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Existence and uniqueness of solution to scalar BSDEs with $L\exp(\mu\sqrt{2\log(1+L)})$-integrable terminal values: the critical case*

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Abstract

In [8], the existence of the solution is proved for a scalar linearly growing backward stochastic differential equation (BSDE) when the terminal value is $L\exp(\mu\sqrt{2\log(1+L)})$-integrable for a positive parameter $\mu > \mu_0$ with a critical value μ_0, and a counterexample is provided to show that the preceding integrability for $\mu < \mu_0$ is not sufficient to guarantee the existence of the solution. Afterwards, the uniqueness result (with $\mu > \mu_0$) is also given in [3] for the preceding BSDE under the uniformly Lipschitz condition of the generator. In this note, we prove that these two results still hold for the critical case: $\mu = \mu_0$.

Keywords: backward stochastic differential equation; $L\exp(\mu\sqrt{2\log(1+L)})$-integrability; existence and uniqueness; critical case.
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1 Introduction

Let us fix a positive integer d and a positive real number $T > 0$. For any two elements x, y in $\mathbb{R}^d$, denote by $x \cdot y$ their scalar inner product. Let $(B_t)_{t\in[0,T]}$ be a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t\in[0,T]}$ its natural filtration augmented by all $\mathbb{P}$-null sets of $\mathcal{F}$. We study the following backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0, T], \tag{1.1}$$

where ξ is a real-valued and $\mathcal{F}_T$-measurable random variable called the terminal condition or terminal value, the function (called the generator) $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is $(\mathcal{F}_t)$-progressively measurable for each (y, z) and continuous in (y, z), and the pair

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of processes $(Y_t, Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}\times\mathbb{R}^d$ is called the solution of (1.1), which is $(\mathcal{F}_t)$-progressively measurable such that $\mathbb{P}-a.s.$, $t\mapsto Y_t$ is continuous, $t\mapsto Z_t$ belongs to $L^2(0,T)$, $t\mapsto g(t,Y_t,Z_t)$ is integrable, and verifies (1.1). By $\mathrm{BSDE}(\xi,g)$, we mean the BSDE with terminal value ξ and generator g.

Denote $g_0 := \int_0^T g(t,0,0)\mathrm{d}t$. It is well known that for any $p>1$, if the generator g satisfies the linear growth condition (see the assumption (H1) in section 2) and the terminal value $\xi+g_0$ is L^p-integrable, then $\mathrm{BSDE}(\xi,g)$ admits a minimal (maximal) solution $(Y_\cdot, Z_\cdot)$ in the space of the processes $\mathcal{S}^p\times\mathcal{M}^p$ (see their definitions in section 2), and the solution is unique in the space $\mathcal{S}^p\times\mathcal{M}^p$ if g further satisfies the uniformly Lipschitz condition (see the assumption (H2) in section 2). See e.g. [10, 5, 9, 1, 7] for more details. However, if the random variable $\xi+g_0$ is only integrable, one needs to restrict the generator g to grow sub-linearly with respect to z, i.e., with some $q\in[0,1)$,

$$|g(\omega,t,y,z)|\le|g_0(\omega,t)|+\beta|y|+\gamma|z|^q,\ \ (\omega,t,y,z)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d,$$

for $\mathrm{BSDE}(\xi,g)$ to have a minimal (maximal) adapted solution and a unique solution when g satisfies (H1) and (H2) respectively. See for example [1, 2, 6] for more details.

Recently, by applying the dual representation of solution to BSDE with convex generator, see for instance [5, 11, 4], to establish some a priori estimate and the localization procedure, the authors in [8] proved the existence of a solution to $\mathrm{BSDE}(\xi,g)$ when the generator g satisfies (H1) and the terminal value $\xi+g_0$ is $L\exp\left(\mu\sqrt{2\log(1+L)}\right)$-integrable for a positive parameter $\mu>\mu_0$ with a critical value $\mu_0=\gamma\sqrt{T}$, and showed by a counterexample that the conventionally expected $L\log L$ integrability and even the preceding integrability for a positive parameter $\mu<\mu_0$ is not enough for the existence of a solution to a BSDE with the generator g satisfying (H1). Furthermore, by establishing some properties of the function $\psi(x,\mu)=x\exp\left(\mu\sqrt{2\log(1+x)}\right)$ and observing the property of the obtained solution Y that $\psi(|Y|,a)$ belongs to class (D) for some $a>0$, the authors in [3] divided the whole interval $[0,T]$ into a finite number of subintervals and proved the uniqueness of the solution to the preceding $\mathrm{BSDE}(\xi,g)$ with the generator g satisfying (H2) and $\mu>\mu_0$.

In this note, we prove that the existence and uniqueness result obtained respectively in [8] and [3] is still true in the critical value case: $\mu=\gamma\sqrt{T}$, see Theorem 3.1 in section 3.

For the existence of the solution to $\mathrm{BSDE}(\xi,g)$, in order to apply the localization procedure put forward initially in [2], the key idea is always to establish some uniform a priori estimate for the first process $Y_\cdot^{n,p}$ in the solution of the approximated BSDEs (see the definition and the a priori estimate of $Y_\cdot^{n,p}$ in the proof of the existence part of Theorem 3.1 after Remark 3.6 in section 3). For this, instead of applying the dual representation of solution to BSDE with convex generator, our whole idea consists in searching for an appropriate function $\phi(s,x;t)$ in order to apply Itô-Tanaka's formula to $\phi(s,|Y_s^{n,p}|;t)$ on the time interval $s\in[t,\tau_m]$ with $(\mathcal{F}_t)$-stopping time τ_m valued in $[t,T]$ (see the proof of Proposition 3.5 in section 3 for details). More specifically, we need to find a positive, continuous, strictly increasing and strictly convex function $\phi(s,x;t):[t,T]\times[0,+\infty)\mapsto(0,+\infty)$ with $t\in(0,T]$ satisfying

$$-\gamma\phi_x(s,x;t)|z|+\frac{1}{2}\phi_{xx}(s,x;t)|z|^2+\phi_s(s,x;t)\ge0,\ \ (s,x,z)\in[t,T]\times[0,+\infty)\times\mathbb{R}^d,\ \ (1.2)$$

where and hereafter, for each $t\in(0,T]$, $\phi_s(\cdot,\cdot;t)$ denotes the first-order partial derivative of $\phi(\cdot,\cdot;t)$ with respect to the first variable, and $\phi_x(\cdot,\cdot;t)$ and $\phi_{xx}(\cdot,\cdot;t)$ respectively the first-order and second order partial derivative of $\phi(\cdot,\cdot;t)$ with respect to the second variable. Observe from the basic inequality $2ab \le a^2+b^2$ that

$$
\begin{aligned}
-\gamma\phi_x(s,x;t)|z| + \frac{1}{2}\phi_{xx}(s,x;t)|z|^2 &= \phi_{xx}(s,x;t)\left(-\frac{\gamma\phi_x(s,x;t)}{\phi_{xx}(s,x;t)}|z| + \frac{1}{2}|z|^2\right)\\
&\ge -\frac{\gamma^2}{2}\frac{\phi_x^2(s,x;t)}{\phi_{xx}(s,x;t)}.
\end{aligned}
$$

Hence, it suffices if for each $t \in (0,T]$, the function $\phi(\cdot,\cdot;t)$ satisfies the following condition:

$$
-\frac{\gamma^2}{2}\frac{\phi_x^2(s,x;t)}{\phi_{xx}(s,x;t)} + \phi_s(s,x;t) \ge 0, \quad (s,x) \in [t,T]\times[0,+\infty). \tag{1.3}
$$

Inspired by the investigation in [8] and [3], we can choose the following function, for each $t \in (0,T]$,

$$
\phi(s,x;t) := (x+e)\exp\left(\mu_s\sqrt{2\log(x+e)} + \int_t^s k_r \mathrm{d}r\right), \ (s,x) \in [t,T]\times[0,+\infty) \tag{1.4}
$$

to explicitly solve the inequality (1.3). We find that (1.3) is satisfied for $\phi(s,x;t)$ when

$$
\mu_s = \gamma\sqrt{s} \ \text{ and } \ k_r = \frac{\gamma}{2}\left(\gamma + \sqrt{\frac{2}{r}}\right). \tag{1.5}
$$

For the uniqueness of the solution to BSDE(ξ, g), by virtue of two useful inequalities obtained in [8], we use a similar idea to that in [3] to divide the whole interval $[0,T]$ into some sufficiently small subintervals and show successively the uniqueness of the solution in these subintervals. However, different from [3], in our case the number of these subintervals, which are $[3T/4,T]$, $[3^2T/4^2, 3T/4]$, $[3^3T/4^3, 3^2T/4^2]$, $\cdots$, $[3^nT/4^n, 3^{n-1}T/4^{n-1}]$, $\cdots$, is infinite. Fortunately, observing that the left end points of these subintervals tend to 0 as $n\to\infty$ and in view of the continuity of the first process in the solution with respect to the time variable, we can obtain the uniqueness of the solution on the whole interval $[0,T]$ by taking the limit.

The rest of this note is organized as follows. In next section, we introduce some notations and assumptions which will be used later, and in section 3 we state and prove the main result.

2 Notations and assumptions

First, for any real number $p\ge 1$, let L^p represent the set of (equivalent classes of) all real-valued and $\mathcal{F}_T$-measurable random variables ξ such that $\mathbb{E}[|\xi|^p] < +\infty$, $\mathcal{L}^p$ the set of (equivalent classes of) all real-valued and $(\mathcal{F}_t)$-progressively measurable processes $(X_t)_{t\in[0,T]}$ such that

$$
\|X\|_{\mathcal{L}^p} := \left\{\mathbb{E}\left[\left(\int_0^T |X_t|\mathrm{d}t\right)^p\right]\right\}^{1/p} < +\infty,
$$

$\mathcal{S}^p$ the set of (equivalent classes of) all real-valued, $(\mathcal{F}_t)$-progressively measurable and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$
\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E}[\sup_{t\in[0,T]}|Y_t|^p]\right)^{1/p} < +\infty,
$$

and $\mathcal{M}^p$ the set of (equivalent classes of) all $\mathbb{R}^d$-valued and $(\mathcal{F}_t)$-progressively measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{\mathcal{M}^p} := \left\{ \mathbb{E}\left[\left(\int_0^T |Z_t|^2 \mathrm{d}t \right)^{p/2} \right] \right\}^{1/p} < +\infty.$$

Secondly, we recall that a real-valued and $(\mathcal{F}_t)$-progressively measurable process $(X_t)_{t\in[0,T]}$ belongs to class (D) if the family of random variables $\{X_\tau : \tau \in \Sigma_T\}$ is uniformly integrable, where and hereafter Σ_T is the set of all $(\mathcal{F}_t)$-stopping times τ valued in $[0,T]$.

Finally, we use the following two assumptions with respect to the generator g. The first one is called the linear growth condition, and the second one is called the uniformly Lipschitz condition, which is obviously stronger than the linear growth condition.

(H1) There exist two positive constants β and γ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t - a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$|g(\omega,t,y,z)| \le |g(\omega,t,0,0)| + \beta|y| + \gamma|z|;$$

(H2) There exist two positive constants β and γ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t - a.e.$, for all $(y^i,z^i)\in\mathbb{R}\times\mathbb{R}^d$, $i=1,2$,

$$|g(\omega,t,y^1,z^1) - g(\omega,t,y^2,z^2)| \le \beta|y^1-y^2| + \gamma|z^1-z^2|.$$

3 Existence and uniqueness

Define the function ψ:

$$\psi(x,\mu) = x\exp\left(\mu\sqrt{2\log(1+x)}\right), \quad (x,\mu)\in[0,+\infty)\times[0,+\infty), \tag{3.1}$$

which is introduced in [8] and [3].

The following existence and uniqueness theorem is the main result of this note.

Theorem 3.1. Let ξ be a terminal condition and g be a generator which is continuous in (y,z). If g satisfies assumption (H1) with parameters β and γ, and

$$\psi\left(|\xi| + \int_0^T |g(t,0,0)|\mathrm{d}t,\ \gamma\sqrt{T}\right) \in L^1,$$

then BSDE(ξ,g) admits a solution $(Y_t,Z_t)_{t\in[0,T]}$ such that $\left(\psi\left(|Y_t|,\gamma\sqrt{t}\right)\right)_{t\in[0,T]}$ belongs to class (D), and $\mathbb{P}-a.s.$, for each $t\in[0,T]$,

$$|Y_t| \le \psi(|Y_t|,\gamma\sqrt{t}) \le C\mathbb{E}\left[\psi\left(|\xi| + \int_0^T |g(t,0,0)|\mathrm{d}t,\ \gamma\sqrt{T}\right) \middle| \mathcal{F}_t \right] + C, \tag{3.2}$$

where C is a positive constant depending only on (β,γ,T).

Furthermore, if g also satisfies assumption (H2), then BSDE(ξ,g) admits a unique solution $(Y_t,Z_t)_{t\in[0,T]}$ such that $\left(\psi\left(|Y_t|,\gamma\sqrt{t}\right)\right)_{t\in[0,T]}$ belongs to class (D).

In order to prove the above theorem, we need the following lemmas and propositions. First, the following lemma have been proved in Proposition 2.3 and Theorem 2.5 of [3].

Lemma 3.2. We have the following assertions on ψ:

(i) For each $x\ge 0$, $\psi(x,\cdot)$ is nondecreasing on $[0,+\infty)$.

(ii) For $\mu \ge 0$, $\psi(\cdot,\mu)$ is a positive, strictly increasing and strictly convex function on $[0,+\infty)$.
(iii) For $c \ge 1$, we have $\psi(cx,\mu) \le \psi(c,\mu)\psi(x,\mu)$, for all $x,\mu \ge 0$.
(iv) For all $x_1,x_2,\mu \ge 0$, we have $\psi(x_1+x_2,\mu) \le \frac{1}{2}\psi(2,\mu)\left[\psi(x_1,\mu)+\psi(x_2,\mu)\right]$.

In order to apply Itô-Tanaka's formula to establish the a priori estimate (see Proposition 3.5), for each $t \in [0,T]$ we define the following function φ:

$$\varphi(s,x;t) := (x+e)\exp\left(\gamma\sqrt{2s\log(x+e)}+\frac{\gamma}{2}\int_t^s\left(\gamma+\sqrt{\frac{2}{r}}\right)\mathrm{d}r\right),\ (s,x)\in[t,T]\times[0,+\infty), \tag{3.3}$$

which is the function ϕ in (1.4) with μ_s and k_r defined in (1.5). We have, for each $t \in (0,T]$ and each $(s,x) \in [t,T]\times[0,+\infty)$,

$$\varphi_x(s,x;t) = \varphi(s,x;t)\ \frac{\gamma\sqrt{s}+\sqrt{2\log(x+e)}}{(x+e)\sqrt{2\log(x+e)}} > 0,$$

$$\varphi_{xx}(s,x;t) = \varphi(s,x;t)\ \frac{\gamma\sqrt{s}\left(2\log(x+e)+\gamma\sqrt{2s\log(x+e)}-1\right)}{(x+e)^2\left(\sqrt{2\log(x+e)}\right)^3} > 0,$$

and

$$\varphi_s(s,x;t) = \frac{\gamma\varphi(s,x;t)}{2}\ \left(\frac{\sqrt{2\log(x+e)}+\sqrt{2}}{\sqrt{s}}+\gamma\right) > 0.$$

Moreover, the following proposition holds.

Proposition 3.3. We have the following assertions on φ:

(i) For $t \in [0,T]$, $\varphi(\cdot,\cdot;t)$ is continuous on $[t,T]\times[0,+\infty)$; And, for all $t \in (0,T]$, $\varphi(\cdot,\cdot;t) \in \mathcal{C}^{1,2}([t,T]\times[0,+\infty))$;
(ii) For all $t \in (0,T]$, $\varphi(\cdot,\cdot;t)$ satisfies the inequality in (1.2), i.e.,

$$-\gamma\varphi_x(s,x;t)|z| + \frac{1}{2}\varphi_{xx}(s,x;t)|z|^2 + \varphi_s(s,x;t) \ge 0,\quad (s,x,z)\in[t,T]\times[0,+\infty)\times\mathbb{R}^d.$$

Proof. The first assertion is obvious. From the introduction we know that the inequality (1.3) implies the inequality (1.2). Then, in order to prove Assertion (ii), it suffices to prove that the inequality (1.3) holds for the function $\varphi(\cdot,\cdot;t)$ with $t \in (0,T]$. In fact, by a simple computation, we have, for each $(s,x)\in[t,T]\times[0,+\infty)$,

$$-\frac{\gamma^2}{2}\frac{\varphi_x^2(s,x;t)}{\varphi_{xx}(s,x;t)} = -\frac{\gamma\varphi(s,x;t)}{2}\frac{\left(\gamma\sqrt{s}+\sqrt{2\log(x+e)}\right)^2\sqrt{2\log(x+e)}}{\sqrt{s}\left(2\log(x+e)+\gamma\sqrt{s}\sqrt{2\log(x+e)}-1\right)}.$$

Define $v=\sqrt{2\log(x+e)}$. Then,

$$-\frac{\gamma^2}{2}\frac{\varphi_x^2(s,x;t)}{\varphi_{xx}(s,x;t)}+\varphi_s(s,x;t) = \frac{\gamma\varphi(s,x;t)}{2}\left[\frac{1}{\sqrt{s}}\left(v-\frac{\left(\gamma\sqrt{s}+v\right)^2 v}{v^2+\gamma\sqrt{s}v-1}\right)+\gamma+\sqrt{\frac{2}{s}}\right].$$

Furthermore, in view of the fact of $v \ge \sqrt{2}$, we know that

$$\begin{aligned}\frac{\left(\gamma\sqrt{s}+v\right)^2 v}{v^2+\gamma\sqrt{s}v-1}-v &= \frac{\gamma\sqrt{s}(v^2+\gamma\sqrt{s}v-1)+v+\gamma\sqrt{s}}{v^2+\gamma\sqrt{s}v-1}\\ &\le \gamma\sqrt{s}+\frac{v+\gamma\sqrt{s}}{\frac{1}{2}(v^2+\gamma\sqrt{s}v)} = \gamma\sqrt{s}+\frac{2}{v} \le \gamma\sqrt{s}+\sqrt{2}.\end{aligned}$$

Hence, for each $t \in (0, T]$,

$$-\frac{\gamma^2}{2}\frac{\varphi_x^2(s,x;t)}{\varphi_{xx}(s,x;t)} + \varphi_s(s,x;t) \geq 0, \quad (s,x) \in [t,T] \times [0,+\infty).$$

Then, Assertion (ii) is proved, and the proof is complete. □

The two functions ψ and φ defined respectively on (3.1) and (3.3) have the following connection.

Proposition 3.4. There exists a universal constant $K > 0$ depending only on γ and T such that for all $t \in [0, T]$ and $(s, x) \in [t, T] \times [0, +\infty)$,

$$\psi(x, \gamma\sqrt{s}) \leq \varphi(s, x; t) \leq K\psi(x, \gamma\sqrt{s}) + K. \tag{3.4}$$

In particular, by letting $s = t$, we have

$$\psi(x, \gamma\sqrt{t}) \leq \varphi(t, x; t) \leq K\psi(x, \gamma\sqrt{t}) + K, \quad (t, x) \in [0, T] \times [0, +\infty). \tag{3.5}$$

Proof. The first inequality in (3.4) is clear, and (3.5) is a direct corollary of (3.4). We now prove the second inequality in (3.4). In fact, for each $t \in [0, T]$ and $(s, x) \in [t, T] \times [1, +\infty)$,

$$\begin{aligned}
\frac{\varphi(s,x;t)}{\psi(x,\gamma\sqrt{s})+1} \quad &= \quad \frac{(x+e)\exp\left(\gamma\sqrt{2s\log(x+e)} + \frac{\gamma}{2}\int_t^s\left(\gamma+\sqrt{\frac{2}{r}}\right)\mathrm{d}r\right)}{x\exp\left(\gamma\sqrt{2s\log(1+x)}\right)+1} \\
&\leq \quad \tfrac{x+e}{x}\exp\left(\gamma\sqrt{T}\left(\sqrt{2\log(x+e)}-\sqrt{2\log(x+1)}\right)+\tfrac{\gamma^2 T}{2}+\gamma\sqrt{2T}\right) \\
&=: \quad H_1(x,\gamma,T).
\end{aligned}$$

And, in the case of $x \in [0, 1]$,

$$\frac{\varphi(s,x;t)}{\psi(x,\gamma\sqrt{s})+1} \leq (e+1)\exp\left(\gamma\sqrt{2T\log(1+e)} + \frac{\gamma^2 T}{2} + \gamma\sqrt{2T}\right) =: H_2(\gamma, T).$$

Hence, for all $x \in [0, +\infty)$, we have

$$\frac{\varphi(s,x;t)}{\psi(x,\gamma\sqrt{s})+1} \leq H_1(x,\gamma,T)\mathbf{1}_{x\geq 1} + H_2(\gamma,T)\mathbf{1}_{0\leq x<1}. \tag{3.6}$$

With inequality (3.6) in hand and in view of the fact that the function $H_1(x, \gamma, T)$ is continuous on $[1, +\infty)$ and tends to

$$\exp\left(\frac{\gamma^2 T}{2} + \gamma\sqrt{2T}\right)$$

as $x \to +\infty$, we obtain the second inequality in (3.4). The proof is complete. □

The following Proposition 3.5 establishes some a priori estimate for the solution to a BSDE with an L^p $(p > 1)$ terminal value and a linear-growth generator.

Proposition 3.5. Let ξ be a terminal condition and g be a generator which is continuous in (y, z). If g satisfies assumption (H1) with parameters β and γ, $(\xi, g(t, 0, 0)) \in L^p \times \mathcal{L}^p$ for some $p > 1$, and $(Y_t, Z_t)_{t\in[0,T]}$ is a solution in $\mathcal{S}^p \times \mathcal{M}^p$ to BSDE(ξ, g), then $\mathbb{P} - a.s.$, for each $t \in [0, T]$, we have

$$|Y_t| \leq \psi(|Y_t|, \gamma\sqrt{t}) \leq C\mathbb{E}\left[\psi\left(|\xi| + \int_0^T |g(t,0,0)|\mathrm{d}t,\ \gamma\sqrt{T}\right)\middle|\mathcal{F}_t\right] + C, \tag{3.7}$$

where C is a positive constant depending only on (β, γ, T), and ψ is defined in (3.1).

Proof. Note first that if $(\xi, g(t,0,0)) \in L^p \times \mathcal{L}^p$ for some $p > 1$, then

$$\psi\left(|\xi| + \int_0^T |g(t,0,0)|\mathrm{d}t,\ \mu\right) \in L^1$$

for any $\mu \geq 0$, which has been shown in Remark 1.2 of [8]. Define

$$\bar{Y}_t := e^{\beta t}|Y_t| + \int_0^t e^{\beta s}|g(s,0,0)|\mathrm{d}s \ \text{ and } \ \bar{Z}_t := e^{\beta t}\mathrm{sgn}(Y_t)Z_t, \quad t \in [0,T], \tag{3.8}$$

where $\mathrm{sgn}(y) = \mathbf{1}_{y>0} - \mathbf{1}_{y\leq 0}$. It then follows from Itô-Tanaka's formula that, with $t \in [0,T]$,

$$\bar{Y}_t = \bar{Y}_T + \int_t^T e^{\beta s}\left(\mathrm{sgn}(Y_s)g(s,Y_s,Z_s) - \beta|Y_s| - |g(s,0,0)|\right)\mathrm{d}s - \int_t^T \bar{Z}_s \cdot \mathrm{d}B_s - \int_t^T e^{\beta s}\mathrm{d}L_s,$$

where $L_\cdot$ denotes the local time of $Y_\cdot$ at 0. Now, fixing $t \in (0,T]$ and applying Itô-Tanaka's formula to the process $\varphi(s, \bar{Y}_s; t)$, where the function $\varphi(\cdot,\cdot;t)$ is defined in (3.3), we derive, in view of assumption (H1),

$$\begin{aligned}
& \mathrm{d}\varphi(s,\bar{Y}_s;t) \\
= \ & e^{\beta s}\varphi_x(s,\bar{Y}_s;t)\left(-\mathrm{sgn}(Y_s)g(s,Y_s,Z_s) + \beta|Y_s| + |g(s,0,0)|\right)\mathrm{d}s + \varphi_x(s,\bar{Y}_s;t)\bar{Z}_s \cdot \mathrm{d}B_s \\
& + e^{\beta s}\varphi_x(s,\bar{Y}_s;t)\mathrm{d}L_s + \frac{1}{2}e^{2\beta s}\varphi_{xx}(s,\bar{Y}_s;t)|Z_s|^2\mathrm{d}s + \varphi_s(s,\bar{Y}_s;t)\mathrm{d}s \\
\geq \ & \left[-\gamma e^{\beta s}\varphi_x(s,\bar{Y}_s;t)|Z_s| + \tfrac{1}{2}e^{2\beta s}\varphi_{xx}(s,\bar{Y}_s;t)|Z_s|^2 + \varphi_s(s,\bar{Y}_s;t)\right]\mathrm{d}s + \varphi_x(s,\bar{Y}_s;t)\bar{Z}_s \cdot \mathrm{d}B_s.
\end{aligned}$$

Furthermore, by letting $x = \bar{Y}_s$ and $z = e^{\beta s}Z_s$ in Assertion (ii) of Proposition 3.3 we get that

$$\mathrm{d}\varphi(s,\bar{Y}_s;t) \geq \varphi_x(s,\bar{Y}_s;t)\bar{Z}_s \cdot \mathrm{d}B_s, \quad s \in [t,T]. \tag{3.9}$$

Let us consider, for each integer $n \geq 1$, the following stopping time

$$\tau_n := \inf\left\{s \in [t,T] : \int_t^s \left[\varphi_x(r,\bar{Y}_r;t)\right]^2 |\bar{Z}_r|^2\mathrm{d}r \geq n\right\} \wedge T,$$

with the convention that $\inf\emptyset = +\infty$. It follows from the inequality (3.9) and the definition of τ_n that for each $t \in (0,T]$ and $n \geq 1$,

$$\varphi(t,\bar{Y}_t;t) \leq \mathbb{E}\left[\varphi(\tau_n,\bar{Y}_{\tau_n};t)\middle|\mathcal{F}_t\right].$$

Thus, thanks to Proposition 3.4, we know the existence of a positive constant K depending only on γ and T such that

$$\psi(\bar{Y}_t,\gamma\sqrt{t}) \leq \varphi(t,\bar{Y}_t;t) \leq \mathbb{E}\left[\varphi(\tau_n,\bar{Y}_{\tau_n};t)\middle|\mathcal{F}_t\right] \leq K\mathbb{E}\left[\psi(\bar{Y}_{\tau_n},\gamma\sqrt{\tau_n})\middle|\mathcal{F}_t\right] + K, \tag{3.10}$$

and, from the definition of $\bar{Y}_t$ in (3.8), we have

$$|Y_t| \leq \bar{Y}_t \quad \text{and} \quad \bar{Y}_{\tau_n} \leq e^{\beta T}\left(|Y_{\tau_n}| + \int_0^{\tau_n}|g(s,0,0)|\mathrm{d}s\right). \tag{3.11}$$

By virtue of (ii) and (iii) in Lemma 3.2 together with the fact that $\psi(x,\mu) \geq x$ for each $x, \mu \geq 0$ due to (3.1), we obtain from (3.10) and (3.11) that for each $t \in (0,T]$ and $n \geq 1$,

$$\begin{aligned}
|Y_t| & \leq \psi\left(|Y_t|,\gamma\sqrt{t}\right) \leq \psi(\bar{Y}_t,\gamma\sqrt{t}) \leq K\mathbb{E}\left[\psi(\bar{Y}_{\tau_n},\gamma\sqrt{\tau_n})\middle|\mathcal{F}_t\right] + K \\
& \leq K\psi\left(e^{\beta T},\gamma\sqrt{\tau_n}\right)\mathbb{E}\left[\psi\left(|Y_{\tau_n}| + \int_0^{\tau_n}|g(s,0,0)|\mathrm{d}s,\ \gamma\sqrt{\tau_n}\right)\middle|\mathcal{F}_t\right] + K,
\end{aligned}$$

from which the inequality (3.7) follows for $t \in (0,T]$ by sending n to infinity. Finally, in view of the continuity of $Y_\cdot$ and the martingale in the right side hand of (3.7) with respect to the time variable t, we know that (3.7) holds still true for $t = 0$. The proposition is then proved. □

Remark 3.6. We specially point out that, to the best of our knowledge, in the critical case: $\mu = \gamma\sqrt{T}$, the method of the dual representation used in [8] can not be applied to obtain the desired a priori estimate as that in (3.7) at the time $t = 0$.

Now, we give the proof of the existence part of Theorem 3.1.

The proof of the existence part of Theorem 3.1. Let us fix two positive integers n and p. Set $\xi^{n,p} := \xi^+ \wedge n - \xi^- \wedge p$, $g^{n,p}(t,0,0) := g^+(t,0,0) \wedge n - g^-(t,0,0) \wedge p$ and $g^{n,p}(t,y,z) := g(t,y,z) - g(t,0,0) + g^{n,p}(t,0,0)$. As the terminal condition $\xi^{n,p}$ and $g^{n,p}(t,0,0)$ are bounded (hence square-integrable) and $g^{n,p}(t,y,z)$ is a continuous and linear-growth generator, in view of the existence result in [9], BSDE$(\xi^{n,p}, g^{n,p})$ admits a minimal solution $(Y_\cdot^{n,p}, Z_\cdot^{n,p})$ in $\mathcal{S}^2 \times \mathcal{M}^2$. It then follows from Proposition 3.5 that there exists a positive constant C depending only on (β, γ, T) such that for each $t \in [0,T]$ and each $n, p \geq 1$,

$$\begin{aligned} |Y_t^{n,p}| \leq \psi(|Y_t^{n,p}|, \gamma\sqrt{t}) \quad &\leq \quad C\mathbb{E}\left[\psi\left(|\xi^{n,p}| + \int_0^T |g^{n,p}(t,0,0)|\mathrm{d}t,\ \gamma\sqrt{T} \right) \middle| \mathcal{F}_t \right] + C \\ &\leq \quad C\mathbb{E}\left[\psi\left(|\xi| + \int_0^T |g(t,0,0)|\mathrm{d}t,\ \gamma\sqrt{T} \right) \middle| \mathcal{F}_t \right] + C. \end{aligned} \tag{3.12}$$

Since $Y_\cdot^{n,p}$ is nondecreasing in n and non-increasing in p by the comparison theorem (see, for example, Theorem 2.3 in [6]), then in view of (3.12) and assumption (H1), by virtue of the localization method put forward in [2], we know that there exists an $(\mathcal{F}_t)$-progressively measurable process $(Z_t)_{t\in[0,T]}$ such that $(Y_\cdot := \inf_p \sup_n Y_\cdot^{n,p},\ Z_\cdot)$ is an adapted solution to BSDE(ξ, g). Finally, sending n and p to infinity in (3.12) yields the inequality (3.2), and then the process $\left(\psi\left(|Y_t|, \gamma\sqrt{t}\right)\right)_{t\in[0,T]}$ belongs to class (D). The proof is complete. □

Remark 3.7. From the above proof, it is easy to see that the linear-growth assumption (H1) in Theorem 3.1 and Proposition 3.5 can be easily weakened to the following one-sided linear-growth assumption: There exist two real constants $\beta \geq 0$, $\gamma > 0$ and a nonnegative, real-valued and $(\mathcal{F}_t)$-progressively measurable process $(f_t)_{t\in[0,T]}$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\mathrm{sgn}(y) g(\omega,t,y,z) \leq f_t(\omega) + \beta|y| + \gamma|z| \ \text{ and } \ |g(\omega,t,y,z)| \leq f_t(\omega) + h(|y|) + \gamma|z|,$$

where $h(\cdot)$ is a deterministic, continuous and nondecreasing function with $h(0) = 0$. In this case, $|g(t,0,0)|$ in the conditions of Theorem 3.1 and Proposition 3.5 only needs to be replaced with f_t.

In order to prove the uniqueness part of Theorem 3.1, we need the following two lemmas, which are Lemmas 2.4 and 2.6 in [8].

Lemma 3.8. For each $x \in \mathbb{R}$, $y \geq 0$ and $\mu > 0$, we have

$$e^x y \leq e^{\frac{x^2}{2\mu^2}} + e^{2\mu^2}\psi(y,\mu),$$

where the function ψ is defined in (3.1) again.

Lemma 3.9. Let $(q_t)_{t\in[0,T]}$ be a d-dimensional and $(\mathcal{F}_t)$-progressively measurable process with $|q_\cdot| \leq \gamma$ almost surely. For each $t \in [0,T]$, if $0 \leq \lambda < \frac{1}{2\gamma^2(T-t)}$, then

$$\mathbb{E}\left[e^{\lambda\left|\int_t^T q_s \cdot \mathrm{d}B_s\right|^2} \middle| \mathcal{F}_t \right] \leq \frac{1}{\sqrt{1 - 2\lambda\gamma^2(T-t)}}.$$

Now, we give the proof of the uniqueness part of Theorem 3.1.

The proof of the uniqueness part of Theorem 3.1. Let g satisfy assumption (H2), and for $i = 1, 2$, let $(Y_t^i, Z_t^i)_{t\in[0,T]}$ be a solution of BSDE(ξ, g) such that $\left(\psi\left(|Y_t^i|, \gamma\sqrt{t}\right)\right)_{t\in[0,T]}$ belongs to class (D). Define $\delta Y_\cdot := Y_\cdot^1 - Y_\cdot^2$ and $\delta Z_\cdot := Z_\cdot^1 - Z_\cdot^2$. Then the pair $(\delta Y_\cdot, \delta Z_\cdot)$ verifies the following BSDE:

$$\delta Y_t = \int_t^T (u_s \delta Y_s + v_s \cdot \delta Z_s)\,\mathrm{d}s - \int_t^T \delta Z_s \cdot \mathrm{d}B_s, \quad t \in [0, T],$$

where $g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2) = u_s \delta Y_s + v_s \cdot \delta Z_s$ with a pair of $(\mathcal{F}_t)$-progressively measurable process $(u_\cdot, v_\cdot)$ such that $|u_s| \le \beta$ and $|v_s| \le \gamma$ by a standard linearization procedure. For each $t \in (0, T]$ and each positive integer $n \ge 1$, define the following stopping times:

$$\sigma_n := \inf\left\{ s \in [t, T] : \ |\delta Y_s| + \int_t^s |\delta Z_r|^2 \mathrm{d}r \ge n \right\} \wedge T,$$

with the convention that $\inf \emptyset = +\infty$. Then,

$$\delta Y_t = \mathbb{E}\left[e^{\int_t^{\sigma_n} u_s \mathrm{d}s + \int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s - \frac{1}{2}\int_t^{\sigma_n} |v_s|^2 \mathrm{d}s} \delta Y_{\sigma_n} \middle| \mathcal{F}_t \right].$$

Therefore,

$$|\delta Y_t| \le e^{\beta T} \mathbb{E}\left[e^{\int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s} |\delta Y_{\sigma_n}| \middle| \mathcal{F}_t \right]. \tag{3.13}$$

Furthermore, by virtue of Lemma 3.8 we know that for each $n \ge 1$,

$$e^{\int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s} |\delta Y_{\sigma_n}| \le e^{\frac{1}{2\gamma^2 t}\left(\int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s\right)^2} + e^{2\gamma^2 t} \psi(|\delta Y_{\sigma_n}|, \gamma\sqrt{t}), \quad t \in (0, T]. \tag{3.14}$$

And, it follows from Lemma 3.9 that for all $n \ge 1$,

$$\mathbb{E}\left[\left| e^{\frac{1}{2\gamma^2 t}\left(\int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s\right)^2} \right|^2 \right] = \mathbb{E}\left[e^{\frac{1}{\gamma^2 t}\left(\int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s\right)^2} \right] \le \frac{1}{\sqrt{1 - \frac{2(T-t)}{t}}} \le \sqrt{3}, \quad t \in [3T/4, T],$$

and, thus, the family of random variables $e^{\frac{1}{2\gamma^2 t}\left(\int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s\right)^2}$ is uniformly integrable on the time interval $[3T/4, T]$. On the other hand, in view of (i), (ii) and (iv) in Lemma 3.2, observe that for all $n \ge 1$,

$$\begin{aligned} e^{2\gamma^2 t} \psi(|\delta Y_{\sigma_n}|, \gamma\sqrt{t}) &\le e^{2\gamma^2 T} \psi(|Y_{\sigma_n}^1| + |Y_{\sigma_n}^2|, \gamma\sqrt{\sigma_n}) \\ &\le \frac{e^{2\gamma^2 T}\psi(2, \gamma\sqrt{T})}{2} \left[\psi(|Y_{\sigma_n}^1|, \gamma\sqrt{\sigma_n}) + \psi(|Y_{\sigma_n}^2|, \gamma\sqrt{\sigma_n}) \right], \quad t \in [0, T]. \end{aligned}$$

Thus, from (3.14) we can conclude that, for $t \in [3T/4, T]$, the family of random variables $e^{\int_t^{\sigma_n} v_s \cdot \mathrm{d}B_s} |\delta Y_{\sigma_n}|$ is uniformly integrable. Consequently, by letting $n \to \infty$ in the inequality (3.13) we have $\delta Y_\cdot = 0$ on the interval $[3T/4, T]$. It is clear that $\delta Z_\cdot = 0$ on the interval $[3T/4, T]$. The uniqueness of the solution on the interval $[3T/4, T]$ is obtained. In a same way, we successively have the uniqueness on the intervals $[3^2T/4^2, 3T/4]$, $[3^3T/4^3, 3^2T/4^2]$, $\cdots$, $[3^pT/4^p, 3^{p-1}T/4^{p-1}]$, $\cdots$. Finally, in view of the continuity of process δY_t with respect to the time variable t, we obtain the uniqueness on the whole interval $[0, T]$ by sending p to infinity. The proof is then complete. □

Remark 3.10. By a similar analysis to Remark 2.6 in [3], we know that the uniformly Lipschitz assumption (H2) in Theorem 3.1 can be relaxed to the following monotone assumption:

$$\mathrm{sgn}(y_1 - y_2)\left(g(\omega, t, y^1, z) - g(\omega, t, y^2, z)\right) \le \beta |y^1 - y^2|$$

and

$$|g(\omega, t, y, z^1) - g(\omega, t, y, z^2)| \le \gamma |z^1 - z^2|.$$

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