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Large deviations of the long term distribution of a non Markov process

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Abstract

We prove that the long term distribution of the queue length process in an ergodic generalised Jackson network obeys the Large Deviation Principle with a deviation function given by the quasipotential. The latter is related to the unique long term idempotent distribution, which is also a stationary idempotent distribution, of the large deviation limit of the queue length process. The proof draws on developments in queueing network stability and idempotent probability.

Keywords: generalised Jackson networks; long term distribution; large deviation principle; idempotent probability.
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1 Introduction and summary

In a seminal contribution, Freidlin and Wentzell [5] obtained the Large Deviation Principle (LDP) for the stationary distribution of a diffusion process and showed that the deviation function, which is often referred to as the action functional or the (tight) rate function, is given by the quasipotential. Their ingenious analysis relied heavily on the strong Markov property and involved an intricate study of attainment times. Shwartz and Weiss [10] adapted the methods of Freidlin and Wentzell [5] to the setting of jump Markov processes. In Puhalskii [8], we suggested a different, arguably, more direct and, as we hope, more robust approach. It was prompted by the analogy between large deviations and weak convergence and sought to identify the deviation function in terms of the stationary idempotent distribution of a large deviation limit. In this paper, the approach is applied to establishing the LDP for the long term distribution of the non Markov process of queue lengths in a generalised Jackson network. It is noteworthy that, in addition to being non Markovian, generalised Jackson networks fall into the category of stochastic systems with discontinuous dynamics, whose analysis is generally more difficult. We show that the deviation function is still given by the quasipotential which is related to the stationary idempotent distribution of the limit idempotent process. That stationary idempotent distribution is also a unique long term idempotent distribution, the uniqueness being proved by a coupling argument. Geometric ergodicity of the queue length process enables us to conclude that the long term idempotent distribution is the large deviation limit of the long term queue length distributions.

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2 The setup and main result

We consider a queueing network with a homogeneous customer population which comprises K single server stations. Customers arrive exogenously at the stations and are served there in the order of arrival, one customer at a time. Upon being served, they either join a queue at another station or leave the network. Let $A_k(t)$ denote the cumulative number of exogenous arrivals at station k by time t, let $S_k(t)$ denote the cumulative number of customers that are served at station k for the first t units of busy time of that station, and let $R_{kl}(m)$ denote the cumulative number of customers among the first m customers departing station k that go directly to station l. Let $A_k = (A_k(t), t \in \mathbb{R}_+)$, $S_k = (S_k(t), t \in \mathbb{R}_+)$, and $R_k = (R_k(m), m \in \mathbb{Z}_+)$, where $R_k(m) = (R_{kl}(m), l \in \mathcal{K})$ and $\mathcal{K} = \{1, 2, \ldots, K\}$. It is assumed that the A_k and S_k are nonzero renewal processes and $R_{kl}(m) = \sum_{i=1}^m 1_{\{\zeta_k^{(i)}=l\}}$, where $\{\zeta_k^{(1)}, \zeta_k^{(2)}, \ldots\}$ is a sequence of i.i.d. random variables assuming values in $\mathcal{K}$, 1_Γ standing for the indicator function of set Γ. The random entities A_k, S_l, R_i and $Q(0)$ are assumed to be defined on common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and be mutually independent, where $k, l, i \in \mathcal{K}$. We denote $p_{kl} = \mathbf{P}(\zeta_k^{(1)} = l)$ and let $P = (p_{kl})_{k,l=1}^K$. The matrix P is assumed to be of spectral radius less than unity so that every arriving customer eventually leaves. Let $Q = (Q(t), t \in \mathbb{R}_+)$ represent the queue length process, where $Q(t) = (Q_k(t), k \in \mathcal{K})$ and $Q_k(t)$ represents the number of customers at station k at time t. All the stochastic processes are assumed to have piecewise constant right–continuous with left–hand limits trajectories. Accordingly, they are considered as random elements of the associated Skorohod spaces.

For $k \in \mathcal{K}$ and $t \in \mathbb{R}_+$, the following equations are satisfied:

$$Q_k(t) = Q_k(0) + A_k(t) + \sum_{l=1}^K R_{lk}\big(D_l(t)\big) - D_k(t), \tag{2.1}$$

where

$$D_k(t) = S_k\big(B_k(t)\big) \tag{2.2}$$

represents the number of departures from station k by time t and

$$B_k(t) = \int_0^t 1_{\{Q_k(u)>0\}}\, du \tag{2.3}$$

represents the cumulative busy time of station k by time t. For given realisations of $Q_k(0)$, A_k, S_k, and R_k, there exist unique $Q_k = (Q_k(t), t \in \mathbb{R}_+)$, $D_k = (D_k(t), t \in \mathbb{R}_+)$ and $B_k = (B_k(t), t \in \mathbb{R}_+)$ that satisfy (2.1), (2.2) and (2.3), see, e.g., Chen and Mandelbaum [4]. The process Q is non Markov unless all A_k and S_k are Poisson processes.

Let, for $k \in \mathcal{K}$, nonnegative random variables ξ_k and η_k represent generic times between exogenous arrivals and service times at station k, respectively. We assume that $\mathbf{P}(\xi_k = 0) = \mathbf{P}(\eta_k = 0) = 0$, $\mathbf{E}\exp(\theta\xi_k) < \infty$ and $\mathbf{E}\exp(\theta\eta_k) < \infty$ for some $\theta > 0$, and the cumulative distribution functions of the ξ_k and η_k are right–differentiable at 0 with positive derivatives. Let $\beta_k = \sup\{\theta \in \mathbb{R}_+ : \mathbf{E}\exp(\theta\xi_k) < \infty\}$ and $\gamma_k = \sup\{\theta \in \mathbb{R}_+ : \mathbf{E}\exp(\theta\eta_k) < \infty\}$. Let also $\pi(u) = u \ln u - u + 1$ if $u > 0$, $\pi(0) = 1$, $\pi(\infty) = \infty$, $0/0 = 0$, and $\infty \cdot 0 = 0$. Let $\mathbb{S}_+^{K\times K}$ represent the set of row–substochastic $K \times K$–matrices and I represent the $K \times K$–identity matrix. Given vectors $\alpha = (\alpha_1, \ldots, \alpha_K)^T \in \mathbb{R}_+^K$ and

$\delta = (\delta_1, \ldots, \delta_K)^T \in \mathbb{R}_+^K$, matrix $\varrho \in \mathbb{S}_+^{K\times K}$ with rows ϱ_k, $k \in \mathcal{K}$, and $J \subset \mathcal{K}$, we define

$$\psi_k^A(\alpha_k) = \sup_{\theta<\beta_k} \big(\theta - \alpha_k \ln \mathbf{E} \exp(\theta \xi_k)\big) ,$$
$$\psi_k^S(\delta_k) = \sup_{\theta<\gamma_k} \big(\theta - \delta_k \ln \mathbf{E} \exp(\theta \eta_k)\big) ,$$
$$\psi_k^R(\varrho_k) = \sum_{l=1}^K \pi\Big(\frac{\varrho_{kl}}{p_{kl}}\Big) p_{kl} + \pi\Big(\frac{1-\sum_{l=1}^K \varrho_{kl}}{1-\sum_{l=1}^K p_{kl}}\Big)\big(1 - \sum_{l=1}^K p_{kl}\big)$$

and

$$\psi_J(\alpha, \delta, \varrho) = \sum_{k=1}^K \psi_k^A(\alpha_k) + \sum_{k\in J^c} \psi_k^S(\delta_k) + \sum_{k\in J} \psi_k^S(\delta_k)\, 1_{\{\delta_k > \mu_k\}} + \sum_{k=1}^K \delta_k\, \psi_k^R(\varrho_k) , \qquad (2.4)$$

where $J^c = \mathcal{K} \setminus J$. Also, for $y \in \mathbb{R}^K$, we let

$$\Psi_J(y) = \inf_{\substack{(\alpha,\delta,\varrho)\in\mathbb{R}_+^K\times\mathbb{R}_+^K\times\mathbb{S}_+^{K\times K}:\\ y=\alpha+(\varrho^T-I)\delta}} \psi_J(\alpha, \delta, \varrho) . \qquad (2.5)$$

If J is a nonempty subset of $\mathcal{K}$, we denote $F_J = \{x = (x_1, \ldots, x_K)^T \in \mathbb{R}_+^K : x_k = 0, k \in J, x_k > 0, k \notin J\}$, $F_\emptyset$ is defined to be the interior of $\mathbb{R}_+^K$. Let, for $x \in \mathbb{R}_+^K$ and $y \in \mathbb{R}^K$,

$$L(x, y) = \sum_{J\subset\mathcal{K}} 1_{F_J}(x)\Psi_J(y) . \qquad (2.6)$$

The function $L(x, y)$ is seen to be nonnegative.

Let
$$\mathbf{I}_x(q) = \int_0^\infty L(q(t), \dot{q}(t))\, dt ,$$
provided $q = (q(t)\, , t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+^K)$ is absolutely continuous with $q(0) = x \in \mathbb{R}_+^K$ and $\mathbf{I}_x(q) = \infty$, otherwise, where $q(t) = (q_1(t), \ldots, q_K(t))^T$.

With large deviations in mind, we will assume in the next theorem that the initial queue length depends on large parameter n, so, superscript "n" will be used to denote the associated random quantities, e.g., $Q^n(t)$ is the queue length vector at time t. Theorem 2.2 in Puhalskii [9] proves the following result.

Theorem 2.1. *If, in addition,* $\mathbf{P}(|Q^n(0)/n - x| > \epsilon)^{1/n} \to 0$ *as* $n \to \infty$*, for all* $\epsilon > 0$*, then the queue length processes* $\{(Q^n(nt)/n\, , t \in \mathbb{R}_+)\, , n \in \mathbb{N}\}$ *obey the LDP in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+^K)$ *for rate* n *with the deviation function* $\mathbf{I}_x(q)$*.*

For $x \in \mathbb{R}_+^K$, we define the quasipotential by

$$\mathbf{V}(x) = \inf_{t\in\mathbb{R}_+} \inf_{\substack{q\in\mathbb{D}(\mathbb{R}_+,\mathbb{R}_+^K):\\ q(t)=x}} \mathbf{I}_0(q) . \qquad (2.7)$$

In order to address an LDP for the stationary queue length distribution, we assume that the network is subcritical:

$$\mu > (I - P^T)^{-1}\lambda \qquad (2.8)$$

where $\mu = (\mu_1, \ldots, \mu_K)^T$, $\lambda = (\lambda_1, \ldots, \lambda_K)^T$, $\mu_k = 1/\mathbf{E}\eta_k$ and $\lambda_k = 1/\mathbf{E}\xi_k$. (Inequalities between vectors or matrices are understood to hold entrywise.) In addition, we assume that

1. there exists number $\overline{\eta} > 0$ such that $\mathbf{E}(\eta_k - u|\eta_k > u) \le \overline{\eta}$, for $k \in \mathcal{K}$ and $u > 0$,
2. $\mathbf{P}(\xi_k > u) > 0$, for $k \in \mathcal{K}$ and $u > 0$,

3. for $k \in \mathcal{K}$, there exist nonnegative function $f_k(u)$ on $\mathbb{R}_+$ with $\int_0^\infty f_k(u)\,du > 0$ and $m_k \in \mathbb{N}$ such that $\mathbf{P}(\sum_{i=1}^{m_k} \xi_{k,i} \in [v,w]) \geq \int_v^w f_k(u)\,du$, provided $0 \leq v \leq w$, where $\xi_{k,1},\ldots,\xi_{k,m_k}$ are i.i.d. and are distributed as ξ_k.

Under these hypotheses, the $Q(t)$ converge in distribution to a random variable $\hat{Q}$, as $t \to \infty$, see Meyn and Down [6]. The convergence holds for arbitrary initial vector $Q(0)$ and the convergence rate is geometric for the metric of total variation. In addition, if the random variables $Q(t)$ are augmented with residual service and interarrival times to produce a Markov process, then that Markov process has a unique stationary distribution, the distribution of $\hat{Q}$ being a marginal distribution. Our main result is the following theorem.

Theorem 2.2. *The sequence $\{\hat{Q}/n\,, n \in \mathbb{N}\}$ obeys the LDP in $\mathbb{R}_+^K$ for rate n with the deviation function $\mathbf{V}(x)$.*

Remark 2.3. Under (2.8), there is no “large deviation cost” for staying at the origin. On taking in (2.5) $J = \mathcal{K}$, $y = 0$, $\alpha = \lambda$, $\varrho = P$, and $\delta = (I - P^T)^{-1}\lambda$ and noting that $\delta < \mu$ by (2.8), one can see by (2.4) that $\psi_{\mathcal{K}}(\alpha,\delta,\varrho) = 0$, so $\Psi_{\mathcal{K}}(0) = 0$ and $L(0,0) = 0$. More generally, $L(q(t),\dot{q}(t)) = 0$ when q is “a fluid limit queue length” or the trajectory of the law of large numbers, i.e., $\dot{q}(t) = \lambda + (P^T - I)\mu + (I - P^T)\phi(t)$, where $\phi(t) \in \mathbb{R}_+^K$ and $\phi_k(t)q_k(t) = 0$, for $k \in \mathcal{K}$, cf. Puhalskii [9]. The converse is also true: if $\Psi_J(y) = 0$, then the infimum in (2.5) is attained at $\alpha = \lambda$, $\delta_k = \mu_k$ when $k \notin J$ and $\varrho = P$. (For a proof, one notes that $\psi_k^A(\alpha_k) = 0$, $\psi_k^S(\delta_k) = 0$, and $\psi_k^R(\varrho_k) = 0$ if and only if $\alpha_k = \lambda_k$, $\delta_k = \mu_k$, and $\varrho_k = (p_{k1},\ldots,p_{kK})$, respectively.) As a byproduct, in (2.7) $\mathbf{I}_0(q)$ can be replaced with $\int_0^t L(q(s),\dot{q}(s))\,ds$.

3 Idempotent probability and the proof of Theorem 2.2

Let us recap some notions of idempotent probability, see, e.g., Puhalskii [7]. Let Υ be a set. Function $\mathbf{\Pi}$ from the power set of Υ to $[0,1]$ is called an idempotent probability if $\mathbf{\Pi}(\Gamma) = \sup_{\upsilon\in\Gamma}\mathbf{\Pi}(\{\upsilon\})$, $\Gamma \subset \Upsilon$ and $\mathbf{\Pi}(\Upsilon) = 1$. The pair $(\Upsilon,\mathbf{\Pi})$ is called an idempotent probability space. For economy of notation, we denote $\mathbf{\Pi}(\upsilon) = \mathbf{\Pi}(\{\upsilon\})$. Property $\mathcal{P}(\upsilon)$, $\upsilon \in \Upsilon$, pertaining to the elements of Υ is said to hold $\mathbf{\Pi}$-a.e. if $\mathbf{\Pi}(\mathcal{P}(\upsilon)$ does not hold$) = 0$, where, in accordance with a tradition of probability theory, we define $\mathbf{\Pi}(\mathcal{P}(\upsilon)$ does not hold$) = \mathbf{\Pi}(\{\upsilon \in \Upsilon : \mathcal{P}(\upsilon)$ does not hold$\})$. Function f from set Υ equipped with idempotent probability $\mathbf{\Pi}$ to set Υ' is called an idempotent variable. The idempotent distribution of the idempotent variable f is defined as the set function $\mathbf{\Pi} \circ f^{-1}(\Gamma) = \mathbf{\Pi}(f \in \Gamma)$, $\Gamma \subset \Upsilon'$. If f is the canonical idempotent variable defined by $f(\upsilon) = \upsilon$, then it has $\mathbf{\Pi}$ as the idempotent distribution. If $f = (f_1,f_2)$, with f_i assuming values in Υ_i', then the (marginal) distribution of f_1 is defined by $\mathbf{\Pi}^{f_1}(\upsilon_1') = \mathbf{\Pi}(f_1 = \upsilon_1') = \sup_{\upsilon:\,f_1(\upsilon)=\upsilon_1'}\mathbf{\Pi}(\upsilon)$. The idempotent variables f_1 and f_2 are said to be independent if $\mathbf{\Pi}(f_1 = \upsilon_1',\, f_2 = \upsilon_2') = \mathbf{\Pi}(f_1 = \upsilon_1')\mathbf{\Pi}(f_2 = \upsilon_2')$ for all $(\upsilon_1',\upsilon_2') \in \Upsilon_1' \times \Upsilon_2'$, so, the joint distribution is the product of the marginal ones. Independence of finite collections of idempotent variables is defined similarly. Collection $(X_t,\, t \in \mathbb{R}_+)$ of idempotent variables on Υ is called an idempotent process. The functions $(X_t(\upsilon),\, t \in \mathbb{R}_+)$ for various $\upsilon \in \Upsilon$ are called trajectories (or paths) of X. Idempotent processes are said to be independent if they are independent as idempotent variables with values in the associated function spaces. The concepts of idempotent processes with independent and (or) stationary increments mimic those for stochastic processes.

If Υ is, in addition, a metric space and the sets $\{\upsilon \in \Upsilon : \mathbf{\Pi}(\upsilon) \geq \kappa\}$ are compact for all $\kappa \in (0,1]$, then $\mathbf{\Pi}$ is called a deviability. Obviously, $\mathbf{\Pi}$ is a deviability if and only if $\mathbf{I}(\upsilon) = -\log\mathbf{\Pi}(\upsilon)$ is a deviation function. If f is a continuous mapping from Υ to another metric space Υ', then $\mathbf{\Pi} \circ f^{-1}$ is a deviability on Υ'. As a matter of fact, for the latter property to hold, one can only require that f be continuous on the sets

$\{\upsilon \in \Upsilon : \mathbf{\Pi}(\upsilon) \geq \kappa\}$ for $\kappa \in (0,1]$. In general, an idempotent variable is said to be Luzin if its idempotent distribution is a deviability.

Let $\{\mathbf{P}_n,\, n \in \mathbb{N}\}$ be a sequence of probability measures on a metric space Υ endowed with the Borel σ-algebra and let $\mathbf{\Pi}$ be a deviability on Υ. The sequence $\{\mathbf{P}_n,\, n \in \mathbb{N}\}$ is said to large deviation converge (LD converge) at rate n to $\mathbf{\Pi}$ as $n \to \infty$ if $\lim_{n\to\infty}\Bigl(\int_\Upsilon f(\upsilon)^n\,\mathbf{P}_n(d\upsilon)\Bigr)^{1/n} = \sup_{\upsilon\in\Upsilon} f(\upsilon)\mathbf{\Pi}(\upsilon)$ for every bounded continuous $\mathbb{R}_+$-valued function f on Υ. Equivalently, one may require that $\lim_{n\to\infty}\mathbf{P}_n(H)^{1/n} = \mathbf{\Pi}(H)$ for every $\mathbf{\Pi}$–continuity set H, which is defined by the requirement that the values of $\mathbf{\Pi}$ on the interior and closure of H are equal to each other. Obviously, the sequence $\{\mathbf{P}_n,\, n \in \mathbb{N}\}$ LD converges at rate n to $\mathbf{\Pi}$ if and only if this sequence obeys the LDP for rate n with deviation function $\mathbf{I}(\upsilon) = -\log\mathbf{\Pi}(\upsilon)$. Similarly, a sequence $\mathbf{\Pi}_n$ of deviabilities on Υ is said to converge weakly to deviability $\mathbf{\Pi}$, as $n \to \infty$, if $\lim_{n\to\infty}\sup_{\upsilon\in\Upsilon} f(\upsilon)\mathbf{\Pi}_n(\upsilon) = \sup_{\upsilon\in\Upsilon} f(\upsilon)\mathbf{\Pi}(\upsilon)$ for every bounded continuous $\mathbb{R}_+$-valued function f on Υ. The analogue of Prohorov's theorem holds: if the sequence $\mathbf{\Pi}_n$ is tight meaning that $\inf_{\Gamma\in\Xi}\limsup_{n\to\infty}\mathbf{\Pi}_n(\Upsilon\setminus\Gamma) = 0$, where Ξ represents the collection of compact subsets of Υ, then the $\mathbf{\Pi}_n$ converge to a deviability along a subsequence.

LD convergence of probability measures can be also expressed as LD convergence in distribution of the associated random variables to idempotent variables. We say that a sequence $\{X_n,\, n \in \mathbb{N}\}$ of random variables defined on respective probability spaces $(\Omega_n, \mathcal{F}_n, \mathbf{P}_n)$ and assuming values in Υ' LD converges in distribution at rate n as $n \to \infty$ to an idempotent variable X defined on an idempotent probability space $(\Upsilon, \mathbf{\Pi})$ and assuming values in Υ' if the sequence of the probability laws of the X_n LD converges to the idempotent distribution of X at rate n. If a sequence $\{\mathbf{P}_n,\, n \in \mathbb{N}\}$ of probability measures on Υ LD converges to deviability $\mathbf{\Pi}$ on Υ, then one has LD convergence in distribution for the canonical setting.

We now return to the setup of generalised Jackson networks and let $\mathbf{\Pi}^Q_x(q) = e^{-\mathbf{I}_x(q)}$. It is proved in Puhalskii [9] that under the hypotheses of Theorem 2.1 there exists a unique deviability $\mathbf{\Pi}_x$ on $\Upsilon = \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+^K \times \mathbb{R}_+^K \times \mathbb{R}_+^K \times \mathbb{R}_+^K \times \mathbb{R}_+^K \times \mathbb{R}_+^{K\times K})$ such that the processes $\bigl((Q^n(nt)/n\,, t \in \mathbb{R}_+), (B^n(nt)/n\,, t \in \mathbb{R}_+), (D^n(nt)/n\,, t \in \mathbb{R}_+), (A^n(nt)/n\,, t \in \mathbb{R}_+), (S^n(nt)/n,\, t \in \mathbb{R}_+), (R^n(nt)/n\,, t \in \mathbb{R}_+)\bigr)$ LD converge at rate n to the canonical idempotent process (q, b, d, a, s, r) on Υ. The component idempotent processes of b, d, a, s, and r have $\mathbf{\Pi}_x$-a.e. absolutely continuous nondecreasing trajectories starting at 0, the component idempotent processes of b grow not faster than at rate 1, and the component idempotent processes of q have $\mathbf{\Pi}_x$-a.e. absolutely continuous trajectories, the idempotent process q has idempotent distribution $\mathbf{\Pi}^Q_x$, the idempotent processes a, s and r are independent with respective idempotent distributions $\mathbf{\Pi}^A$, $\mathbf{\Pi}^S$ and $\mathbf{\Pi}^R$ defined as follows, where, by virtue of our working in a canonical setting, identical pieces of notation are used for denoting idempotent processes and their sample trajectories:

$$\mathbf{\Pi}^A(a) = \prod_{k=1}^K \mathbf{\Pi}^A_k(a_k)\,,\ \mathbf{\Pi}^A_k(a_k) = \exp\Bigl(-\int_0^\infty \psi^A_k(\dot{a}_k(t))\,dt\Bigr)\,, \tag{3.1}$$

$$\mathbf{\Pi}^S(s) = \prod_{k=1}^K \mathbf{\Pi}^S_k(s_k)\,,\ \mathbf{\Pi}^S_k(s_k) = \exp\Bigl(-\int_0^\infty \psi^S_k(\dot{s}_k(t))\,dt\Bigr)\,, \tag{3.2}$$

$$\mathbf{\Pi}^R(r) = \prod_{k=1}^K \mathbf{\Pi}^R_k(r_k)\,,\ \mathbf{\Pi}^R_k(r_k) = \exp\Bigl(-\int_0^\infty \psi^R_k(\dot{r}_k(t))\,dt\Bigr)\,, \tag{3.3}$$

where $a = (a(t)\,, t \in \mathbb{R}_+) = (a_1, \ldots, a_K)^T$, $a_k = (a_k(t)\,, t \in \mathbb{R}_+)$, $a(t) = (a_1(t), \ldots, a_K(t))^T$, $s = (s(t)\,, t \in \mathbb{R}_+) = (s_1, \ldots, s_K)^T$, $s_k = (s_k(t)\,, t \in \mathbb{R}_+)$, $s(t) = (s_1(t), \ldots, s_K(t))^T$, $r = (r(t)\,, t \in \mathbb{R}_+) = (r_1, \ldots, r_K)^T$, $r_k = (r_k(t)\,, t \in \mathbb{R}_+) = (r_{k1}, \ldots, r_{kK})$, $r_{kl} = (r_{kl}(t)\,, t \in \mathbb{R}_+)$,

$r_k(t) = (r_{k1}(t), \ldots, r_{kK}(t))$, $r(t) = (r_1(t), \ldots, r_K(t))^T$, the functions a_k, s_k, and r_{kl} being absolutely continuous with $a_k(0) = 0$, $s_k(0) = 0$, $r_{kl}(0) = 0$, $\dot{a}_k(t) \in \mathbb{R}_+$ a.e., $\dot{s}_k(t) \in \mathbb{R}_+$ a.e., and $\dot{r}(t) \in \mathbb{S}_+^{K\times K}$ a.e.

Also $\mathbf{\Pi}_x$-a.e. the following equations hold for $t \in \mathbb{R}_+$ and $k \in \mathcal{K}$:

$$q_k(t) = x_k + a_k(t) + \sum_{l=1}^{K} r_{lk}\big(d_l(t)\big) - d_k(t), \tag{3.4}$$

$$d_k(t) = s_k\big(b_k(t)\big), \tag{3.5}$$

$$\int_0^t q_k(u)\, db_k(u) = \int_0^t q_k(u)\, du, \tag{3.6}$$

where $b = (b(t),\, t \in \mathbb{R}_+)$, $b(t) = (b_1(t), \ldots, b_K(t))^T$, $d = (d(t),\, t \in \mathbb{R}_+)$, and $d(t) = (d_1(t), \ldots, d_K(t))^T$. Equations (3.4) and (3.5) are obtained by taking large deviation limits in (2.1) and (2.2), respectively. whereas, in order to derive (3.6), one notes that (2.3) implies that $\int_0^t Q_k(u)\, dB_k(u) = \int_0^t Q_k(u)\, du$ and passes to the large deviation limit in the latter equation. It is noteworthy that since in (3.1), (3.2) and (3.3) the sample trajectories enter the deviabilities only through their derivatives, the idempotent processes a, s and r have independent and stationary increments.

By (3.4), $q(0) = x$ $\mathbf{\Pi}_x$-a.e. In order to allow the initial value $q(0)$ to have a nondegenerate idempotent distribution, we introduce

$$\mathbf{\Pi}(\upsilon) = \sup_{x\in\mathbb{R}_+^K} \mathbf{\Pi}_x(\upsilon)\tilde{\mathbf{\Pi}}^{Q_0}(x)\,, \tag{3.7}$$

where $\tilde{\mathbf{\Pi}}^{Q_0}$ is a deviability on $\mathbb{R}_+^K$. One can see that $\mathbf{\Pi}$ is a deviability on Υ. Obviously, $\mathbf{\Pi}(q(0) = x) = \tilde{\mathbf{\Pi}}^{Q_0}(x)$ and $q(0)$, a, s and r are independent under $\mathbf{\Pi}$. Also, the marginal idempotent distribution of q is given by

$$\mathbf{\Pi}^Q(q) = \sup_{x\in\mathbb{R}_+^K} \mathbf{\Pi}_x^Q(q)\tilde{\mathbf{\Pi}}^{Q_0}(x)\,.$$

By (3.4), $\mathbf{\Pi}$–a.e.,

$$q_k(u) = q_k(0) + a_k(u) + \sum_{l=1}^{K} r_{lk}(d_l(u)) - d_k(u)\,. \tag{3.8}$$

Let

$$\mathbf{\Pi}_{x,t}(y) = \sup_{q:q(t)=y} \mathbf{\Pi}_x^Q(q)\,,\ \mathbf{\Pi}_{x,t}(\Gamma) = \sup_{y\in\Gamma} \mathbf{\Pi}_{x,t}^Q(y)\,,\ \text{where } \Gamma \subset \mathbb{R}_+^K\,.$$

The definition implies the semigroup property that

$$\mathbf{\Pi}_{x,u+v}(y) = \sup_{z\in\mathbb{R}_+^K} \mathbf{\Pi}_{x,u}(z)\mathbf{\Pi}_{z,v}(y)\,. \tag{3.9}$$

For $J \subset \mathcal{K}$, we will denote by $\mathbf{1}_J$ the vector with unity entries whose dimension equals the number of elements in J. For compactness of notation, we let $\mathbf{1} = \mathbf{1}_{\mathcal{K}}$.

Lemma 3.1. *Given $\kappa > 0$ and $\epsilon > 0$, there exists $T > 0$ such that*

$$\mathbf{\Pi}\big(\cup_{u\geq T}\{\{a(u) \notin [(\lambda - \epsilon\mathbf{1})u, (\lambda + \epsilon\mathbf{1})u]\} \cup \{s(u) \notin [(\mu - \epsilon\mathbf{1})u, (\mu + \epsilon\mathbf{1})u]\} \cup \{r(u) \notin [(P - \epsilon I)u, (P + \epsilon I)u]\}\}\big) < \kappa\,.$$

Proof. By the maxitivity property that $\mathbf{\Pi}(\cup_i \Gamma_i) = \sup_i \mathbf{\Pi}(\Gamma_i)$, for arbitrary collection of sets Γ_i, it suffices to work with $\mathbf{\Pi}(a(u) \notin [(\lambda - \epsilon\mathbf{1})u, (\lambda + \epsilon\mathbf{1})u])$ only. By the LD convergence in distribution of $(A(nt)/n\,, t \in \mathbb{R}_+)$ to a and Lemma A.1 relegated to the appendix, whose assertion can be also found in Appendix A of Bell and Williams [1], for some $\sigma \in (0,1)$,

$$\mathbf{\Pi}(a(u) \notin [(\lambda - \epsilon\mathbf{1})u, (\lambda + \epsilon\mathbf{1})u]) = \mathbf{\Pi}^A(a(u) \notin [(\lambda - \epsilon\mathbf{1})u, (\lambda + \epsilon\mathbf{1})u]) \\ \le \liminf_{n\to\infty} \mathbf{P}\bigl(|\frac{A(nu)}{n} - \lambda u| > \epsilon u\bigr)^{1/n} \le \sigma^u . \quad \square$$

Lemma 3.2. *Given a bounded set $B \subset \mathbb{R}_+^K$ and $\kappa \in (0,1)$, there exists $\hat{T} > 0$ such that if $\mathbf{\Pi}^Q(q) > \kappa$ and $q(0) \in B$, then $\min_{u\in[0,\hat{T}]}|q(u)| = 0$.*

Proof. The idea is to draw on the proof of the stability of fluid models of queueing networks in Bramson [2, 3] in order to show that, owing to condition (2.8), the function $\mathbf{1} \cdot (I - P^T - \epsilon I)^{-1} q(u)$ decreases linearly with u, provided ϵ is small enough, which implies that the function must attain 0.

By (2.8), there exists $\epsilon > 0$ such that $(I - P^T - \epsilon I)^{-1}(\lambda + \epsilon\mathbf{1}) \le \mu - \epsilon\mathbf{1}$ and $(I - P^T + \epsilon I)^{-1}(\lambda - \epsilon\mathbf{1}) \le \mu - \epsilon\mathbf{1}$. (In the course of the proof, potentially smaller ϵ will be needed. Yet, there exists ϵ that satisfies all the requirements. Importantly, it depends neither on κ nor on B.) By Lemma 3.1, there exists $T > 0$ such that $\mathbf{\Pi}(a(u) \ge (\lambda + \epsilon\mathbf{1})u$ for some $u \ge T) < \kappa$, $\mathbf{\Pi}(a(u) \le (\lambda - \epsilon\mathbf{1})u$ for some $u \ge T) < \kappa$, $\mathbf{\Pi}(s(u) \le (\mu - \epsilon\mathbf{1})u$ for some $u \ge T) < \kappa$, $\mathbf{\Pi}(s(u) \ge (\mu + \epsilon\mathbf{1})u$ for some $u \ge T) < \kappa$, $\mathbf{\Pi}(r(u) \ge (P + \epsilon I)u$ for some $u \ge T) < \kappa$, and $\mathbf{\Pi}(r(u) \le (P - \epsilon I)u$ for some $u \ge T) < \kappa$.

Let

$$\Gamma_\kappa = \{\upsilon :\ (\lambda - \epsilon\mathbf{1})u \le a(u) \le (\lambda + \epsilon\mathbf{1})u\,, (P - \epsilon I)u \le r(u) \le (P + \epsilon I)u\,, \\ \text{and } (\mu - \epsilon\mathbf{1})u \le s(u) \le (\mu + \epsilon\mathbf{1})u\,,\ \text{for } u \ge T\}.$$

We have that $\mathbf{\Pi}(\Gamma_\kappa^c) < \kappa$ and that on Γ_κ, provided $u \ge T$, by (3.4),

$$q(0) + (\lambda - \epsilon\mathbf{1})u + (P^T - \epsilon I)d(u) - d(u) \le q(u) \le q(0) + (\lambda + \epsilon\mathbf{1})u + (P^T + \epsilon I)d(u) - d(u)\,. \tag{3.10}$$

Let us show that there exists $S \ge 2T$ such that $b_k(S) \ge T$ for all k on Γ_κ. Intuitively, this is the case because otherwise some $s_k(b_k(u))$ would be “bounded” whereas $a_k(u)$ can be arbitrarily great for great u pushing $b_k(u)$ past T. Formally, assuming that $\epsilon < \lambda_k$, for all k, let $S \ge 2T$ be such that $(\lambda_k - \epsilon)(S - T) - (\mu_k + \epsilon)T > 0$, for all k. If $b_k(S) < T$, for some k, then, by (3.5), $d_k(u) \le s_k(T)$ for $u \in [0, S]$. By (3.8), on Γ_κ, for $u \in [S - T, S]$, $q_k(u) \ge a_k(u) - s_k(T) \ge (\lambda_k - \epsilon)u - (\mu_k + \epsilon)T > 0$. Therefore, by (3.6), $\dot{b}_k(u) = 1$ a.e. when $u \in [S - T, S]$, so $b_k(S) \ge T$, which contradicts the assumption that $b_k(S) < T$. It is worth noting that whereas both T and S may depend on either ϵ or κ, neither of them depends on $q(0)$.

We now assume that q is piecewise linear, which assumption is to be disposed of later. Let us suppose that $q_k(u) > 0$ for u in a right neighborhood of S for some k on Γ_κ. Then, $b_k(u) = b_k(S) + u - S \ge T + u - S$, for $u \ge S$, until $q_k(u)$ hits zero. Accordingly, $d_k(u) = s_k(b_k(u)) \ge (\mu_k - \epsilon)(u - S)$. Hence, if $q(u) > 0$ entrywise in a right neighborhood of S, then

$$d(u) \ge (I - (P^T + \epsilon I))^{-1}(\lambda + \epsilon\mathbf{1})(u - S) = \bigl(\nu + \epsilon(I - (P^T + \epsilon I))^{-1}(\nu + \mathbf{1})\bigr)(u - S)\,,$$

where we denote

$$\nu = (I - P^T)^{-1}\lambda\,. \tag{3.11}$$

As a consequence, for some $\rho > 0$, which is dependent on ϵ only, while $q(u) > 0$ entrywise,

$$d(u) \ge (\nu + \rho\mathbf{1})(u - S)\,. \tag{3.12}$$

By the righthand inequality in (3.10) and (3.11),

$$\begin{aligned}\mathbf{1}\cdot(I-P^T-\epsilon I)^{-1}q(u) &\le \mathbf{1}\cdot(I-P^T-\epsilon I)^{-1}q(0)+\mathbf{1}\cdot(I-P^T-\epsilon I)^{-1}(\lambda+\epsilon\mathbf{1})u-\mathbf{1}\cdot d(u)\\ &= \mathbf{1}\cdot(I-P^T-\epsilon I)^{-1}q(0)+\mathbf{1}\cdot(\nu u-d(u))+\epsilon\mathbf{1}\cdot(I-P^T)^{-1}\mathbf{1}u\\ &\qquad+\epsilon\mathbf{1}\cdot(I-P^T-\epsilon I)^{-1}(I-P^T)^{-1}(\lambda+\epsilon\mathbf{1})u\,. \end{aligned} \tag{3.13}$$

By (3.12), there exist $\hat{\rho}>0$ and $\gamma>0$ such that, provided ϵ is small enough, if $u\ge S$, then, while $q(u)$ stays entrywise positive,

$$\mathbf{1}\cdot(I-P^T-\epsilon I)^{-1}q(u)\le \mathbf{1}\cdot(I-P^T-\epsilon I)^{-1}q(0)+\gamma S-\hat{\rho}u\,. \tag{3.14}$$

Let us show that similar inequalities hold on Γ_κ for all $u\ge S$. Given $v\ge S$, let O denote a possibly empty set of indices k such that $q_k(u)=0$ on some interval $[v,v+\eta]$ and $q_k(u)>0$ if $k\notin O$ and $u\in(v,v+\eta)$. Such η exists because $q(u)$ is piecewise linear. We assume that $q(u)\ne 0$ on $[v,v+\eta]$, so O is a proper subset of $\mathcal{K}$. By the lefthand inequality in (3.10), on Γ_κ,

$$q_k(0)+(\lambda_k-\epsilon)u+((P^T-\epsilon I)d(u))_k-d_k(u)\le 0\,,\ \text{for } k\in O\,,u\in[v,v+\eta]\,. \tag{3.15}$$

Therefore, using subscript O and O^c to denote restrictions of vectors to indices in O and O^c respectively, and using subscripts OO and OO^c to denote restrictions of matrices to entries with both indices in OO and OO^c, respectively, we have that

$$d_O(u)\ge q_O(0)+(\lambda_O-\epsilon\mathbf{1}_O)u+(P^T-\epsilon I)_{OO}d_O(u)+(P^T-\epsilon I)_{OO^c}d_{O^c}(u)\,,$$

so, assuming ϵ is small enough,

$$d_O(u)\ge(I-(P^T-\epsilon I))_{OO}^{-1}\big(q_O(0)+(\lambda_O-\epsilon\mathbf{1}_O)u+(P^T-\epsilon I)_{OO^c}d_{O^c}(u)\big)\,. \tag{3.16}$$

On the other hand, by (3.11), $\lambda_O=(I-P^T)_{OO}\nu_O-P^T_{OO^c}\nu_{O^c}$. Substitution in (3.16) and rearranging yield

$$d_O(u)\ge\nu_O u+(I-(P^T-\epsilon I))_{OO}^{-1}\big(P^T_{OO^c}(d_{O^c}(u)-\nu_{O^c}u)-\epsilon\nu_O u-\epsilon\mathbf{1}_O u-\epsilon I_{OO^c}d_{O^c}(u)\big)\big)\,.$$

In analogy with the derivation of (3.12), one obtains that, for some $\rho_O>0$,

$$d_{O^c}(u)\ge(\nu_{O^c}+\rho_O\mathbf{1}_{O^c})(u-S)\,. \tag{3.17}$$

Therefore, for $u\in[v,v+\eta]$,

$$\begin{aligned}d_O(u)\ge\nu_O u+(I-(P^T-\epsilon I))_{OO}^{-1}\big(&-P^T_{OO^c}(\nu_{O^c}+\rho_O\mathbf{1}_{O^c})S-\epsilon\nu_O u-\epsilon\mathbf{1}_O u\\ &-\epsilon I_{OO^c}d_{O^c}(u)\big)\,.\end{aligned} \tag{3.18}$$

Since $O^c\ne\emptyset$, by (3.17), (3.18) and the bound $d(u)\le(\mu+\epsilon)u$ when $u\ge S$, there exist $\tilde{\rho}>0$ and $\gamma>0$ which do not depend on O such that, assuming ϵ is small enough, for $u\in[v,v+\eta]$,

$$\mathbf{1}\cdot d(u)=\mathbf{1}_O\cdot d_O(u)+\mathbf{1}_{O^c}\cdot d_{O^c}(u)\ge\mathbf{1}\cdot\nu u+\tilde{\rho}u-\gamma S\,.$$

By (3.13), we obtain that (3.14) still holds, for suitable $\hat{\rho}>0$ which does not depend on O and $u\in[v,v+\eta]$, provided ϵ is small enough. We can repeat the same argument over and over again, so, (3.14) holds until $q(u)=0$. Hence, one can take

$$\hat{T}=\frac{1}{\hat{\rho}}\,(|(I-P^T-\epsilon I)^{-1}\mathbf{1}|\sup_{x\in B}|x|+\gamma S)$$

as the time by which q is bound to hit the origin.

Suppose now that q is not necessarily piecewise linear and $\mathbf{\Pi}^Q(q) > \kappa$. By Lemmas 4.1–4.4 in Puhalskii [9], there exist piecewise linear q^n which converge to q as $n \to \infty$ such that $\mathbf{\Pi}^Q(q^n) \to \mathbf{\Pi}^Q(q)$. By what's been proved, there exist t^n from $[0, \hat{T}]$ such that $|q^n(t^n)| = 0$. Since $|q^n(t^n) - q(t^n)| \to 0$, it follows that $|q(t')| = 0$ where t' represents a subsequential limit of the t^n. □

Theorem 3.3. *There exists deviability $\hat{\mathbf{\Pi}}$ on $\mathbb{R}_+^K$ such that, for every bounded set $B \subset \mathbb{R}_+^K$,*

$$\lim_{t\to\infty} \sup_{x\in B} \sup_{y\in\mathbb{R}_+^K} |\mathbf{\Pi}_{x,t}(y) - \hat{\mathbf{\Pi}}(y)| = 0\,. \tag{3.19}$$

Furthermore, given y, $\mathbf{\Pi}_{x,t}(y) = \hat{\mathbf{\Pi}}(y)$ for all t great enough and all $x \in B$. The deviability $\hat{\mathbf{\Pi}}$ is a unique stationary deviability for the semigroup $\mathbf{\Pi}_{x,t}$ meaning that, for all $y \in \mathbb{R}_+^K$ and $t \in \mathbb{R}_+$,

$$\hat{\mathbf{\Pi}}(y) = \sup_{x\in\mathbb{R}_+^K} \mathbf{\Pi}_{x,t}(y)\hat{\mathbf{\Pi}}(x)\,.$$

Proof. One can see that $\mathbf{\Pi}_{0,t}(y)$ is a nondecreasing function of t. Indeed, let $u \le t$. Given function q such that $q(0) = 0$ and $q(u) = y$, we can associate with it function $\tilde{q}$ such that $\tilde{q}(v) = 0$ for $v \in [0, t-u]$ and $\tilde{q}(v) = q(v - (t-u))$ for $v \in [t-u, t]$. It follows that $\mathbf{I}_0(\tilde{q}) = \mathbf{I}_0(q)$ yielding the desired monotonicity. We let

$$\hat{\mathbf{\Pi}}(y) = \lim_{t\to\infty} \mathbf{\Pi}_{0,t}(y)\,. \tag{3.20}$$

Let us show that $\mathbf{\Pi}_{0,t}(y)$ levels off eventually as a function of t. Let $\kappa > 0$. We define t' as $\hat{T}$ in the statement of Lemma 3.2 with $\{x \in \mathbb{R}_+^K : |x| \le 1\}$ as the set B. Suppose that $\mathbf{\Pi}_{0,t}(y) > \kappa$, where $t \ge t' + 1$. Let q be such that $q(0) = 0$, $q(t) = y$ and $\mathbf{\Pi}_0^Q(q) = \mathbf{\Pi}_{0,t}(y)$. Let $\tilde{t} = \inf\{s : |q(s)| \ge 1\} \wedge 1$. Then $|q(\tilde{t})| \le 1$ and $0 < \tilde{t} \le 1$. By Lemma 3.2, there exists $\breve{t} \in [\tilde{t}, t' + 1]$ such that $q(\breve{t}) = 0$. On defining $\tilde{q}(s) = q(s + \breve{t})$, for $s \in \mathbb{R}_+$, we have that $\mathbf{\Pi}_0^Q(\tilde{q}) \ge \mathbf{\Pi}_0^Q(q)$. On the other hand, since $\breve{t} \le t$, we have that $\tilde{q}(t - \breve{t}) = y$ which implies that $\mathbf{\Pi}_0^Q(\tilde{q}) \le \mathbf{\Pi}_{0,t-\breve{t}}(y) \le \mathbf{\Pi}_{0,t}(y) = \mathbf{\Pi}_0^Q(q)$, so, $q(u) = 0$ on $[0, \breve{t}]$, for Remark 2.3 implies that if $q(0) = 0$ and $q(u) = 0$ for some $u > 0$, then $\int_0^u L(q(v), \dot{q}(v))\, dv = 0$ if and only if $q(v) = 0$ on $[0, u]$. Hence, $\breve{t} = \tilde{t} = 1$, so, $\mathbf{\Pi}_{0,t-1}(y) = \mathbf{\Pi}_{0,t}(y)$. This proves that if $\mathbf{\Pi}_{0,t}(y) > \kappa$ and $t \ge t' + 1$, then $\mathbf{\Pi}_{0,t'+1}(y) = \mathbf{\Pi}_{0,t}(y)$. We also have that $\mathbf{\Pi}_{0,t}(y) \le \kappa \vee \mathbf{\Pi}_{0,t'+1}(y)$, for all t and y. Hence, the net of deviabilities $\mathbf{\Pi}_{0,t}$ is tight, so, $\hat{\mathbf{\Pi}}$ is a deviability too.

Let us prove that

$$\lim_{t\to\infty} \sup_{x\in B} \sup_{y\in\mathbb{R}_+^K} |\mathbf{\Pi}_{x,t}(y) - \mathbf{\Pi}_{0,t}(y)| = 0\,. \tag{3.21}$$

A coupling argument is employed. We prove, at first, that, for arbitrary $\kappa > 0$,

$$\mathbf{\Pi}_{x,t}(y) \le \mathbf{\Pi}_{0,t}(y) + \kappa \text{ for all } x \in B \text{ and } y \in \mathbb{R}^K \text{ provided } t \text{ is great enough}\,. \tag{3.22}$$

By Lemma 3.2, there exists $\hat{T}$ such that if $q(0) \in B$ and $\mathbf{\Pi}^Q(q) > \kappa$, then $q(u) = 0$ for some $u \in [0, \hat{T}]$. Let us fix $x \in B$ and $y \in \mathbb{R}_+^K$. One can assume that $t \ge \hat{T}$ and that $\mathbf{\Pi}_{x,t}(y) > \kappa$. Let trajectory $\hat{q}$ be such that $\hat{q}(0) = x$, $\hat{q}(t) = y$ and $\mathbf{\Pi}_{x,t}(y) = \mathbf{\Pi}^Q(\hat{q})$. By Lemma 3.2, there exists $\hat{T}_1 \in [0, \hat{T}]$ such that $\hat{q}(\hat{T}_1) = 0$. We define $\tilde{q}$ by letting $\tilde{q}(u) = 0$ when $u \le \hat{T}_1$ and $\tilde{q}(u) = \hat{q}(u)$ when $u \ge \hat{T}_1$. By Remark 2.3, $\mathbf{\Pi}^Q(\hat{q}) \le \mathbf{\Pi}^Q(\tilde{q}) \le \mathbf{\Pi}_{0,t}(y)$, proving (3.22).

On the other hand, given $t \ge \hat{T} + 1$, $x \in B$, and q such that $q(0) = 0$, $q(t) = y$, $\mathbf{\Pi}^Q(q) = \mathbf{\Pi}_{0,t}(y) > \kappa$, and $q(u) = 0$ for all $u \in [0, \hat{T}]$ (the latter can be always assumed as we have seen), we define $\hat{q}$ with $\hat{q}(0) = x$ by letting it follow the law of large numbers until it hits zero at some $\hat{T}_1 \in [0, \hat{T}]$ and by letting $\hat{q}(u) = q(u - \hat{T}_1)$, for $u \ge \hat{T}_1$. Since

$\mathbf{\Pi}^Q(\hat{q}) = \mathbf{\Pi}^Q(q)$ by Remark 2.3, we obtain that $\mathbf{\Pi}_{0,t}(y) = \mathbf{\Pi}^Q(\hat{q}) \le \mathbf{\Pi}_{x,t}(y)$, which concludes the proof of (3.21).

We have shown that $\mathbf{\Pi}_{x,t}(y) \to \hat{\mathbf{\Pi}}(y)$, as $t \to \infty$, uniformly over $y \in \mathbb{R}_+^K$ and over x from bounded sets. It follows that, for arbitrary initial deviability $\tilde{\mathbf{\Pi}}^{Q_0}$,

$$\lim_{t\to\infty} \sup_{y\in\mathbb{R}_+^K} |\mathbf{\Pi}(q(t) = y) - \hat{\mathbf{\Pi}}(y)| = 0\,.$$

Letting $u \to \infty$ in (3.9) implies that $\hat{\mathbf{\Pi}}$ is a unique stationary initial deviability. (For, if $\mathbf{\Pi}'$ is another stationary deviability, then

$$|\hat{\mathbf{\Pi}}(y) - \mathbf{\Pi}'(y)| = |\hat{\mathbf{\Pi}}(y) \sup_{x\in\mathbb{R}_+^K} \mathbf{\Pi}'(x) - \sup_{x\in\mathbb{R}_+^K} \mathbf{\Pi}'(x)\mathbf{\Pi}_{x,t}(y)|$$
$$\le \sup_{x\in\mathbb{R}_+^K} |\mathbf{\Pi}'(x)\hat{\mathbf{\Pi}}(y) - \mathbf{\Pi}'(x)\mathbf{\Pi}_{x,t}(y)| \le \sup_{\substack{x\in\mathbb{R}_+^K:\\ \mathbf{\Pi}'(x)\ge\kappa}} |\hat{\mathbf{\Pi}}(y) - \mathbf{\Pi}_{x,t}(y)| \vee \kappa\,,$$

where $\kappa \in (0, 1]$, and one can let $t \to \infty$.)

□

Remark 3.4. The proof shows that the value of t where the $\mathbf{\Pi}_{0,t}(y)$ level off can be chosen uniformly over y such that $\hat{\mathbf{\Pi}}(y) \ge \kappa$.

Proof of Theorem 2.2. Let $\mathbf{Q}^n$ denote the distribution of $\hat{Q}/n$ and let $\mathbf{Q}^n_{0,t}$ denote the distribution of $Q(nt)/n$ for $Q(0) = 0$. Let $H \subset \mathbb{R}^K$ be a $\hat{\mathbf{\Pi}}$–continuity set. We have that

$$|\mathbf{Q}^n(H)^{1/n} - \hat{\mathbf{\Pi}}(H)| \le |\mathbf{Q}^n(H) - \mathbf{Q}^n_{0,t}(H)|^{1/n} + |\mathbf{Q}^n_{0,t}(H)^{1/n} - \mathbf{\Pi}_{0,t}(H)| + |\mathbf{\Pi}_{0,t}(H) - \hat{\mathbf{\Pi}}(H)|\,. \quad (3.23)$$

By Theorem 4.1 in Meyn and Down [6], there exist $A > 1$ and $\rho \in (0, 1)$ such that $|\mathbf{Q}^n(H) - \mathbf{Q}^n_{0,t}(H)| \le A\rho^{nt}$. Given $\epsilon > 0$, let t be such that $A\rho^t < \epsilon$ and $|\mathbf{\Pi}_{0,t}(H) - \hat{\mathbf{\Pi}}(H)| < \epsilon$. Since, by Theorem 2.1, for all n great enough, $|\mathbf{Q}^n_{0,t}(H)^{1/n} - \mathbf{\Pi}_{0,t}(H)| < \epsilon$, it follows that $|\mathbf{Q}^n(H)^{1/n} - \hat{\mathbf{\Pi}}(H)| < 3\epsilon$, for all n great enough. (Alternatively, one may let $n \to \infty$ and then let $t \to \infty$ in (3.23).) Finally, $\hat{\mathbf{\Pi}}(x) = e^{-\mathbf{V}(x)}$ by (2.7) and (3.20). □

Remark 3.5. Since $\mathbf{\Pi}_{0,t}(H) \uparrow \hat{\mathbf{\Pi}}(H)$, as $t \to \infty$, one can see by (3.23), that, more generally, geometric ergodicity of $\mathbf{Q}_{0,t}$, as $t \to \infty$, for the metric of total variation and a sample path LDP for $(Q(nt)/n\,, t \in \mathbb{R}_+)$ with $Q(0) = 0$, imply an LDP for $\mathbf{Q}^n$.

A Appendix

Lemma A.1. *Let $(N(t)\,, t \in \mathbb{R}_+)$ be a renewal process with rate λ. Suppose that certain exponential moments of the inter-renewal times are finite. Then, given arbitrary $\epsilon > 0$, there exists $\sigma \in (0, 1)$ such that, for all $t \in \mathbb{R}_+$,*

$$\limsup_{n\to\infty} \mathbf{P}(|\frac{N(nt)}{n} - \lambda t| > \epsilon t)^{1/n} \le \sigma^t\,.$$

Proof. Let $\vartheta_1, \vartheta_2, \ldots$ denote successive inter-renewal times, $\overline{\vartheta}_i = \vartheta_i - 1/\lambda$, and $\alpha > 0$. Then,

$$\mathbf{P}(\frac{N(nt)}{n} - \lambda t > \epsilon t) \le \mathbf{P}\big(\sum_{i=1}^{\lfloor n(\lambda+\epsilon)t\rfloor} \overline{\vartheta}_i \le nt - \frac{\lfloor n(\lambda+\epsilon)t\rfloor}{\lambda}\big)$$
$$\le \exp\big(\lfloor n(\lambda+\epsilon)t\rfloor \ln \mathbf{E} \exp(-\alpha\overline{\vartheta}_1) - \alpha(\frac{\lfloor n(\lambda+\epsilon)t\rfloor}{\lambda} - nt)\big)\,.$$

Hence,

$$\limsup_{n\to\infty} \mathbf{P}\big(\frac{N(nt)}{n} - \lambda t > \epsilon t\big)^{1/(nt)} \le \exp\big((\lambda+\epsilon)\ln \mathbf{E}\exp(-\alpha(\vartheta_1 - \frac{1}{\lambda})) - \alpha\frac{\epsilon}{\lambda}\big)\,.$$

Since $\mathbf{E}\overline{\vartheta}_1 = 0$, the latter righthand side is less than unity for α small enough. The probability $\mathbf{P}(N(nt)/n - \lambda t < -\epsilon t)$ is dealt with similarly. □

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