{-3z_2^2/8}\, dz_1\, dz_2 \\ &= C' \int 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}z)\, e^{-3z^2/20}\, dz. \end{aligned}$$

We have used the convolution property for independent Gaussians. We define Gaussian random variables $g_1 \sim \mathcal{N}(0, 4t/3)$ and $g_2 \sim \mathcal{N}(0, 10/3)$. Substituting the last expression into (4.29), we obtain

$$\begin{aligned} C t^{-\lambda_0} &\int_0^{e^{-s^*}t} w^{-\lambda_0} \bigg[\iint 1_{B(x,r)}(\sqrt{t}y)\, 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}z)\, e^{-3z^2/20}\, e^{-3y^2/8} dz\, dy \bigg] dw \\ &= C' t^{-\lambda_0} \int_0^{e^{-s^*}t} w^{-\lambda_0} \Big[P\big(g_1 \in B(x,r), g_1 + \sqrt{w} g_2 \in B(x,r)\big) \Big] dw \\ &\le C t^{-\lambda_0} \int_0^{e^{-s^*}t} w^{-\lambda_0} \Big[P\big(g_1 \in B(x,r)\big)\, P\big(\sqrt{w} g_2 \in B(0, 2r)\big) \Big] dw \\ &= C t^{-\lambda_0} P\big(g_1 \in B(x,r)\big) \int_0^{e^{-s^*}t} w^{-\lambda_0} P\big(g_2 \in B(0, 2r w^{-1/2})\big) dw. \end{aligned} \tag{4.30}$$

Suppose that $4r^2 < e^{-s^*}t$. If $2rw^{-1/2} > 1$, we bound the probability in the integral above by 1. If $2rw^{-1/2} \le 1$, the probability is simply bounded by the diameter of the ball,

$4rw^{-1/2}$. Thus (4.30), and hence (4.29), is bounded above by

$$\begin{aligned} & Ct^{-\lambda_0} P(g_1 \in B(x,r)) \left[\int_0^{4r^2} w^{-\lambda_0} dw + 4r \int_{4r^2}^{e^{-s^*}t} w^{-\lambda_0 - 1/2} dw \right] \\ & \quad = Ct^{-\lambda_0} P(g_1 \in B(x,r)) \left[4^{1-\lambda_0} \frac{r^{2-2\lambda_0}}{1-\lambda_0} + \frac{4r}{\lambda_0 - 1/2} \left((4r^2)^{-(\lambda_0 - 1/2)} - (te^{-s^*})^{-(\lambda_0 - 1/2)} \right) \right] \\ & \quad \le Ct^{-\lambda_0} P(g_1 \in B(x,r))\, r^{2-2\lambda_0}. \end{aligned} \tag{4.31}$$

Finally, note that if $4r^2 \ge e^{-s^*}t$, then (4.30) is bounded above by

$$\begin{aligned} Ct^{-\lambda_0} P(g_1 \in B(x,r)) \int_0^{e^{-s^*}t} w^{-\lambda_0} dw &\le Ct^{-\lambda_0} P(g_1 \in B(x,r)) (e^{-s^*}t)^{1-\lambda_0} \\ &\le Ct^{-\lambda_0} P(g_1 \in B(x,r)) r^{2-2\lambda_0}, \end{aligned}$$

so the upper bound for (4.29) obtained in (4.31) holds in this case as well.

Case 2: $w \in (e^{-s^*}t, t]$**.**

In this case we simply bound the exponential term in (4.28) above by 1, effectively ignoring the killing, in which case $Y_{\log(t/w)} \sim m$. We also use (2.5) with $\delta = \frac{1}{4}$. Hence the contribution to (4.28) from the $w \in (e^{-s^*}t, t]$ case is bounded above by

$$\begin{aligned} & C \int_{e^{-s^*}t}^{t} w^{-2\lambda_0} \Big[\iiint 1_{B(x,r)}(\sqrt{t}y)\, 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}(z_2 - z_1)) \\ & \qquad\qquad \times e^{z_1^2/4}\, e^{z_2^2/4}\, dm(z_1)\, dm(z_2)\, dm(y) \Big] dw \\ & \le C \int_{e^{-s^*}t}^{t} w^{-2\lambda_0} \Big[\iiint 1_{B(x,r)}(\sqrt{t}y)\, 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}(z_2 - z_1)) \\ & \qquad\qquad \times e^{-z_1^2/4}\, e^{-z_2^2/4}\, dz_1\, dz_2\, dm(y) \Big] dw \\ & = C \int_{e^{-s^*}t}^{t} w^{-2\lambda_0} \Big[\iint 1_{B(x,r)}(\sqrt{t}y)\, 1_{B(x,r)}(\sqrt{t}y + \sqrt{w}z)\, e^{-z^2/2}\, dz\, dm(y) \Big] dw \\ & \le Ct^{-\lambda_0} P(g_3 \in B(x,r)) \int_{e^{-s^*}t}^{t} w^{-2\lambda_0} P(g_4 \in B(0, 2rw^{-1/2}))\, dw. \end{aligned}$$

In the above, $g_3 \sim \mathcal{N}(0,t)$ and $g_4 \sim \mathcal{N}(0,1)$. The third line follows by the convolution property of Gaussians. We again bound the probability in the integral by the size diameter of the ball, which gives the following upper bound for the above:

$$\begin{aligned} & Ct^{-\lambda_0} P(g_3 \in B(x,r))\, 4r \int_{e^{-s^*}t}^{t} w^{-2\lambda_0 - 1/2}\, dw. \\ & \quad \le Ct^{-\lambda_0} P(g_3 \in B(x,r))\, 4r\, (e^{-s^*}t)^{-2\lambda_0 + 1/2} \\ & \quad = Ct^{-3\lambda_0 + 1/2} P(g_3 \in B(x,r))\, r. \end{aligned} \tag{4.32}$$

By combining (4.31) and (4.32) and interpreting the Gaussian probabilities in terms of Brownian motion, we obtain the first inequality of the result. The second bound is obtained by bounding the Brownian density above by its maximum value. □

5 Proof of Theorem 1.4

The proof of Theorem 1.4 is split up into two main parts. In the first, we obtain representations for $\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h))$ in terms of solutions to (3.4), in particular the family $V^{\lambda,\lambda'}$ introduced in Section 3. In the second part, using these representations, we establish convergence of $\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h))$ as $\lambda, \lambda' \to \infty$. The proof of a technical lemma (Lemma 5.5) is given in Section 6.

5.1 PDE representations and preliminary bounds

We begin by deriving an expression for second moments of L_t^λ under the canonical measure. In particular, we study $\mathbb{N}_0((L_t^\lambda \times, L_t^{\lambda'})(h))$ for $\lambda, \lambda' > 0$. The formula we obtain is naturally suggested by a branching particle heuristic. Its proof uses PDE methods. Let E_x^B denote the expectation of a Brownian motion started at x. $E_{(x,y)}^{B^1,B^2}$ denotes the law of two independent Brownian motions B^1 and B^2 started from points x and y respectively. We recall the definition of $V_t^{\lambda,\lambda'}$ from (3.20).

Proposition 5.1 (PDE representation for second moments)**.** *Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel function and $\lambda, \lambda', t > 0$. Then*

$$\begin{aligned}\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)) = (\lambda\lambda')^{2\lambda_0} \int_0^t E_0^B \Bigg(E_{(0,0)}^{B^1,\,B^2} \Bigg[& h(B_s + B_{t-s}^1, B_s + B_{t-s}^2) \\ & \times \exp\left(- \int_0^s V_{t-u}^{\lambda,\lambda'} (B_{t-s}^1 + B_s - B_u, B_{t-s}^2 + B_s - B_u)\, du \right) \\ & \times \exp\left(- \int_0^{t-s} V_r^{\lambda,\lambda'} (B_r^1, B_r^1 + B_{t-s}^2 - B_{t-s}^1)\, dr \right) \\ & \times \exp\left(- \int_0^{t-s} V_r^{\lambda,\lambda'} (B_r^2 + B_{t-s}^1 - B_{t-s}^2, B_r^2)\, dr \right) \Bigg] \Bigg) ds.\end{aligned}$$

The proof of Proposition 5.1 requires the following lemma.

Lemma 5.2. *Let $\varphi \in \mathcal{M}_F(\mathbb{R})$ and $\varphi_1, \varphi_2 \in \mathcal{L}^1(\mathbb{R})$ be non-negative and continuous. Then*

$$\begin{aligned}\mathbb{N}_0\left(X_t(\varphi_1) X_t(\varphi_2) e^{-X_t(\varphi)} \right) = & \int_0^t E_0^B \Bigg(\exp\left(- \int_0^s V_{t-u}^\varphi (B_u)\, ds \right) \\ & \times \prod_{i=1,2} E_0^{B^i} \left[\exp\left(- \int_0^{t-s} V_{t-s-r}^\varphi (B_s + B_r^i)\, dr \right) \varphi_i(B_s + B_{t-s}^i) \right] \Bigg) ds.\end{aligned}$$

Proof. Let $\epsilon_1, \epsilon_2 > 0$ and φ, φ_1 and φ_2 be as in the statement. Viewing φ_1 and φ_2 as the density functions of the finite measures they induce (ie. $\varphi_i(A) = \int_A \varphi_i(x)\, dx$), let $V_t^{\varphi,\epsilon_1,\epsilon_2}$ denote the solution to (3.4) when $\phi = \varphi + \epsilon_1\varphi_1 + \epsilon_2\varphi_2 \in \mathcal{M}_F(\mathbb{R})$. By (3.3) and the discussion below (3.4),

$$\mathbb{N}_0(1 - e^{-X_t(\varphi + \epsilon_1\varphi_1 + \epsilon_2\varphi_2)}) = V_t^{\varphi,\epsilon_1,\epsilon_2}(0).$$

We differentiate this expression once with respect to ϵ_1 and once with respect to ϵ_2. The derivatives of the inner expression of the left hand side are bounded above by integrable quantities (i.e. $X_t(\varphi_1)$ and $X_t(\varphi_1)X_t(\varphi_2)$) so we can take the differentiation inside the expectation in the probabilistic representation, and the derivatives of the right hand side exist. The resulting equation is the following:

$$\mathbb{N}_0\left(X_t(\varphi_1)\, X_t(\varphi_2)\, e^{-X_t(\varphi + \epsilon_1\varphi_1 + \epsilon_2\varphi_2)} \right) = -\frac{\partial^2}{\partial\epsilon_1 \partial\epsilon_2} V_t^{\varphi,\epsilon_1,\epsilon_2}(0). \tag{5.1}$$

We note that the limit of the left hand side as $\epsilon_1, \epsilon_2 \downarrow 0$ is the desired expression. We now obtain an expression for the first derivatives of $V_t^{\varphi,\epsilon_1,\epsilon_2}(0)$ with respect to ϵ_1 and ϵ_2. Consider the following partial differential equation:

$$\frac{\partial u_t}{\partial t} = \frac{\Delta}{2} u_t - V_t^{\varphi,\epsilon_1,\epsilon_2} u_t \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \qquad u_t \to \varphi_1 \;\text{ as } t \downarrow 0, \tag{5.2}$$

where the $u_t \to \varphi_1$ in the sense of weak convergence of measures. The above can be obtained heuristically by formally differentiating (3.4) with respect to ϵ_1 when the initial conditions are $\varphi + \epsilon_1\varphi_1 + \epsilon_2\varphi_2$. By Lemmas 2.3 and 2.5 of [14], (5.2) has a unique solution, which we denote by $U_t^{1,\epsilon_1,\epsilon_2}$, which satisfies

$$V_t^{\varphi,\epsilon_1,\epsilon_2}(x) = V_t^{\varphi,0,\epsilon_2}(x) + \int_0^{\epsilon_1} U_t^{1,\epsilon,\epsilon_2}(x)\,d\epsilon.$$

Thus $U_t^{1,\epsilon_1,\epsilon_2} = \frac{\partial}{\partial\epsilon_1} V_t^{\phi,\epsilon_1,\epsilon_2}$. We can apply the same argument to obtain a similar representation for $\frac{\partial}{\partial\epsilon_2} V_t^{\phi,\epsilon_1,\epsilon_2}$, which we denote by $U_t^{2,\epsilon_1,\epsilon_2}$. Both $U_t^{1,\epsilon_1,\epsilon_2}$ and $U_t^{2,\epsilon_1,\epsilon_2}$ have Feynman-Kac representations; for example, see Theorem 7.6 of Karatzas and Shreve [8] (on p. 366). For $i = 1, 2$ we have

$$U_t^{i,\epsilon_1,\epsilon_2}(x) = E_x^B\left(\varphi_i(B_t)\exp\left(-\int_0^t V_{t-s}^{\varphi,\epsilon_1,\epsilon_2}(B_s)\,ds\right)\right). \tag{5.3}$$

We take the expression for $i = 1$ and differentiate it with respect to ϵ_2. We obtain

$$\begin{aligned}
&-\frac{\partial^2}{\partial\epsilon_2\partial\epsilon_1} V_t^{\varphi,\epsilon_1,\epsilon_2}(x)\\
&\quad= E_x^B\left(\varphi_1(B_t)\exp\left(-\int_0^t V_{t-s}^{\varphi,\epsilon_1,\epsilon_2}(B_s)\,ds\right)\int_0^t U_{t-s}^{2,\epsilon_1,\epsilon_2}(B_s)\,ds\right)\\
&\quad= E_x^B\left(\varphi_1(B_t)\exp\left(-\int_0^t V_{t-s}^{\varphi,\epsilon_1,\epsilon_2}(B_s)\,ds\right)\right.\\
&\qquad\left.\times\int_0^t E_0^{B^2}\left(\varphi_2(B_s+B_{t-s}^2)\exp\left(-\int_0^{t-s} V_{t-s-r}^{\varphi,\epsilon_1,\epsilon_2}(B_s+B_r^2)\,dr\right)ds\right)\right),
\end{aligned}$$

where the final line follows from another application of (5.3), this time with $i = 2$. First we note that all the terms are non-negative, so we can take the internal integral over time outside the expectation. For $s < t$, the integrand then describes one Brownian motion started at 0 and run to time t, and a second which branches from the first at time s and evolves independently. By applying the Markov property at time s we equivalently view it as a Brownian path that branches at time s into two independent Brownian motions B^1 and B^2 which themselves run for a duration of $t - s$. This formulation combined with the independence of the Brownian motions gives us

$$\begin{aligned}
\frac{\partial^2}{\partial\epsilon_2\partial\epsilon_1} V_t^{\varphi,\epsilon_1,\epsilon_2}(x) &= -\int_0^t E_x^B\left(\exp\left(-\int_0^s V_{t-u}^{\varphi,\epsilon_1,\epsilon_2}(B_u)\,ds\right)\right.\\
&\quad\left.\times\prod_{i=1,2} E_0^{B^i}\left[\varphi_i(B_s+B_{t-s}^i)\exp\left(-\int_0^{t-s} V_{t-s-r}^{\varphi,\epsilon_1,\epsilon_2}(B_s+B_r^i)\,dr\right)\right]ds\right).
\end{aligned}$$

The derivatives in ϵ_1 and ϵ_2 are one-sided at 0 so we cannot exactly evaluate at $\epsilon_1 = \epsilon_2 = 0$. However, $V_t^{\varphi,\epsilon_1,\epsilon_2}(x)$ is continuous in ϵ_1 and ϵ_2 and the integrand is bounded above by $\|\varphi_1\|_\infty\|\varphi_2\|_\infty$ so we can apply bounded convergence. As $\epsilon_1, \epsilon_2 \downarrow 0$, $V_t^{\varphi,\epsilon_1,\epsilon_2} \to V_t^\varphi$ by Lemma 2.1(d) of [14]. We also take $\epsilon_1, \epsilon_2 \downarrow 0$ in the left hand side of (5.1) and apply Dominated Convergence. Evaluating at $x = 0$ gives the result. □

Proof of Proposition 5.1. We will prove the result for functions of product form, ie. $h(x, y) = \phi_1(x)\,\phi_2(y)$, and then use a monotone class theorem. Let $x_1, x_2 \in \mathbb{R}$ and $\lambda, \lambda' > 0$. Consider the expression from Lemma 5.2 with $\varphi = \lambda\delta_{x_1} + \lambda'\delta_{x_2}$. For now we simply let φ_1 and φ_2 be functions satisfying the assumptions of Lemma 5.2, but we will shortly choose them to be approximate identities at x_1 and x_2. Applying Lemma 5.2, we have

$$\mathbb{N}_0\left(X_t(\varphi_1)X_t(\varphi_2)e^{-\lambda X(t,x_1)-\lambda' X(t,x_2)}\right) = \int_0^t E_0^B\left(\exp\left(-\int_0^s V_{t-u}^{\lambda,\lambda'}(B_u - x_1, B_u - x_2)\,du\right)\right.$$
$$\left.\times \prod_{i=1,2} E_0^{B^i}\left[\exp\left(-\int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'}(B_s + B_r^i - x_1, B_s + B_r^i - x_2)\,dr\right)\varphi_i(B_s + B_{t-s}^i)\right]\right) ds, \tag{5.4}$$

where we have also used (3.21), ie. translation invariance of $V^{\lambda,\lambda'}(x_1,x_2)$. Now let $\varphi_i = p_\delta(\cdot - x_i)$, where we recall that $p_\delta(\cdot)$ denotes the Gaussian density of variance δ. Let ϕ_1, ϕ_2 be bounded, continuous functions and integrate $\phi_1(x_1)\phi_1(x_2)$ multiplied by the above over x_1 and x_2. The left hand side is then

$$\iint \phi_1(x_1)\,\phi_2(x_2)\,\mathbb{N}_0\left(X_t(p_\delta(\cdot - x_1))X_t(p_\delta(\cdot - x_2))e^{-\lambda X(t,x_1)-\lambda' X(t,x_2)}\right) dx_1\,dx_2. \tag{5.5}$$

The absolute value of (5.5) is bounded above by

$$\|\phi_1\|_\infty \|\phi_2\|_\infty \mathbb{N}_0\left(\int X_t(p_\delta(\cdot - x_1))dx_1 \int X_t(p_\delta(\cdot - x_2))\,dx_2\right), \tag{5.6}$$

where the change of order of integration follows because all the terms are non-negative once we bound $|\phi_i(x_i)|$ by $\|\phi_i\|_\infty$. Now we note that

$$\int X_t(p_\delta(\cdot - x_i))dx_i = \iint X(t,y)p_\delta(x_i - y)\,dy\,dx_i = \int X(t,y)\left(\int p_\delta(x_i - y)\,dx_i\right) dy$$
$$= X_t(1).$$

Combined with (5.6), this implies that the expression in (5.5) is integrable and its absolute value is bounded above by $\|\phi_1\|_\infty\|\phi_2\|_\infty \mathbb{N}_0(X_t(1)^2)$. We note that $\mathbb{N}_0(X_t(1)^2) < \infty$. To see this, first observe that $E_{\delta_0}^X(X_t(1)^2)$ is finite and in fact equals $1+t$, which follows from the martingale problem for super-Brownian motion (see Section II.5 of [17]). By the cluster decomposition (1.16), $E_{\delta_0}^X(X_t(1)^2)$ is equal to the mean of a Poisson random variable multiplied by $\mathbb{N}_0(X_t(1)^2 | X_t > 0)$, and so the latter quantity, and hence $\mathbb{N}_0(X_t(1)^2)$, are also finite. Thus we can apply Fubini and rewrite (5.5) as

$$\mathbb{N}_0\left(\iint \phi_1(x_1)\,\phi_2(x_2)\,X_t(p_\delta(\cdot - x_1))X_t(p_\delta(\cdot - x_2))e^{-\lambda X(t,x_1)-\lambda' X(t,x_2)}dx_1\,dx_2\right). \tag{5.7}$$

As noted, the absolute value of the expression inside $\mathbb{N}_0$ is bounded above by $\|\phi_1\|_\infty\|\phi_2\|_\infty X_t(1)^2$, which is integrable under $\mathbb{N}_0$, for all δ. We take $\delta \downarrow 0$ and apply Dominated Convergence to obtain that the limit of (5.7) as $\delta \downarrow 0$ is equal to

$$\mathbb{N}_0\left(\lim_{\delta\to 0^+}\iint \phi_1(x_1)\,\phi_2(x_2)X_t(p_\delta(\cdot - x_1))X_t(p_\delta(\cdot - x_2))e^{-\lambda X(t,x_1)-\lambda' X(t,x_2)}\,dx_1\,dx_2\right)$$
$$= \mathbb{N}_0\left(\lim_{\delta\to 0^+}\left(\int \phi_1(x_1)X_t(p_\delta(\cdot - x_1))e^{-\lambda X(t,x_1)}dx_1\right)\right.$$
$$\left.\times\left(\int \phi_2(x_2)X_t(p_\delta(\cdot - x_2))e^{-\lambda' X(t,x_2)}dx_2\right)\right). \tag{5.8}$$

We know that

$$X_t(p_\delta(\cdot - x_i)) = \int X(t,y)p_\delta(y - x_i)\,dy = X_t * p_\delta(x_i).$$

Moreover, $X(t,\cdot) \in C_c(\mathbb{R})$ (ie. $X(t,\cdot)$ is continuous with compact support) $\mathbb{N}_0$-a.s. and $\{p_\delta\}_{\delta>0}$ are an approximate identity family, which together with the above imply that

$X_t(p_\delta(\cdot - x_i)) \to X(t,x_i)$ as $\delta \downarrow 0$. We note that for all $\delta > 0$, $|X_t(p_\delta(\cdot - x_i)| = |X_t * p_\delta(x_i)| \le \|X(t,\cdot)\|_\infty$. Choose $K \in \mathbb{N}$ such that $\mathrm{supp}(X_t) \subseteq [-K,K]$. On the set $[-2K,2K]$ we bound $|X_t(p_\delta(\cdot - x_i)|$ above by $\|X(t,\cdot)\|_\infty$. For $|x_i| > 2K$, a short calculation shows that $|X_t(p_\delta(\cdot - x_i))| \le C(K) X_t(1)\, p_1(|x_i| - K)$ for some constant $C(K) > 0$ for all $0 < \delta \le 1/2$. Hence $|X_t(p_\delta(\cdot - x_i))|$ has an upper bound which is a bounded function with Gaussian tails, uniformly for $0 < \delta \le 1/2$. Using this bound and boundedness of ϕ_i, we can apply Dominated Convergence in (5.8), which gives that the limit of (5.5) as $\delta \downarrow 0$ equals

$$\mathbb{N}_0\bigg(\bigg(\int \phi_1(x_1)\, X(t,x)\, e^{-\lambda X(t,x_1)} dx_1\bigg)\bigg(\int \phi_2(x_2)\, X(t,x_2) e^{-\lambda' X(t,x_2)} dx_2\bigg)\bigg).$$

When rescaled by $(\lambda\lambda')^{2\lambda_0}$ this is equal to $\mathbb{N}_0(L_t^{\lambda}(\phi_1)\, L_t^{\lambda'}(\phi_2))$. We now turn our attention to the right hand side of (5.4). With $\varphi_i = p_\delta(\cdot - x_i)$, integrating against $\phi(x_1)\phi(x_2)dx_1dx_2$, we have

$$\begin{aligned}
&\iint \phi_1(x_1)\, \phi_2(x_2)\bigg(\int_0^t E_0^B\bigg(E_{(0,0)}^{B^1,B^2}\bigg[\exp\bigg(-\int_0^s V_{t-u}^{\lambda,\lambda'}(B_u - x_1, B_u - x_2) du\bigg) \\
&\times \exp\bigg(-\int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'}(B_s + B_r^1 - x_1, B_s + B_r^1 - x_2) dr\bigg) p_\delta(B_s + B_{t-s}^1 - x_1) \\
&\times \exp\bigg(-\int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'}(B_s + B_r^2 - x_1, B_s + B_r^2 - x_2) dr\bigg) p_\delta(B_s + B_{t-s}^2 - x_2)\bigg]\bigg) ds\bigg) dx_1 dx_2.
\end{aligned}$$

Since the above is equal to (5.5), which we have shown is integrable, we can take the spatial integrals inside the expectations. At this point we note that we are integrating a bounded function of x_1 and x_2 with respect to the densities $p_\delta(B_s + B_{t-s}^1 - x_i)$, which, because p_δ is the kernel of the Brownian semigroup, is the same as viewing x_i as $B_s + B_{t-s+\delta}^i$. Hence the above is equal to

$$\begin{aligned}
&\int_0^t E_0^B\bigg(E_{(0,0)}^{B^1,B^2}\bigg[\phi_1(B_s + B_{t-s+\delta}^1)\, \phi_2(B_s + B_{t-s+\delta}^2) \\
&\times \exp\bigg(-\int_0^s V_{t-u}^{\lambda,\lambda'}(B_u - B_s - B_{t-s+\delta}^1, B_u - B_s - B_{t-s+\delta}^2)\, du\bigg) \\
&\times \exp\bigg(-\int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'}(B_r^1 - B_{t-s+\delta}^1, B_r^1 - B_{t-s+\delta}^2)\, dr\bigg) \\
&\times \exp\bigg(-\int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'}(B_r^2 - B_{t-s+\delta}^1, B_r^2 - B_{t-s+\delta}^2)\, dr\bigg)\bigg]\bigg) ds. \qquad (5.9)
\end{aligned}$$

Taking $\delta \downarrow 0$ and applying Dominated Convergence, we note that because $B_{t-s+\delta}^i \to B_{t-s}^i$ and ϕ_1, ϕ_2 and $V_s^{\lambda,\lambda'}$ are continuous, the limit is equal to the above with $\delta = 0$. To obtain the desired form we make a time reversal of the Brownian motions. Define $\hat{B}_u^i = B_{t-s}^i - B_{t-s-u}^i$. We note that the $\hat{B}^i$ are standard Brownian motions and that $\hat{B}_{t-s}^i = B_{t-s}^i$, $\hat{B}_0^i = 0$ and $B_r^i - B_{t-s}^i = -\hat{B}_{t-s-r}^i$. Making this substitution shows that (5.9) with $\delta = 0$ is equal to

$$\begin{aligned}
&\int_0^t E_0^B\bigg(E_{(0,0)}^{\hat{B}^1,\hat{B}^2}\bigg[\phi_1(B_s + \hat{B}_{t-s}^1)\, \phi_2(B_s + \hat{B}_{t-s}^2) \\
&\times \exp\bigg(-\int_0^s V_{t-u}^{\lambda,\lambda'}(B_u - B_s - \hat{B}_{t-s}^1, B_u - B_s - \hat{B}_{t-s}^2)\, du\bigg) \\
&\times \exp\bigg(-\int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'}(-\hat{B}_{t-s-r}^1, -\hat{B}_{t-s-r}^1 + \hat{B}_{t-s}^1 - B_{t-s}^2)\, dr\bigg) \\
&\times \exp\bigg(-\int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'}(-\hat{B}_{t-s-r}^2 + \hat{B}_{t-s}^2 - B_{t-s}^1, -\hat{B}_{t-s-r}^2)\, dr\bigg)\bigg]\bigg) ds.
\end{aligned}$$

The time index of the Brownian motions now matches the time index of the function $V^{\lambda,\lambda'}$ in the last two lines, allowing us to reverse the time of the integrals for a simpler expression. To obtain the desired expression we now use (3.22), ie. $V_t^{\lambda,\lambda'}(a,b) = V_t^{\lambda,\lambda'}(-a,-b)$, and relabel $\hat{B}^i$ to be simply B^i. This proves the result for $h(x_1,x_2) = \phi_1(x_1)\,\phi_2(x_2)$ when ϕ_1,ϕ_2 are bounded and continuous. The result for general bounded measurable $h:\mathbb{R}^2\to\mathbb{R}$ now follows from a standard monotone class argument such as Corollary 4.4 in the Appendix of Ethier and Kurtz [5]. □

Definition. *Let $\Gamma^{\lambda,\lambda'}(s)$ denote the integrand in Proposition 5.1, so that the proposition states that*

$$\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)) = (\lambda\lambda')^{2\lambda_0}\int_0^t \Gamma^{\lambda,\lambda'}(s)ds. \tag{5.10}$$

$\Gamma^{\lambda,\lambda'}(s)$ also depends on h, but we omit this. The next lemma changes variables to obtain an expression involving Ornstein-Uhlenbeck processes. We first introduce some notation. For bounded and measurable $h:\mathbb{R}^2\to\mathbb{R}$ and a (continuous) path $(B_u : u\in[0,s])$, define $\Psi_{B,s}^{\lambda,\lambda'}(\cdot,\cdot)$ by

$$\Psi_{B,s}^{\lambda,\lambda'}(x,y) = h(x+B_s, y+B_s)\exp\left(-\int_0^s V_{t-u}^{\lambda,\lambda'}(x+B_s-B_u, y+B_s-B_u)\,du\right). \tag{5.11}$$

We define H_u^c as a scaling of $V_t^{\lambda,\lambda'}$:

$$H_u^c(x,y) = uV_u^{1,c}(\sqrt{u}x,\sqrt{u}y) = V_1^{\sqrt{u},\sqrt{u}c}(x,y). \tag{5.12}$$

The scaling in the following lemma cannot be done uniformly for all $s\in[0,t]$ because it requires $\lambda^2 > (t-s)^{-1}$ and $\lambda'^2 > (t-s)^{-1}$. We derive an expression for $\Gamma^{\lambda,\lambda'}(s)$ in terms of two independent Ornstein-Uhlenbeck processes which we denote Y^1 and Y^2, for which we denote the joint (independent) expectation $E_{(x,y)}^{Y^1,Y^2}$.

Lemma 5.3. *Let $0<s<t$, $T_1 = T_1(s) = \log(\lambda^2(t-s))$, $T_2 = T_2(s) = \log(\lambda'^2(t-s))$. Then for all $\lambda > (t-s)^{-1/2}$ and $\lambda' > (t-s)^{-1/2}$, we have*

$$\begin{aligned}\Gamma^{\lambda,\lambda'}(s) = {} & E_0^B\Bigg(E_{(0,0)}^{B^1,B^2}\Bigg[E_{(B_1^1,B_1^2)}^{Y^1,Y^2}\Bigg(\Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}\,Y_{T_1}^1,\sqrt{t-s}\,Y_{T_2}^2)\\ &\times\exp\left(-\int_0^1 V_u^{1,\lambda'/\lambda}(B_u^1, B_u^1+e^{T_1/2}(Y_{T_2}^2-Y_{T_1}^1)) + V_u^{1,\lambda/\lambda'}(B_u^2,B_u^2+e^{T_2/2}(Y_{T_1}^1-Y_{T_2}^2))du\right)\\ &\times\exp\left(-\int_0^{T_1} H_{e^u}^{\lambda'/\lambda}(Y_u^1, Y_u^1+e^{(T_1-u)/2}(Y_{T_2}^2-Y_{T_1}^1))du\right)\\ &\times\exp\left(-\int_0^{T_2} H_{e^u}^{\lambda/\lambda'}(Y_u^2, Y_u^2+e^{(T_2-u)/2}(Y_{T_1}^1-Y_{T_2}^2))du\right)\Bigg)\Bigg]\Bigg).\end{aligned}$$

Proof. We begin with the expression from Proposition 5.1. We observe that $\Psi_{B,s}^{\lambda,\lambda'}$ appears and we may write the quantities in the first two lines as $\Psi_{B,s}^{\lambda,\lambda'}(B_{t-s}^1,B_{t-s}^2)$. In the third and fourth lines we apply (3.23) to obtain

$$\begin{aligned}\Gamma^{\lambda,\lambda'}(s) = E_0^B\Bigg(E_{(0,0)}^{B^1,B^2}\Bigg[&\Psi_{B,s}^{\lambda,\lambda'}(B_{t-s}^1,B_{t-s}^2)\\ &\times\exp\left(-\int_0^{t-s}\lambda^2 V_{\lambda^2 r}^{1,\lambda'/\lambda}(\lambda B_r^1, \lambda(B_r^1+B_{t-s}^2-B_{t-s}^1))\,dr\right)\\ &\times\exp\left(-\int_0^{t-s}\lambda'^2 V_{\lambda'^2 r}^{\lambda/\lambda',1}(\lambda'(B_r^2+B_{t-s}^1-B_{t-s}^2),\lambda' B_r^2)\,dr\right)\Bigg]\Bigg).\end{aligned}$$

We define $\hat{B}^1_u = \lambda B^1_{\lambda^{-2}u}$ and $\hat{B}^2_u = \lambda' B^2_{\lambda'^{-2}u}$, which are both standard Brownian motions. Making a time change in the integrals (ie. letting $u = \lambda^2 r$ or $\lambda'^2 r$) gives

$$\begin{aligned}\Gamma^{\lambda,\lambda'}(s) = E^B_0\Bigg(E^{\hat{B}^1,\hat{B}^2}_{(0,0)}\Bigg[&\Psi^{\lambda,\lambda'}_{B,s}(\lambda^{-1}\hat{B}^1_{\lambda^2(t-s)}, \lambda'^{-1}\hat{B}^2_{\lambda'^2(t-s)}) \\ &\times \exp\left(-\int_0^{\lambda^2(t-s)} V^{1,\lambda'/\lambda}_u(\hat{B}^1_u, \hat{B}^1_u + \frac{\lambda}{\lambda'}\hat{B}^2_{\lambda'^2(t-s)} - \hat{B}^1_{\lambda^2(t-s)})\,du\right) \\ &\times \exp\left(-\int_0^{\lambda'^2(t-s)} V^{\lambda/\lambda',1}_u(\hat{B}^2_u + \frac{\lambda'}{\lambda}\hat{B}^1_{\lambda^2(t-s)} - \hat{B}^2_{\lambda'^2(t-s)}, \hat{B}^2_u)\,du\right)\Bigg]\Bigg).\end{aligned}$$

Because we have assumed $\lambda, \lambda' > (t-s)^{-1/2}$, the upper bounds of integration in the integrals are greater than 1. We now apply the Markov property for $\hat{B}^i$ at time $u = 1$. We collect the portions of the integrals from the second and third lines on the interval $[0,1]$, leaving the integrals from 1 to $\lambda^2(t-s)$ and $\lambda'^2(t-s)$. Conditional on $\hat{B}^i_1$, the Brownian motions in the integrands' arguments are Brownian motions with initial position $\hat{B}^i_1$. If we denote these by $\tilde{B}^i_u$ (in which case, essentially, $\tilde{B}^i_u = \hat{B}^i_{u+1}$), we obtain

$$\begin{aligned}\Gamma^{\lambda,\lambda'}(s) = E^B_0\Bigg(E^{\hat{B}^1,\hat{B}^2}_{(0,0)}\Bigg[E^{\tilde{B}^1,\tilde{B}^2}_{(\hat{B}^1_1,\hat{B}^2_1)}\Bigg(&\Psi^{\lambda,\lambda'}_{B,s}(\lambda^{-1}\tilde{B}^1_{\lambda^2(t-s)-1}, \lambda'^{-1}\tilde{B}^2_{\lambda'^2(t-s)-1}) \\ \times \exp\Bigg(-\int_0^1 &V^{1,\lambda'/\lambda}_u(\hat{B}^1_u, \hat{B}^1_u + \frac{\lambda}{\lambda'}\tilde{B}^2_{\lambda'^2(t-s)-1} - \tilde{B}^1_{\lambda^2(t-s)-1}) \\ &+ V^{\lambda/\lambda',1}_u(\hat{B}^2_u + \frac{\lambda'}{\lambda}\tilde{B}^1_{\lambda^2(t-s)-1} - \tilde{B}^2_{\lambda'^2(t-s)-1}, \hat{B}^2_u)\,du\Bigg) \\ \times \exp\Bigg(-\int_0^{\lambda^2(t-s)-1} &V^{1,\lambda'/\lambda}_{u+1}(\tilde{B}^1_u, \tilde{B}^1_u + \frac{\lambda}{\lambda'}\tilde{B}^2_{\lambda'^2(t-s)-1} - \tilde{B}^1_{\lambda^2(t-s)-1})\,du\Bigg) \\ \times \exp\Bigg(-\int_0^{\lambda'^2(t-s)-1} &V^{\lambda/\lambda',1}_{u+1}(\tilde{B}^2_u + \frac{\lambda'}{\lambda}\tilde{B}^1_{\lambda^2(t-s)-1} - \tilde{B}^2_{\lambda'^2(t-s)-1}, \tilde{B}^2_u)\,du\Bigg)\Bigg)\Bigg]\Bigg).\end{aligned} \tag{5.13}$$

Recall that if a process Y is defined by

$$Y_r = e^{-r/2}B_{e^r-1}$$

where B is a standard Brownian motion, then Y is a standard one-dimensional Ornstein-Uhlenbeck process with $Y_0 = B_0$. For $i = 1,2$ we let $Y^i_r = e^{-r/2}\tilde{B}^i_{e^r-1}$. Recall that $T_1 = \log(\lambda^2(t-s))$ and $T_2 = \log(\lambda'^2(t-s))$. We therefore have that

$$\tilde{B}^1_{\lambda^2(t-s)-1} = e^{T_1/2}Y^1_{T_1}, \quad \tilde{B}^2_{\lambda'^2(t-s)-1} = e^{T_2/2}Y^2_{T_2}.$$

Expressing λ and λ' in terms of T_1, T_2 shows that

$$\frac{\lambda}{\lambda'}\tilde{B}^2_{\lambda^2(t-s)-1} = e^{T_1/2}Y^2_{T_2}, \quad \frac{\lambda'}{\lambda}\tilde{B}^1_{\lambda'^2(t-s)-1} = e^{T_2/2}Y^1_{T_1}.$$

Likewise, we express the argument of $\Psi^{\lambda,\lambda'}_{B,s}$ in terms of Y^i and T_i. We substitute $u = e^r - 1$ and apply the above in (5.13) to obtain

$$\begin{aligned}\Gamma^{\lambda,\lambda'}(s) = E^B_0\Bigg(E^{\hat{B}^1,\hat{B}^2}_{(0,0)}\Bigg[E^{Y^1,Y^2}_{(\hat{B}^1_1,\hat{B}^2_1)}\Bigg(&\Psi^{\lambda,\lambda'}_{B,s}(\sqrt{t-s}\,Y^1_{T_1}, \sqrt{t-s}\,Y^2_{T_2}) \\ \times \exp\Bigg(-\int_0^1 V^{1,\lambda'/\lambda}_u(\hat{B}^1_u, \hat{B}^1_u + e^{T_1/2}(Y^2_{T_2} - Y^1_{T_1})) &+ V^{\lambda/\lambda',1}_u(\hat{B}^2_u + e^{T_2/2}(Y^1_{T_1} - Y^2_{T_2}), \hat{B}^2_u)\,du\Bigg)\end{aligned}$$

$$\times \exp\bigg(- \int_0^{T_1} e^r V_{e^r}^{1,\lambda'/\lambda}(e^{r/2}Y_r^1, e^{r/2}Y_r^1 + e^{T_1/2}(Y_{T_2}^2 - Y_{T_1}^1))\, dr\bigg)$$
$$\times \exp\bigg(- \int_0^{T_2} e^r V_{e^r}^{\lambda/\lambda',1}(e^{r/2}Y_r^2 + e^{T_2/2}(Y_{T_1}^1 - Y_{T_2}^2)), e^{r/2}Y_r^2)\, dr\bigg)\bigg)\bigg]\bigg).$$

We now apply (3.23) and (5.12) in the third and fourth lines. In the third line this gives

$$\begin{aligned} &e^r V_{e^r}^{1,\lambda'/\lambda}(e^{r/2}Y_r^1, e^{r/2}Y_r^1 + e^{T_1/2}(Y_{T_2}^2 - Y_{T_1}^1)) \\ &\quad = V_1^{e^{r/2}, e^{r/2}\lambda'/\lambda}(Y_r^1, Y_r^1 + e^{(T_1-r)/2}(Y_{T_2}^2 - Y_{T_1}^1)) \\ &\quad = H_{e^r}^{\lambda'/\lambda}(Y_r^1, Y_r^1 + e^{(T_1-r)/2}(Y_{T_2}^2 - Y_{T_1}^1)), \end{aligned}$$

and similar in the fourth. Noting that $V_t^{c,d}(a,b) = V_t^{d,c}(b,a)$, we have obtained the desired expression. ☐

We now obtain an upper bound for $\Gamma^{\lambda,\lambda'}(s)$ and show that the contribution to $\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h))$ from the integral over $[t-\epsilon, t]$ vanishes as $(\epsilon, \lambda') \to (0, \infty)$.

Lemma 5.4. *Suppose $\lambda^2 t \geq 1$, and let $h : \mathbb{R}^2 \to \mathbb{R}$ be bounded and measurable. There is a constant $C_{5.4} > 0$ such that the following hold.*
(a) For all $\lambda' > (t-s)^{-1/2}$,

$$(\lambda\lambda')^{2\lambda_0} \big|\Gamma^{\lambda,\lambda'}(s)\big| \leq C_{5.4} \|h\|_\infty t^{-\lambda_0}(t-s)^{-\lambda_0}.$$

(b) For $0 < \epsilon < t$,

$$(\lambda\lambda')^{2\lambda_0} \int_{t-\epsilon}^t \big|\Gamma^{\lambda,\lambda'}(s)\big|\, ds \leq C_{5.4}\|h\|_\infty t^{-\lambda_0}(\epsilon^{1-\lambda_0} + \lambda'^{-2(1-\lambda_0)}).$$

Proof. To begin we use $|h| \leq \|h\|_\infty$ and apply monotonicity (Proposition 3.1(a)), ie. $V^\lambda(x), V^{\lambda'}(y) \leq V^{\lambda,\lambda'}(x,y)$, to obtain

$$\begin{aligned} &(\lambda\lambda')^{2\lambda_0}\big|\Gamma^{\lambda,\lambda'}(s)\big| \\ &\quad \leq \|h\|_\infty (\lambda\lambda')^{2\lambda_0} E_0^B\bigg(E_{(0,0)}^{B^1,B^2}\bigg[\exp\bigg(-\int_0^s V_{t-u}^\lambda(B_{t-s}^1 + B_s - B_u)\, du\bigg) \\ &\qquad \times \exp\bigg(-\int_0^{t-s} V_r^\lambda(B_r^1)\, dr\bigg)\exp\bigg(-\int_0^{t-s} V_r^{\lambda'}(B_r^2)\, dr\bigg)\bigg]\bigg) \\ &\quad = \|h\|_\infty(\lambda\lambda')^{2\lambda_0} E_0^{B^1}\bigg(\exp\bigg(-\int_0^t V_u^\lambda(B_u^1)\, du\bigg)\bigg) E_0^{B^2}\bigg(\exp\bigg(-\int_0^{t-s} V_r^{\lambda'}(B_r^2)\, dr\bigg)\bigg), \end{aligned}$$

where the final line follows from a time reversal of B and concatenating the time-reversed B with B^1. Applying (3.6) twice and changing the time variable, the above is equal to

$$\begin{aligned} \|h\|_\infty(\lambda\lambda')^{2\lambda_0} &E_0^{B^1}\bigg(\exp\bigg(-\int_0^{\lambda^2 t} V_u^1(\lambda B_{\lambda^{-2}u}^1)\, du\bigg)\bigg) \\ &\times E_0^{B^2}\bigg(\exp\bigg(-\int_0^{\lambda'^2(t-s)} V_u^1(\lambda' B_{\lambda'^{-2}u}^2)\, du\bigg)\bigg). \end{aligned}$$

The rescaled Brownian motions in the above are themselves standard Brownian motions which we will denote by $\hat{B}^1, \hat{B}^2$. We next let $e^r = u$ in both integrals and apply (3.6) to see that the above equals

$$\begin{aligned}
&\|h\|_\infty(\lambda\lambda')^{2\lambda_0}E_0^{\hat B^1}\left(\exp\left(-\int_{-\infty}^{\log(\lambda^2t)}V_1^{e^{r/2}}(e^{-r/2}\hat B^1_{e^r})\,dr\right)\right)\\
&\quad\times\ E_0^{\hat B^2}\left(\exp\left(-\int_{-\infty}^{\log(\lambda'^2(t-s))}V_1^{e^{r/2}}(e^{-r/2}\hat B^2_{e^r})\,dr\right)\right)ds\\
&\le\|h\|_\infty(\lambda\lambda')^{2\lambda_0}E_m^{Y^1}\left(\exp\left(-\int_0^{\log(\lambda^2t)}V_1^{e^{r/2}}(Y^1_r)\,dr\right)\right)\\
&\quad\times\ E^{Y^2}\left(\exp\left(-\int_{-\infty}^{\log(\lambda'^2(t-s))}V_1^{e^{r/2}}(Y^2_r)\,dr\right)\right),
\end{aligned}\tag{5.14}$$

where $Y^i_r=e^{-r/2}\hat B^i_{e^r}$, which makes Y^i_u a stationary Ornstein-Uhlenbeck process for $u\in\mathbb{R}$, and we recall our assumption that $\lambda^2t\ge1$. We condition on the value of Y^1_0, which has distribution m.

We first use the above to prove (a). Assuming that $\lambda'>(t-s)^{-1/2}$, the upper endpoint of the second integral is positive, so by (5.14) we have

$$\begin{aligned}
(\lambda\lambda')^{2\lambda_0}\left|\Gamma^{\lambda,\lambda'}(s)\right|\le\|h\|_\infty(\lambda\lambda')^{2\lambda_0}E_m^{Y^1}\left(\exp\left(-\int_0^{\log(\lambda^2t)}V_1^{e^{r/2}}(Y^1_r)\,dr\right)\right)\\
\times E_m^{Y^2}\left(\exp\left(-\int_0^{\log(\lambda'^2(t-s))}V_1^{e^{r/2}}(Y^2_r)\,dr\right)\right),
\end{aligned}\tag{5.15}$$

where we have also conditioned on Y^2_0. In order to approximate the expectations above with survival probabilities for killed Ornstein-Uhlenbeck processes, we add and subtract $F(Y^i_u)$ in the integrals. Recalling the definition of $Z_T(Y)$ from (3.16), we define $Z^1_T(Y^1),Z^2_T(Y^2)$ in the same way. Thus (5.15) is equal to

$$\begin{aligned}
&\|h\|_\infty(\lambda\lambda')^{2\lambda_0}E_m^{Y^1}\left(Z^1_{\log(\lambda^2t)}(Y^1)\exp\left(-\int_0^{\log(\lambda^2t)}F(Y^1_r)\,du\right)\right)\\
&\times E_m^{Y^2}\left(Z^2_{\log(\lambda'^2(t-s))}(Y^2)\exp\left(-\int_0^{\log(\lambda'^2(t-s))}F(Y^2_r)\,dr\right)\right)\\
&\quad\le\|h\|_\infty(\lambda\lambda')^{2\lambda_0}C_ZE_m^{Y^1}\left(\exp\left(-\int_0^{\log(\lambda^2t)}F(Y^1_r)\,du\right)\right)\\
&\qquad\times C_ZE_m^{Y^2}\left(\exp\left(-\int_0^{\log(\lambda'^2(t-s))}F(Y^2_r)\,dr\right)\right)ds\\
&\quad=C\|h\|_\infty(\lambda\lambda')^{2\lambda_0}P_m^{Y^1}(\rho^F>\log(\lambda^2t))\,P_m^{Y^2}(\rho^F>\log(\lambda'^2(t-s))).
\end{aligned}\tag{5.16}$$

In the first inequality we have used (3.18) twice, and the second equality follows by recognizing the expectations as survival probabilities of killed Ornstein-Uhlenbeck processes killed at rate $F(Y^i_r)$. By (2.9), we have

$$P_m^{Y^1}(\rho^F>\log(\lambda^2t))\le Ct^{-\lambda_0}\lambda^{-2\lambda_0},\quad P_m^{Y^2}(\rho^F>\log(\lambda'^2(t-s)))\le C(t-s)^{-\lambda_0}\lambda'^{-2\lambda_0}.$$

Using the above in (5.16), which is an upper bound for $(\lambda\lambda')^{2\lambda_0}\left|\Gamma^{\lambda,\lambda'}(s)\right|$, proves (a).

We now show (b). Let $0<\epsilon<t$. Using (5.14) we obtain that

$$\begin{aligned}
(\lambda\lambda')^{2\lambda_0}\int_{t-\epsilon}^t\left|\Gamma^{\lambda,\lambda'}(s)\right|ds\ \le\ \|h\|_\infty(\lambda\lambda')^{2\lambda_0}E_m^{Y^1}\left(\exp\left(-\int_0^{\log(\lambda^2t)}V_1^{e^{r/2}}(Y^1_r)\,dr\right)\right)\\
\times\int_{t-\epsilon}^tE^{Y^2}\left(\exp\left(-\int_{-\infty}^{\log(\lambda'^2(t-s))}V_1^{e^{r/2}}(Y^2_r)\,dr\right)\right)ds.
\end{aligned}\tag{5.17}$$

We can approximate the first expectation with the survival probability of Y^1, just as we did in the proof of (a), and bound it above by $C\lambda^{-2\lambda_0}t^{-\lambda_0}$. Furthermore, by the proof of part (a), we know that when $\lambda' > (t-s)^{-1/2}$ the expectation in the integral above is bounded above by $C(\lambda')^{-2\lambda_0}(t-s)^{-\lambda_0}$. When this is not the case we bound it above by 1. Thus the right hand side of (5.17) is bounded above by

$$\begin{aligned} &C\|h\|_\infty t^{-\lambda_0}\Bigg[1(\lambda' \geq \epsilon^{-1/2})\int_{t-\epsilon}^{t-\lambda'^{-2}} (t-s)^{-\lambda_0}\,ds \\ &\qquad\qquad + (\lambda')^{2\lambda_0}\int_{t-\lambda'^{-2}}^{t} E^{Y^2}\left(\exp\left(-\int_{-\infty}^{\log(\lambda'^2(t-s))} V_1^{e^{r/2}}(Y^2_r)\,dr\right)\right)ds\Bigg] \\ &\quad \leq C\|h\|_\infty t^{-\lambda_0}\left[\epsilon^{1-\lambda_0} + \lambda'^{-2(1-\lambda_0)}\right]. \end{aligned}$$

The result now follows. □

5.2 Convergence

We now show that the expressions for $\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)$ obtained in the previous section converge as $\lambda,\lambda' \to \infty$. We do so by computing the limit explicitly. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be bounded and measurable. Clearly we may assume without loss of generality that $h \geq 0$. We recall from (5.10) and Proposition 5.1 that

$$\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)) = \int_0^t (\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)\,ds,$$

where $h \geq 0$ implies $\Gamma^{\lambda,\lambda'}(s) \geq 0$. Our strategy is to compute the limit of $(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)$ as $\lambda,\lambda' \to \infty$ and pass the limit through the integral. However, the scaling we use cannot be done uniformly in s. In order to handle this and the singularity at $s = t$, we fix $\epsilon > 0$ and analyse the integral on $[t-\epsilon, t]$ separately. We have

$$\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)) = \int_0^{t-\epsilon} (\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)\,ds + (\lambda\lambda')^{2\lambda_0}\int_{t-\epsilon}^{t} \Gamma^{\lambda,\lambda'}(s)\,ds. \tag{5.18}$$

By Lemma 5.4(b), the limit superior of the absolute value of the second term as $\lambda' \to \infty$ is bounded above by $C\|h\|_\infty t^{-\lambda_0}\epsilon^{1-\lambda_0}$. Hence, if

$$\lim_{\lambda,\lambda'\to\infty}\int_0^{t-\epsilon} (\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)\,ds$$

exists for all $\epsilon > 0$, then by the Cauchy condition $\lim_{\lambda,\lambda'\to\infty}\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'}(h))$ exists and is the limit of the above as $\epsilon \downarrow 0$. Thus it suffices to fix $\epsilon > 0$ and establish the convergence of, and find the limit of, the first term of (5.18), first as $\lambda,\lambda' \to \infty$ and then as $\epsilon \downarrow 0$. By Lemma 5.4(a), we have

$$(\lambda\lambda')^{2\lambda_0}|\Gamma^{\lambda,\lambda'}(s)| \leq g(s) \quad \text{for all} \;\; s \in [0, t-\epsilon]$$

for all $\lambda,\lambda' > \epsilon^{-1/2}$ for a function $g(s) \geq 0$ satisfying $\int_0^{t-\epsilon} g(s)ds < \infty$. It follows that, if $(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)$ converges as $\lambda,\lambda' \to \infty$, then Dominated Convergence implies that

$$\begin{aligned} \lim_{\lambda,\lambda'\to\infty}\mathbb{N}_0((L^\lambda \times L_t^{\lambda'})(h)) &= \lim_{\epsilon\to 0^+}\lim_{\lambda,\lambda'\to\infty}\int_0^{t-\epsilon}(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)\,ds \\ &= \lim_{\epsilon\to 0^+}\int_0^{t-\epsilon}\lim_{\lambda,\lambda'\to\infty}(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)\,ds, \end{aligned} \tag{5.19}$$

and so it suffices to find the limit of $(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)$ as $\lambda, \lambda' \to \infty$.

Let $s \in (0,t)$ and assume $\lambda, \lambda' > (t-s)^{-1/2}$. By Lemma 5.3,

$$\begin{aligned}
&(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s) = (\lambda\lambda')^{2\lambda_0} E_0^B\Bigg(E_{(0,0)}^{B^1,B^2}\Bigg(E_{B_1^1,B_1^2}^{Y^1,Y^2}\Bigg[\Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}\,Y^1_{T_1}, \sqrt{t-s}\,Y^2_{T_2})\\
&\times \exp\Bigg(-\int_0^1 V_u^{1,\lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(Y^2_{T_2} - Y^1_{T_1})) + V_u^{1,\lambda/\lambda'}(B_u^2, B_u^2 + e^{T_2/2}(Y^1_{T_1} - Y^2_{T_2}))du\Bigg)\\
&\times \exp\Bigg(-\int_0^{T_1} H_{e^u}^{\lambda'/\lambda}(Y_u^1, Y_u^1 + e^{\frac{T_1-u}{2}}(Y^2_{T_2} - Y^1_{T_1}))\,du\Bigg)\\
&\times \exp\Bigg(-\int_0^{T_2} H_{e^u}^{\lambda/\lambda'}(Y_u^2, Y_u^2 + e^{\frac{T_2-u}{2}}(Y^1_{T_1} - Y^2_{T_2}))\,du\Bigg)\Bigg)\Bigg)\Bigg], \qquad (5.20)
\end{aligned}$$

where $T_1 = T_1(s) = \log(\lambda^2(t-s))$, $T_2 = T_2(s) = \log(\lambda'^2(t-s))$. Inside the integral in the third term we add and subtract $F(Y_u^i)$ and decompose as follows

$$\begin{aligned}
&\exp\Bigg(-\int_0^{T_1} H^1_{e^u}(Y_u^1, Y_u^1 + e^{\frac{T-u}{2}}(Y^2_{T_2} - Y^1_{T_1}))\,du\Bigg)\\
&\quad = \exp\Bigg(-\int_0^{T_1} F(Y_u^1)\,du\Bigg)\exp\Bigg(\int_0^{T_1} F(Y_u^1) - H_{e^u}^{\lambda'/\lambda}(Y_u^1, Y_u^1 + e^{\frac{T_1-u}{2}}(Y^2_{T_2} - Y^1_{T_1}))\,du\Bigg).
\end{aligned}$$

We do the same to the fourth term with the obvious changes of indices. The first term in the above is the probability that the Ornstein-Uhlenbeck process Y^1 with killing function F survives until time T_1. We extract a similar term from the symmetric term corresponding to Y^2 and T_2. Weighting the expectation of a functional with this survival probability is equivalent to restricting the expectation to the event that the process survives; in our case, we restrict to the event that Y^1 and Y^2 survive until T_1 and T_2, respectively. Thus (5.20) is equal to

$$\begin{aligned}
&(\lambda\lambda')^{2\lambda_0} E_0^B\Bigg(E_{(0,0)}^{B^1,B^2}\Bigg(E_{B_1^1,B_1^2}^{Y^1,Y^2}\Bigg[\Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}\,Y^1_{T_1}, \sqrt{t-s}\,Y^2_{T_2})\\
&\times \exp\Bigg(-\int_0^1 V_u^{1,\lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(Y^2_{T_2} - Y^1_{T_1})) + V_u^{1,\lambda/\lambda'}(B_u^2, B_u^2 + e^{T_2/2}(Y^1_{T_1} - Y^2_{T_2}))du\Bigg)\\
&\times \exp\Bigg(\int_0^{T_1} F(Y_u^1) - H_{e^u}^{\lambda'/\lambda}(Y_u^1, Y_u^1 + e^{\frac{T_1-u}{2}}(Y^2_{T_2} - Y^1_{T_1}))\,du\Bigg)\\
&\times \exp\Bigg(\int_0^{T_2} F(Y_u^2) - H_{e^u}^{\lambda/\lambda'}(Y_u^2, Y_u^2 + e^{\frac{T_2-u}{2}}(Y^1_{T_1} - Y^2_{T_2}))\,du\Bigg)1(\rho_1 > T_1)1(\rho_2 > T_2)\Bigg]\Bigg)\Bigg), \qquad (5.21)
\end{aligned}$$

where $\rho_i = \rho_i^F$ is the lifetime of the killed process Y^i. Recall the transition density $q_t(\cdot,\cdot)$ (with respect to m) of the killed diffusion. We condition on the endpoints $Y^i_{T_i} = z_i$ (recall from Lemma 2.3(a) that the regular conditional distributions exist for all $z_i \in \mathbb{R}$) and integrate against $q_{T_i}(\cdot, z_i)\,dm(z_i)$ to obtain that (5.21) is equal to

$$\begin{aligned}
&(\lambda\lambda')^{2\lambda_0} E_0^B\Bigg(E_{(0,0)}^{B^1,B^2}\Bigg(\iint \Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}z_1, \sqrt{t-s}z_2)\\
&\times \exp\Bigg(-\int_0^1 V_u^{1,\lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(z_2 - z_1)) + V_u^{1,\lambda/\lambda'}(B_u^2, B_u^2 + e^{T_2/2}(z_1 - z_2))\,du\Bigg)\\
&\times E_{B_1^1}^{Y^1}\Bigg(\exp\Bigg(\int_0^{T_1} F(Y_u^1) - H_{e^u}^{\lambda'/\lambda}(Y_u^1, Y_u^1 + e^{\frac{T_1-u}{2}}(z_2 - z_1))\,du\Bigg)\Bigg|\,\rho_1 > T_1, Y^1_{T_1} = z_1\Bigg)\\
&\times E_{B_1^2}^{Y^2}\Bigg(\exp\Bigg(\int_0^{T_2} F(Y_u^2) - H_{e^u}^{\lambda/\lambda'}(Y_u^2, Y_u^2 + e^{\frac{T_2-u}{2}}(z_1 - z_2))\,du\Bigg)\Bigg|\,\rho_2 > T_2, Y^2_{T_2} = z_2\Bigg)
\end{aligned}$$

$$\times\, q_{T_1}(B_1^1, z_1)\, q_{T_2}(B_1^2, z_2)\, dm(z_1)\, dm(z_2)\Big)\Big)$$

$$=: \iint (\lambda\lambda')^{2\lambda_0}\Big[E_0^B\Big(E_{(0,0)}^{B^1,B^2}\Big(G(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2) \\ \times\, q_{T_1}(B_1^1, z_1)\, q_{T_2}(B_1^2, z_2)\Big)\Big)\Big]dm(z_1)\, dm(z_2). \tag{5.22}$$

The function G is defined implicitly. The conditional probabilities that appear are the same that are defined in Section 2, in particular Lemma 2.3. We have used that the terms in the third and fourth lines are independent conditional on the endpoints. Hereafter, Y^1 and Y^2, and their respective laws, refer to killed Ornstein-Uhlenbeck processes with killing function F. Furthermore, after this point we will suppress the conditioning on $\rho_i > T_i$, as it is implicit in the conditioning $Y_{T_i}^i = z_i$ that $\rho_i > T_i$.

We introduce notation for the terms appearing in $G(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)$. We define

$$Q(\lambda,\lambda',B^1,B^2,z_1,z_2) \\ := \exp\Big(-\int_0^1 V_u^{1,\lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(z_2-z_1)) + V_u^{1,\lambda/\lambda'}(B_u^2, B_u^2 + e^{T_2/2}(z_1-z_2))\, du\Big), \tag{5.23}$$

and

$$\tilde{Z}_{T_1}^1 = \tilde{Z}_{T_1}^1(Y^1, z_1, z_2, \lambda'/\lambda) \\ := \exp\Big(\int_0^{T_1} F(Y_u^1) - H_{e^u}^{\lambda'/\lambda}(Y_u^1, Y_u^1 + e^{\frac{T_1-u}{2}}(z_2-z_1))\, du\Big), \tag{5.24}$$

$$\tilde{Z}_{T_2}^2 = \tilde{Z}_{T_2}^2(Y^2, z_2, z_1, \lambda/\lambda') \\ := \exp\Big(\int_0^{T_2} F(Y_u^2) - H_{e^u}^{\lambda/\lambda'}(Y_u^2, Y_u^2 + e^{\frac{T_2-u}{2}}(z_1-z_2))\, du\Big). \tag{5.25}$$

Recall that $\Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}z_1, \sqrt{t-s}z_2)$ was defined in (5.11). From (5.22) we have

$$G(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2) \\ = \Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}z_1, \sqrt{t-s}z_2) Q(\lambda,\lambda',B^1,B^2,z_1,z_2)\, E_{B_1^1}^{Y^1}(\tilde{Z}_{T_1}^1 \big| Y_{T_1}^1 = z_1) E_{B_1^2}^{Y^2}(\tilde{Z}_{T_2}^2 \big| Y_{T_2}^2 = z_2). \tag{5.26}$$

We note that $\tilde{Z}_{T_1}^1$ and $\tilde{Z}_{T_2}^2$ are perturbations of the corresponding $Z_{T_i}^i$ terms. In particular, we defined $Z_{T_i}^i$ by

$$Z_{T_i}^i(Y^i) = \exp\Big(\int_0^{T_i} F(Y_u^i) - V_1^{e^{u/2}}(Y_u^i)\, du\Big). \tag{5.27}$$

By Proposition 3.1(a) and (5.12), we have that $H_{e^u}^c(x,y) \geq V_1^{e^{u/2}}(x)$, and hence

$$\tilde{Z}_{T_i}^i \leq Z_{T_i}^i(Y^i) \leq C_Z, \tag{5.28}$$

where the second inequality is by (3.18). Using $Q(\lambda,\lambda',B^1,B^2,z_1,z_2) \leq 1$ and $|\Psi_{B,s}^{\lambda,\lambda'}| \leq \|h\|_\infty$, both of which are obvious from these terms' definitions, we therefore obtain that for a constant $C_1 > 0$,

$$|G(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)| \leq C_1 \tag{5.29}$$

uniformly in its arguments. We now define $\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)$ as the function in the square-bracketed term in (5.22) multiplied by the scaling factor $(\lambda\lambda')^{2\lambda_0}$. That is,

$$\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2) \\ := G(\lambda,\lambda',s,B,B_1^1,B_1^2,z_1,z_2)\, (\lambda\lambda')^{2\lambda_0}\, q_{T_1}(B_1^1,z_1)\, q_{T_2}(B_1^2,z_2). \tag{5.30}$$

Note from (5.22) that

$$\Gamma^{\lambda,\lambda'}(s) = \iint E_0^B(E_{(0,0)}^{B^1,B^2}(\,\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)))\,dm(z_1)\,dm(z_2). \tag{5.31}$$

Recall that $T_1 = \log(\lambda^2(t-s))$ and $T_2 = \log(\lambda'^2(t-s))$. Taking $s^*(1/8)$ as in Theorem 2.1(c), we note that if $\lambda,\lambda' > e^{s^*/2}(t-s)^{-1/2}$, then $T_1,T_2 \geq s^*(1/8)$. We define $\bar\lambda(s)$ as

$$\bar\lambda(s) := \left[e^{s^*(1/8)/2}(t-s)^{-1/2}\right] \vee 1 \tag{5.32}$$

and $\tau(s)$ by

$$\tau(s) = \log(\bar\lambda(s)^2(t-s)). \tag{5.33}$$

Applying (2.2) with $\delta = 1/8$, we obtain

$$q_{T_1}(b_1,z_1)\,q_{T_2}(b_2,z_2) \leq C(t-s)^{-2\lambda_0}(\lambda\lambda')^{-2\lambda_0}e^{1/8(b_1^2+b_2^2+z_1^2+z_2^2)}$$

for all $T_1,T_2 > \tau(s)$ (equivalently, $\lambda,\lambda' > \bar\lambda(s)$). Using the above and (5.29), we obtain

$$\begin{aligned}&|\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)|\\ &\quad\leq C\,(t-s)^{-2\lambda_0}\exp\left(\left[(B_1^1)^2+(B_1^2)^2+z_1^2+z_2^2\right]/8\right) \quad \text{for all } \lambda,\lambda' > \bar\lambda(s).\end{aligned} \tag{5.34}$$

Since $B_1^i \sim m$, (5.34) implies that Θ has a (uniform in $\lambda,\lambda' > \bar\lambda(s)$) upper bound which is integrable with respect to $dP_0^B dP_0^{B^1} dP_0^{B^2} dm(z_1)\,dm(z_2)$. From (5.31), this implies that $(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)$ is bounded for $\lambda,\lambda' > \bar\lambda(s)$ (for fixed $s < t$). Moreover, if $\lim_{\lambda,\lambda'\to\infty}\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)$ exists for $P_0^B \otimes P_{(0,0)}^{B^1,B^2}$-a.a. ω and Lebesgue-a.a. $z_1,z_2 \in \mathbb{R}$, then by (5.31) and Dominated Convergence (using (5.34)), we have

$$\begin{aligned}&\lim_{\lambda,\lambda'\to\infty}(\lambda\lambda')^{2\lambda_0}\Gamma^{\lambda,\lambda'}(s)\\ &\quad= \iint E_0^B(E_{(0,0)}^{B^1,B^2}[\lim_{\lambda,\lambda'\to\infty}\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)])\,dm(z_1)\,dm(z_2).\end{aligned} \tag{5.35}$$

In view of (5.19), the above implies the following:

If $\lim_{\lambda,\lambda'\to\infty}\Theta(\lambda,\lambda',s,B,B^1,B^2,z_2,z_2)$ exists $P_0^B \otimes P_{(0,0)}^{B^1,B^2}$-a.s. for a.e. $z_1,z_2 \in \mathbb{R}$,

then $\lim_{\lambda,\lambda'\to\infty}\mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h))$

$$= \int_0^t \left[\iint E_0^B(E_{(0,0)}^{B^1,B^2}[\lim_{\lambda,\lambda'\to\infty}\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)])\,dm(z_1)\,dm(z_2)\right]ds. \tag{5.36}$$

As $h \geq 0$, and hence $\Gamma^{\lambda,\lambda'}(s) \geq 0$, the right hand side of the above is equal to the last expression of (5.19) (provided Θ converges) by Monotone Convergence. Thus it suffices to compute the limit of $\Theta(\lambda,\lambda',s,B,B^1,B^2,z_2,z_2)$ as $\lambda,\lambda' \to \infty$. As we only need to find the limit a.e. in (z_1,z_2), we will hereafter assume that $z_1 \neq z_2$. We also take this opportunity to reiterate our assumptions about λ and λ'. Originally we assumed $\lambda,\lambda' > (t-s)^{-1/2}$; in view of the above, we augment the assumption to $\lambda,\lambda' > \bar\lambda(s)$, or equivalently, $T_1,T_2 > \tau(s)$ (see (5.32) and (5.33)). This implies that $\lambda,\lambda' > 1$ and $T_1,T_2 > s^*(1/8)$.

Θ is the product of the function G and the rescaled transition densities, that is, $\lambda^{2\lambda_0}q_{T_1}(B_1^1,z_1)$ and $\lambda'^{2\lambda_0}q_{T_2}(B_1^2,z_2)$. We will show that both of these approach finite limits as $\lambda,\lambda' \to \infty$. First, let us handle the transition densities. By Lemma 2.2,

$$\lim_{T_i\to\infty} e^{\lambda_0 T_i} q_{T_i}(B_1^i,z_i) = \psi_0(B_1^i)\,\psi_0(z_i)$$

for $i = 1, 2$. Using the definitions of T_1 and T_2 (e.g. $T_1 = \log(\lambda^2(t-s))$), we readily obtain from the above that

$$\lambda^{2\lambda_0} q_{T_1}(B_1^1, z_1) \to (t-s)^{-\lambda_0}\psi_0(B_1^1)\,\psi_0(z_1) \;\text{ as }\; \lambda \to \infty, \quad \text{and}$$
$$\lambda'^{2\lambda_0} q_{T_2}(B_1^2, z_2) \to (t-s)^{-\lambda_0}\psi_0(B_1^2)\,\psi_0(z_2) \;\text{ as }\; \lambda' \to \infty \tag{5.37}$$

for all $B_1^1, B_1^2, z_1, z_2 \in \mathbb{R}$.

We now compute the limit of G, whose definition we recall from (5.26). We proceed by computing the limit of each constituent term. The analysis is most technical for the conditional expectations of $\tilde{Z}^i_{T_i}$, for $i = 1, 2$, which were defined in (5.24) and (5.25). These two quantities are essentially the same with different parameters. To avoid excessive and cumbersome notation we define a variable $\tilde{Z}_T$. Let $\lambda, \lambda' > 0$ and $T = \log(\lambda^2(t-s))$, and $x, z_2, z_2 \in \mathbb{R}$. For a killed Ornstein-Uhlenbeck process Y, we define

$$\tilde{Z}_T = \tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda) := \exp\left(\int_0^T F(Y_u) - H^{\lambda'/\lambda}_{e^u}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\,du\right). \tag{5.38}$$

In order to characterize the limit of the conditional expectation of $\tilde{Z}_T$ (as in (5.26)), we introduce a quantity $W_S(Y, z)$. For $S > 0$ and $z \in \mathbb{R}$, we define

$$W_S(Y, z) = \exp\left(\int_0^S F(Y_u) - F_2(Y_u, Y_u - e^{u/2}(z - Y_0))\,du\right), \tag{5.39}$$

where we recall from (3.24) that $F_2(a, b) = V_1^{\infty,\infty}(a, b)$. By Proposition 3.1(a), $F(a) \le F_2(a, b)$ for all $a, b \in \mathbb{R}$, so the integrand is non-positive and hence $W_S(Y, z) \le 1$ and is non-increasing in S. Since it is bounded below by 0, we can define its monotone limit as $W_\infty(Y, z) = \lim_{S\to\infty} W_S(Y, z) \le 1$.

Lemma 5.5. *Let $x, z_1, z_2 \in \mathbb{R}$ such that $z_1 \ne z_2$. Then*

$$\lim_{\lambda,\lambda'\to\infty} E_x^Y\left(\tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda)\,\middle|\, Y_T = z_1\right) = E_x^{Y,\infty}(Z_\infty(Y))\,E_{z_1}^{Y,\infty}(W_\infty(Y, z_2)).$$

Recall that $E_x^{Y,\infty}$ is the expectation under the law of the killed process Y with $Y_0 = x$ conditioned to survive for all time, as defined in Theorem 2.1(e). $Z_T(Y)$ is as defined in (3.16) and we recall from (3.17) and (3.18) that $Z_\infty(Y) = \lim_{T\to\infty} Z_T(Y)$ exists and is bounded by C_Z.

Heuristically, the Z_∞ term in the limiting expression in Lemma 5.5 comes from the early (small u) part of the integral in $\tilde{Z}_T$, and the W_∞ term comes from the tail part (ie. u near T), and these two contributions are asymptotically independent. Since the time at which we condition is T and T goes to infinity, in the limit the expectations are computed under the measure of the process conditioned to survive forever.

Section 6 is devoted to the proof of Lemma 5.5. For now, we carry on with the proof of Theorem 1.4. Returning to $\tilde{Z}^1_{T_1}$ and $\tilde{Z}^2_{T_2}$, it follows from Lemma 5.5 that

For all $B_1^1, B_1^2 \in \mathbb{R}$ and all $z_1, z_2 \in \mathbb{R}$ such that $z_1 \ne z_2$,

$$\lim_{\lambda,\lambda'\to\infty} E^{Y^1}_{B_1^1}\left(\tilde{Z}^1_{T_1}(Y^1, z_1, z_2, \lambda'/\lambda)\,\middle|\, Y^1_{T_1} = z_1\right) = E^{Y,\infty}_{B_1^1}(Z_\infty(Y))\,E^{Y,\infty}_{z_1}(W_\infty(Y, z_2)), \text{ and}$$
$$\lim_{\lambda,\lambda'\to\infty} E^{Y^2}_{B_1^2}\left(\tilde{Z}^2_{T_2}(Y^2, z_2, z_1, \lambda/\lambda')\,\middle|\, Y^2_{T_2} = z_2\right) = E^{Y,\infty}_{B_1^2}(Z_\infty(Y))\,E^{Y,\infty}_{z_2}(W_\infty(Y, z_1)). \tag{5.40}$$

To find the limit of G it remains to identify the limits of $\Psi^{\lambda,\lambda'}_{B,s}(\sqrt{t-s}z_1, \sqrt{t-s}z_2)$ and $Q(\lambda_1, \lambda_2, B^1, B^2, z_1, z_2)$. From (5.11) we recall that the former prelimit is defined as

$$\Psi^{\lambda,\lambda'}_{B,s}(\sqrt{t-s}z_1, \sqrt{t-s}z_2) = h(\sqrt{t-s}z_1 + B_s, \sqrt{t-s}z_2 + B_s)$$
$$\times \exp\left(-\int_0^s V^{\lambda,\lambda'}_{t-u}(\sqrt{t-s}z_1 + B_s - B_u, \sqrt{t-s}z_2 + B_s - B_u)\,du\right).$$

For all $z_1, z_2 \in \mathbb{R}$ and all Brownian paths $(B_u, u \in [0,s])$, the obvious limit of the above as $\lambda, \lambda' \to \infty$ is obtained by replacing $V^{\lambda,\lambda'}_{t-u}$ with $V^{\infty,\infty}_{t-u}$. By monotonicity (in λ, λ') of the integral and continuity of the exponential we can take the limit inside. Denoting the limit by $\Psi^{\infty,\infty}_{B,s}(\sqrt{t-s}z_1, \sqrt{t-s}z_2)$, we have

$$\lim_{\lambda,\lambda'\to\infty} \Psi^{\lambda,\lambda'}_{B,s}(\sqrt{t-s}z_1, \sqrt{t-s}z_2) = \Psi^{\infty,\infty}_{B,s}(\sqrt{t-s}z_1, \sqrt{t-s}z_2). \tag{5.41}$$

This leaves $Q(\lambda, \lambda', B^1, B^2, z_1, z_2)$, which we recall from (5.23) is defined by

$$\begin{aligned} &Q(\lambda, \lambda', B^1, B^2, z_1, z_2) \\ &= \exp\bigg(- \int_0^1 V^{1,\lambda'/\lambda}_u(B^1_u, B^1_u + e^{T_1/2}(z_2 - z_1)) + V^{1,\lambda/\lambda'}(B^2_u, B^2_u + e^{T_2/2}(z_1 - z_2))\, du\bigg). \end{aligned}$$

The integrand is the sum of two terms that are very similar; for now we restrict our attention to the first. In particular, we will show that

$$\begin{aligned} \lim_{T_1\to\infty} \sup_{\lambda' > (t-s)^{-1/2}} \bigg| \exp\bigg(- \int_0^1 V^{1,\lambda'/\lambda}_u(B^1_u, B^1_u + e^{T_1/2}(z_2 - z_1))\, du\bigg) \\ - \exp\bigg(- \int_0^1 V^1_u(B^1_u)\, du\bigg)\bigg| = 0. \end{aligned} \tag{5.42}$$

We claim that since the second argument of the integrand goes to infinity, asymptotically the function resembles $V^1_u(B^1_u)$. To see this use both parts of Proposition 3.1 to conclude that

$$0 \le \Big[V^{1,c}_u(B^1_u, B^1_u + e^{T_1/2}(z_2 - z_1)) - V^1_u(B^1_u)\Big] \le V^\infty_u(B^1_u + e^{T_1/2}(z_2 - z_1))$$

for all $c > 0$. P^B_0-a.s., there is a constant $R(\omega) > 0$ such that $|B^1_u(\omega)| \le R(\omega)$ for all $u \in [0,s]$. Provided $z_1 \ne z_2$, for sufficiently large λ, $e^{T_1/2}|z_2 - z_1| \ge 2R$. Then for λ sufficiently large and $\lambda, \lambda' > \bar{\lambda}(s)$,

$$\begin{aligned} \bigg| \exp\bigg(- \int_0^1 V^{1,\lambda'/\lambda}_u(B^1_u, B^1_u + e^{T_1/2}(z_2 - z_1))\, du\bigg) - \exp\bigg(- \int_0^1 V^1_u(B^1_u)\, du\bigg)\bigg| \\ \le \int_0^1 V^\infty_u(e^{T_1/2}(z_2 - z_1) - R)\, du. \end{aligned}$$

The integrand is bounded above by $V^\infty_u(R)$. Since $V^\infty_u(R) = u^{-1}F(u^{-1/2}R)$, (from (3.13)) by (3.14)(iii) we have $V^\infty_u(R) \le cu^{-3/2}Re^{-u^{-1}R/2}$, which is bounded on $[0,1]$. We take $\lambda \to \infty$ and apply Dominated Convergence; since $V^\infty_u(y) = u^{-1}F(u^{-1/2}y)$, applying (3.14)(iii) again gives that $\lim_{|y|\to\infty} V^\infty_u(y) = 0$, and hence limit of the above as $\lambda \to \infty$ (ie. as $T_1 \to \infty$) is zero. Thus (5.42) holds. We handle the second term in the integral in $Q(\lambda, \lambda', B^1, B^2, z_1, z_2)$ in an identical fashion, now with the roles of λ and λ' reversed, thereby establishing that

For all $z_1, z_2 \in \mathbb{R}$ such that $z_1 \ne z_2$, $dP^{B^1}_0 dP^{B^2}_0$-a.s., (5.43)

$$\lim_{\lambda,\lambda'\to\infty} Q(\lambda, \lambda', B^1, B^2, z_1, z_2) = \exp\bigg(- \int_0^1 V^1_u(B^1_u)\, du\bigg) \exp\bigg(- \int_0^1 V^1_u(B^2_u)\, du\bigg).$$

We have therefore found the limit of G and hence of Θ. In particular, recall the definitions (5.26) and (5.30). From (5.37), (5.40), (5.41) and (5.43), we have shown that $dP^B_0 dP^{B^1}_0 dP^{B^2}_0$-a.s., for all $z_1, z_2 \in \mathbb{R}$ such that $z_1 \ne z_2$,

$$\begin{aligned} \lim_{\lambda,\lambda'\to\infty} \Theta(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2) &= (t-s)^{-2\lambda_0} \Psi^{\infty,\infty}_{B,s}(\sqrt{t-s}z_1, \sqrt{t-s}z_2) \\ &\times E^{Y,\infty}_{z_1}(W_\infty(Y, z_2)) E^{Y,\infty}_{z_2}(W_\infty(Y, z_1))\, E^{Y,\infty}_{B^1_1}(Z_\infty(Y)) E^{Y,\infty}_{B^2_1}(Z_\infty(Y)) \\ &\times \exp\bigg(\int_0^1 -V^1_u(B^1_u) - V^1_u(B^2_u)\, du\bigg)\, \psi_0(B^1_1)\, \psi_0(B^2_1)\, \psi_0(z_1)\, \psi_0(z_2). \end{aligned} \tag{5.44}$$

Thus by (5.36), $\lim_{\lambda,\lambda'\to\infty} \mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h))$ exists and satisfies

$$\lim_{\lambda,\lambda'\to\infty} \mathbb{N}_0((L_t^\lambda \times L_t^{\lambda'})(h)) = \int_0^t \left[\iint E_0^B (E_{(0,0)}^{B^1,B^2}[\lim_{\lambda,\lambda'\to\infty} \Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)])\, dm(z_1)\, dm(z_2)\right] ds. \quad (5.45)$$

To obtain the desired expression, we note that the terms in (5.44) that depend on B^1 and B^2 can be collected in a constant. In particular, we define a constant $C_{1.4} > 0$ by

$$\begin{aligned} C_{1.4}^2 &= E_{(0,0)}^{B^1,B^2}\Bigg(\exp\Bigg(-\int_0^1 V_u^1(B_u^1) + V_u^1(B_u^2)\, du\Bigg) \\ &\qquad \times E_{B_1^1}^{Y,\infty}(Z_\infty(Y)) E_{B_1^2}^{Y,\infty}(Z_\infty(Y))\psi_0(B_1^1)\psi_0(B_1^2)\Bigg) \\ &= \left[E_0^B \left(\exp\left(-\int_0^1 V_u^1(B_u)\, du\right) E_{B^1}^{Y,\infty}(Z_\infty(Y))\,\psi_0(B_1)\right)\right]^2. \end{aligned} \quad (5.46)$$

We also define a function $\rho(\cdot,\cdot)$ by

$$\rho(z_1,z_2) = E_{z_1}^{Y,\infty}(W_\infty(Y,z_2)) E_{z_2}^{Y,\infty}(W_\infty(Y,z_1)). \quad (5.47)$$

It is clear that $\rho(\cdot,\cdot)$ is jointly continuous and bounded by 1 from the definition of $W_\infty(Y,z)$. Thus by (5.44),

$$\begin{aligned} &E_{(0,0)}^{B^1,B^2}\left[\lim_{\lambda,\lambda'\to\infty} \Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)\right] \\ &\quad = C_{1.4}^2 (t-s)^{-2\lambda_0}\, \Psi_{B,s}^{\infty,\infty}(\sqrt{t-s}z_1, \sqrt{t-s}z_2)\, \rho(z_1,z_2)\, \psi_0(z_1)\, \psi_0(z_2). \end{aligned}$$

Substituting the above into (5.45) completes the proof of Theorem 1.4. □

6 Proof of Lemma 5.5

Recalling the statement of the lemma, we will show that

$$\lim_{\lambda,\lambda'\to\infty} E_x^Y\left(\tilde{Z}_T(Y,z_1,z_2,\lambda'/\lambda) \middle| Y_T = z_1\right) = E_x^{Y,\infty}(Z_\infty(Y))\, E_{z_1}^{Y,\infty}(W_\infty(Y,z_2)) \quad (6.1)$$

for all $x, z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$, where the law of Y on the left hand side is that of a killed Ornstein-Uhlenbeck process with killing function F, with $Y_0 = x$. Recall that $E_x^{Y,\infty}$ is the expectation under the law of the killed process Y with $Y_0 = x$ conditioned to survive for all time. For convenience, we now recall the definitions of the quantities above: from (3.16), (5.38) and (5.39),

$$\begin{aligned} \tilde{Z}_T(Y,z_1,z_2,\lambda'/\lambda) &= \exp\left(\int_0^T F(Y_u) - H_{e^u}^{\lambda'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\, du\right), \\ W_S(Y,z) &= \exp\left(\int_0^S F(Y_u) - F_2(Y_u, Y_u - e^{u/2}(z - Y_0))\, du\right), \\ Z_T(Y) &= \exp\left(\int_0^T F(Y_u) - V_1^{e^{u/2}}(Y_u)\, du\right). \end{aligned}$$

As we have previously discussed, $Z_\infty(Y)$ and $W_\infty(Y,z)$ are the respective limits of $Z_T(Y)$ and $W_S(Y,z)$ as $T,S\to\infty$, both of which exist. Recall from (3.18) that $Z_\infty(Y) \le C_Z$. Because z_1 and z_2 are fixed, we will hereafter suppress the dependence of $\tilde{Z}_T(Y,z_1,z_2,\lambda'/\lambda)$ on z_1 and z_2 and simply write $\tilde{Z}_T(Y,\lambda'/\lambda)$. Finally, we recall our working assumption that $\lambda,\lambda' > \bar{\lambda}(s)$, (see (5.32)) which implies that $\lambda,\lambda' > 1$ and $T > s^*(1/8)$.

Let $0 < K < T/2$. We apply the Markov property to $E^Y_x(\tilde{Z}_T(Y,\lambda'/\lambda)\big|\, Y_T = z_1)$ at times K and $T-K$ and expand in terms of the joint density of (Y_K, Y_{T-K}). As in (2.11), the joint density of (Y_K, Y_{T-K}) at (w,y) with respect to $m \times m$ under $P^Y_x(Y \in \cdot\,|Y_T = z_1)$ is

$$\frac{q_K(x,w)q_{T-2K}(w,y)q_K(y,z_1)}{q_T(x,z_1)}.$$

Thus we obtain the following:

$$\begin{aligned}
&E^Y_x\left(\tilde{Z}_T(Y,\lambda'/\lambda)\,\big|\, Y_T = z_1\right)\\
&\quad = E^Y_x\bigg(\exp\bigg(\int_0^T F(Y_u) - H^{\lambda'/\lambda}_{e^u}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\,du\bigg)\bigg|\, Y_T = z_1\bigg)\\
&\quad = \iint E^Y_x\bigg(\exp\bigg(\int_0^K F(Y_u) - H^{\lambda'/\lambda}_{e^u}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\,du\bigg)\bigg|\, Y_K = w\bigg)\\
&\qquad \times E^Y_w\bigg(\exp\bigg(\int_0^{T-2K} F(Y_u) - H^{\lambda'/\lambda}_{e^{K+u}}(Y_u, Y_u + e^{\frac{T-K-u}{2}}(z_2 - z_1))\,du\bigg)\bigg|\, Y_{T-2K} = y\bigg)\\
&\qquad \times E^Y_y\bigg(\exp\bigg(\int_0^K F(Y_u) - H^{\lambda'/\lambda}_{e^{T-K+u}}(Y_u, Y_u + e^{\frac{K-u}{2}}(z_2 - z_1))\,du\bigg)\bigg|\, Y_K = z_1\bigg)\\
&\qquad \times \frac{q_K(x,w)q_{T-2K}(w,y)q_K(y,z_1)}{q_T(x,z_1)}\,dm(w)\,dm(y). \qquad (6.2)
\end{aligned}$$

Denote the three conditional expectations by $A_1(x,w,\lambda,\lambda',K)$, $A_2(w,y,\lambda,\lambda',K)$ and $A_3(y,z_1,\lambda,\lambda',K)$. That is,

$$\begin{aligned}
&A_1(x,w,\lambda,\lambda',K) \qquad (6.3)\\
&\quad = E^Y_x\bigg(\exp\bigg(\int_0^K F(Y_u) - H^{\lambda'/\lambda}_{e^u}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\,du\bigg)\bigg|\, Y_K = w\bigg),\\
&A_2(w,y,\lambda,\lambda',K) \qquad (6.4)\\
&\quad = E^Y_w\bigg(\exp\bigg(\int_0^{T-2K} F(Y_u) - H^{\lambda'/\lambda}_{e^{K+u}}(Y_u, Y_u + e^{\frac{T-K-u}{2}}(z_2 - z_1))\,du\bigg)\bigg|\, Y_{T-2K} = y\bigg),\\
&A_3(y,z_1,\lambda,\lambda',K) \qquad (6.5)\\
&\quad = E^Y_y\bigg(\exp\bigg(\int_0^K F(Y_u) - H^{\lambda'/\lambda}_{e^{T-K+u}}(Y_u, Y_u + e^{\frac{K-u}{2}}(z_2 - z_1))\,du\bigg)\bigg|\, Y_K = z_1\bigg).
\end{aligned}$$

We observe that A_1, A_2 and A_3 all depend on z_1 and z_2 in addition to their listed arguments, as these values appear in their integrands. Again, as z_1 and z_2 are fixed, we omit this additional dependence. Noting that the integrand is bounded above by $F(Y_u) - V_1^{e^{u/2}}(Y_u)$ in each case, from (3.18) we have $A_i \le C_Z$ for $i = 1,2,3$. In terms of the A_i, (6.2) can be rewritten as

$$\begin{aligned}
E^Y_x\left(\tilde{Z}_T(Y,\lambda'/\lambda)\,\big|\, Y_T = z_1\right) &= \iint A_1(x,w,\lambda,\lambda',K)\,A_2(w,y,\lambda,\lambda',K)\,A_3(y,z_1,\lambda,\lambda',K)\\
&\quad \times \frac{q_K(x,w)\,q_{T-2K}(w,y)\,q_K(y,z_1)}{q_T(x,z_1)}\,dm(w)\,dm(y). \qquad (6.6)
\end{aligned}$$

There are two main contributions in the A_i. The first comes from F and the first argument of the H function, and is approximately equal to $F(Y_u) - V_1^{e^{u/2}}(Y_u)$; the second comes from the second argument of the H function. We will see that, asymptotically, A_1 is only affected by the first contribution and gives the $Z_\infty(Y)$ term in (6.1); A_3 is only affected by the second contribution and gives the $W_\infty(Y,z_2)$ term in (6.1). The contribution of

A_2 is will be seen to be negligible. We first show that A_2 is arbitrarily close to 1 as K is made large, uniformly in T sufficiently large depending on K. Define $Z^a_T(Y,\lambda'/\lambda,K)$ as $\tilde{Z}_T(Y,\lambda'/\lambda,K)$ with A_2 replaced by 1; that is,

$$\begin{aligned} Z^a_T(Y,\lambda'/\lambda,K) = \exp\bigg(\int_0^K F(Y_u) - H^{\lambda'/\lambda}_{e^u}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2-z_1))\,du\bigg) \\ \times \exp\bigg(\int_{T-K}^T F(Y_u) - H^{\lambda'/\lambda}_{e^u}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2-z_1))\,du\bigg). \end{aligned} \tag{6.7}$$

As in (6.2) and (6.6), we therefore have

$$\begin{aligned} &E^Y_x\big(Z^a_T(Y,\lambda'/\lambda,K)\,\big|\,Y_T=z_1\big) \\ &\quad= \iint A_1(x,w,\lambda,\lambda',K)A_3(y,z_1,\lambda,\lambda',K)\frac{q_K(x,w)\,q_{T-2K}(w,y)\,q_K(y,z_1)}{q_T(x,z_1)}\,dm(w)\,dm(y). \end{aligned} \tag{6.8}$$

By monotonicity (Proposition 3.1(a)) and (3.15) we have

$$F(Y_u) - H^{\lambda'/\lambda}_{e^{K+u}}(Y_u, Y_u + e^{\frac{T-K-u}{2}}(z_2-z_1)) \le F(Y_u) - V_1^{e^{(K+u)/2}}(Y_u) \le Ce^{-(K+u)(2\lambda_0-1)/2} \tag{6.9}$$

uniformly in $T>2K$. Integrating this over u shows that the exponent in A_2 is bounded above $C'e^{-(2\lambda_0-1)K/2}$ for a constant C', uniformly in $T>2K$. We choose K large enough so that exponent in A_2 is smaller than 2. Then by (6.6) and (6.7), applying the mean value theorem, we have

$$\begin{aligned} &\big|E^Y_x\big(\tilde{Z}_T(Y,\lambda'/\lambda) - Z^a_T(Y,\lambda'/\lambda,K)\,\big|\,Y_T=z_1\big)\big| \\ &\le \frac{1}{q_T(x,z_1)}\iint A_1(x,w,\lambda,\lambda',K)\Big|A_2(w,y,\lambda,\lambda',K)-1\Big|A_3(y,z_1,\lambda,\lambda',K) \\ &\qquad\qquad \times q_K(x,w)q_{T-2K}(w,y)q_K(y,z_1)\,dm(w)\,dm(y) \\ &\le \frac{e^2C_Z^2}{q_T(x,z_1)}\iint E^Y_w\bigg(\int_0^{T-2K}|F(Y_u) - H^{\lambda'/\lambda}_{e^{K+u}}(Y_u,Y_u+e^{\frac{T-K-u}{2}}(z_2-z_1))|\,du\,\bigg|\,Y_{T-2K}=y\bigg) \\ &\qquad\qquad \times q_K(x,w)q_{T-2K}(w,y)q_K(y,z_1)\,dm(w)\,dm(y) \end{aligned} \tag{6.10}$$

uniformly for all $T>2K$, where we have also used $A_1A_3 \le C_Z^2$. The term in the absolute value inside the integral can be positive or negative; (6.9) provides an upper bound for $F - H^{\lambda'/\lambda}_{e^{K+u}}$. To obtain a lower bound, we note that $H^{\lambda'/\lambda}_{e^{K+u}}(a,b) \le F_2(a,b) \le F(a)+F(b)$ by Proposition 3.1 (using part (a) and then part (b)). This bound implies that

$$F(Y_u) - H^{\lambda'/\lambda}_{e^{K+u}}(Y_u, Y_u + e^{\frac{T-K-u}{2}}(z_2-z_1)) \ge -F(Y_u + e^{\frac{T-K-u}{2}}(z_2-z_1)). \tag{6.11}$$

Together, (6.9) and (6.11) imply that the absolute value appearing in the integral in (6.10) is bounded above by

$$Ce^{-(K+u)(2\lambda_0-1)/2} + F(Y_u + e^{\frac{T-K-u}{2}}(z_2-z_1)).$$

We have already noted that when integrated over u, the first term is bounded by $C'e^{-K(2\lambda_0-1)/2}$ (uniformly in T). The first term has no dependence on the spatial parameters w and y, so in (6.10) the transition densities and can be integrated and cancelled with the denominator. We get that for all $T>2K$, (6.10) is bounded above by

$$\begin{aligned} C'e^{-K(2\lambda_0-1)/2} + \frac{C}{q_T(x,z_1)}\iint E^Y_w\bigg(\int_0^{T-2K} F(Y_u + e^{\frac{T-K-u}{2}}(z_2-z_1))\,du\,\bigg|\,Y_{T-2K}=y\bigg) \\ \times q_K(x,w)q_{T-2K}(w,y)q_K(y,z_1)\,dm(w)\,dm(y). \end{aligned}$$

We consider the time reversed process in the above and apply (2.12), which implies that the above is equal to, and hence for all $T > 2K$, (6.10) is bounded above by

$$C' e^{-K(2\lambda_0-1)/2} + \frac{C}{q_T(x,z_1)} \iint E_y^Y \left(\int_0^{T-2K} F(Y_u + e^{\frac{K+u}{2}}(z_2 - z_1))\, du \,\middle|\, Y_{T-2K} = w \right) \times q_K(x,w) q_{T-2K}(w,y) q_K(y,z_1)\, dm(w)\, dm(y). \tag{6.12}$$

We recall the asymptotic behaviour of F from (3.14)(iii), ie. that $F(x) \sim c_1 |x| e^{-x^2/2}$ as $|x| \to \infty$. This implies there is a constant $c_2 > 0$ such that

$$F(x) \le c_2(1 + |x|)e^{-x^2/2} \quad \text{for all} \quad x \in \mathbb{R}. \tag{6.13}$$

In order for this to give a useful upper bound in (6.12), we'll need to show that the argument of F is large in absolute value. It is enough to show that $|Y_u| \ll e^{\frac{K+u}{2}}|z_2 - z_1|$ with high probability when conditioned on its endpoint. Recall that we have assumed $z_1 \neq z_2$. We bound the integrand over the two cases mentioned above and exchange the integral and expectation, which is justifiable since F is positive. We have

$$\begin{aligned} &E_y^Y \left(\int_0^{T-2K} F(Y_u + e^{\frac{K+u}{2}}(z_2 - z_1))\, du \,\middle|\, Y_{T-2K} = w \right) \\ &\le E_y^Y \left(\int_0^{T-2K} F(e^{\frac{K+u}{4}}|z_2 - z_1|) + F(0) 1(|Y_u| \ge e^{\frac{K+u}{4}}|z_2 - z_1|)\, du \,\middle|\, Y_{T-2K} = w \right) \\ &\le c_2 \int_0^\infty (1 + e^{\frac{K+u}{4}}|z_2 - z_1|) e^{-e^{\frac{K+u}{2}}|z_2 - z_1|^2/2} du \\ &\qquad + F(0) \int_0^{T-2K} P_y^Y(|Y_u| \ge e^{\frac{K+u}{4}}|z_2 - z_1| \,|\, Y_{T-2K} = w)\, du, \end{aligned} \tag{6.14}$$

where we have used (6.13) and the fact that F is radially decreasing. A simple substitution shows that

$$\begin{aligned} &c_2 \int_0^\infty (1 + e^{\frac{K+u}{4}}|z_2 - z_1|) e^{-e^{\frac{K+u}{2}}|z_2 - z_1|^2/2} du \\ &\le 4c_1 \int_{e^{K/4}|z_2 - z_1|}^\infty (1 + a^{-1}) e^{-a^2/2} da \\ &\le C \int_{e^{K/4}|z_2 - z_1|}^\infty e^{-a^2/2} da + C 1(e^{K/4}|z_2 - z_1| < 1) \int_{e^{K/4}|z_2 - z_1|}^1 a^{-1} e^{-a^2/2} da \\ &\le C \int_{e^{K/4}|z_2 - z_1|}^\infty e^{-a^2/2} da - C \left[\log(e^{K/4}|z_2 - z_1|) \wedge 0 \right]. \end{aligned} \tag{6.15}$$

To bound the second term in (6.14) we expand the probability of the large excursion in terms of the transition densities. There are two cases, which we handle in the following lemma. In what follows, $s^* = s^*(1/8)$ from Theorem 2.1(c).

Lemma 6.1. *Let $M > 0$ and $w, y \in \mathbb{R}$.*
(a) There is a constant $C > 0$ such that for $S, u > 0$ satisfying $u, S - u \ge s^*$,

$$P_y^Y(|Y_u| \ge M \,|\, Y_S = w) \le \frac{C}{q_S(y,w)} e^{-\lambda_0 S} e^{(y^2 + w^2)/8} \left[\frac{e^{-M^2/4}}{M} \wedge 1 \right].$$

(b) For fixed $u_0 > 0$ the families

$$\{P_y^Y(Y_u \in \cdot \,|\, Y_S = w) : S \ge u_0, 0 \le u \le u_0\} \text{ and}$$
$$\{P_y^Y(Y_{S-u} \in \cdot \,|\, Y_S = w) : S \ge u_0, 0 \le u \le u_0\}$$

are tight.

Proof. To see (a), we use (2.11) and (2.2) with $\delta = 1/8$ to obtain that for $u, S-u \geq s^*$,

$$\begin{aligned} P^Y_y(Y_u \geq M \,|\, Y_S = w) &= \int_M^\infty \frac{q_u(y,a)q_{S-u}(a,w)}{q_S(y,w)} dm(a) \\ &\leq \frac{c^2_{1/8}}{q_S(y,w)} e^{-\lambda_0 S} \int_M^\infty e^{(y^2+2a^2+w^2)/8}\, dm(a) \\ &\leq \frac{C}{q_S(y,w)} e^{-\lambda_0 S} e^{(y^2+w^2)/8} \left[\frac{e^{-M^2/4}}{M} \wedge 1\right], \end{aligned}$$

where the last line uses a standard upper bound on Gaussian tails and bounds the integral above by a constant when M is small. The bound for $Y_u < -M$ is the same. The first family in part (b) is tight as a consequence of Lemma 2.3(c). To see that the second family is tight we consider the time reversal of Y and use (2.12), from which tightness now also follows from Lemma 2.3(c). □

Applying Lemma 6.1(a), using (6.15) and separating the integrals depending if $u, S-u \geq s^*$ or not, we have that (6.14) is bounded above by

$$\begin{aligned} & C \int_{e^{K/4}|z_2-z_1|}^\infty e^{-a^2/2} da - C\left[\log(e^{K/4}|z_2-z_1|) \wedge 0\right] \\ &+ \frac{C}{q_{T-2K}(y,w)} e^{-\lambda_0(T-2K)} e^{y^2/8} e^{w^2/8} \int_{s^*}^{T-2K-s^*} \left[\frac{e^{-e^{\frac{K+u}{2}}|z_2-z_1|^2/4}}{e^{\frac{K+u}{4}}|z_2-z_1|} \wedge 1\right] du \\ &+ C\left(\int_0^{s^*} + \int_{T-2K-s^*}^{T-2K}\right) P^Y_y(|Y_u| \geq e^{\frac{K+u}{4}}|z_2-z_1| \,|\, Y_{T-2K} = w)\, du. \end{aligned} \tag{6.16}$$

As the above is an upper bound for the expectation appearing in the second term of (6.12), and (6.12) is an upper bound for (6.10), we have

$$\begin{aligned} & \left|E^Y_x\left(\tilde{Z}_T(Y,\lambda'/\lambda) - Z^a_T(Y,\lambda'/\lambda,K) \,|\, Y_T = z_1\right)\right| \\ &\quad \leq Ce^{-K(2\lambda_0-1)/2} + \frac{C}{q_T(x,z_1)} \iint \Bigg[\int_{e^{K/4}|z_2-z_1|}^\infty e^{-a^2/2} da - C\left[\log(e^{K/4}|z_2-z_1|) \wedge 0\right] \\ &\quad + \frac{1}{q_{T-2K}(y,w)} e^{-\lambda_0(T-2K)} e^{y^2/8} e^{w^2/8} \int_{s^*}^{T-2K-s^*} \left[\frac{e^{-e^{\frac{K+u}{2}}|z_2-z_1|^2/4}}{e^{\frac{K+u}{4}}|z_2-z_1|} \wedge 1\right] du \\ &\quad + \left(\int_0^{s^*} + \int_{T-2K-s^*}^{T-2K}\right) P^Y_y(|Y_u| \geq e^{\frac{K+u}{4}}|z_2-z_1| \,|\, Y_{T-2K} = w)\, du\Bigg] \\ &\qquad \times q_K(x,w) q_{T-2K}(w,y) q_K(y,z_1)\, dm(w)\, dm(y). \end{aligned} \tag{6.17}$$

Note that the first two terms in the integral with respect to y and w are independent of these variables. We can therefore integrate them out; using the fact that

$$\iint q_K(x,w) q_{T-2K}(w,y) q_K(y,z_1)\, dm(w)\, dm(y) = q_T(x,z_1)$$

(and an obvious cancellation) we obtain that

$$\begin{aligned} & \left|E^Y_x\left(\tilde{Z}_T(Y,\lambda'/\lambda) - Z^a_T(Y,\lambda'/\lambda,K) \,|\, Y_T = z_1\right)\right| \\ &\quad \leq Ce^{-K(2\lambda_0-1)/2} + C\int_{e^{K/4}|z_2-z_1|}^\infty e^{-a^2/2} da - C\left[\log(e^{K/4}|z_2-z_1|) \wedge 0\right] \\ &\quad + \frac{C}{q_T(x,z_1)} e^{-\lambda_0(T-2K)} \iint e^{y^2/8} e^{w^2/8} q_K(x,w) q_K(y,z_1)\, dm(w)\, dm(y) \end{aligned}$$

$$\times \left(\int_{s^*}^{T-2K-s^*} \left[\frac{e^{-e^{\frac{K+u}{2}}|z_2-z_1|^2/4}}{e^{\frac{K+u}{4}}|z_2-z_1|} \wedge 1 \right] du \right)$$
$$+ \frac{C}{q_T(x,z_1)} \iint \left[\left(\int_0^{s^*} + \int_{T-2K-s^*}^{T-2K} \right) P_y^Y\left(|Y_u| \geq e^{\frac{K+u}{4}}|z_2-z_1| \,\middle|\, Y_{T-2K}=w\right) du \right]$$
$$\times q_K(x,w) q_{T-2K}(w,y) q_K(y,z_1)\, dm(w)\, dm(y)$$
$$=: \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5, \tag{6.18}$$

where $\delta_i = \delta_i(T,K,z_1,z_2)$ and is defined by the obvious correspondence. We first note that

$$\delta_i(T,K,z_1,z_2) \to 0 \text{ as } K \to \infty \text{ (uniformly in } T \geq 2K) \text{ for } i = 1,2,3.$$

Turning to δ_4 and δ_5, we observe that by Lemma 2.2, $e^{\lambda_0 T} q_T(x,z_1) \to \psi_0(x)\psi_0(z_1)$ as $\lambda \to \infty$. Since $T \to q_T(x,z_1)$ is continuous, $q_T(x,z_1) > 0$ for all $T \geq \tau(s)$ and $\psi_0(x)\psi_0(z_1) > 0$, this implies that there exists $\beta(x,z_1) = \beta > 0$ such that

$$q_T(x,z_1) \geq \beta e^{-\lambda_0 T} \psi_0(x)\psi_0(z_1) \qquad \forall\, T \geq \tau(s). \tag{6.19}$$

Applying (2.2) twice with $\delta = 1/8$ and using (6.19), we have

$$|\delta_4(T,K,z_1,z_2)|$$
$$\leq \frac{C\beta^{-1}}{\psi_0(x)\psi_0(z_1)} e^{2\lambda_0 K} e^{-2\lambda_0 K} e^{(x^2+z_1^2)/8} \iint e^{y^2/4} e^{w^2/4} dm(w)\, dm(y)$$
$$\times \left(\int_{s^*}^{T-2K-s^*} \frac{e^{-e^{\frac{K+u}{2}}|z_2-z_1|^2/4}}{e^{\frac{K+u}{4}}|z_2-z_1|}\, du \right) \tag{6.20}$$
$$\leq \frac{Ce^{(x^2+z_1^2)/8}}{\psi_0(x)\psi_0(z_1)} \left(\int_{s^*}^{T-2K-s^*} \frac{e^{-e^{\frac{K+u}{2}}|z_2-z_1|^2/4}}{e^{\frac{K+u}{4}}|z_2-z_1|}\, du \right)$$
$$= \frac{4Ce^{(x^2+z_1^2)/8}}{\psi_0(x)\psi_0(z_1)} \int_{e^{s^*/4}e^{K/4}|z_2-z_1|}^{\infty} \frac{e^{-a^2/4}}{a^2}\, da, \tag{6.21}$$

where the last line follows from a simple substitution. Thus we have $\delta_4(T,K,z_1,z_2) \to 0$ as $K \to \infty$, and again we note that convergence is uniform in $T > 2K$. It remains to handle δ_5. By three applications of (2.2) with $\delta = 1/8$ and (6.19), we have

$$|\delta_5(T,K,z_1,z_2)|$$
$$\leq \frac{C}{\psi_0(x)\psi_0(z_1)} \iint \left[\left(\int_0^{s^*} + \int_{T-2K-s^*}^{T-2K} \right) P_y^Y\left(|Y_u| \geq e^{\frac{K+u}{4}}|z_2-z_1| \,\middle|\, Y_{T-2K}=w\right) du \right]$$
$$\times e^{x^2/8} e^{w^2/4} e^{y^2/4} e^{z_1^2/8}\, dm(w)\, dm(y).$$

The square bracketed term vanishes as $K \to \infty$ uniformly in $T \geq 2K + s^*(1/8)$ by Lemma 6.1(b). The probabilities are bounded so the integrand obviously has a uniformly integrable upper bound. By Dominated Convergence, we have that $\delta_5(T,K,z_1,z_2) \to 0$ as $K \to \infty$, uniformly in $T \geq 2K + s^*(1/8)$. We have therefore shown that $\sum_{i=1}^5 \delta_i(T,K,z_1,z_2)$ is arbitrarily small as $K \to \infty$, uniformly in $T \geq 2K + s^*(1/8)$ and in $\lambda' > \bar{\lambda}(s)$, where we recall that we have assumed $\lambda, \lambda' > \bar{\lambda}(s)$. From (5.33), $\lambda > \bar{\lambda}(s)$ is equivalent to $T > \tau(s)$. As $\tau(s) \geq s^*(1/8)$, $T \geq 2K + \tau(s)$ implies that $T \geq 2K + s^*(1/8)$. Thus by (6.18) we have proved the following. Recall that $\tilde{Z}_T(Y, \lambda'/\lambda) = \tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda)$.

Lemma 6.2. *For all* $x, z_1, z_2 \in \mathbb{R}$ *such that* $z_1 \neq z_2$,

$$\delta_a^{\sim}(K) = \sup_{T \geq 2K+\tau(s),\, \lambda' > \bar{\lambda}(s)} \left| E_x^Y\left(\tilde{Z}_T(Y, \lambda'/\lambda) - Z_T^a(Y, \lambda'/\lambda, K) \,\middle|\, Y_T = z_1 \right) \right|$$

satisfies $\lim_{K\to\infty} \delta_a^{\sim}(K) = 0$.

Given this lemma, it suffices to find the limit of (the conditional expectation of) $Z_T^a(Y, \lambda'/\lambda, K)$, and so A_2 has been replaced by 1.

Next we consider $A_3(y, z_1, \lambda, \lambda', K)$, which we recall from (6.5) is defined as

$$E_y^Y\left(\exp\left(\int_0^K F(Y_u) - H^{\lambda'/\lambda}_{e^{T-K+u}}(Y_u, Y_u + e^{\frac{K-u}{2}}(z_2 - z_1))\, du \right) \middle| Y_K = z_1 \right).$$

We will show that in the limit as $\lambda, \lambda' \to \infty$, the integrand will be $F - F_2$. Define $A_3^*(y, z_1, K)$ by

$$A_3^*(y, z_1, K) = E_y^Y\left(\exp\left(\int_0^K F(Y_u) - F_2(Y_u, Y_u + e^{\frac{K-u}{2}}(z_2 - z_1))\, du \right) \middle| Y_K = z_1 \right). \quad (6.22)$$

The difference between the integrands of A_3 and A_3^* is equal to $(F_2 - H^{\lambda'/\lambda}_{e^{T-K+u}})(Y_u, Y_u + e^{\frac{K-u}{2}}(z_2 - z_1))$, which is non-negative by monotonicity. To obtain an upper bound we apply Lemma 3.2 to obtain

$$\begin{aligned}
&(F_2 - H^{\lambda'/\lambda}_{e^{T-K+u}})(Y_u, Y_u + e^{\frac{K-u}{2}}(z_2 - z_1)) \\
&\quad \le C\left[e^{-(T-K+u)(2\lambda_0-1)/2} + \left(\frac{\lambda'}{\lambda}\right)^{-(2\lambda_0-1)} e^{-(T-K+u)(2\lambda_0-1)/2} \right] \\
&\quad \le Ce^{(K-u)(2\lambda_0-1)/2}(t-s)^{-(2\lambda_0-1)/2}\left[\lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)} \right]. \qquad (6.23)
\end{aligned}$$

The first line uses the definition of H^c, which we recall from (5.12), and in the second line we have used that $T = \log(\lambda^2(t-s))$. Since $\lambda, \lambda' > \bar{\lambda}(s) \ge 1$, the last expression in (6.23) is bounded by $Ce^{K/2}(t-s)^{-(2\lambda_0-1)/2}$ for all $u \in [0, K]$. Thus, using $|e^x - e^y| \le (e^x \vee e^y)|x-y|$ and (6.23), we have

$$\begin{aligned}
&|A_3^*(y, z_1, K) - A_3(y, z_1, \lambda, \lambda', K)| \\
&\quad \le \exp\left(CKe^{K/2}(t-s)^{-(2\lambda_0-1)/2} \right) \\
&\qquad \times E_y^Y\left(\int_0^K (F_2 - H^{\lambda'/\lambda}_{e^{T-K+u}})(Y_u, Y_u + e^{K/2}(z_2 - z_1))\, du \,\middle|\, Y_K = z_1 \right) \\
&\quad \le \exp\left(CKe^{K/2}(t-s)^{-(2\lambda_0-1)/2} \right)(t-s)^{-(2\lambda_0-1)/2} \\
&\qquad \times \left[\lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)} \right] \int_0^K Ce^{(K-u)(2\lambda_0-1)/2} du \\
&\quad \le C(K, t-s)\left[\lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)} \right] \qquad (6.24)
\end{aligned}$$

for some constant $C(K, t-s) > 0$. Define $Z_T^b(Y, \lambda'/\lambda, K)$ as we defined $Z_T^a(Y, \lambda'/\lambda, K)$ in (6.7) but with $F - F_2$ replacing the integrand in the second term. That is,

$$\begin{aligned}
Z_T^b(Y, \lambda'/\lambda, K) = &\exp\left(\int_0^K F(Y_u) - H^{\lambda'/\lambda}_{e^u}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\, du \right) \\
&\times \exp\left(\int_{T-K}^T F(Y_u) - F_2(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\, du \right). \qquad (6.25)
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&E_x^Y\left(Z_T^b(Y, \lambda'/\lambda, K) \,\middle|\, Y_T = z_1 \right) \qquad (6.26) \\
&\quad = \iint A_1(x, w, \lambda, \lambda', K) A_3^*(y, z_1, K) \frac{q_K(x, w)\, q_{T-2K}(w, y)\, q_K(y, z_1)}{q_T(x, z_1)}\, dm(w)\, dm(y).
\end{aligned}$$

Because (6.24) is uniform in y and z_1 and $|A_1| \le C_Z$, we can integrate out the transition densities to obtain the following.

Lemma 6.3. *For $K > 0$ and $s \in [0,t)$, there is a constant $C(K, t-s)$ such that*

$$\begin{aligned}\delta_b^a(K,\lambda,\lambda') &= \left|E_x^Y\left(Z_T^a(Y,\lambda'/\lambda,K) - Z_T^b(Y,\lambda'/\lambda,K)\,\middle|\,Y_T = z_1\right)\right| \\ &\le C(K,t-s)\left[\lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)}\right]\end{aligned}$$

for all $\lambda, \lambda' > \bar{\lambda}(s)$.

We now analyse A_3^* in greater detail. In particular, we perform a time reversal on the process Y. By (2.12), we have

$$A_3^*(y,z_1,K) = E_{z_1}^Y\left(\exp\left(\int_0^K F(Y_u) - F_2(Y_u, Y_u + e^{\frac{u}{2}}(z_2-z_1))\,du\right)\middle| Y_K = y\right).$$

This is the term that in (6.1) we claimed converges to $E_{Z_1}^{Y,\infty}(W_\infty(Y,z_2))$, defined in (5.39), in the limit. However, the above expectation is still conditional on the endpoint. We now show that the contribution from the tail of the integral is vanishing, making the quantity asymptotically independent of the endpoint y. Let $0 < M < K$. Define $A_3^*(y,z_1,M,K)$ by truncating the integral in (6.22) at time M. That is,

$$A_3^*(y,z_1,M,K) = E_{z_1}^Y\left(\exp\left(\int_0^M F(Y_u) - F_2(Y_u, Y_u + e^{\frac{u}{2}}(z_2-z_1))\,du\right)\middle| Y_K = y\right). \quad (6.27)$$

We now define $Z^c(Y,\lambda'/\lambda,M,K)$ by truncating the corresponding integral in $Z^b(Y,\lambda'/\lambda,K)$ (the integral over $[T-K,T]$ in (6.25) becomes the integral over $[T-M,T]$) so that $A_3^*(y,z_1,M,K)$ replaces $A_3^*(y,z_1,K)$ in the conditional expectation.

Lemma 6.4. *For all $x, z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$,*

$$\delta_c^b(M) = \sup_{K \ge M + s^*(1/8)} \; \sup_{T \ge 2K+\tau(s),\, \lambda' > \bar{\lambda}(s)} \left|E_x^Y\left(Z_T^b(Y,\lambda'/\lambda,K) - Z_T^c(Y,\lambda'/\lambda,M,K)\,\middle|\,Y_T = z_1\right)\right|$$

satisfies $\lim_{M\to\infty}\delta_c^b(M) = 0$.

Proof. Using the inequality $|e^{-x} - e^{-y}| \le |x-y|$ for $x,y \ge 0$, we have

$$\begin{aligned}&|A_3^*(y,z_1,K) - A_3^*(y,z_1,M,K)| \\ &\quad\le E_{z_1}^Y\left(\int_M^K |F(Y_u) - F_2(Y_u, Y_u + e^{u/2}(z_2-z_1))|\,du\,\middle|\, Y_K = y\right).\end{aligned} \quad (6.28)$$

By Proposition 3.1(b), the absolute value of the above integrand is at most $F(Y_u + e^{\frac{u}{2}}(z_2 - z_1))$ (for a similar argument see (6.11)). Exchanging expectation and integration, we proceed as in (6.14), (6.15), and (6.16), and apply Lemma 6.1 to bound (6.28) above by

$$\begin{aligned}&c_1\int_0^{K-M}(1 + e^{\frac{M+u}{4}}|z_2-z_1|)e^{-e^{\frac{M+u}{2}}|z_2-z_1|^2/2}\,du \\ &+ F(0)\int_M^K P_{z_1}^Y(|Y_u| \ge e^{\frac{u}{4}}|z_2-z_1|\,|\,Y_K = y)\,du \\ &\quad\le 4c_1\int_{e^{M/4}|z_2-z_1|}^\infty (1+a^{-1})e^{-a^2/2}\,da \\ &\qquad + \frac{Ce^{-\lambda_0 K}}{q_K(z_1,y)}e^{(z_1^2+y^2)/8}\int_0^{K-M-s^*}\left[\frac{e^{-e^{\frac{M+u}{2}}|z_2-z_1|^2}}{e^{\frac{M+u}{4}}|z_2-z_1|}\wedge 1\right]du \\ &\qquad + C\int_{K-s^*}^K P_{z_1}^Y(|Y_u| \ge e^{\frac{M+u}{4}}|z_2-z_1|\,|\,Y_K = y)\,du.\end{aligned} \quad (6.29)$$

Expanding in terms of transition densities and using $|A_1| \leq C_Z$, we have

$$\begin{aligned}&\left|E_x^Y\left(Z_T^b(Y,\lambda'/\lambda,K) - Z_T^c(Y,\lambda'/\lambda,M,K)\,\middle|\,Y_T = z_1\right)\right| \\ &\quad\leq \frac{C}{q_T(x,z_1)}\int |A_3^*(y,z_1,K) - A_3^*(y,z_1,M,K)| q_{T-K}(x,y)\,q_K(y,z_1)\,dm(y).\end{aligned}$$

Using (6.29) as an upper bound for the integrand, we obtain an expression which closely resembles (6.18); in particular, four terms appear, directly corresponding to $\delta_2, \delta_3, \delta_4$ and δ_5 of that expression. Moreover, they can be handled using the exact same arguments, as in the proof of Lemma 6.2, but with (M,K) playing the roles of (K,T). Because the arguments are the same, we omit them. □

We now establish the limit of $A_3^*(y,z_1,M,K)$ as $K\to\infty$. Recalling (6.27) and the definition of W_M in (5.39), we have

$$\begin{aligned}A_3^*(y,z_1,M,K) &= E_{z_1}^Y\left(\exp\left(\int_0^M F(Y_u) - F_2(Y_u, Y_u + e^{u/2}(z_2 - z_1))\,du\right)\middle|Y_K = y\right) \\ &= E_{z_1}^Y(W_M(Y,z_2)\,|\,Y_K = y).\end{aligned}$$

The functional $W_M(Y,z_2)$ is a bounded continuous function of $Y|_{[0,M]}$. By Lemma 2.3(b), we have

$$\forall\, M>0,\ \lim_{K\to\infty} A_3^*(y,z_1,M,K) = E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right). \tag{6.30}$$

We define $Z_T^d(Y,\lambda'/\lambda,M,K)$ by

$$\begin{aligned}&Z_T^d(Y,\lambda'/\lambda,M,K) \\ &\quad:= \exp\left(\int_0^K F(Y_u) - H_{e^{K+u}}^{\lambda'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\,du\right) E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right).\end{aligned} \tag{6.31}$$

Note that the second term is now deterministic; it no longer depends on the original Ornstein-Uhlenbeck process Y or the spatial variable y. We then have

$$\begin{aligned}&E_x^Y\left(Z_T^d(Y,\lambda'/\lambda,M,K)\,\middle|\,Z_T = z_1\right) \\ &\quad= E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right)\int A_1(x,w,\lambda,\lambda',K)\frac{q_K(x,w)\,q_{T-K}(w,z_1)}{q_T(x,z_1)}\,dm(w).\end{aligned}$$

Bounding $A_1 \leq C_Z$ and integrating out the transition densities, by (6.30) we obtain the following.

Lemma 6.5. *For all* $x, z_1, z_2 \in \mathbb{R}$*,*

$$\delta_d^c(M,K) = \sup_{T\geq 2K+\tau(s),\,\lambda'>\bar\lambda(s)} \left|E_x^Y\left(Z_T^c(Y,\lambda'/\lambda,M,K) - Z_T^d(Y,\lambda'/\lambda,M,K)\,\middle|\,Y_T = z_1\right)\right|$$

satisfies $\lim_{K\to\infty}\delta_d^c(M,K) = 0$ *for each fixed* $M>0$*.*

From our starting expression for $\tilde Z_T(Y,\lambda'/\lambda)$ in (6.6), all that remains to be handled in $Z_T^d(Y,\lambda'/\lambda,M,K)$ is the A_1 term, whose definition we recall from (6.3) is

$$A_1(x,w,\lambda,\lambda',K) = E_x^Y\left(\exp\left(\int_0^K F(Y_u) - H_{e^u}^{\lambda'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\,du\right)\middle|\,Y_K = w\right).$$

The dominant contribution to the integral in A_1 resembles $F(Y_u) - V_1^{e^{u/2}}(Y_u)$. By Proposition 3.1 we have the following upper and lower bounds for the difference of this quantity and the integrand:

$$\begin{aligned}0 &\leq \left[F(Y_u) - V_1^{e^{u/2}}(Y_u)\right] - \left[F(Y_u) - H_{e^u}^{\lambda'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))\right] \\ &\leq F(Y_u + e^{\frac{T-u}{2}}(z_2 - z_1)).\end{aligned} \tag{6.32}$$

Recall from (3.16) that $Z_K(Y)$ is defined as

$$Z_K(Y) = \exp\left(\int_0^K F(Y_u) - V_1^{e^{u/2}}(Y_u)\,du\right).$$

Because the exponentials in both A_1 and $Z_K(Y)$ are bounded above by C_Z, by (6.32) we have

$$\begin{aligned}\big|A_1(x,w,\lambda,\lambda',K) - E_x^Y(Z_K(Y)\,|\,Y_K=w)\big| \\ \le C_Z\,E_x^Y\left(\int_0^K F(Y_u + e^{\frac{T-u}{2}}(z_2-z_1))\,du\,\middle|\,Y_K=w\right).\end{aligned} \tag{6.33}$$

We define $Z^e(Y,M,K)$ by

$$Z^e(Y,M,K) = Z_K(Y)\times E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right). \tag{6.34}$$

Using (6.33) and proceeding as in the proofs of Lemmas 6.2 and 6.4, we obtain the following.

Lemma 6.6. *For all $x, z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$,*

$$\delta_e^d(T,M,K) = \sup_{\lambda'>\bar\lambda(s)}\left|E_x^Y\left(Z_T^d(Y,\lambda'/\lambda,M,K) - Z^e(Y,M,K)\,\middle|\,Y_T=z_1\right)\right|$$

satisfies $\lim_{T\to\infty}\delta_e^d(T,M,K)=0$ *for all fixed M and K such that $0<M<K$.*

From (6.34), we have

$$E_x^Y(Z^e(Y,M,K)\,|\,Y_T=z_1) = E_x^Y(Z_K(Y)\,|\,Y_T=z_1)\,E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right).$$

Thus by Lemma 2.3(b) and the fact that $Z_K(Y)\le C_Z$ (and is a continuous function of Y) we have the following.

Lemma 6.7. *For all $x, z_1, z_2 \in \mathbb{R}$,*

$$\delta_f^e(T,M,K) = \left|E_x^Y(Z^e(Y,M,K)\,|\,Y_T=z_1) - E_x^{Y,\infty}(Z_K(Y))E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right)\right|$$

satisfies $\lim_{T\to\infty}\delta_f^e(T,M,K)=0$ *for all fixed M and K such that $0<M<K$.*

We are now ready to establish the limiting form of $E_x^Y(\tilde Z_T(Y,\lambda'/\lambda)\,|\,Y_T=z_1)$ (provided $z_1\neq z_2$). Let $M>0$, $K>M$, $T\ge 2K+\tau(s)$ and $\lambda'>\bar\lambda(s)$. Bounding above by the sum of the δ terms in Lemmas 6.2-6.7, we have that

$$\begin{aligned}\left|E_x^Y(\tilde Z_T(Y,\lambda'/\lambda)\,|\,Y_T=z_1) - E_x^{Y,\infty}(Z_K(Y))E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right)\right| \\ \le \delta_a^{\sim}(K)+\delta_b^a(K,\lambda,\lambda')+\delta_c^b(M)+\delta_d^c(M,K)+\delta_e^d(T,M,K)+\delta_f^e(T,M,K).\end{aligned} \tag{6.35}$$

Let $\epsilon>0$. By Lemma 6.4, we can choose $M_0>0$ to be sufficiently large such that $\delta_c^b(M)<\epsilon/4$ for all $M\ge M_0$, and choose some $M\ge M_0$. By Lemma 6.2 and Lemma 6.5, we can then choose K_0 to be large enough such that $\delta_a^{\sim}(K)+\delta_d^c(M,K)<\epsilon/4$ for all $K\ge K_0$. Fix $K>K_0$. Next, by Lemmas 6.6 and 6.7 we can choose $T_0>2K+\tau(s)$ such that for all $T\ge T_0$, $\delta_e^d(T,M,K)+\delta_f^e(T,M,K)<\epsilon/4$. Finally, Lemma 6.3 allows us to choose $\lambda(\epsilon)>\bar\lambda(s)$ such that $T=\log(\lambda^2(t-s))\ge T_0$ and $\delta_b^a(K,\lambda,\lambda')<\epsilon/4$ for all $\lambda,\lambda'\ge\lambda(\epsilon)$. We therefore obtain from (6.35) that

$$\limsup_{\lambda,\lambda'\to\infty}\left|E_x^Y(\tilde Z_T(Y,\lambda'/\lambda)\,|\,Y_T=z_1) - E_x^{Y,\infty}(Z_K(Y))E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right)\right|<\epsilon$$

for the M and K chosen above. This holds for all $\epsilon>0$ for sufficiently large M and K (with $M<K$). It therefore holds that if $\lim_{M,K\to\infty,K>M}E_x^{Y,\infty}(Z_K(Y))E_{z_1}^{Y,\infty}\left(W_M(Y,z_2)\right)$

exists, then $\lim_{\lambda,\lambda'\to\infty} E_x^Y(\tilde{Z}_T(Y,\lambda'/\lambda)\,|\,Y_T = z_1)$ exists and is equal to it. Thus it suffices to find the limit of $E_x^{Y,\infty}(Z_K(Y))E_{z_1}^{Y,\infty}(W_M(Y,z_2))$ as $M, K \to \infty$ with $M < K$. As the first term depends only on K and the second depends only on M, we can consider the limits independently. First consider $E_x^{Y,\infty}(Z_K(Y))$. By (3.18), $Z_K \uparrow Z_\infty \le C_Z$, so the limit of the first term as $K \to \infty$ is $E_x^{Y,\infty}(Z_\infty(Y))$ by Monotone Convergence. We recall the definition of W_M from (5.39). The integral in W_M is monotone in M and hence converges to the integral on $[0,\infty]$ as $M \to \infty$. Using the fact that $|W_M(Y,z_2)| \le 1$ for all M and continuity of the exponential, we can bring the limit inside, and $E_{z_1}^{Y,\infty}(W_M(Y,z_2)) \to E_{z_1}^{Y,\infty}(W_\infty(Y,z_2))$ as $M \to \infty$. Thus we have shown that

$$\lim_{\lambda,\lambda'\to\infty} E_x^Y(\tilde{Z}_T(Y,\lambda'/\lambda)\,|\,Y_T = z_1) = E_x^{Y,\infty}(Z_\infty(Y))E_{z_1}^{Y,\infty}(W_\infty(Y,z_2)).$$

This is (6.1), which is what we wanted to show, so the proof of Lemma 5.5 is complete. □

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