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On some càdlàguity moment estimates of stochastic jump processes

R. Mikulevičius* Fanhui Xu†

Abstract

Using X. Fernique's results on the compactness of distributions of càdlàg random functions, we derive some embedding type càdlàguity moment estimates for stochastic processes with jumps.

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1 Introduction

As suggested by Kolmogorov, it was proved in [2] (1956) that if $X_t,\ 0 \le t \le 1$, is a separable real valued process (see [5]) such that

$$\mathbf{E}\left[|X_{t_1} - X_{t_2}|^p \wedge |X_{t_2} - X_{t_3}|^p\right] \le C\,|t_1 - t_3|^{1+r} \tag{1.1}$$

with $r > 0, p > 0$, and C independent of t, then X has no discontinuities of the second kind with probability 1. If

$$\mathbf{E}\left[|X_{t_1} - X_{t_2}|^p\right] \le C\,|t_1 - t_2|^{1+r} \tag{1.2}$$

is assumed instead of of (1.1), then X paths are Hölder continuous (Kolmogorov, 1934). It can be shown (e.g. [8], [11]), that under (1.2), the Hölder continuity is a consequence of the well-known Sobolev embedding theorem. In that case, X Hölder norm moment estimates can be derived. Although a càdlàg Hölder coefficient of X is not a semi-norm, in this note we provide an embedding type estimate of the moments of time supremum and càdlàg Hölder coefficient of X in terms of some integrated time differences of X from which, using assumption (1.1), we can derive the classical claim about the existence of a càdlàg modification of X. On the other hand, the estimate obtained could be helpful in the construction of the solutions to SPDEs driven by jump processes when the method of characteristics with a time reversal is used (see [4]). Some different type moment estimates were derived in [10] by imposing assumptions on the cumulative distribution function of the quantities introduced in [3] (see [7], Section 4 of Chapter III, as well).

Our note is organized as follows. In Section 2, we introduce some notation and state the main claim. Some auxiliary results are presented in Section 3, and the main theorem is proved in Section 4.

*University of Southern California, United States of America. E-mail: mikulvcs@math.usc.edu
†University of Southern California, United States of America. E-mail: fanhuixu@usc.edu

2 Notation and main result

Let E be a Polish space with distance d and $D([0,1], E)$ be the standard space of E-valued càdlàg functions on $[0,1]$. For $0 \leq s \leq t \leq u \leq 1$, denote $\Delta(f; s,t,u) = d(f(s), f(t)) \wedge d(f(t), f(u))$. For $\sigma < \tau$, let us introduce the standard modulus of càdlàguity

$$\Delta(f; (\sigma,\tau)) = \sup_{\sigma \leq s \leq t \leq u \leq \tau} d(f(s), f(t)) \wedge d(f(t), f(u)).$$

For $\mu \in (0,1)$, we define μ-Hölder càdlàg function space $D^\mu([0,1], E)$ to be the set of of all $f \in D([0,1], E)$ such that

$$[f]_\mu + |f]_\mu + [f|_\mu < \infty,$$

where

$$\begin{aligned} [f]_\mu &= \sup_{0\leq s\leq t\leq u\leq 1} \frac{d(f(s), f(t)) \wedge d(f(t), f(u))}{|u-s|^\mu} = \sup_{0\leq s\leq t\leq u\leq 1} \frac{\Delta(f; s,t,u)}{|u-s|^\mu}, \\ |f]_\mu &= \sup_{t\in(0,1]} \frac{d(f(0), f(t))}{t^\mu},\ [f|_\mu = \sup_{t\in[0,1)} \frac{d(f(1), f(t))}{|1-t|^\mu}. \end{aligned}$$

For $\mu \in (0,1), p > 1, f \in D([0,1], E)$, let

$$\begin{aligned} [[f]]_{\mu,p} &= \left(\int\int\int_{s<t<u} \frac{|\Delta(f; s,t,u)|^p}{|u-s|^{\mu p+3}} ds dt du \right)^{1/p}, \\ ||f]]_{\mu,p} &= \left(\int \frac{d(f(0), f(t))^p}{t^{\mu p+1}} dt \right)^{1/p},\ [[f||_{\mu,p} = \left(\int \frac{d(f(1), f(t))^p}{|1-t|^{\mu p+1}} \right)^{1/p}. \end{aligned}$$

Our main result is the following estimate.

Theorem 2.1. *Let $p > 1,\ \mu \in (0,1)$. There is $C > 0$ such that for any $f \in D^\mu([0,1], E)$*

$$[f]_\mu + |f]_\mu + [f|_\mu \leq C\left([[f]]_{\mu,p} + ||f]]_{\mu,p} + [[f||_{\mu,p}\right).$$

Moreover, if $E = \mathbf{R}^d, d(x,y) = |x-y|, x, y \in \mathbf{R}^d$, then

$$\sup_{0\leq t\leq 1} |f(t)| \leq C\left(|f|_{L_p([0,1])} + [[f]]_{\mu,p} + ||f]]_{\mu,p} + [[f||_{\mu,p} \right).$$

Remark 2.2. Obviously, for any E-valued measurable function f on $[0,1]$,

$$\begin{aligned} &[[f]]^p_{\mu,p} \\ \leq &\int\int\int_{s<t<u} \frac{d(f(s), f(t))^{\frac{p}{2}} d(f(t), f(u))^{\frac{p}{2}}}{|u-s|^{\mu p+3}} ds dt du. \end{aligned}$$

Corollary 2.3. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $X : [0,1] \times \Omega \to E$ be a measurable function, $p > 1, r > 0$. Assume that*

$$\begin{aligned} \mathbf{E}\left[\Delta(X; s,t,u)^p\right] &\leq C_0 |u-s|^{1+r}, 0 \leq s < t < u \leq 1, \\ \mathbf{E}\left[d(X(1), X(t))^p\right] &\leq C_0 (1-t)^r, 0 \leq t \leq 1, \\ \mathbf{E}\left[d(X(0), X(t))^p\right] &\leq C_0 t^r, 0 \leq t \leq 1, \end{aligned}$$

for some $C_0 > 0$. Then for each $\mu \in (0,1), \mu < r/p$, there is a constant $N = N(\mu, r, p)$ so that

$$\mathbf{E}\left([[X]]^p_{\mu,p} + ||X]]^p_{\mu,p} + [[X||^p_{\mu,p} \right) \leq N C_0. \tag{2.1}$$

If $E = \mathbf{R}^k, d(x,y) = |x-y|, x, y \in \mathbf{R}^k$*, and* $X_\cdot \in D^\mu([0.1], E)$ *a.s. with* $\mu \in (0,1), \mu < r/p$*, then, in addition,*

$$\mathbf{E}\left[\sup_{0\le t\le 1} |X_t|^p\right] \le N[C_0 + \mathbf{E}\int_0^1 |X_t|^p\, dt].$$

Proof. Let $r - \mu p > 0$. Then

$$\mathbf{E}\left([[X]]_{\mu,p}^p\right) \le C_0 \int\int_{s<u} \frac{|u-s|^{2+r}}{|u-s|^{\mu p+3}} ds du \le NC_0 \int_0^1 u^{r-\mu p} du.$$

Similarly, the other terms can be estimated.

If $E = \mathbf{R}^k, d(x,y) = |x-y|, x, y \in \mathbf{R}^k$, and $X_\cdot \in D^\mu([0.1], E)$ a.s. with $\mu \in (0,1), \mu < r/p$, then the last estimate obviously follows by Theorem 2.1 and (2.1). □

Corollary 2.4. *Let* $X_t, t \in [0,1]$*, be a real valued and stochastically continuous process. Assume that*

$$\begin{aligned} \mathbf{E}\left[|X_t - X_s|^p \wedge |X_t - X_u|^p\right] &\le C|u-s|^{1+r}, \\ \mathbf{E}\left[|X_0 - X_t|^p\right] &\le Ct^r, \\ \mathbf{E}\left[|X_1 - X_t|^p\right] &\le C|1-t|^r \end{aligned}$$

for all $0 \le s < t < u \le 1$*, and some* $r > 0, p > 1$*. Then* X *has a càdlàg modification.*

Proof. Let

$$X_t^n = X_{\pi_n(t)}, \quad t \in [0,1],$$

where $\pi_n(t) = k/2^n$ if $k/2^n \le s < (k+1)/2^n$, $n = 1, 2, \ldots, k = 0, \ldots, 2^n - 1$, and $\pi_n(1) = 1 - 1/2^n$. Obviously the sequence $X_\cdot^n \in D^\beta([0,1])$ a.s. for any $\beta \in (0,1)$. Let $\mu \in (0,1), \mu p < r$. It is enough to show that

$$\sup_n \mathbf{E}\left[[[X^n]]_{\mu,p}^p + ||X^n]]_{\mu,p}^p + [[X^n||_{\mu,p}^p\right] < \infty. \tag{2.2}$$

Indeed, every X_n induces a probability measure $X^n(\mathbf{P})$ on $D([0,1])$. The estimate (2.2) implies that the sequence of measures $\{X^n(\mathbf{P}), n \ge 1\}$ is weakly relatively compact (see [7], [9]). Any weak limit of a weakly converging subsequence X^{n_k} has càdlàg paths with probability 1 and, obviously, the same finite-dimensional distributions as X. Therefore X has a càdlàg modification according to Lemma 2.24 in [9].

In order to show (2.2), we estimate, using assumptions imposed,

$$\begin{aligned} & \mathbf{E}\left[\int_0^1 \frac{|X_0^n - X_t^n|^p}{t^{\mu p+1}} dt\right] \\ \le\ & \sum_{k=1}^{2^n-1} \int_{k/2^n}^{(k+1)/2^n} \frac{\mathbf{E}\left[\left|X_0 - X_{k/2^n}\right|^p\right]}{(k/2^n)^{\mu p+1}} dt \le C2^{-n} \sum_{k=1}^{2^n-1} \left(\frac{k}{2^n}\right)^{r-\mu p-1} \\ =\ & C\left(2^{-n}\right)^{r-\mu p} \sum_{k=1}^{2^n-1} k^{r-\mu p-1} \le C, n \ge 1. \end{aligned}$$

Similarly,

$$\mathbf{E}\left[\int \frac{|X_1^n - X_t^n|^p}{|1-t|^{\mu p+1}} dt\right] \le C, n \ge 1.$$

In the same vein,

$$\begin{aligned}&\mathbf{E}\int\int\int_{s<t<u}\frac{|X_t^n-X_s^n|^p\wedge|X_t^n-X_u^n|^p}{|u-s|^{\mu p+3}}dsdtdu\\ \leq\quad & C\sum_{i=0}^{2^n-3}\sum_{j=i+1}^{2^n-2}\sum_{k=j+1}^{2^n-1}\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}\int_{\frac{j}{2^n}}^{\frac{j+1}{2^n}}\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}}\frac{\left(\frac{k}{2^n}-\frac{i}{2^n}\right)^{1+r}}{(u-s)^{\mu p+3}}dsdtdu.\end{aligned}$$

Note that $r>\mu p$ and for every set of $\{n,i,j,k\}$, $1/2^n\leq(k-i-1)/2^n\leq u-s\leq(k-i+1)/2^n$. Hence, $(k-i)/2^n\leq 2(u-s)$, and

$$\begin{aligned}&\sum_{i=0}^{2^n-3}\sum_{j=i+1}^{2^n-2}\sum_{k=j+1}^{2^n-1}\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}\int_{\frac{j}{2^n}}^{\frac{j+1}{2^n}}\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}}\frac{\left(\frac{k}{2^n}-\frac{i}{2^n}\right)^{1+r}}{(u-s)^{\mu p+3}}dsdtdu\\ \leq\quad & C\int_0^1\int_0^u\frac{1}{(u-s)^{1+\mu p-r}}dsdu\leq C,\end{aligned}$$

which shows that

$$\mathbf{E}\int\int\int_{s<t<u}\frac{|X_t^n-X_s^n|^p\wedge|X_t^n-X_u^n|^p}{|u-s|^{\mu p+3}}dsdtdu\leq C, n\geq 1.$$

Thus (2.2) follows, and the statement is proved. □

3 Auxiliary results

Following [6], for $0\leq\sigma<\tau\leq 1$, we introduce another modulus of càdlàguity

$$N(f;(\sigma,\tau))=\inf_{\sigma<\theta\leq\tau}\sup_{s\in[\sigma,\theta),u\in[\theta,\tau]}[d(f(\sigma),f(s))\vee d(f(u),f(\tau))].$$

Denote for $\eta>0$,

$$N(f;\eta)=\sup_{\sigma<\tau\leq\sigma+\eta}N(f;(\sigma,\tau))=\sup_{0<\tau-\sigma\leq\eta}N(f;(\sigma,\tau)).$$

Clearly, $N(f;\eta)$ is increasing in η.

Remark 3.1. According to Lemma 1.0 in [6],

(a) For any $\sigma<\tau$,

$$\frac{1}{2}N(f;(\sigma,\tau))\leq\Delta(f;(\sigma,\tau))\leq 2N(f;(\sigma,\tau)).$$

In particular,

$$\Delta(f;(0,1))=\sup_{0\leq s\leq t\leq u\leq 1}d(f(s),f(t))\wedge d(f(t),f(u))\leq 2N(f;(0,1)).$$

(b) For $(s,t)\subseteq(\sigma,\tau)$,

$$N(f;(s,t))\leq 2N(f;(\sigma,\tau)),\Delta(f;(s,t))\leq\Delta(f;(\sigma,\tau)).$$

For $\mu\in(0,1)$, define

$$[f]_{\tilde{}\,\mu}:=\sup_{\eta>0}\frac{N(f;\eta)}{\eta^\mu}, f\in D^\mu([0,1]).$$

Remark 3.2. Obviously,

$$[f]_\mu = \sup_{0\le s\le t\le u\le 1}\frac{\Delta(f;s,t,u)}{|u-s|^\mu} = \sup_{a>0}\sup_{\substack{0\le s\le t\le u\le 1,\\ |u-s|\le a}}\frac{\Delta(f;s,t,u)}{|u-s|^\mu}$$
$$= \sup_{a>0}\frac{\sup_{|u-s|\le a, s\le t\le u}\Delta(f;s,t,u)}{a^\mu},$$

and

$$[f]_\mu \le \sup_{s\le t\le u,|u-s|\le 1/2}\frac{\Delta(f;s,t,u)}{|u-s|^\mu} + 2^\mu\Delta(f;(0,1)); \tag{3.1}$$

also,

$$[f]_{\tilde{}\mu} = \sup_{\eta>0}\frac{N(f;\eta)}{\eta^\mu} = \sup_{a>0}\sup_{\eta\le a}\frac{N(f;\eta)}{\eta^\mu} = \sup_{a>0}\frac{\sup_{\eta\le a}N(f;\eta)}{a^\mu}.$$

We show that $[f]_\mu$ and $[f]_{\tilde{}\mu}$ are equivalent.

Lemma 3.3. *Let $\mu\in(0,1)$. For any $f\in D^\mu([0,1],E)$,*

$$\frac{1}{2}[f]_{\tilde{}\mu} \le [f]_\mu \le 2[f]_{\tilde{}\mu}.$$

Proof. Since for each $\sigma\le r\le\tau,\tau\le\sigma+\eta$,

$$\frac{d(f(\sigma),f(r))\wedge d(f(r),f(\tau))}{\eta^\mu} \le \frac{\Delta(f;(\sigma,\tau))}{\eta^\mu} \le \frac{\Delta(f;(\sigma,\tau))}{(\tau-\sigma)^\mu}$$
$$\le \sup_{\sigma\le s\le t\le u\le\tau}\frac{\Delta(f;s,t,u)}{|u-s|^\mu} \le [f]_\mu$$

it follows that

$$\sup_{\sigma\le r\le\tau,\tau\le\sigma+\eta}\frac{d(f(\sigma),f(r))\wedge d(f(r),f(\tau))}{\eta^\mu} \le \sup_{\sigma\le\tau,\tau\le\sigma+\eta}\frac{\Delta(f;(\sigma,\tau))}{\eta^\mu} \le [f]_\mu,$$

and, using Remark 3.2,

$$[f]_\mu = \sup_{\eta>0}\sup_{\sigma\le\tau,\tau\le\sigma+\eta}\frac{\Delta(f;(\sigma,\tau))}{\eta^\mu} \tag{3.2}$$

Hence for any $\eta>0$, by Remark 3.1(a),

$$\frac{1}{2}N(f;\eta) = \frac{1}{2}\sup_{\sigma<\tau\le\sigma+\eta}N(f;(\sigma,\tau)) \le \sup_{\sigma<\tau\le\sigma+\eta}\Delta(f;(\sigma,\tau)) \le 2N(f;\eta),$$

and by (3.2),

$$\frac{1}{2}\sup_{\eta>0}\frac{N(f;\eta)}{\eta^\mu} \le \sup_{\eta>0}\sup_{\sigma\le\tau,\tau\le\sigma+\eta}\frac{\Delta(f;(\sigma,\tau))}{\eta^\mu} \le 2\sup_{\eta>0}\frac{N(f;\eta)}{\eta^\mu}.$$ □

The following key estimate was pointed out in [6], Lemma 1.2.4, as an extraction from Theorem 12.5 in [1] (cf. inequality 12.76 in [1]). For the sake of completeness we provide its proof.

Lemma 3.4. *(Lemma 1.2.4 in [6]) For any $f\in D(0.1],E)$ and every triplet $0\le\sigma<t<\tau\le 1$,*

$$N(f;(\sigma,\tau)) \le N(f;(\sigma,t))\vee N(f;(t,\tau)) + \Delta(f;(\sigma,t,\tau)). \tag{3.3}$$

Proof. Let $0 \le \sigma < t < \tau \le 1$. By the definition of N, for each $\varepsilon \in (0,1)$, there exist $\sigma < \theta_1 \le t < \theta_2 \le \tau$ such that

$$d(f(\sigma), f(s)) \vee d(f(u), f(t)) \le N(f; (\sigma, t)) + \varepsilon, s \in [\sigma, \theta_1), u \in [\theta_1, t],$$
$$d(f(t), f(s)) \vee d(f(u), f(\tau)) \le N(f; (t, \tau)) + \varepsilon, s \in [t, \theta_2), u \in [\theta_2, \tau].$$

Assume $d(f(\sigma), f(t)) \le d(f(t), f(\tau))$, i.e. $\Delta(f; \sigma, t, \tau) = d(f(\sigma), f(t))$. Obviously,

$$N(f; (\sigma, \tau)) \le \sup_{s \in [\sigma, \theta_2)} d(f(\sigma), f(s)) \vee \sup_{u \in [\theta_2, \tau]} d(f(u), f(\tau)), \tag{3.4}$$

and

$$\sup_{u \in [\theta_2, \tau]} d(f(u), f(\tau)) \le N(f; (t, \tau)) + \varepsilon. \tag{3.5}$$

Now,

$$d(f(\sigma), f(s)) \le N(f; (\sigma, t)) + \varepsilon, \text{ if } s \in [\sigma, \theta_1),$$

and

$$\begin{aligned} d(f(\sigma), f(s)) &\le d(f(s), f(t)) + d(f(\sigma), f(t)) \\ &\le N(f; (\sigma, t)) + \varepsilon + \Delta(f; \sigma, t, \tau), \text{ if } s \in [\theta_1, t], \\ d(f(\sigma), f(s)) &\le d(f(s), f(t)) + d(f(\sigma), f(t)) \\ &\le N(f; (t, \tau)) + \varepsilon + \Delta(f; \sigma, t, \tau), \text{ if } s \in (t, \theta_2). \end{aligned}$$

Hence

$$\sup_{s \in [\sigma, \theta_2)} d(f(\sigma), f(s)) \le N(f; (\sigma, t)) \vee N(f; (t, \tau)) + \Delta(f; \sigma, t, \tau) + \varepsilon. \tag{3.6}$$

Using (3.4)-(3.6), we have

$$N(f; (\sigma, \tau)) \le N(f; (\sigma, t)) \vee N(f; (t, \tau)) + \Delta(f; \sigma, t, \tau) + \varepsilon. \tag{3.7}$$

Modifying the proof above in an obvious way, we see that (3.7) holds if $d(f(\sigma), f(t)) > d(f(t), f(\tau))$ as well. Since $\varepsilon \in (0,1)$ is arbitrary, the statement follows. □

For $\mu \in (0,1), f \in D([0,1], E)$, define

$$[f]_{\hat{}\mu} = \sup_{0 < s < u \le 1} \frac{\Delta\left(f; s, \frac{s+u}{2}, u\right)}{|u - s|^\mu} = \sup_{a > 0} \frac{\sup_{|\sigma - \tau| \le a} \Delta\left(f; \sigma, \frac{\sigma+\tau}{2}, \tau\right)}{a^\mu}.$$

We will need the following equivalence claim.

Lemma 3.5. *Let $\mu \in (0,1)$. For any $f \in D^\mu([0,1], E)$,*

$$[f]_{\hat{}\mu} \le [f]_\mu \le 2[f]_{\check{}\mu} \le \frac{2}{1 - 2^{-\mu}} [f]_{\hat{}\mu}.$$

Proof. Let $K = [f]_{\check{}\mu}$. According to Remark 3.2, for any $a > 0$,

$$N(f; a/2) \le K a^\mu 2^{-\mu}.$$

By (3.3), for every $\sigma < \tau \le 1$ such that $|\tau - \sigma| \le a$, we have, by Lemma 3.4,

$$\begin{aligned} &N(f; (\sigma, \tau)) \\ \le\ & N\left(f; \left(\sigma, \frac{\sigma+\tau}{2}\right)\right) \vee N\left(f; \left(\frac{\sigma+\tau}{2}, \tau\right)\right) + \Delta\left(f; \sigma, \frac{\sigma+\tau}{2}, \tau\right) \\ \le\ & N(f; a/2) + \Delta\left(f; \sigma, \frac{\sigma+\tau}{2}, \tau\right) \\ \le\ & K a^\mu 2^{-\mu} + \Delta\left(f; \sigma, \frac{\sigma+\tau}{2}, \tau\right). \end{aligned}$$

Hence

$$N(f;a) \le K a^{\mu} 2^{-\mu} + \sup_{0<\tau-\sigma\le a} \Delta\left(f;\left(\sigma, \frac{\sigma+\tau}{2}, \tau\right)\right)$$

and

$$\frac{N(f;a)}{a^{\mu}} \le 2^{-\mu} K + \frac{\sup_{0<\tau-\sigma\le a} \Delta\left(f;\left(\sigma, \frac{\sigma+\tau}{2}, \tau\right)\right)}{a^{\mu}}$$

Taking $\sup$ in $a>0$ on both sides, we see that

$$K \le 2^{-\mu} K + \sup_{a>0} \frac{\sup_{0<\tau-\sigma\le a} \Delta\left(f;\sigma, \frac{\sigma+\tau}{2}, \tau\right)}{a^{\mu}}$$

or

$$\begin{aligned} [f]_{\check{}\mu} &= K \le \frac{1}{1-2^{-\mu}} \sup_{a>0} \frac{\sup_{0<\tau-\sigma\le a} \Delta\left(f;\sigma, \frac{\sigma+\tau}{2}, \tau\right)}{a^{\mu}} \\ &= \frac{1}{1-2^{-\mu}} [f]_{\hat{}\mu}. \end{aligned}$$

The statement follows by Lemma 3.3. □

4 Proof of Theorem 2.1

First we show that there is $C = C(\mu, p)$ so that with any $\delta \in (0,1)$,

$$d(f(0), f(t)) \le C t^{\mu} \left([f]_{\mu} \delta^{\mu} + \delta^{-1/p} ||f]]_{\mu,p}\right), t \le 3/4. \tag{4.1}$$

Assume $0 < t \le 3/4$. Taking $\varepsilon < \frac{1}{4} t$, we have for $\tau' \in (t-\varepsilon, t), \tau'' \in (t, t+\varepsilon)$,

$$d(f(0), f(t)) \le \Delta(f;\tau', t, \tau'') + d(f(0), f(\tau')) + d(f(0), f(\tau'')).$$

Integrating with respect to τ', τ'' over $\bar{Q} = [t-\varepsilon, t] \times [t, t+\varepsilon]$,

$$\begin{aligned} & \varepsilon^2 d(f(0), f(t)) \\ \le\ & \varepsilon \int_{t-\varepsilon}^{t} d(f(0), f(\tau')) d\tau' + \varepsilon \int_{t}^{t+\varepsilon} d(f(0), f(\tau'')) d\tau'' \\ & + \int\int_{\bar{Q}} \Delta(f;\tau', t, \tau'') d\tau' d\tau'' = A_1 + A_2 + B. \end{aligned} \tag{4.2}$$

Now

$$B \le [f]_{\mu} \int_{t-\varepsilon}^{t} \int_{t}^{t+\varepsilon} (\tau'' - \tau')^{\mu} d\tau'' d\tau' \le C [f]_{\mu} \varepsilon^{2+\mu}.$$

Taking $\varepsilon = \frac{1}{4} \delta t$ with any $\delta \in (0,1)$, we have

$$\varepsilon^{-2} B \le C [f]_{\mu} \delta^{\mu} t^{\mu}. \tag{4.3}$$

By Hölder inequality, $\kappa = \mu + 1/p, 1/p + 1/q = 1$,

$$\begin{aligned} A_1 &= \varepsilon \int_{t-\varepsilon}^{t} \frac{d(f(0), f(\tau'))}{\tau'} \tau' d\tau' \\ &\le \varepsilon \left(\int_{t-\varepsilon}^{t} \frac{d(f(0), f(\tau'))^{p}}{\tau'^{\kappa p}} d\tau'\right)^{1/p} \left(\int_{t-\varepsilon}^{t} \tau'^{\kappa q} d\tau'\right)^{1/q} \\ &= C\varepsilon \left[t^{1+\kappa q} - (t-\varepsilon)^{1+\kappa q}\right]^{1/q} ||f]]_{\mu,p} \le C t^{\kappa} \varepsilon^{1+1/q} ||f]]_{\mu,p}. \end{aligned}$$

Taking $\varepsilon = \frac{1}{4}\delta t$, with any $\delta \in (0,1)$,

$$\varepsilon^{-2} A_1 \leq C t^{\kappa} \varepsilon^{1/q-1} A_1 = C t^{\kappa} \varepsilon^{-1/p} A_1 = C t^{\mu} \delta^{-1/p} ||f]]_{\mu,p} \tag{4.4}$$

and, the same way,

$$\varepsilon^{-2} A_2 \leq C t^{\mu} \delta^{-1/p} ||f]]_{\mu,p}. \tag{4.5}$$

The inequality (4.1) follows from (4.2)-(4.5).

Similarly, with obvious changes, we prove that

$$d\left(f\left(1\right), f\left(t\right)\right) \leq C\left(1-t\right)^{\mu}\left(\left[f\right]_{\mu}\delta^{\mu} + \delta^{-1/p}[[f||_{\mu,p}\right), t \geq 1/4, \tag{4.6}$$

for some $C = C(\mu, p)$ with any $\delta \in (0,1)$.

Finally, (4.1), (4.6) imply that there is $C = C(\mu, p)$ so that with any $\delta \in (0,1)$,

$$\begin{aligned} d\left(f\left(1\right), f\left(0\right)\right) &\leq d\left(f\left(\frac{1}{2}\right), f\left(0\right)\right) + d\left(f\left(1\right), f\left(\frac{1}{2}\right)\right) \\ &\leq C\left(\frac{1}{2}\right)^{\mu}\left(\left[f\right]_{\mu}\delta^{\mu} + \delta^{-1/p}[[f||_{\mu,p}\right) + C\left(\frac{1}{2}\right)^{\mu}\delta^{-1/p}||f]]_{\mu,p}. \end{aligned} \tag{4.7}$$

Now,

$$\begin{aligned} d\left(f\left(1\right), f\left(t\right)\right) &\leq d\left(f\left(1\right), f\left(0\right)\right) + d\left(f\left(0\right), f\left(t\right)\right) \text{ if } t \in \left(0, 1/4\right), \\ d\left(f\left(0\right), f\left(t\right)\right) &\leq d\left(f\left(1\right), f\left(0\right)\right) + d\left(f\left(1\right), f\left(t\right)\right) \text{ if } t \in \left(3/4, 1\right). \end{aligned}$$

Hence, by (4.1), (4.6) and (4.7), there is $C = C(\mu, p)$ so that for all $\delta \in (0,1), t \in (0,1)$,

$$[f|_{\mu} + |f]_{\mu} \leq C\left(\left[f\right]_{\mu}\delta^{\mu} + \delta^{-1/p}[[f||_{\mu,p} + \delta^{-1/p}||f]]_{\mu,p}\right). \tag{4.8}$$

Now we estimate $\Delta(f; s, t, u)$ with $0 \leq s < t < u \leq 1$ and $t = \frac{s+u}{2}$.

(i) Assume $|s| > \frac{1}{4}|u-s|$ and $|1-u| > \frac{1}{4}|u-s|$.

Let $\varepsilon < \frac{1}{4}(u-s), s' \in (s-\varepsilon, s), s'' \in (s, s+\varepsilon), t' \in (t-\varepsilon, t), t'' \in (t, t+\varepsilon), u' \in (u-\varepsilon, u), u'' \in (u, u+\varepsilon)$, and

$$\begin{aligned} A &= \Delta\left(f; s', s, s''\right) + \Delta\left(f; t', t, t''\right) + \Delta\left(f; u', u, u''\right), \\ B &= \Delta\left(f; s', t', u''\right) + \Delta\left(f; s', t', u'\right) + \Delta\left(f; s', t'', u''\right) + \Delta\left(f; s', t'', u'\right) \\ &\quad + \Delta\left(f; s'', t', u''\right) + \Delta\left(f; s'', t', u'\right) + \Delta\left(f; s'', t'', u''\right) + \Delta\left(f; s'', t'', u'\right). \end{aligned}$$

Let $Q = (s-\varepsilon, s) \times (s, s+\varepsilon) \times (t-\varepsilon, t) \times (t, t+\varepsilon) \times (u-\varepsilon, u) \times (u, u+\varepsilon)$. Then $|Q| = \varepsilon^6$, and

$$\Delta(f; s, t, u) \leq A + B. \tag{4.9}$$

Let

$$\begin{aligned} \tilde{A} &= \varepsilon^{-4}\int_Q A = \int_{s-\varepsilon}^{s}\int_{s}^{s+\varepsilon}\Delta\left(f; s', s, s''\right) ds'' ds' + \int_{t-\varepsilon}^{t}\int_{t}^{t+\varepsilon}\Delta\left(f; t', t, t''\right) dt'' dt' \\ &\quad + \int_{u-\varepsilon}^{u}\int_{u}^{u+\varepsilon}\Delta\left(f; u', u, u''\right) du'' du', \end{aligned}$$

and

$$\tilde{B} = \varepsilon^{-3} \int_Q B = \int_{s-\varepsilon}^{s} \int_{t-\varepsilon}^{t} \int_{u}^{u+\varepsilon} \Delta\left(f; s', t', u''\right) du'' dt' ds'$$
$$+ \int_{s-\varepsilon}^{s} \int_{t-\varepsilon}^{t} \int_{u-\varepsilon}^{u} \Delta\left(f; s', t', u'\right) du' dt' ds'$$
$$+ \int_{s-\varepsilon}^{s} \int_{t}^{t+\varepsilon} \int_{u}^{u+\varepsilon} \Delta\left(f; s', t'', u''\right) du'' dt'' ds'$$
$$+ \int_{s-\varepsilon}^{s} \int_{t}^{t+\varepsilon} \int_{u-\varepsilon}^{u} \Delta\left(f; s', t'', u'\right) du' dt'' ds'$$
$$+ \int_{s}^{s+\varepsilon} \int_{t-\varepsilon}^{t} \int_{u}^{u+\varepsilon} \Delta\left(f; s'', t', u''\right) du'' dt' ds''$$
$$+ \int_{s}^{s+\varepsilon} \int_{t-\varepsilon}^{t} \int_{u-\varepsilon}^{u} \Delta\left(f; s'', t', u'\right) du' dt' ds''$$
$$+ \int_{s}^{s+\varepsilon} \int_{t}^{t+\varepsilon} \int_{u}^{u+\varepsilon} \Delta\left(f; s'', t'', u''\right) du'' dt'' ds''$$
$$+ \int_{s}^{s+\varepsilon} \int_{t}^{t+\varepsilon} \int_{u-\varepsilon}^{u} \Delta\left(f; s'', t'', u'\right) du' dt'' ds'' = \tilde{B}_1 + \ldots + \tilde{B}_8.$$

Integrating (4.9) over Q,

$$\Delta\left(f; s, t, u\right) \leq \varepsilon^{-2} \tilde{A} + \varepsilon^{-3} \tilde{B}. \tag{4.10}$$

Now,

$$\int_{s-\varepsilon}^{s} \int_{s}^{s+\varepsilon} \Delta\left(f; s', s, s''\right) ds' ds'' \leq [f]_\mu \int_{s-\varepsilon}^{s} \int_{s}^{s+\varepsilon} (s'' - s')^\mu ds' ds'' \leq C\, [f]_\mu\, \varepsilon^{2+\mu}.$$

Similarly we estimate the other two terms in $\tilde{A}$ and see that

$$\varepsilon^{-2} \tilde{A} \leq C_1\, [f]_\mu\, \varepsilon^\mu. \tag{4.11}$$

Using Hölder inequality, $1/q + 1/p = 1, p > 1$, with $\kappa = \mu + 3/p$,

$$\varepsilon^{-3} \tilde{B}_1 = \varepsilon^{-3} \int_{s}^{s+\varepsilon} \int_{t-\varepsilon}^{t} \int_{u}^{u+\varepsilon} \frac{\Delta\left(f; s', t', u''\right)}{|s'' - u''|^\kappa} |s'' - u''|^\kappa\, ds' dt' du''$$
$$\leq \quad \varepsilon^{-3} \left(\int_{s-\varepsilon}^{s} \int_{t-\varepsilon}^{t} \int_{u}^{u+\varepsilon} (u'' - s')^{\kappa q} ds' dt' du'' \right)^{1/q} [[f]]_{\mu,p}.$$

Since

$$\int_{s-\varepsilon}^{s} \int_{t-\varepsilon}^{t} \int_{u}^{u+\varepsilon} (u'' - s')^{\kappa q} ds' dt' du'' \leq C\varepsilon^3 \left(u - s\right)^{\kappa q},$$

we have

$$\varepsilon^{-3} \tilde{B}_1 \leq C\varepsilon^{-3+\frac{3}{q}} \left(u - s\right)^\kappa [[f]]_{\mu,p}\,.$$

Similarly estimating the other terms in $\tilde{B}$, we get

$$\begin{aligned} \varepsilon^{-3} \tilde{B} \quad &\leq \quad C\varepsilon^{-3+3/q} \left(u - s\right)^\kappa [[f]]_{\mu,p} = C\varepsilon^{-3/p} \left(u - s\right)^\kappa [[f]]_{\mu,p}. \\ &= \quad C\delta^{-3/p} \left(u - s\right)^{\kappa - 3/p} [[f]]_{\mu,p} = C\delta^{-3/p} \left(u - s\right)^\mu [[f]]_{\mu,p}. \end{aligned}$$

Hence by (4.10) and (4.11), for some $C = C\left(\mu, p\right)$,

$$\Delta\left(f; s, t, u\right) \leq C_1\, [f]_\mu\, \varepsilon^\mu + C\varepsilon^{-3/p} \left(u - s\right)^\kappa [[f]]_{\mu,p} \tag{4.12}$$

if $|s| > \frac{1}{4}|u-s|$ and $|1-u| > \frac{1}{4}|u-s|$. Taking $\varepsilon = \frac{1}{4}\delta(u-s)$ with any $\delta \in (0,1)$, we have

$$\Delta(f;s,t,u) \leq C(u-s)^{\mu}\left([f]_{\mu}\delta^{\mu} + \delta^{-\frac{3}{p}}[[f]]_{\mu,p}\right) \tag{4.13}$$

for some $C = C(\mu,p)$ if $|s| > \frac{1}{4}|u-s|$ and $|1-u| > \frac{1}{4}|u-s|$.

(ii) Assume $|s| \leq \frac{1}{4}|u-s|$ or $|1-u| \leq \frac{1}{4}|u-s|$.

If $s \leq \frac{1}{4}|u-s|$ (recall $t = \frac{s+u}{2}$), then $s \leq 1/4$ and

$$t = s + \frac{u-s}{2} \leq \frac{3}{4}(u-s) \leq \frac{3}{4}.$$

By (4.1), there is $C = c(\mu,p)$ so that for any $\delta \in (0,1)$, $s \leq \frac{1}{4}|u-s|$,

$$\begin{aligned}\Delta(f;s,t,u) &\leq d(f(s),f(0)) \wedge d(f(u),f(0)) + d(f(t),f(0)) \\ &\leq C|u-s|^{\mu}\left([f]_{\mu}\delta^{\mu} + \delta^{-1/p}||f]]_{\mu,p}\right).\end{aligned}$$

If $1-u \leq \frac{1}{4}|u-s|$, then $u \geq \frac{3}{4}$ and $t \geq 1/4$ because

$$1-t = 1 - \frac{u+s}{2} = 1-u+\frac{u-s}{2} \leq \frac{3}{4}(u-s) \leq \frac{3}{4}.$$

By (4.6), there is $C = C(\mu,p)$ so that for any $\delta \in (0,1)$, $1-u \leq \frac{1}{4}|u-s|$, we have

$$\begin{aligned}\Delta(f;s,t,u) &\leq d(f(s),f(1)) \wedge d(f(u),f(1)) + d(f(t),f(1)) \\ &\leq C|u-s|^{\mu}\left([f]_{\mu}\delta^{\mu} + \delta^{-1/p}[[f||_{\mu,p}\right).\end{aligned}$$

Hence

$$\Delta(f;s,t,u) \leq C(u-s)^{\mu}\left([f]_{\mu}\delta^{\mu} + \delta^{-1/p}[[f||_{\mu,p} + \delta^{-1/p}||f]]_{\mu,p}\right) \tag{4.14}$$

if $|s| \leq \frac{1}{4}|u-s|$ or $|1-u| \leq \frac{1}{4}|u-s|$.

According to (4.13) and (4.14), there is $C = C(\mu,p)$ so that

$$\begin{aligned}&\Delta(f;s,t,u) \\ \leq\ & C(u-s)^{\mu}\left([f]_{\mu}\delta^{\mu} + +\delta^{-\frac{3}{p}}[[f]]_{\mu,p} + \delta^{-1/p}[[f||_{\mu,p} + \delta^{-1/p}||f]]_{\mu,p}\right),\end{aligned}$$

for any $0 \leq s < t < u \leq 1, t = (s+u)/2$. Hence for all $\delta \in (0,1)$,

$$[f]\hat{}_{\mu} \leq C\left([f]_{\mu}\delta^{\mu} + [[f]]_{\mu,p}\delta^{-3/p}\right) \tag{4.15}$$

for some $C = C(\mu,p)$. Then by Lemma 3.5 and (4.8), (4.15), there is $C_1 = C_1(\mu,p)$ so that for all $\delta \in (0,1)$ we have

$$\begin{aligned}&[f]_{\mu} + |f]_{\mu} + [f|_{\mu} \\ \leq\ & C_1\left([f]_{\mu}\delta^{\mu} + \delta^{-\frac{3}{p}}[[f]]_{\mu,p} + \delta^{-\frac{1}{p}}||f]]_{\mu,p} + \delta^{-\frac{1}{p}}[[f||_{\mu,p}\right).\end{aligned}$$

Choosing δ so that $\delta^{\mu}C_1 \leq 1/2$ we see that for some $C = C(\mu,p)$,

$$[f]_{\mu} + |f]_{\mu} + [f|_{\mu} \leq C([[f]]_{\mu,p} + ||f]]_{\mu,p} + [[f||_{\mu,p}), f \in D^{\mu}([0,1],E).$$

If $E = \mathbf{R}^k, d(x,y) = |x-y|, x,y \in \mathbf{R}^k$, then we can estimate the supremum of f. For each t,

$$\begin{aligned}|f(t)| &\leq |f(\tau) - f(0)| + |f(t) - f(0)| + |f(\tau)| \\ &\leq 2|f]_{\mu} + |f(\tau)|, \tau \in [0,1].\end{aligned}$$

Hence $|f(t)| \leq 2|f]_{\mu} + \int_0^1 |f(\tau)|\,d\tau$,and

$$\sup_{0\leq t\leq 1}|f(t)| \leq 2|f]_{\mu} + \left(\int_0^1 |f(\tau)|^p\,d\tau\right)^{1/p}.$$

The claim of Theorem 2.1 follows.

References

[1] Billingsley, P.: Convergence of Probability Measures. *John Wiley & Sons, Inc.*, New York-London-Sydney, 1968. xii+253 pp. MR-0233396

[2] Chentsov, N. N.: Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov-Smirnov tests. *Teor. Veroyatn. Primen.* **1**, (1956), 140–144. MR-0084220

[3] Cramer, H.: On stochastic processes whose trajectories have no discontinuities of the second kind. *Annali di Matematica Pura ed Applicata* **71**, (1966), 85–92. MR-0207034

[4] Da Prato, G., Menaldi, J.-L., and Tubaro, L.: Some results of backward Ito formula. *Stochastic Analysis and Applications* **25**, (2007), 679–703. MR-2321903

[5] Dellacherie, C. and Meyer, P.-A.: Probabilties and Potential A. *North-Holland Publishing Co.*, Amsterdam-New York, 1978. viii+189 pp. MR-0521810

[6] Fernique X.: Compactness of distributions of càdlàg random functions. *Lith. Math. J.* **34**, (1994), 231–243. MR-1355239

[7] Gihman, I. I. and Skorokhod, A. V.: Theory of Stochastic Processes I. *Springer-Verlag*, New York-Heidelberg, 1980. viii+574 pp. MR-0636254

[8] Ibragimov, I. A.: Properties of sample functions of stochastic processes and embedding theorems. *Teor. Veroyatn. Primen.* **18**, (1973), 468–480. MR-0326827

[9] Kallenberg, O.: Foundations of Modern Probability. *Springer-Verlag*, New York, 1997. xii+523 pp. MR-1464694

[10] Peszat, S. and Zabczyk, J.: Time regularity of solutions to linear equations with Levy noise in infinite dimensions. *Stochastic Processes and Appl.* **123**, (2013), 719–751. MR-3005003

[11] Schilling, R. S.: Sobolev embedding for stochastic processes. *Expositiones Mathematicae* **18**, (2000), 239–242. MR-1763890