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Bi-log-concavity: some properties and some remarks towards a multi-dimensional extension

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Abstract

Bi-log-concavity of probability measures is a univariate extension of the notion of log-concavity that has been recently proposed in a statistical literature. Among other things, it has the nice property from a modelisation perspective to admit some multimodal distributions, while preserving some nice features of log-concave measures. We compute the isoperimetric constant for a bi-log-concave measure, extending a property available for log-concave measures. This implies that bi-log-concave measures have exponentially decreasing tails. Then we show that the convolution of a bi-log-concave measure with a log-concave one is bi-log-concave. Consequently, infinitely differentiable, positive densities are dense in the set of bi-log-concave densities for L_p-norms, $p \in [1, +\infty]$. We also derive a necessary and sufficient condition for the convolution of two bi-log-concave measures to be bi-log-concave. We conclude this note by discussing a way of defining a multi-dimensional extension of the notion of bi-log-concavity.

Keywords: bi-log-concavity; isoperimetric constant; log-concavity.
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1 Introduction

Bi-log-concavity (of a probability measure on the real line) is a property recently introduced by Dümbgen, Kolesnyk and Wilke ([5]), that aims at bypassing some restrictive aspects of log-concavity while preserving some of its nice features. More precisely, bi-log-concavity amounts to log-concavity of both F and $1 - F$, where F is a cumulative distribution function, and a simple application of Prékopa's theorem on stability of log-concavity through marginalization ([10], see also [13] for a discussion on the various proofs of this fundamental theorem) shows that log-concave measures are also bi-log-concave (see [1] for a more direct, elementary proof of this latter fact).

From a modelisation perspective, bi-log-concavity and log-concavity may be seen as shape constraints. In statistics, when they are available, shape constraints represent an interesting alternative to more classical parametric, semi-parametric or non-parametric approaches and constitute an active contemporary line of research ([14, 12]). Bi-log-concavity was indeed proposed in the aim to contribute to this research area ([5]). It was used in [5] to construct efficient confidence bands for the cumulative distribution

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function and some functionals of it. The authors highlight that bi-log-concave measures admit multi-modal measures while it is well-known that log-concave measures are unimodal. Furthermore, Dümbgen et al. [5] establish the following characterization of bi-log-concave distributions. For a (cumulative) distribution function F, denote

$$J(F) \equiv \{x \in \mathbb{R} : 0 < F(x) < 1\}$$

and call "non-degenerate", the functions F such that $J(F) \neq \emptyset$.

Theorem 1.1 (Characterization of bi-log-concavity, [5])**.** *Let F be a non-degenerate distribution function. The following four statements are equivalent:*

(i) *F is bi-log-concave, i.e. F and $1-F$ are log-concave functions in the sense that their logarithm is concave.*

(ii) *F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$ such that, for all $x \in J(F)$ and $t \in \mathbb{R}$,*

$$1-(1-F(x))\exp\left(-\frac{f(x)}{1-F(x)}t\right) \leq F(x+t) \leq F(x)\exp\left(\frac{f(x)}{F(x)}t\right).$$

(iii) *F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$ such that the hazard function $f/(1-F)$ is non-decreasing and reverse hazard function f/F is non-increasing on $J(F)$.*

(iv) *F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with bounded and strictly positive derivative $f = F'$. Furthermore, f is locally Lipschitz continuous on $J(F)$ with L_1-derivative $f' = F''$ satisfying*

$$\frac{-f^2}{1-F} \leq f' \leq \frac{f^2}{F}.$$

Note that if one includes degenerate measures – that is Dirac masses – it is easily seen that the set of bi-log-concave measures is closed under weak limits.

Just as s-concave measures generalize log-concave ones, Laha and Wellner [8] proposed the concept of bi-s^*-concavity, that generalizes bi-log-concavity and that includes s-concave densities. Some characterizations of bi-s^*-concavity, that extend the previous theorem, are derived in [8].

On the probabilistic side, even if some characterizations are available, many important questions remain about the properties of bi-log-concave measures. Indeed, log-concave measures satisfy many nice properties (see for instance [7, 13, 4] and references therein) and it is natural to ask whether some of those are extended to bi-log-concave measures. Answering this question is the primary object of this note.

We show in Section 2 that the isoperimetric constant of a bi-log-concave measure is simply equal to two times the value of its density with respect to the Lebesgue measure – that indeed exists – at its median, thus extending a property available for log-concave measures. We deduce that a bi-log-concave measure has exponential tails, also extending a property valid in the log-concave case.

In Section 3, we show that the convolution of a log-concave measure and a bi-log-concave measure is bi-log-concave. As a consequence, we get that any bi-log-concave measure can be approximated by a sequence of bi-log-concave measures having regular densities. Furthermore, we give a necessary and sufficient condition for the convolution of two bi-log-concave measures to be bi-log-concave.

Finally, we discuss in Section 3.1 a possible way to obtain a multivariate notion of bi-log-concavity, extending the univariate notion. In particular, log-concave vectors are bi-log-concave and the proposed definition ensures stability through convolution by any log-concave measure. The question of providing a nice definition of bi-log-concavity in higher dimension, that would also impose existence of some exponential moments, remains open.

2 Isoperimetry and concentration for bi-log-concave measures

Let $F(x) = \mu((-\infty, x])$ be the distribution function of a probability measure μ on the real line. Assume that μ is non-degenerate (in the sense of its distribution function being non-degenerate) and let f be the density of its absolutely continuous part.

Recall the following formula for the isoperimetric constant $Is(\mu)$ of μ, due to Bobkov and Houdré [3],

$$Is(\mu) = \text{ess} \inf_{x \in J(F)} \frac{f(x)}{\min\{F(x), 1 - F(x)\}} .$$

The following theorem extends a well-known fact related to the isoperimetric constant of a log-concave measure to the case of a bi-log-concave measure.

Theorem 2.1. *Let μ be a probability measure with non-degenerate distribution function F being bi-log-concave. Then μ admits a density $f = F'$ on $J(F)$ and it holds*

$$Is(\mu) = 2f(m) ,$$

where m is the median of μ.

In general, the isoperimetric constant is hard to compute, but in the bi-log-concave case Theorem 2.1 provides a straightforward formula, that extends a formula valid for log-concave measures (see for instance [13]).

In the following, we will also use the notation $J(F) = (a, b)$.

Proof. Note that the median m is indeed unique by Theorem 1.1 above. For $x \in (a, m]$,

$$I_F(x) := \frac{f(x)}{\min\{F(x), 1 - F(x)\}} = \frac{f(x)}{F(x)} .$$

As μ is bi-log-concave, I_F is thus non-increasing on $(a, m]$. For $x \in [m, b)$,

$$I_F(x) = \frac{f(x)}{1 - F(x)} .$$

Thus, I_F is non-decreasing on $[m, b)$. Consequently, the minimum of $I_F(x)$ is attained on m and its value is $Is(\mu) = 2f(m)$. □

Corollary 2.2. *Let μ as above be a bi-log-concave measure with median m. Then $f(m) > 0$ and μ satisfies the following Poincaré inequality: for any square integrable function $g \in L_2(\mu)$ with derivative $g' \in L_2(\mu)$,*

$$f^2(m) \operatorname{Var}_\mu(g) \leq \int (g')^2 \, d\mu , \tag{2.1}$$

where $\operatorname{Var}_\mu(g) = \int g^2 d\mu - \left(\int g d\mu\right)^2$ is the variance of g with respect to μ. Consequently, μ has bounded Ψ_1 Orlicz norm and achieves the following exponential concentration inequality,

$$\alpha_\mu(r) \leq \exp(-rf(m)/3) , \tag{2.2}$$

where α_μ is the concentration function of μ, defined by $\alpha_\mu(r) = \sup\{1 - \mu(A_r) : A \subset \mathbb{R}, \mu(A) \geq 1/2\}$, where $r > 0$ and $A_r = \{x \in \mathbb{R} : \exists y \in A, |x - y| < r\}$ is the (open) r-neighborhood of A.

As it is well-known (see [9] for instance), inequality (2.2) implies that for any 1-Lipschitz function g,

$$\mu\left(g \geq m_g + r\right) \leq \exp\left(-rf\left(m\right)/3\right) \ ,$$

where m_g is a median of g, that is $\mu\left(g \geq m_g\right) \geq 1/2$ and $\mu\left(g \leq m_g\right) \geq 1/2$.

Proof. The fact that $f\left(m\right) > 0$ is given by point **(iii)** of Theorem 1.1 above. Then Inequality (2.1) is a consequence of Theorem 2.1 via Cheeger's inequality for the first eigenvalue of the Laplacian (see for instance Inequality 3.1 in [9]). Inequality (2.2) is a classical consequence of Inequality (2.1) as well (see Theorem 3.1 in [9]). □

We shortly describe now another proof of the fact that log-concave measures are bi-log-concave. Indeed, by Theorem 1.1 above, bi-log-concavity of μ reduces to non-increasingness of the functions f/F and $-f/\left(1-F\right)$, which is equivalent to non-increasingness of $I\left(p\right)/p$ and $-I\left(p\right)/\left(1-p\right)$, with $I\left(p\right) = f\left(F^{-1}\left(p\right)\right)$. Furthermore, following Bobkov [2], for a log-concave probability measure μ on $\mathbb{R}$ having a positive density f on $J\left(F\right)$, the function I is concave. As $I\left(0\right) = I\left(1\right) = 0$, concavity of I implies non-increasingness of the ratios $I\left(p\right)/p$ and $-I\left(p\right)/\left(1-p\right)$. Hence, the conclusion follows.

Example 2.3. The function $I\left(p\right) = f\left(F^{-1}\left(p\right)\right)$ is in general hard to compute. But a few easy examples exist. For instance, for the logistic distribution, $F\left(x\right) = 1/\left(1+\exp\left(-x\right)\right)$, we have $I\left(p\right) = p\left(1-p\right)$. For the Laplace distribution, $f\left(x\right) = \exp\left(-\left|x\right|\right)/2$, $I\left(p\right) = \min\left\{p, 1-p\right\}$.

3 Stability through convolution

Take X and Y two independent random variables with respective distribution functions F_X and F_Y that are bi-log-concave. Hence X and Y have densities, denoted by f_X and f_Y. Then

$$F_{X+Y}\left(x\right) = \mathbb{P}\left(X+Y \leq x\right) = \mathbb{E}\left[\mathbb{P}\left(X \leq x-Y\left|Y\right.\right)\right] = \int F_X\left(x-y\right)f_Y\left(y\right)dy \ . \tag{3.1}$$

In addition,

$$1 - F_{X+Y}\left(x\right) = \int\left(1-F_X\left(x-y\right)\right)f_Y\left(y\right)dy \ . \tag{3.2}$$

Proposition 3.1. *If X is bi-log-concave, Y is log-concave and X is independent of Y, then $X+Y$ is bi-log-concave.*

Proof. By using formulas (3.1) and (3.2), this is a direct application of the stability through convolution of the log-concavity property (also known as Prékopa's theorem, [10]). □

Corollary 3.2. *Take a (non-degenerate) bi-log-concave measure on $\mathbb{R}$, with density f. Then there exists a sequence of infinitely differentiable bi-log-concave densities, positive on $\mathbb{R}$, that converge to f in $L_p\left(\text{Leb}\right)$, for any $p \in [1, +\infty]$.*

Corollary 3.2 is also an extension of an approximation result available in the set of log-concave distributions, see [13, Section 5.2].

Proof. Note first that the density f is uniformly bounded on $\mathbb{R}$. Indeed, by point (**iii**) of Theorem 1.1 above, the ratio f/F is non-increasing, so that for any $x \in J(F)$, $x \geq m$, $f(x) \leq f(x)/F(x) \leq 2f(m)$. Symmetrically, as the ratio $f/(1-F)$ is non-decreasing, we deduce that $f(x) \leq 2f(m)$ for every $x \in (-\infty, m) \cap J(F)$. This gives that $\left\|f\right\|_\infty = \sup_{x\in\mathbb{R}}\left|f(x)\right| \leq 2f(m)$. Hence, the density f belongs to $L_1\left(\text{Leb}\right) \bigcap L_\infty\left(\text{Leb}\right)$, so it belongs to any $L_p\,(\text{Leb})$, $p \in [1,+\infty]$. It suffices now to consider the convolution of f with a sequence of centered Gaussian densities with variances converging to zero. Indeed, a simple application of classical theorems about convolution in L_p (see for instance [11, p. 148]) allows to check that the approximations converge to f in any $L_p\,(\text{Leb})$, $p \in [1,+\infty]$. □

More generally, the following theorem gives a necessary and sufficient condition for the convolution of two bi-log-concave measures to be bi-log-concave.

Theorem 3.3. *Take X and Y two independent bi-log-concave random variables with respective densities f_X and f_Y and cumulative distribution functions F_X and F_Y. Denote $w(x,y) = f_Y(y) F_X(x-y)$ and $\bar{w}(x,y) = f_Y(y)(1-F_X)(x-y)$ and consider for any $x \in J(F_{X+Y})$, the following measures on $\mathbb{R}$,*

$$dm_x(y) = \frac{w(x,y)\,dy}{\int w(x,y)\,dy} = \frac{w(x,y)\,dy}{F_{X+Y}(x)}$$

and

$$d\bar{m}_x(y) = \frac{\bar{w}(x,y)\,dy}{\int \bar{w}(x,y)\,dy} = \frac{\bar{w}(x,y)\,dy}{1-F_{X+Y}(x)}\,.$$

Then $X+Y$ is bi-log-concave if and only if for any $x \in J(F_{X+Y})$,

$$\mathrm{Cov}_{m_x}\left((-\log f_Y)', (-\log F_X)'(x-\cdot)\right) \geq 0 \tag{3.3}$$

and

$$\mathrm{Cov}_{\bar{m}_x}\left((-\log f_Y)', (-\log(1-F_X))'(x-\cdot)\right) \geq 0\,. \tag{3.4}$$

Theorem 3.3 allows to formulate the question of stability through convolution of two bi-log-concave measures as a problem of covariance inequalities. For instance, as the functions $(-\log F_X)'(x-\cdot)$ and $(-\log(1-F_X))'(x-\cdot)$ are non-decreasing for any $x \in J(F_{X+Y})$, an application of the FKG inequality ([6]) shows that conditions (3.3) and (3.4) are satisfied if $(-\log f_Y)'$ is non-decreasing, which means that f_Y is log-concave, in which case we recover Proposition 3.1 above. But Theorem 3.3 is more general. Indeed, it is easily checked by direct computations that the convolution of the Gaussian mixture $2^{-1}\mathcal{N}(-1.34,1) + 2^{-1}\mathcal{N}(1.34,1)$ – which is bi-log-concave but not log-concave, see [5, Section 2] – with itself is bi-log-concave.

To prove Theorem 3.3, we will use the following lemma.

Lemma 3.4. *Take $p,q \in [1,+\infty]$ such that $p^{-1}+q^{-1}=1$ and a measure ν on $\mathbb{R}$ with absolutely continuous density $f = \exp(-\phi)$ and $f' \in L_p(\nu)$. Take $g \in L_q(\nu)$ Lipschitz continuous such that $g' \in L_1(\nu)$ and*

$$\lim_{x\to+\infty} f(x)(g(x) - \mathbb{E}_\nu[g]) = \lim_{x\to-\infty} f(x)(g(x) - \mathbb{E}_\nu[g]) = 0,$$

then

$$\mathbb{E}_\nu[g'] = \mathrm{Cov}_\nu(g,\phi')\,.$$

In the case where ν is a Gaussian measure, Lemma 3.4 is known as Stein's lemma.

Proof of Lemma 3.4. This is a simple integration by parts: from the assumptions, we have

$$\mathbb{E}_\nu[g'] = \int g' f dx = -\int (g - \mathbb{E}_\nu[g]) f' dx = \int (g - \mathbb{E}_\nu[g])\,\phi' f dx.$$ □

Proof of Theorem 3.3. Recall that we have

$$F_{X+Y}(x) = \int f_Y(y) F_X(x-y)\,dy = \int w(x,y)\,dy\ .$$

Our first goal is to find some conditions such that F_{X+Y} is log-concave. It is sufficient to prove that, for any $x \in J(F_{X+Y})$,

$$\frac{\left(F'_{X+Y}(x)\right)^2}{F_{X+Y}(x)} - F''_{X+Y}(x) \geq 0\,,$$

or equivalently,

$$\left(\frac{F'_{X+Y}(x)}{F_{X+Y}(x)}\right)^2 - \frac{F''_{X+Y}(x)}{F_{X+Y}(x)} \geq 0\,.$$

Denote $\rho_X = (\log F_X)'$. We have

$$\begin{aligned} F_{X+Y}(x) &= \int w(x,y)\,dy \\ f_{X+Y}(x) &= F'_{X+Y}(x) = \int \rho_X(x-y)\,w(x,y)\,dy \\ F''_{X+Y}(x) &= \int \left(\rho'_X(x-y) + \rho_X^2(x-y)\right) w(x,y)\,dy \end{aligned}$$

Furthermore, we get

$$\left(\frac{F'_{X+Y}(x)}{F_{X+Y}(x)}\right)^2 - \frac{\int w\rho_X^2(x-y)\,dy}{F_{X+Y}(x)} = -\operatorname{Var}_{m_x}\left(\rho_X(x-\cdot)\right)\ .$$

Now, by Lemma 3.4, it holds,

$$\begin{aligned} \frac{\int \rho'_X(x-y)\,w(x,y)\,dy}{F_{X+Y}(x)} &= \mathbb{E}_{m_x}\left[\rho'_X(x-\cdot)\right] \\ &= \operatorname{Cov}_{m_x}\left(-\rho_X(x-\cdot), (-\log f_Y)' + \rho_X(x-\cdot)\right)\ . \end{aligned}$$

Gathering the equations, we get

$$\begin{aligned} \left(\frac{F'_{X+Y}(x)}{F_{X+Y}(x)}\right)^2 - \frac{F''_{X+Y}(x)}{F_{X+Y}(x)} &= \operatorname{Cov}_{m_x}\left(-\rho_X(x-\cdot), (-\log f_Y)'\right) \\ &= \operatorname{Cov}_{m_x}\left(-\log F_X(x-\cdot), (-\log f_Y)'\right)\ , \end{aligned}$$

which gives condition (3.3). Likewise condition (3.4) arises from the same type of computations when studying log-concavity of $(1 - F_{X+Y})$. □

3.1 Towards a multivariate notion of bi-log-concavity

We consider the following multidimensional extension of the univariate notion of bi-log-concavity defined in [5] and studied above.

Definition 3.5. *Let μ be a probability measure on $\mathbb{R}^d$, $d \geq 1$. Then μ is said to be bi-log-concave if for every line $\ell \subset \mathbb{R}^d$, the (Euclidean) projection measure μ_ℓ of μ onto the line ℓ is a (one-dimensional) bi-log-concave measure on ℓ (that can be possibly degenerate). More explicitly, for any $x \in \ell$ and any Borel set $B \subset \mathbb{R}$,*

$$\mu_\ell(x + Bu) = \mu\left\{y \in \mathbb{R}^d : (y-x)\cdot u \in B\right\}$$

where u is a unit directional vector of the line ℓ.

Note that log-concave measures on $\mathbb{R}^d$ are also bi-log-concave in the sense of Definition 3.5. The following result states that our mutivariate notion of bi-log-concavity is stable through convolution by log-concave measures.

Proposition 3.6. *The convolution of a log-concave measure on $\mathbb{R}^d$ with a bi-log-concave one is bi-log-concave.*

Proof. The formula $(X+Y)\cdot u = X\cdot u + Y\cdot u$ shows that the projection of the convolution of two measures on a line is the convolution of the projections of measures on this line. This allows to reduce the stability through convolution by a log-concave measure to dimension one and concludes the proof. □

It is moreover directly seen that the proposed multivariate notion of bi-log-concavity is stable by affine transformations of the space.

Actually, in addition to containing log-concave measures and being stable through convolution by a log-concave measure, there are at least two other properties that one would naturally require for a convenient multidimensional concept of bi-log-concavity: existence of a density with respect to the Lebesgue measure on the convex hull of its support and existence of a finite exponential moment for the (Euclidean) norm. We can express this latter remark through the following open problem, that concludes this note.

Open Problem: Find a nice characterization of probability measures on $\mathbb{R}^d$ that are bi-log-concave in the sense of Definition 3.5, that admit a density with respect to the Lebesgue measure on the convex hull of their support and whose associated random vector has an Euclidean norm with exponentially decreasing tails.

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