Electron. Commun. Probab. **24** (2019), no. 55, 1–10.
https://doi.org/10.1214/19-ECP261
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Exponential convergence to equilibrium for the d-dimensional East model*

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Abstract

Kinetically constrained models (KCMs) are interacting particle systems on $\mathbb{Z}^d$ with a continuous-time constrained Glauber dynamics, which were introduced by physicists to model the liquid-glass transition. One of the most well-known KCMs is the one-dimensional East model. Its generalization to higher dimension, the d-dimensional East model, is much less understood. Prior to this paper, convergence to equilibrium in the d-dimensional East model was proven to be at least stretched exponential, by Chleboun, Faggionato and Martinelli in 2015. We show that the d-dimensional East model exhibits exponential convergence to equilibrium in all settings for which convergence is possible.

Keywords: interacting particle systems; Glauber dynamics; kinetically constrained models; East model; convergence to equilibrium.
AMS MSC 2010: 60K35.
Submitted to ECP on May 2, 2019, final version accepted on August 19, 2019.

1 Introduction

Kinetically constrained models (KCMs) are interacting particle systems on graphs, in which each vertex (or *site*) of the graph has state (or *spin*) 0 or 1. Each site tries at rate 1 to *update* its spin, that is to replace it by 1 with probability p and by 0 with probability $1-p$, but the update is accepted only if a certain *constraint* is satisfied, the constraint being of the form "there are enough sites with spin zero around this site".

KCMs were introduced by physicists to model the liquid-glass transition, which is an important open problem in condensed matter physics (see [16, 11]). In addition to their physical interest, they are also mathematically challenging because the presence of the constraints gives them a very different behavior from classical Glauber dynamics and renders most of the usual tools ineffective.

A key feature of KCMs is the existence of blocked spin configurations, which makes the large-time behavior of KCMs hard to study, especially their relaxation to equilibrium when starting out of equilibrium. Indeed, worst case analysis does not help and standard coercive inequalities of the log-Sobolev type also fail. Furthermore, the dynamics of KCMs is not attractive, so coupling arguments that have proven very useful for other types of Glauber dynamics are here inefficient. Because of these difficulties, convergence

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to equilibrium has been proven only in a few models and under particular conditions (see [6, 3, 7, 15]).

There is only one model for which exponentially fast relaxation to equilibrium was proven under general conditions (apart from some models on trees that use the same proof): the East model, whose base graph is $\mathbb{Z}$ and in which an update is accepted when the site at the left has spin 0. Introduced by physicists in [13], the East model is the most well-understood KCM (see [9] for a review).

A natural generalization of the East model to $\mathbb{Z}^d$, introduced in [1], is to accept updates at a site x when $x - e$ has spin 0 for some e in the canonical basis of $\mathbb{R}^d$. The higher dimension makes this d-dimensional East model much harder to study than the unidimensional one, and until now the relaxation to equilibrium was only proved to be at least stretched exponential ([7]).

In this article, we prove that the relaxation to equilibrium in the d-dimensional East dynamics is exponentially fast as soon as the initial configuration is not blocked. This also allowed us to prove that the *persistence function*, which is the probability that a given site has not yet been updated, decays exponentially with time.

Our results, which are the first to hold for a KCM in dimension greater than 1 and for any p, may help to understand further the out-of-equilibrium behavior of the d-dimensional East model. Indeed, such an exponential relaxation result was key to proving "shape theorems" in one-dimensional models in [2, 10, 4].

This paper is organized as follows: we begin by presenting the notations and stating our results in Section 2, then we prove the exponential relaxation to equilibrium in Section 3, and finally we show the exponential decay of the persistence function in Section 4.

2 Notations and results

We fix $d \in \mathbb{N}^*$. For any $\Lambda \subset \mathbb{Z}^d$, the d-dimensional East model (in the following, we will just call it "East model") in Λ is a dynamics on $\{0,1\}^\Lambda$. The elements of Λ will be called sites and the elements of $\{0,1\}^\Lambda$ will be called configurations. For any $\eta \in \{0,1\}^\Lambda$, $x \in \Lambda$, the value of η at x will be called the spin of η at x and denoted by $\eta(x)$.

If $f : \{0,1\}^\Lambda \mapsto \mathbb{R}$ is a function and $\Lambda' \subset \Lambda$, we say the support of f is contained in Λ' and we write $\mathrm{supp}(f) \subset \Lambda'$ when for any $\eta, \eta' \in \{0,1\}^\Lambda$ coinciding in Λ', $f(\eta) = f(\eta')$. Moreover, the ℓ^∞-norm of f, denoted by $\|f\|_\infty$, is $\sup_{\eta \in \{0,1\}^\Lambda} |f(\eta)|$.

We denote $\{e_1, \dots, e_d\}$ the canonical basis of $\mathbb{R}^d$. For any $r \in \mathbb{R}^+$, we denote $\Lambda(r) = (\prod_{i=1}^d \{0, \dots, \lfloor r \rfloor\}) \setminus \{(0, \dots, 0)\}$.

For any set A, $|A|$ will denote the cardinal of A. For $\rho, \rho' \in \mathbb{R}$, we will use the abbreviation $\rho \wedge \rho' = \min(\rho, \rho')$.

To define the East dynamics in $\Lambda \subset \mathbb{Z}^d$, we begin by fixing $p \in]0,1[$. Informally, the East dynamics can be seen as follows: each site x, independently of all others, waits for a random time with exponential law of mean 1, then tries to update its spin, that is to replace it by 1 with probability p and by 0 with probability $1-p$, but the update is accepted if and only if one of the $x - e_i$ is at zero. Then x waits for another random time with exponential law, etc.

More rigorously, independently for each $x \in \Lambda$, we consider a sequence $(B_{x,n})_{n \in \mathbb{N}^*}$ of independent random variables with Bernoulli law of parameter p, and a sequence of times $(t_{x,n})_{n \in \mathbb{N}^*}$ such that, denoting $t_{x,0} = 0$, the $(t_{x,n} - t_{x,n-1})_{n \in \mathbb{N}^*}$ are independent random variables with exponential law of parameter 1, independent from $(B_{x,n})_{n \in \mathbb{N}^*}$. The dynamics is continuous-time, denoted by $(\eta_t)_{t \in \mathbb{R}^+}$, and evolves as follows. For each $x \in \Lambda$, $n \in \mathbb{N}^*$, if there exists $i \in \{1, \dots, d\}$ such that $\eta_{t_{x,n}}(x - e_i) = 0$, then the spin at x is replaced by $B_{x,n}$ at time $t_{x,n}$. We then say there was an update at x at time $t_{x,n}$, or that x was updated at time $t_{x,n}$. (If there are sites $x - e_i$, $x \in \Lambda$, $i \in \{1, \dots, d\}$ that are not in Λ, we need to fix the state of their spins in order to run the dynamics.) One can use the arguments in Section 4.3 of [17] to see that this dynamics is well-defined.

For any $\eta \in \{0,1\}^{\Lambda}$, we denote the law of the dynamics starting from the configuration η by $\mathbb{P}_\eta$, and the associated expectation by $\mathbb{E}_\eta$. If the initial configuration follows a law ν on $\{0,1\}^{\Lambda}$, the law and expectation of the dynamics will be respectively denoted by $\mathbb{P}_\nu$ and $\mathbb{E}_\nu$. In the remainder of this work, we will always consider the dynamics on $\mathbb{Z}^d$ unless stated otherwise.

For any $t \geq 0$ and $\Lambda \subset \mathbb{Z}^d$, we denote $\mathcal{F}_{t,\Lambda} = \sigma(t_{x,n}, B_{x,n}, x \in \Lambda, t_{x,n} \leq t)$ the σ-algebra of the exponential times and Bernoulli variables in the domain Λ between time 0 and time t. We notice that if η_0 is deterministic, for any $x \in \mathbb{Z}^d$, $\eta_t(x)$ depends only on the $t_{x,n}, B_{x,n}$ with $t_{x,n} \leq t$ and on the state of sites "below" x: $x - e_1, \dots, x - e_d$, which in turn depends only on the $\eta_0(x - e_i)$, $t_{x-e_i,n}, B_{x-e_i,n}$ with $t_{x-e_i,n} \leq t$ and on the state of the sites "below" the $x - e_i$, etc. Therefore $\eta_t(x)$ depends only on η_0 and on the $t_{y,n}, B_{y,n}$ with $t_{y,n} \leq t$ and $y \in x + (-\mathbb{N})^d$, hence $\eta_t(x)$ is $\mathcal{F}_{t,x+(-\mathbb{N})^d}$-measurable.

We will call μ the product $\mathrm{Bernoulli}(p)$ measure on the configuration space $\{0,1\}^{\Lambda}$. The expectation with respect to μ of a function $f : \{0,1\}^{\Lambda} \mapsto \mathbb{R}$, if it exists, will be denoted $\mu(f)$. μ is the equilibrium measure of the dynamics, which can be seen using reversibility, since the detailed balance is satisfied.

We say that a measure ν on $\{0,1\}^{\mathbb{Z}^d}$ satisfies Condition $(\mathcal{C})$ when

$$(\mathcal{C}) : \exists\, a, A > 0, \forall\, \ell \geq 0, \nu(\forall\, x \in \{-\lfloor \ell \rfloor, \dots, 0\}^d, \eta(x) = 1) \leq A e^{-a\ell}.$$

Remark 2.1. The set of measures satisfying $(\mathcal{C})$ includes

- the δ_η for any $\eta \in \{0,1\}^{\mathbb{Z}^d}$ such that there exists $x = (x_1, \dots, x_d) \in (-\mathbb{N})^d$ with $\eta(x) = 0$. This is the minimal condition on η for which to expect convergence to equilibrium, since if the initial configuration contains only ones, there can be no updates, hence the dynamics is blocked.
- the product $\mathrm{Bernoulli}(p')$ measures with $p' \in [0,1[$, which are particularly relevant for physicists (see [14]).

We can now state the main result of the paper, the convergence of the dynamics to equilibrium:

Theorem 2.2. *For any measure ν on $\{0,1\}^{\mathbb{Z}^d}$ satisfying $(\mathcal{C})$, there exist constants $\chi = \chi(p) > 0$, $c_1 = c_1(p,\nu) > 0$ and $C_1 = C_1(p,\nu) > 0$ such that, for any $t \geq 0$ and any $f : \{0,1\}^{\mathbb{Z}^d} \mapsto \mathbb{R}$ with $\mathrm{supp}(f) \subset \Lambda(\chi t^{1/d})$,*

$$\int_{\{0,1\}^{\mathbb{Z}^d}} |\mathbb{E}_\eta(f(\eta_t)) - \mu(f)| \,\mathrm{d}\nu(\eta) \leq C_1 \|f\|_\infty e^{-c_1 t}.$$

Remark 2.3. With only minor modifications in the proof, one can also show exponential convergence of the quantity $\int_{\{0,1\}^{\mathbb{Z}^d}} |\mathbb{E}_\eta(f(\eta_t)) - \mu(f)|^\gamma \,\mathrm{d}\nu(\eta)$ for any $\gamma > 0$.

Another quantity of interest is the persistence function. If ν is the law of the initial configuration and $x \in \mathbb{Z}^d$, the corresponding persistence function can be defined as $F_{\nu,x}(t) = \mathbb{P}_\nu(\tau_x > t)$ for any $t \geq 0$, where τ_x is the first time there is an update at x. The persistence function is a "measure of the mobility of the system": the more the spin at x can change, the faster it will decrease. Theorem 2.2 allows to prove exponential decay of the persistence function:

Corollary 2.4. *For any measure ν on $\{0,1\}^{\mathbb{Z}^d}$ satisfying $(\mathcal{C})$, there exist constants $\chi = \chi(p) > 0$, $c_2 = c_2(p,\nu) > 0$ and $C_2 = C_2(p,\nu) > 0$ such that for any $t \geq 0$ and any $x \in \Lambda(\chi t^{1/d})$, $F_{\nu,x}(t) \leq C_2 e^{-c_2 t}$.*

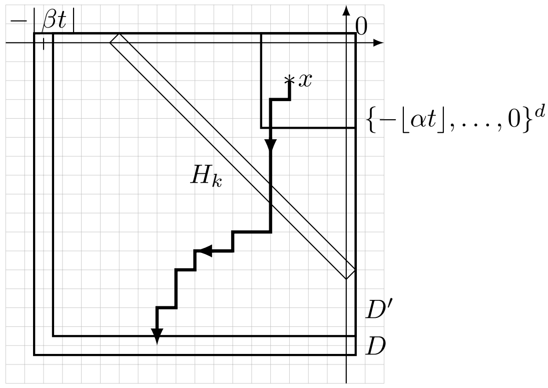

Figure 1: The setting of the proof of Proposition 3.1 for $d = 2$. The thick squares represent $\{-\lfloor\alpha t\rfloor,\dots,0\}^d$, D' and D (from smallest to largest). An oriented path joining $x \in \{-\lfloor\alpha t\rfloor,\dots,0\}^d$ to $D \setminus D'$ (thick arrows) intersects any H_k with $k \in \{d\lfloor\alpha t\rfloor,\dots,\lfloor\beta t\rfloor\}$ (sloped rectangle).

Remark 2.5. The decay of the persistence function can not be faster than exponential, because $\tau_x \ge t_{x,1}$, thus $F_{\nu,x}(t) \ge \mathbb{P}_\nu(t_{x,1} \ge t) = e^{-t}$. Moreover, since the spin of a site x will remain in its initial state until τ_x, the convergence to equilibrium can not be faster than exponential. Consequently, the exponential speed is the actual speed.

Remark 2.6. In Theorem 2.2 and Corollary 2.4, one could replace $\Lambda(\chi t^{1/d})$ with any box of the form $(\prod_{i=1}^d\{0,\dots,a_i\}) \setminus \{(0,\dots,0)\}$, $a_1,\dots,a_d \in \mathbb{N}$, $\prod_{i=1}^d(a_i+1) - 1 \le 2^d\chi^d t$.

3 Proof of Theorem 2.2

The proof of the theorem can be divided in three steps. Firstly, we use a novel argument to find a site of $(-\mathbb{N})^d$ at distance $O(t)$ from the origin that remains at zero for a total time $\Omega(t)$ between time 0 and time t (Section 3.1). Afterwards, we use sequentially a result of [7] to prove that the origin also stays at zero for a time $\Omega(t)$ (Section 3.2). Finally, we end the proof of the theorem with the help of a formula derived in [7].

3.1 Finding a site that stays at zero for a time $\Omega(t)$

For any $t \ge 0$ and $\alpha > 0$, we denote $D = D(t,\alpha) = \{-\lfloor 2d\alpha t\rfloor,\dots,0\}^d$. For any $x \in \mathbb{Z}^d$, we denote $\mathcal{T}_t(x) = \int_0^t \mathbb{1}_{\{\eta_s(x)=0\}}\mathrm{d}s$ the time that x spends at zero between time 0 and time t. We also define $\mathcal{G} = \{\exists x \in D \,|\, \mathcal{T}_t(x) \ge \frac{1-p}{4}t\}$. We then have

Proposition 3.1. *For any $\alpha > 0$, there exist constants $c_3 = c_3(p,\alpha) > 0$ and $C_3 = C_3(p,\alpha) > 0$ such that for any $t \ge 0$, for any $\eta \in \{0,1\}^{\mathbb{Z}^d}$ such that there exists $x \in \{-\lfloor\alpha t\rfloor,\dots,0\}^d$ with $\eta(x) = 0$, $\mathbb{P}_\eta(\mathcal{G}^c) \le C_3 e^{-c_3 t}$.*

Proof. The setting of the proof is illustrated in Figure 1. We set $\alpha > 0$. It is enough to prove the proposition for $t \ge 1/(2d\alpha - \alpha)$, so we fix $t \ge 1/(2d\alpha - \alpha)$. Let $\eta \in \{0,1\}^{\mathbb{Z}^d}$ with $x \in \{-\lfloor\alpha t\rfloor,\dots,0\}^d$ such that $\eta(x) = 0$ be the initial configuration. We define $E = \{y \in D \,|$ there was an update at y in the time interval $[0,t/2]\}$. Moreover, an *oriented path* will be a sequence of sites $(x^{(1)},\dots,x^{(n)})$ with $n \in \mathbb{N}^*$ such that for any $k \in \{1,\dots,n-1\}$, there exists $i \in \{1,\dots,d\}$ with $x^{(k+1)} = x^{(k)} - e_i$. Furthermore, writing $\beta = 2d\alpha$, we can define $D' = \{-\lfloor\beta t\rfloor + 1,\dots,0\}^d$. Since $t \ge 1/(2d\alpha - \alpha)$, $2d\alpha t - 1 \ge \alpha t$, so $-\lfloor\beta t\rfloor + 1 \le -\lfloor\alpha t\rfloor$, thus $x \in D'$.

The proof of Proposition 3.1 relies on the following auxiliary lemma, whose proof will be postponed until after the proof of Proposition 3.1:

Lemma 3.2. *If no site in D stays at zero during the time interval $[0, t/2]$, then there exists an oriented path in E joining x to $D \setminus D'$.*

This auxiliary lemma implies that we either get a site satisfying $\mathcal{G}$, or a path of $\Omega(t)$ sites that were updated before time $t/2$. In the latter case, the orientation of the model allows us to use a conditioning which yields that the probabilty that none of the sites of the path stays at zero for a time $\frac{1-p}{4}t$ is the product of the probabilities for each of the sites not to stay at zero for a time $\frac{1-p}{4}t$, and we can prove that this probabilty is strictly smaller than one. We now make this argument precise to prove Lemma 3.1.

For any $k \in \{0, \dots, d\lfloor \beta t \rfloor\}$, we define the "diagonal hyperplane" $H_k = \{(x_1, \dots, x_d) \in D \,|\, x_1 + \cdots + x_d = -k\}$ (see Figure 1) and we denote $\mathcal{U}_k = \{H_k \cap E \neq \emptyset\}$. If $\mathcal{G}^c$ occurs, no site of D can stay at zero during the whole time interval $[0, t/2]$, hence by Lemma 3.2 there exists an oriented path in E joining x to $D \setminus D'$. Since $x \in \bigcup_{k=0}^{d\lfloor \alpha t \rfloor} H_k$ and $D \setminus D' \subset \bigcup_{k=\lfloor \beta t \rfloor}^{d\lfloor \beta t \rfloor} H_k$, E intersects all the H_k for $k \in \{d\lfloor \alpha t \rfloor, \dots, \lfloor \beta t \rfloor\}$. This implies $\mathcal{G}^c \subset \bigcap_{k=d\lfloor \alpha t \rfloor}^{\lfloor \beta t \rfloor} \mathcal{U}_k$. Furthermore, for any $k \in \{0, \dots, d\lfloor \beta t \rfloor\}$, we may define $\mathcal{G}_k = \{\exists x \in H_k, \mathcal{T}_t(x) \geq \frac{1-p}{4}t\}$, then $\mathcal{G}^c \subset \bigcap_{k=d\lfloor \alpha t \rfloor}^{\lfloor \beta t \rfloor} \mathcal{G}_k^c$. We deduce $\mathcal{G}^c \subset \bigcap_{k=d\lfloor \alpha t \rfloor}^{\lfloor \beta t \rfloor} (\mathcal{U}_k \cap \mathcal{G}_k^c)$, so

$$\mathbb{P}_\eta(\mathcal{G}^c) \leq \mathbb{E}_\eta \left(\prod_{k=d\lfloor \alpha t \rfloor}^{\lfloor \beta t \rfloor} (\mathbb{1}_{\mathcal{U}_k} \mathbb{1}_{\mathcal{G}_k^c}) \right).$$

For all $k \in \{d\lfloor \alpha t \rfloor, \dots, \lfloor \beta t \rfloor\}$, we define a σ-algebra $\mathcal{F}_k = \sigma(\mathcal{F}_{t,\Lambda}, \sigma(t_{x,n}, x \in H_k, t_{x,n} \leq t/2))$, where $\Lambda = \{(x_1, \dots, x_d) \in (-\mathbb{N})^d \,|\, x_1 + \cdots + x_d < -k\}$. For any $\ell \in \{d\lfloor \alpha t \rfloor, \dots, \lfloor \beta t \rfloor\}$ with $\ell > k$, one can see that everything that happens at the sites in H_ℓ between times 0 and t is $\mathcal{F}_k$-measurable, thus $\mathcal{U}_\ell$ and $\mathcal{G}_\ell^c$ are $\mathcal{F}_k$-measurable. Moreover, for any $x \in H_k$, the spins of the $x - e_i$, $i \in \{1, \dots, d\}$ in the time interval $[0, t/2]$ are $\mathcal{F}_k$-measurable and the $t_{x,n} \leq t/2$ are also $\mathcal{F}_k$-measurable. Therefore the event {there was an update at x between time 0 and time $t/2$} is $\mathcal{F}_k$-measurable, hence $\mathcal{U}_k$ is $\mathcal{F}_k$-measurable. Consequently,

$$\mathbb{P}_\eta(\mathcal{G}^c) \leq \mathbb{E}_\eta \left(\mathbb{E}_\eta \left(\mathbb{1}_{\mathcal{G}_{d\lfloor \alpha t \rfloor}^c} \middle| \mathcal{F}_{d\lfloor \alpha t \rfloor} \right) \mathbb{1}_{\mathcal{U}_{d\lfloor \alpha t \rfloor}} \prod_{k=d\lfloor \alpha t \rfloor+1}^{\lfloor \beta t \rfloor} (\mathbb{1}_{\mathcal{U}_k} \mathbb{1}_{\mathcal{G}_k^c}) \right).$$

Therefore, if we can find a constant $c_3' = c_3'(p) > 0$ such that

$$\forall k \in \{d\lfloor \alpha t \rfloor, \dots, \lfloor \beta t \rfloor\}, \mathbb{1}_{\mathcal{U}_k} \mathbb{E}_\eta \left(\mathbb{1}_{\mathcal{G}_k^c} | \mathcal{F}_k \right) \leq e^{-c_3'} \tag{3.1}$$

then we have

$$\mathbb{P}_\eta(\mathcal{G}^c) \leq e^{-c_3'} \mathbb{E}_\eta \left(\prod_{k=d\lfloor \alpha t \rfloor+1}^{\lfloor \beta t \rfloor} (\mathbb{1}_{\mathcal{U}_k} \mathbb{1}_{\mathcal{G}_k^c}) \right),$$

so by a simple induction $\mathbb{P}_\eta(\mathcal{G}^c) \leq e^{-c_3'(\lfloor \beta t \rfloor + 1 - d\lfloor \alpha t \rfloor)} \leq e^{-c_3'(\beta t - d\alpha t)} = e^{-c_3' d\alpha t}$, which is Proposition 3.1.

Consequently, we only need to prove (3.1). Let $k \in \{d\lfloor \alpha t \rfloor, \dots, \lfloor \beta t \rfloor\}$. For any $x \in H_k$, if the state of the $x - e_i$, $i \in \{1, \dots, d\}$ between time 0 and time t is known, and if the $t_{x,n} \leq t/2$ are also known, the state of x between time 0 and time t depends only on the $t/2 < t_{x,n} \leq t$ and on the $B_{x,n}$ such that $t_{x,n} \leq t$. Therefore, conditionally on $\mathcal{F}_k$, the state of x between time 0 and time t depends only on $\{t/2 < t_{x,n} \leq t\} \cup \{B_{x,n} \,|\, t_{x,n} \leq t\}$. Moreover, these sets for $x \in H_k$ are mutually independent conditionally on $\mathcal{F}_k$, hence the states of the $x \in H_k$ between time 0 and time t are mutually independent conditionally on $\mathcal{F}_k$, which implies

$$\mathbb{1}_{\mathcal{U}_k}\mathbb{E}_\eta\left(\mathbb{1}_{\mathcal{G}_k^c}|\mathcal{F}_k\right) = \mathbb{1}_{\mathcal{U}_k}\prod_{x\in H_k}\mathbb{P}_\eta\left(\mathcal{T}_t(x) < \frac{1-p}{4}t\,\middle|\,\mathcal{F}_k\right) \leq \mathbb{1}_{\mathcal{U}_k}\prod_{x\in H_k\cap E}\mathbb{P}_\eta\left(\mathcal{T}_t(x) < \frac{1-p}{4}t\,\middle|\,\mathcal{F}_k\right). \tag{3.2}$$

In addition, for $x \in H_k \cap E$, we have the following (in the second inequality we use the Markov inequality):

$$\begin{aligned}\mathbb{P}_\eta\left(\mathcal{T}_t(x) < \frac{1-p}{4}t\,\middle|\,\mathcal{F}_k\right) &\leq \mathbb{P}_\eta\left(\int_{t/2}^t \mathbb{1}_{\{\eta_s(x)=0\}}\mathrm{d}s < \frac{1-p}{4}t\,\middle|\,\mathcal{F}_k\right)\\ &\leq \frac{\mathbb{E}_\eta\left(\int_{t/2}^t\mathbb{1}_{\{\eta_s(x)=1\}}\mathrm{d}s\,\middle|\,\mathcal{F}_k\right)}{\frac{t}{2}-\frac{1-p}{4}t} = \frac{\int_{t/2}^t\mathbb{P}_\eta(\eta_s(x)=1|\mathcal{F}_k)\mathrm{d}s}{\left(1-\frac{1-p}{2}\right)\frac{t}{2}}.\end{aligned}$$

Furthermore, for $s \in [t/2, t]$, since $x \in H_k \cap E$, conditionally on $\mathcal{F}_k$ we know that there was an update at x before time s, but not the associated Bernoulli variable, hence $\mathbb{P}_\eta(\eta_s(x) = 1|\mathcal{F}_k) = p$. This implies

$$\mathbb{P}_\eta\left(\mathcal{T}_t(x) < \frac{1-p}{4}t\,\middle|\,\mathcal{F}_k\right) \leq \frac{\int_{t/2}^t p\mathrm{d}s}{\left(1-\frac{1-p}{2}\right)\frac{t}{2}} = \frac{p}{1-\frac{1-p}{2}}.$$

Moreover, $\frac{p}{1-\frac{1-p}{2}} = \frac{2p}{1+p} < 1$, hence if we write $c_3' = -\ln(\frac{p}{1-\frac{1-p}{2}})$, we have $c_3' > 0$ and $\mathbb{P}_\eta(\mathcal{T}_t(x) < \frac{1-p}{4}t|\mathcal{F}_k) \leq e^{-c_3'}$. Consequently, (3.2) yields

$$\mathbb{1}_{\mathcal{U}_k}\mathbb{E}_\eta\left(\mathbb{1}_{\mathcal{G}_k^c}|\mathcal{F}_k\right) \leq \mathbb{1}_{\mathcal{U}_k}\prod_{x\in H_k\cap E} e^{-c_3'} = \mathbb{1}_{\mathcal{U}_k}e^{-c_3'|H_k\cap E|}.$$

Finally, $\mathcal{U}_k$ indicates that $H_k \cap E \neq 0$, thus $\mathbb{1}_{\mathcal{U}_k}\mathbb{E}_\eta(\mathbb{1}_{\mathcal{G}_k^c}|\mathcal{F}_k) \leq \mathbb{1}_{\mathcal{U}_k}e^{-c_3'} \leq e^{-c_3'}$ with $c_3' > 0$ depending only on p, which is (3.1). □

Proof of Lemma 3.2. Let us suppose that no site of D stays at zero during the time interval $[0, t/2]$. Then E contains x, because $x \in D$ and if there was no update at x between time 0 and time $t/2$, the spin of x would stay during this whole time interval at its initial state of 0, which does not happen by assumption. We are going to show that if we have an oriented path in E starting from x that does not reach $D \setminus D'$, we can add a site at its end in a way we still have an oriented path in E. This is enough, because from the path composed only of x we can do at most $d\lfloor \beta t\rfloor$ steps before reaching $D \setminus D'$. Thus we consider an oriented path in E starting from x that does not reach $D \setminus D'$. Let us call y its last site; we have $y \in D'$. Since $y \in E$, y was updated between time 0 and time $t/2$. This implies that one of the $y - e_i$, $i \in \{1, \dots, d\}$, that we may call y', was at zero at the moment of the update. Moreover, $y \in D'$, hence $y' \in D$. There are two possibilities:

- Either the spin of y' was not zero in the initial configuration. Then there was an update at y' before the update at y, hence before time $t/2$, so since $y' \in D$, $y' \in E$.
- Or the spin at y' was zero in the initial configuration. In this case, if there was no update at y' before time $t/2$, y' stayed at 0 during the whole time interval $[0, t/2]$. However $y' \in D$, so this is impossible by assumption. Therefore there was an update at y' before time $t/2$, which implies $y' \in E$.

Therefore $y' \in E$ in both cases, which allows to add a site to the path and ends the proof of Lemma 3.2. □

3.2 Proving the origin stays at zero for a time $\Omega(t)$

In this section, we will use Proposition 3.1 to prove the following result:

Lemma 3.3. *There exist constants $\delta = \delta(p) \in]0,1[$, $\alpha = \alpha(p) > 0$, $c_4 = c_4(p) > 0$ and $C_4 = C_4(p) > 0$ such that for any $t \geq 0$, for any $\eta \in \{0,1\}^{\mathbb{Z}^d}$ such that there exists $x \in \{-\lfloor \alpha t \rfloor, \dots, 0\}^d$ with $\eta(x) = 0$, $\mathbb{P}_\eta(\mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t) \leq C_4 e^{-c_4 t}$.*

Proof. Let $t \geq 0$. Thanks to Proposition 3.1, for any $\alpha > 0$ and $\eta \in \{0,1\}^{\mathbb{Z}^d}$ such that there exists $x \in \{-\lfloor \alpha t \rfloor, \dots, 0\}^d$ with $\eta(x) = 0$, we have $\mathbb{P}_\eta(\mathcal{G}^c) \leq C_3 e^{-c_3 t}$ with $c_3, C_3 > 0$ depending only on p and α. Therefore, it is enough to find $\delta = \delta(p) \in]0,1[$, $\alpha = \alpha(p) > 0$, $C_4' = C_4'(p) > 0$ and $c_4' = c_4'(p) > 0$ depending only on p such that for $\eta \in \{0,1\}^{\mathbb{Z}^d}$ we have $\mathbb{P}_\eta(\mathcal{G}, \mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t) \leq C_4' e^{-c_4' t}$.

Moreover, for any $\delta \in]0,1[$, $\alpha > 0$ and $\eta \in \{0,1\}^{\mathbb{Z}^d}$, we have

$$\mathbb{P}_\eta\left(\mathcal{G}, \mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t\right) \leq \sum_{y \in D} \mathbb{P}_\eta\left(\mathcal{T}_t(y) \geq \frac{1-p}{4} t, \mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t\right). \tag{3.3}$$

For $y = (y_1, \dots, y_d) \in D$, we define the following sequence of sites: $y^{(0)} = y, y^{(1)} = (0, y_2, \dots, y_d), y^{(2)} = (0, 0, y_3, \dots, y_d), \dots, y^{(d)} = (0, \dots, 0)$. We then have

$$\begin{aligned} &\mathbb{P}_\eta\left(\mathcal{T}_t(y) \geq \frac{1-p}{4} t, \mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t\right) \\ &\leq \sum_{i=1}^{d} \mathbb{P}_\eta\left(\mathcal{T}_t(y^{(i-1)}) \geq \delta^{i-1}\frac{1-p}{4} t, \mathcal{T}_t(y^{(i)}) \leq \frac{1-p}{4}\delta^i t\right). \end{aligned} \tag{3.4}$$

To deal with this expression, we are going to use Lemma 4.9 of [7]. This lemma yields that there exist constants $\delta \in]0,1[$ and $c > 0$ depending only on p such that for any $i \in \{1, \dots, d\}$, defining $I_i = \{(0, \dots, j, y_{i+1}, \dots, y_d) | j \in \{y_i + 1, \dots, 0\}\}$ if $y_i \neq 0$ and $I_i = \emptyset$ if $y_i = 0$,

$$\mathbb{P}_\eta(\mathcal{T}_t(y^{(i)}) \leq \delta \mathcal{T}_t(y^{(i-1)}) | \mathcal{F}_{t, I_i^c}) \leq \frac{1}{(p \wedge (1-p))^{|y_i|}} e^{-c\mathcal{T}_t(y^{(i-1)})}.$$

(Actually, this lemma was proven for a dynamics in $\mathbb{N}^d$, but the proof works in $\mathbb{Z}^d$ with only minor modifications.)

Therefore we can set δ to the value given by [7], and obtain the following (in the first inequality we use that $\mathcal{T}_t(y^{(i-1)})$ is $\mathcal{F}_{t, y^{(i-1)} + (-\mathbb{N})^d}$-measurable, hence $\mathcal{F}_{t, I_i^c}$-measurable):

$$\begin{aligned} &\mathbb{P}_\eta\left(\mathcal{T}_t(y^{(i-1)}) \geq \delta^{i-1}\frac{1-p}{4} t, \mathcal{T}_t(y^{(i)}) \leq \frac{1-p}{4}\delta^i t\right) \\ &\leq \mathbb{E}_\eta(\mathbb{1}_{\{\mathcal{T}_t(y^{(i-1)}) \geq \delta^{i-1}\frac{1-p}{4}t\}} \mathbb{P}_\eta(\mathcal{T}_t(y^{(i)}) \leq \delta \mathcal{T}_t(y^{(i-1)}) | \mathcal{F}_{t, I_i^c})) \\ &\leq \mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(y^{(i-1)}) \geq \delta^{i-1}\frac{1-p}{4}t\}} \frac{1}{(p \wedge (1-p))^{|y_i|}} e^{-c\mathcal{T}_t(y^{(i-1)})}\right) \leq \frac{1}{(p \wedge (1-p))^{|y_i|}} e^{-c\delta^{i-1}\frac{1-p}{4}t}. \end{aligned}$$

Moreover, since $y \in D$, $|y_i| \leq \lfloor 2d\alpha t \rfloor \leq 2d\alpha t$, so if we set $\alpha = \frac{c(1-p)\delta^{d-1}}{-16d\ln(p \wedge (1-p))}$ (which is positive and depends only on p), we obtain $(p \wedge (1-p))^{|y_i|} \geq e^{-\frac{c(1-p)\delta^{d-1}}{8}t}$, hence the last term in the display is bounded by $e^{-\frac{c(1-p)\delta^{d-1}}{8}t}$. Therefore, by (3.4),

$$\mathbb{P}_\eta\left(\mathcal{T}_t(y) \geq \frac{1-p}{4} t, \mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t\right) \leq d e^{-\frac{c(1-p)\delta^{d-1}}{8}t},$$

so by (3.3)

$$\mathbb{P}_\eta\left(\mathcal{G}, \mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t\right) \leq |D| d e^{-\frac{c(1-p)\delta^{d-1}}{8}t} = (\lfloor 2d\alpha t \rfloor + 1)^d d e^{-\frac{c(1-p)\delta^{d-1}}{8}t}$$

with $\frac{c(1-p)\delta^{d-1}}{8} > 0$ depending only on p and α depending only on p, so we get a suitable bound on $\mathbb{P}_\eta(\mathcal{G}, \mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t)$. $\square$

3.3 Ending the proof of Theorem 2.2

Let ν a measure on $\{0,1\}^{\mathbb{Z}^d}$ satisfying $(\mathcal{C})$, $t \geq 0$ and $f : \{0,1\}^{\mathbb{Z}^d} \mapsto \mathbb{R}$ non constant with $\|f\|_\infty < \infty$. We denote $\mathcal{N}(\eta) = \{\exists x \in \{-\lfloor \alpha t \rfloor, \dots, 0\}^d, \eta(x) = 0\}$, where $\alpha = \alpha(p) > 0$ is given by Lemma 3.3. We also denote $g = \frac{f-\mu(f)}{\|f-\mu(f)\|_\infty}$. Then

$$\int_{\{0,1\}^{\mathbb{Z}^d}} |\mathbb{E}_\eta(f(\eta_t)) - \mu(f)| \, \mathrm{d}\nu(\eta) = \|f-\mu(f)\|_\infty \int_{\{0,1\}^{\mathbb{Z}^d}} |\mathbb{E}_\eta(g(\eta_t))| \, \mathrm{d}\nu(\eta)$$
$$\leq 2\|f\|_\infty \left(\int_{\{0,1\}^{\mathbb{Z}^d}} |\mathbb{E}_\eta(g(\eta_t))| \mathbb{1}_{\mathcal{N}(\eta)^c} \mathrm{d}\nu(\eta) + \int_{\{0,1\}^{\mathbb{Z}^d}} |\mathbb{E}_\eta(g(\eta_t))| \mathbb{1}_{\mathcal{N}(\eta)} \mathrm{d}\nu(\eta) \right).$$

Moreover, since $\|\mu(g)\|_\infty = 1$ and ν satisfies $(\mathcal{C})$, we can see that we have the following: $\int_{\{0,1\}^{\mathbb{Z}^d}} |\mathbb{E}_\eta(g(\eta_t))| \mathbb{1}_{\mathcal{N}(\eta)^c} \mathrm{d}\nu(\eta) \leq \nu(\mathcal{N}(\eta)^c) \leq A e^{-a\alpha t}$ with $A, a > 0$ depending only on ν.

Therefore, to prove Theorem 2.2, it is enough to find $\chi > 0$ depending only on p such that for any $f : \{0,1\}^{\mathbb{Z}^d} \mapsto \mathbb{R}$ non constant (if f is constant the theorem is trivially true) with support in $\Lambda(\chi t^{1/d})$ (which automatically gives $\|f\|_\infty < \infty$) and any $\eta \in \{0,1\}^{\mathbb{Z}^d}$ such that $\mathcal{N}(\eta)$, $|\mathbb{E}_\eta(g(\eta_t))| \leq C_1' e^{-c_1' t}$ with $C_1', c_1' > 0$ depending only on p. For $\chi > 0$, we set such f and η. Since $\|g\|_\infty = 1$, for δ as in Lemma 3.3 we have

$$|\mathbb{E}_\eta(g(\eta_t))| \leq \mathbb{P}_\eta\left(\mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t\right) + \left|\mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0) > \frac{1-p}{4}\delta^d t\}} g(\eta_t)\right)\right|.$$

In addition, since there is $x \in \{-\lfloor \alpha t \rfloor, \dots, 0\}^d$ such that $\eta(x) = 0$, by Lemma 3.3 we have $\mathbb{P}_\eta(\mathcal{T}_t(0) \leq \frac{1-p}{4}\delta^d t) \leq C_4 e^{-c_4 t}$ with $C_4, c_4 > 0$ depending only on p. Consequently, it is enough to bound $|\mathbb{E}_\eta(\mathbb{1}_{\{\mathcal{T}_t(0) > \frac{1-p}{4}\delta^d t\}} g(\eta_t))|$.

Writing $\Lambda = \Lambda(\chi t^{1/d})$ for short, we notice that the event $\{\mathcal{T}_t(0) > \frac{1-p}{4}\delta^d t\}$ is $\mathcal{F}_{t,(-\mathbb{N})^d}$-measurable hence $\mathcal{F}_{t,\Lambda^c}$-measurable, which implies

$$\left|\mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0) > \frac{1-p}{4}\delta^d t\}} g(\eta_t)\right)\right| = \left|\mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0) > \frac{1-p}{4}\delta^d t\}} \mathbb{E}_\eta\left(g(\eta_t) | \mathcal{F}_{t,\Lambda^c}\right)\right)\right|,$$

therefore

$$\left|\mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0) > \frac{1-p}{4}\delta^d t\}} g(\eta_t)\right)\right|$$
$$\leq \frac{1}{\min_{\sigma \in \{0,1\}^\Lambda} \mu(\sigma)} \mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0) > \frac{1-p}{4}\delta^d t\}} \sum_{\sigma \in \{0,1\}^\Lambda} \mu(\sigma) \mathbb{E}_{\sigma\cdot\eta}\left(g(\eta_t) | \mathcal{F}_{t,\Lambda^c}\right)\right),$$

where $\sigma \cdot \eta$ is the configuration equal to σ in Λ and to η in Λ^c. Furthermore, the reasoning of Equation (4.2) of [7] and of the paragraphs around it yields that

$$\sum_{\sigma \in \{0,1\}^\Lambda} \mu(\sigma) \mathbb{E}_{\sigma\cdot\eta}\left(g(\eta_t) | \mathcal{F}_{t,\Lambda^c}\right) \leq e^{-\lambda \mathcal{T}_t(0)}$$

where λ is the *spectral gap* of the East dynamics in Λ where the spin of the origin is fixed at 0 and the other spins outside Λ are at 1 (see Chapter 2 of [12] for the definition of the spectral gap and Section 2.4 of [5] for an introduction to the spectral gap in the particular context of kinetically constrained models). Moreover, one can use the argument of Section 6.2.2 of [8] on our Λ instead of on a cube to obtain that λ is bigger than the spectral gap λ' of the one-dimensional East dynamics in $\{1, \dots, d\lfloor \chi t^{1/d} \rfloor\}$ with the origin fixed at zero. To do that, one can use a forest instead of a tree and apply the fact that the spectral gap of a product dynamics is the minimum of the spectral gaps of the component dynamics (Theorem 2.5 of [12]). Furthermore, Equation (3.3) of [8] yields that λ' is bigger than the spectral gap λ'' of the East dynamics in $\mathbb{Z}$, which depends only on p and is positive by Theorem 6.1 of [5].

Consequently, we have

$$\begin{aligned}\left|\mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0)>\frac{1-p}{4}\delta^d t\}}g(\eta_t)\right)\right| &\leq \frac{1}{\min_{\sigma\in\{0,1\}^\Lambda}\mu(\sigma)}\mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0)>\frac{1-p}{4}\delta^d t\}}e^{-\lambda''\mathcal{T}_t(0)}\right)\\ &\leq \frac{1}{(p\wedge(1-p))^{|\Lambda|}}e^{-\lambda''\frac{1-p}{4}\delta^d t}.\end{aligned}$$

Moreover, $|\Lambda| \leq (\chi t^{1/d}+1)^d$ and we can suppose $\chi t^{1/d} \geq 1$, since if $\chi t^{1/d} < 1$, $|\Lambda|$ is empty and there is no non constant function with support in Λ. Therefore we get $|\Lambda| \leq (2\chi t^{1/d})^d = 2^d\chi^d t$. Now, if we set $\chi = \frac{1}{2}(\frac{\lambda''(1-p)\delta^d}{-8\ln(p\wedge(1-p))})^{1/d}$, χ is positive and depends only on p, and we have $(p\wedge(1-p))^{|\Lambda|} \geq e^{-\frac{\lambda''(1-p)\delta^d}{8}t}$, thus

$$\left|\mathbb{E}_\eta\left(\mathbb{1}_{\{\mathcal{T}_t(0)>\frac{1-p}{4}\delta^d t\}}g(\eta_t)\right)\right| \leq e^{-\frac{\lambda''(1-p)\delta^d}{8}t}$$

with $\frac{\lambda''(1-p)\delta^d}{8}$ positive depending only on p, which ends the proof of Theorem 2.2.

4 Proof of Corollary 2.4

This proof is inspired from the proof of Lemma A.3 of [8].

Let ν a measure on $\{0,1\}^{\mathbb{Z}^d}$ satisfying $(\mathcal{C})$, χ as in Theorem 2.2, $t \geq 0$, $x \in \Lambda(\chi t^{1/d})$. For any $\eta \in \{0,1\}^{\mathbb{Z}^d}$, we have

$$\begin{aligned}\mathbb{E}_\eta(\eta_t(x)) &= \mathbb{E}_\eta(\eta_t(x)|\tau_x \leq t)\mathbb{P}_\eta(\tau_x \leq t) + \mathbb{E}_\eta(\eta_t(x)|\tau_x > t)\mathbb{P}_\eta(\tau_x > t)\\ &= p\mathbb{P}_\eta(\tau_x \leq t) + \eta(x)\mathbb{P}_\eta(\tau_x > t) = p - p\mathbb{P}_\eta(\tau_x > t) + \eta(x)\mathbb{P}_\eta(\tau_x > t)\end{aligned}$$

since if $\tau_x \leq t$, $\eta_t(x)$ is a Bernoulli random variable of parameter p. Therefore,

$$|\mathbb{E}_\eta(\eta_t(x)) - p| = |\eta(x)-p|\mathbb{P}_\eta(\tau_x > t) \geq (p\wedge(1-p))\mathbb{P}_\eta(\tau_x > t),$$

and we deduce

$$\begin{aligned}F_{\nu,x}(t) = \mathbb{P}_\nu(\tau_x > t) &= \int_{\{0,1\}^{\mathbb{Z}^d}}\mathbb{P}_\eta(\tau_x > t)\mathrm{d}\nu(\eta)\\ &\leq \frac{1}{p\wedge(1-p)}\int_{\{0,1\}^{\mathbb{Z}^d}}|\mathbb{E}_\eta(\eta_t(x)) - p|\mathrm{d}\nu(\eta) \leq \frac{1}{p\wedge(1-p)}C_1e^{-c_1 t}\end{aligned}$$

by Theorem 2.2 with $C_1 > 0$ and $c_1 > 0$ depending only on p and ν.

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Acknowledgments. I would like to thank my PhD advisor, Cristina Toninelli.