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Optimal stopping of oscillating Brownian motion

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Abstract

We solve optimal stopping problems for an oscillating Brownian motion, i.e. a diffusion with positive piecewise constant volatility changing at the point $x = 0$. Let σ_1 and σ_2 denote the volatilities on the negative and positive half-lines, respectively. Our main result is that continuation region of the optimal stopping problem with reward $((1 + x)^+)^2$ can be disconnected for some values of the discount rate when $2\sigma_1^2 < \sigma_2^2$. Based on the fact that the skew Brownian motion in natural scale is an oscillating Brownian motion, the obtained results are translated into corresponding results for the skew Brownian motion.

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1 Introduction

The optimal stopping problems of diffusions with exceptional points have attracted interest in recent years. These include cases where the underlying diffusion has sticky points, skew points, or discontinuities in the diffusion coefficients. One of the first findings is that in the presence of sticky points the classical smooth fit principle does not necessarily hold, even for differentiable payoff functions (as found in Crocce and Mordecki [4] and Salminen and Ta [19]). A second finding is that if the diffusion has a skew point, it can be the case that this point is in the continuation region for all discount values, as found by Alvarez and Salminen [1] and Presman [16]. A third one is that the continuation region in these cases can be disconnected, as observed in [1] for the skew Brownian motion, and also found recently by Mordecki and Salminen [12] for a diffusion with discontinuous drift and payoff function $(1 + x)^+$. General verification results for diffusions with discontinuous coefficients were obtained by Rüschendorf and Urusov [17]. An exposition of the general theory of optimal stopping (including historical comments) can be found in Shiryaev [20] and Peskir and Shiryaev [15].

In this paper the focus is on the case when the underlying diffusion has discontinuous infinitesimal variance. We then consider the optimal stopping problem for the oscillating Brownian motion (OBM), a diffusion with positive piecewise constant volatility changing at the origin. For details and further results on OBM, see Keilson and Wellner [7], Lejay and Pigato [10], and the references therein. Our main results are the following:

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Firstly, for the payoff $(1+x)^+$ the solution of the optimal stopping problem for the OBM is one sided for all values of the parameters, but for the payoff $((1+x)^+)^2$ the continuation region is disconnected for some values of the parameters. Hence, this latter situation is similar to the one in [12]. Secondly, based on the fact that the skew Brownian motion (SBM) in natural scale is an OBM, we obtain a result that connects the solutions of the respective optimal stopping problems for SBM and OBM, finding that the non-differentiability of the scale function of SBM at the origin plays a key role in understanding some of the phenomena that appear in the solutions of these problems.

2 Diffusions and optimal stopping

Consider a conservative and regular one-dimensional (or linear) diffusion $X = (X_t)_{t\geq 0}$ taking values in $\mathbb{R}$, in the sense of Itô and McKean [6] (see also Borodin and Salminen [2]). Let $\mathbb{P}_x$ and $\mathbb{E}_x$ denote the probability and the expectation associated with X when starting from x, respectively; m denotes the speed measure and S the scale function. For $r \geq 0$ let φ_r (ψ_r) be the decreasing (increasing) positive fundamental solution of the generalized ODE

$$\frac{d}{dm}\frac{d}{dS}u = ru, \tag{2.1}$$

satisfying the appropriate boundary conditions (see [2] II.10 p. 18). Denote by $\mathcal{M}$ the set of all stopping times in the filtration $(\mathcal{F}_t)_{t\geq 0}$, the usual augmentation of the natural filtration generated by X. Given a continuous reward function $g\colon \mathbb{R} \to [0,\infty)$ and a discount factor $r \geq 0$, consider the optimal stopping problem consisting in finding a function V_r and a stopping time $\tau^* \in \mathcal{M}$, such that

$$V_r(x) = \mathbb{E}_x[\mathrm{e}^{-r\tau^*} g(X_{\tau^*})] = \sup_{\tau\in\mathcal{M}} \mathbb{E}_x[\mathrm{e}^{-r\tau} g(X_\tau)], \tag{2.2}$$

where on the set $\{\tau = \infty\}$

$$\mathrm{e}^{-r\tau} g(X_\tau) := \limsup_{t\to\infty} \mathrm{e}^{-rt} g(X_t).$$

The *value function* V_r and the *optimal stopping time* τ^* constitute the solution of the problem. The optimal stopping time τ^* in (2.2), can be characterized (see Theorem 3, Section 3.3 in [20]) as the first entrance time into the stopping region

$$\Gamma_r := \{x\colon V_r(x) = g(x)\}. \tag{2.3}$$

The set $\mathrm{C}_r := \mathbb{R} \setminus \Gamma_r$ is called the continuation region.

Our main tools to solve the optimal stopping problem for OBM are the representation theory for excessive functions, and the following two results from the theory of optimal stopping. The first one (Theorem 2.1) — formulated here for a left boundary point of the stopping region — is the smooth fit theorem, proof of which can be found in [18] or [14]; the second one (Proposition 2.2) is a verification result, for the proof see Corollary on p. 124 in [20].

Theorem 2.1. *Let z be a left boundary point of Γ_r, i.e., $[z, z+\varepsilon_1) \subset \Gamma_r$ and $(z-\varepsilon_2, z) \subset \mathrm{C}_r$ for some positive ε_1 and ε_2. Assume that the reward function g and the fundamental solutions φ_r and ψ_r are differentiable at z. Then the value function V_r in (2.2) is differentiable at z and it holds $V_r'(z) = g'(z)$.*

Proposition 2.2. *Let $A \subset \mathbb{R}$ be a nonempty Borel subset of $\mathbb{R}$ and*

$$H_A := \inf\{t : X_t \in A\}.$$

Assume that the function

$$\widehat{V}(x) := \mathbb{E}_x \left[e^{-r H_A} g(X_{H_A}) \right]$$

is r-excessive and dominates g. Then $\widehat{V}$ coincides with the value function of OSP (2.2) and H_A is an optimal stopping time.

3 Oscillating Brownian motion

Consider the diffusion satisfying the stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s) dW_s,$$

where

$$\sigma(x) = \begin{cases} \sigma_1, & x < 0, \\ \sigma_2, & x \geq 0, \end{cases}$$

$\sigma_1 > 0$, $\sigma_2 > 0$, and $(W_t)_{t \geq 0}$ is a standard Brownian motion. The diffusion X is called an oscillating Brownian motion (OBM). Notice that this process is in natural scale, i.e. the scale function is $S(x) = x$, and the speed measure is

$$m(dx) = \begin{cases} (2/\sigma_1^2) dx, & x < 0, \\ (2/\sigma_2^2) dx, & x > 0. \end{cases}$$

(by definition there is no mass at $x = 0$). Let

$$\lambda_1^{\pm} = \pm \frac{\sqrt{2r}}{\sigma_1}, \qquad \lambda_2^{\pm} = \pm \frac{\sqrt{2r}}{\sigma_2}.$$

The decreasing fundamental solution is

$$\varphi_r(x) = \begin{cases} A_1 \exp(\lambda_1^- x) + A_2 \exp(\lambda_1^+ x), & x < 0, \\ \exp(\lambda_2^- x), & x \geq 0, \end{cases} \tag{3.1}$$

where the constants A_1 and A_2 are determined so that φ_r is continuous and differentiable at 0. Hence,

$$A_1 = \frac{\lambda_1^+ - \lambda_2^-}{\lambda_1^+ - \lambda_1^-} = \frac{1 + \sigma_1/\sigma_2}{2}, \qquad A_2 = \frac{\lambda_2^- - \lambda_1^-}{\lambda_1^+ - \lambda_1^-} = \frac{1 - \sigma_1/\sigma_2}{2}.$$

Analogously, the increasing solution is

$$\psi_r(x) = \begin{cases} \exp(\lambda_1^+ x), & x < 0, \\ B_1 \exp(\lambda_2^+ x) + B_2 \exp(\lambda_2^- x), & x \geq 0, \end{cases} \tag{3.2}$$

with

$$B_1 = \frac{\lambda_1^+ - \lambda_2^-}{\lambda_2^+ - \lambda_2^-} = \frac{1 + \sigma_2/\sigma_1}{2}, \qquad B_2 = \frac{\lambda_2^+ - \lambda_1^+}{\lambda_2^+ - \lambda_2^-} = \frac{1 - \sigma_2/\sigma_1}{2}.$$

4 Optimal stopping of OBM

We first analyze the optimal stopping problem (2.2) for the diffusion introduced above and the reward function

$$g_1(x) = \begin{cases} 0, & x \leq -1, \\ 1 + x, & x > -1. \end{cases}$$

The following result shows that the solution of this problem is one sided.

Proposition 4.1. *Consider the OSP problem* (2.2) *with payoff* g_1. *For all values of* $r > 0$, σ_1 *and* σ_2 *the continuation region is given by*

$$\mathrm{C}_r = (-\infty, c),$$

where $c = c(r) > -1$ *is the unique solution of the equation*

$$\psi_r'(x)(1+x) - \psi_r(x) = 0. \tag{4.1}$$

Furthermore

$$2r \lesseqgtr \sigma_1^2 \quad \Rightarrow \quad c(r) \gtreqless 0. \tag{4.2}$$

Proof. To prove the first statement, consider for $x > -1$ the functions (cf. [18])

$$H_-(x) := \psi_r'(x)g(x) - \psi_r(x)g'(x) = \psi_r'(x)(1+x) - \psi_r(x), \tag{4.3}$$
$$H_+(x) := \varphi_r(x)g'(x) - \varphi_r'(x)g(x) = \varphi_r(x) - \varphi_r'(x)(1+x), \tag{4.4}$$

and their derivatives which for $x \neq 0$ can be expressed as

$$\begin{aligned} H_-'(x) &= m(x)\frac{d}{dm}H_-(x) = m(x)\psi_r(x)\left(r(1+x) - \frac{d}{dm}\frac{d}{dx}(1+x)\right) \\ &= m(x)\psi_r(x)r(1+x), \end{aligned}$$

where m is the density of the speed measure, and, similarly,

$$H_+'(x) = -m(x)\varphi_r(x)r(1+x),$$

where it is used that φ_r and ψ_r solve (2.1). Observe now that the function H_- in (4.3) has a unique positive root, since for $x > -1$ the derivative is strictly positive, $H_-(-1) = -\psi_r(-1) < 0$, and $H_-(x) \to \infty$ as $x \to \infty$. Therefore, equation (4.1) has a unique solution as claimed. The rest of the proof is standard, see for instance [18] or the detailed proof of Proposition 4.2 below. Statement (4.2) follows since $H_-(0) = \sqrt{2r}/\sigma_1 - 1$. □

A key rôle in the construction of the solution to the OSP is played by the sign of the derivative of the function H_- in (4.3). For a general payoff g, the derivative of the corresponding function H has the same sign as the function

$$x \mapsto rg(x) - \frac{d}{dm}\frac{d}{dS}g(x). \tag{4.5}$$

We remark that this function appears in the expression for the density of the representing measure of the smallest r-excessive majorant of g both in the Martin kernel approach (see Proposition 3.3 in [18]) and in the Green kernel approach (see (5.11) in [12] or (4) in [3]). It is also worth noting that this density can be traced back to formula (8.30) in [5], for the cases when the limit therein can be interchanged with the integral. The monotonicity of the function in (4.5) usually ensures a one-sided solution to the considered OSP. Since we are interested in problems for which the solution is not one-sided, i.e., the stopping set is unconnected, we focus on OSP with the payoff function

$$g_2(x) = \begin{cases} (1+x)^2, & x > -1, \\ 0, & x \leq -1. \end{cases} \tag{4.6}$$

The sign of the function in (4.5) with this g_2 can be seen from Figure 1. Possible applications of these type of rewards include the pricing of perpetual power options. There is a large literature on optimal stopping for power-like and polynomial rewards.

First contributions to this type of payoffs, for random walks and Lévy process where given in [13] and [8], respectively.

We turn now to study the OSP (2.2) for OBM with $0 < \sigma_1 \leq \sigma_2$ and the reward function g_2 given in (4.6). In this situation it is seen that, for some specific values of the parameters, the continuation region is disconnected. The approach to analyze this problem is similar to the one in [12]. Let

$$\begin{aligned} G_-(x) :=& \psi_r'(x)g_2(x) - \psi_r(x)g_2'(x) \\ =& (1+x)\left(\psi_r'(x)(1+x) - 2\psi_r(x)\right), \end{aligned} \tag{4.7}$$

$$\begin{aligned} G_+(x) :=& \varphi_r(x)g_2'(x) - \varphi_r'(x)g_2(x) \\ =& 2\varphi_r(x)(1+x) - \varphi_r'(x)(1+x)^2. \end{aligned} \tag{4.8}$$

These functions are used below to verify the excessivity of the proposed value function. The derivatives for $x > -1$ and $x \neq 0$ are

$$G_-'(x) = m(x)\psi_r(x) \begin{cases} r(1+x)^2 - \sigma_1^2, & x < 0, \\ r(1+x)^2 - \sigma_2^2, & x > 0, \end{cases} \tag{4.9}$$

$$G_+'(x) = m(x)\varphi_r(x) \begin{cases} \sigma_1^2 - r(1+x)^2, & x < 0, \\ \sigma_2^2 - r(1+x)^2, & x > 0. \end{cases} \tag{4.10}$$

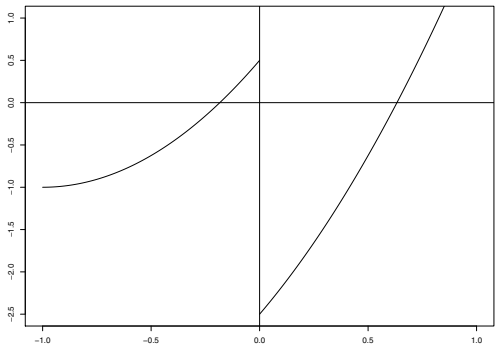

Figure 1: The sign of the derivative G_-' is ruled by the above depicted function $r(1+x)^2 - \sigma(x)^2$. Here the parameters are $r = 1.5$, $\sigma_1 = 1$, $\sigma_2 = 2$.

Proposition 4.2. *In case $0 < r \leq \sigma_1^2 \leq \sigma_2^2$ the continuation region for OSP (2.2) is given by*

$$\mathrm{C}_r = (-\infty, c),$$

where $c = c(r)$ is the unique positive solution of the equation

$$\psi_r'(x)(1+x) - 2\psi_r(x) = 0, \quad x \geq -1. \tag{4.11}$$

Proof. We show first that equation (4.11) has a unique positive solution. To this end, consider for $x > -1$ the function G_- defined in (4.7). The claim is equivalent with the statement that G_- has a unique positive zero. In fact, we claim a bit more; namely that the function G_- attains the global minimum at $x_0 := \sigma_2/\sqrt{r} - 1 > 0$, is negative and decreasing for $x \leq x_0$, is increasing for $x > x_0$, and has, therefore, a unique zero.

Analyzing G'_- as given in (4.9), it is straightforward to deduce, since $0 < r \leq \sigma_1^2 \leq \sigma_2^2$, the claimed properties of G_-.

Let

$$H_c := \inf\{t : X_t \geq c\},$$

where c is the unique solution of (4.11), and define

$$\widehat{V}(x) := \mathbb{E}_x\big[\mathrm{e}^{-rH_c} g_2(X_{H_c})\big] = \begin{cases} \dfrac{\psi_r(x)}{\psi_r(c)}\, g_2(c), & x \leq c, \\ g_2(x), & x \geq c. \end{cases} \tag{4.12}$$

If $\widehat{V}$ is an r-excessive majorant of g_2 it follows from Proposition 2.2 that $\widehat{V}$ is the value function of OSP (2.2). The excessivity can be checked with the method based on the representation theory of excessive functions (cf. [18] Section 3). This boils down to study for $x \neq -1$ the functions

$$I_V(x) := \psi_r'(x)\widehat{V}(x) - \psi_r(x)\widehat{V}'(x),$$
$$D_V(x) := \varphi_r(x)\widehat{V}'(x) - \widehat{V}(x)\varphi_r'(x).$$

Clearly, $I_V(x) = 0$ for $x \leq c$ and increasing for $x > c$. Notice that $I_V = G_-$ on $[c, +\infty)$. Studying the derivative (w.r.t. the speed measure) of D_V it is easily seen that D_V is positive and decreasing to 0 on $[c, +\infty)$. Consequently, I_V and D_V induce a (probability) measure which represents $\widehat{V}$ proving that $\widehat{V}$ is r-excessive. To prove that $\widehat{V}$ is a majorant of g_2 consider for $-1 < x < c$

$$\widehat{V}(x) \geq g_2(x) \quad \Leftrightarrow \quad \frac{\psi_r(x)}{g_2(x)} \geq \frac{\psi_r(c)}{g_2(c)}.$$

The inequality on the right hand side holds since the derivative of ψ_r/g_2 is $G_-(x)$ which is negative for $-1 < x < c$, as is shown above. □

If the volatilities are close enough the problem is one sided for all discount values. This is made precise in the next result.

Proposition 4.3. *In case $0 < \sigma_1^2 \leq \sigma_2^2 \leq 2\sigma_1^2$ the continuation region for the OSP (2.2) is given by*

$$\mathrm{C}_r = (-\infty, c),$$

where $c = c(r)$ is the unique solution of equation (4.11). As r increases from 0 to $+\infty$, $c(r)$ decreases monotonically from $+\infty$ to -1. In particular, $c(r) = 0$ for $r = 2\sigma_1^2$.

Proof. If $r \leq \sigma_1^2$ the statement is the same as in Proposition 4.2. We assume next that $r \geq \sigma_2^2$. The proof in this case is very similar to the proof of Proposition 4.2. It can be seen that G_- attains the global minimum at $x_1 := \sigma_1/\sqrt{r} - 1 < 0$, is negative and decreasing for $x \leq x_1$, is increasing for $x > x_1$, and has, therefore, a unique zero. Consequently, this root can be taken to be an optimal stopping point $c = c(r)$ and the analogous function $\widehat{V}$ as in (4.12) can be proved to be the value of OSP (2.2). Finally, assume $\sigma_1^2 < r < \sigma_2^2 \leq 2\sigma_1^2$ In this case, G_- has a local maximum at 0, which is negative since

$$G_-(0) = \psi_r'(0) - 2\psi_r(0) = \lambda_1^+ - 2 = \frac{\sqrt{2r}}{\sigma_1} - 2 \leq 0. \tag{4.13}$$

Clearly, $G_-(0) = 0$ (and then $c(r) = 0$) when $r = 2\sigma_1^2$. Hence, equation (4.11) has a unique positive root and the proof can be completed as in the previous cases. □

Proposition 4.4. *Assume $0 < 2\sigma_1^2 < \sigma_2^2$. For $r \geq 2\sigma_1^2$ there exist A and B such that the function*

$$F(x) := \begin{cases} A\exp(\lambda_1^+ x) + B\exp(\lambda_1^- x), & x \leq 0, \\ (1+x)^2, & x \geq 0, \end{cases} \tag{4.14}$$

satisfies the principle of smooth fit at 0, i.e., $F'(0-) = F'(0+) = 2$. The function F is r-harmonic (and positive) on $(-\infty, 0)$ but not r-excessive if $r < \sigma_2^2$. For $r < 2\sigma_1^2$ the coefficient B is negative and the function $F(x) \to -\infty$ as $x \to -\infty$ (and the function is not r-excessive).

Proof. We study only the case $r = r_0 := 2\sigma_1^2$ and leave the details of the other cases to the reader. In this case $\lambda_1^+ = \sqrt{2r}/\sigma_1 = 2$, and, obviously,

$$F(x) := \begin{cases} \mathsf{e}^{2x}, & x \leq 0, \\ (1+x)^2, & x \geq 0, \end{cases}$$

satisfies smooth fit at 0. Consequently, F is r_0-harmonic (and positive) on $(-\infty, 0)$ and it remains to prove that F is not r_0-excessive. For this, consider the representing function (this corresponds G_- in (4.7))

$$x \mapsto \psi'_{r_0}(x)F(x) - \psi_{r_0}(x)F'(x).$$

The claim is that this function is not non-decreasing. Indeed, differentiate w.r.t. the speed measure to obtain

$$\begin{aligned} \frac{d}{dm}\Big(\psi'_{r_0}(x)F(x) - \psi_{r_0}(x)F'(x)\Big) & \\ &= F(x)\frac{d}{dm}\frac{d}{dx}\psi_{r_0}(x) - \psi_{r_0}(x)\frac{d}{dm}\frac{d}{dx}F(x) \\ &= \psi_{r_0}(x)\begin{cases} 0, & x < 0, \\ r_0(1+x)^2 - \sigma_2^2, & x > 0. \end{cases} \end{aligned}$$

Since $r_0 = 2\sigma_1^2 < \sigma_2^2$ this derivative is negative, e.g., for small positive x-values; therefore, F is not r_0-excessive. □

For the theorem to follow, which can be seen as our main result concerning OSP (2.2), we need the following technical result.

Lemma 4.5. *Consider a family $\{h_r \colon \mathbb{R} \to [0,\infty)\,;\, r \in I\}$ such that for each $r \in I \subset \mathbb{R}$ the function h_r is r-excessive. Assume that this family is dominated by a function $\widehat{h}$ (i.e. $h_r \leq \widehat{h}$) such that $\mathbb{E}_x(\widehat{h}(X_t)) < \infty$ for all $t \geq 0$ and $x \in \mathbb{R}$. Then, if for some r_0 the limit*

$$\lim_{r \to r_0} h_r(x) \to h_0(x)$$

exists for all $x \in \mathbb{R}$, the function h_0 is r_0-excessive.

Proof. Consider

$$\begin{aligned} \mathbb{E}_x\left[\mathsf{e}^{-r_0 t}h_0(X_t)\right] = \mathbb{E}_x\big[\lim_{r\to r_0}\mathsf{e}^{-rt}h_r(X_t)\big] &= \lim_{r\to r_0}\mathbb{E}_x\left[\mathsf{e}^{-rt}h_r(X_t)\right] \\ &\leq \lim_{r\to r_0} h_r(x) = h_0(x), \end{aligned}$$

where in the second step we use the dominated convergence theorem which is applicable since $\mathsf{e}^{-rt}h_r(X_t) \leq \widehat{h}(X_t)$. □

The theorem below states that if $r \in (2\sigma_1^2, \sigma_2^2)$ but is close enough to σ_2^2 then C_r has a bubble (i.e. an isolated bounded interval in the continuation region). However, the bubble disappears when r becomes larger than σ_2^2 or tends to $2\sigma_1^2$.

Theorem 4.6. *In case $0 < 2\sigma_1^2 < \sigma_2^2$ there exists $r_0 \in (2\sigma_1^2, \sigma_2^2)$ with the following properties:*

(a) *If $r \in [r_0, \sigma_2^2)$ the continuation region is given by*

$$\mathrm{C}_r = (-\infty, c_1) \cup (c_2, c_3),$$

where $c_i = c_i(r)$, $i = 1, 2, 3$, are such that $-1 < c_1 \leq c_2 \leq 0 < c_3$. In particular, for $r = r_0$ it holds $c_1 = c_2 < 0$.

(b) *If $r \geq \sigma_2^2$ the continuation region is explicitly given by*

$$\mathrm{C}_r = (-\infty, c_-),$$

where

$$c_- = c_-(r) = \frac{2\sigma_1}{\sqrt{2r}} - 1 < 0,$$

i.e. c_- is the unique solution of (4.11).

(c) *If $r < r_0$ the continuation region is given by*

$$\mathrm{C}_r = (-\infty, c_+),$$

where $c_+ = c_+(r) > 0$ is the unique solution of (4.11).

Proof. The proof of (b) is as the proof of Proposition 4.3 when $r \geq \sigma_2^2$. Notice, however, that in the present case $c(r) < 0$ for all $r \geq \sigma_2^2$.

We consider next (c) in case $r \leq 2\sigma_1^2$. Studying G_-' and $G_-(0)$ in (4.9) and (4.13) respectively, it is seen, as in the proof of Proposition 4.2, that equation $G_-(x) = 0$ has for $r < 2\sigma_1^2$ one (and only one) root $\rho = \rho(r) > \sigma_2/\sqrt{r} - 1 > 0$. In case $r = 2\sigma_1^2$ there are two roots $\rho_1 = 0$ and $\rho_2 > \sigma_2/(\sqrt{2}\sigma_1) - 1 > 0$. Proceeding as in the proof of Proposition 4.2 it is seen that the stopping region is as claimed with $c_+ = \rho$ if $r < 2\sigma_1^2$ and $c_+ = \rho_2$ if $r = 2\sigma_1^2$.

Finally we consider (a). Assume now that there does not exist a bubble for any $r \in [2\sigma_1^2, \sigma_2^2]$. Then for all $r \in [2\sigma_1^2, \sigma_2^2]$ we can find $c = c(r)$ such that $\Gamma_r = [c, +\infty)$. Knowing that $c(r) > 0$ for $r = 2\sigma_1^2$ and $c(r) < 0$ for $r = \sigma_2^2$ we remark first that there does not exists r such that $c(r) = 0$. Indeed, by Theorem 2.1, the value should satisfy the smooth fit principle at 0 but from Proposition 4.4 we know that such functions are not r-excessive. Next, using $\Gamma_{r_1} \subseteq \Gamma_{r_2}$ for $r_1 < r_2$ (cf. Proposition 1 in [12]) it is seen that $r \mapsto c(r)$ is non-increasing, and has, hence, left and right limits. Consquently, there exists a unique point r_0 such that

$$\hat{c}_+ := \lim_{r \uparrow r_0} c(r) > 0 \quad \text{and} \quad \hat{c}_- := \lim_{r \downarrow r_0} c(r) < 0.$$

Under the assumption that there is no bubble the value function is of the form given in (4.12), i.e.,

$$\begin{aligned} V_r(x) &= \begin{cases} \psi_r(x) \frac{(1+c(r))^2}{\psi_r(c(r))}, & x \leq c(r), \\ (1+x)^2, & x \geq c(r). \end{cases} \\ &= \mathbb{E}_x\big(\mathrm{e}^{-rH_c}\,(1 + X_{H_c})^2\big), \end{aligned} \tag{4.15}$$

where $H_c := \inf\{y : X_t \geq c(r)\}$. For $\widehat{r}$ small enough, there exists an excessive majorant $h_{\widehat{r}}$, so that we can apply Proposition 4.5 with $\widehat{h} := h_{\widehat{r}}$, since $\mathbb{E}_x \mathrm{e}^{-\widehat{r}t} h_{\widehat{r}}(X_t) \leq h_{\widehat{r}}(x) < \infty$. Then, letting in (4.15) $r \uparrow r_0$ yields an r_0-excessive function which by Proposition 2.2 is the value of the corresponding OSP (2.2). Similarly, letting $r \downarrow r_0$ yields an r_0-excessive function which should also be the value of the same OSP. However, the functions are clearly different and since the value is unique we have reached a contradiction showing that there exists at least one bubble, i.e. a bounded open interval $(x_1, x_2) \subseteq \mathrm{C}_r$ with endpoints x_1 and x_2 in the stopping set (see [12]). Proceeding similarly as in Proposition 6 in [12], it can be seen that if there is a bubble, there is at most one bubble, and this contains the origin or the origin is its left end point. This completes the proof. □

We end this section by considering the case $\sigma_1^2 \geq \sigma_2^2$ with quadratic reward (4.6), and show that the stopping region in this case is always one sided.

Proposition 4.7. *Consider the OSP problem* (2.2) *for OBM with* $\sigma_1^2 \geq \sigma_2^2$, $r > 0$ *and* $g_2(x)$ *as given in* (4.6). *For all values of* $r > 0$, *the continuation region is given by*

$$\mathrm{C}_r = (-\infty, c),$$

where $c = c(r) > -1$ *is the unique solution of the equation* (4.11). *Furthermore*

$$r \lessgtr 2\sigma_1^2 \quad \Rightarrow \quad c(r) \gtrless 0.$$

Proof. The proof follows the general lines of [18] developed in the previous proofs of Propositions 4.1 and 4.2. Consider then the functions defined in (4.7) and (4.8), with their respective derivatives in (4.9) and (4.10). As $\sigma_1^2 \geq \sigma_2^2$, the derivative $G'_-(x)$ changes sign only once, from negative to positive. Hence, equation (4.11) has only one root $c(r) > -1$, and the first claim is proved. For the second claim, notice that the root $c(r)$ equals 0 when $r = 2\sigma_1^2$, the result then follows from the monotonicity of the function $c(r)$. □

5 OSP for skew Brownian motion

Consider a skew Brownian motion (SBM) $(\widehat{X}_t)_{t\geq 0}$ with index $\beta \in (0,1)$ starting at $\widehat{x} \in \mathbb{R}$ (see [6], [22], [9]). This diffusion can be characterized by scale function

$$\widehat{S}(x) = \begin{cases} x/(2(1-\beta)), & x < 0, \\ x/(2\beta), & x \geq 0, \end{cases}$$

and speed measure

$$m(dx) = \begin{cases} 4(1-\beta)dx, & x < 0, \\ 4\beta dx, & x \geq 0. \end{cases}$$

It is known (see e.g. [10]) that $(\widehat{S}(\widehat{X}_t))_{t\geq 0} \stackrel{d}{=} (X_t)_{t\geq 0}$, i.e., the composition of SBM with its scale function, has the same law as OBM with $\sigma_1 = 1/(2(1-\beta))$ and $\sigma_2 = 1/(2\beta)$ and starting point $x = \widehat{S}(\widehat{x})$. In other words, we can say that SBM in natural scale is OBM. We use this relationship to obtain conclusions about the OSP problem (2.2). Given a payoff function $g\colon \mathbb{R} \to [0, \infty)$ we introduce the payoff function

$$\widehat{g}(x) = (g \circ \widehat{S})(x). \tag{5.1}$$

Due to the fact that $\widehat{S}$ is not differentiable at the origin, both functions g and $\widehat{g}$ can not be differentiable at the origin. The next result connects the optimal stopping problems for OBM and SBM.

Proposition 5.1. *For $\beta \in (0,1)$ the optimal stopping problem* (2.2) *for OBM X with parameters $\sigma_1 = 1/(2(1-\beta))$, $\sigma_2 = 1/(2\beta)$ and continuous reward g has value function $V(x)$ and stopping region Γ if and only if the optimal stopping problem for SBM $\widehat{X}$ with index $\beta \in (0,1)$ and reward $\widehat{g}$ in* (5.1) *has stopping region $\widehat{\Gamma} = \widehat{S}^{-1}(\Gamma)$ and value function $\widehat{V}(y) = V(\widehat{S}(y))$.*

Proof. It holds

$$\begin{aligned} V(x) &= \sup_\tau \mathbb{E}_x \left(\mathrm{e}^{-r\tau} g(X_\tau)\right) = \sup_\tau \mathbb{E}_x \left(\mathrm{e}^{-r\tau} \widehat{g}(\widehat{S}^{-1}(X_\tau))\right) \\ &= \sup_\tau \mathbb{E}_{\widehat{S}^{-1}(x)} \left(\mathrm{e}^{-r\tau} \widehat{g}(\widehat{X}_\tau)\right) = \widehat{V}(\widehat{S}^{-1}(x)), \end{aligned}$$

and this yields $\widehat{V}(y) = V(\widehat{S}(y))$. Let $\widehat{\Gamma} = \{y\colon \widehat{g}(y) = \widehat{V}(y)\}$, and consider

$$\begin{aligned} \widehat{S}(\widehat{\Gamma}) &= \{x\colon \exists y \in \widehat{\Gamma} \text{ such that } x = \widehat{S}(y)\} \\ &= \{x\colon \widehat{g}(\widehat{S}^{-1}(x)) = \widehat{V}(\widehat{S}^{-1}(x))\} \\ &= \{x\colon g(x) = V(x)\} = \Gamma. \end{aligned}$$

This concludes the proof. □

From this proposition it follows that $0 \notin \Gamma$ if and only if $0 \notin \widehat{\Gamma}$, because $\widehat{S}(0) = 0$. Furthermore, Γ is disconnected if and only if $\widehat{\Gamma}$ is disconnected, as the function $\widehat{S}$ is strictly increasing and continuous.

Example 5.2. Consider the problem (2.2) for SBM with $\beta > 1/2$ and reward $\widehat{g}(x) = (1+x)^+$ (cf. [1]). The corresponding OSP for OBM has $\sigma_1 = 1/(2(1-\beta))$, $\sigma_2 = 1/(2\beta)$, and reward

$$g(x) = \begin{cases} (1 + 2(1-\beta)x)^+, & x < 0, \\ 1 + 2\beta x, & x \geq 0. \end{cases} \tag{5.2}$$

Notice that $g'(0-) = 2(1-\beta) < 2\beta = g'(0+)$ (see Figure 2). As the scale function of OBM is $S(x) = x$, any r-excessive function h satisfies (see p. 93 in [18])

$$h'(x-) \geq h'(x+), \text{ for all } x \in \mathbb{R}. \tag{5.3}$$

If $0 \in \Gamma$ for some $r \geq 0$, then $V(0) = g(0)$, hence $V'(0-) < V'(0+)$ violating condition (5.3). We conclude that $0 \notin \Gamma$ for any value of r, hence $0 \notin \widehat{\Gamma}$ for any value of r for the SBM problem. This is a particular case of the result obtained in Proposition 1 in [1].

6 Conclusions

In this paper we consider the optimal stopping problem for the oscillating Brownian motion having a larger infinitesimal variance on the positive side than on the negative side. Our main findings are that for the classical payoff

$$g_1(x) = (1+x)^+$$

the solution is one-sided, that is, there exists an optimal threshold that can be determined using different classical methods. Here we rely on equation (4.1), but, e.g., the principle of smooth pasting can also be applied.

More interesting, when the reward is quadratic, i.e. has the form

$$g_2(x) = ((1+x)^+)^2,$$

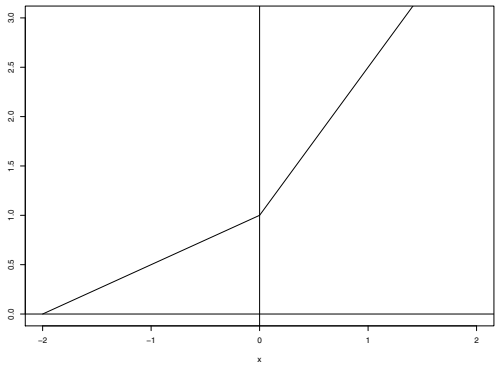

Figure 2: The reward g in (5.2) for OBM, with $\sigma_1 = 2$ and $\sigma_2 = 2/3$ ($\beta = 3/4$).

then for some values of the volatilities and the discount the optimal stopping region becomes disconnected. Such results have previously been recorded in [1] for a skew Brownian motion and in [12] for a diffusion with a piecewise constant drift changing at zero.

A key rôle in the analysis is played by the representation theory of excessive functions as discussed in [18]. This boils down to study the sign of a function that is the candidate for the density of the representing measure of the r-excessive value of the problem. This function involves the infinitesimal generator of the process. Since the generator is a second order differential operator without a first order term the discontinuity of the infinitesimal variance is not noticed in case of a linear payoff and the solution is one-sided.

Our analysis also suggests that the phenomenon of the one-sided solution happens only when the second derivative of the payoff vanishes at the discontinuity point. Consequently, e.g., for a general power payoff $g_\alpha(x) = ((1+x)^+)^\alpha$, $\alpha \neq 1$, a disconnected stopping region can arise similarly as in the quadratic case.

The results in the present paper and in [12] show that if the drift and/or the diffusion coefficient of a diffusion have discontinuities the stopping region could be disconnected even for nice monotonic payoff functions, and special care should be taken when calculating optimal stopping rules. Such considerations may have also economical relevance when applying the real options theory.

The concrete optimal stopping problem for diffusions with several discontinuities both in the drift and the diffusion coefficient is technically challenging but, undoubtedly, important. It is our hope that the joint analysis of the results obtained in [12] and in the present paper would provide some necessary tools for further research.

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